CM 1606 Computational Mathematics

Matrices I

Week 7| Ganesha Thondilege

ROBERT GORDON UNIVERSITY ABERDEEN

INFORMATICS INSTITUTE OF TECHNOLOGY

Learning Outcomes

- Covers LO1 for CM 1606
- On completion of this lecture, students are expected to be able to:
  - Recognize how matrices are used to store and process data
  - Discuss the applications of Matrices in Machine Learning and Deep Learning

CONTENT

- Introduction
- Types of Matrices
- .Addition and Subtraction
- Multiplication
- Transpose
- Determinant – Order 2
- Matrix properties

Matrix

- A matrix is any doubly subscripted array of elements arranged in rows and columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots \\ a_{m1} \ldots\ldots\ldots a_{mn} \end{bmatrix}_{m \times n}$$

Any element is a real number

$a_{ij}; i - row\ number$ and $j - column\ number$

Types of Matrices

Row matrix

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}_{1\times n}$$

Column matrix

$$B = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{bmatrix}_{m\times 1}$$

Square matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n}$$

$$where$$

$$m = n$$

Equal Matrices

Example: $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given that $A = B$, find a, b, c and d.

if $A = B$, then $a = 1$, $b = 0$, $c =$ -4 and $d = 2$.

Zero Matrix

- Every element of a matrix is zero, it is called a zero matrix, i.e.,

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

Lower and Upper Triangular Matrices

- Upper triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

- Lower triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Diagonal Matrices

- All off diagonal entries are zero

$$D = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

Simply

$$D = \mathrm{diag}[a_{11}, a_{22}, \dots, a_{nn}]$$

Identity Matrix

- If all diagonal entries are 1 for a diagonal matrix, the matrix is called identity matrix.
- Properties: $AI = IA = A$

Examples of identity matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix addition and subtraction

- Applied for same order matrices

$$\{C_{ij}\} = \{A_{ij}\} \pm \{B_{ij}\}$$

- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ then

$$C = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}$$

Matrix Multiplication

Suppose matrices A and B have these dimensions:

[m x n] and [p x q]

Matrix Multiplication

Matrices A and B can be multiplied if:

$$[m \times n] \text{ and } [p \times q]$$

$$n = p$$

Matrix Multiplication

The resulting matrix will have the dimensions:

[m x n] and [p x q]

m x q

Matrix Multiplication

- Computation A x B = C

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$AB \neq BA$ in general

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

Matrix Multiplication

- Computation A x B = C

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

[3 x 2] [2 x 3]

A and B can be multiplied

$$C = \begin{bmatrix} 2*1+3*1=5 & 2*1+3*0=2 & 2*1+3*2=8 \\ 1*1+1*1=2 & 1*1+1*0=1 & 1*1+1*2=3 \\ 1*1+0*1=1 & 1*1+0*0=1 & 1*1+0*2=1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 8 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

[3 x 3]

Transpose of a Matrix

- Obtained by interchanging the rows and columns of a matrix A
- Denoted as A^T

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Symmetric Matrices

- A matrix A such that $A^T = A$ is called symmetric

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

- $A^T + A?$

Determinant – Order 2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- Determinant of A, denoted $|A|$ or $\det(A)$,
- Is a real number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant – Order 2

- Only for order 2 matrices

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

- +

Ex: $$A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

Properties

- Following properties are true for determinants of *any* order.

1. If every element of a row (column) is zero, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1\times 0 - 2\times 0 = 0$, then $|A| = 0$.

2. $|A^T| = |A|$

   determinant of a matrix = that of its transpose

3. $|AB| = |A||B|$

Exercises

1) Suppose $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix}$.

   Evaluate the following.

   i. $A + 2B - C$

   ii. AB and BA

   iii. $A^T, A^T B$ and $(A + A^T)B$

   iv. $|A|, |B|, |C|$ and $|ABC|$

   v. $A - B, A + B$ and $A^2 - B^2$

Matrix properties

- $A+B=B+A$
- $A+(B+C)=(A+B)+C$
- $a(A+B)=aA+aB$; a-constant
- $(ab)A=a(bA)=b(aA)$; a,b- constants

Matrix properties

- $(AB)C = A(BC)=ABC$
- $A(B+C) = AB + AC$
- $A(aB) = a(AB) = (aA)B$

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$