CM 2607 Advanced Mathematics for Data Science

Lecture 03

Differentiation III

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TEF Gold

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Learning Outcomes

- Covers LO1 and LO2 for CM2607
- On completion of this lecture on differentiation, students are expected to be able to:
  - Understand the geometric application of differentiation in detail
  - Relate differentiation with finding critical points/stationary points of a function

Geometric applications of differentiation

Content

- Straight lines
- Tangents and normal
- Increasing, decreasing and the curvature
- Critical points/Stationary points

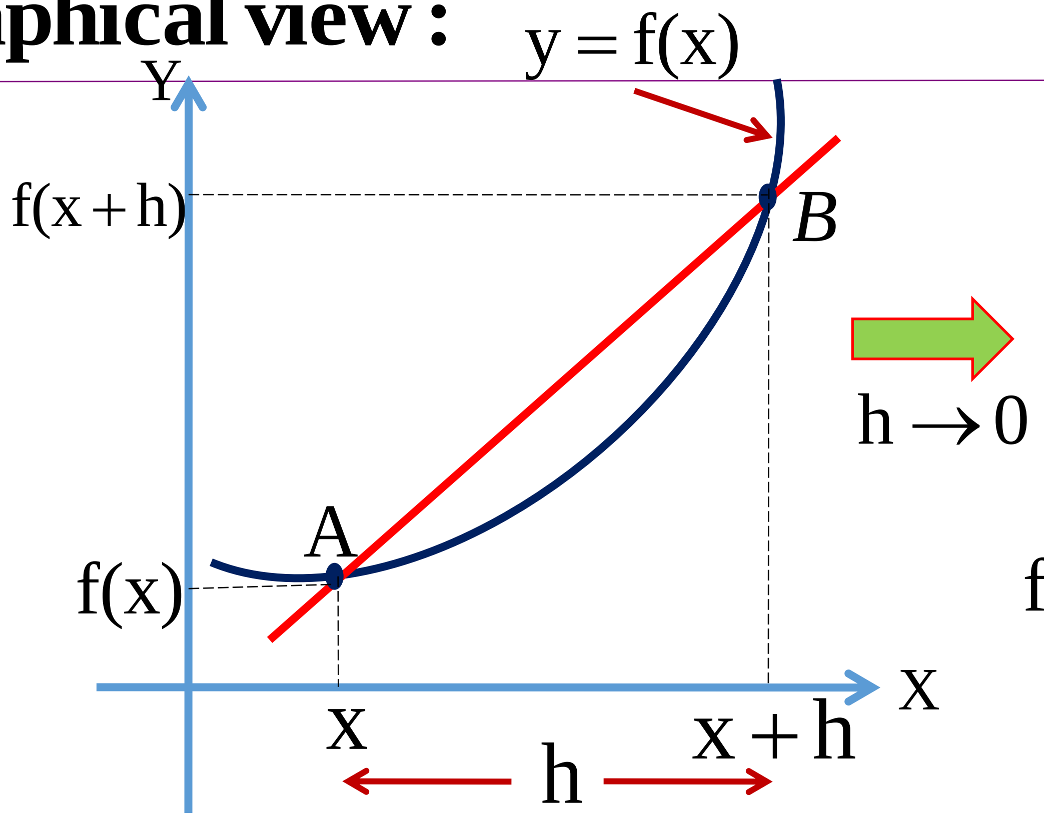
Geometric applications of differentiation

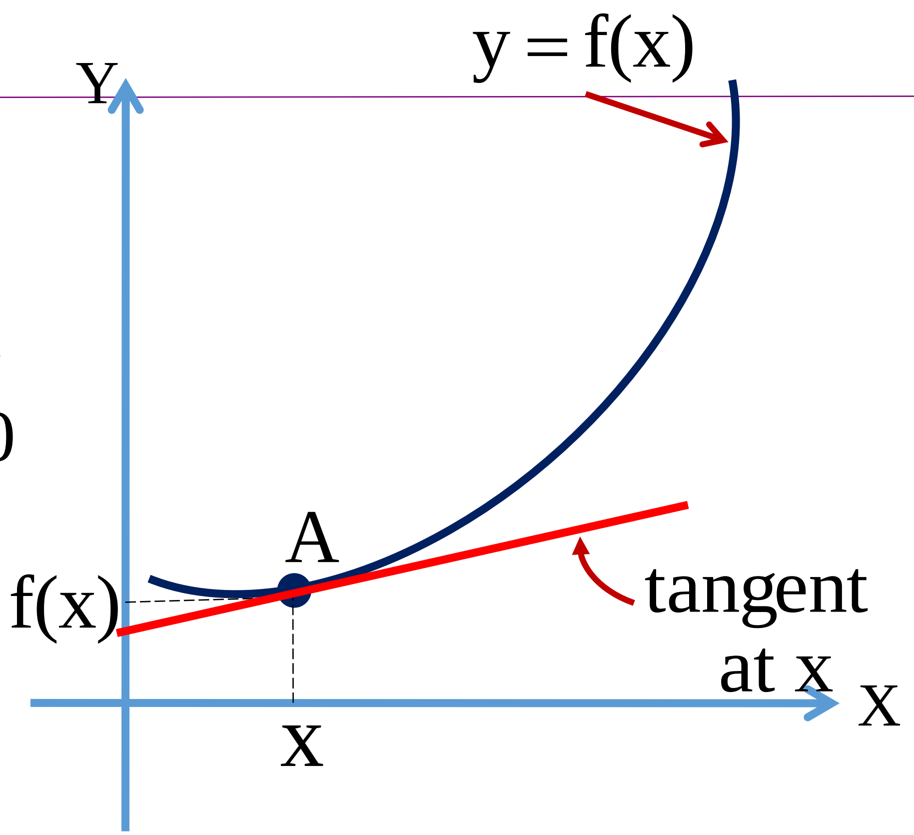
Let us start with the geometric context of the derivative.

Derivative of a function $f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Graphical view :





$$\text{slope (gradient) of AB} = \frac{f(x+h) - f(x)}{h}$$

(average change)

$$\text{slope of tangent at x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'(x)$$

(instantaneous change)

Application 1 : Straight lines

Equation of a straight line : $y = mx + c$

$m -$ gradient

$c -$ intercept

What does $\frac{dy}{dx}$ give ?

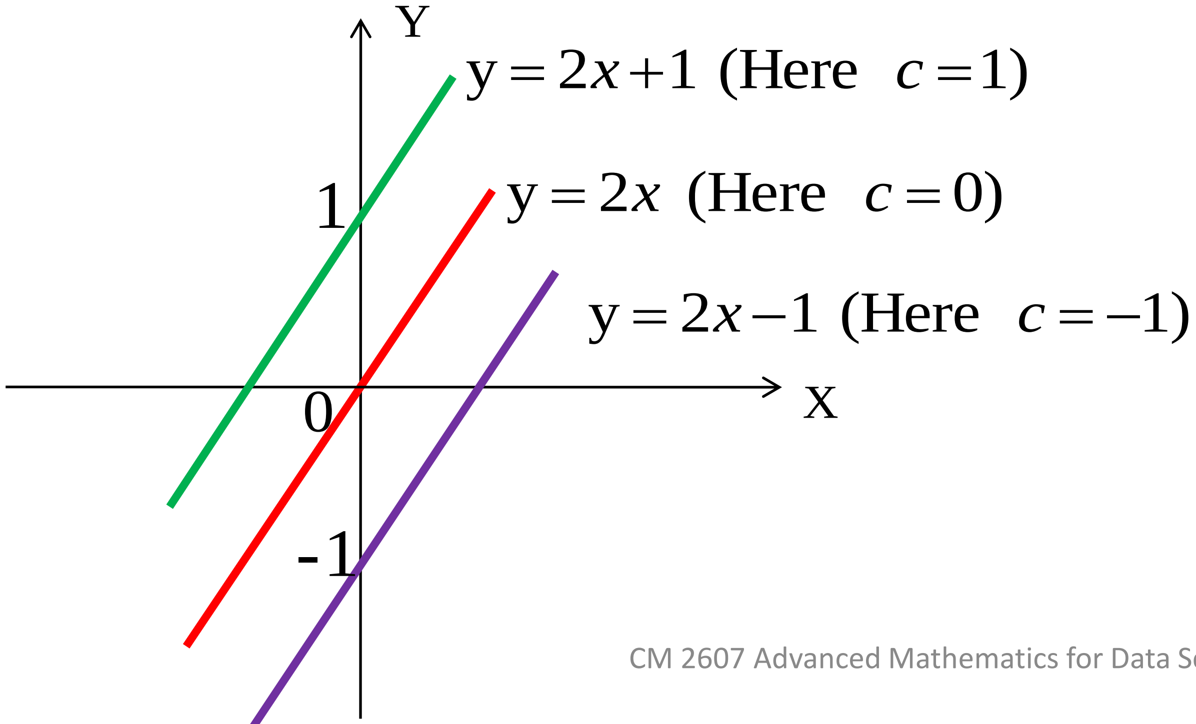
$$\frac{dy}{dx} = m \text{ (gradient)}.$$

Thus, if we want to plot a function whose derivative is a constant throughout the domain, then the choice is a straight line.

Application 1 : Straight lines ctd.

eg. Curve satisfying $\dfrac{dy}{dx} = 2$ is a straight line with gradient 2.

(ie. $y = 2x + c$, c is an arbitary constant)



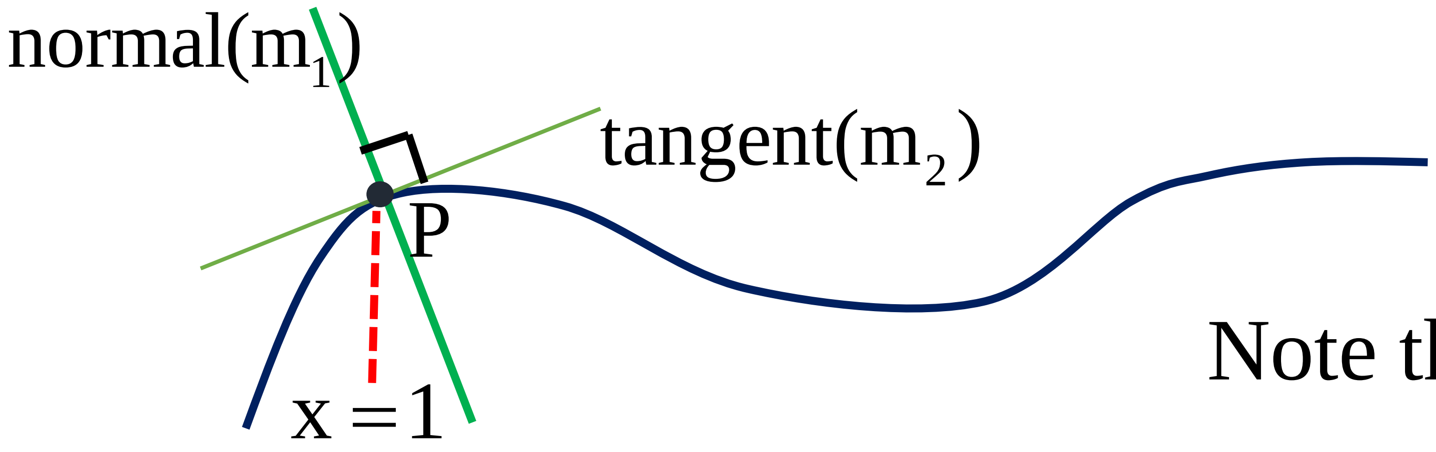
Application 2 : Tangents and normals.

Recall that the gradient of the tangent at a point P of curve $y = f(x)$ is given by the derivative of $f(x)$ evaluated at P.

Normal at P is the straight line through P perpendicular to the tangent at P.

Application 2 : Tangents and normals.

eg 1. Find the equations of tangent and normal to the curve $y = 2x^3 + 3x^2 - 2x - 3$ at $x = 1$.



Note that

$$m_1 . m_2 = -1$$

$$\frac{dy}{dx} = 6x^2 + 6x - 2$$

$$\text{Gradient of the tangent at } x = 1 \quad = \left.\frac{dy}{dx}\right|_{x=1}$$

$$= 6(1)^2 + 6(1) - 2$$

$$= 10$$

$$x = 1 \quad \Rightarrow \quad y = 2(1)^3 + 3(1)^2 - 2(1) - 3 = 0$$

$$\therefore \quad \mathrm{P} \equiv (1,0)$$

$$\therefore \quad \text{Equation of the tangent at P} : \quad \frac{y-0}{x-1} = 10$$

$$y = 10x - 10$$

As

$$m_1.m_2 = -1$$

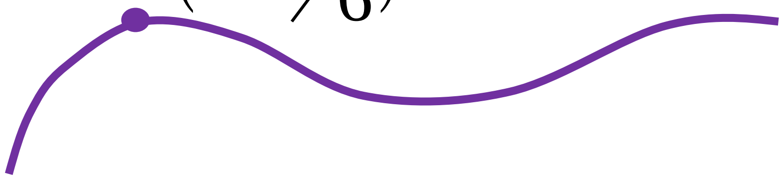
$$\text{Gradient of the normal} = \frac{-1}{\text{Gradient of tangent}} = \frac{-1}{10}$$

$$\therefore \quad \text{Equation of the normal at P}: \quad \frac{y-0}{x-1} = \frac{-1}{10}$$

$$10y + x = 1$$

eg 2. Find the equations of tangent and normal to the curve $y = \cos 2t$, $x = \sin t$ at the point given by $t = \pi/6$.

$\mathrm{P}\left(t = \pi/6\right)$



$$x = \sin \pi/6 = 1/2$$

$$y = \cos 2\pi/6 = \cos \pi/3 = 1/2$$

$$\therefore \ \mathrm{P} \equiv \left(1/2, 1/2\right)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= -2\sin 2t \ . \ \frac{1}{dx/dt} = -2\sin 2t \ . \ \frac{1}{\cos t}$$

$$= \frac{-4\sin t \cos t}{\cos t}$$

$$= -4\sin t$$

$$\text{Gradient of the tangent at P} = \left.\frac{dy}{dx}\right|_{t=\pi/6}$$

$$= -4\sin \pi/6$$

$$= -4\left(1/2\right) = -2$$

$$\text{Tangent at P}: \frac{y - \frac{1}{2}}{x - \frac{1}{2}} = -2$$

$$y - \tfrac{1}{2} = -2x + 1$$

$$y + 2x = \tfrac{3}{2} \qquad \Rightarrow \qquad 2y + 4x = 3$$

$$\text{Gradient of the normal at P} = \frac{-1}{-2} = \frac{1}{2}$$

$$\text{Normal at P}: \frac{y - \frac{1}{2}}{x - \frac{1}{2}} = \tfrac{1}{2}$$

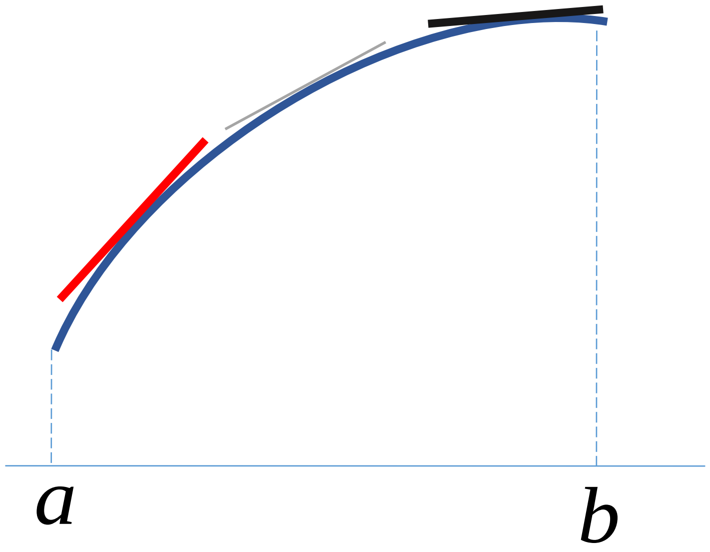
$$y - \tfrac{1}{2} = \tfrac{1}{2}x - \tfrac{1}{4}$$

$$4y = 2x + 1$$

Application 3 : Increasing & decreasing (strictly) functions and the curvature

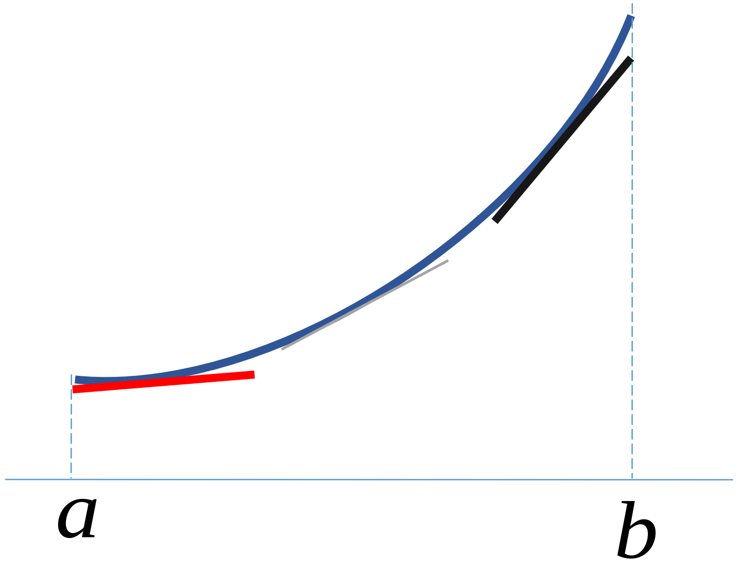
Let $f(x)$ be a function defined on $[a,b]$.

* If $f'(x) > 0 \ \forall \ x \in [a,b]$, then $f(x)$ is increasing.



f – increasing

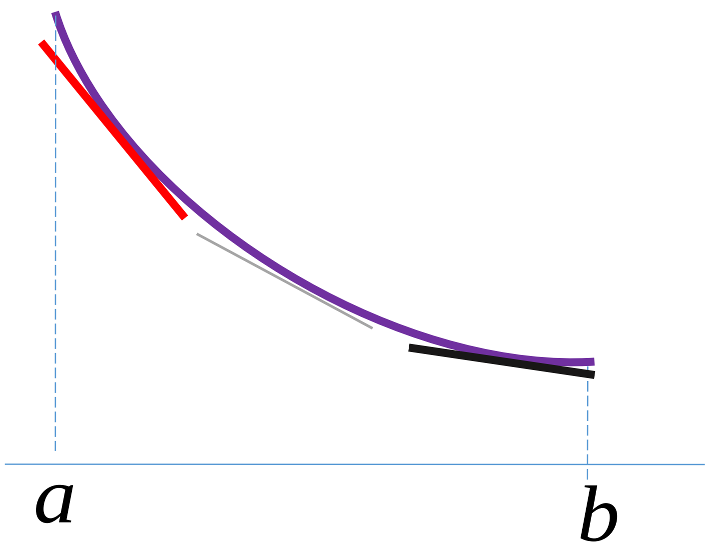
f' – decreasing



f – increasing

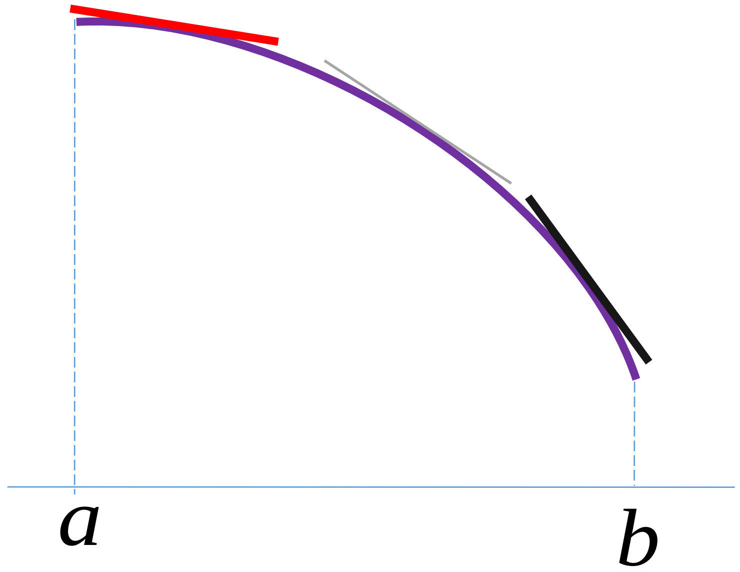
f' – increasing

* If $f'(x) < 0 \;\; \forall \;\; x \in [a, b]$, then $f(x)$ is decreasing.



f – decreasing

f' – increasing



f – decreasing

f' – decreasing

* If $f'(x) = 0 \;\; \forall \;\; x \in [a, b]$, then $f(x)$ is constant.

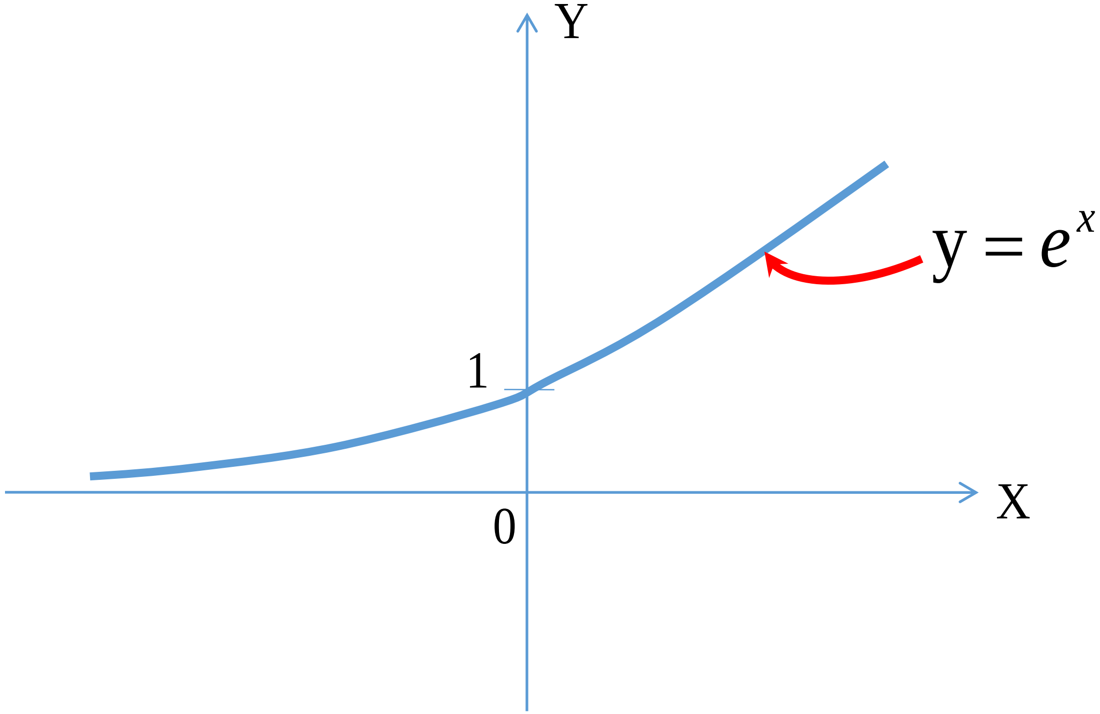
Thus, we can identify the curvature (type of bending) by the behaviour of f' or f''.

$$\boxed{\begin{array}{l} f'>0 \\ f'<0 \\ f'=0 \\ f''>0 \\ f''<0 \\ f''=0 \end{array}} \quad \begin{array}{l} \Rightarrow \quad f \;-\; \text{increasing} \\ \Rightarrow \quad f \;-\; \text{decreasing} \\ \Rightarrow \quad f \;-\; \text{constant} \\ \Rightarrow \quad f' \;-\; \text{increasing} \;\; (f - \text{concave upwards}) \\ \Rightarrow \quad f' \;-\; \text{decreasing} \; (f - \text{concave downwards}) \\ \Rightarrow \quad f' \;-\; \text{constant} \; (f - \text{no bending/ straight line}) \end{array}$$

These conditions must be satisfied $\forall\, x$ in considered interval.

$$\text{eg.}\quad f(x) = e^x \ \text{ in } \ \Re. - \ \text{increasing}\,(\because\ f'(x) = e^x > 0$$

$$\forall\ x \in \Re)$$



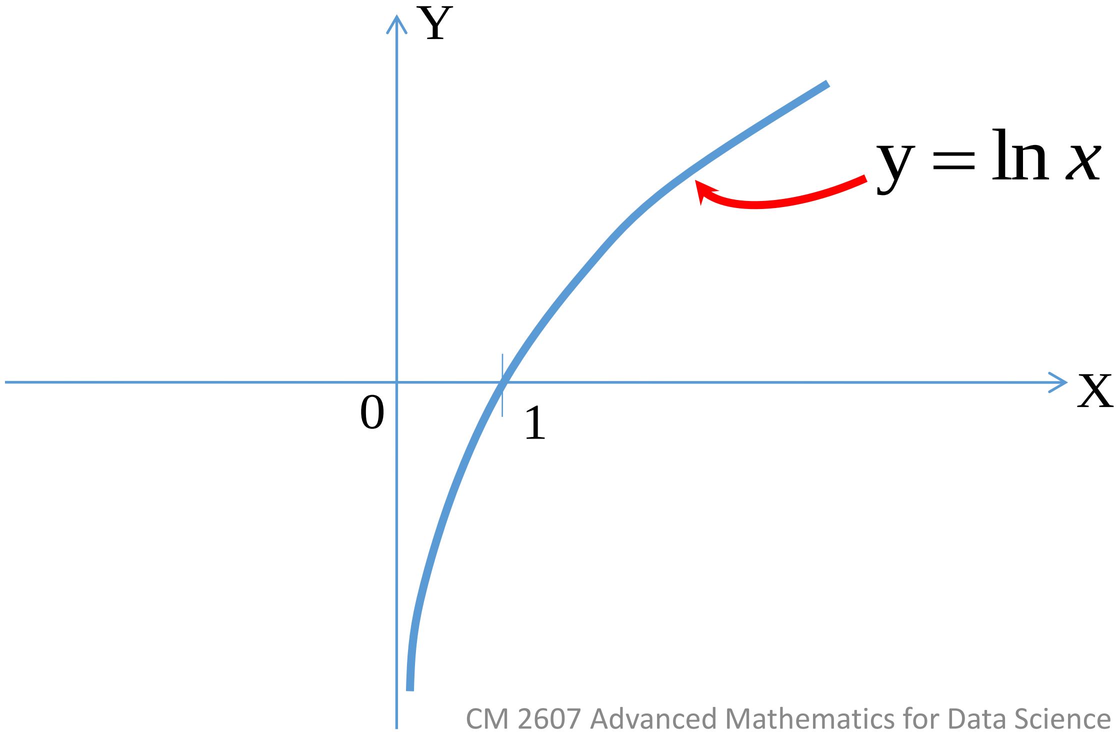
$$f''(x) = e^x > 0 \quad \rightarrow \quad f'(x) \ \text{ increasing}$$

$$(\text{curvature: concave upwards})$$

eg. $f(x) = \ln x \;;\; x > 0$ – increasing $(\because f'(x) = 1/x > 0 \;\; \forall x > 0)$

$$f''(x) = -1/x^2 < 0 \quad \rightarrow \quad f'(x) \text{ decreasing}$$

(curvature: concave downwards)



Critical points/Stationary points

A <u>critical point</u> of a function is a value in the domain where, the derivative of the function is either zero or not differentiable (derivative does not exist).

ie. If $f'(x_0) = 0$ or $f'(x_0)$ does not exist then x_0(in the domain) is a critical point of function f.
Image of the critical point $(f(x_0))$ is called a critical value of f.

Usually, once we come to the graph $(x_0, f(x_0))$ is considered as the critical point.

A <u>stationary point</u> of a function is a value in the domain where the derivative of the function is zero.

ie. If $f'(x_0) = 0$ then x_0 (in the domain) is a stationary point of function f.

Again in the graph $(x_0, f(x_0))$ is taken as the stationary point.

Function is neither increasing nor decreasing at a stationary point.

Remark : Any stationary point is a critical point but converse is not always true.

eg. Let $f(x) = \dfrac{2x}{1+x^2}$.

i. Find the critical points of $f(x)$.

ii. What are the corresponding critical values?

iii. Are all critical points of $f(x)$ stationary points?

i. $$f'(x) = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Here $f'(x)$ exists for any x. So, critical points are given by $f'(x) = 0$.

$$\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}=0$$

$$\Rightarrow \quad x=\pm 1$$

Critical points are 1 and -1.

ii. Critical value for $x=1$: $f(1) = \dfrac{2}{1+1} = 1$

Critical value for $x=-1$: $f(-1) = \dfrac{2(-1)}{1+1} = -1$

iii. All critical points (± 1) are stationary points since $f'(x)$ is zero at these points.

eg. Find the critical points and critical values of $f(x) = |x|$.

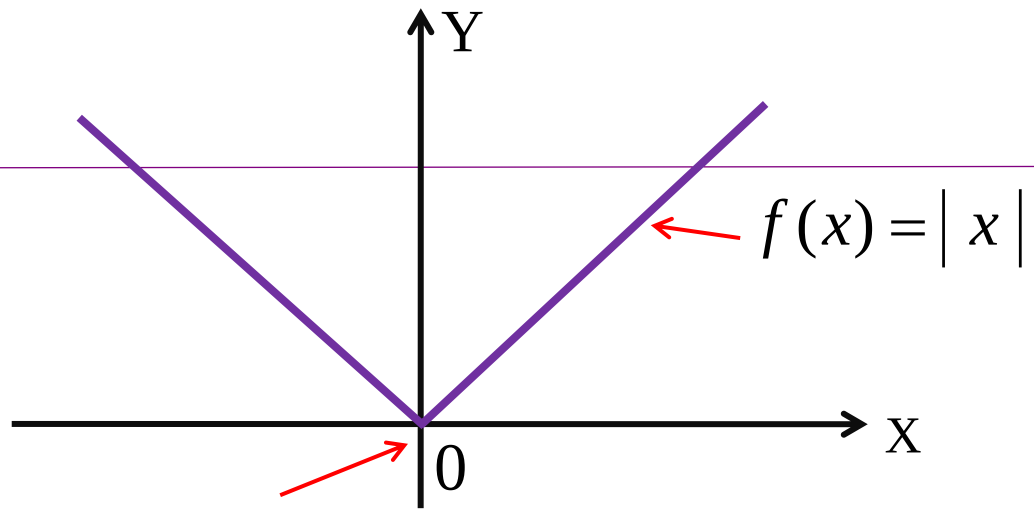
$$f(x) = \begin{cases} x & ; \quad x \geq 0 \\ -x & ; \quad x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & ; \quad x > 0 \\ -1 & ; \quad x < 0 \end{cases}$$, but $f'(x)$ does not exist at $x = 0$.

So, 0 is the only critical point of $f(x) = |x|$.

Critical value: $f(0) = 0$.

(Note that 0 is not a stationary point.)



not differentiable at $x = 0$.

Ex. Determine the critical / stationary points and critical values of the following functions.

i. $f(x) = x^2 - 4x + 1$

ii. $f(x) = x^2(1+x)^3$

iii. $f(x) = 2|x+1| - 1$

Local maximizer/local minimizer

Local maximizer : $x = a$ is called a local maximizer of a function $f(x)$ if we can find an interval included $x = a$ in which $f(a)$ is the maximum value in that interval.

If $f(a)$ is the maximum for the entire domain, then $x = a$ is called a global maximizer.

Local maximizer

Local maximizer $x = a$ can be expressed in different ways;

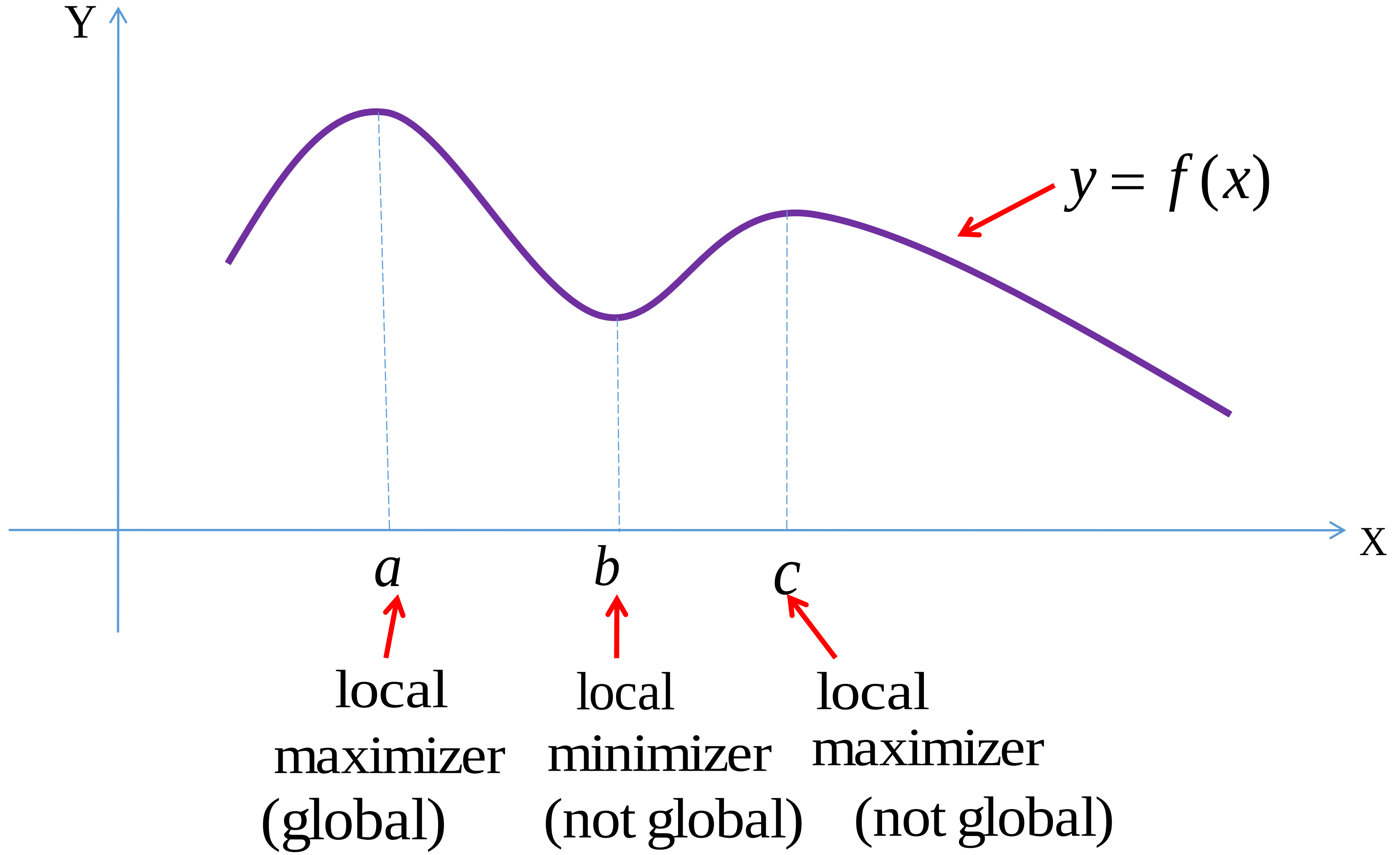
* $f(x)$ has a local maximum at $x = a$
* $(a, f(a))$ is a local maximum point of $f(x)$
* a local maximum for $f(x)$ occurs at $x = a$

Local minimizer

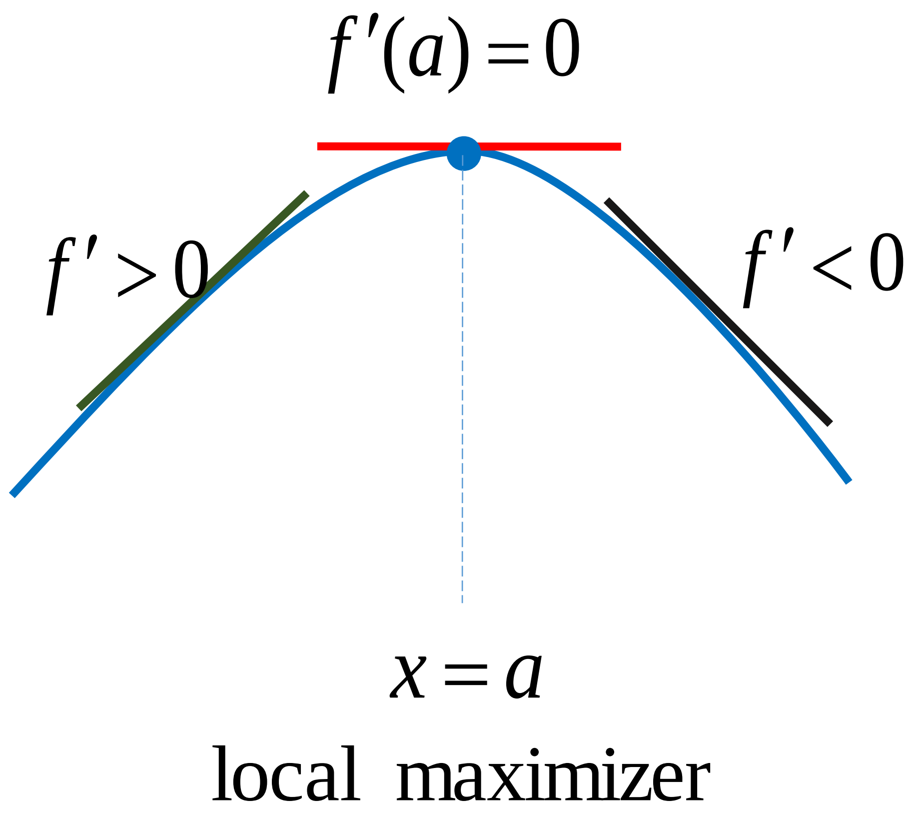
Local minimizer : $x = a$ is called a local minimizer of a function $f(x)$ if we can find an interval included $x = a$ in which $f(a)$ is the minimum value in that interval.

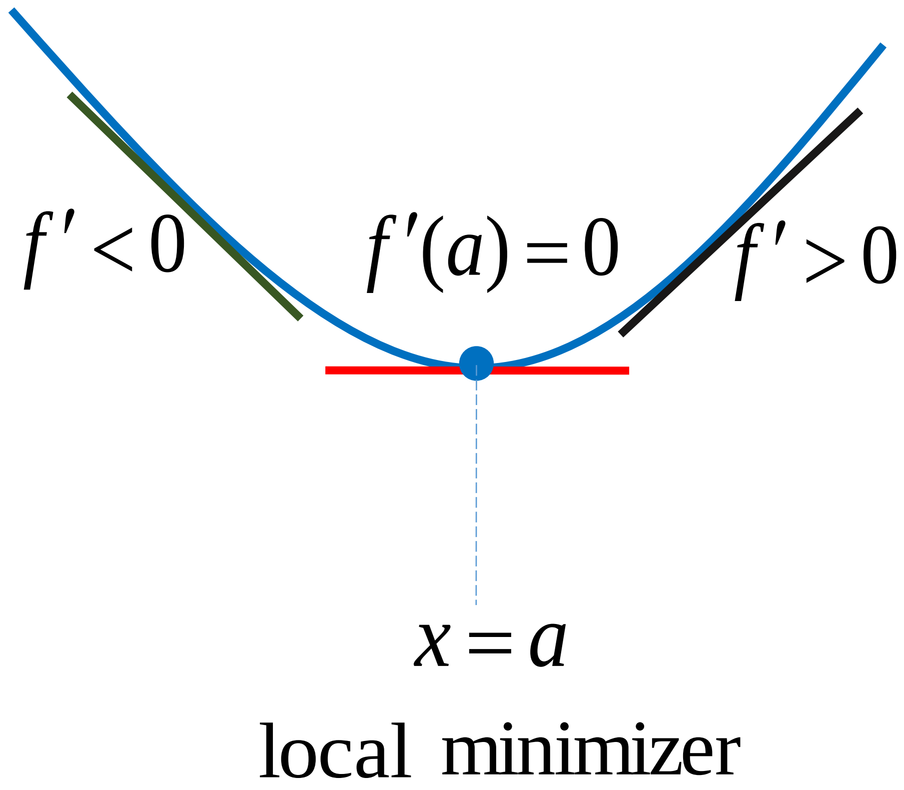
If $f(a)$ is the minimum for the entire domain, then $x = a$ is called a global minimizer.

Local maximizer/local minimizer



Stationary points (maximizers and minimizers



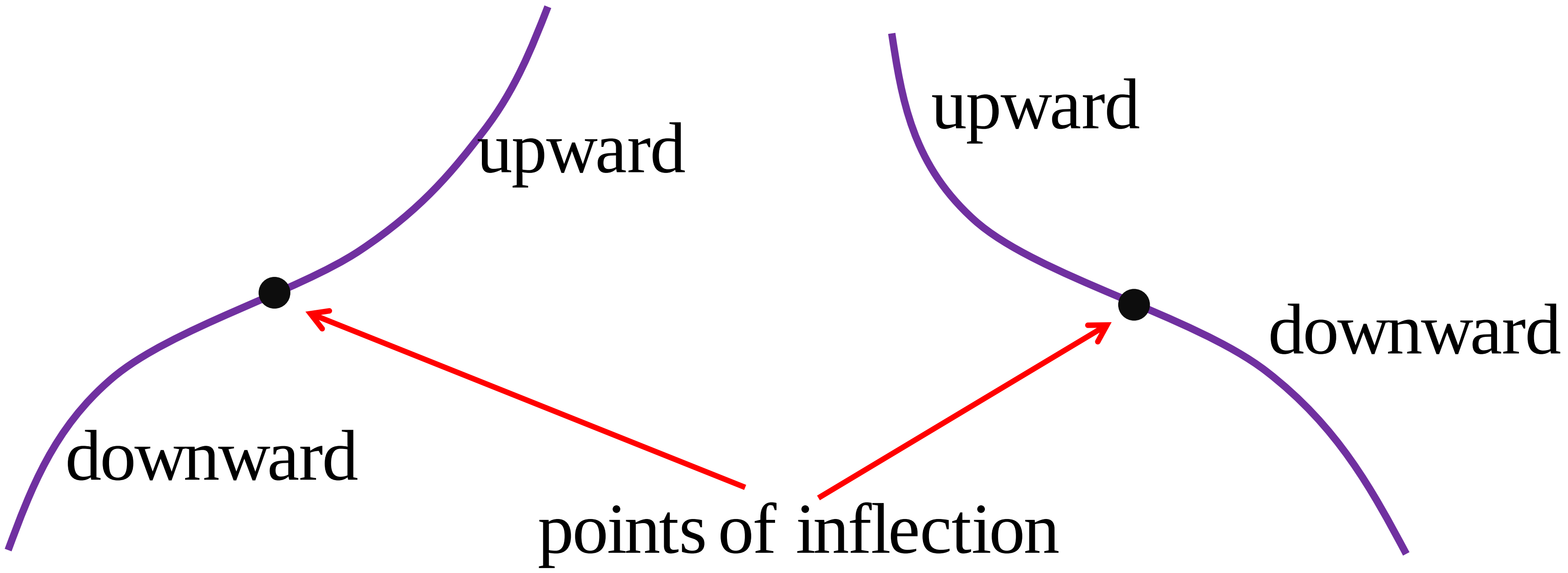


Point of inflection

A point of inflection is a point on a curve at which the direction of bending changes.

(ie. concave nature changes from upward $(f'' > 0)$ to downward $(f'' < 0)$ or vice versa once we go through the point)

Point of inflection



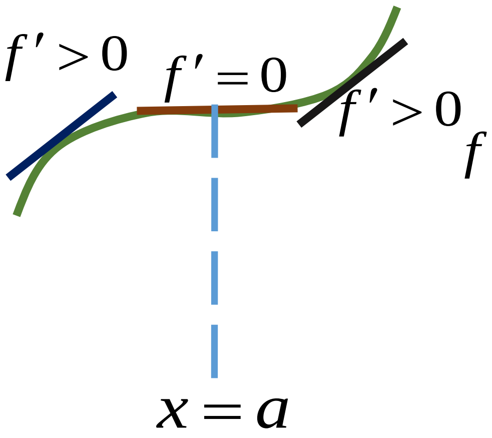
Note that the derivative at a point of inflection can either be zero or not.

Classification of a point of inflection:

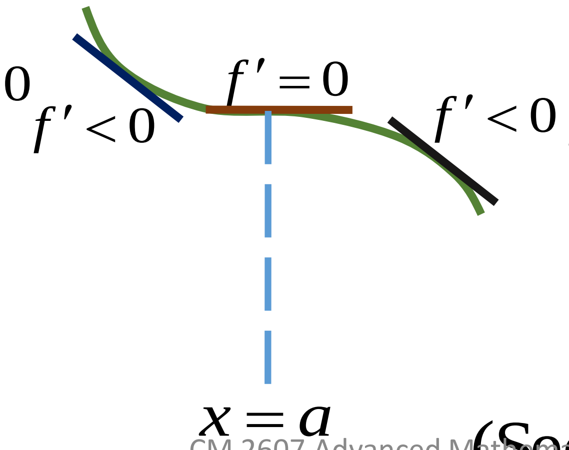
point of inflection at $x = a$

$f'(a) = 0$
(stationary point)

$f' > 0$ around $x = a$

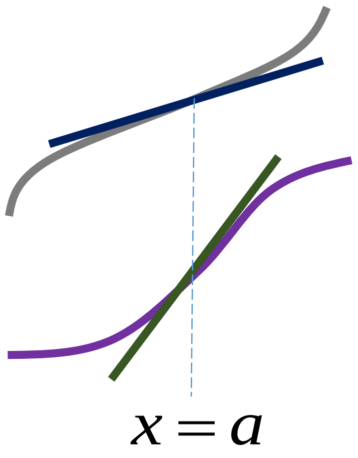


$f' < 0$ around $x = a$

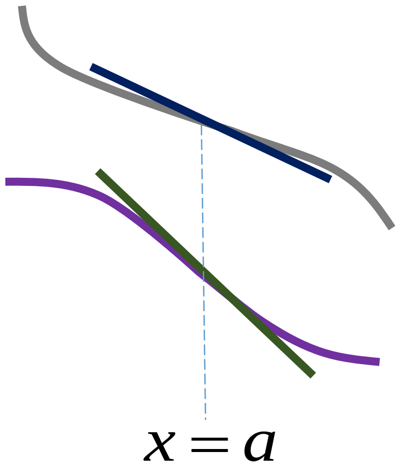


$f'(a) \neq 0$
(not a stationary point)

$f'(a) > 0$



$f'(a) < 0$



(See downward $\Leftrightarrow$ upward changes)

Tests for stationary points

1^{st} derivative test

Let $x = a$ be a stationary point of $f(x)$. (ie. $f'(a) = 0$)

Then, the following cases occur as x increases through $x = a$.

i. If f' changes sign from positive to negative, then $x = a$ is a local maximizer.

ii. If f' changes sign from negative to positive, then $x = a$ is a local minimizer.

iii. If f' deos not change the sign, then a point of inflection occurs at $x = a$.

eg. Determine the stationary points and their nature for

$f(x) = 4x^5 - 5x^4 - \frac{40}{3}x^3 + 2$ using the 1st derivative test.

$$f(x) = 4x^5 - 5x^4 - \frac{40}{3}x^3 + 2$$

For stationary points: $f'(x) = 20x^4 - 20x^3 - 40x^2 = 0$

$$20x^2(x^2 - x - 2) = 0$$

$$x = 0 \quad \text{or} \quad x^2 - x - 2 = 0$$

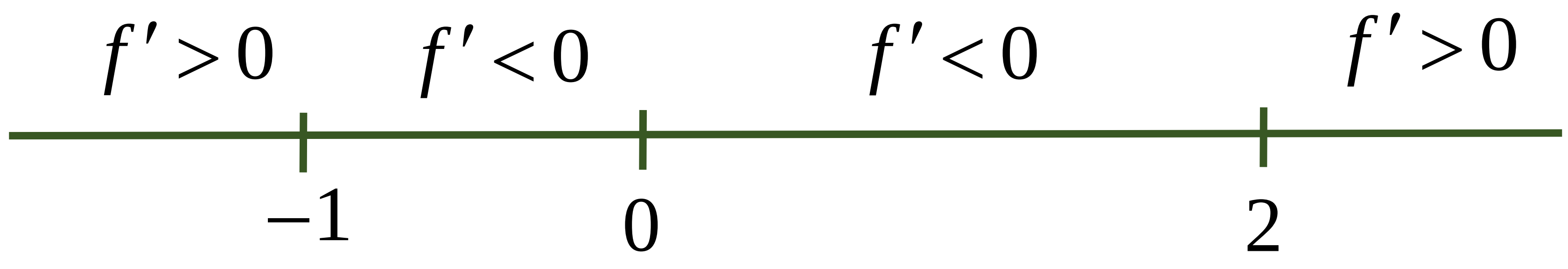
$$(x-2)(x+1) = 0$$

$$x = 2 \quad \text{or} \quad x = -1$$

$\therefore$ Stationary points are -1 , 0 and 2.

Determining the nature:

$$f'(x) = 20x^2\,(x-2)\,(x+1)$$



For $x = -1$, f' changes from $+$ to $-$.

$\therefore$ $x = -1$ is a local maximizer.

For $x = 0$, f' does not change the sign.

$\therefore$ $f(x)$ has a point of inflection at $x = a$.

For $x = 2$, f' changes from $-$ to $+$.

$\therefore$ $x = 2$ is a local minimizer.

2^{nd} derivative test

Let $x = a$ be a stationary point of $f(x)$. (ie. $f'(a) = 0$)

i. If $f''(a) < 0$, then $x = a$ is a local maximizer.

ii. If $f''(a) > 0$, then $x = a$ is a local minimizer.

iii. If $f''(a) = 0$, it is not enough to decide the nature.

But if f'' changes sign as x increases through

$x = a$, then a point of inflection occurs at $x = a$.

Ex : Find out the reasons for above three claims in the

2^{nd} derivative test.

Note that :

The 3rd claim is an important clue on finding points of inflection. Thus, we want to find points with $f''(x) = 0$ and test the possibility of sign change.

eg. Determine the stationary points and their nature

for $f(x) = 4x^5 - 5x^4 - \frac{40}{3}x^3 + 2$ using the

2^{nd} derivative test.

From previous example we have

$f'(x) = 20x^2(x-2)(x+1)$ with stationry points

$-1, 0$ and 2.

Now, $f''(x) = 80x^3 - 60x^2 - 80x$

$$\text{For} \quad x=-1, \quad f''(-1)=80(-1)-60(1)-80(-1)=-60<0$$

$$\therefore \quad x=-1 \text{ is a local maximizer.}$$

$$\text{For} \quad x=2, \quad f''(2)=80(8)-60(4)-80(2)=240>0$$

$$\therefore \quad x=2 \text{ is a local minimizer.}$$

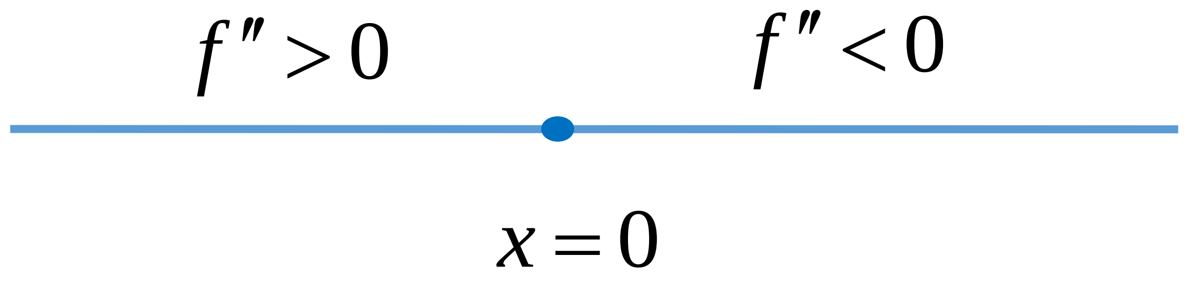
$$\text{For} \quad x=0, \quad f''(0)=80(0)-60(0)-80(0)=0 \quad ?$$

Testing the sign change......

$$f''(x)=80x^3-60x^2-80x$$

$$=20x\,(4x^2-3x-4) \;=20x\,.4\,(x^2-\frac{3}{4}x-1)$$

$$=80x\left[x-\left(\frac{3+\sqrt{73}}{8}\right)\right]\left[x-\left(\frac{3-\sqrt{73}}{8}\right)\right]$$



So, f'' changes sign as x increases through $x = 0$.

Hence, $f(x)$ has a point of inflection at $x = 0$.

Ex : Do the followings for the given functions.

i. Find the stationary points.

ii. Determine the nature of the stationary points (by using both derivative tests)

iii. Find all inflections by testing $f''(x)$.

1. $f(x) = \dfrac{2x}{1+x^2}$
2. $f(x) = 2x^3 - 12x^2 + 18x + 1$
3. $f(x) = x^3$
4. $f(x) = \sin x \quad ; \; 0 \le x \le 2\pi$
5. $f(x) = \tan x \quad ; \; -\pi/2 < x < \pi/2$
6. $f(x) = \sinh x$

Example

- Use the partial derivatives and find the critical point of the function $f(x, y) = -x^2 + y^2$.
  - Determine the Hessian matrix for $f(x, y)$
  - **Self Study: Search on how you can use the Hessian matrix to determine the nature of critical points for several variable functions.**