CM 2607 Advanced Mathematics for Data Science

Integration II

ROBERT GORDON UNIVERSITY ABERDEEN

INFORMATICS INSTITUTE OF TECHNOLOGY

Indefinite and Definite Integrals ctd.

Content

- Integration of improper and proper algebraic functions
- Integration by parts
- Integration by substitution
- Applications

Standard form II

$$\int \frac{px+q}{ax^2+bx+c}\,dx;\ p, a \neq 0$$

$$\text{case I}: b^2 - 4ac > 0$$

Use partial fractions and write the anti-derivative using the standard integral

$$\int \frac{1}{px \pm q}\,dx = \frac{1}{p}\ln\left|px \pm q\right| + C$$

$eg:$

$$i)\int \frac{2x+1}{x^2-x-2}\,dx$$

case II : $b^2 - 4ac \le 0$

Find λ and μ such that

Numerator $= \lambda$(Derivative of the denominator) $+ \mu$

Then

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{\lambda(2ax+b)}{ax^2+bx+c} + \frac{\mu}{ax^2+bx+c}$$

$$= \lambda \ln\left|ax^2+bx+c\right| + \mu\left(\text{Standard form I}\right)$$

eg :

$i)\int \frac{2x+7}{4x^2-4x+1}\,dx \qquad ii)\int \frac{7-2x}{x^2-x+4}\,dx$

Integration of improper algebraic functions

$$\int \frac{p_n(x)}{ax^2+bx+c}\,dx; a \neq 0 \text{ and } n \geq 2$$

Express the fraction as a mixed fraction by division. Then using standard form, I or II write the anti-derivative.

$eg:$

$$i)\int \frac{x^2}{x^2+4}\,dx \qquad ii)\int \frac{x^2+8x}{x^2+6x+10}\,dx$$

Integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$eg:$

$$i) \int xe^x dx = \int x \frac{de^x}{dx} dx$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + c$$

$$ii) \int x \sin x dx$$

$$iii) \int x \ln x dx$$

Integration by parts ctd.

Furthermore

$$\int u\frac{dv}{dx}dx = uv - u'v_1 + u''v_2 - u'''v_3 + \dots\dots\dots$$

$$Where$$

$$v_1 = \int v dx,\; v_2 = \int v_1 dx,\; v_3 = \int v_2 dx, \dots\dots\dots\dots\text{and}$$

$$u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2}, \dots\dots\dots\dots$$

Integration by parts ctd.

$eg:$

$$i)\int x^2 \cos x dx = \int x^2 \frac{d}{dx}\sin x dx$$

$$= (x^2)(\sin x) - (2x)(-\cos x) + (2)(-\sin x)$$

Evaluate each of the following definite intagrals.

$$i)\int_1^e x \ln x dx \qquad \text{ii)}\int_0^1 x^2 e^{2x} dx$$

Integration by substitution

If

$$I = \int f(x)dx \quad \text{then since} \ \frac{dI}{dx} = f(x)$$

by the chain rule

$$\frac{dI}{du} = \frac{dI}{dx} . \frac{dx}{du} = f(x)\frac{dx}{du}$$

by integrating w.r.t. u

$$I = \int f(x)\frac{dx}{du}du$$

$eg:$

$$i)\int (x+3)\sqrt{x-2}\,dx=\int (u^2+5)u\frac{dx}{du}\,du$$

where $\sqrt{x-2}=u, then \quad \dfrac{du}{dx}=\dfrac{1}{2u}$

$$\int (x+3)\sqrt{x-2}\,dx=\int (u^2+5)2u^2\,du$$

$$=2\left[\frac{u^5}{5}+5\frac{u^3}{3}\right]+c$$

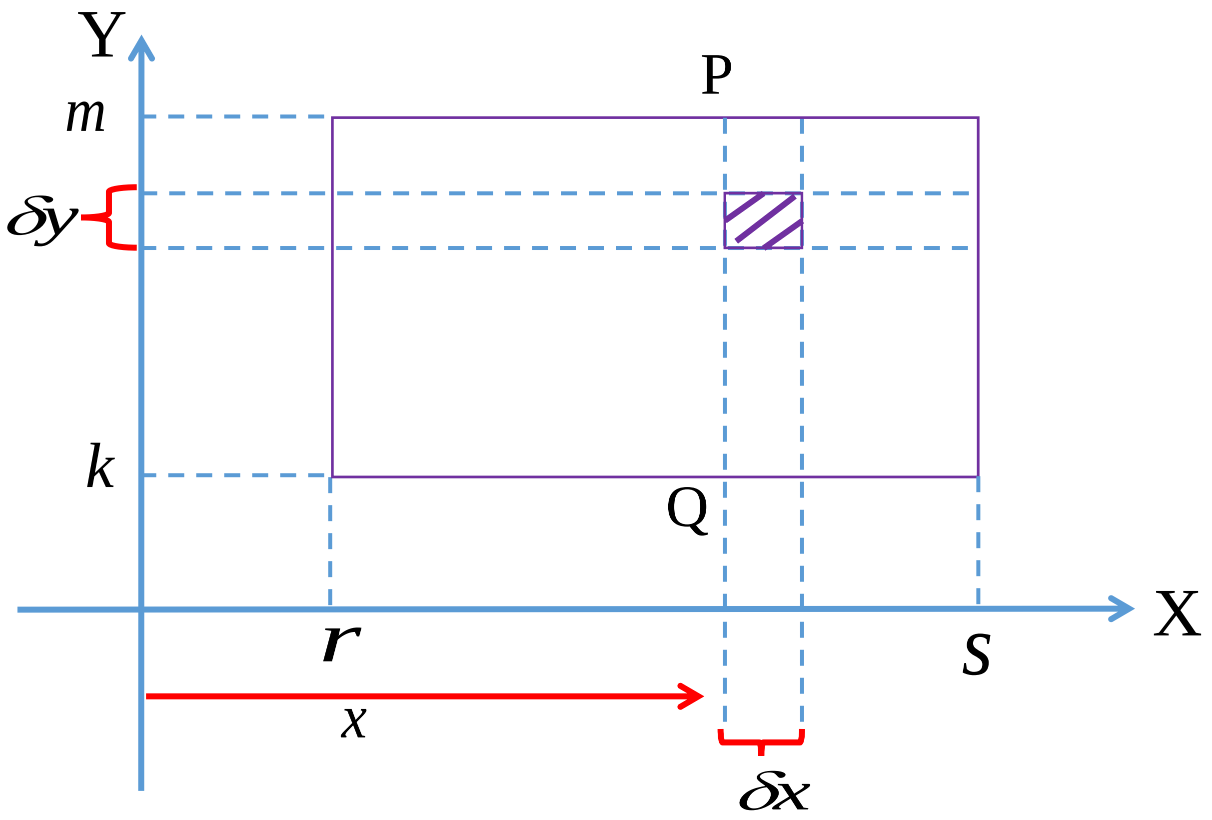
Replace u in terms of x.

$$i)\int \frac{x^2}{\sqrt{x+2}}\,dx \qquad\qquad ii)\int x^3\sqrt{x^2-4}\,dx$$

Applications

Multiple integrals: Summation in two directions

Let us consider a rectangle with the following boundaries.



Multiple integrals: Summation in two directions

Then, the area of the shaded element, $\delta a = \delta y \cdot \delta x$

Area of the vertical strip PQ, $\delta A = \sum_{y=k}^{y=m} \delta y \cdot \delta x$

$\left(\text{Note that during this summation in the y-direction, } \delta x \text{ is constant}\right)$

Total area of the rectangle, $A = \sum_{x=r}^{x=s} \left(\text{all strips like PQ}\right)$

$$A = \sum_{x=r}^{x=s} \left[\sum_{y=k}^{y=m} \delta y \cdot \delta x \right]$$

$$A = \sum_{x=r}^{x=s} \sum_{y=k}^{y=m} \delta y \cdot \delta x$$

Multiple integrals: Summation in two directions

By taking $\delta y \to 0,\ \delta x \to 0$

$$A = \int_{x=r}^{x=s} \int_{y=k}^{y=m} dy\, dx$$

To evaluate this expression, we start from the inside and work outwards.

$$A = \int_{x=r}^{x=s} \left(\int_{y=k}^{y=m} dy \right) dx = \int_{x=r}^{x=s} \left[y\right]_k^m dx$$

$$= \int_{x=r}^{x=s} (m-k)\, dx$$

Multiple integrals: Summation in two directions

Since, m & k are constants,

$$A = (m-k)\int_{x=r}^{x=s} dx$$

$$= (m-k)[x]_r^s$$

$$A = (m-k)(s-r)$$

Note that these are the lengths of the two sides of the rectangle.

Now, let us extend this two - way summation to the double integrals.

Double integrals

The expression $\int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y)\, dx\, dy$ is called a double integral and indicates that :

i) $f(x, y)$ is first integrated w.r.t. x between the limits $x = x_1$ and $x = x_2$ regarding y as being constant.

ii) the result in (i) is then integrated w.r.t. y between the limits $y = y_1$ and $y = y_2$.

Example

eg. 1)Evaluate $I = \int_1^2 \int_2^4 (x+2y)\, dx\, dy$

$$I = \int_1^2 \left[\int_2^4 (x+2y)\, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^2}{2} + 2xy \right]_2^4 dy$$

$$= \int_1^2 \left[(8+8y) - (2+4y) \right] dy$$

$$= \int_1^2 (6+4y)\, dy = \left[6y + 2y^2 \right]_1^2 = 20 - 8 = 12$$

Ex. 2) Evaluate $I = \int_1^2 \int_0^3 x^2 y\, dx\, dy$

Example

eg. Evaluate $I = \int_1^2 \int_0^\pi (3+\sin\theta)\, d\theta\, dr$

$$
\begin{aligned}
I &= \int_1^2 [3\theta - \cos\theta]_0^\pi \, dr \\
&= \int_1^2 [(3\pi+1)-(0-1)]\, dr \\
&= \int_1^2 (3\pi+2)\, dr \\
&= (3\pi+2)[r]_1^2 \\
&= (3\pi+2)(2-1) \\
&= (3\pi+2)
\end{aligned}
$$