CM2607 Advanced mathematics for data science

Sequences and series

Week No 07

ROBERT GORDON UNIVERSITY ABERDEEN
INFORMATICS INSTITUTE OF TECHNOLOGY

Learning Outcomes

- Covers LO3 for Module
- On completion of this lecture, students are expected to be able to:
  - Define sequences and series
  - Define series and the sequence of terms of a series
  - Arithmetic progressions
  - Geometric progressions

Sequences

- Definition: An enumerated collection of objects in which repetitions are allowed an order matter.

- Examples:
  - The alphabet: A, B, C, …, Z
  - Natural numbers: 1, 2, 3, 4, 5, …
  - The Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, …
  - Any others?

Defining a sequence

- Each element in a sequence can be numbered as follows:

  - 1st Element: a_1
  - 2nd Element: a_2

    .

    .

    .
  - nth element: a_n

- Series: sum of terms of the sequence.
  - $S_n = a_1 + a_2 + \cdots + a_n$

Defining a sequence

- A sequence can defined as a function of the position of each element.
- Example: $a_n = n^2$-1

  The resulting elements would be:

  $a_1 = 1^2$–1 = 0, $a_2 = 2^2$-1 = 3, …., $a_n = n^2$-1

  resulting sequence: 0, 3, 8, 15, …

Defining a sequence

- A sequence can also be defined recursively, as a function of previous elements.
- Example: Fibonacci sequence. The elements are defined as:

$$a_1 = 1, a_2 = 1$$
$$a_n = a_{n-1} + a_{n-2}$$

- Therefore, $a_3 = a_2 + a_1 = 1 + 1 = 2$
- The rest of the element can be calculated the same way.

Arithmetic progression

- Definition: A sequence of numbers where the difference between consecutive terms is constant.
- Example: 1, 5, 9, 13, …
- Formula:

$$a_n = a_1 + (n - 1)d$$

where

a_1 is the first term of the sequence,

d is the difference between two consecutive terms

Arithmetic series

- Definition: The sum of terms of an arithmetic progression
- Example: 1 + 5 + 9 + 13
- Formula:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

where

S_n is the sum of the first n terms of the series
n is the number of terms of the series
a_1 is the first term of the sequence,
a_n is the last term of the series

Arithmetic series

Derivation of formula:

$S_n = \ a_1 + (a_1 + d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$

Writing the terms in terms of a_n instead of a_1 :

$S_n = (a_n \ - (n-1)d) + (a_n \ - (n-2)d) + \ \cdots + (a_n - \mathrm{d}) + a_n$

Adding the two gives:

$$2S_n = n(a_1 + a_n)$$

$$\therefore S_n = \frac{n(a_1 + a_n)}{2}$$

Arithmetic series

- Alternative Formula:

$$S_n = \frac{n(2a_1 + (n-1)d)}{2}$$

where

S_n is the sum of the first n terms of the series
n is the number of terms of the series
a_1 is the first term of the sequence,
d is the difference between two consecutive terms

Note that this is derived by substituting $a_n = a_1 + (n-1)d$

Geometric progression

- Definition: A sequence of non-zero numbers where the ratio between two consecutive numbers is constant.
- Example: 1, 2, 4, 8, …
- Formula:

$$a_n = ar^{n-1}$$

where

a is the first term of the sequence,

r is the ratio between two consecutive terms

Note: r can be negative. E.g.: 1, -2, 4, -8, …

Geometric series

- Definition: The sum of terms of a geometric progression
- Example: 1 + 2 + 4 + 8
- Formula:

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

where

S_n is the sum of the first n terms of the series
n is the number of terms of the series
a is the first term of the sequence,
r is the ratio between two consecutive terms

Constant sequence

- A constant sequence is a sequence where every term of the sequence is the same real number.
- Example: 1, 1, 1, 1, …
- Some sequences are eventually constant, i.e., 1, 4, 6, 8, 8, 8, …

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Monotonic sequences

- A monotonic sequence is defined as a sequence where either

$$a_{i+1} \leq a_i \quad for\ every\ i \geq 1$$

Or

$$a_{i+1} \geq a_i \quad for\ every\ i \geq 1$$

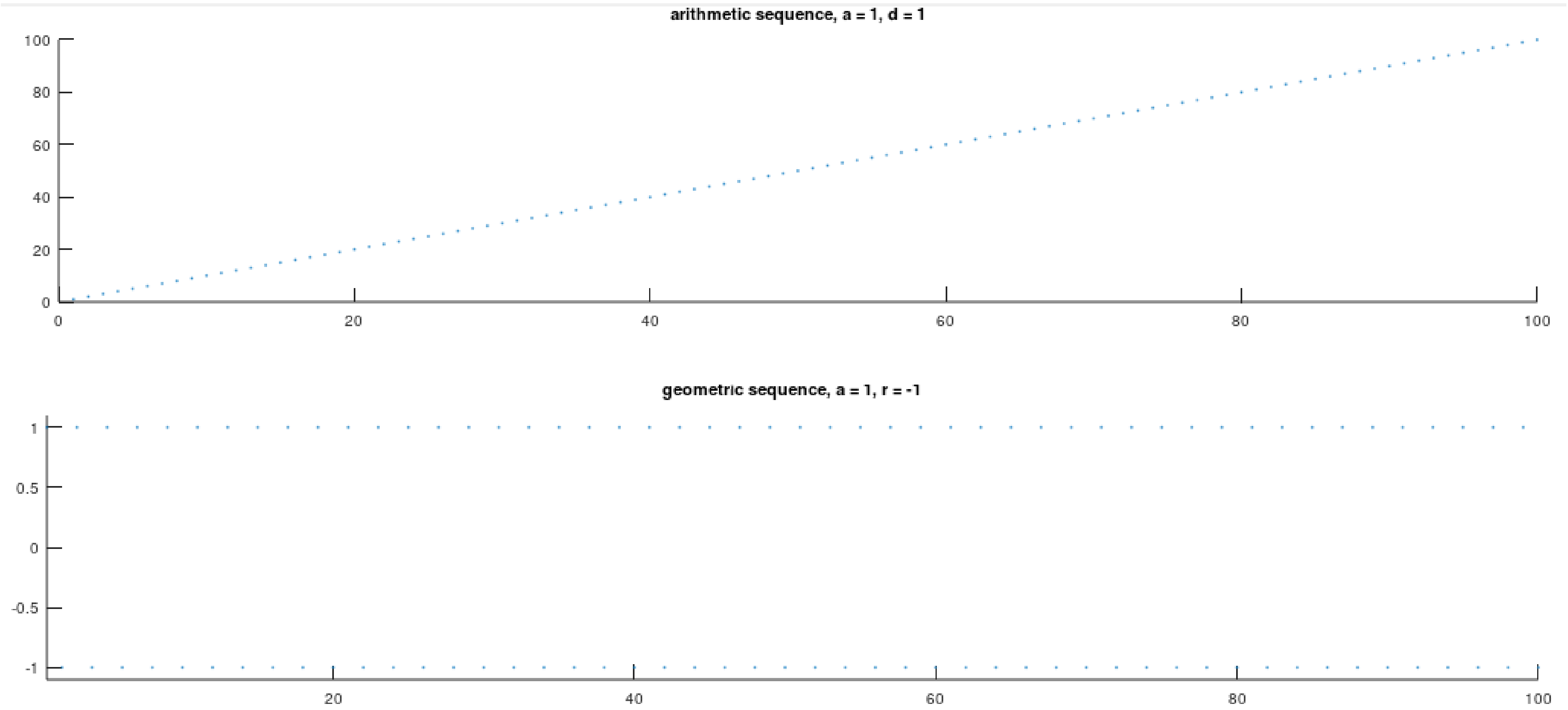
- A monotonic sequence is bounded when either

$$There\ exists\ N\ when\ a_n \leq N\ for\ every\ n \geq 1$$

Or

$$There\ exists\ N\ when\ a_n \geq N\ for\ every\ n \geq 1$$

Are these sequences monotonic or bounded?



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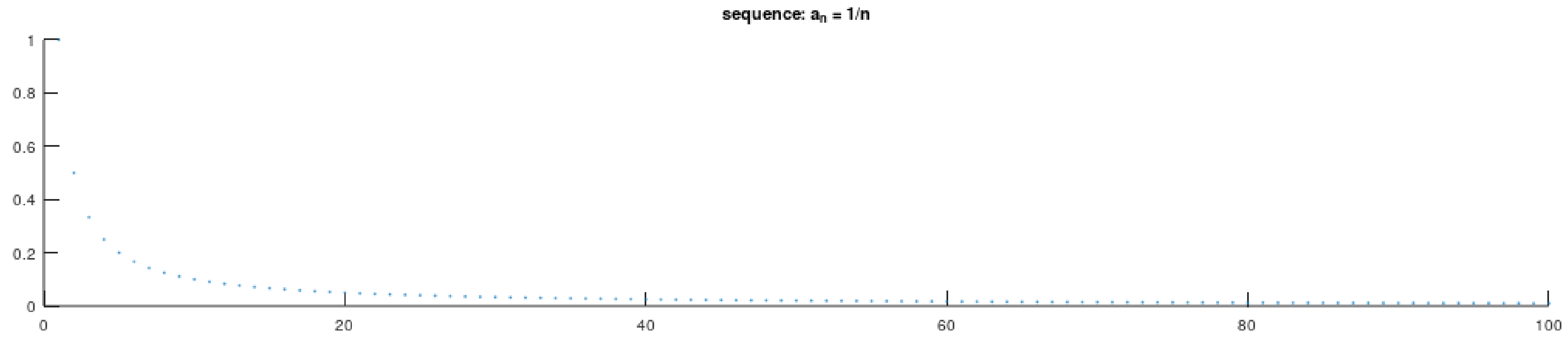

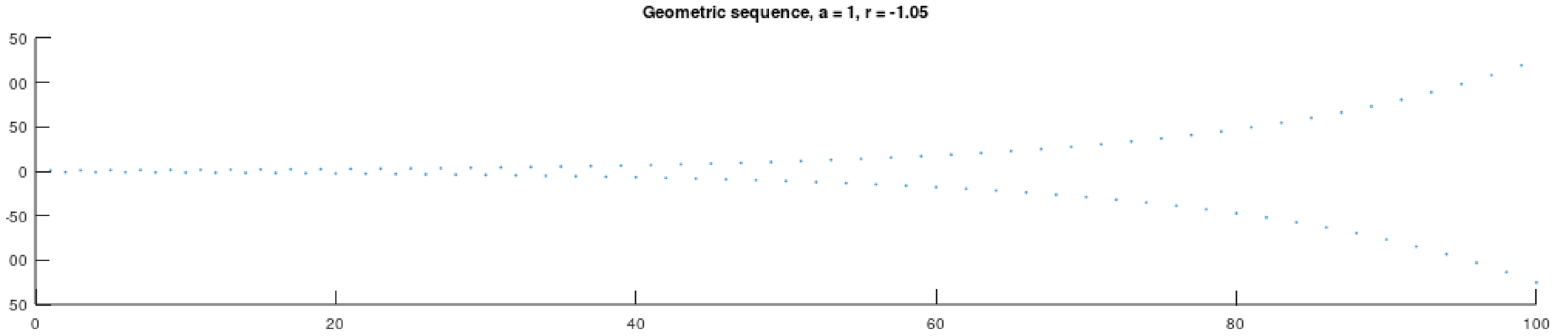

Convergence and divergence of sequences

- The limit of a sequence is defined as

$$\lim_{n \to \infty} a_n$$

- If this limit exists, the sequence is said to be convergent.
- If the sequence is defined by a function, its limit can be found by finding the limit of the function.
- Constant sequences converge to the constant value of its elements.
- Bounded (both upper and lower bounded) and monotonic sequences are also convergent.

Convergence and divergence of sequences

- If $\lim_{n \to \infty} a_n$ does not exist, the sequence is divergent.
- Divergence can happen in one of the following ways:
  - Diverge to +∞: 1 + 2 + 4 + 8 + …
  - Diverge to -∞: -1 – 2 – 4 – 8 – …
  - Oscillate finitely: 1 – 1 + 1 – 1 + …
  - Oscillate infinitely: 1 – 2 + 4 – 8 + …

Convergence and divergence of sequences

Example: $a_n = \frac{1}{n}$

Limit of the sequence:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

As the limit exists, this series converges.

Convergence and divergence of sequences

Example: $a_n = n$

Limit of the sequence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$$

This limit is infinite, therefore the sequence diverges.

Example: $a_n = (-1)^n$

Limit of the sequence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n$$

This limit is undefined. Therefore, this series does not converge.

Properties of limits

- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} c \cdot a_n \; = c. \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} a_n \cdot b_n \; = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad given \lim_{n\to\infty} b_n \neq 0 \; and b_n \neq 0 \; for \; all \; n \in N$
- $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right|$

Squeeze rule of limits

- Squeeze rule:

$$If\ a_n \leq c_n \leq b_n\ for\ all\ n > N\ for\ some\ N, and$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L,$$
$$then\ \lim_{n\to\infty} c_n = L$$

Squeeze rule example

Consider the sequence $c_n = \frac{\sin(n)}{n}$

we know that $-1 \leq \sin(n) \leq 1$ for all n.

Take $a_n = \frac{-1}{n}$ and $b_n = \frac{1}{n}$, such that $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

Calculate limits for a_n and b_n: $\lim_{n\to\infty} \frac{-1}{n} = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$

From the squeeze rule, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$