CM2607 Advanced Mathematics for Data Science

Fourier Transform

Week No 10

ROBERT GORDON UNIVERSITY ABERDEEN
INFORMATICS INSTITUTE OF TECHNOLOGY

Learning Outcomes

- Covers LO3 for Module
- On completion of this lecture, students are expected to be able to:
  - Fourier Sine and Cosine Series
  - Understand Fourier transform
  - Derive the Fourier transform of a simple function
  - Apply Fourier transform to common functions and combinations of common functions

Half range Fourier Series

- If a function is defined over the half range $[0, L]$ instead of the full range $[-L, L]$

its period function can be represented as in terms of Sine terms only or in terms of Cosine terms only.

- The series produced is called as Half range Fourier Series.
- Even functions – Fourier Cosine Series
- Odd functions – Fourier Sine Series

Fourier Cosine Series

- Let $f(x)$ defined over $[0, L]$, Then the Fourier Cosine Series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where,

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\,, n = 1,2,3, \dots \dots \dots \dots$$

Fourier Sine Series

- Let $f(x)$ defined over $[0, L]$, Then the Fourier Sine Series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx, n = 1,2,3, \dots \dots \dots \dots \dots \dots \dots .$$

$$a_0 = 0, a_n = 0$$

Example

Consider the half range function $f(x) = x; [0, \pi]$.

1) Show that the Fourier Cosine Series is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} \frac{+ \cos 5x}{5} + \cdots\cdots\right)$$

2) Find the Fourier Sine Series

Fourier Transform

- Derived from the Fourier series
- Is derived from the special case when $T \to \infty$ (or $2L \to \infty$ in our Fourier series calculations)
- Not limited to periodic functions (though periodic functions can be represented)
- Fourier transform converts a function into its frequency domain representation.

Fourier transform applications

- Mainly used because many functions (mainly differentiation, integration, convolution) are easier to perform in the frequency domain.
- Therefore, can be used to reduce computational complexity.
- Is also useful for filtration purposes (removing either high-frequency or low-frequency components)

Fourier transform equation

This transforms the function from time domain to frequency domain.

Fourier transform, radian frequency:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

Radian frequency, symmetric form:

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\, dt$$

Hertz frequency:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt$$

Inverse Fourier transform

This transforms the function from frequency domain to time domain.

Inverse Fourier transform, radian frequency:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t}\, dt$$

Radian frequency, symmetric form:

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t}\, dt$$

Hertz frequency:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t}\, dt$$

Notes

- $\omega = 2\pi f$. This quantity is called angular frequency or radian frequency.
- The most common form is the radian frequency form (first equation)
- The Fourier transform uses $e^{i2\pi ft}$, derived from Euler's formula

$$e^{i2\pi ft} = \cos(2\pi ft) + i\sin(2\pi ft)$$

  as it simplifies many formulas.
- For many standard functions, you will not have to calculate the Fourier transform. You can refer to Fourier transform pairs instead.

Example: One sided decaying exponential

Find the Fourier transform of the following function:

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-t}, & t \geq 0 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{0} 0 \cdot e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-t} e^{-i\omega t}\, dt$$

$$F(\omega) = \int_{0}^{\infty} e^{t(-1-i\omega)}\, dt = \left[\frac{e^{(-t-i\omega t)}}{-1-i\omega}\right]_0^{\infty}$$

$$= 0 - \frac{1}{-1\ - i\omega} = \frac{1}{1 + i\omega}$$

Example: Rectangular pulse

- Find the Fourier transform of the following function:

$$f(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & |t| > T \end{cases}$$

$$F(\omega) = \int_{-T}^{T} e^{-i\omega t}\, dt = \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-T}^{T}$$

$$F(\omega) = \frac{-1}{i\omega}\left(e^{-i\omega T} - e^{i\omega T}\right)$$

Simplifying using Euler's formula ($e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$) gives:

$$F(\omega) = \frac{2\sin(\omega T)}{\omega}$$

Some standard functions:

These are some standard functions that you need to know.

- Dirac delta function or unit impulse: This is a function that is infinite at t = 0, and zero everywhere else. The area under the function is 1. It is denoted $\delta(t)$.
- $\Pi(t)$: unit rectangular pulse. This is a rectangular pulse, defined as:

$$\Pi(t) = \begin{cases} 0, & |t| > \frac{1}{2} \\ 1, & |t| \leq \frac{1}{2} \end{cases}$$

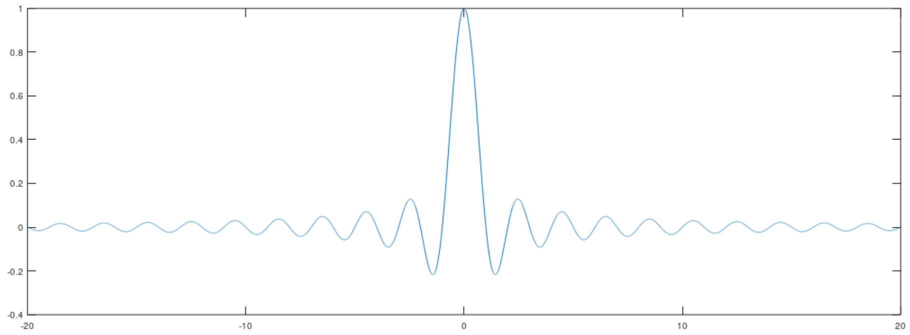
Some standard functions

- $\Lambda(t)$: Unit triangle: a triangular pulse, that can also be thought of as the convolution of two rectangular pulses: $\Lambda(t) = \Pi(t) * \Pi(t)$

$$\Lambda(t) = \begin{cases} 1+t, & -1 \le t < 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & otherwise \end{cases}$$

- sinc(t): This is a common function in Fourier analysis.

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$



Some standard functions

- Heaviside step function:

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

This is the integral of the Dirac delta function (or alternatively, the Heaviside step function is the derivative of the Dirac delta function. Sometimes written as $\gamma(t)$.

Properties of Fourier transform

- Linearity: if $x(t) = a.x_1(t) + b.x_2(t)$ for any complex numbers a and b, then $X(\omega) = a.X_1(\omega) + b.X_2(w)$
- Time shifting: $x(t\ - a)$ in the time domain becomes $X(\omega)e^{-i\omega a}$
- Frequency shifting: $x(t)e^{i\omega_0 t}$ becomes $X(\omega - \omega_0)$
- Time scaling: $x\left(\frac{t}{a}\right)$ becomes $aX(\omega a)$
- Time reversal: This is a special case of time scaling. $x(-t)$ becomes $X(-\omega)$

Properties of Fourier transform (contd)

- Convolution in time domain $x_1(t) * x_2(t)$ becomes multiplication in frequency domain: $X_1(\omega)X_2(\omega)$
- Differentiation: $\frac{d^n x(t)}{dt^n}$ in time domain becomes $(i\omega)^n X(\omega)$
- Integration: $\int_{-\infty}^{t} f(\tau)\ d\tau$ in time domain becomes $\frac{F(\omega)}{i\omega} + \pi \cdot F(0) \cdot \delta(\omega)$

Common Fourier transform pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t}\, d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\, dt$
$\delta(t)$	1
$\Pi(t) = \begin{cases} 0, & \lvert t\rvert > \frac{1}{2} \\ 1, & \lvert t\rvert \leq \frac{1}{2} \end{cases}$	$sinc\left(\frac{\omega}{2\pi}\right)$
1	$2\pi\delta(\omega)$
$e^{i\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$

Common Fourier transform pairs

$\Lambda(t) = \begin{cases} 1+t, & -1 \le t < 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & otherwise \end{cases}$	$sinc^2\left(\frac{\omega}{2\pi}\right)$
$\gamma(t)$	$\frac{1}{i\omega} + \pi\delta(\omega)$
$\cos(\omega_0 t)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin(\omega_0 t)$	$-\mathrm{i}\pi\delta(\omega - \omega_0) + i\pi\delta(\omega + \omega_0)$

Example: Fourier transform properties

Find the Fourier transform of

$$f(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t < 0 \end{cases}$$

To solve this, we can write $f(t) = 2\gamma(t) - 1$ where $\gamma(t)$ is the Heaviside step function.

Substituting the Fourier transform for $\gamma(t)$ and 1:

$$F(\omega) = 2\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) - 2\pi\delta(\omega) = \frac{2}{i\omega}$$