CM1606: Computational Mathematics

Conditional Probability

Week No 03| Prashan Rathnayaka *(prashan.r@iit.ac.lk)*

Learning Outcomes

On completion of this lecture,

students are expected to be able to:

- demonstrate the knowledge associated with basic probability concepts
- Apply the ideas of conditional probability to solve real-world problems

Conditional Probability

- Conditional probability is a very important area in probability.
- It is relevant when partial information is known.
- It is assumed students are familiar with the concept of ‘sample space’, which is the ‘set of all possible outcomes’ in a random experiment.

Example 01

- Nimal rolls a balanced die and sees the exact outcome.
- But he only tells Kamal that he got an '**odd**' outcome.
- Kamal knows that it has to be either '1' or '3' or '5'.
- What is the probability that the outcome has been a prime?

without any information before-hand, $P(\text{prime}) = \frac{3}{6} = \frac{1}{2}$;

but with the updated knowledge $P(\text{prime}) = 2/3$.

- In general, if A and B are two events in a sample space S, the probability of event A, when B has been revealed, is $P\ (A \mid B)$.

- It is pronounced **conditional probability of A given B**.

Formula

for $P(B) > 0$

$$P\,(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Back to Example 01:

Let $A = \{\text{outcome has been a prime}\}$, and $B = \{\text{outcome has been odd}\}$.

$$P(A) = \frac{3}{6} = \frac{1}{2}, \qquad P(B) = \frac{3}{6} = \frac{1}{2}, \qquad \text{and } P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

Now, $P\,(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3}$.

Note

When calculating conditional probability, the original sample space is restricted by the given information.

In Example 01,

original sample space: $S = \{1, 2, 3, 4, 5, 6\}$ and the restricted sample space, S^* is a subset of S, and $S^* = \{1, 3, 5\}$.

Multiplication Rule

For any two events A, and B, in sample space S

$$P(A \cap B) = P\ (A \mid B)\ .\ P(B) \text{ if } P(B) > 0$$

Similarly,

$$P(B \cap A) = P\ (B|\ A)\ .\ P(A) \text{ if } P(A) > 0$$

Therefore,

$$P\ (B|\ A)\ .\ P(A) = P\ (B|\ A)\ .\ P(A) \text{ if } P(A) > 0 \text{ and } P(B) > 0.$$

Independence of Events

Let A and B be two events in a sample space. Then, **$\boldsymbol{A}$ and $\boldsymbol{B}$ are independent if and only if $\boldsymbol{P(A \cap B) = P(A).P(B)}$**

Example 02:

Consider tossing a fair coin twice. Let $\mathrm{A} = \{\text{heads on the 1st toss}\}$, and $\mathrm{B} = \{\text{heads on the 2nd toss}\}$

Are the above events A and B independent?

Total Probability Law

$B_1, B_2, \dots, B_k$ is said to be a **disjoint partition** of S, if $S = B_1 \cup B_2 \cup \cdots \cup B_k$ and $P(B_i \cap B_j) = 0$ for $i \neq j$.

Let A be any event in S

Now,

$$
\begin{aligned}
P(A) = P(A \cap S) & \\
&= P\{ A \cap (B_1 \cup B_2 \cup \cdots \cup B_k)\} \\
&= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k) \\
&= P(A|B_1).P(B_1) + P(A|B_2).P(B_2) + \cdots + P(A|B_k).P(B_k)
\end{aligned}
$$

(above result called '**total probability law**' has many applications in statistics)

Example 03

Suppose in a given region 10% of the adult population is known to be infected with Covid-19. Assume that 2% of this population is 'Covid infected without showing any symptoms.

If a randomly selected person is known to have Covid-19, what is the probability of him not showing any symptoms?

Example 04

There are two boxes.

Box 1 contains 5 Red and 4 Blue marbles. Box 2 contains 3 Red and 2 Blue marbles.

A marble is drawn at random from Box 1, its colour noted, and a marble of the same colour is added to Box 2. Now, a marble is drawn at random from Box 2.

What's the probability that it is Red?

Example 05

In a certain university, 70% of the students get news from social media. Out of these, 20% watch TV news.

What percentage of students see both?

Exercise 06

If A and B are independent events in a sample space, S,

show that

(i) A and B-complement are also independent

(ii) A-complement and B are also independent,

(iii) A-complement and B-complement are also independent.

(Note that they are both events in S)

Example 07

A mobile phone manufacturer gets his batteries from three different suppliers.

Suppose supplier1 accounts for 30%, supplier2 40%, and supplier3, 30% of the total demand.

The rate of battery life ending before the warranty period is: .01 for supplier1; .005 for supplier2; and .008 for supplier3.

A randomly selected battery was found to be faulty before warranty period.

What is the probability that it came from supplier1? supplier2? supplier3?

Exercise 08

Suppose an airplane has two engines that work or fail independently. Assume that each has a probability 0.02 of failing during a given flight. The plane can fly safely as long as at least one engine works.

What is the probability for the next flight to be safe (assuming a crash can only be due to engine failure)?

READING

Introduction to Probability and Statistics by Tim Swartz