CM 2607 Advanced Mathematics for Data Science

Lecture 02

Differentiation part II

ROBERT GORDON UNIVERSITY ABERDEEN
TEF Gold
INFORMATICS INSTITUTE OF TECHNOLOGY

Learning Outcomes

- Covers LO1 and LO2 for CM2607
- On completion of this series of lectures on differentiation, students are expected to be able to:
  - Understand the concept of logarithmic differentiation
  - Find the partial derivatives
  - Find the Hessian matrix for a several variable function
  - Differentiate implicit functions

Content

- Logarithmic Differentiation
- Partial derivatives
- Differentiate implicit functions

Logarithmic Differentiation

Consider the derivative $\frac{d}{dx}\ln x = \frac{1}{x}$ and Let $F(x) be\ a\ function\ of\ x$

Result: $\frac{d}{dx}\ln(F(x)) = \frac{1}{F(x)}\frac{d}{dx}F(x)$

Eg: If $y = \frac{uv}{w}$-----------→$lny = lnu + lnv - lnw$

Differentiate w.r.t. (x)→$\frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{u}\frac{\mathrm{du}}{\mathrm{dx}} + \frac{1}{v}\frac{\mathrm{dv}}{\mathrm{dx}} - \frac{1}{w}\frac{\mathrm{dw}}{\mathrm{dx}}$

Example

1) Find the derivative of $y = \dfrac{x^2 e^{3x}}{\cos 2x}$

2) Find the derivative of $y = x^{-4} \sin^2 x \, (2x^3 - 1)^3$

3) Redo the example in lecture 01, $\dfrac{d^2\left(\frac{y}{\sin y}\right)}{dy^2}$ using logarithmic differentiation.

Note: Discuss the use of logarithmic differentiation when there are more than two functions involved in product or quotient or both

Partial Derivatives

Partial derivatives of several variable functions

If a function has more than one independent variable then it is called as a several variable function.

eg. $f(x, y) = x^2 + \sin y - 1$

Two independent variables x and y

eg. $f(x, y, z) = xy + e^z$

Three independent variables x, y and z

eg. $f(x,y,t) = x \sin t + y \cos t$

Three independent variables x, y and t

Differentiation of several variable functions is considered 'partially' with respect to each independent variable.

Notations:

eg. If $z = f(x,y)$

$$\frac{\partial z}{\partial x} \rightarrow 1^{st} \text{ partial derivative of } f(x,y) \text{ w.r.t. } x \rightarrow z_x$$

$$\frac{\partial z}{\partial y} \rightarrow 1^{st} \text{ partial derivative of } f(x,y) \text{ w.r.t. } y \rightarrow z_y$$

$$\frac{\partial^2 z}{\partial x^2} \rightarrow 2^{nd} \text{ partial derivative of } f(x,y) \text{ w.r.t. } x \rightarrow z_{xx}$$

Notations

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{yx}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{xy}$$

In determining partial derivative w.r.t. one independent variable, all the other independent variables are considered as constants.

eg. Find all 1^{st} and 2^{nd} order partial derivatives of $z = x + x^3 y^2$.

$$\frac{\partial z}{\partial x} = \frac{\partial x}{\partial x} + \frac{\partial (x^3 y^2)}{\partial x} = 1 + y^2 \frac{\partial x^3}{\partial x}$$

$$= 1 + y^2 (3x^2)$$

$$= 1 + 3y^2 x^2$$

$$\frac{\partial z}{\partial y} = 0 + x^3 (2y) = 2x^3 y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial(1+3y^2x^2)}{\partial x} = \frac{\partial(1)}{\partial x} + 3y^2\frac{\partial x^2}{\partial x}$$

$$= 0 + 3y^2(2x)$$

$$= 6y^2x$$

Find $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial(2x^3 y)}{\partial y} = 2x^3\frac{\partial y}{\partial y}$$

$$= 2x^3$$

Ex.

1) Find all 1st and 2nd order partial derivatives of the following functions.

i. $z = y^{-3} + \sin x$

ii. $z = xy^4 - ye^x$

iii. $z = \cos(xy)$

iv. $z = e^{x^2 y}$

v. $z = \dfrac{x + y^2 + 1}{\ln x}$

2) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x^2}$ for the following parametric case of z.

$$z = sin\,2t + cos\,3t$$

$$x = t^2$$

$$y = 1 + t \qquad\qquad t - \text{parameter}$$

3) Find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial h}$ and $\frac{\partial z}{\partial t}$ for the implicit case

$$z = h\,sin\left(zr^2 t\right)$$

Hessian Matrix

- The Hessian of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined to be the matrix

$$\mathbf{H}f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} & \cdots \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} & \cdots \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

Find the Hessian matrix for the function $f = x^3 - 2xy + y^3$ and compute it at the point (1,3)

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}$$

$$f_x(x,y) = 3x^2 - 2y \; and \; f_y(x,y) = -2x + 3y^2$$

$$Hf(x,y) = \begin{pmatrix} 6x & -2 \\ -2 & 6y \end{pmatrix}$$

$$Hf(1,3) = \begin{pmatrix} 6 & -2 \\ -2 & 18 \end{pmatrix}$$

Rules of differentiation

4. Differentiating implicit functions

If y is completely defined in terms of x, then y is called an <u>explicit function</u> of x (ie. $y = f(x)$ form can be achieved).

But, if it is not possible to seperate y completely in terms of x, then y is called an <u>implicit function</u> of x.

eg. $xy + 2 = \sin x \quad \rightarrow \quad y = \dfrac{\sin x - 2}{x}$ – explicit

$x^2 + y^2 = 25 \quad \rightarrow \quad y = \boxed{\;?\;}$ – implicit

Derivative of implicit functions

In differentiating implicit functions, we use chain rule by considering y as a seperate function.

eg. Find $\dfrac{dy}{dx}$ for $x^2 + y^2 = 25.$

Differentiating both sides w.r.t. (with respect to) x

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$

$$2x + \frac{dy^2}{dy}\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Notice that derivatives of implicit functions may contain both x and y.

eg. If $x^2+y^2-2x-6y+5=0$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Evaluate these derivatives at $x=3$, $y=2$.

$$\frac{d}{dx}\left(x^2+y^2-2x-6y+5\right) = \frac{d}{dx}(0)$$

$$2x+2y\frac{dy}{dx}-2-6\frac{dy}{dx}+0 = 0$$

$$\frac{dy}{dx} = \frac{2-2x}{2y-6} = \frac{1-x}{y-3}$$

At $x=3, y=2$ $\quad \dfrac{dy}{dx} = \dfrac{1-3}{2-3} = 2$

Differentiating $\dfrac{dy}{dx}$ w.r.t. x again;

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-x}{y-3}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(y-3)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(y-3)}{(y-3)^2}$$

$$= \frac{(y-3)(-1)-(1-x)\frac{dy}{dx}}{(y-3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-(y-3)-(1-x)\left(\frac{1-x}{y-3}\right)}{(y-3)^2}$$

$$= \frac{-(y-3)^2-(1-x)^2}{(y-3)^3}$$

At $\quad x=3\,,\, y=2 \qquad \frac{d^2y}{dx^2} = \frac{-(2-3)^2-(1-3)^2}{(2-3)^3}$

$$= \frac{-1-4}{-1} = 5$$

Ex. Find $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ for the following implicit cases.

1. $y^3 + x = e^x$
2. $x^2 + xy + 3y^2 = 4$
3. $e^{x^2 y} = y$
4. $ln(x+y) = 3\,sin\,y$