CM 2607 Advanced Mathematics for Data Science

Lecture 01

Differentiation part I

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Learning Outcomes

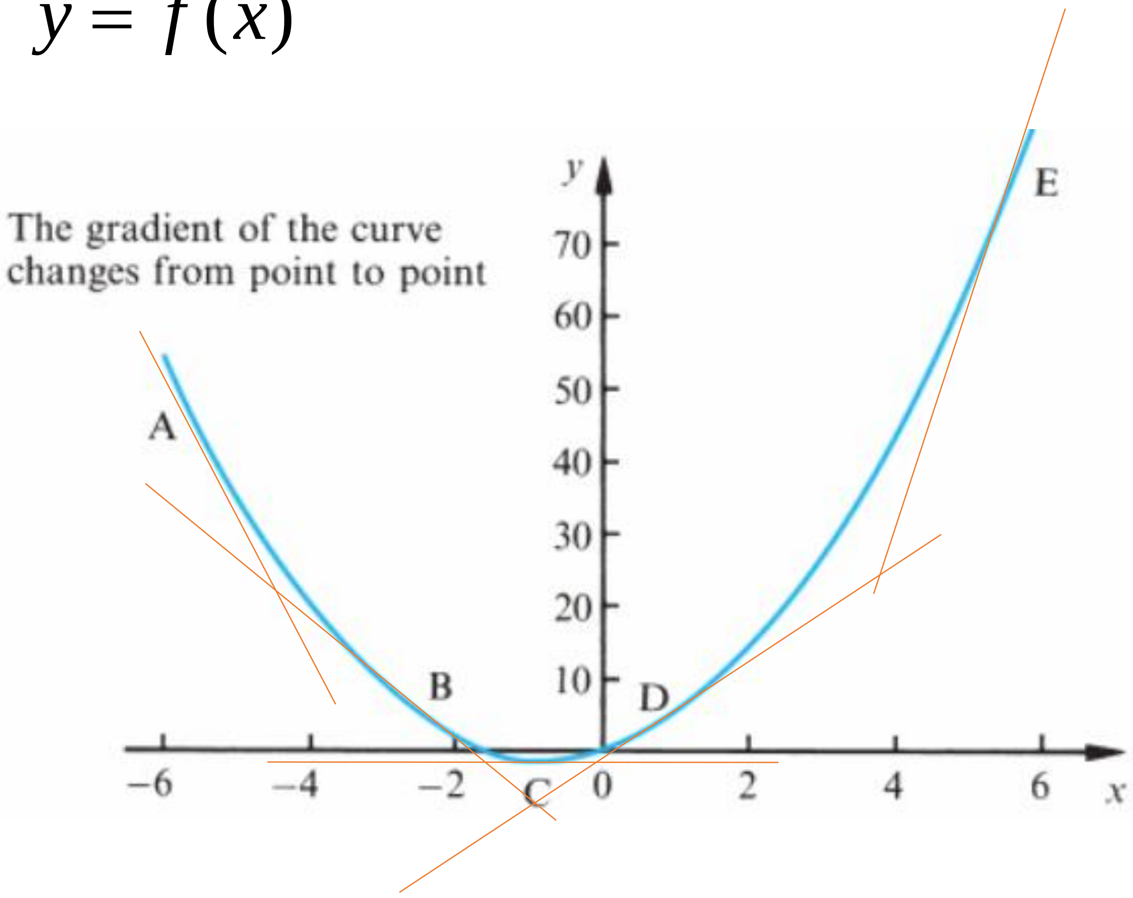
- Covers LO1 and LO2 for CM2607
- On completion of this series of lectures on differentiation, students are expected to be able to:
  - Identify and explain the gradient function
  - Find the gradient function for single variable polynomial functions
  - Evaluate the gradient function at a given point
  - Identify some standard derivatives

CONTENT

- Gradient of a function
- Gradient function
- Gradient function of $y = x^n$
- Finding gradient at any point
- Rules on differentiation
- Some standard derivatives
- Higher order derivatives
- Derivative of a composition functions

Gradient of a function

- Consider the one variable function $y = f(x)$
- Slope of the function at any point



Gradient of a function ctd.

- At point A – Graph is falling rapidly
- At point B - Graph is falling but less rapidly than point A
- At point C – lies at the bottom of the dip
- At point D – Graph is rising
- At point E - Graph is rising but more quickly than D

Note: Gradient of the curve changes from point to point

$$\text{Gradient of the tangent drawn} \simeq \text{Gradient of the curve at the point contact}$$

Gradient function

- Written as $\dfrac{dy}{dx}$ for the function $y = f(x)$
- Read as ‘dee y by dee x’
- Denoted as y' or $f'(x)$
- Knowing the gradient function, gradient of the function at any point can be found
- Measures how rapidly a graph is changing

Gradient function of $y = x^n$

- For any function of the form $y = x^n$, the gradient function is

$$y' = nx^{n-1}$$

Eg: Find the gradient functions

$$i)\, y = x \qquad ii)\, y = x^5 \qquad iii)\, y = x^{-4} \qquad iv)\, y = x^{\frac{3}{2}} \qquad v)\, y = x^{-\frac{1}{3}}$$

Finding gradient at any point

- Consider the point $A(x_0, y_0)$ lies on the curve $y = f(x)$
- Finding the gradient at A, of the function $y = f(x)$

  - Find the gradient function y'
  - Substitute the x-coordinate of the point A in then gradient function y'

  So,

$$\text{The gradient at } A(x_0, y_0) = y'(x_0) = f'(x_0)$$

Example

1) Find the gradient of the function $y = x^2$ at the points

$$i)x = -2 \qquad ii)x = 0 \qquad iii)x = 3 \qquad iv)x = \frac{5}{2}$$

2) Find the gradient function of $y = x^{-4}$. Hence find the gradient at the point A(-1,2)

3) Find the gradient function of $y = 1$. Hence find the gradient function of any constant function $y = k$.

Gradient function ctd.

Gradient function is AKA

- First derivative
- Derivative (the process of obtaining the derivative is differentiation)
- rate of change

Note: Gradient functions of wide range of functions can be found by using standard results obtained for some main functions and rules.

Eg: Trigonometric functions

Exponential function and natural log functions

Rules on Differentiation I

Let f and g be functions of x.

1. Addition

$$\frac{d}{dx}(f+g)(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f' + g'$$

2. Substraction

$$\frac{d}{dx}(f-g)(x) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f' - g'$$

Example

1) Find the gradient function of $y = x^5 + x^{-4}$

$$y = x^5 + x^{-4}$$

$$= 5x^4 - 4x^{-5}$$

Similarly,

$$y = x^5 + x^{-4} + x$$

$$= 5x^4 - 4x^{-5} + 1$$

Try the same example with subtraction.

Rules on Differentiation II

Let f be a functions of x.

1. Multiply by a constant c

$$\frac{d}{dx} cf(x) = c\frac{d}{dx} f(x) = cf'$$

Let f and g be functions of x and two constants α and β

2. A linear combination of functions

$$\frac{d}{dx}\left(\alpha f(x) \pm \beta g(x)\right) = \alpha\frac{d}{dx} f(x) \pm \beta \frac{d}{dx} g(x) = \alpha f' \pm \beta g'$$

Example

1) Find the derivative of the functions given.

$$i)\, y = -3x^4 + 2x^{-4} - 5x$$

$$ii)\, y = \frac{5}{2}x^3 - \frac{2}{x^4} + \frac{1}{x}$$

$$iii)\, y = (3 - 2x)^2$$

Standard derivatives

Function: $y = f(x)$	Derivative: $\dfrac{dy}{dx} = y'$
1. a (constant)	0
2. x^n	$n\,x^{n-1}$
3. e^x	e^x
4. e^{kx}	$\mathrm{k}e^{kx}; k - \text{constant}$
5. $\ln x$	$\dfrac{1}{x}$
6. $\ln kx$	$\dfrac{1}{x}; k - \text{constant}$

Standard derivatives ctd.

	Function: $y = f(x)$	Derivative: $\frac{dy}{dx} = y'$
7.	$\sin x$	$\cos x$
8.	$\cos x$	$-\sin x$
9.	$\sin kx$	$k\cos kx; k - \text{constant}$
10.	$\cos kx$	$-k\sin kx; k - \text{constant}$
11.	$\tan x$	$\sec^2 x$

Standard derivatives ctd.

Function: $y = f(x)$		Derivative: $\dfrac{dy}{dx}$
12.	$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
13.	$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}$
14.	$\tan^{-1} x$	$\dfrac{1}{1+x^2}$

Example

1) Gradient of the function $y = 3x^2 - px + 5$ at (-1,3) is 6. Find the value of p.

Higher derivatives

- First derivative of $f(x) = f' = \frac{df}{dx}$
- Second derivative of $f(x) = f'' = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$
- nth derivative of $f(x) = f^{(n)} = \frac{d^n f}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} f}{dx^{n-1}}\right)$

Higher derivatives

Example

1)second derivative of $x_3{}^{-2}$

$$= \frac{d^2(x^{-2})}{dx^2} = \frac{d}{dx}\left(\frac{dx^{-2}}{dx}\right) = \frac{d}{dx}\left(-2x^{-3}\right) = -2\frac{d}{dx}\left(x^{-3}\right)$$

$$= -2(-3)x^{-4} = 6x^{-4}$$

2)Find $\dfrac{d^2(4\sqrt{x})}{dx^2}$

3)Find $\dfrac{d^3(x^3+2x^2+6)}{dx^3}$

Rules of differentiation

Let f and g be functions of x.

1. Multiplication (Product)

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

2. Division (Quotient)

$$\frac{d}{dx}\left(f/g\right) = \frac{g\dfrac{df}{dx} - f\dfrac{dg}{dx}}{g^2} \quad ; \quad g \neq 0$$

Rules of differentiation

Example

$$1)\ \frac{d(y^3.e^y)}{dy} = y^3 \frac{d(e^y)}{dy} + e^y \frac{d(y^3)}{dy}$$

$$= y^3.e^y + e^y(3y^2)$$

$$2)\ \frac{d(\sqrt{x}e^x)}{dx}$$

Rules of differentiation

Example

$$1)\ \frac{d\left(\dfrac{y}{\sin y}\right)}{dy} = \frac{\sin y \dfrac{dy}{dy} - y \dfrac{d \sin y}{dy}}{\sin^2 y}$$

$$= \frac{\sin y - y \cos y}{\sin^2 y}$$

$$2)\ \frac{d^2\left(\dfrac{y}{\sin y}\right)}{dy^2}$$

Rules of differentiation

3. Chain rule (on function composition)

Let $y = (g \circ f)(x)$,

then $\dfrac{dy}{dx} = \dfrac{dy}{df} \cdot \dfrac{df}{dx}$

eg. $y = e^{\sin x}$. Here $f(x) = \sin x$ & $g(x) = e^x$

which yeild $y = (g \circ f)(x)$.

Then, $\dfrac{dy}{dx} = \dfrac{dy}{df} \cdot \dfrac{df}{dx}$

$= e^{\sin x} . \cos x$

Rules of differentiation

Exercise

Find the first derivative of following functions

$$1)(2x-3)^{10}$$

$$2)\sin^4 x$$

$$3)\sin 2x$$

$$4)5\cos^2 3x$$