CM2607 Advanced Mathematics for data science

Fourier Transformation

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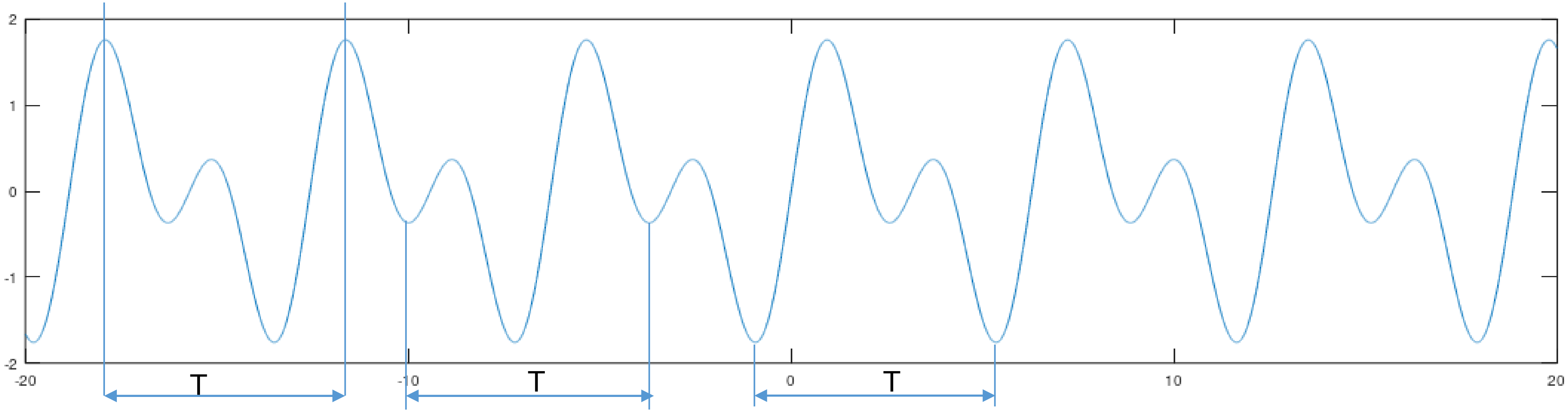
Learning Outcomes

- Covers LO3 for Module
- On completion of this lecture, students are expected to be able to:
  - Understand Fourier series
  - Represent periodic function as Fourier series
  - Understand Fourier transform
  - Apply Fourier transform to common functions

Periodic functions

- A periodic function is a function that repeats its values at regular intervals.
- Mathematically, a function is periodic, if for a non-zero constant P,
$$f(x+P)=f(x)$$
for all x in the domain.
- The smallest possible value for P is the function's period, T.



Odd and Even functions

- Even functions are functions that are symmetric about the y-axis. i.e.,

$$f(-x) = f(x)$$

  Example: $\cos(x)\,, x^2$

- Odd functions are functions that satisfy

$$f(-x) = -f(x)$$

  Examples: $\sin(x)\,, x^3$

- Not every function is even or odd.

  Examples: $e^x, \sqrt{x}$

Fourier Analysis

Application:

Fourier analysis which mathematically separates an image into its spatial frequency components.

The Fourier series is a method to express a periodic function in terms of a combination of sines and cosines – i.e., waveforms.
It essentially decomposes a signal into its component frequencies.

Fourier series

The Fourier series is used in many applications, including:

- Image processing
- Signal processing
- Control applications, robotics, automation,
- And many more

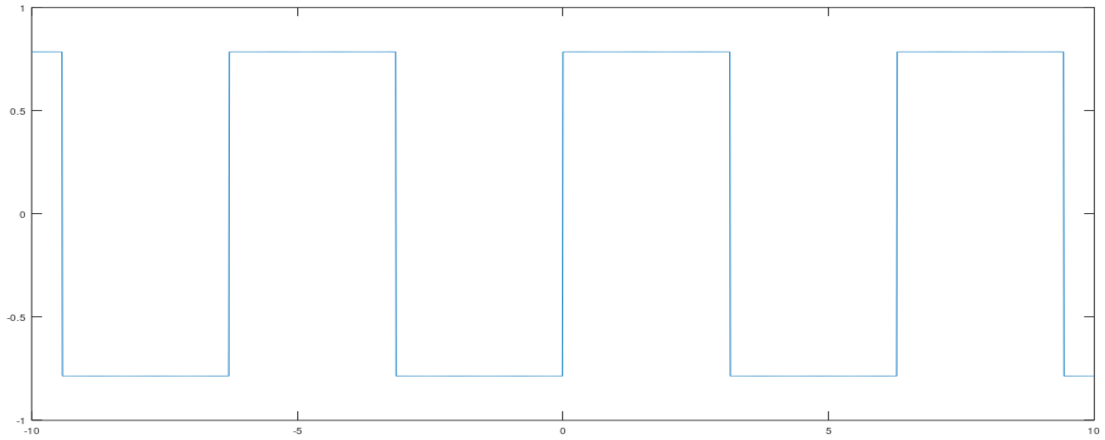
It is primarily used because it can reduce the complexity of some calculations drastically (more on that later).

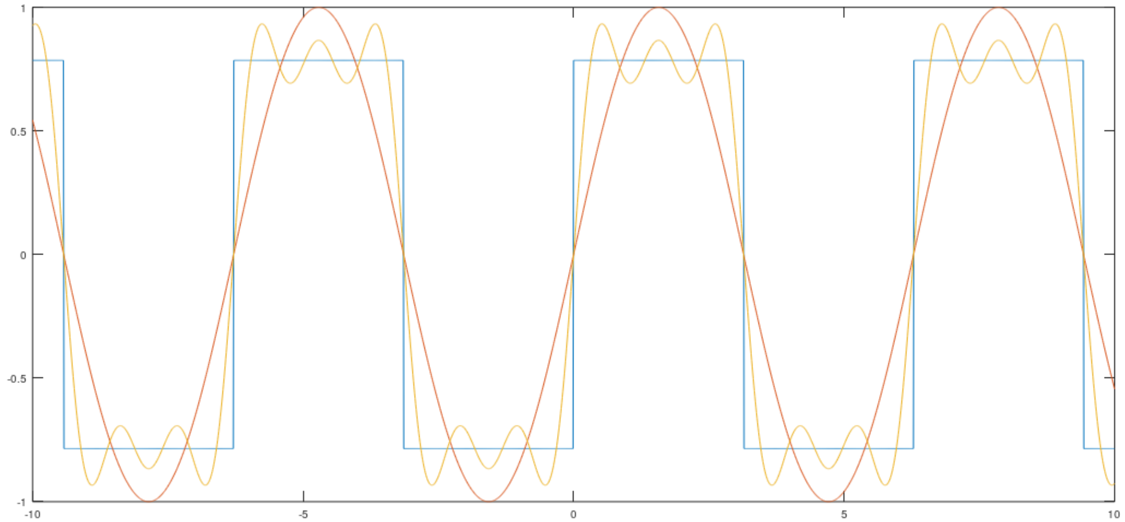
Example: square wave

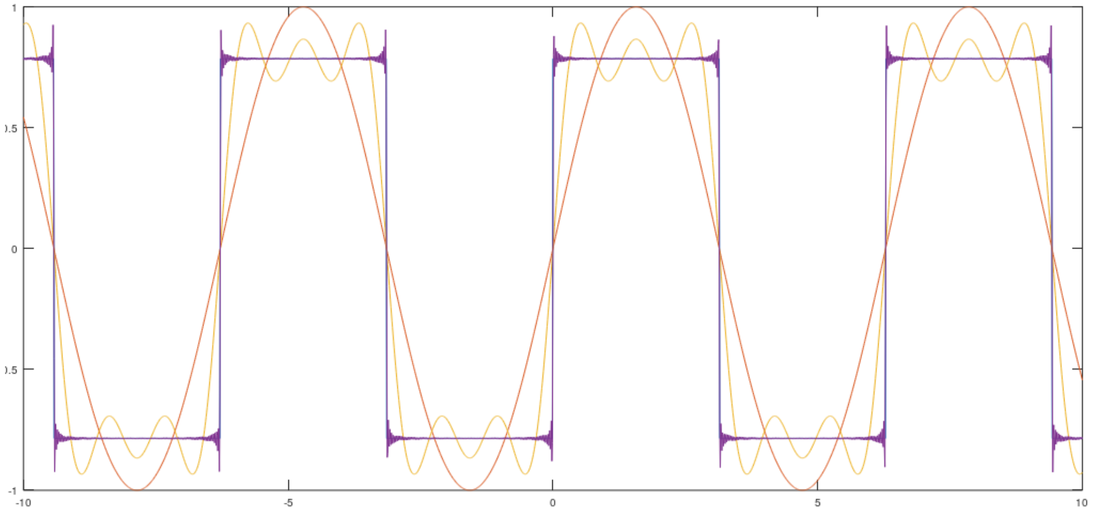
- A simple square wave with period 2π, 50% duty cycle, amplitude π/4
- Can be approximated as:

$$\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots$$

- Graphs: square wave, 3 terms (up to 5x), and 100 terms of series.







General formula – Fourier series

- It is possible to calculate the Fourier series for any periodic function

With period $2L$

General formula:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where

L = half the period(2L) of the function

Coefficients a_0, a_n, b_n are constants and must be calculated (next slide)

General formula – Fourier series

- Coefficient a_0: offset. If we wanted the previous square wave from 0-π/2 instead of –π/4 to π/4, we could add $a_0 = \frac{\pi}{4}$.
- General formula:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

General formula – Fourier series

- Coefficients a_n and b_n can be calculated as:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

- You can also apply this to any periodic function.

Example

- Find the Fourier series expansion for the periodic function given below.

$$f(x) = \begin{cases} 0; & 0 < x < \pi \\ 1; & \pi < x < 2\pi \end{cases}$$

Even and odd functions

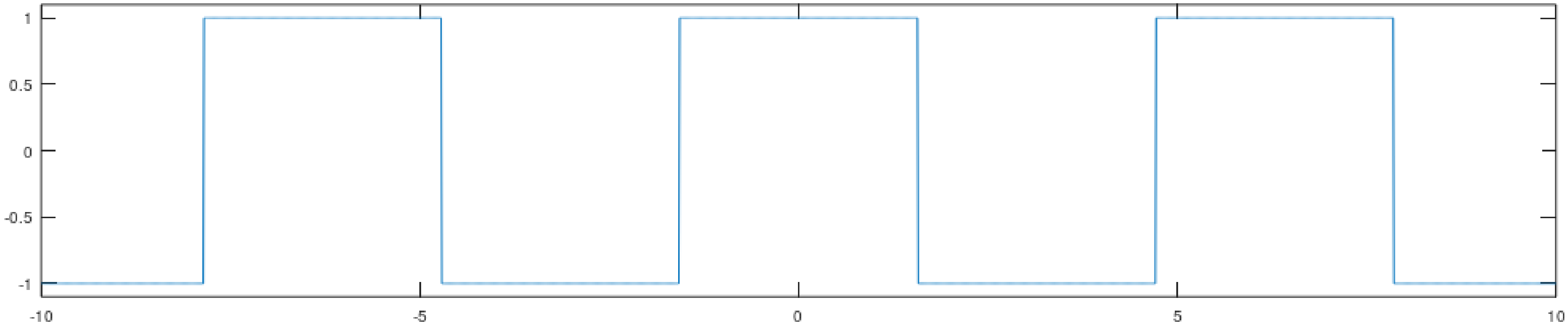
- In the previous example, an odd function, all $a_n = 0$ for n=1,2,….
- Note: cos(x) is an even function, sin(x) is an odd function.
- The Fourier series of odd functions contain only sine components, and that of even functions contain only cosine components.
- Functions that are neither even nor odd contain both components.

Example: Even functions

- Consider the following square wave

$$x(t) = \begin{cases} 1, & |t| \leq \dfrac{\pi}{2} \\ -1, & |t| > \dfrac{\pi}{2} \end{cases}$$

for one cycle, with period 2π.



Calculating of coefficients

- Since this function is even, we can assume that the sine components (b_n) will be zero.

$$a_0 = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\frac{-\pi}{2}} -1\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\ dx + \int_{\frac{\pi}{2}}^{\pi} -1\,dx = 0$$

This is expected, as the function is symmetrical about the x-axis.

Calculation of coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} -\cos(nx)\,dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(nx)\,dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} -\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(-\left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{-\frac{\pi}{2}} + \left[\frac{1}{n}\sin(nx)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left[\frac{1}{n}\sin(nx)\right]_{\frac{\pi}{2}}^{\pi}\right)$$

$$= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

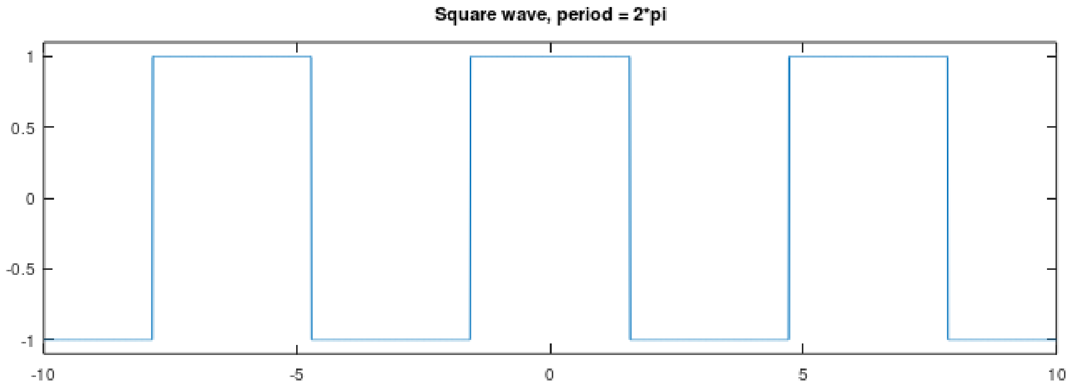
Fourier series: Even function

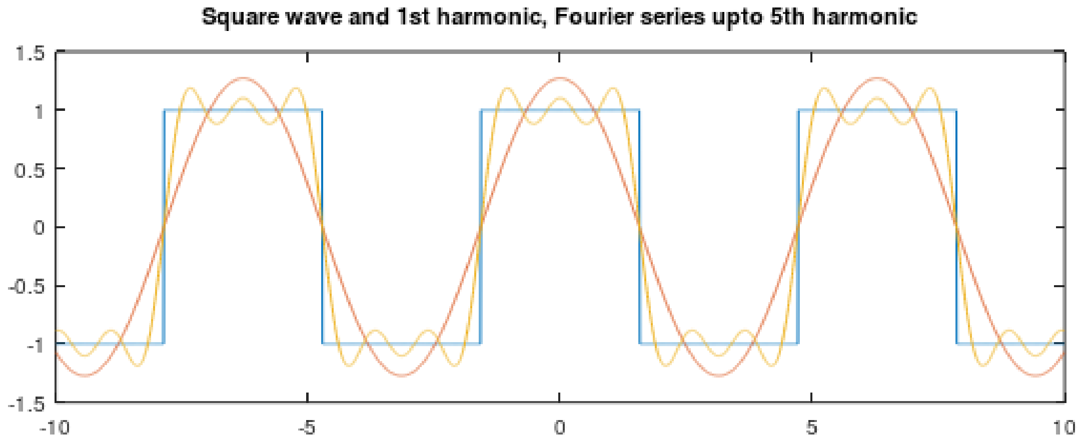
- Substituting the calculated values:

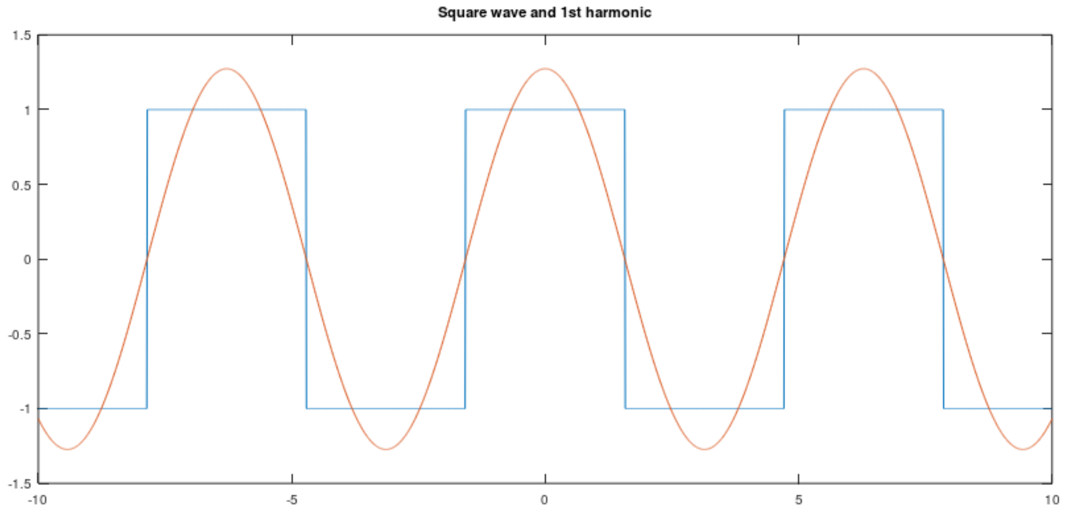
$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right).\cos(nx)$$

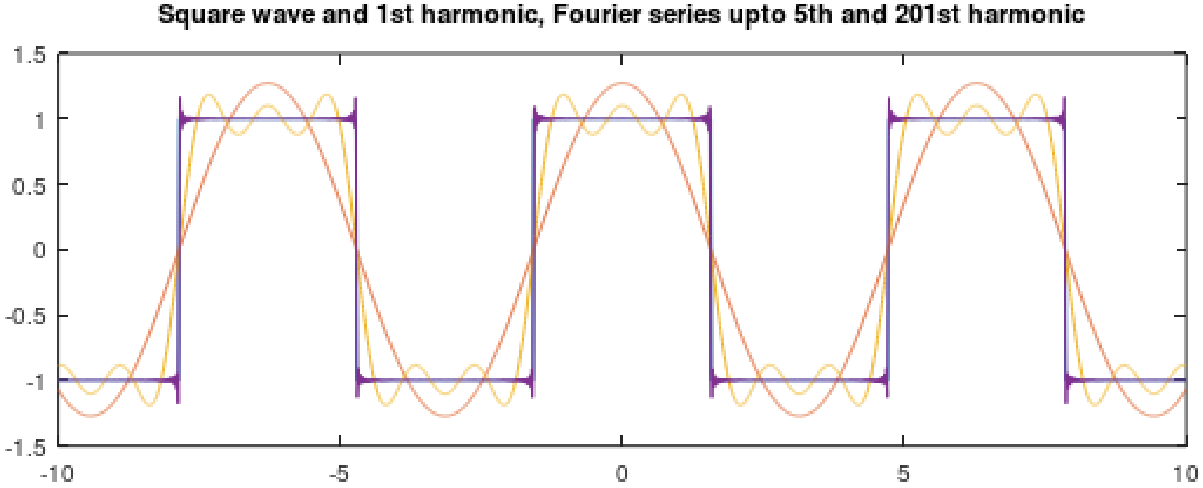
Note that $a_n = 0$ for even values of n, and that $\sin\left(\frac{n\pi}{2}\right)$ will alternate between 1 and -1 for odd values.

Fourier series: even function







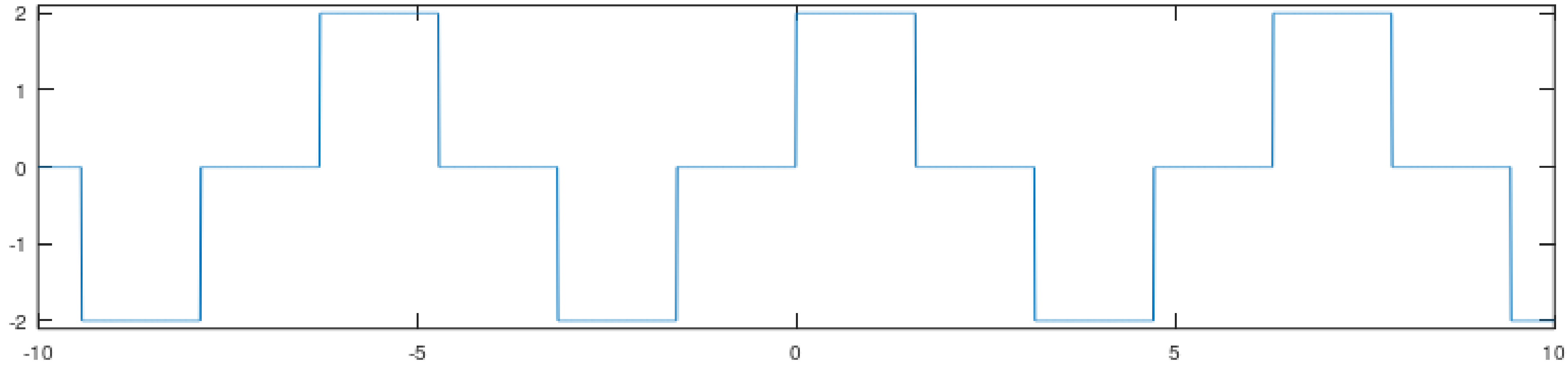


Fourier series: Example 3

- The following function is neither even nor odd.

$$f(x) = \begin{cases} -2, & -\pi \le x < -\frac{\pi}{2} \\ 0, & -\frac{\pi}{2} \le x < 0 \\ 2, & 0 \le x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

This is the sum of two square waves, amplitude 1, one odd, one even.



Example 3: coefficients

- This example has both a_n and b_n.
- $a_0 = 0$

$$a_n = \frac{1}{\pi}\left(-2\int_{-\pi}^{-\frac{\pi}{2}} \cos(nx)\ dx + 2\int_{0}^{\frac{\pi}{2}} \cos(nx)\ \ dx\right)$$

$$= \frac{2}{\pi}\left(-\left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{-\frac{\pi}{2}} + \left[\frac{1}{n}\sin(nx)\right]_{0}^{\frac{\pi}{2}}\right) = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Again, $\sin\left(\frac{n\pi}{2}\right)$ is 0 for even n, and either 1 or -1 for odd n.

Example 3: coefficients

$$b_n = \frac{1}{\pi}\left(-2\int_{-\pi}^{-\frac{\pi}{2}} \sin(nx)\,dx + 2\int_{0}^{\frac{\pi}{2}} \sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(-\left[\frac{-1}{n}\cos(nx)\right]_{-\pi}^{-\frac{\pi}{2}} + \left[\frac{-1}{n}\cos(nx)\right]_{0}^{\frac{\pi}{2}}\right) = \frac{-2}{n\pi}(\cos(n\pi) - 1)$$

$\cos(n\pi)$ is 1 for even values of n, making b_n zero.

$\cos(n\pi)$ is -1 for odd values of n, therefore $b_n = \frac{4}{n\pi}$

Example 3: Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx) + \sum_{n=1}^{\infty} \frac{-2}{n\pi} (\cos(n\pi) - 1) \sin(nx)$$

- In this case, too, the even coefficients (n = 2, 4, …) are zero.
- The reason is the function's half wave symmetry, i.e., $f(x) = -f\left(x - \frac{T}{2}\right)$.

Example 3: Fourier series