CM2607 Advanced Mathematics for Data Science

Discrete Fourier Transform / Discrete Cosine Transform

Week No 11

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Learning Outcomes

- Covers LO3 for Module
- On completion of this lecture, students are expected to be able to:
  - Understand Discrete Fourier transform
  - Understand Discrete Cosine transform

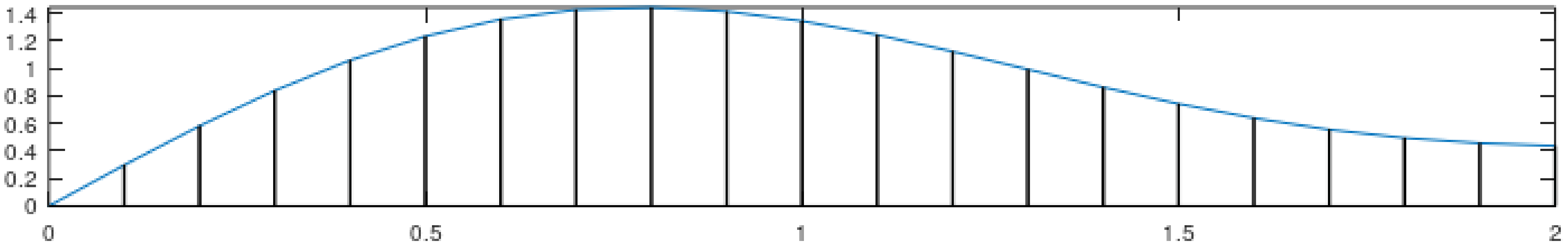
Discrete Fourier transform

- This is the Fourier transform as applied to discrete data
- Used because many real data is sampled discretely, such as at a fixed frequency, and therefore, a smooth waveform is not available for analysis.
- Examples:
  - Sampled audio from a microphone
  - The brightness values of pixels of an image
  - Temperature readings from a thermometer
- Can be implemented easily in computers as it deals with discrete data

Discrete signals

Consider the waveform below. It is a continuous signal 2s long. However, in most cases, we will be sampling the signal at fixed intervals (0.1 s in this case) as shown by the dark bars.

Sampled output: [0 0.2977 0.5816 0.8390 1.0588 1.2327 1.3553 1.4247 1.4423 1.4127 1.3432 1.2429 1.1223 0.9921 0.8624 0.7422 0.6383 0.5553 0.4950 0.4568 0.4378]



Frequency analysis

- We can't apply the Fourier transform directly to this signal.
- Solution: Discrete Fourier transform.
- Equivalent to the continuous Fourier transform, but for signals where only N samples are available, separated by time T (0.1 s in previous example)
- If the original signal is $f(t)$, and the samples are $f(0), f(1), \cdots, f(k), \cdots f(N-1)$, its Fourier transform would be:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \; dt$$

Discrete Fourier transform

- To derive the Discrete Fourier transform, we can regard every sample $f(k)$ as an impulse with area $f(k)$.
- The integral becomes:

$$F(\omega) = f(0)e^{-i0} + f(1)e^{-i\omega T} + \cdots + f(k)e^{-i\omega kT} + \cdots + f(N-1)e^{-i\omega(N-1)T}$$

$$F(\omega) = \sum_{k=0}^{N-1} f(k)e^{-i\omega kT}$$

Discrete Fourier transform

- The previous equation can be evaluated for any ω, but as we have only N sample points, N outputs will be sufficient.
- Due to the limited number of data points, the DFT treats the signal as if it were periodic (similar to the Fourier series).
- Therefore, we evaluate DFT for the fundamental frequency and its harmonics, i.e.,

$$0, \quad \frac{2\pi}{NT}, \quad \frac{2\pi}{NT} \times 2, \cdots, \frac{2\pi}{NT} \times n, \cdots, \frac{2\pi}{NT} \times (N-1)$$

General Equation:

$$F[n] = \sum_{k=0}^{N-1} f(k) e^{-i\frac{2\pi}{N}nk} \quad for\; n = [0: N-1]$$

Discrete Fourier transform, matrix form

- DFT can also be written in matrix form:

$$\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\frac{2\pi}{N}} & e^{-i\frac{2\pi}{N}.2} & \cdots & e^{-i\frac{2\pi}{N}.(N-1)} \\ 1 & e^{-i\frac{2\pi}{N}.2} & e^{-i\frac{2\pi}{N}.2.2} & \cdots & e^{-i\frac{2\pi}{N}.(N-1).2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & e^{-i\frac{2\pi}{N}.(N-1)} & e^{-i\frac{2\pi}{N}.2.(N-1)} & \cdots & e^{-i\frac{2\pi}{N}.(N-1).(N-1)} \end{bmatrix} \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{bmatrix}$$

Discrete Fourier transform: Example

- Let's take a sample, N = 4, T = 0.2 as follows:
  $f(0) = 1.2327, f(1) = 1.4247, f(2) = 1.4127, f(3) = 1.2429$

$$\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-i\frac{2\pi}{4}} & e^{-i\frac{2\pi}{4}.2} & e^{-i\frac{2\pi}{4}.3} \\ 1 & e^{-i\frac{2\pi}{4}.2} & e^{-i\frac{2\pi}{4}.2.2} & e^{-i\frac{2\pi}{4}.3.2} \\ 1 & e^{-i\frac{2\pi}{4}.3} & e^{-i\frac{2\pi}{4}.2.3} & e^{-i\frac{2\pi}{4}.3.3} \end{bmatrix} \begin{bmatrix} 1.2327 \\ 1.4247 \\ 1.4247 \\ 1.2429 \end{bmatrix}$$

Discrete Fourier transform: Example

- Simplifying using Euler's formula: $e^{ix} = \cos(x) + i\,\sin(x)$

$$\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1.2327 \\ 1.4247 \\ 1.4247 \\ 1.2429 \end{bmatrix} = \begin{bmatrix} 5.3129 \\ -0.1801 - 0.1818i \\ -0.0222 \\ -0.1801 + 0.1818i \end{bmatrix}$$

Inverse Discrete Fourier transform

- Equation:

$$f(k) = \frac{1}{N}\sum_{n=1}^{N-1} F(n)e^{i\frac{2\pi}{N}nk}$$

- i.e., the matrix form is 1/N time the complex conjugate of the forward transform.
- NOTE: F(n) is usually complex, unlike f(k) which is usually real (there are exceptions).

Discrete Fourier transform: problems

- In the previous example, you may have noticed that F(1) and F(3) are complex conjugates of each other, i.e., $F(1) = F^*(3)$
- In general, DFT results in $F(n) = F^*(N - n)$
- Out of these, the lower frequency $\left(n \leq \frac{N}{2}\right)$ is considered valid.
- The higher component $\left(n > \frac{N}{2}\right)$ is called the aliasing frequency.
- What this means is frequencies above $\frac{2}{T}$, or $2f_s$ where f_s is the sampling frequency are not captured accurately and result in errors.

Aliasing

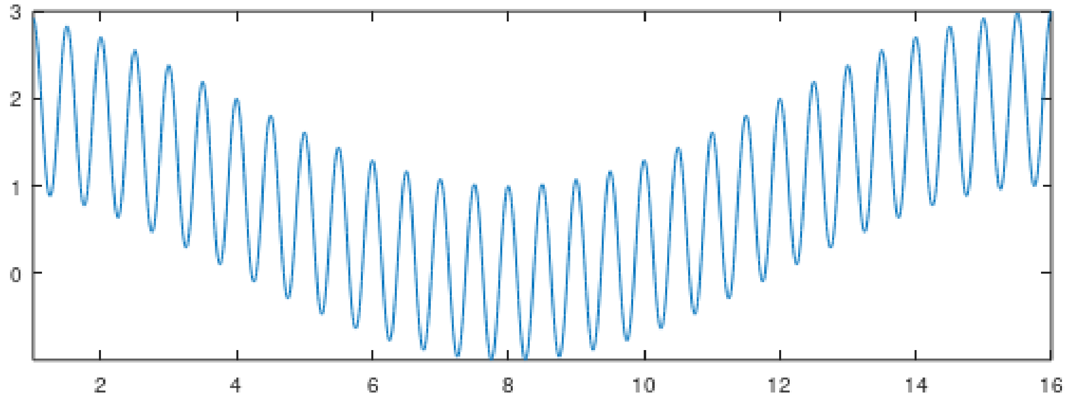
- F(n) for $n > \frac{N}{2}$ are not accurate.
- What this means is frequencies above $\frac{2}{T}$, or $2f_s$ where f_s is the sampling frequency, are not captured accurately.
- The solution is to either
  - Filter out high-frequency components of the signal

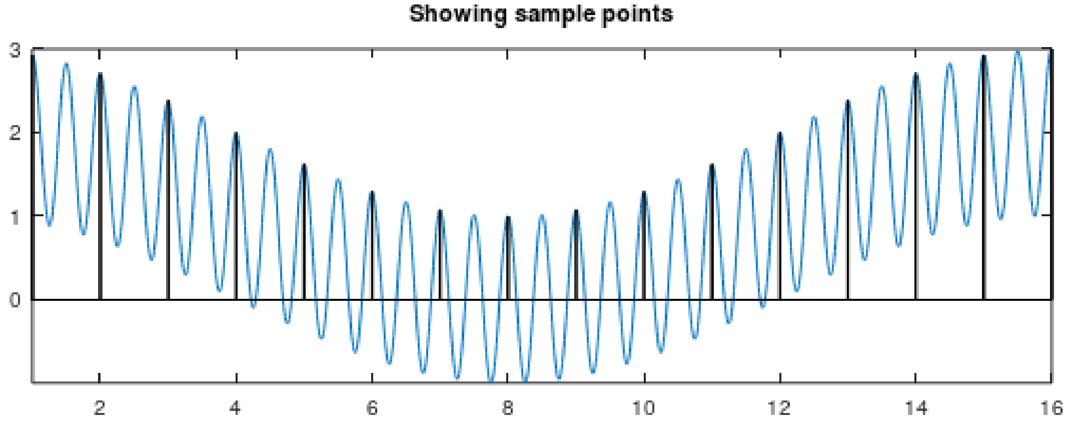
    or

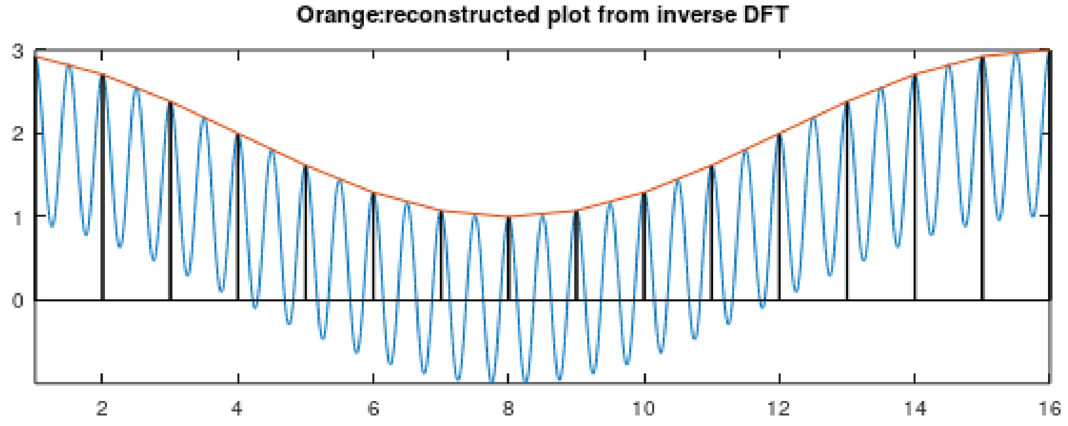
  - Increase sampling rate

Aliasing

- The following plots are of the function $y(t) = 1 + \cos\left(\frac{\pi t}{8}\right) + + \cos(4\pi t)$
- 16 samples are taken at 1s intervals. Note that there is a component with a frequency higher than $2f_s$, $\cos(4\pi t)$

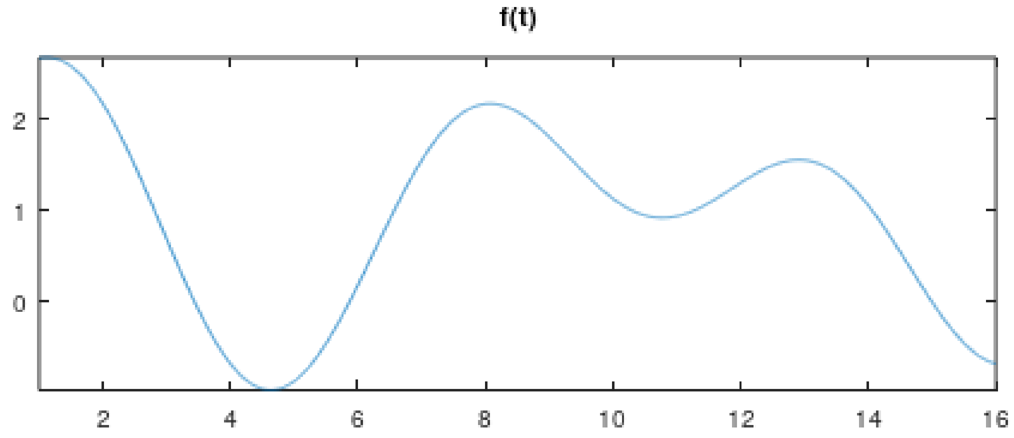


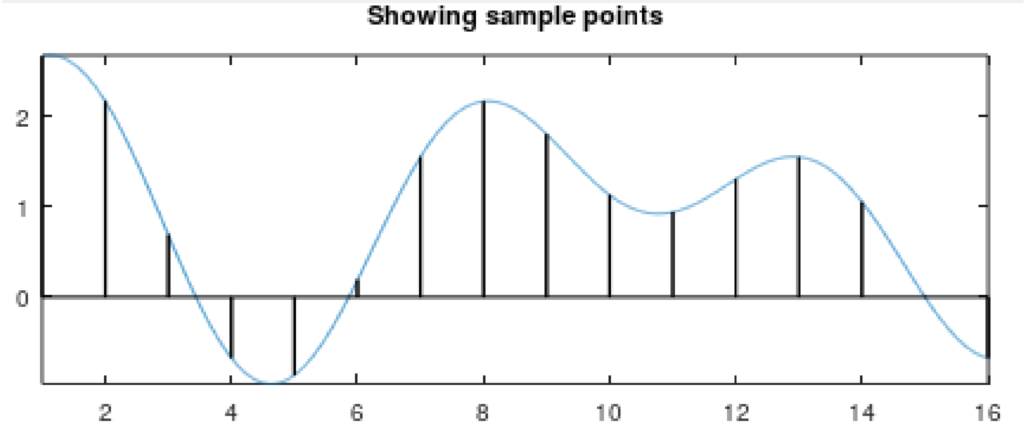


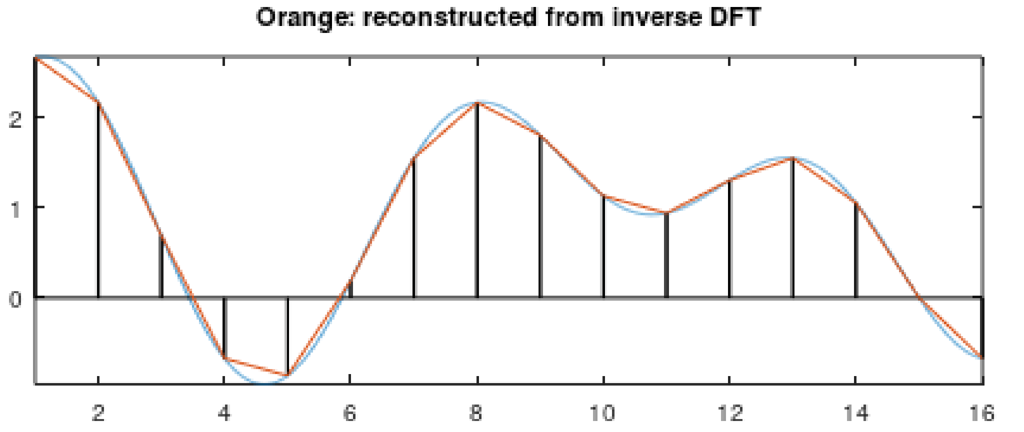


Leakage

- This occurs when we try to apply DFT to signal over a non-integer number of cycles of the input signal.
- 16 samples are taken at 1 s intervals, of the function $f(t) = 1 + \cos\left(\frac{\pi t}{5}\right) + \sin\left(\frac{\pi t}{3}\right)$.







Discrete cosine transform (DCT)

- Widely used in image processing (e.g. JPEG, MPEG)
- Lossy compression, so can result in artifacts
- Expresses the function as a set of cosines
- Unlike DFT which is a periodic extension of the original function, DCT assumes that the function is even
    - Similar to the difference between the Fourier series and the Fourier cosine series
- Does not result in a complex component like DFT

Discrete cosine transform

- Many version exist
- Most common: known as DCT II

$$F[n] = e(n) \sum_{n=0}^{N-1} f(k) \cdot \cos\left(\frac{(2k+1)\pi n}{2N}\right)$$

- Inverse:

$$f(k) = \frac{2}{N} \sum_{n=0}^{N-1} e(n) \cdot F(n) \cdot \cos\left(\frac{(2k+1)\pi n}{2N}\right)$$

$$e(n) = \begin{cases} \frac{1}{\sqrt{2}}, & if\ n = 0 \\ 1, & otherwise \end{cases}$$