CM 1606 Computational Mathematics

Tensors

Week 09| Ganesha Thondilege

ROBERT GORDON UNIVERSITY ABERDEEN

INFORMATICS INSTITUTE OF TECHNOLOGY

Learning Outcomes

- Covers LO1 for Module
- On completion of this lecture, students are expected to be able to:
  - Recall direction cosines of vectors
  - Identify tensors
  - Identify the way that high dimensional input data can specify using tensors
  - Discuss applications in Machine learning

CONTENT

- Introduction
- What is a tensor
- Rank of a tensor
- Transformation rules
- Coordinate transformation
- Summation convention

Introduction

Temperature – Colombo city

303 K

- Single component to specify the temperature
- Scalar
- Component – Zero basis vectors

Displacement from Homagama to 30th floor of the WTC west tower

32km

$B: WTC$ west tower 30th floor

$A: Homagama$

$Displacement \quad \overrightarrow{AB} = 32km$

$Direction: From\ A\ to\ B = \overrightarrow{AB}$

- Displacement from Homagama to 30th floor of the WTC west tower

  \- 32km

$B : WTC$ west tower 30th floor

$A : Homagama$

$$Displacement \quad \vec{AB} = 32km$$

$$Direction : From\ A\ to\ B = \vec{AB}$$

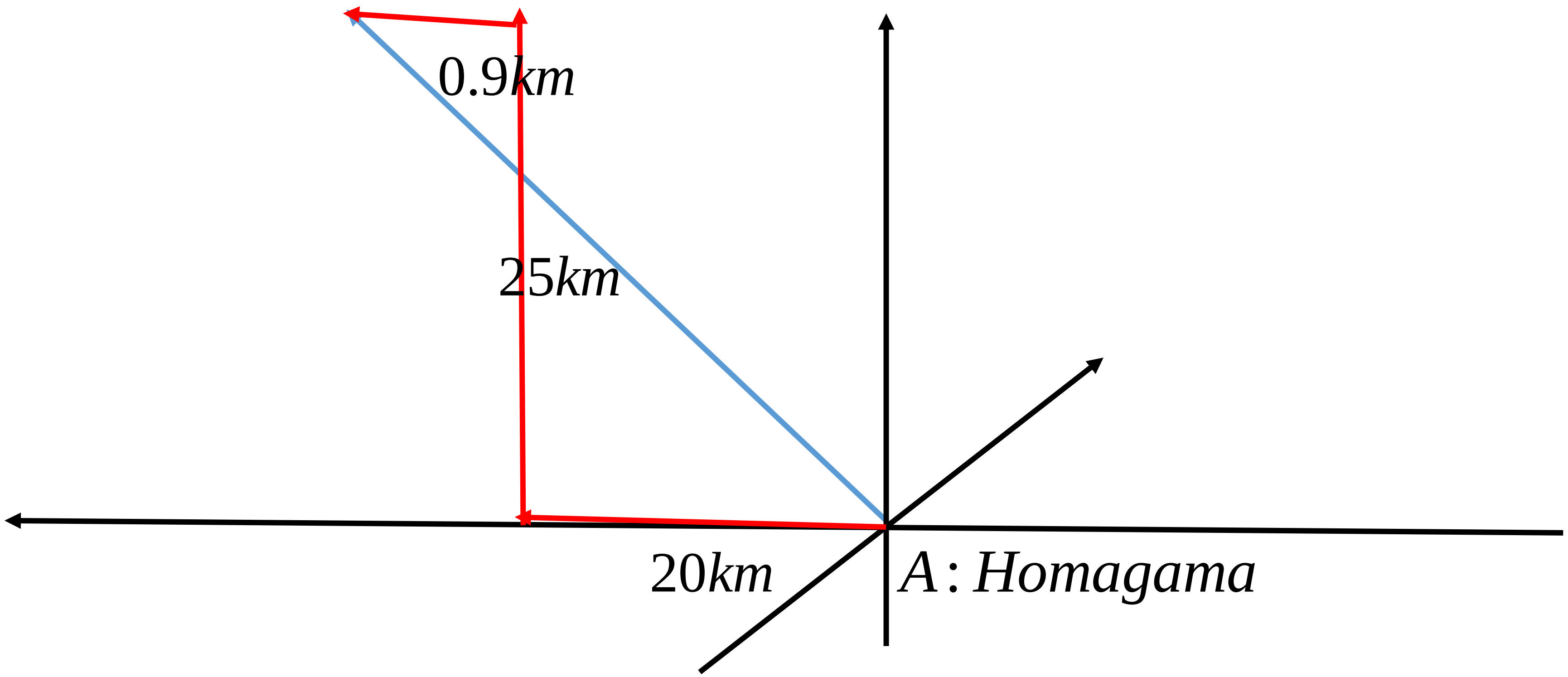
$$20km - To\ west$$

$$25km - To\ North$$

$$0.9km - up$$

- Displacement using basis vectors

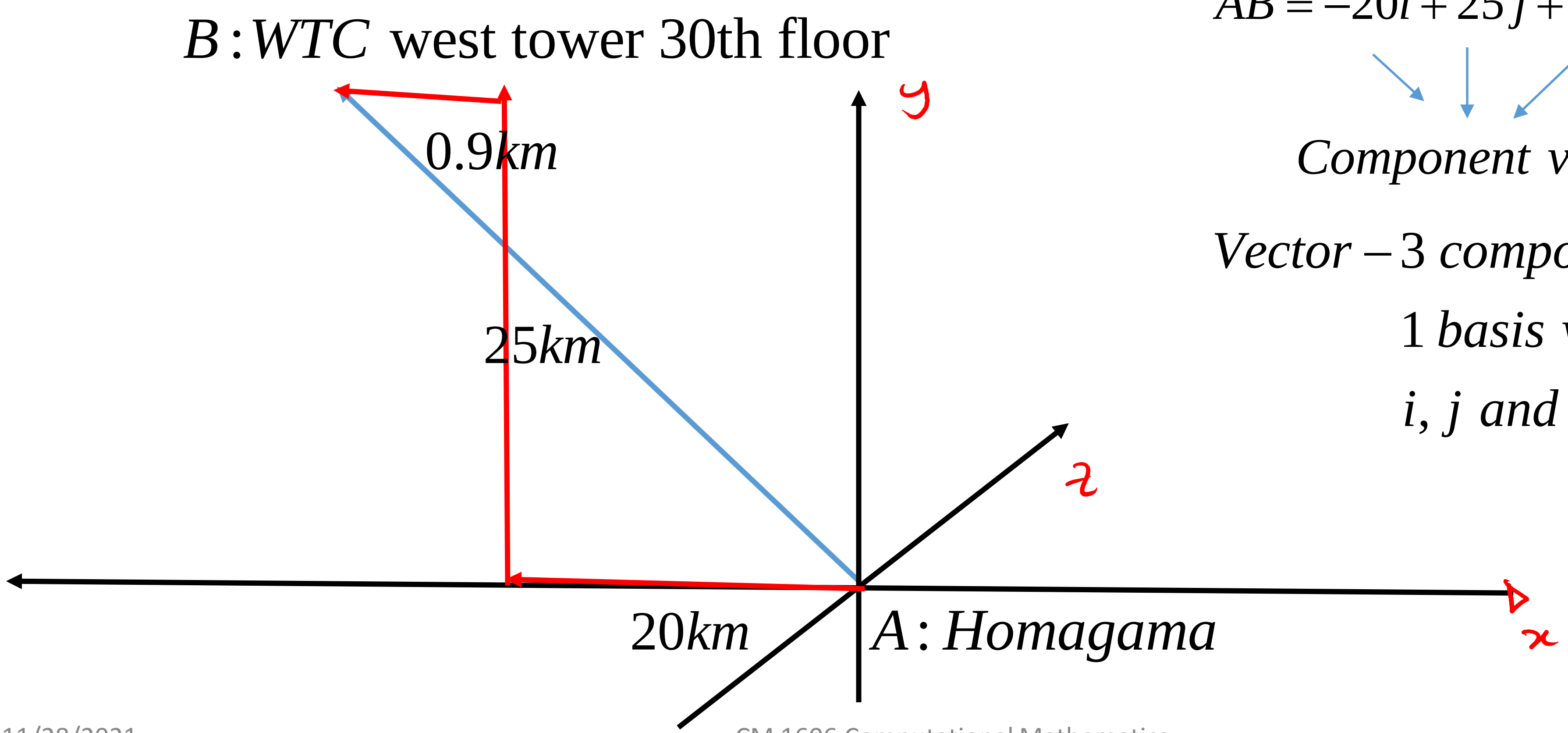
$B : WTC$ west tower 30th floor



$$\vec{AB} = -20i + 25j + 0.9k$$

Component vectors

- Displacement using basis vectors



$$\vec{AB} = -20i + 25j + 0.9k$$

Component vectors

Vector – 3 *componets*

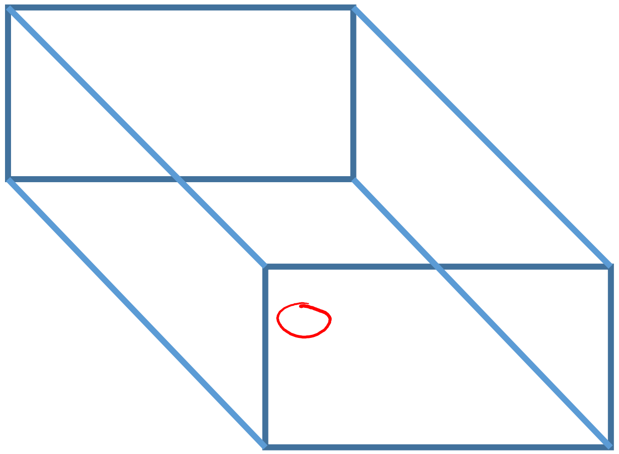
1 *basis vector per component*

i, *j and k*

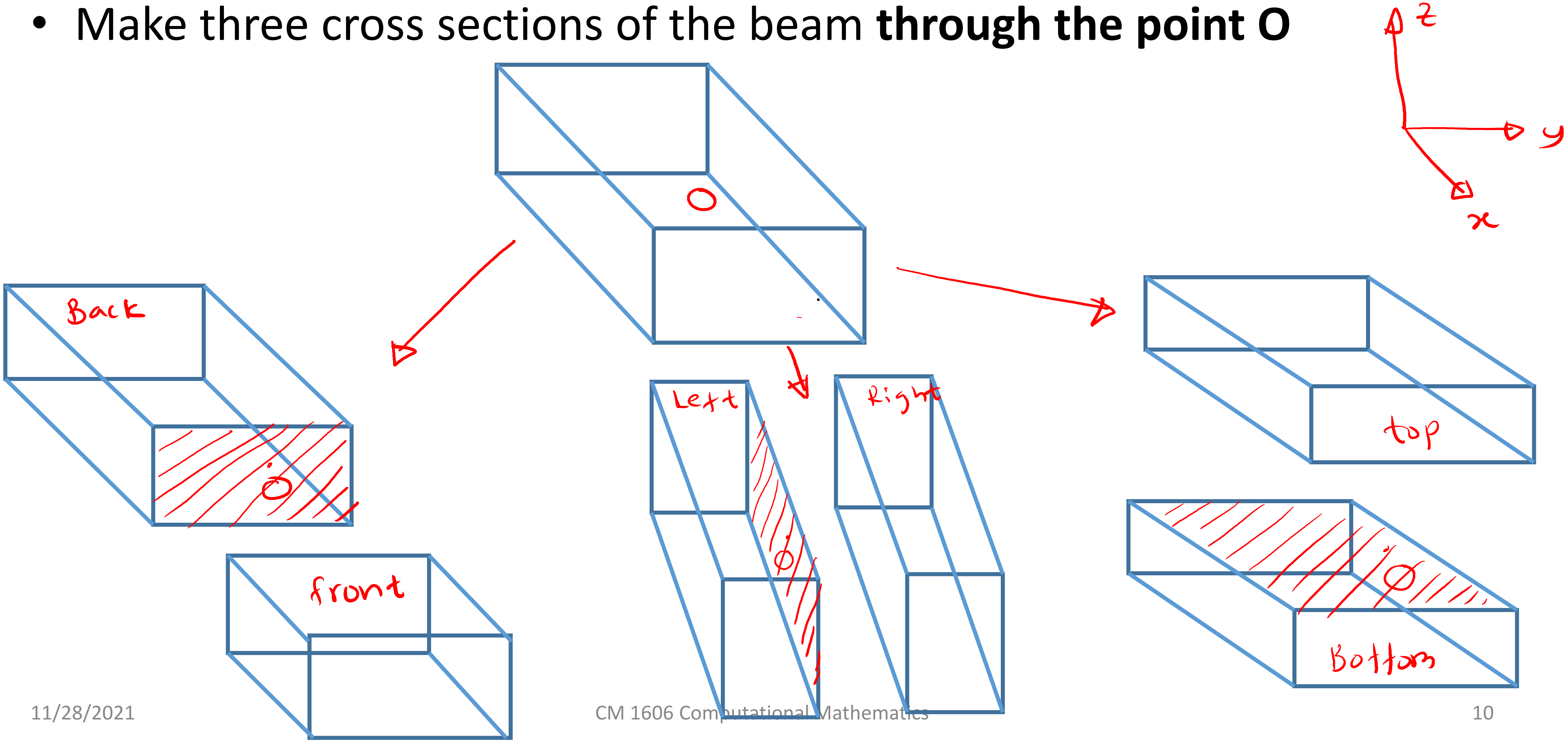
Introduction ctd.

- Consider a rectangular steel beam and the point O inside the beam
- How we can specify all the stresses act on the point O

Stresses on O?



- Make three cross sections of the beam **through the point O**



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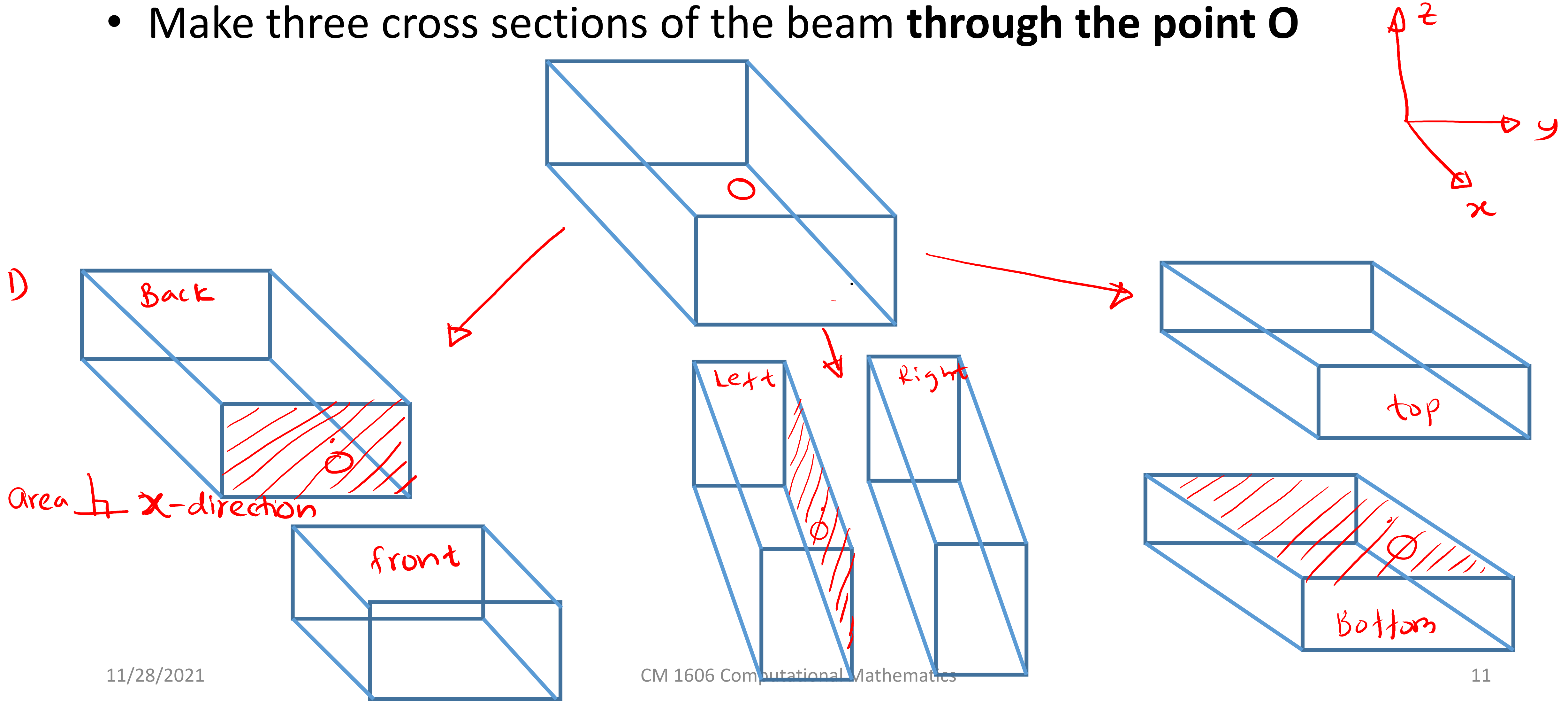

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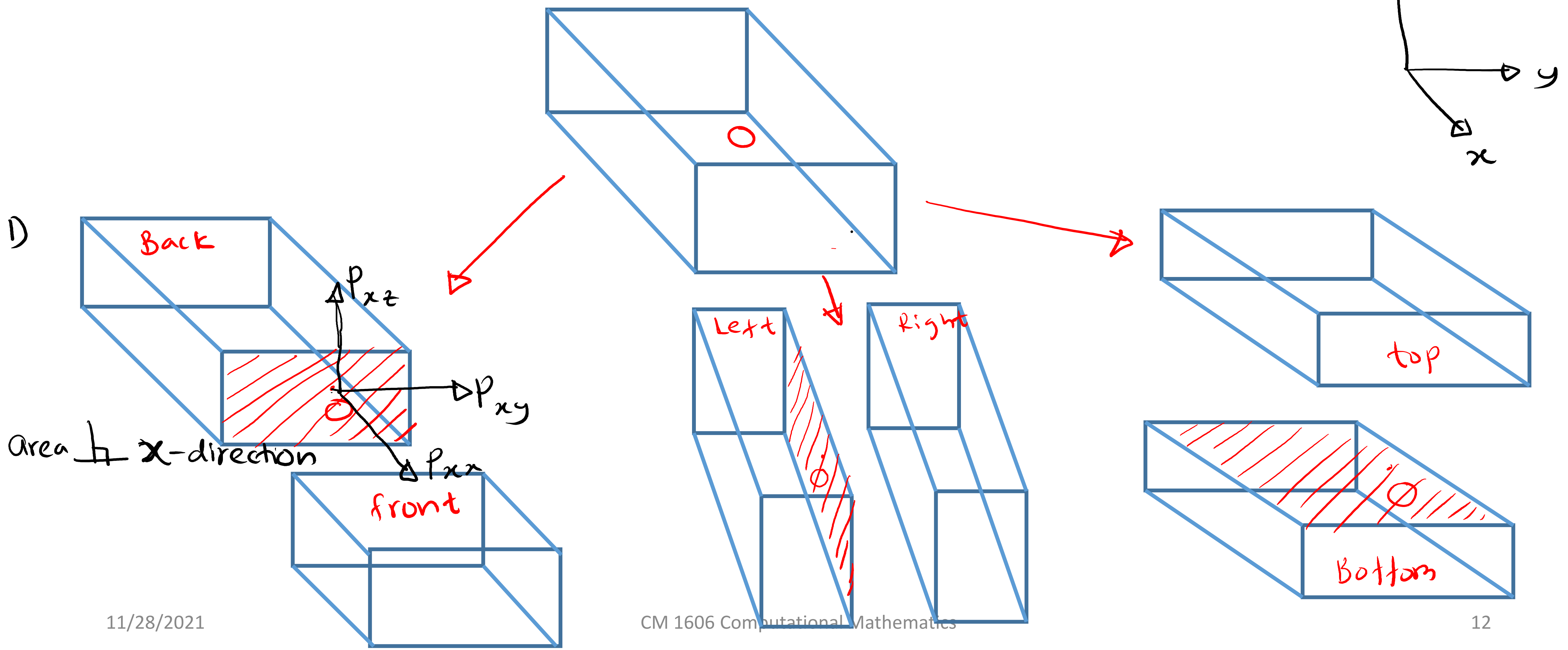
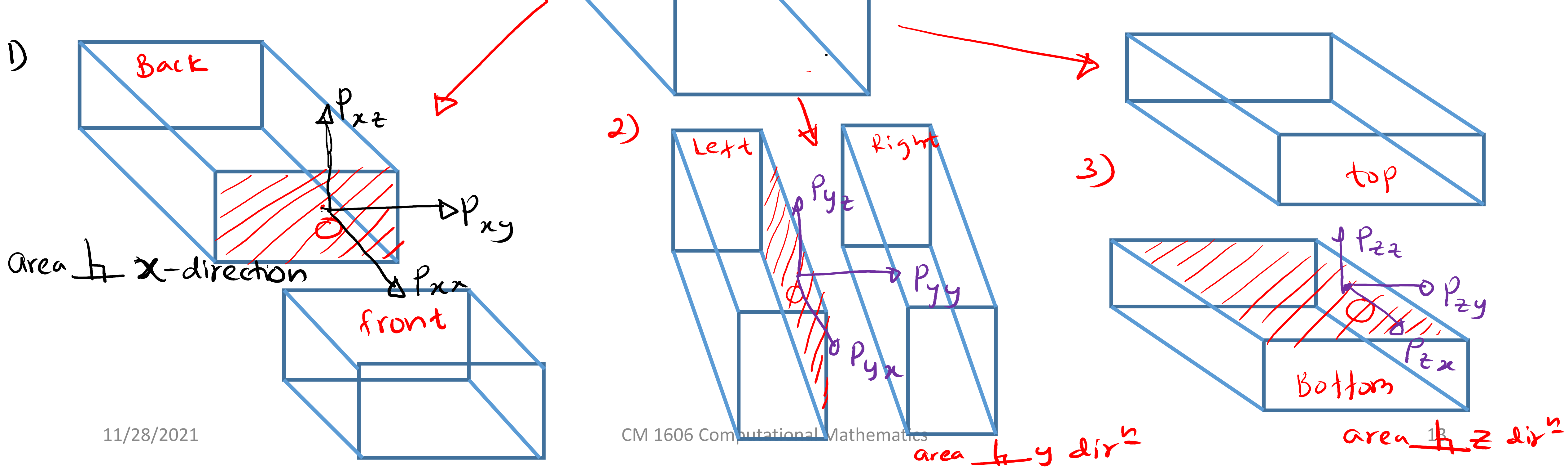


- Make three cross sections of the beam **through the point O**



- We can combine all these components in a 3*3 matrix as

$$P=\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ p_{yx} & P_{yy} & Pyz \\ P_{zx} & p_{zy} & P_{zz} \end{pmatrix}$$

Use this matrix P to specify all stresses acting on the point O

Tensors

- P has 9 components and 2 basis vectors per component

  2 basis vectors – 1 for the cross-sectional area

  1 for the direction of force

- This concept is known as Tensors

In an m dimensional space, a tensor of rank n is a mathematical object that has n indices, m^n components.

- obeys some transformation rules
- Generally, m=3

Rank of a Tensor

Number of basis vectors needed to fully specify a component of a tensor.

- Scalar

  Component – Zero basis vectors – A tensor of rank zero

- $Vector - 3\ componets$

  $1\ basis\ vector\ per\ component$

  $i,\ j\ and\ k$

So, a vector is a tensor of rank 1

- P is a tensor of rank 2 – Stress tensor

m dimensional space, rank n Tensor ➔ m^n components

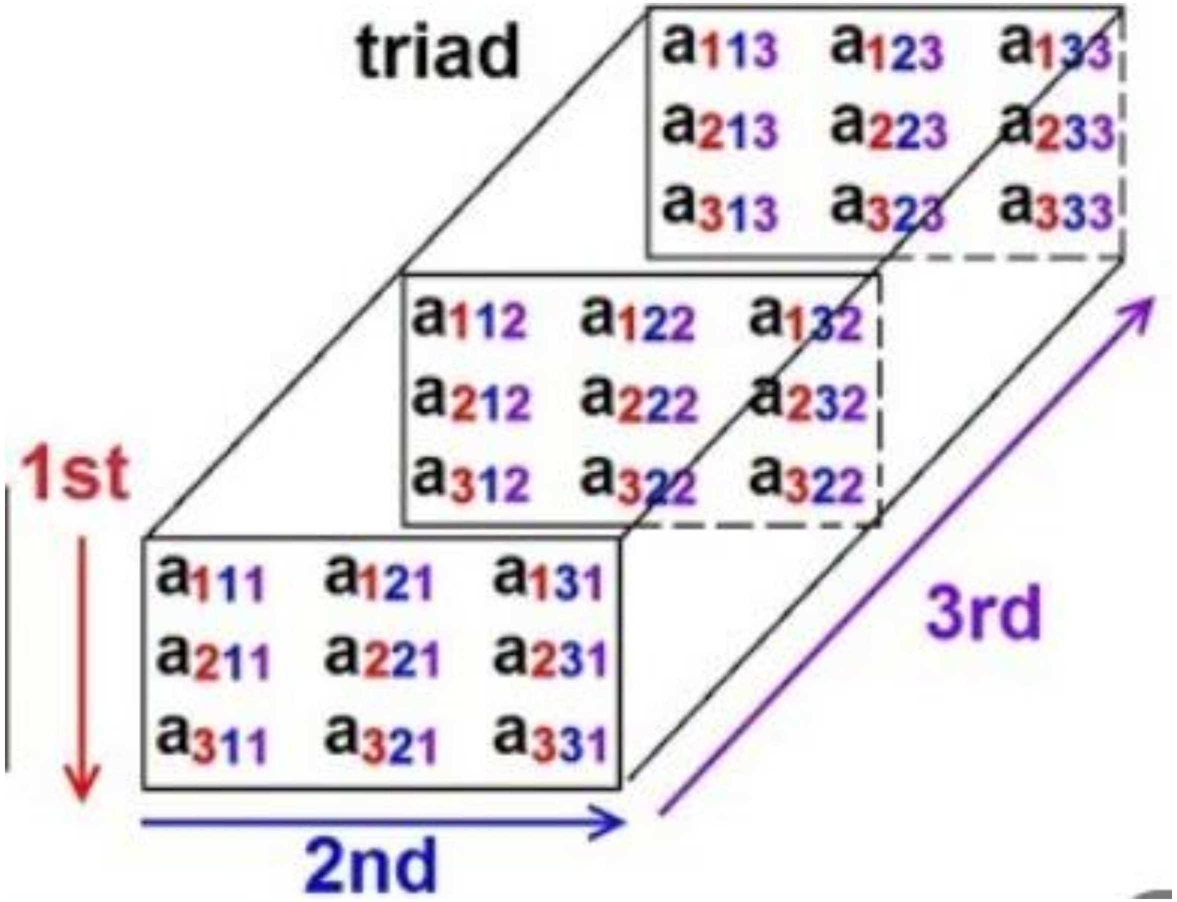
Scalar	Vector	Stress Tensor(P)
1 component	3 components	9 components
3^(0)	3^(1)	3^(2)
rank 0	rank 1	rank 2

Tensor of rank 3

- 27 components=>3^3 $-$ rank 3 tensor
- How can we specify this 27 components?

Rank 3 tensor - Triad

- Any component a_{xyz}
- 3 pages

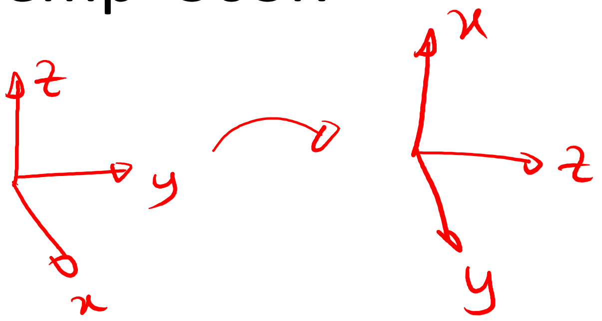


Transformation rules

- A tensor is an object that transforms like a tensor
- A tensor is an object that is invariant under a change of coordinate systems, with components that change according to a special set of mathematical formulae.

Rank zero tensor(scalar)

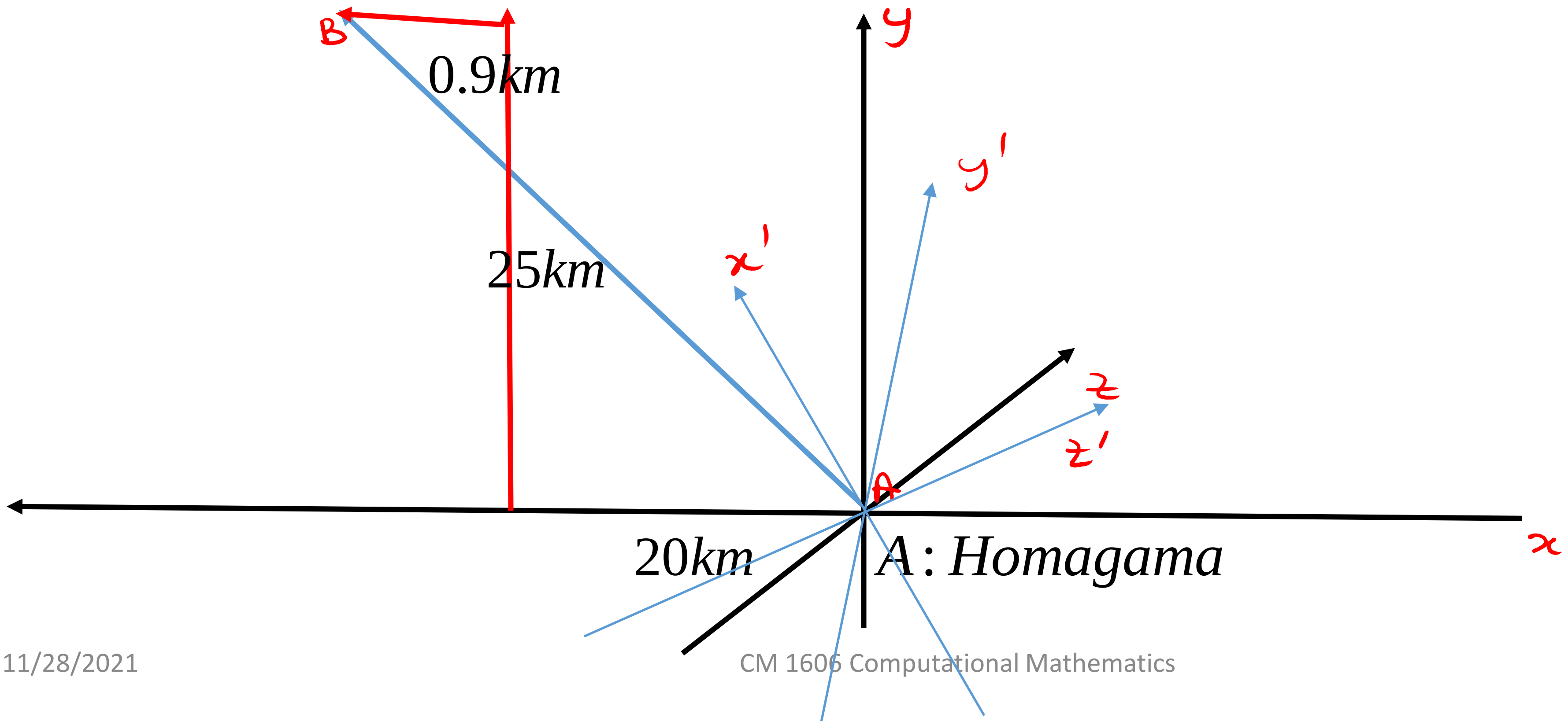
Temp=303K



Change of coordinate system does not change the temp.

- Vector – Rank 1 tensor
- Displacement $= \overrightarrow{AB} = -20\mathrm{i} + 25j + 0.9k$

$B : WTC$ west tower 30th floor



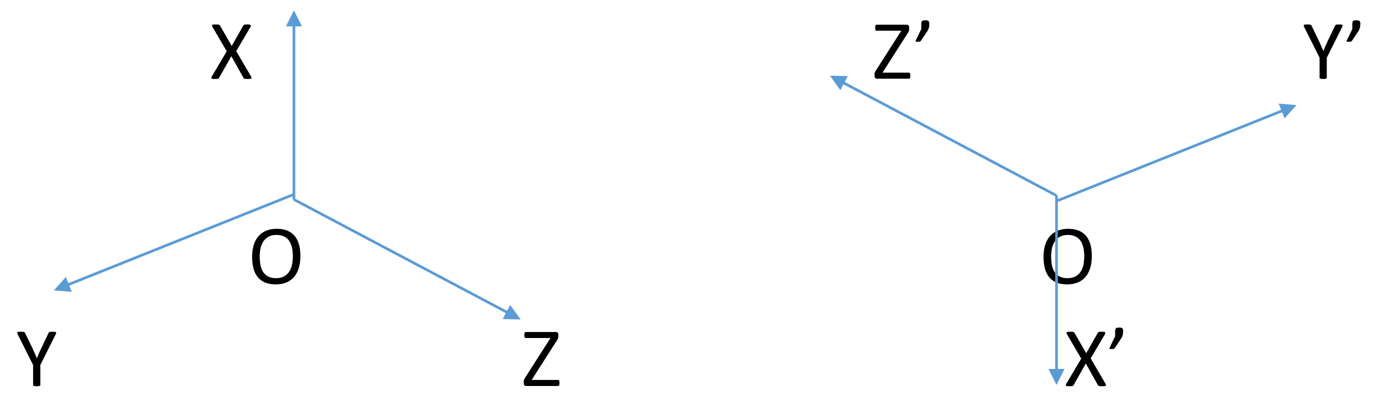
- It change the way that we write the $\overrightarrow{AB}$ using new basis vectors $\mathrm{i}', \mathrm{j}', \text{and } \mathrm{k}'$

- Does it change the vector itself?

  No. vector still from A to B

Coordinate transformation

$$\mathrm{OXYZ} \rightarrow OX'Y'Z'$$



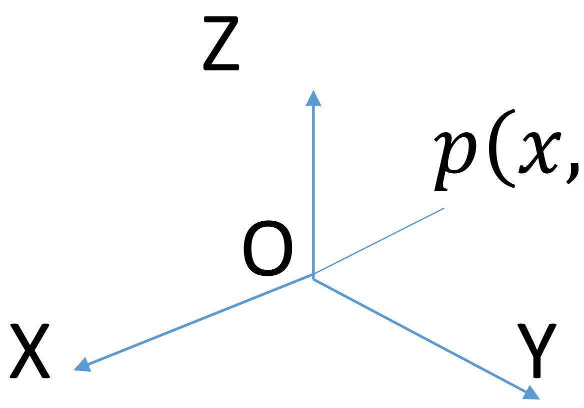
Direction cosines of Ox’, OY’ & OZ’ relative to OXYZ are

$$\mathrm{OX}' \rightarrow l_1, m_1, n_1$$

$$\mathrm{OY}' \rightarrow l_2, m_2, n_2$$

$$\mathrm{OZ}' \rightarrow l_3, m_3, n_3$$

Coordinate transformation



$$p(x, y, z) \rightarrow p(x', y', z')$$

O

X’

Then two equivalent systems of transformation equations

$$\mathrm{x}' = l_1 x + m_1 y + n_1 z$$
$$\mathrm{y}' = l_2 x + m_2 y + n_2 z$$
$$\mathrm{z}' = l_3 x + m_3 y + n_3 z$$

Coordinate transformation

$$x = l_1 x' + m_1 y' + n_1 z'$$
$$y = l_2 x' + m_2 y' + n_2 z'$$
$$z = l_3 x' + m_3 y' + n_3 z'$$

This transformation can be written as

$$\begin{array}{ccc} x & y & z \\ \end{array}$$
$$\begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}$$

Summation convention

The sum of

$a_1x_1 + a_2x_2 + a_3x_3 \dots \dots + a_nx_n$ can be written as

$$\sum_{i=1}^{n} a_i x_i$$

The compact notation for this is

$$a_i x^{\mathrm{i}} \quad (Einstein\ Notation)$$

Example

$$Write\ a_{rs}x^{s} = b_r(r, s = 1,2,3, \ldots \ldots \ldots \ldots \ldots, n) in\ full.$$

Hint: Substitute values for r first and then substitute for s.

Example

- Write all the tensors in

$$T = a_{ij}x^{\mathrm{i}}\ taking\ n = 3$$

Tensor Addition

- Take the element wise addition of two tensors with same dimensions and results a new tensor with the same dimensions

Tensor subtraction

- Take the element wise subtraction of two tensors with same dimensions and results a new tensor with the same dimensions

Reference

- https://www.youtube.com/watch?v=f5liqUk0ZTw