CM 2607 Advanced Mathematics for Data Science

Integration III

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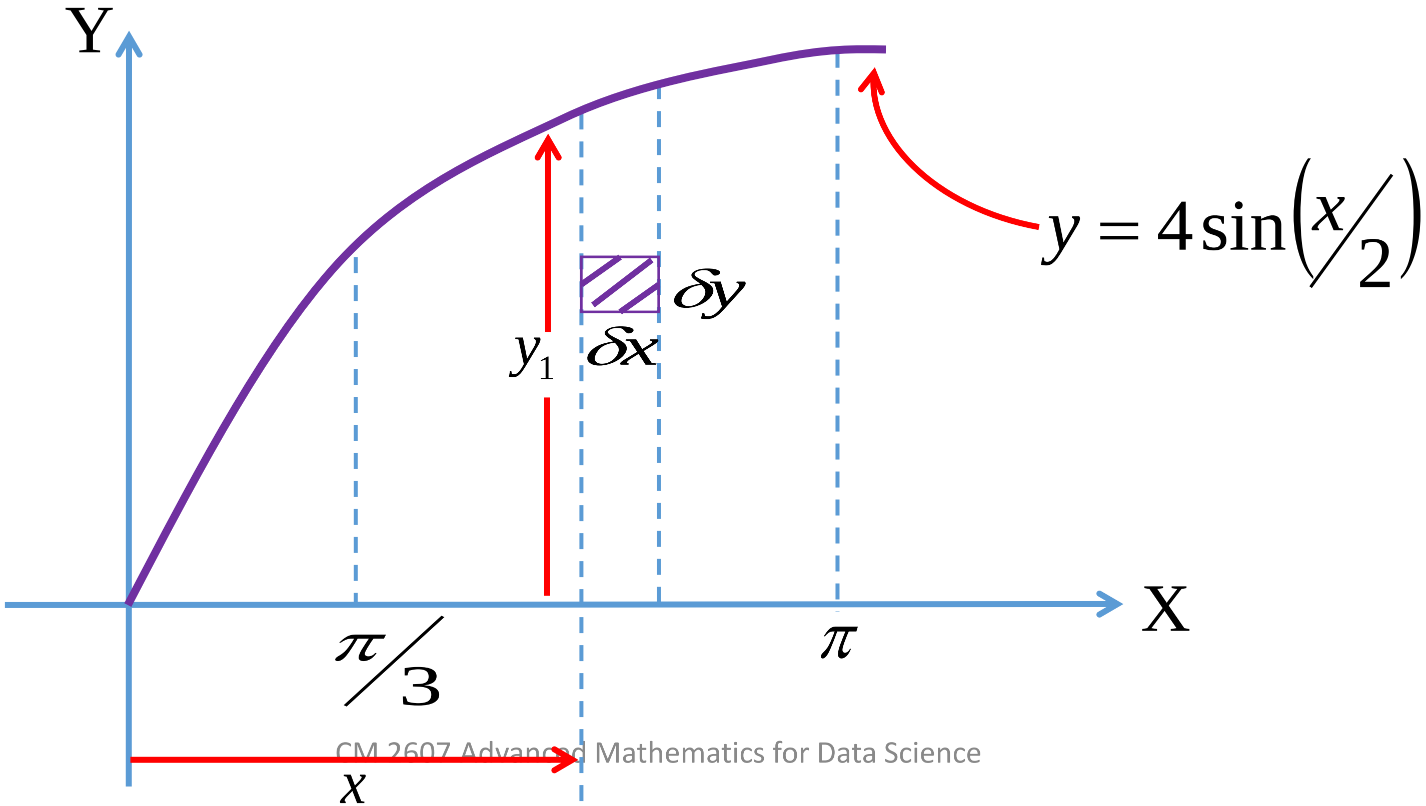
Application 1: Finding the area of a plane figure

Find the area under the curve $y = 4\sin\left(x/2\right)$ between $x = \pi/3$ and $x = \pi$ by the double integral method.



$$\text{Total area, } A \approx \sum_{x=\pi/3}^{x=\pi} \left[\sum_{y=0}^{y=y_1} \delta y \cdot \delta x \right]$$

$$\text{By } \delta y \to 0, \ \delta x \to 0$$

$$A = \int_{\pi/3}^{\pi} \int_{0}^{y_1} dy\, dx$$

$$= \int_{\pi/3}^{\pi} [y]_0^{y_1}\, dx = \int_{\pi/3}^{\pi} y_1\, dx$$

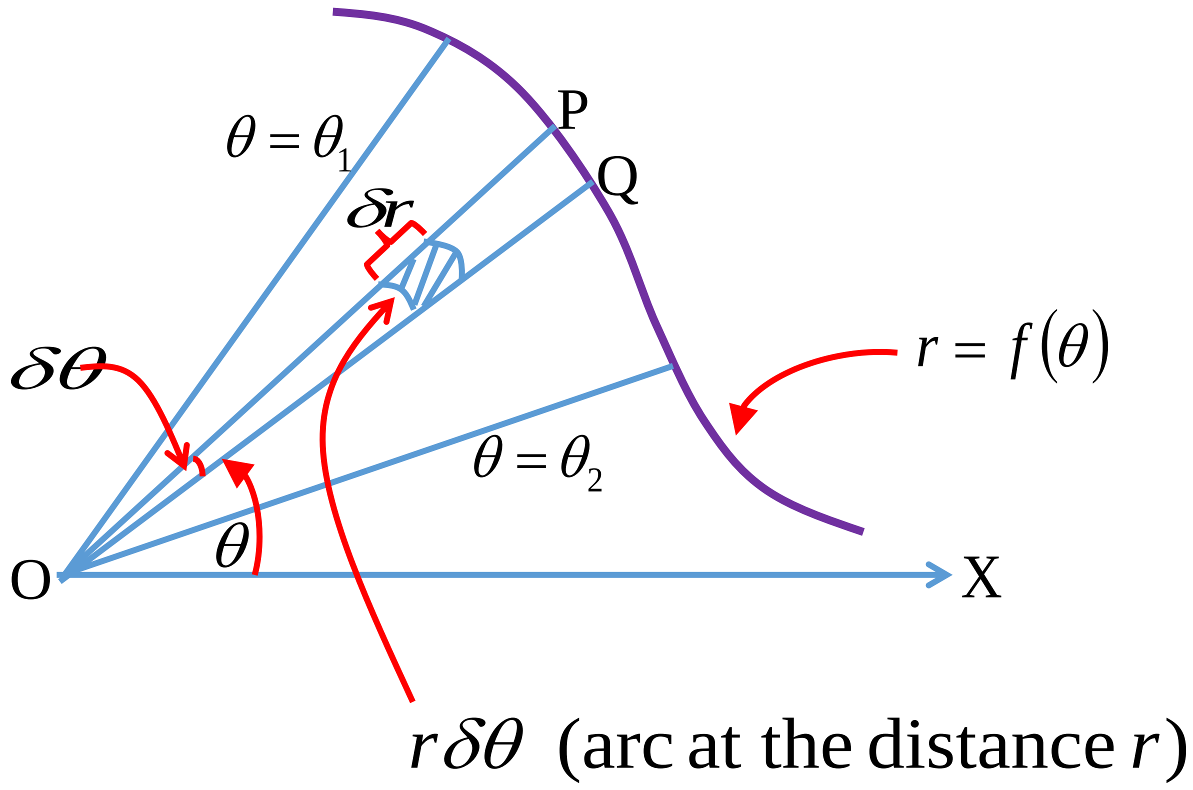
$$= \int_{\pi/3}^{\pi} 4\sin x/2\, dx$$

(Have you seen this before?)

$$= \left[-8\cos\left(x/2\right)\right]_{\pi/3}^{\pi} = 4\sqrt{3}$$

$$\boxed{\text{In general, } A = \int_{x=a}^{x=b} f(x)\, dx}$$

Find the area of the plane figure bounded by the polar curve $r = f(\theta)$ and the radius vectors at $\theta = \theta_1$ and $\theta = \theta_2$.



$$\text{Area of the element} \approx \delta r \cdot r\delta\theta$$

$$\text{Area of the sector POQ} \approx \sum_{r=0}^{r=r_1} \delta r \cdot r\delta\theta$$

$$\text{Total area, } A = \sum_{\theta=\theta_1}^{\theta=\theta_2} (\text{all such sectors})$$

$$A = \sum_{\theta=\theta_1}^{\theta=\theta_2} \left[\sum_{r=0}^{r=r_1} r\, \delta r\, \delta\theta \right]$$

$$A = \sum_{\theta=\theta_1}^{\theta=\theta_2} \sum_{r=0}^{r=r_1} r\, \delta r\, \delta\theta$$

$$\text{By } \delta r \to 0,\ \delta\theta \to 0$$

$$A = \int_{\theta_1}^{\theta_2} \int_{0}^{r_1} r \, dr \, d\theta$$

$$A = \int_{\theta_1}^{\theta_2} \left[\frac{r^2}{2} \right]_0^{r_1} d\theta$$

$$A = \int_{\theta_1}^{\theta_2} \frac{r_1^2}{2} \, d\theta$$

In general, $\boxed{A = \int_{\theta_1}^{\theta_2} \frac{1}{2} \left[f(\theta) \right]^2 \, d\theta}$

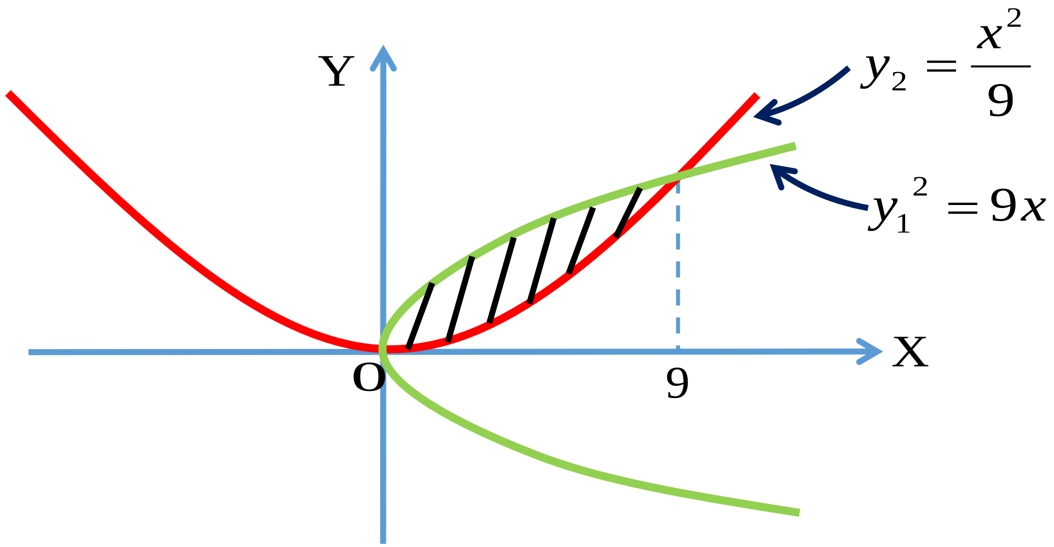
Example

Using the double integrals, find the area enclosed by the polar curve $r = 4(1+\cos\theta)$ and the radius vectors at $\theta = 0$ and $\theta = \pi$.

$(\text{Answer: } 12\pi)$

Example

Find the area enclosed by the curves $y_1^2 = 9x$ and $y_2 = \dfrac{x^2}{9}$ as shown in the following figure.



Note: Recall Probability density functions and its properties using integral

Application: Triple integrals

This is the extension of multiple integrals to 3-dimensional case. Thus, it resembles the context of a volume contrasting to the situation of an area in double integrals.

$$\text{eg. Evaluate } I = \int_{1}^{3}\int_{-1}^{1}\int_{0}^{2}(x+2y-z)\,dx\,dy\,dz$$

$$I = \int_{1}^{3}\int_{-1}^{1}\left[\frac{x^2}{2}+2yx-zx\right]_0^2 dy\,dz \quad (\text{w.r.t. } x)$$

$$= \int_{1}^{3}\int_{-1}^{1}(2+4y-2z)\,dy\,dz$$

$$
\begin{aligned}
I &= \int_{1}^{3} \left[2y + 2y^2 - 2zy\right]_{-1}^{1} dz \quad (\text{w.r.t. } y) \\
&= \int_{1}^{3} \left[(2+2-2z)-(-2+2+2z)\right] dz \\
&= \int_{1}^{3} (4-4z)\, dz \\
&= \left[4z - 2z^2\right]_{1}^{3} \\
&= (12-18) - (4-2) \\
&= 8
\end{aligned}
$$

Examples

Ex. Evaluate the following triple integrals.

i. $$\int_{1}^{2}\int_{0}^{3}\int_{0}^{1}\left(x^2+y^2-z^2\right)dx\,dy\,dz$$

ii. $$\int_{0}^{a}\int_{0}^{b}\int_{0}^{c}\left(x^2+y^2\right)dx\,dy\,dz$$

iii. $$\int_{0}^{\pi}\int_{0}^{\pi/2}\int_{0}^{1}x^2\sin\theta\,dx\,d\theta\,d\phi$$

Alternative notation

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y)\, dx\, dy$$ could have been written

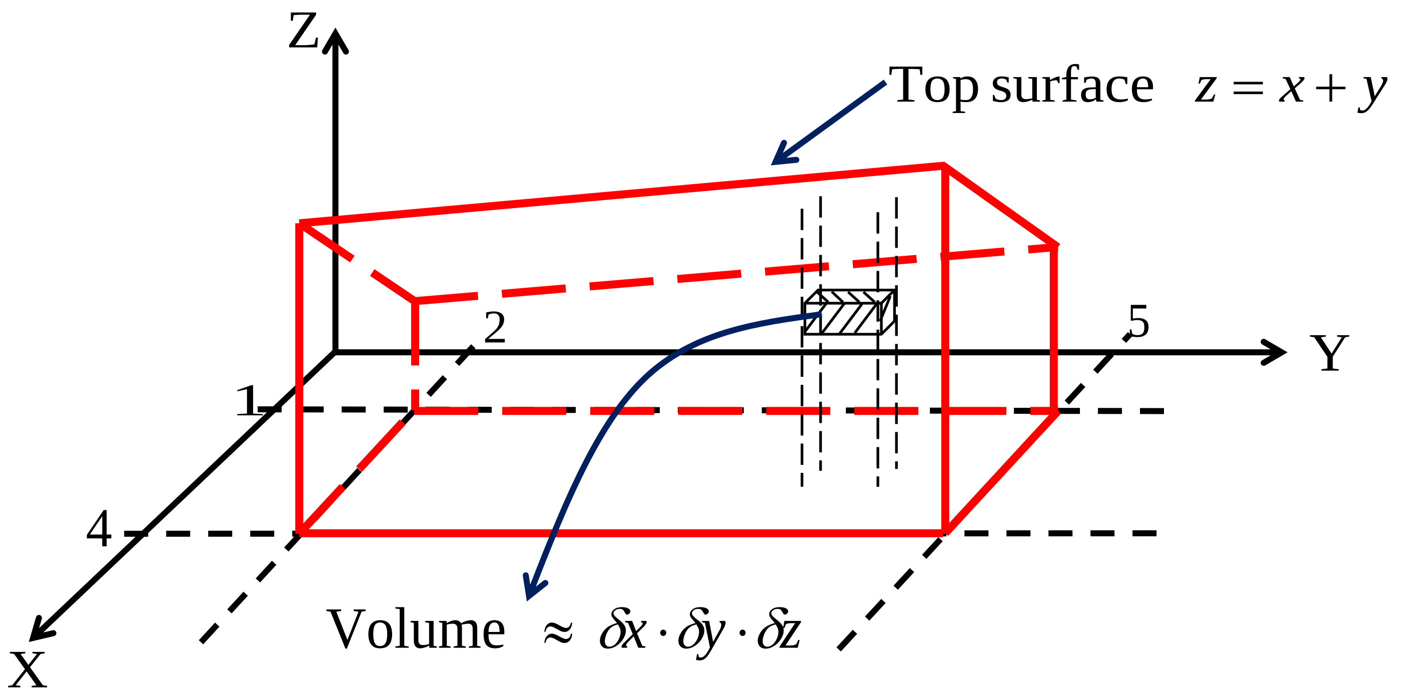
$$\int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x, y)\, dx$$

Now, we start working from the right-hand side integral and gradually work back towards the front. Of course, we get the same result.

Let us use this alternative case with triple integral in the following application.

Application: Finding volume using triple integrals

A solid is enclosed by the planes $z=0,\ x=1,\ x=4, y=2,\ y=5$ and the surface $z=x+y$. Find the volume of the solid.



Volume of the column touching top & bottom $\approx \delta x\, \delta y \sum_{z=0}^{z=x+y} \delta z$

Volume of the slice parallel to y-axis $\approx \delta x \sum_{y=2}^{y=5} \delta y \sum_{z=0}^{z=x+y} \delta z$

$$\text{Total volume, } (V) \approx \sum_{x=1}^{x=4} \delta x \sum_{y=2}^{y=5} \delta y \sum_{z=0}^{z=x+y} \delta z$$

$$\text{By } \delta x \to 0, \quad \delta y \to 0, \quad \delta z \to 0$$

$$V = \int_1^4 dx \int_2^5 dy \int_0^{x+y} dz$$

$$V = \int_1^4 dx \int_2^5 dy \,(x+y)$$

$$V = \int_1^4 dx \int_2^5 (x+y)\,dy$$

$$= \int_1^4 dx \left[xy + \frac{y^2}{2} \right]_2^5$$

$$= \int_1^4 dx \left[5x + \frac{25}{2} - 2x - 2 \right]$$

$$= \int_1^4 \left[3x + \frac{21}{2} \right] dx$$

$$= \left[\frac{3x^2}{2} + \frac{21x}{2} \right]_1^4 \quad = \frac{1}{2}[132 - 24]$$

$$= 54 \ \text{(in cubic units)}$$

Example

Find the volume of the solid bounded by the planes

$z = 0,\ x = 1,\ x = 2, y = -1,\ y = 1$ and the surface

$z = x^2 + y^2$. (Answer: $16/3$)

Probability Density functions

1) Find the constant c such that the function given below is a probability density function.

$$f(x) = \begin{cases} cx^2; 0 \leq x \leq 3 \\ 0; Otherwise \end{cases}$$

Hence find the P$(1 \leq x \leq 2)$.

2) For the given probability density function, find the value of k.

$$f(x) = \frac{k}{1 + x^2}; -\infty < x < \infty$$