

UNIVERSITY OF THESSALY



NEURO-FUZZY COMPUTING

ECE447

1st Problem Set

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Problem 1

The contour lines of $f(x, y)$ are plotted with the following MATLAB code and are presented in figure 1.

```
function [Z] = plot_contour(start_num , end_num)

    x = linspace(start_num , end_num , 100);
    y = x;
    [X, Y] = meshgrid(x, y);
    Z = X.^2 + 4*X.*Y + Y.^2;
    contour(X, Y, Z, 40);
    xlabel('X');
    ylabel('Y');
end
```

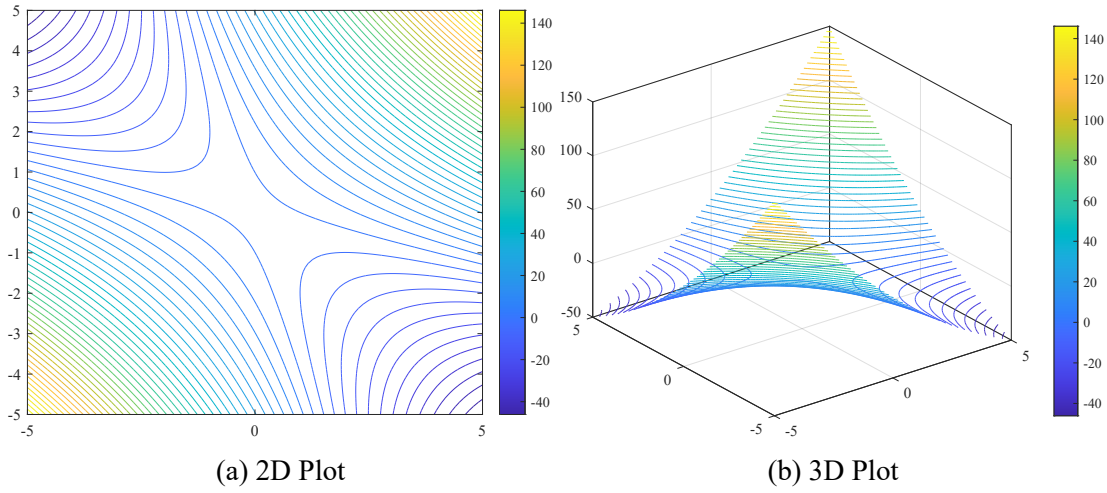


Figure 1: Contour lines of $f(x, y)$

A general formula of a quadratic equation is $f(x, y) = ax^2 + 2bxy + cy^2$. Writing our formula in the previous form, we find that $a = 1$, $b = 2$, $c = 1$. Calculation of the discriminant can help us calculate the location of the function's local minimum/maximum.

$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - f_{xy}^2 = 2 \times 2 - 4^2 = -12 < 0, \quad \text{where} \quad (1)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2, \quad f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 4D = 2 \times 2 - 4^2 = -12 < 0.$$

So, we only have to find the point at which $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are equal to 0. This point will be a saddle point at which the gradients in each orthogonal direction are 0, but this point is neither a local minimum nor a maximum. More precisely:

$$\begin{cases} \frac{\partial f}{\partial x} = 2x + 4y = 0 \\ \frac{\partial f}{\partial y} = 4x + 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (2)$$

Thus, the point $(x, y) = (0, 0)$ is the saddle point mentioned for the given function and this can be justified using the plotted contour lines.

Problem 2

Gradient Descent is an optimization algorithm used to minimize a function by iteratively moving in the direction of steepest descent defined by the negative value of the gradient. It is mostly used to update the parameters of models.

In this exercise we need to execute two iterations of the Gradient Descent to the function

$$f(x_1, x_2) = (x_1 + 2 \cdot x_2 - 7)^2 + (2 \cdot x_1 + x_2 - 5)^2$$

with initial point $x_0 = (-9.5, 9.5)$.

The steps that we must use each time are specific.

FIRST ITERATION $k = 0$

Step1: Calculate the Gradient

$$\nabla f(x_1, x_2) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2 \cdot (x_1 + 2 \cdot x_2 - 7) + 4 \cdot (2 \cdot x_1 + x_2 - 5) \\ 4 \cdot (x_1 + 2 \cdot x_2 - 7) + 2 \cdot (2 \cdot x_1 + x_2 - 5) \end{pmatrix} = \begin{pmatrix} 10 \cdot x_1 + 8 \cdot x_2 - 34 \\ 8 \cdot x_1 + 10 \cdot x_2 - 38 \end{pmatrix}$$

where at the point $x_0 = (-9.5, 9.5)$ we have $(\nabla f(x) = \begin{pmatrix} -53 \\ -19 \end{pmatrix})$

Step2: Calculate the norm $\|f\|$.

$$\|\nabla f(x_1, x_2)\| = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2} = \sqrt{(10 \cdot x_1 + 8 \cdot x_2 - 34)^2 + (8 \cdot x_1 + 10 \cdot x_2 - 38)^2}$$

we want the $\|f(x_1, x_2)\|$ at the point $x_0 = (-9.5, 9.5)$. so by substituting we have $\|f(x_1, x_2)\| = 56.3028$.

Step3: Find the direction of descending at $x_0 = (-9.5, 9.5)$

$$S_0 = -\frac{\nabla f(x_1, x_2)}{\|f\|} = -\frac{1}{56.3028} \cdot \begin{pmatrix} -53 \\ -19 \end{pmatrix} \rightarrow S_0 = \begin{pmatrix} 0.9413 \\ 0.3375 \end{pmatrix}$$

Step4: Find next step and optimal λ

Each step is characterized by $x_{k+1} = x_k + \lambda_k \cdot S_k$

$$x_1 = x_0^T + \lambda_0 \cdot S_0 = \begin{pmatrix} -9.5 \\ 9.5 \end{pmatrix} + \lambda_0 \cdot \begin{pmatrix} 0.9413 \\ 0.3375 \end{pmatrix} \rightarrow x_1 = (-9.5 + 0.9413 \cdot \lambda_0, 9.5 + 0.3375 \cdot \lambda_0) \quad (3)$$

Now we replace to the $f(x_1, x_2)$ function the x_1 from (3).

$$f(x_1) = (-9.5 + 0.9413 \cdot \lambda_0 + 12 + 2 \cdot \lambda_0 \cdot 0.3375)^2 + (9.5 + 2 \cdot \lambda_0 \cdot 0.9413 + 0.3375 \cdot \lambda_0 - 24)^2 \quad (4)$$

In order to find the optimal λ we must follow some more steps.

We should set the derivative of $f(x_1)$ relative to λ_0 equal to zero.

$$\begin{aligned}
\frac{\partial f(x_1)}{\partial \lambda_0} &= 2 \cdot (-9.5 + 0.9413 \cdot \lambda_0 + 12 + 2 \cdot \lambda_0 \cdot 0.3375) \cdot (0.9413 + 2 \cdot 2 \cdot 0.3375) \\
&\quad + 2 \cdot (9.5 + 2 \cdot \lambda_0 \cdot 0.9413 + 0.3375 \cdot \lambda_0 - 24) \cdot (2 \cdot 0.9413 + 0.3375) \\
&= 2 \cdot [1.6163 \cdot (1.6163 \cdot \lambda_0 + 2.5) + 2.2201 \cdot (2.2201 \cdot \lambda_0 - 14.5)] \\
&\rightarrow \frac{\partial f(x_1)}{\partial \lambda_0} = 7.5412 \cdot \lambda_0 - 28.1506
\end{aligned}$$

We set $\frac{\partial f(x_1)}{\partial \lambda_0} = 0$ and because $\lambda \geq 0$,

$$7.5412 \cdot \lambda_0 - 28.1506 = 0 \rightarrow \boxed{\lambda_0 = 3.7329}$$

and from equation 3, by replacing the λ_0 , we get $x_1 = (-5.9862, 10.7599)$

SECOND ITERATION $k = 1$

We will follow all 4 steps again to complete the second iteration for $x_1 = (-5.9862, 10.7599)$.

Step1- Gradient at x_1 : $\nabla f(x_1) = \begin{pmatrix} -7.7828 \\ 21.7094 \end{pmatrix}$

Step2- Norm at x_1 : $(\|\nabla f(x_1)\|) = 23.0623$

Step3- Direction of descending at x_1 : $S_1 = -\frac{\nabla f(x_1)}{\|\nabla f(x_1)\|} = -\frac{1}{23.0623} \cdot \begin{pmatrix} -7.7828 \\ 21.7094 \end{pmatrix} \rightarrow S_1 = \begin{pmatrix} 0.3375 \\ -0.9413 \end{pmatrix}$

Step4- Find next step and optimal λ

$$\begin{aligned}
x_2 &= x_1^T + \lambda_1 \cdot S_1 = \begin{pmatrix} -5.9862 \\ 10.7599 \end{pmatrix} + \lambda_1 \cdot \begin{pmatrix} 0.3375 \\ -0.9413 \end{pmatrix} \\
&\rightarrow x_2 = (-5.9862 + 0.3375 \cdot \lambda_1, 10.7599 - 0.9413 \cdot \lambda_1)
\end{aligned} \tag{5}$$

Again we replace to the $f(x_1, x_2)$ function the x_2 from (5).

$$f(x_2) = (8.5336 + 0.3375 \cdot \lambda_1 - 2 \cdot \lambda_1 \cdot 0.9413)^2 + (-0.9413 \cdot \lambda_1 + 2 \cdot \lambda_1 \cdot 0.3375 - 6.2125)^2 \tag{6}$$

Subsequently, we will find the derivative of $f(x_2)$

$$\begin{aligned}
\frac{\partial f(x_2)}{\partial \lambda_1} &= 2 \cdot [(8.5336 + 0.3375 \cdot \lambda_1 - 2 \cdot \lambda_1 \cdot 0.9413) \cdot (-1.5451) + \\
&\quad \cdot (-0.9413 \cdot \lambda_1 + 2 \cdot \lambda_1 \cdot 0.3375 - 6.2125) \cdot (-0.2663)] \\
&= 2 \cdot [-1.5451 \cdot (-1.5451 \cdot \lambda_1 + 8.5336) + (-0.2663) \cdot (-0.2663 \cdot \lambda_1 - 6.2125)] \\
&\rightarrow \frac{\partial f(x_2)}{\partial \lambda_1} = 2 \cdot [2.4582 \cdot \lambda_1 - 11.5309]
\end{aligned}$$

We set $\frac{\partial f(x_2)}{\partial \lambda_1} = 0$ and because $\lambda \geq 0$,

$$2 \cdot [2.4582 \cdot \lambda_1 - 11.5309] = 0 \rightarrow \boxed{\lambda_1 = 4.6908}$$

and from equation 5, by replacing the λ_1 , we get $x_2 = (-4.4031, 6.3444)$.

The results of the execution of the two iterations are shown in the following table.

| # of iterations | $\mathbf{x}_k + \mathbf{1}$ | λ_k |
|-----------------|-----------------------------|-------------|
| k = 0 | (-5.9862, 10.7599) | 3.7329 |
| k = 1 | (-4.4031, 6.3444) | 4.6908 |

Table 1: Results

Problem 3

Henon map is a dynamic system described by the recursive equation

$$x_{k+1} = 1 - ax_k^2 + bx_{k-1}$$

A lot of dynamic systems transition into chaos as gain or control is increased to a certain point and this system falls into this category. In this problem, we will plot the trajectories of sequences $x_0 \dots x_i$ and describe it. The first parameters are $(a, b) = (0.3, 0.4)$ and the trajectories of the sequences are presented in figure 2.

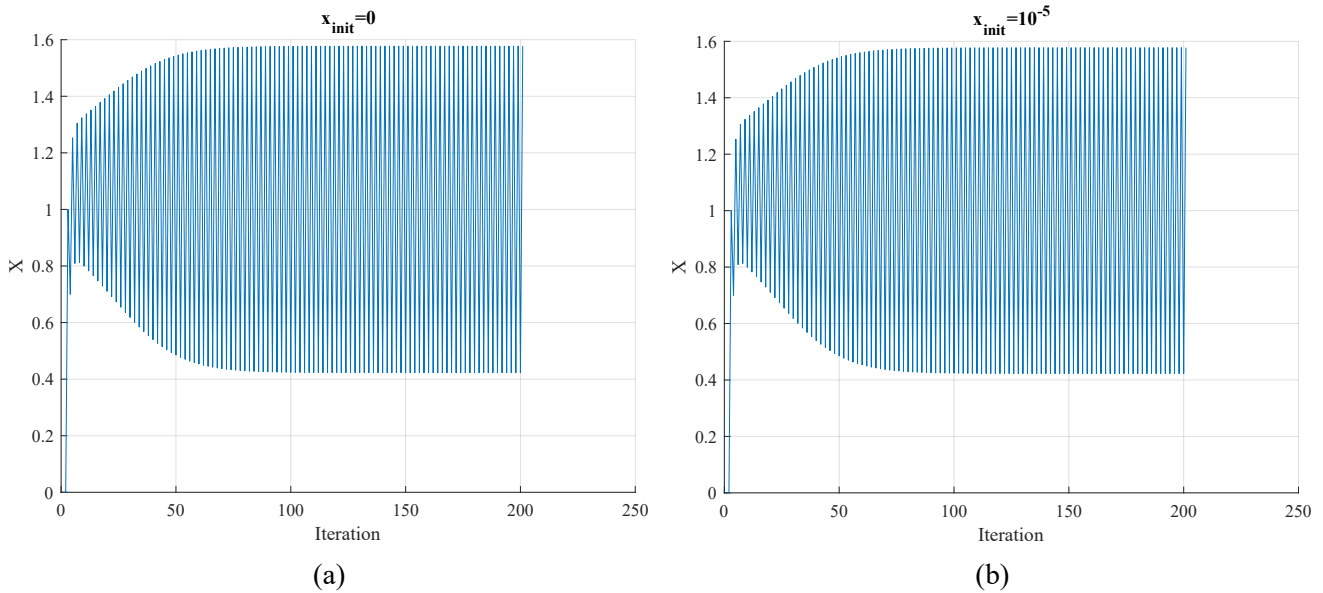


Figure 2: Trajectories of Henon map's sequence with $(a, b) = (0.3, 0.4)$

From the produced plots, this system with parameters $(0.3, 0.4)$ is periodic after converging.

(a) Multiple a values

Figure 3 shows output of the sequence for different values of a and $b = 0$. When $a \leq 0.6$, the output swings for a bit and then stabilizes to a fixed number, with that number decreasing while a approaches 0.6.

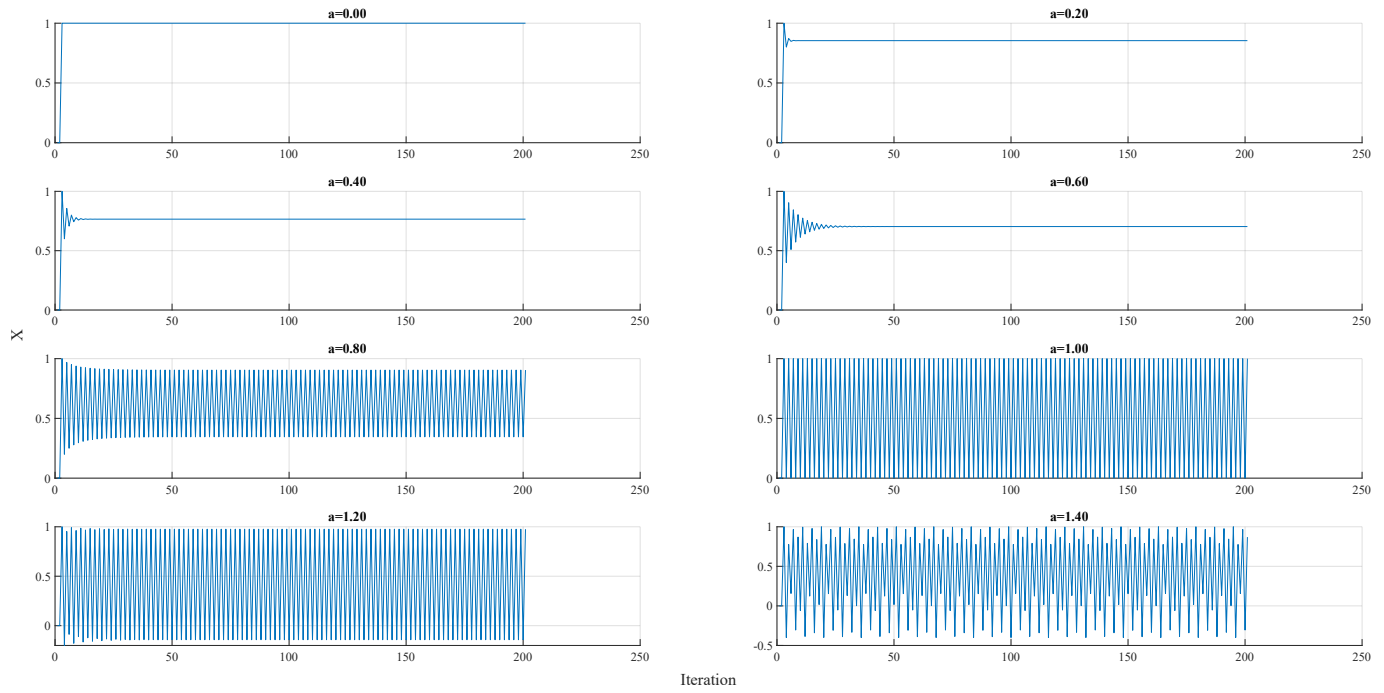


Figure 3: Sequence output for different a and $b = 0$

When $a \geq 0.8$, output starts to oscillate. As a approaches 1, the oscillation's amplitude is getting bigger until it reaches value 1. After a surpasses 1, the oscillation starts to break down, as shown in the last two sub-figures. The greater a , the greater the disturbance on the oscillation thus chaos is created.

(b) Multiple initial x values

Figure 4 shows the trajectory of sequences with different x_{init} and b , with $a = c$, where $c = 0.3$ from previous question. Inspecting this figure, we can understand some things about those parameters and what they affect on the system, as even with $b = 0$, the system performs a damped oscillation for the first 50 (at most) x_i . For a given x_{init} (ie. 0), increasing b increases the amplitude of oscillation as well as the overshoot percentage (if we see it as a system response).

PS: Overshoot % is the percentage of system's maximum value that surpasses its steady-state value over the latter.

(c) $(a,b) = (0.3675, 0.3)$

By setting these values to the system in figure 5, we observe that the sequence starts to oscillate with a very high frequency. As the sequence progresses, the observed oscillation's amplitude dampens and settles around a fixed number, whilst still oscillating.

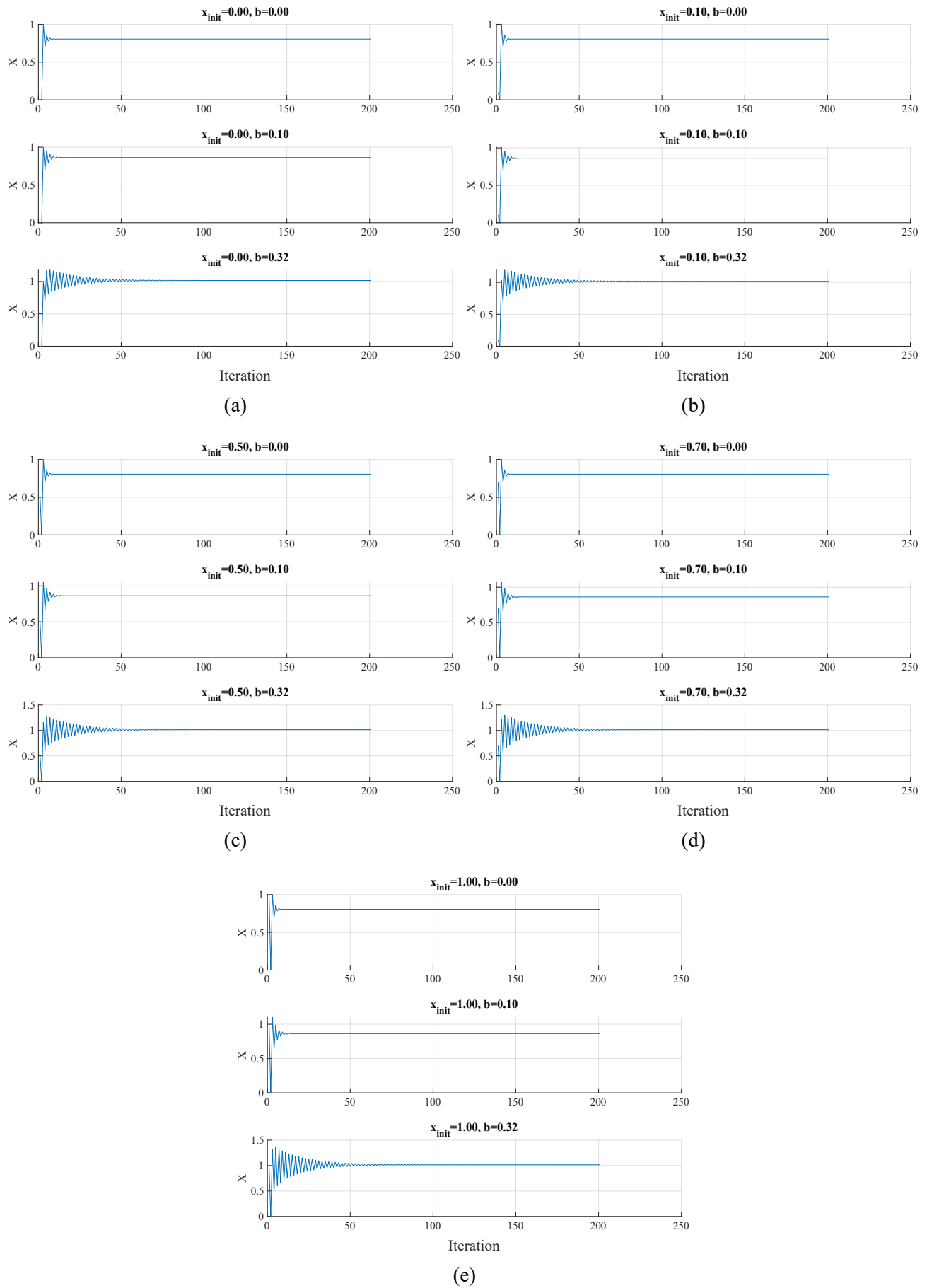


Figure 4

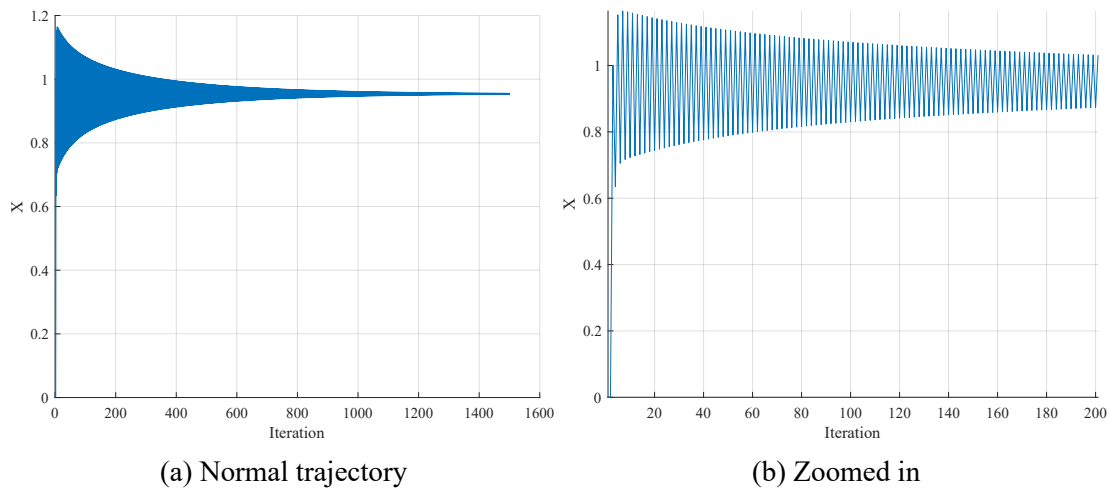


Figure 5: Sequence's trajectory with (0.3675, 0.3)

(d) Multiple a values

By changing a parameter, we can recognize and characterize the output of the system. By observing the figure 6, we can understand that by increasing the parameter's value the oscillation duration increases. At first, (when a is smaller to b), the system oscillates for a bit and then dampens around a fixed number. But the more a approaches b , the longer the oscillation lasts until the moment when $a = b$. From this value and beyond, the oscillation continues for all x_i . As a increases beyond b , the oscillations begin to deteriorate, and beyond a certain point of a , chaos ensues.

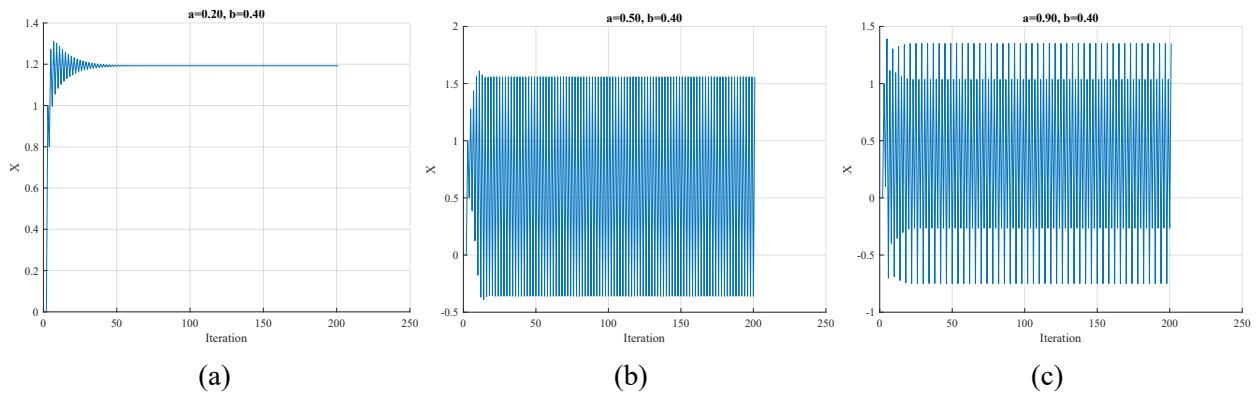


Figure 6

Problem 4

In this problem, we will express the derivatives in respect to the matching activation function.

(a) LogSig

This activation function is expressed as

$$S(x) = \frac{1}{1 + e^{-x}}$$

Multiplying itself with $(1 + e^{-x})$ gives

$$(1 + e^{-x}) S(x) = 1 \Leftrightarrow e^{-x} = \frac{1}{S(x)} - 1$$

So, activation function's derivative will be

$$\begin{aligned} \frac{dS}{dx} &= \frac{d\left((1 + e^{-x})^{-1}\right)}{dx} = (1 + e^{-x})^{-2} e^{-x} = S^2(x) \left(\frac{1}{S(x)} - 1\right) \\ &= S(x) - S^2(x) = S(x) (1 - S(x)) \end{aligned}$$

(b) TanSig

Activation function is

$$S(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Its derivative is

$$\begin{aligned} \frac{dS}{dx} &= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = 1 - S^2(x) \end{aligned} \quad (7)$$

(c) Swish

Activation function is

$$S(x) = \frac{x}{1 + e^{-x}}$$

The derivative in respect to x is

$$\frac{dS}{dx} = \frac{1 + e^{-x} + xe^{-x}}{(1 + e^{-x})^2} = \frac{1 + e^{-x}}{(1 + e^{-x})^2} + \frac{xe^{-x}}{(1 + e^{-x})^2} = \frac{1}{1 + e^{-x}} + x \frac{e^{-x}}{(1 + e^{-x})^2}$$

Rewriting the function gives us

$$\frac{S(x)}{x} = \frac{1}{1 + e^{-x}} \quad \text{and} \quad e^{-x} = \frac{x - S(x)}{S(x)}$$

So, continuing with the derivative:

$$\begin{aligned} \frac{dS}{dx} &= \frac{1}{1 + e^{-x}} + x \frac{e^{-x}}{(1 + e^{-x})^2} = \frac{S(x)}{x} + \frac{e^{-x}}{1 + e^{-x}} \frac{x}{1 + e^{-x}} = \frac{S(x)}{x} + S(x) \frac{x}{1 + e^{-x}} = \\ &= \frac{S(x)}{x} + S(x) \frac{S(x)}{x} e^{-x} = \frac{S(x)}{x} (1 + S(x) e^{-x}) = \frac{S(x)}{x} \left(1 + S(x) \frac{x - S(x)}{S(x)}\right) = \\ &= \frac{S(x)}{x} (1 + x - S(x)) = \frac{S(x)}{x} + S(x) - \frac{S^2(x)}{x} \end{aligned}$$

Let $\sigma = \frac{1}{1 + e^{-x}}$, thus

$$\frac{dS}{dx} = \frac{S(x)}{x} + S(x) - \frac{S^2(x)}{x} = \sigma + S(x) - \sigma S(x) = S(x) + \sigma (1 - S(x)) \quad (8)$$

(d) Custom tanh

Activation function is

$$S(x) = x \tanh(\ln(1 + e^x)) = x \tanh(g(x)), \text{ where } g(x) = \ln(1 + e^x).$$

By calculating the derivative of this function as is, we get

$$\begin{aligned} \frac{dS}{dx} &= \tanh(g(x)) + x \tanh'(g(x)) \frac{dg}{dx} = \tanh(g(x)) + x \tanh'(g(x)) \frac{1}{1 + e^{-x}} = \\ &= \tanh(g(x)) + \tanh'(g(x)) x \frac{\text{Swish}(x)}{x} = \frac{S(x)}{x} + \tanh'(g(x)) \text{Swish}(x) = \\ &= \frac{S(x)}{x} + (1 - \tanh^2(g(x))) \text{Swish}(x) = \frac{S(x)}{x} + \text{Swish}(x) - \tanh^2(g(x)) \text{Swish}(x) \end{aligned}$$

So, $\phi(x, S)$ is

$$\frac{S(x)}{x} + \text{Swish}(x) - \frac{S^2(x)}{x^2} \text{Swish}(x), \quad \text{where} \quad (9)$$

$S(x)$ is the activation function and $\text{Swish}(x)$ is the Swish activation function from before

Problem 5

The given neural network consists of two layers and of three neurons. On the first one, activation function is logsig or swish and on the second one is purelin. On the left side of figure 7 we see the sketches for all outputs when the activation function is logsig and on the right side all outputs with activation function being swish.

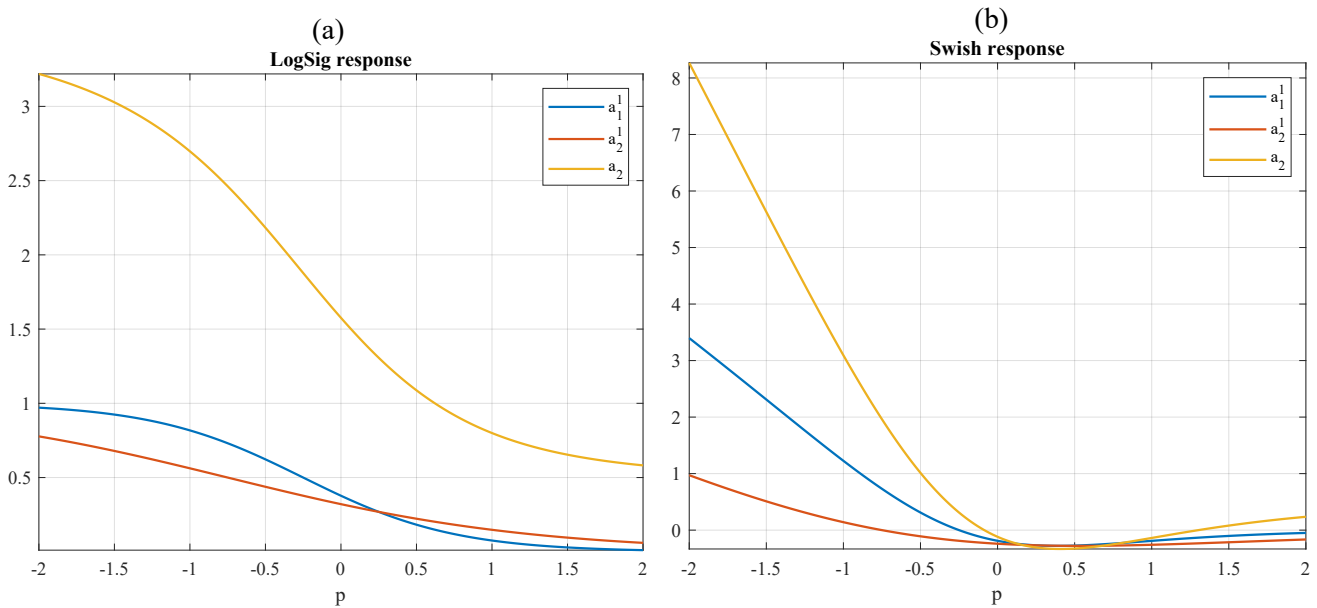


Figure 7: Responses of different outputs of the neural network

Problem 6

We are interested in comparing the gradient descent optimizer for different values of the batch size

$n_b = 1, n, \phi(n) \ll n$.

These observations can be explained by understanding the trade-offs between the different variants of gradient descent:

- **Stochastic Gradient Descent** ($n_b = 1$): SGD updates the parameters for each training example, which introduces a lot of noise into the learning process. This noise can help to avoid shallow local minima, which can lead to better convergence to the global minimum. However, the learning process can be slow due to the high variance in the updates.
- **Batch Gradient Descent** ($n_b = n$): GD updates the parameters based on all training examples, reducing noise in the learning process and giving a clear direction to the minimum. However, it can get stuck in flat, non-optimal minima and converges slower than SGD when the data set is large.
- **Mini-Batch Gradient Descent** ($n_b = \phi(n) \ll n$): BGD is a compromise between SGD and GD. It updates the parameters based on a subset of the training examples, which reduces noise compared to SGD and enables faster calculation compared to GD. This can lead to faster convergence and a smaller value of the objective function.

The observation that increasing the batch size from $n_b = 1$ to $n_b = \frac{n}{10}$ improves the results can be explained by the fact that BGD can benefit from the advantages of both SGD and GD. It can avoid flat minima caused by minibatch noise (like SGD) and it can calculate the gradient more accurately than SGD by averaging over a minibatch.

The observation that convergence slows down when the batch size is increased from $\frac{n}{10}$ to n can be explained by the fact that GD can get stuck in shallow minima and may not explore the parameter space as effectively as BGD or SGD. In addition, GD can be computationally intensive with large data sets, which can slow down the learning process.

Problem 7

(a) Question A

Let a generic activation function be

$$S(x) = \begin{cases} x^k, & x > L \\ x^k \frac{(L+x)^m}{(L+x)^m + (L-x)^m}, & |x| \leq L \\ 0, & x < -L \end{cases} \quad (10)$$

This function has some parameters k, L, m called hyperparameters. Using them, this function can approximate others.

(a). i Swish

In order for our activation function to approximate Swish, we have to see the latter first. Swish's function is

$$S(x) = \frac{x}{1 + e^{-x}} = x \frac{e^x}{1 + e^x}$$

and is plotted in figure 8. In order to determine the parameters, we have to take a look in the plotted function. When $x > 0$, Swish seems to be almost $y = x$, thus $k = 1$. If we look at $x < 0$, we can see that Swish curves until it settles back to 0. That point x_1 where $Swish(x_1) = 0$ is parameter L . After trying multiple values, we opted with $2e$ as this gives the smaller overall error in that region. After filling k, L , only parameter m is left to be found. In general, x^e is a good approximation of e^x , but an approximation only. Just trying $m = e$ and plotting the created function gets us the plot in figure 8 which is really close to Swish. Finally, the activation function can be written

$$S(x) = \begin{cases} x, & x > 2e \\ x \frac{(2e + x)^e}{(2e + x)^e + (2e - x)^e}, & |x| \leq 2e \\ 0, & x < -2e \end{cases}$$

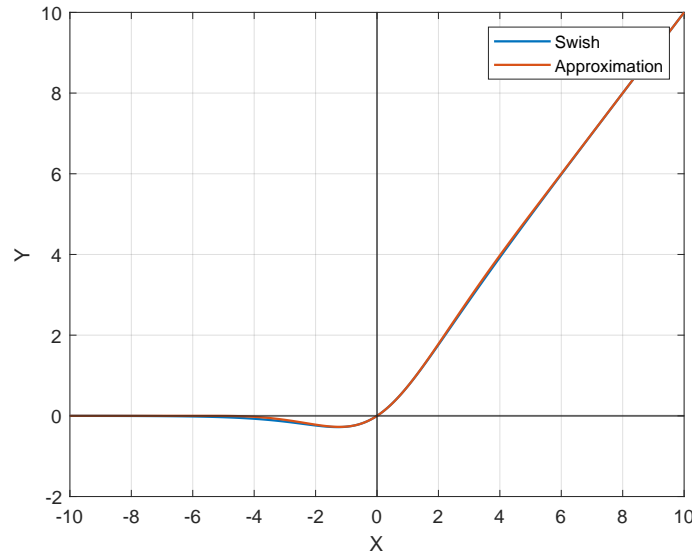


Figure 8: Swish activation function and its approximation.

(a). ii Sigmoid

Sigmoid's function is

$$S(x) = \frac{1}{1 + e^{-x}} = \frac{e^x}{1 + e^x}$$

Just from this type, we understand that is $\frac{Swish(x)}{x}$. So, selecting parameters is easier this time as m and L will not change. Because of the previous equation, parameter k must be decreased by 1, thus $k = 0$. Figure 9 shows both the sigmoid and its approximation.

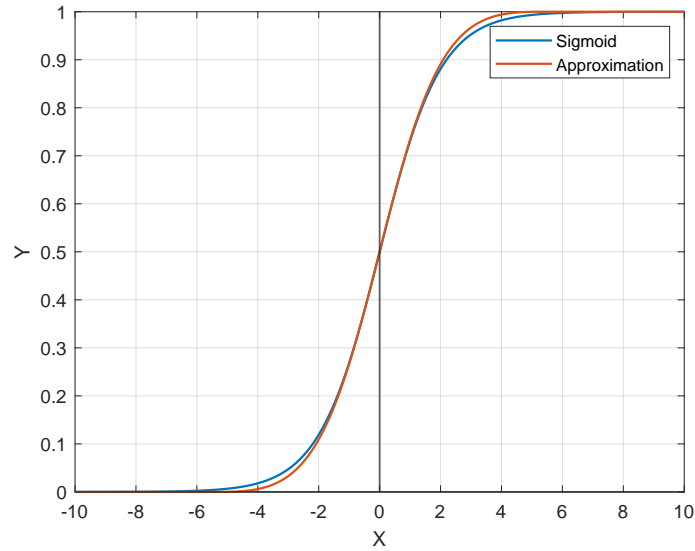


Figure 9: Sigmoid activation function and its approximation.

(a). iii ReLU

ReLU is defined as

$$ReLU(x) = \max(0, x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

From this, we can already determine parameter k to be $k = 1$, as we want $x^k = x \Rightarrow k = 1$. Setting $L = 0$ ensures that the transition point between the linear and non-linear parts of the function occurs at $x = 0$, which is consistent with the behavior of ReLU. Also, setting $m = 0$ means that the denominator term becomes 1 for both the linear and non-linear parts of the function, simplifying the expression. The plotted functions are presented in figure 10

With parameters those mentioned above, $S(x)$ is the following

$$S(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

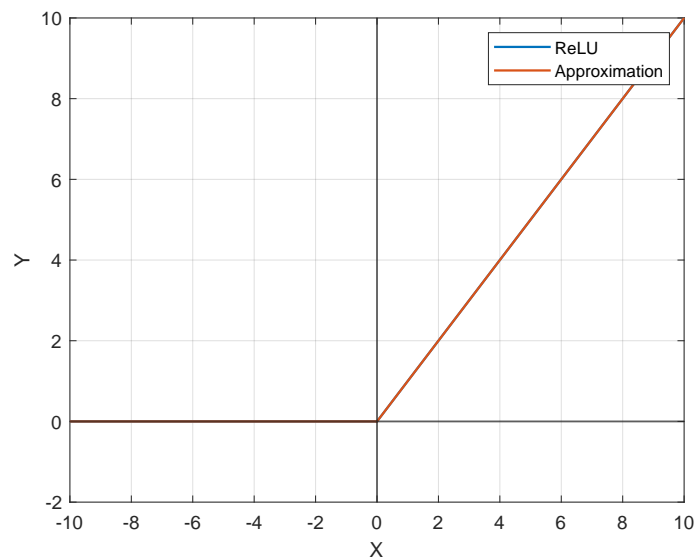


Figure 10: ReLU activation function and its approximation.

(a). iv ELU

ELU's equation is

$$S(x) = \begin{cases} x, & x \geq 0 \\ \alpha(e^x - 1), & x < 0 \end{cases}, \quad \text{with } \alpha = (0.1 \dots 0.3)$$

From the equation above we understand that this activation function is almost the same as ReLU, with only difference being treatment of negative numbers. So, we can choose the same parameters from ReLU and still be a fairly good approximation to ELU, thus $k = 1$, $L = 0$, $m = 0$. Both ELU and it's approximation are presented in figure 11.

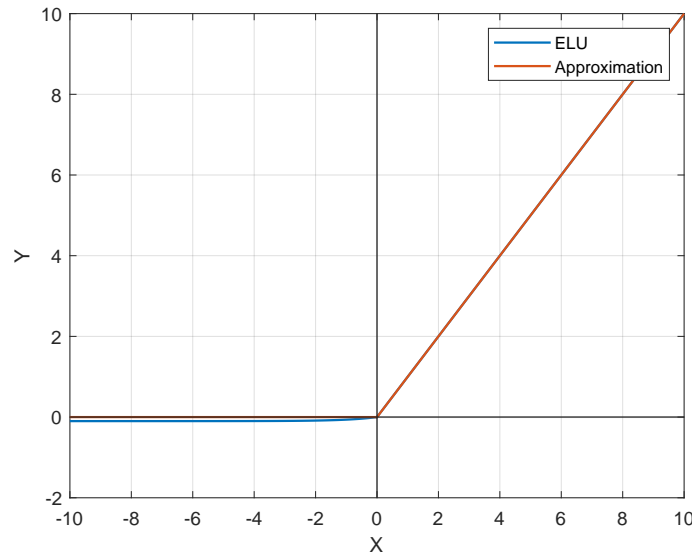


Figure 11: ELU activation function and its approximation

(b) Question B

In order to make our calculations a bit easier, we set $g(x) = L + x$ and $i(x) = L - x$, with $\frac{dg(x)}{dx} = 1$, $\frac{di(x)}{dx} = -1$, $\frac{dg(x)}{dL} = 1$ and $\frac{di(x)}{dL} = 1$.

(b). i Derivative with respect to x

In $x \in [-L, L]$, the derivative is calculated as

$$\begin{aligned} \frac{dS(x)}{dx} &= kx^{k-1} \frac{g^m(x)}{(g(x) + i(x))^m} + \\ &+ x^k \frac{mg^{m-1}(x)(g^m(x) + i^m(x)) - g^m(x)(mg^{m-1}(x) - mi^{m-1}(x))}{(g^m(x) + i^m(x))^2} = \\ &= kx^{k-1} \frac{g^m(x)}{(g(x) + i(x))^m} + x^k \frac{mg^m(x)i^m(x) \left(\frac{1}{g(x)} + \frac{1}{i(x)} \right)}{(g^m(x) + i^m(x))^2} \end{aligned}$$

So, activation's function derivative is

$$\frac{dS(x)}{dx} = \begin{cases} kx^{k-1}, & x > L \\ kx^{k-1} \frac{g^m(x)}{(g(x) + i(x))^m} + x^k \frac{mg^m(x)i^m(x) \left(\frac{1}{g(x)} + \frac{1}{i(x)} \right)}{(g^m(x) + i(x))^2}, & |x| \leq L \\ 0, & x < L \end{cases} \quad (11)$$

(b). ii Derivative with respect to k

We know from algebra that $\frac{d}{dx}(a^x) = a^x \ln a$. So

$$\frac{dS}{dk} = x^k \ln(k) \frac{(L+x)^m}{(L+x)^m + (L-x)^m}, \quad \text{for } x \in [-L \dots L]$$

Thus, the derivative is

$$\frac{dS(x)}{dk} = \begin{cases} x^k \ln(k), & x > L \\ x^k \ln(k) \frac{g^m(x)}{g^m(x) + i^m(x)}, & |x| \leq L \\ 0, & x < L \end{cases} \quad (12)$$

(b). iii Derivative with respect to L

This derivative is similar to that in respect to x , so calculating it gives us

$$\frac{dS}{dL} = x^k \frac{mg^{m-1}(x)(g^m(x) + i^m(x)) - g^m(x)(mg^{m-1}(x) + mi^{m-1}(x))}{(g^m(x) + i^m(x))^2} = 0$$

So,

$$\frac{dS(x)}{dL} = 0, \quad \text{for all different L} \quad (13)$$

(b). iv Derivative with respect to m

$$\begin{aligned} \frac{dS}{dm} &= x^k \frac{g^{2m}(x) \ln(g(x)) + g^m(x)i^m(x) \ln(g(x)) - (g^{2m}(x) \ln(g(x)) + g^m(x)i^m(x) \ln(i(x)))}{(g^m(x) + i^m(x))^2} = \\ &= \frac{g^m(x)i^m(x) \ln(g(x)) - g^m(x)i^m(x) \ln(i(x))}{(g^m(x) + i^m(x))^2} = \frac{g^m(x)i^m(x) \ln\left(\frac{g(x)}{i(x)}\right)}{(g^m(x) + i^m(x))^2} \end{aligned}$$

Thus, the derivative of $S(x)$ with respect to m is

$$\frac{dS}{dm} = \begin{cases} 0, & x > L \\ \frac{g^m(x)i^m(x) \ln\left(\frac{g(x)}{i(x)}\right)}{(g^m(x) + i^m(x))^2}, & |x| \leq L \\ 0, & x < L \end{cases} \quad (14)$$

Problem 8

ADALINE is a single-layer artificial neural network that can learn and adapt to non-linear relationships between inputs and outputs.

It consists of a single neuron with a linear activation function. Each input of the neuron has a corresponding weight, which is adapted during training to minimize the error between the network's output and the desired output. To adjust the weights the algorithm uses the learning rule α .

Suppose that we have the following three reference patterns and their targets:

$$\left\{ p_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, t_1 = [26] \right\} \quad \left\{ p_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, t_2 = [26] \right\} \quad \left\{ p_3 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, t_3 = [-26] \right\}$$

The probability of vector p_1 is $P_1 = 0.20$, the probability of vector p_2 is $P_2 = 0.70$, and the probability of vector p_3 is $P_3 = 0.10$.

(a) Question a

The number of inputs to an ADALINE network for each neural network is determined by the dimensionality of our data, not by the number of patterns we have. In our case, each pattern is a 2-dimensional vector. Therefore, our ADALINE network has two inputs, one for each dimension.

In an ADALINE network, the number of weights is equal to the number of inputs. In our case, we have two inputs. Thus, our neural network has two weights, one for each dimension of the input.

From theory, the output of an ADALINE network is $a = \text{purelin}(W_p + b)$.

So, the network diagram for the given ADALINE network with no bias that will be trained with these patterns is shown in figure 12.

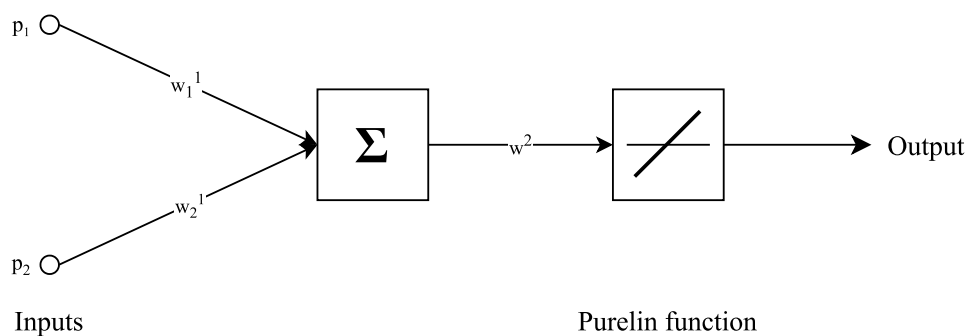


Figure 12: ADALINE neural network architecture

(b) Question b

In order to sketch the contour plot of the mean square error performance index, we first must calculate the various terms of the quadratic function.

Recall that, $F(x) = c - 2 \cdot x^T \cdot h + x^T \cdot R \cdot x$ where,

- c: A scalar constant term. It shifts the function up or down along the y-axis.
- R: Correlation matrix of the input data. It determines the curvature of the function
- h: The cross-correlation between the input data and its associated target. It determines the slope of the function.
- x: The vector of variables (or weights).

These parameters define the shape of the quadratic function. So, we must calculate c, h, R in relation to

$$x = \begin{bmatrix} W_{11} & W_{12} \end{bmatrix}$$

The calculations:

$$c = E[t^2] = t_1^2 \cdot p_1 + t_2^2 \cdot p_2 + t_3^2 \cdot p_3 = 26^2 \cdot 0.2 + 26^2 \cdot 0.7 + (-26)^2 \cdot 0.1 \\ \rightarrow c = 676$$

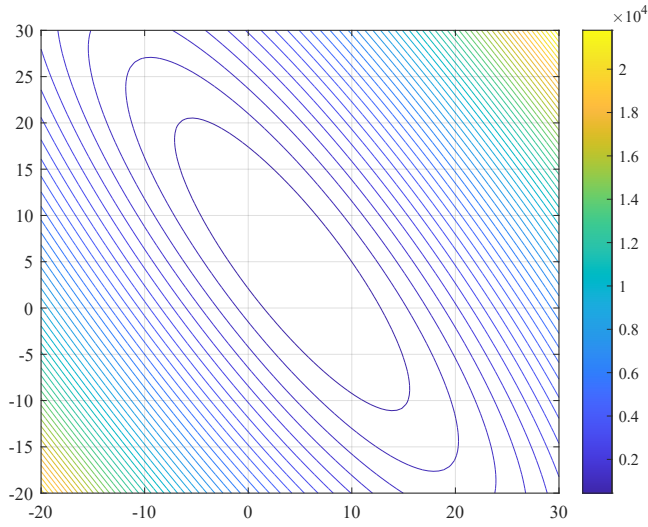
$$h = E[t \cdot p] = P_1 \cdot t_1 \cdot p_1 + P_2 \cdot t_2 \cdot p_2 + P_3 \cdot t_3 \cdot p_3 = \\ 0.2 \cdot 26 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 0.7 \cdot 26 \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 0.1 \cdot (-26) \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix} \\ \rightarrow h = \begin{bmatrix} 88.4 \\ 62.4 \end{bmatrix}$$

$$R = E[p \cdot p^T] = P_1 \cdot p_1 \cdot p_1^T + P_2 \cdot p_2 \cdot p_2^T + P_3 \cdot p_3 \cdot p_3^T \\ = 0.2 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix}^T + 0.7 \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}^T + 0.1 \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix}^T \\ \rightarrow R = \begin{bmatrix} 12.4 & 7.6 \\ 7.6 & 6.4 \end{bmatrix}$$

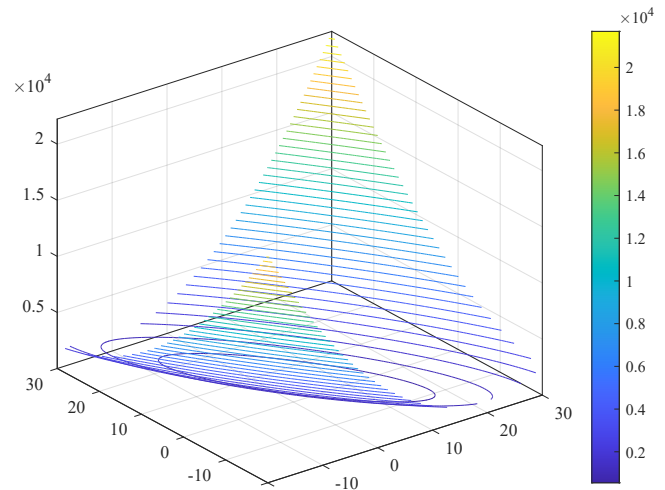
In conclusion the Mean Square Error (MSE) performance index is:

$$F(x) = 676 - 2 \cdot \begin{bmatrix} W_{11} & W_{12} \end{bmatrix} \cdot \begin{bmatrix} 88.4 \\ 62.4 \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} \end{bmatrix} \cdot \begin{bmatrix} 12.4 & 7.6 \\ 7.6 & 6.4 \end{bmatrix} \cdot \begin{bmatrix} W_{11} \\ W_{12} \end{bmatrix} \rightarrow \\ F(x) = 676 - 176.8 \cdot W_{11} - 124.8 \cdot W_{12} + 12.4 \cdot W_{11}^2 + 15.2 \cdot W_{11} \cdot W_{12} + 6.4 \cdot W_{12}^2$$

By inserting this function into Matlab we get the 2D and 3D contour plots of the MSE index as shown in figures 13a and 13b



(a) 2D plot of MSE index



(b) 3D plot of MSE index

(c) Question c

A decision boundary is a hypersurface that separates different classes in a classification problem. The optimal decision boundary is the one that minimizes a certain transfer function. Here it is a line that can be described by the equation

$$f(x) = W^T \cdot x^*,$$

where $W = \begin{bmatrix} W_{11} & W_{12} \end{bmatrix}$ and x^* the minimum square error
 x^* is the strong minimum, a stationary point that is indeed the center of the cycle of Figure 13a.

So, in order to find the optimal decision boundary we follow the mentioned steps:

- $f(x) = W^T \cdot x^*$
- Set $f(x) = 0$
- Calculate $x^* = R^{-1} \cdot h$
- Replace the value in $f(x)$

By calculating $x^* = R^{-1} \cdot h$ we get that $x^* = \begin{bmatrix} 4.2370 \\ 4.7185 \end{bmatrix}$

and the optimal decision boundary is $f(W_{11}, W_{12}) = 4.2370 \cdot W_{11} + 4.7185 \cdot W_{12}$

With the help of Matlab we get the following figure 14

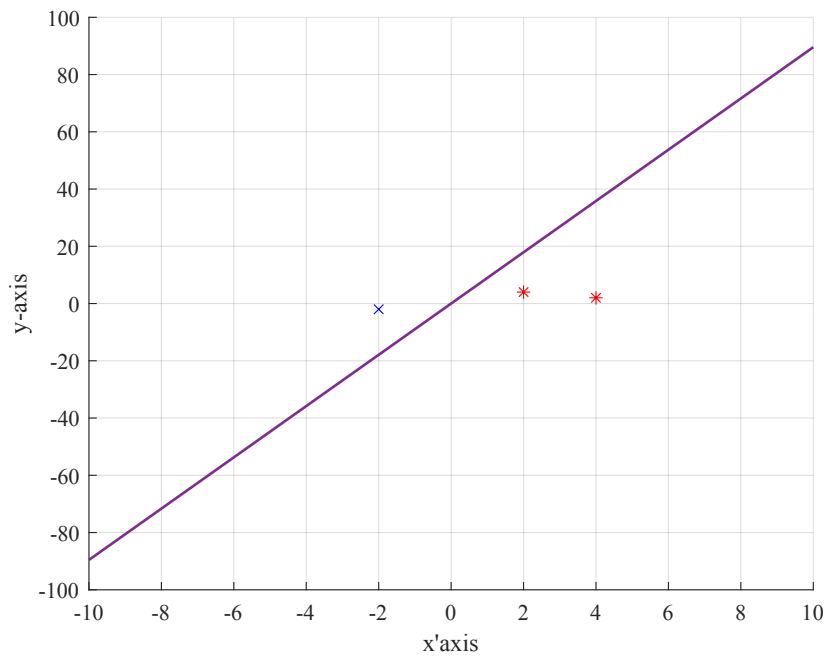


Figure 14: Optimal Decision Boundary line and the patterns

From the figure 14 it is obvious that the optimal decision boundary separates and correctly classifies the patterns. For a binary classification problem, as in our case, all points (*) on one side of the decision boundary are predicted to belong to one class, while points (x) on the other side are predicted to belong to the other class.

(d) Question d

The maximum stable learning rate for the LMS algorithm can be calculated by $lr_{max} = \frac{1}{\lambda_{max}}$, where λ_{max} is the largest eigenvalue of the correlation matrix R. With the help of matlab i calculated and found that the eigenvalue matrix of R is

$$eigenvalue = \begin{bmatrix} 1.2293 \\ 17.5707 \end{bmatrix}$$

Thus the largest eigenvalue is $\lambda_{max} = 17.5707$ and in conclusion the maximum learning rate is $lr_{max} = \frac{1}{17.5707} \rightarrow \boxed{lr_{max} = 0.0569}$

As we have already mentioned, the learning rate is related to the correlation matrix R. The matrix R depends only on the properties of the input data $-R = E[p \cdot p^T]$ - and not on the output. The learning rate does not depend on the target values but on the properties of the input data.

This means that, changing the target values could potentially affect the convergence of the LMS algorithm and the final solution. However, the maximum stable learning rate itself, as a parameter of the algorithm, would not be directly affected by changes in the target values.

(e) Question e

To perform one iteration of the LMS algorithm we will need a learning rate, the initial weight values and an input. It is given that

- Input data: pattern $p_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- $w(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Learning rate: $lr = 0.05$

For a single-layer ADALINE network we recall the rule

$$W_{k+1} = W_k + 2 \cdot lr \cdot e_k \cdot p_k^T$$

where e_k is the error on that step.

To begin with, we must follow these steps for the iterations. For $k=0$ we get:

- Step1: Find output a_k
 $a_0 = \text{purelin}(W(0) \cdot p_1) = \text{purelin}(w(0)) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$
- Step2: Calculate error e_k
 $e_0 = t_0 - a_0 = t_1 - a_0 = 26 - 0 = 26$
- Step3: Calculate weight W_{k+1}
 $W_1 = W_0 + 2 \cdot lr \cdot e_0 \cdot p_0^T = w(0) + 2 \cdot 0.05 \cdot e_0 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5.2000 & 10.4000 \end{bmatrix}$

Problem 9

Again, suppose that we have the following two reference patterns and their targets:

$$\left\{ p_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, t_1 = [-1] \right\} \quad \left\{ p_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, t_2 = [1] \right\}$$

The probabilities of vectors p_1 and p_2 are equiprobable, so that means $P_1 = P_2 = 0.5$.

(a) Question a

In order to sketch the contour plot of the Mean Square Error performance, we ought to calculate the arguments of the quadratic function $F(x)$. As we did on Problem8-Question (b), with the help of Matlab we calculate the c,R,h and we get the following results

$$\bullet R = \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}$$

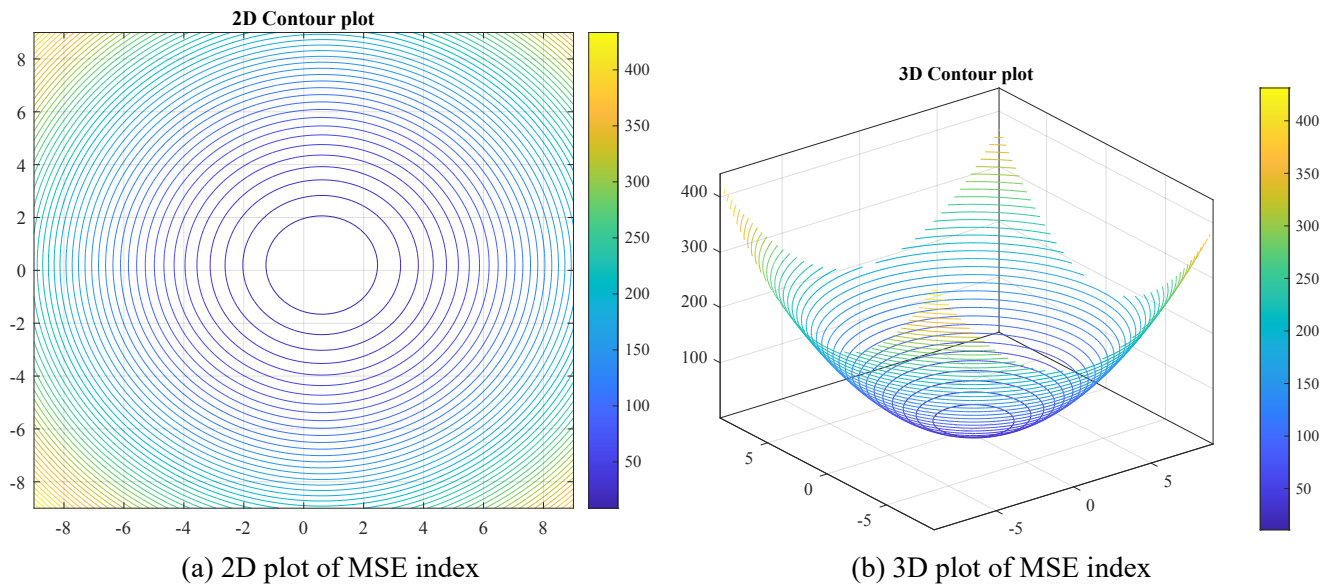
- $h = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$

- $c = 1$

and the MSE function is: $F(x) = 1 - 3 \cdot w_{11} - w_{12} + 2.5 \cdot w_{11}^2 + 2.5 \cdot w_{12}^2$

From the form of the equation we can understand that it is a circle with a center of $x^* = R^{-1} \cdot h$.

By the Matlab code that i provided to calculate the $F(x)$, we get the 2D and 3D contour plots of the MSE index as shown in figures 15a and 15b



(b) Question b

Working with the same method as i did in Problem8-Question (c) but this time with the help of Matlab code, we got the following sketch about the optimal decision boundary.

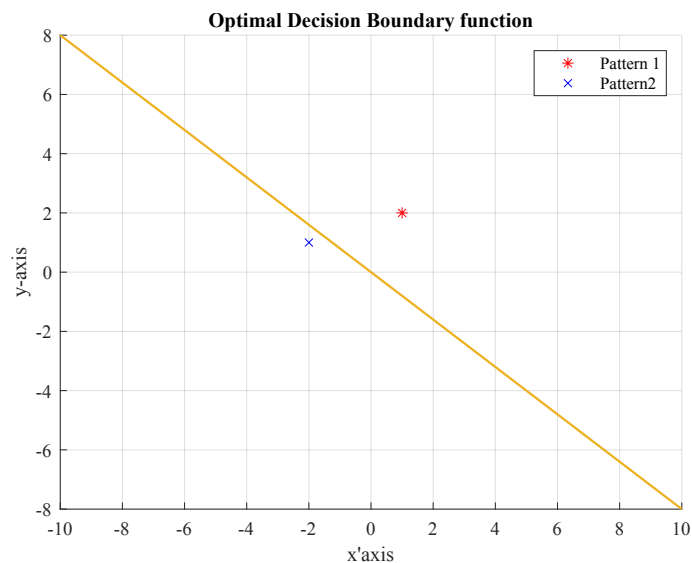


Figure 16: Optimal decision Boundary plot

We can verify that indeed the optimal decision boundary separates and correctly classifies the patterns.

(c) Question c

If we want to sketch the trajectory of the LMS algorithm on our contour plot, we follow the methodology of Problem8- Question(e), but for the number of iterations $N = 100$.

Here our learning rate is $lr = 0.025$ and the initial weight is $w(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Using Matlab, we exported this trajectory of the LMS algorithm to our 2D contour plot of the mean square error performance index.

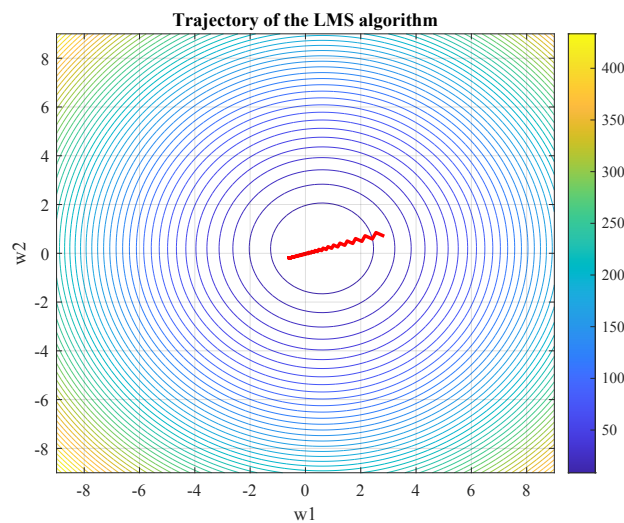


Figure 17: Trajectory of the LMS algorithm

Problem 10

(a) Question A

The patterns that we want to separate are plotted in figure 18.

We can clearly see that there can be a straight line that can separate the two classes, thus an ADALINE neural network can work in classification for this system.

(b) Question B

The designed ADALINE neural network will be of the following architecture

- **Input Layer:** Since the patterns are two-dimensional (each pattern has two values), the input layer will have two nodes.
- **Output Layer:** The output layer will have one node. This is because the task is a binary classification. The output node will use a linear activation function, as is standard in ADALINE networks.
- **Weights and Bias:** There will be two weights (one for each input node) and one bias. The weights and bias are parameters that the network will learn during the training process.

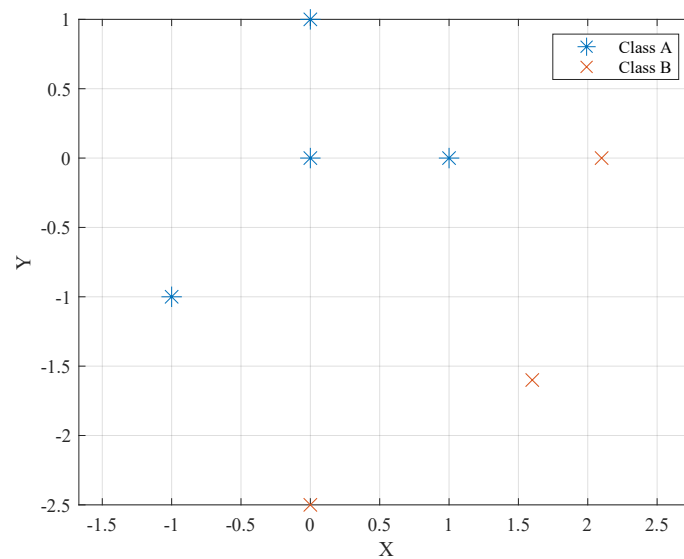


Figure 18: Plot of patterns

- **Learning Rule:** The network will use the LMS learning rule (*Least Mean Square*) to update the weights and bias. This rule minimizes the mean square error between the network's output and the target output.

This architecture described beforehand is shown in figure 19.

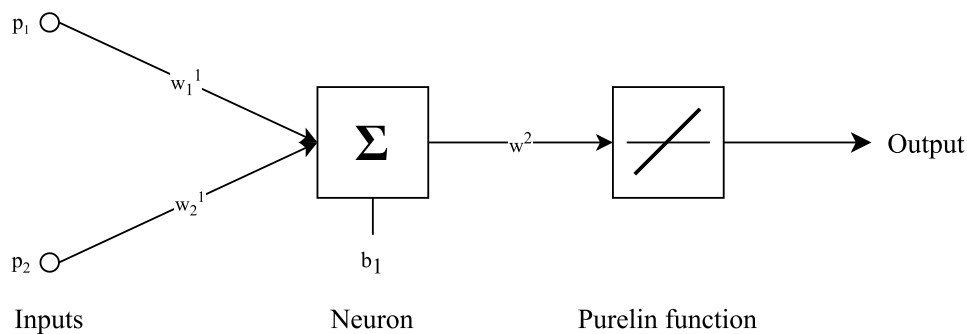


Figure 19: ADALINE neural network architecture

(c) Question c

The ADALINE neural network mentioned above was coded in MATLAB. During training, we plotted its weights and bias in order to check their progression. Maximum iteration value was defined in code to be 10^4 and minimum error to end train and consider the solutions converged is *epsilon* of the machine, specifically $eps = 2.2204 \cdot 10^{-16}$.

After converging, the final weights and bias are presented in table 2 below. Also, the decision boundary lines with the patterns are shown in figure 20c.

| Weight 1 | Weight 2 | Bias |
|----------|----------|--------|
| -0.6574 | 0.6052 | 0.8407 |

Table 2: Final table of weights and bias

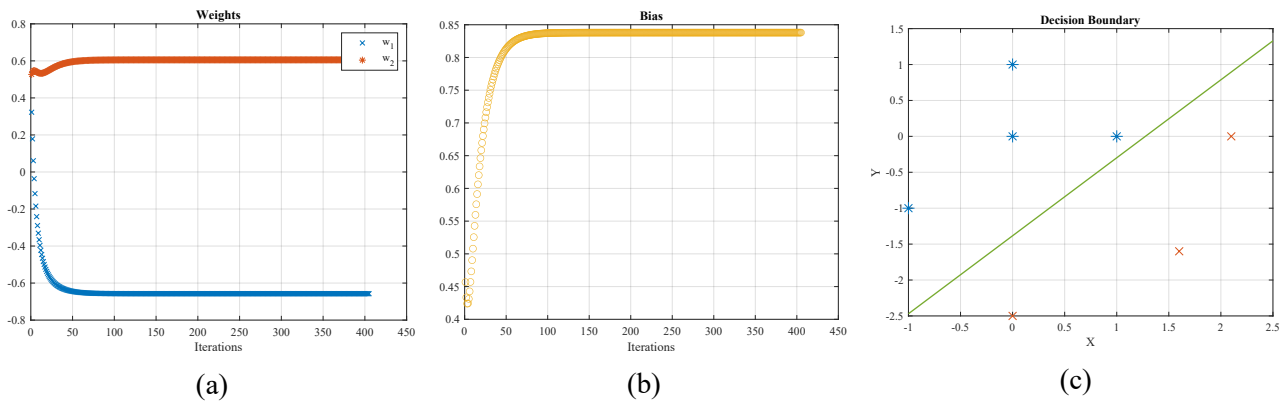


Figure 20: Plots of weights, biases and decision boundary

Problem 11

Fuzzy logic is a type of logic that deals with vague, imprecise, or uncertain information. It is based on the concept of fuzzy sets, which are sets that can have any degree of membership between 0 and 1. The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. This means that an element can be a member of a fuzzy set to some degree, rather than all or nothing.

The uniqueness of fuzzy logic is that fuzzy logic can handle imprecise and uncertain information, which makes it a valuable tool for dealing with real-life problems that are inherently vague or fuzzy.

On this exercise, we are dealing with the linguistic variable *Truth* with a possible membership set:

$$T = \text{Absolutely false, Very false, False, Fairly true, True, Very true, Absolutely true}$$

Based on that set we may define the membership function of truth as:

$$\text{True}(u) = u \quad \text{False}(u) = 1-u$$

for each $u \in [0, 1]$.

Problem 12

In order to evaluate the expression "not ($A(x)$ OR $B(x)$)", we must first take a look at how fuzzy logic differs from binary logic at operation level. In binary logic we have three basic operations: AND(x, y), OR(x, y) and NOT(x). But, in fuzzy logic, where a function can have a value in the range of $[0...1]$, things are slightly different. The binary operation AND(x, y) is equivalent to MIN(x, y) from fuzzy logic, OR(x, y) to MAX(x, y) and NOT(x) to $1-x$.

We need to find the proper x for which the previous expression has the maximum value. First, we calculate the expression and then find the correct x . To achieve this, we need to divide our calculations into ranges. Starting for $x \leq 2$, " $A(x)$ AND $B(x)$ " is equal to " $\max(A(x), B(x))$ " = 1. Applying De Morgan's law, we therefore have $\max(A(x), B(x)) \Rightarrow \text{not}(A(x) \text{ or } B(x)) = 0$. Exactly the same result is obtained with $x \geq 7$.

Things are a bit different in $2 \leq x \leq 7$. The function $A(x)$ starts to fall while $B(x)$ starts to rise. The point at which the two functions cross is important for the definition of the required expression and can be obtained by solving the equation:

$$A(x_{crit}) = B(x_{crit}) \Leftrightarrow 1 - \frac{x_{crit} - 2}{3} = \frac{x_{crit} - 3}{4} \Rightarrow x_{crit} = \frac{29}{7}$$

For $2 \leq x \leq \frac{29}{7}$, $\max(A(x), B(x)) = 1 - \frac{x-2}{3} = A(x)$, because in this region $A(x)$ lies above $B(x)$. Thus, $\text{not}(A(x) \text{ or } B(x)) = \frac{x-2}{3}$.

Using the same logic, we find out that for $\frac{29}{7} \leq x \leq 7$, $\text{not}(A(x) \text{ or } B(x)) = 1 - \frac{x-3}{4}$.

Therefore, the expression $f(x) = \text{not}(A(x) \text{ or } B(x))$ is summarized below:

$$f(x) = \begin{cases} 0 & x \leq 2, \\ \frac{x-2}{3} & 2 \leq x \leq \frac{29}{7}, \\ 1 - \frac{x-3}{4} & \frac{29}{7} \leq x \leq 7, \\ 0 & x \geq 7 \end{cases} \quad (15)$$

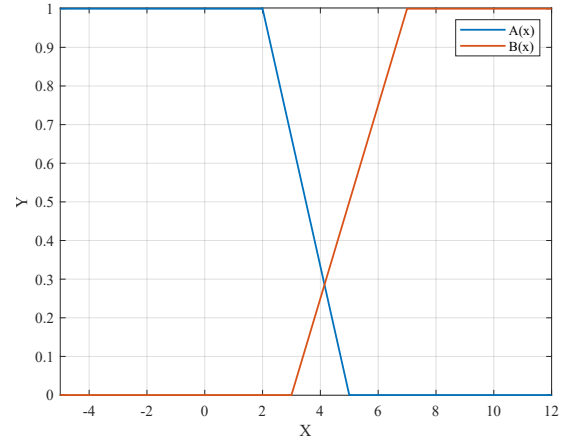


Figure 21: Plot of A(x), B(x)

By plotting this function in figure 22, we can clearly see that the maximum occurs at $x = x_{crit} = \frac{29}{7}$ and its value is $\frac{10}{14}$ or 0.715465.

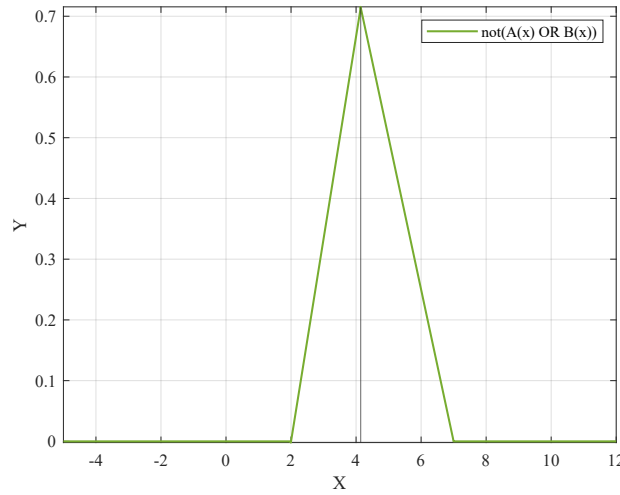


Figure 22: Expression's plot