Fast Fourier Transform in Heston Stochastic Volatility Model

Nikos Matsavelas

1/23/2020

Heston Stochastic Volatility Model :

The stochastic differential equations of Heston's Model are :

$$dX_t = (r - \frac{1}{2}V_t)d_t + \sqrt{V_t}dW_t, X_0 = x$$

$$dV_t = k(\theta - V_t)d_t + \eta\sqrt{V_t}d\tilde{W}_t, V_0 = v$$

Characteristic Function

The main purpose of this exercise is to transform the characteristic function of the log-spot X_t which is provided by:

$$\mathbb{E}_{u,v}\left[e^{(u_1V_1+u_2X_t)}\right] = e^{\{\phi(t,u_1,u_2)+\psi_1(t,u_1,u_2)\cdot v+\psi_2(t,u_1,u_2)\cdot x\}}$$

Riccati Equations

The Riccati system of equations are :

$$\begin{aligned}
F(u_1, u_2) &= \kappa\theta u_1 + ru_2 \\
R(u_1, u_2) &= -\kappa u_1 - \frac{1}{2}u_2 + \frac{1}{2}u_2^2 + \frac{1}{2}\eta^2 u_1^2 + \eta\rho u_1 u_2
\end{aligned}$$

At maturity ($t = 0$), the value of $X_t = \log S_t$ is known, so the expectation in the characteristic function will disappear, and consequently the right-hand side will reduce to simply

$$e^{\{\phi(t,u_1,u_2)+\psi_1(t,u_1,u_2)\cdot v\}}.$$

This implies that the initial conditions at maturity are $\psi_1(0, u_1, u_2) = u_1$ and $\phi(0, u_1, u_2) = 0$.

Finally, when we compute the characteristic function, we use x_t as the log spot price of the underlying asset, and v_t as its unobserved initial variance.

This last quantity is the parameter v_0 must be estimated.

Solution of Riccati Equations

According to G.S.Vasilev (February 2014) paper of Time-dependent Heston model (see attached) the solutions of Riccati ODE are:

(My effort for this exercise was to reproduce this parer and implement it in R)

For notation convenience we change the $\phi = A, \psi_1 = B, \psi_2 = C$,the system of ordinary differential equations ODE becomes:

$$
\begin{aligned}
\frac{\partial A}{\partial t} &= aB, A(0, u_1, u_2) \\
\frac{\partial B}{\partial t} &= \alpha - \beta B + \gamma B^2, B(0, u_1, u_2)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= k\theta \\
\alpha &= -\frac{1}{2}(u_1^2 + iu_2) \\
b &= \kappa - \rho\eta i u_1 u_2 \\
\gamma &= \frac{1}{2}\eta^2
\end{aligned}
$$

Black Scholes Formula Evaluation of Call Option via Fast Fourier Transform

For the Call Option :

$$
\log\left(\frac{S_0 * e^{rT}}{K}\right)
$$

and $a = k\theta$ the parameters become:

$$
\begin{aligned}
b &= k - \rho * \eta\theta \\
\alpha &= -\frac{1}{2}(u_1^2 i + iu_2) \\
\beta &= \kappa - \rho\eta - \rho\eta i u_1 u_2 \\
\gamma &= \frac{1}{2}\eta^2 \\
d &= \sqrt{\beta^2 - 4\alpha\gamma} \\
r^+ &= \frac{(\beta + d)}{2\gamma} \\
r^- &= \frac{(\beta - d)}{2\gamma} \\
g &= \frac{r^-}{r^+} \\
D &= r^- \frac{(1 - e^{(-d \cdot t)})}{(1 - g e^{(-dt)})} \\
C &= k\left(\frac{r^+ t - 2}{\eta^2}\right)\log\left(\frac{1 - g e^{-dt}}{1 - g}\right)
\end{aligned}
$$

and finaly

$$
Re\left[\frac{C\theta + Dv_0 + iux}{i - u}\right]
$$

```
kappa = 1              #mean-reversion speed
theta = 0.09           #long-term average volatility
eta   = 1              #volatility of vol process
rho   = -0.3           #correlation between stock and vol
v0    = 0.09           #initial volatility
r     = 0              #risk-free interest rate
tau   = 1              #time to maturity
```

```r
S0    = 100             #initial share price
K     = c(80,100,120)  #strike price


HestonFastFourier <-
  function(kappa, theta, eta, rho, v0, r, tau, S0, K) {

    #  Calculation  of characteristic function
    #  given the solution of Riccati Equations

    PIntegrand <- function(u, kappa, theta, eta, rho, v0, r, tau, S0, K, j) {
      F <- S0*exp(r*tau)
      x <- log(F/K)
      a <- kappa * theta

      if (j == 1) {
        b <- kappa - rho* eta
        alpha <- - u^2/2 - u/2 * 1i + 1i * u
        beta <- kappa - rho * eta - rho * eta * 1i * u
      } else {
        b <- kappa
        alpha <- - u^2/2 - u/2 * 1i
        beta <- kappa - rho * eta * 1i * u
      }

      gamma <- eta^2/2
      d <- sqrt(beta^2 - 4*alpha*gamma)
      rplus <- (beta + d)/(2*gamma)
      rminus <- (beta - d)/(2*gamma)
      g <- rminus / rplus

      D <- rminus * (1 - exp(-d*tau))/(1-g*exp(-d*tau))
      C <- kappa * (rminus * tau - 2/(eta^2) * log( (1-g*exp(-d*tau))/(1-g) ) )

      top <- exp(C*theta + D*v0 + 1i*u*x)
      bottom <- (1i * u)
      Re(top/bottom)
    }



    #   Inverse of Fast Fourier Transform integral calculation

    P <- function(kappa, theta, eta, rho, v0, r, tau, S0, K, j) {
      value <- integrate(PIntegrand, lower = 0, upper = Inf,
                         kappa, theta, eta, rho, v0, r, tau,
                         S0, K, j, subdivisions=1000)$value
      0.5 + 1/pi * value
    }


    # Black Scholes Evaluation of Call Price
```

```r
    A <- S0*P(kappa, theta, eta, rho, v0, r, tau, S0, K, 1)
    B <- K*exp(-r*tau)*P(kappa, theta, eta, rho, v0, r, tau, S0, K, 0)
    A - B

  }
HestonFastFourier(kappa, theta, eta, rho, v0, r, tau, S0, K=K[1])
```

```
## [1] 23.47143
```

```r
HestonFastFourier(kappa, theta, eta, rho, v0, r, tau, S0, K=K[2])
```

```
## [1] 9.77379
```

```r
HestonFastFourier(kappa, theta, eta, rho, v0, r, tau, S0, K=K[3])
```

```
## [1] 3.494616
```

Comparison of Methods

K values	Euler-MC	Fast-Fourier
K = 80	23.92	23.47
K = 100	10.21	9.77
K =120	3.8041	3.49

Table 1: Comparison of Methods of Black Scholes Formula Calculations on Call parity

Black Scholes Formula Evaluation of Put Option via Fast Fourier Transform

$$
\begin{aligned}
b &= k \\
\alpha &= -\frac{1}{2}(u_1^2 i + i u_2) \\
\beta &= \kappa - \rho \eta i u_1 u_2
\end{aligned}
$$

```r
HestonFastFourier <-
  function(kappa, theta, eta, rho, v0, r, tau, S0, K) {
    PIntegrand <- function(u, kappa, theta, eta, rho, v0, r, tau, S0, K, j) {
      F <- S0*exp(r*tau)
      x <- log(K/F)
      a <- kappa * theta

      if (j == 1) {
        b <- kappa - rho* eta
        alpha <- - u^2/2 - u/2 * 1i + 1i * u
        beta <- kappa - rho * eta - rho * eta * 1i * u
      } else {
        b <- kappa
        alpha <- - u^2/2 - u/2 * 1i
        beta <- kappa - rho * eta * 1i * u
      }

      gamma <- eta^2/2
      d <- sqrt(beta^2 - 4*alpha*gamma)
```

```r
      rplus <- (beta + d)/(2*gamma)
      rminus <- (beta - d)/(2*gamma)
      g <- rminus / rplus

      D <- rminus * (1 - exp(-d*tau))/(1-g*exp(-d*tau))
      C <- kappa * (rminus * tau - 2/(eta^2) * log( (1-g*exp(-d*tau))/(1-g) ) )

      top <- exp(C*theta + D*v0 + 1i*u*x)
      bottom <- (1i * u)
      Re(top/bottom)
    }

    P <- function(kappa, theta, eta, rho, v0, r, tau, S0, K, j) {
      value <- integrate(PIntegrand, lower = 0, upper = Inf,
                         kappa, theta, eta, rho, v0, r, tau,
                         S0, K, j, subdivisions=1000)$value
      0.5 + 1/pi * value
    }

    A <- S0*P(kappa, theta, eta, rho, v0, r, tau, S0, K, 0)
    B <- K*exp(-r*tau)*P(kappa, theta, eta, rho, v0, r, tau, S0, K, 1)
    B-A

  }
HestonFastFourier(kappa, theta, eta, rho, v0, r, tau, S0, K=K[1])
```

```
## [1] 2.233936
```

```r
HestonFastFourier(kappa, theta, eta, rho, v0, r, tau, S0, K=K[2])
```

```
## [1] 9.77379
```

```r
HestonFastFourier(kappa, theta, eta, rho, v0, r, tau, S0, K=K[3])
```

```
## [1] 24.97049
```

K values	Put Option
K = 80	2.23
K = 100	9.77
K = 120	24.97

Table 2: Comparison of Different K values of Black Scholes Formula Put Option