# The Binomial Distribution

## Introduction

A situation in which an experiment (or trial) is repeated a fixed number of times can be modelled, under certain assumptions, by the binomial distribution. Within each trial we focus attention on a particular outcome. If the outcome occurs we label this as a success. The binomial distribution allows us to calculate the probability of observing a certain number of successes in a given number of trials.

You should note that the term 'success' (and by implication 'failure') are simply labels and as such might be misleading. For example counting the number of defective items produced by a machine might be thought of as counting successes if you are looking for defective items! Trials with two possible outcomes are often used as the building blocks of random experiments and can be useful to engineers. Two examples are:

1. A particular mobile phone link is known to transmit 6% of 'bits' of information in error. As an engineer you might need to know the probability that two bits out of the next ten transmitted are in error.

2. A machine is known to produce, on average, 2% defective components. As an engineer you might need to know the probability that 3 items are defective in the next 20 produced.

The binomial distribution will help you to answer such questions.

# 1. The binomial model

We have introduced random variables from a general perspective and have seen that there are two basic types: discrete and continuous. We examine four particular examples of distributions for random variables which occur often in practice and have been given special names. They are the **binomial** distribution, the **Poisson** distribution, the **Hypergeometric** distribution and the **Normal** distribution. The first three are distributions for discrete random variables and the fourth is for a continuous random variable. In this Section we focus attention on the binomial distribution.

The binomial distribution can be used in situations in which a given experiment (often referred to, in this context, as a **trial**) is repeated a number of times. For the binomial model to be applied the following four criteria must be satisfied:

- the trial is carried out a fixed number of times $n$
- the outcomes of each trial can be classified into two 'types' conventionally named 'success' or 'failure'
- the probability $p$ of success remains constant for each trial
- the individual trials are independent of each other.

For example, if we consider throwing a coin 7 times what is the probability that exactly 4 Heads occur? This problem can be modelled by the binomial distribution since the four basic criteria are assumed satisfied as we see.

- here the trial is 'throwing a coin' which is carried out 7 times
- the occurrence of Heads on any given trial (i.e. throw) may be called a 'success' and Tails called a 'failure'
- the probability of success is $p = \frac{1}{2}$ and remains constant for each trial
- each throw of the coin is independent of the others.

The reader will be able to complete the solution to this example once we have constructed the general binomial model.

The following two scenarios are typical of those met by engineers. The reader should check that the criteria stated above are met by each scenario.

1. An electronic product has a total of 30 integrated circuits built into it. The product is capable of operating successfully only if at least 27 of the circuits operate properly. What is the probability that the product operates successfully if the probability of any integrated circuit failing to operate is 0.01?

2. Digital communication is achieved by transmitting information in "bits". Errors do occur in data transmissions. Suppose that the number of bits in error is represented by the random variable $X$ and that the probability of a communication error in a bit is 0.001. If at most 2 errors are present in a 1000 bit transmission, the transmission can be successfully decoded. If a 1000 bit message is transmitted, find the probability that it can be successfully decoded.

Before developing the *general* binomial distribution we consider the following examples which, as you will soon recognise, have the basic characteristics of a binomial distribution.

## Example 7

In a box of floppy discs it is known that 95% will work. A sample of three of the discs is selected at random.
Find the probability that (a) none (b) 1, (c) 2, (d) all 3 of the sample will work.

**Example 8**

A worn machine is known to produce 10% defective components. If the random variable $X$ is the number of defective components produced in a run of 3 components, find the probabilities that $X$ takes the values 0 to 3.

We now develop the binomial distribution from a more general perspective. If you find the theory getting a bit heavy simply refer back to this example to help clarify the situation.
First we shall find it convenient to denote the probability of failure on a trial, which is $1-p$, by $q$, that is:

$$q = 1 - p.$$

What we shall do is to calculate probabilities of the number of 'successes' occurring in $n$ trials, beginning with $n = 1$.

$\underline{\boldsymbol{n=1}}$ With only one trial we can observe either 1 success (with probability $p$) or 0 successes (with probability $q$).

$\underline{\boldsymbol{n=2}}$ Here there are 3 possibilities: We can observe 2, 1 or 0 successes. Let $S$ denote a success and $F$ denote a failure. So a failure followed by a success would be denoted by $FS$ whilst two failures followed by one success would be denoted by $FFS$ and so on.
Then

$$\mathsf{P}(\text{2 successes in 2 trials}) = \mathsf{P}(SS) = \mathsf{P}(S)\mathsf{P}(S) = p^2$$

(where we have used the assumption of independence between trials and hence multiplied probabilities). Now, using the usual rules of basic probability, we have:

$$\mathsf{P}(\text{1 success in 2 trials}) = \mathsf{P}[(SF) \cup (FS)] = \mathsf{P}(SF) + \mathsf{P}(FS) = pq + qp = 2pq$$

$$\mathsf{P}(\text{0 successes in 2 trials}) = \mathsf{P}(FF) = \mathsf{P}(F)\mathsf{P}(F) = q^2$$

The three probabilities we have found $-$ $q^2,\ 2qp,\ p^2$ $-$ are in fact the terms which arise in the binomial expansion of $(q+p)^2 = q^2 + 2qp + p^2$. We also note that since $q = 1-p$ the probabilities sum to 1 (as we should expect):

$$q^2 + 2qp + p^2 = (q+p)^2 = ((1-p)+p)^2 = 1$$

**Task**

List the outcomes for the binomial model for the case $n = 3$, calculate their probabilities and display the results in a table.

Note that the probabilities you have obtained:

$$q^3,\ 3q^2p,\ 3qp^2,\ p^3$$

are the terms which arise in the binomial expansion of $(q+p)^3 = q^3 + 3q^2p + 3qp^2 + p^3$

Repeat the previous Task for the binomial model for the case with $n = 4$.

Again we explore the connection between the probabilities and the terms in the binomial expansion of $(q+p)^4$. Consider this expansion

$$(q+p)^4 = q^4 + 4q^3p + 6q^2p^2 + 4qp^3 + p^4$$

Then, for example, the term $4p^3q$, is the probability of 3 successes in the four trials. These successes can occur anywhere in the four trials and there must be one failure hence the $p^3$ and $q$ components which are multiplied together. The remaining part of this term, 4, is the number of ways of selecting three objects from 4.

Similarly there are ${}^4C_2 = \dfrac{4!}{2!2!} = 6$ ways of selecting two objects from 4 so that the coefficient 6 combines with $p^2$ and $q^2$ to give the probability of two successes (and hence two failures) in four trials.

The approach described here can be extended for any number $n$ of trials.

## Key Point 4

**The Binomial Probabilities**

Let $X$ be a discrete random variable, being the number of successes occurring in $n$ independent trials of an experiment. If $X$ is to be described by the binomial model, the probability of exactly $r$ successes in $n$ trials is given by

$$\mathsf{P}(X = r) = {}^nC_r p^r q^{n-r}.$$

Here there are $r$ successes (each with probability $p$), $n - r$ failures (each with probability $q$) and ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ is the number of ways of placing the $r$ successes among the $n$ trials.

## Notation

If a random variable $X$ follows a binomial distribution in which an experiment is repeated $n$ times each with probability $p$ of success then we write $X \sim B(n, p)$.

## Example 9

A worn machine is known to produce 10% defective components. If the random variable $X$ is the number of defective components produced in a run of 4 components, find the probabilities that $X$ takes the values 0 to 4.

**Example 10**

In a box of switches it is known 10% of the switches are faulty. A technician is wiring 30 circuits, each of which needs one switch. What is the probability that (a) all 30 work, (b) at most 2 of the circuits do not work?

**Example 11**

A University Engineering Department has introduced a new software package called SOLVIT. To save money, the University's Purchasing Department has negotiated a bargain price for a 4-user licence that allows only four students to use SOLVIT at any one time. It is estimated that this should allow 90% of students to use the package when they need it. The Students' Union has asked for more licences to be bought since engineering students report having to queue excessively to use SOLVIT. As a result the Computer Centre monitors the use of the software. Their findings show that on average 20 students are logged on at peak times and 4 of these want to use SOLVIT. Was the Purchasing Department's estimate correct?

**Task**

Using the binomial model, and assuming that a success occurs with probability $\frac{1}{5}$ in each trial, find the probability that in 6 trials there are
(a) 0 successes (b) 3 successes (c) 2 failures.

Let $X$ be the number of successes in 6 independent trials.

# 2. Expectation and variance of the binomial distribution

For a binomial distribution $X \sim B(n, p)$, the mean and variance, as we shall see, have a simple form. While we will not prove the formulae in general terms - the algebra can be rather tedious - we will illustrate the results for cases involving small values of $n$.

### The case $n = 2$

Essentially, we have a random variable $X$ which follows a binomial distribution $X \sim B(2, p)$ so that the values taken by $X$ (and $X^2$ - needed to calculate the variance) are shown in the following table:

| $x$ | $x^2$ | $\mathsf{P}(X = x)$ | $x\mathsf{P}(X = x)$ | $x^2\mathsf{P}(X = x)$ |
|---|---|---|---|---|
| 0 | 0 | $q^2$ | 0 | 0 |
| 1 | 1 | $2qp$ | $2qp$ | $2qp$ |
| 2 | 4 | $p^2$ | $2p^2$ | $4p^2$ |

We can now calculate the mean of this distribution:

$$\mathsf{E}(X) = \sum x\mathsf{P}(X = x) = 0 + 2qp + 2p^2 = 2p(q + p) = 2p \qquad \text{since } q + p = 1$$

Similarly, the variance $V(X)$ is given by

$$V(X) = \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 = 0 + 2qp + 4p^2 - (2p)^2 = 2qp$$

**Task**

Calculate the mean and variance of a random variable $X$ which follows a binomial distribution $X \sim B(3, p)$.

**Your solution**

From the results given above, it is reasonable to asert the following result in Key Point 5.

**Key Point 5**

**Expectation and Variance of the Binomial Distribution**

If a random variable $X$ which can assume the values $0, 1, 2, 3, \ldots, n$ follows a binomial distribution $X \sim B(n, p)$ so that

$$\mathsf{P}(X = r) = {}^nC_r p^r q^{n-r} = {}^nC_r p^r (1-p)^{n-r}$$

then the expectation and variance of the distribution are given by the formulae

$$\mathsf{E}(X) = np \qquad \text{and} \qquad V(X) = np(1-p) = npq$$

**Task**

A die is thrown repeatedly 36 times in all. Find $\mathsf{E}(X)$ and $V(X)$ where $X$ is the number of sixes obtained.

# The Poisson Distribution

## Introduction

In this Section we introduce a probability model which can be used when the outcome of an experiment is a random variable taking on positive integer values and where the only information available is a measurement of its average value. This has widespread applications, for example in analysing traffic flow, in fault prediction on electric cables and in the prediction of randomly occurring accidents. We shall look at the Poisson distribution in two distinct ways. Firstly, as a distribution in its own right. This will enable us to apply statistical methods to a set of problems which cannot be solved using the binomial distribution. Secondly, as an approximation to the binomial distribution $X \sim B(n, p)$ in the case where $n$ is large and $p$ is small. You will find that this approximation can often save the need to do much tedious arithmetic.

# 1. The Poisson approximation to the binomial distribution

The probability of the outcome $X = r$ of a set of Bernoulli trials can always be calculated by using the formula

$$\mathsf{P}(X = r) = {}^nC_r q^{n-r} p^r$$

given above. Clearly, for very large values of $n$ the calculation can be rather tedious, this is particularly so when very small values of $p$ are also present. In the situation when $n$ is large and $p$ is small and the product $np$ is constant we can take a different approach to the problem of calculating the probability that $X = r$. In the table below the values of $\mathsf{P}(X = r)$ have been calculated for various combinations of $n$ and $p$ under the constraint that $np = 1$. You should try some of the calculations for yourself using the formula given above for some of the **smaller** values of $n$.

Probability of X successes

| $n$ | $p$ | $X=0$ | $X=1$ | $X=2$ | $X=3$ | $X=4$ | $X=5$ | $X=6$ |
|---|---|---|---|---|---|---|---|---|
| 4 | 0.25 | 0.316 | 0.422 | 0.211 | 0.047 | 0.004 | | |
| 5 | 0.20 | 0.328 | 0.410 | 0.205 | 0.051 | 0.006 | 0.000 | |
| 10 | 0.10 | 0.349 | 0.387 | 0.194 | 0.058 | 0.011 | 0.001 | 0.000 |
| 20 | 0.05 | 0.359 | 0.377 | 0.189 | 0.060 | 0.013 | 0.002 | 0.000 |
| 100 | 0.01 | 0.366 | 0.370 | 0.185 | 0.061 | 0.014 | 0.003 | 0.001 |
| 1000 | 0.001 | 0.368 | 0.368 | 0.184 | 0.061 | 0.015 | 0.003 | 0.001 |
| 10000 | 0.0001 | 0.368 | 0.368 | 0.184 | 0.061 | 0.015 | 0.003 | 0.001 |

Each of the binomial distributions given has a mean given by $np = 1$. Notice that the probabilities that $X = 0, 1, 2, 3, 4, \dots$ approach the values $0.368, 0.368, 0.184, \dots$ as $n$ increases.

If we have to determine the probabilities of success when large values of $n$ and small values of $p$ are involved it would be very convenient if we could do so without having to construct tables. In fact we can do such calculations by using the Poisson distribution which, under certain constraints, may be considered as an approximation to the binomial distribution.

By considering simplifications applied to the binomial distribution subject to the conditions

1. $n$ is large
2. $p$ is small
3. $np = \lambda$ ($\lambda$ a constant)

we can derive the formula

$$\mathsf{P}(X = r) = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!} \quad \text{as an approximation to } \mathsf{P}(X = r) = {}^nC_r q^{n-r} p^r.$$

This is the Poisson distribution given previously. We now show how this is done. We know that the binomial distribution is given by

$$(q+p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2!}q^{n-2}p^2 + \cdots + \frac{n(n-1)\dots(n-r+1)}{r!}q^{n-r}p^r + \cdots + p^n$$

Condition (2) tells us that since $p$ is small, $q = 1 - p$ is approximately equal to 1. Applying this to the terms of the binomial expansion above we see that the right-hand side becomes

$$1 + np + \frac{n(n-1)}{2!}p^2 + \cdots + \frac{n(n-1)\dots(n-r+1)}{r!}p^r + \cdots + p^n$$

Applying condition (1) allows us to approximate terms such as $(n-1), (n-2), \dots$ to $n$ (mathematically, we are allowing $n \to \infty$ ) and the right-hand side of our expansion becomes

$$1 + np + \frac{n^2}{2!}p^2 + \cdots + \frac{n^r}{r!}p^r + \dots$$

Note that the term $p^n \to 0$ under these conditions and hence has been omitted.
We now have the series

$$1 + np + \frac{(np)^2}{2!} + \cdots + \frac{(np)^r}{r!} + \dots$$

which, using condition (3) may be written as

$$1 + \lambda + \frac{(\lambda)^2}{2!} + \cdots + \frac{(\lambda)^r}{r!} + \dots$$

You may recognise this as the expansion of $\mathsf{e}^\lambda$.

If we are to be able to claim that the terms of this expansion represent probabilities, we must be sure that the sum of the terms is 1. We divide by $\mathsf{e}^\lambda$ to satisfy this condition. This gives the result

$$\begin{aligned}\frac{\mathsf{e}^\lambda}{\mathsf{e}^\lambda} = 1 &= \frac{1}{\mathsf{e}^\lambda}(1 + \lambda + \frac{(\lambda)^2}{2!} + \cdots + \frac{(\lambda)^r}{r!} + \dots) \\ &= \mathsf{e}^{-\lambda} + \mathsf{e}^{-\lambda}\lambda + \mathsf{e}^{-\lambda}\frac{\lambda^2}{2!} + \mathsf{e}^{-\lambda}\frac{\lambda^3}{3!} + \cdots + \mathsf{e}^{-\lambda}\frac{\lambda^r}{r!} + \cdots +\end{aligned}$$

The terms of this expansion are very good approximations to the corresponding binomial expansion under the conditions

1. $n$ is large
2. $p$ is small
3. $np = \lambda$ ($\lambda$ constant)

The Poisson approximation to the binomial distribution is summarized below.

## Key Point 6

**Poisson Approximation to the Binomial Distribution**

Assuming that $n$ is large, $p$ is small and that $np$ is constant, the terms

$$\mathsf{P}(X = r) = {}^nC_r(1-p)^{n-r}p^r$$

of a binomial distribution may be closely approximated by the terms

$$\mathsf{P}(X = r) = \mathsf{e}^{-\lambda}\frac{\lambda^r}{r!}$$

of the Poisson distribution for corresponding values of r.

## Example 12

We introduced the binomial distribution by considering the following scenario. A worn machine is known to produce 10% defective components. If the random variable $X$ is the number of defective components produced in a run of 3 components, find the probabilities that $X$ takes the values 0 to 3.

Suppose now that a similar machine which is known to produce 1% defective components is used for a production run of 40 components. We wish to calculate the probability that two defective items are produced. Essentially we are assuming that $X \sim B(40, 0.01)$ and are asking for $\mathrm{P}(X = 2)$. We use both the binomial distribution and its Poisson approximation for comparison.

## Practical considerations

In practice, we can use the Poisson distribution to very closely approximate the binomial distribution provided that the product $np$ is constant with

$n \geq 100$ and $p \leq 0.05$

Note that this is not a hard-and-fast rule and we simply say that

'the larger $n$ is the better and the smaller $p$ is the better provided that $np$ is a sensible size.'

The approximation remains good provided that $np < 5$ for values of $n$ as low as 20.

**Task**

Mass-produced needles are packed in boxes of 1000. It is believed that 1 needle in 2000 on average is substandard. What is the probability that a box contains 2 or more defectives? The correct model is the binomial distribution with $n = 1000,\ p = \dfrac{1}{2000}$ (and $q = \dfrac{1999}{2000}$).

(a) Using the binomial distribution calculate $\mathsf{P}(X = 0)$, $\mathsf{P}(X = 1)$ and hence $\mathsf{P}(X \geq 2)$:

In the above Task we have obtained the same answer to 4 d.p., as the exact binomial calculation, essentially because $p$ was so small. We shall not always be so lucky!

**Example 13**

In the manufacture of glassware, bubbles can occur in the glass which reduces the status of the glassware to that of a 'second'. If, on average, one in every 1000 items produced has a bubble, calculate the probability that exactly six items in a batch of three thousand are seconds.

**Example 14**

A manufacturer produces light-bulbs that are packed into boxes of 100. If quality control studies indicate that 0.5% of the light-bulbs produced are defective, what percentage of the boxes will contain:

(a) no defective? (b) 2 or more defectives?

# 2. The Poisson distribution

The Poisson distribution is a probability model which can be used to find the probability of a single event occurring a given number of times in an interval of (usually) time. The occurrence of these events must be determined by chance alone which implies that information about the occurrence of any one event cannot be used to predict the occurrence of any other event. It is worth noting that only the *occurrence* of an event can be counted; the *non-occurrence* of an event cannot be counted. This contrasts with Bernoulli trials where we know the number of trials, the number of events occurring and therefore the number of events not occurring.

The Poisson distribution has widespread applications in areas such as analysing traffic flow, fault prediction in electric cables, defects occurring in manufactured objects such as castings, email messages arriving at a computer and in the prediction of randomly occurring events or accidents. One well known series of accidental events concerns Prussian cavalry who were killed by horse kicks. Although not discussed here (death by horse kick is hardly an engineering application of statistics!) you will find accounts in many statistical texts. One example of the use of a Poisson distribution where the events are not necessarily time related is in the prediction of fault occurrence along a long weld - faults may occur anywhere along the length of the weld. A similar argument applies when scanning castings for faults - we are looking for faults occurring in a volume of material, not over an interval if time.

The following definition gives a theoretical underpinning to the Poisson distribution.

## Definition of a Poisson process

Suppose that events occur at random throughout an interval. Suppose further that the interval can be divided into subintervals which are so small that:

1. the probability of more than one event occurring in the subinterval is zero

2. the probability of one event occurring in a subinterval is proportional to the length of the subinterval

3. an event occurring in any given subinterval is independent of any other subinterval

then the random experiment is known as a **Poisson process**.

The word 'process' is used to suggest that the experiment takes place over time, which is the usual case. If the average number of events occurring in the interval (not subinterval) is $\lambda$ $(> 0)$ then the random variable $X$ representing the actual number of events occurring in the interval is said to have a Poisson distribution and it can be shown (we omit the derivation) that

$$\mathsf{P}(X = r) = \mathsf{e}^{-\lambda}\frac{\lambda^r}{r!} \qquad r = 0, 1, 2, 3, \ldots$$

The following Key Point provides a summary.

## Key Point 7

### The Poisson Probabilities

If $X$ is the random variable

'number of occurrences in a given interval'

for which the average rate of occurrence is $\lambda$ then, according to the **Poisson** model, the probability of $r$ occurrences in that interval is given by

$$\mathsf{P}(X = r) = e^{-\lambda}\,\frac{\lambda^r}{r!} \qquad r = 0, 1, 2, 3, \ldots$$

**Task**

Using the Poisson distribution $\mathsf{P}(X = r) = \mathrm{e}^{-\lambda}\dfrac{\lambda^r}{r!}$ write down the formulae for $\mathsf{P}(X = 0),\ \mathsf{P}(X = 1),\ \mathsf{P}(X = 2)$ and $\mathsf{P}(X = 6)$, noting that $0! = 1$.

**Task**

Calculate $P(X=0)$ to $P(X=5)$ when $\lambda = 2$, accurate to 4 d.p.

Notice how the values for $P(X=r)$ in the above answer increase, stay the same and then decrease relatively rapidly (due to the significant increase in $r!$ with increasing $r$). Here two of the probabilities are equal and this will always be the case when $\lambda$ is an integer.

In this last Task we only went up to $P(X=5)$ and calculated each entry separately. However, each probability need not be calculated directly. We can use the following relations (which can be checked from the formulae for $P(X=r)$) to get the next probability from the previous one:

$$P(X=1) = \frac{\lambda}{1}\,P(X=0)\ , \quad P(X=2) = \frac{\lambda}{2}\,P(X=1), \qquad P(X=3) = \frac{\lambda}{3}\,P(X=2)\ ,\ \text{etc.}$$

## Key Point 8

### Recurrence Relation for Poisson Probabilities

In general, for ease of calculation the **recurrence relation** below can be used

$$P(X=r) = \frac{\lambda}{r}\,P(X=r-1) \qquad \text{for}\ \ r \geq 1.$$

**Example 15**

Calculate the value for $\mathsf{P}(X = 6)$ to extend the Table in the previous Task using the recurrence relation and the value for $\mathsf{P}(X = 5)$.

We now look further at the Poisson distribution by considering an example based on traffic flow.

**Example 16**

Suppose it has been observed that, on average, 180 cars per hour pass a specified point on a particular road in the morning rush hour. Due to impending roadworks it is estimated that congestion will occur closer to the city centre if more than 5 cars pass the point in any one minute. What is the probability of congestion occurring?

**Example 17**

The mean number of bacteria per millilitre of a liquid is known to be 6. Find the probability that in 1 ml of the liquid, there will be:

(a) 0, (b) 1, (c) 2, (d) 3, (e) less than 4, (f) 6 bacteria.

A Council is considering whether to base a recovery vehicle on a stretch of road to help clear incidents as quickly as possible. The road concerned carries over 5000 vehicles during the peak rush hour period. Records show that, on average, the number of incidents during the morning rush hour is 5. The Council won't base a vehicle on the road if the probability of having more than 5 incidents in any one morning is less than 30%. Based on this information should the Council provide a vehicle?

# 3. Expectation and variance of the poisson distribution

The expectation and variance of the Poisson distribution can be derived directly from the definitions which apply to any discrete probability distribution. However, the algebra involved is a little lengthy. Instead we derive them from the binomial distribution from which the Poisson distribution is derived.

### Intuitive Explanation

One way of deriving the mean and variance of the Poisson distribution is to consider the behaviour of the binomial distribution under the following conditions:

1. $n$ is large    2. $p$ is small    3. $np = \lambda$ (a constant)

Recalling that the expectation and variance of the binomial distribution are given by the results

$$\mathsf{E}(X) = np \qquad \text{and} \qquad \mathsf{V}(X) = np(1-p) = npq$$

it is reasonable to assert that condition (2) implies, since $q = 1 - p$, that $q$ is approximately 1 and so the expectation and variance are given by

$$\mathsf{E}(X) = np \qquad \text{and} \qquad \mathsf{V}(X) = npq \approx np$$

In fact the algebraic derivation of the expectation and variance of the Poisson distribution shows that these results are in fact *exact*.

Note that the expectation and the variance are equal.

## Key Point 9

**The Poisson Distribution**

If $X$ is the random variable $\{$number of occurrences in a given interval$\}$

for which the average rate of occurrences is $\lambda$ and $X$ can assume the values $0, 1, 2, 3, \ldots$ and the probability of $r$ occurrences in that interval is given by

$$\mathsf{P}(X = r) = \mathsf{e}^{-\lambda}\frac{\lambda^r}{r!}$$

then the expectation and variance of the distribution are given by the formulae

$$\mathsf{E}(X) = \lambda \qquad \text{and} \qquad \mathsf{V}(X) = \lambda$$

For a Poisson distribution the Expectation and Variance are equal.