The Uniform Distribution

Introduction

This Section introduces the simplest type of continuous probability distribution which features a continuous random variable X with probability density function $f(x)$ which assumes a constant value over a finite interval.

1. The uniform distribution

The Uniform or Rectangular distribution has random variable X restricted to a finite interval $[a, b]$ and has $f(x)$ a constant over the interval. An illustration is shown in Figure 3:

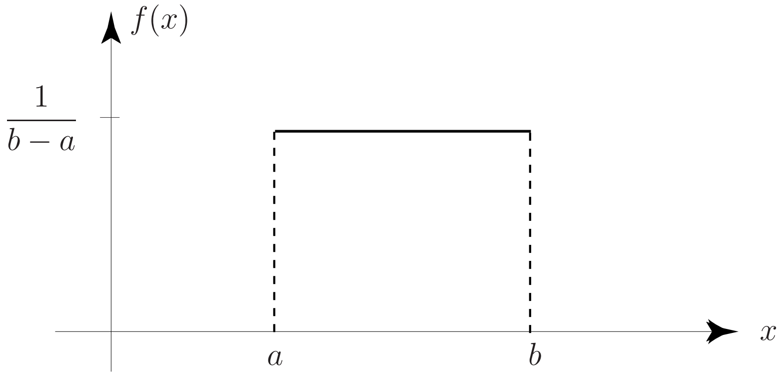

Figure 3

The function $f(x)$ is defined by:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ \\ 0 & \text{otherwise} \end{cases}$$

Mean and variance of a uniform distribution

Using the definitions of expectation and variance leads to the following calculations. As you might expect, for a uniform distribution, the calculations are not difficult.
Using the basic definition of expectation we may write:

$$\begin{aligned} \mathsf{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{2(b-a)} \left[x^2 \right]_a^b \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

Using the formula for the variance, we may write:

$$\begin{aligned} \mathsf{V}(X) &= \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 \\ &= \int_a^b x^2 . \frac{1}{b-a}\, dx - \left(\frac{b+a}{2} \right)^2 = \frac{1}{3(b-a)} \left[x^3 \right]_a^b - \left(\frac{b+a}{2} \right)^2 \\ &= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2} \right)^2 \\ &= \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

Key Point 3

The **Uniform** random variable X whose density function $f(x)$ is defined by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ & \\ 0 & \text{otherwise} \end{cases}$$

has expectation and variance given by the formulae

$$\mathsf{E}(X) = \frac{b+a}{2} \qquad \text{and} \qquad \mathsf{V}(X) = \frac{(b-a)^2}{12}$$

Example 2

The current (in mA) measured in a piece of copper wire is known to follow a uniform distribution over the interval $[0, 25]$. Write down the formula for the probability density function $f(x)$ of the random variable X representing the current. Calculate the mean and variance of the distribution and find the cumulative distribution function $F(x)$.

The thickness x of a protective coating applied to a conductor designed to work in corrosive conditions follows a uniform distribution over the interval $[20, 40]$ microns. Find the mean, standard deviation and cumulative distribution function of the thickness of the protective coating. Find also the probability that the coating is less than 35 microns thick.

Your solution

The Exponential Distribution

Introduction

If an engineer is responsible for the quality of, say, copper wire for use in domestic wiring systems, he or she might be interested in knowing both the number of faults in a given length of wire and also the distances between such faults. While the number of faults may be analysed by using the Poisson distribution, the distances between faults along the wire may be shown to give rise to the exponential distribution defined and used in this Section.

1. The exponential distribution

The exponential distribution is defined by

$$f(t) = \lambda \mathrm{e}^{-\lambda t} \qquad t \geq 0 \qquad \lambda \text{ a constant}$$

or sometimes (see the Section on Reliability in HELM 46) by

$$f(t) = \frac{1}{\mu}\mathrm{e}^{-t/\mu} \qquad t \geq 0 \qquad \mu \text{ a constant}$$

The advantage of this latter representation is that it may be shown that the mean of the distribution is μ.

Example 3

The lifetime T (years) of an electronic component is a continuous random variable with a probability density function given by

$$f(t) = \mathrm{e}^{-t} \qquad t \geq 0 \qquad (\text{i.e. } \lambda = 1 \text{ or } \mu = 1)$$

Find the lifetime L which a typical component is 60% certain to exceed. If five components are sold to a manufacturer, find the probability that at least one of them will have a lifetime less than L years.

Commonly, car cooling systems are controlled by electrically driven fans. Assuming that the lifetime T in hours of a particular make of fan can be modelled by an exponential distribution with $\lambda = 0.0003$ find the proportion of fans which will give at least 10000 hours service. If the fan is redesigned so that its lifetime may be modelled by an exponential distribution with $\lambda = 0.00035$, would you expect more fans or fewer to give at least 10000 hours service?

Your solution