UNIVERSITY OF THESSALY

NEURO-FUZZY COMPUTING

ECE447

1st Problem Set

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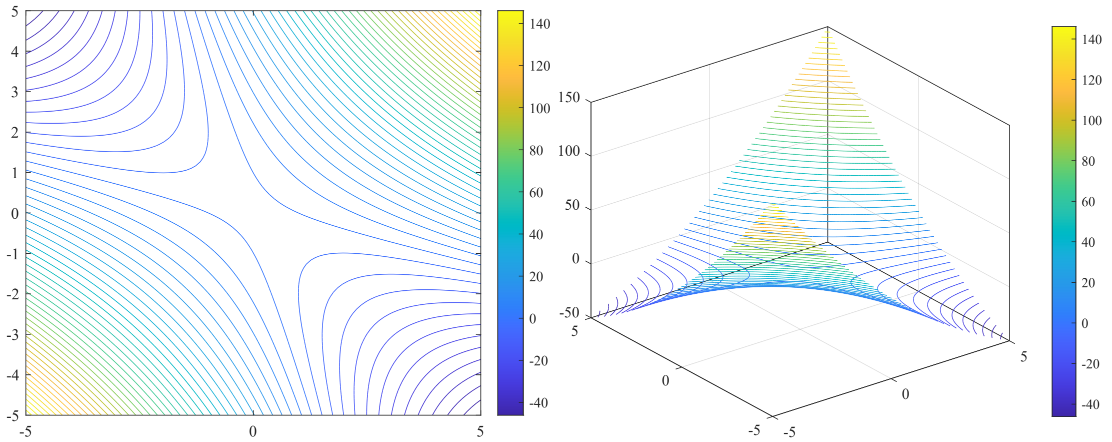
1 Problem 1

The contour lines of $f(x,y)$ are plotted with the following MATLAB code and are presented in figure 1.

```
function [Z] = plot_contour(start_num, end_num)

        x = linspace(start_num, end_num, 100);
        y = x;
        [X, Y] = meshgrid(x, y);
        Z = X.^2 + 4*X.*Y + Y.^2;
        contour(X, Y, Z, 40);
        xlabel('X');
        ylabel('Y');
end
```



(a) 2D Plot (b) 3D Plot

Figure 1: Contour lines of $f(x,y)$

A general formula of a quadratic equation is $f(x,y) = ax^2 + 2bxy + cy^2$. Writing our formula in the previous form, we find that $a = 1,\ b = 2,\ c = 1$. Calculation of the discriminant can help us calculate the location of the function's local minimum/maximum.

$$
\begin{gathered}
D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - f_{xy}^2 = 2 \times 2 - 4^2 = -12 < 0, \quad \text{where} \\
f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2, \quad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4D = 2 \times 2 - 4^2 = -12 < 0.
\end{gathered} \tag{1}
$$

So, we only have to find the point at which $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are equal to 0. This point will be a saddle point at which the gradients in each orthogonal direction are 0, but this point is neither a local minimum nor a maximum. More precisely:

$$
\begin{cases} \dfrac{\partial f}{\partial x} = 2x + 4y = 0 \\ \dfrac{\partial f}{\partial y} = 4x + 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \tag{2}
$$

Thus, the point $(x,y) = (0,0)$ is the saddle point mentioned for the given function and this can be justified using the plotted contour lines.

2 Problem 4

In this problem, we will express the derivatives in respect to the matching activation function.

2.1 LogSig

This activation function is expressed as

$$S(x) = \frac{1}{1+e^{-x}}$$

Multiplying itself with $(1+e^{-x})$ gives

$$\left(1+e^{-x}\right)\ S(x) = 1 \Leftrightarrow e^{-x} = \frac{1}{S(x)} - 1$$

So, activation function's derivative will be

$$\begin{aligned}\frac{dS}{dx} &= \frac{d\left(\left(1+e^{-x}\right)^{-1}\right)}{dx} = \left(1+e^{-x}\right)^{-2}\ e^{-x} = S^2(x)\ \left(\frac{1}{S(x)} - 1\right) \\ &= S(x) - S^2(x) = S(x)\ \left(1 - S(x)\right)\end{aligned}$$

2.2 TanSig

Activation function is

$$S(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Its derivative is

$$\begin{aligned}\frac{dS}{dx} &= \frac{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right) - \left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2} = \\ = \frac{\left(e^x+e^{-x}\right)^2 - \left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2} &= 1 - \left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2 = 1 - S^2(x)\end{aligned} \tag{3}$$

2.3 Swish

Activation function is

$$S(x) = \frac{x}{1+e^{-x}}$$

The derivative in respect to x is

$$\frac{dS}{dx} = \frac{1+e^{-x}+xe^{-x}}{\left(1+e^{-x}\right)^2} = \frac{1+e^{-x}}{\left(1+e^{-x}\right)^2} + \frac{xe^{-x}}{\left(1+e^{-x}\right)^2} = \frac{1}{1+e^{-x}} + x\frac{e^{-x}}{\left(1+e^{-x}\right)^2}$$

Rewriting the function gives us

$$\frac{S(x)}{x} = \frac{1}{1+e^{-x}} \quad \text{and} \quad e^{-x} = \frac{x - S(x)}{S(x)}$$

So, continuing with the derivative:

$$\frac{dS}{dx} = \frac{1}{1+e^{-x}} + x\frac{e^{-x}}{(1+e^{-x})^2} = \frac{S(x)}{x} + \frac{e^{-x}}{1+e^{-x}}\frac{x}{1+e^{-x}} = \frac{S(x)}{x} + S(x)\frac{x}{1+e^{-x}} =$$
$$= \frac{S(x)}{x} + S(x)\frac{S(x)}{x}e^{-x} = \frac{S(x)}{x}\left(1+S(x)e^{-x}\right) = \frac{S(x)}{x}\left(1+S(x)\frac{x-S(x)}{S(x)}\right) =$$
$$= \frac{S(x)}{x}\left(1+x-S(x)\right) = \frac{S(x)}{x} + S(x) - \frac{S^2(x)}{x}$$

Let $\sigma = \dfrac{1}{1+e^{-x}}$, thus

$$\frac{dS}{dx} = \frac{S(x)}{x} + S(x) - \frac{S^2(x)}{x} = \sigma + S(x) - \sigma S(x) = S(x) + \sigma\left(1-S(x)\right) \tag{4}$$

2.4 Custom tanh

Activation function is

$$S(x) = x \times \tanh\left(\ln\left(1+e^x\right)\right) = \frac{x\left(1+e^x\right) - \dfrac{x}{1+e^x}}{1+e^x+\dfrac{1}{1+e^x}}$$

After some calculations, we can observe a relationship to the Swish function:

$$S(x) = \frac{x\left(1+e^x\right) - T(-x)}{1+e^x+\dfrac{1}{1+e^x}} = \frac{\dfrac{-x^2}{T(-x)} - T(-x)}{\dfrac{-x}{T(-x)} + \dfrac{T(-x)}{x}} = \frac{-x^3 - T^2(-x)}{-x^2+T^2(-x)}$$

3 Problem 11

Fuzzy logic is a type of logic that deals with vague, imprecise, or uncertain information. It is based on the concept of fuzzy sets, which are sets that can have any degree of membership between 0 and 1. The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership.This means that an element can be a member of a fuzzy set to some degree, rather than all or nothing.
The uniqueness of fuzzy logic is that fuzzy logic can handle imprecise and uncertain information,which makes it a valuable tool for dealing with real-life problems that are inherently vague or fuzzy.
On this exercise, we are dealing with the linguistic variable *Truth* with a possible membership set:

T = Absolutely false, Very false, False, Fairly true, True, Very true, Absolutely true

Based on that set we may define the membership function of truth as:

True(u) = u *False(u) = 1-u*

for each $u \in [0, 1]$.

4 Problem 12

In order to evaluate the expression ”`not(A(x) OR B(x))`”, we must first take a look at how fuzzy logic differs from binary logic at operation level. In binary logic we have three basic operations: `AND(x,y)`, `OR(x,y)` and `NOT(x)`. But, in fuzzy logic, where a function can have a value in the range of $[0...1]$, things are slightly different. The binary operation `AND(x,y)` is equivalent to `MIN(x,y)` from fuzzy logic, `OR(x,y)` to `MAX(x,y)` and `NOT(x)` to `1-x`.

We need to find the proper x for which the previous expression has the maximum value. First, we calculate the expression and then find the correct x. To achieve this, we need to divide our calculations into ranges.

Starting for $x \leq 2$, ”$A(x)$ AND $B(x)$” is equal to ”$max\,(A(x), B(x))$” $= 1$. Applying De Morgan's law, we therefore have $max\,(A(x), B(x)) \Rightarrow not\,(A(x)\ or\ B(x)) = 0$. Exactly the same result is obtained with $x \geq 7$.

Things are a bit different in $2 \leq x \leq 7$. The function $A(x)$ starts to fall while $B(x)$ starts to rise. The point at which the two functions cross is important for the definition of the required expression and can be obtained by solving the equation:

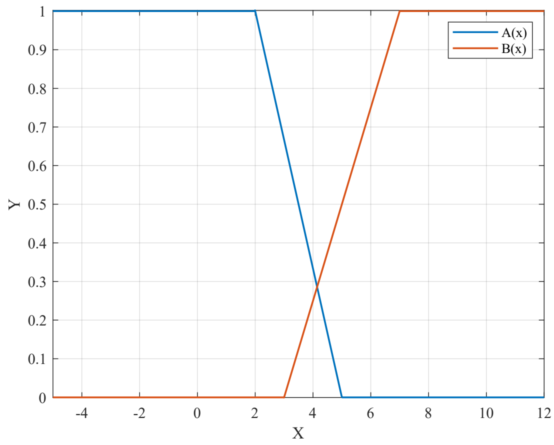


Figure 2: Plot of A(x), B(x)

$$A(x_{crit}) = B(x_{crit}) \Leftrightarrow 1 - \frac{x_{crit} - 2}{3} = \frac{x_{crit} - 3}{4} \Rightarrow x_{crit} = \frac{29}{7}$$

For $2 \leq x \leq \frac{29}{7}$, $max\,(A(x), B(x)) = 1 - \dfrac{x-2}{3} = A(x)$, because in this region $A(x)$ lies above $B(x)$. Thus, $not\,(A(x)\ or\ B(x)) = \dfrac{x-2}{3}$.

Using the same logic, we find out that for $\frac{29}{7} \leq x \leq 7$, $not\,(A(x)\ or\ B(x)) = 1 - \dfrac{x-3}{4}$.

Therefore, the expression $f(x) = not\,(A(x)\ or\ B(x))$ is summarized below:

$$f(x) = \begin{cases} 0 & x \leq 2, \\ \dfrac{x-2}{3} & 2 \leq x \leq \frac{29}{7}, \\ 1 - \dfrac{x-3}{4} & \frac{29}{7} \leq x \leq 7, \\ 0 & x \geq 7 \end{cases} \tag{5}$$

By plotting this function in figure 3, we can clearly see that the maximum occurs at $x = x_{crit} = \dfrac{29}{7}$ and its value is 0.715465.

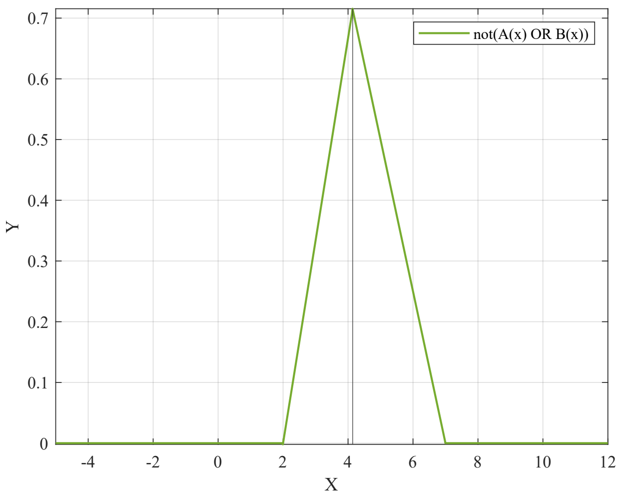


Figure 3: Expression's plot