CM2607 Advanced Mathematics for Data Science

Series, Convergence, Divergence

Week No 08

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Learning Outcomes

- Covers LO3 for Module
- On completion of this lecture, students are expected to be able to:
  - Define series
  - Calculate the sum to infinity of a series
  - Determine convergence and divergence of a series

Series

- Series of the n terms in the sequence

$$a_1, a_2, a_3, a_{4,}\ldots a_n$$

$$a_1 + a_2 + a_3 + a_4\ldots a_n = \sum_{r=1}^{n} a_r$$

Infinite series

$$\sum_{r=1}^{\infty} a_r$$

N^{th} partial sum

- Definition: The sum of terms in a sequence
- The N^{th} partial sum of a series is the sum to n terms.
- Can be found using standard formulae in some cases
- For arithmetic series:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

- For geometric series

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

Other standard formulae

- Some other useful standard formulae:

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Other standard formulae

- Some other useful standard formulae:

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2 = \left(\sum_{r=1}^{n} r \right)^2$$

Applying standard formulae

- Example:

$$\sum_{1}^{5} 2r^2 + 3r + 1$$

$$= 2\sum_{1}^{5} r^2 + 3\sum_{1}^{5} r + \sum_{1}^{5} 1$$

$$= 2\left(\frac{5}{6} \times (5+1) \times (2 \times 5 + 1)\right) + 3\left(\frac{1}{2} \times 5 \times (5+1)\right) + 5 = 160$$

Differencing

- Some series can be summed using partial fractions, based on most terms cancelling out.

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

Sum to infinity

- Sum of all terms in the series.
- Arithmetic series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(2a_1 + (n-1)d)}{2}$$

$$\lim_{n\to\infty} S_n \to \infty.$$

- Geometric series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{(1 - r)}$$

for $|r| > 1, \lim_{n\to\infty} r^n \to \infty, \lim_{n\to\infty} S_n \to \infty$

for $|r| < 1, \lim_{n\to\infty} r^n \to 0, \lim_{n\to\infty} S_n \to \frac{a}{(1-r)}$

Convergence

- An infinite series is the sum of an infinite sequence of numbers
- An infinite series converges when it has a limit as the number of terms approaches infinity.
- i.e.

$$S_n = \sum_{r=1}^{n} a_r \; converges\; when\; S_n has\; a\; limit\; when\; n \to \infty$$

Divergence

- An infinite series diverges if it does not converge
- Divergence can happen in one of the following ways:
  - Diverge to $+\infty$: $1 + 2 + 4 + 8 + \ldots$
  - Diverge to $-\infty$: $-1 - 2 - 4 - 8 - \ldots$
  - Oscillate finitely: $1 - 1 + 1 - 1 + \ldots$
  - Oscillate infinitely: $1 - 2 + 4 - 8 + \ldots$

Tests for convergence

- N[th] term test:

$$If \sum_{n=1}^{\infty} a_n \, converges, \qquad then \lim_{n\to\infty} a_n = 0$$

- However, $\lim_{n\to\infty} a_n = 0$ does not imply that a series converges.
- Also,

$$If \lim_{n\to\infty} a_n \neq 0 \; or \; this \; limit \; does \; not \; exist, \sum_{n=1}^{\infty} a_n \; diverges.$$

D'Alembert's ratio test

D'Alembert's ratio test:

A series of the form $\sum_{n=1}^{\infty} a_n$ converges when

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

The test also states that the series diverges when

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$$

The test does NOT imply anything when the ratio = 1.

Comparison test

Let$\sum_{n=1}^{\infty} a_n$and $\sum_{n=1}^{\infty} b_n$ be two infinite series such that $0 < a_n < b_n$ for all n.

Then,

If $\sum_{n=1}^{\infty} a_n$ is divergent, $\sum_{n=1}^{\infty} b_n$ is also divergent

If $\sum_{n=1}^{\infty} b_n$ is convergent, $\sum_{n=1}^{\infty} a_n$ is also convergent

Example – test for convergence

Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

- $\lim_{n\to\infty} \frac{1}{n} = 0$. This doesn't tell us anything.
- D'Alembert's test ratio:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

The ratio = 1, therefore, the ratio test fails as well.

Example – test for convergence

- Write out the first few terms:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

- $a_1>\frac{1}{2}, a_2=\frac{1}{2}, a_3+a_4>\frac{1}{2}, a_5+a_6+a_7+a_8>\frac{1}{2},\cdots$
- Therefore, every subsequent 2^k terms increase the sum by ½.
- Therefore, the series cannot converge to a limit, so it diverges.
- Note that this is a comparison test to a divergent series.

Power series

- The power series is an infinite series of the form

$$\sum_{x=0}^{\infty} c_n(x - a)^n = c_0(x - a)^0 + c_1(x - a)^1 + c_2(x - a)^2 + \cdots$$

Where c_n and a are coefficients.

When x = a, a power series always converges.

For power series, there exists a number R such that the power series will converge if $|x - a| < R$.

Taylor series

- The Taylor series is used to represent a function in terms of polynomials
- Used in many applications to simplify analysis.
- Used to find the power series expansion of a function about a given point a.
- Taylor series expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Taylor series

- Proof:

Assume function f(x) has a power series expansion, and that it has derivates of every order that can be found.

$$f(x) = c_0 + c_1(x - a)^1 + c_2(x - a)^2 + \cdots$$

Setting $x = a$ gives $f(a) = c_0 \times 1 + c_1 \times 0 + c_2 \times 0 + \cdots$

$$\therefore c_0 = f(a)$$

The first derivative of the power series is:

$$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \cdots$$

Setting $x = a$ gives $c_1 = f'(a)$

Taylor series

Second derivative of the power series:

$$f''(x) = 2c_2 + 3(2)c_3(x-a) + 4(3)c_4(x-a)^2 + \cdots$$

Setting $x = a$ gives $f''(x) = 2c_2$ which gives $c_2 = \frac{f''(x)}{2}$

Third derivative of power series:

$$f^{(3)}(x) = 3(2)c_3 + 4(3)(2)(x-a) + 5(4)(3)(x-a)^2 + \cdots$$

Setting $x = a$ gives $f^{(3)}(x) = 3(2)c_3$ which gives $c_3 = \frac{f^{(3)}(x)}{3(2)}$

Continuing the process gives the general formula for coefficients,

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Example – Taylor series

Find the quadratic approximation using theTaylor series expansion of $\frac{1}{x}$ at $x = 2$.

$$f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$$

$$f'(x) = \frac{-1}{x^2}, \qquad f''(x) = \frac{2}{x^3}$$

Substituting:

$$f(x) \approx \frac{1}{2} + \frac{-1}{2^2}(x-2) + \frac{\frac{2}{2^3}}{2!}(x-2)^2 = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$$

Maclaurin series

- Maclaurin series is a special case of the Taylor series where a = 0.
- Formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Example – Maclaurin series

Find the Maclaurin series for sin(x).

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$f(0) = \sin(0) = 0, \qquad f'(0) = \cos(0) = 1,$$
$$f''(0) = -\sin(0) = 0, \qquad f^3(0) = -\cos(0) = -1,$$
$$f^{(4)}(0) = \sin(0) = 0, etc.$$

Substituting and ignoring all terms that are zero:

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{(2n+1)}}{(2n+1)!} + \ldots\ldots.$$

Example – Maclaurin series ctd.

- Find the Maclaurin series for e^x
- Find the Maclaurin series for $\cos x$

- Identify the relation between three Maclaurin series found above.

Taylor series for two variable function – self study

- $u(x, y)$
- $u_x(x, y), u_y(x, y)$ Forward differencing
- $u_x(x, y), u_y(x, y)$ backward differencing

You may use the grid below.

	$u(x - \Delta x, y)$	
$u(x, y - \Delta y)$	$u(x, y)$	$u(x, y + \Delta y)$
	$u(x + \Delta x, y)$	