CM 1606 Computational Mathematics

Matrices II

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Learning Outcomes

- Covers LO1 for CM 1606
- On completion of this lecture, students are expected to be able to:
  - Recognize how matrices are used to store and process data
  - Discuss the applications of Matrices in Machine Learning and Deep Learning

CONTENT

- Determinant - Higher order
- Diagonal method for 3rd order
- Cofactor matrix
- Adjoint matrix
- Inverse
- Application – Simultaneous equations

Determinants of Third Order

- $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- Select a raw or a column for the expansion
- Find Minors for every element of the selected raw(column)
- Find cofactors and then the determinant
- For higher order determinant

Determinants of Third Order ctd.

Minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Try to get other minors....

Determinants of Third Order ctd.

- Cofactor of a_{ij} th element = $(-1)^{i+j}$ Minor of a_{ij}
- Place signs, $(-1)^{i+j}$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

- Expansion over 1st raw

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Determinants of Third Order - Example

Expanding by 1st row

$$\begin{vmatrix} 2 & 1 & 3 \\ 6 & 0 & -2 \\ -1 & 4 & 5 \end{vmatrix} = 2\begin{vmatrix} 0 & -2 \\ 4 & 5 \end{vmatrix} - 1\begin{vmatrix} 6 & -2 \\ -1 & 5 \end{vmatrix} + 3\begin{vmatrix} 6 & 0 \\ -1 & 4 \end{vmatrix}$$

$$= 2(0-(-8)) - 1(30-2) + 3(24-0) = 16 - 28 + 72 = 60$$

Expanding by 3rd column

$$\begin{vmatrix} 2 & 1 & 3 \\ 6 & 0 & -2 \\ -1 & 4 & 5 \end{vmatrix} = 3\begin{vmatrix} 6 & 0 \\ -1 & 4 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 6 & 0 \end{vmatrix}$$

$$= 3(24-0) + 2(8-(-1)) + 5(0-6) = 72 + 18 - 30 = 60$$

Diagonal Method for third order

- Only for third order matrices

$$\begin{vmatrix} 2 & 1 & 3 \\ 6 & 0 & -2 \\ -1 & 4 & 5 \end{vmatrix}$$

- Find the multiplication of diagonal entries as given, by adding 1st two columns as next two columns

0 -(-16) -30

$$\left[\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 6 & 0 & -2 & 6 & 0 \\ -1 & 4 & 5 & -1 & 4 \end{array}\right]$$

0 +2 +72

Diagonal Method for third order ctd.

- Sum of diagonal multiplications separately

0 + -(-16) + (-30) = -14

$$\left[\begin{array}{ccc} 2 & 1 & 3 \\ 6 & 0 & -2 \\ -1 & 4 & 5 \end{array}\right]\begin{array}{cc} 2 & 1 \\ 6 & 0 \\ -1 & 4 \end{array}$$

0 + 2 + 72 = 74

- Determinant = 74-14=60

Determinants of Third Order ctd.

- Evaluate the following determinants

$$i.\ \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -3 & 4 & 1 \end{vmatrix} \qquad ii.\ \begin{vmatrix} -3 & 4 & 2 \\ 2 & 1 & -4 \\ 0 & 9 & 0 \end{vmatrix}$$

$$iii.\ \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 3 & 1 \end{vmatrix}$$

Cofactor Matrix

- Cofactor Matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

The cofactor for each element of matrix A with the sign :

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

Adjoint matrix

- Cofactor Matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is given by $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$

- Adjoint matrix is denoted as adj(A)
- Defined as

$$\text{adj}(A) = (\text{Cofactor Matrix})^T$$

Inverse of a Matrix

- Denoted as A^{-1}
- If $AB = BA = I$, then $B = A^{-1}$; and $A = B^{-1}$
- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Then

$$AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans: Note that

Can you show the details?

Properties

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^T = A$ and $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Inverse

- Written as $A^{-1} = \frac{1}{|A|} adj(A)$

Where $|A| \neq 0$.

- For a matrix A of order 2 inverse can be written easily as

$$A^{-1} = \frac{1}{|A|} \underbrace{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}_{adj(A)}$$

Inverse for third order matrices

Inverse matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

Solutions for simultaneous equations

eg. $2x+5y=1$

$$x+3y=4$$

$$\Rightarrow \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \text{(matrix representation of the system)}$$

This is of the type $\mathrm{Ax} = \mathrm{b}$, where

$$\mathrm{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \quad \mathrm{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathrm{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Solutions for simultaneous equations ctd.

$$\mathrm{A}\mathrm{x} = \mathrm{b}$$

multiply both sides by A^{-1}

$$\mathrm{A}^{-1}\mathrm{A}\mathrm{x} = \mathrm{A}^{-1}\mathrm{b}$$

$$(\mathrm{A}^{-1}\mathrm{A})\mathrm{x} = \mathrm{A}^{-1}\mathrm{b}$$

$$\mathrm{I}\,\mathrm{x} = \mathrm{A}^{-1}\mathrm{b}$$

$$\mathrm{x} = \mathrm{A}^{-1}\mathrm{b}$$

Then, the solution of the system can be obtained by $\mathrm{A}^{-1}\mathrm{b}$

Then, $\mathrm{x} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -17 \\ 7 \end{pmatrix}$

i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -17 \\ 7 \end{pmatrix} \rightarrow \quad x = -17\,, \quad y = 7.$

Exercises

Solve the given systems using inverse

i) $$2x-3y=-8$$
$$3x+4y=5$$

ii) $$x+2y+z=4$$
$$3x-4y-2z=2$$
$$5x+3y+5z=-1$$

iii) $$2x_1-x_2+3x_3=2$$
$$x_1+3x_2-x_3=11$$
$$2x_1-2x_2+5x_3=3$$