CM 1606 Computational Mathematics

Vectors

Week 9| Ganesha Thondilege

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Learning Outcomes

- Covers LO1 for CM 1606
- On completion of this lecture, students are expected to be able to:
  - Define and represent a vector.
  - Identify component of a vector and addition
  - Set up coordinate system for representing vectors.
  - Calculate the scalar product
  - Calculate the vector product and triple vector product
  - Direction cosines

CONTENT

- Physical quantities
- Vector representation
- Properties
- Types of vectors
- Vector addition
- Components of a vector
- 3D Vectors
- Product of vectors
- Direction cosines

Physical quantities

Physical quantity

<u>Scalar quantity</u>

refers only about the size or magnitude

e.g. speed, length, time

<u>Vector quantity</u>

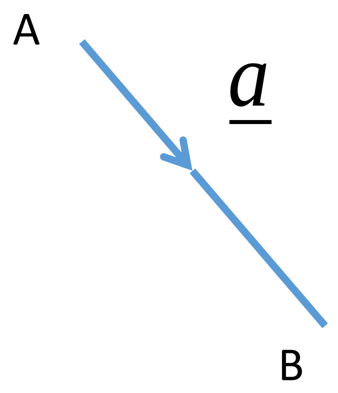
refers not only about the magnitude but also the direction in which it operates

e.g. Velocity, Force, Acceleration

e.g. Wind blows at a speed of 10km/h. (about a scalar quantity)
Wind has a velocity of 10km/h towards north. (about a vector quantity)

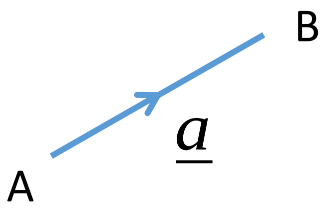
Vector representation

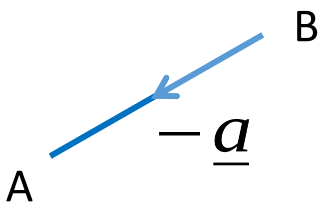
- Represented by a line segment with an arrowhead
- Length of the line segment denotes the magnitude
- Arrowhead denotes the direction
- Vector quantity denoted as $\overline{\mathrm{AB}}$ or $\underline{a}$
- Magnitude denoted as $|\overline{\mathrm{AB}}|$ or $|\underline{a}|$



Properties

- Same magnitude but opposite direction





vector $\overline{\mathrm{AB}}$ vector $\overline{\mathrm{BA}}$ or $-\overline{\mathrm{AB}}$ or $-\underline{a}$

- Same direction but different magnitudes (multiplication by a scalar)

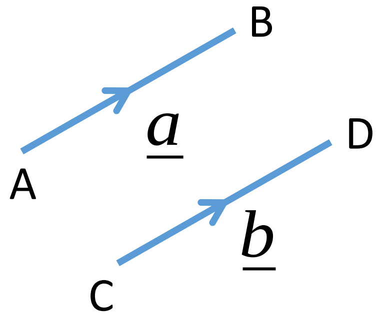
If k>1
(length increases)

If k<1
(length decreases)

if k<0?

Properties ctd.

- Equal vectors – Same direction and same magnitude

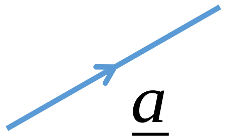


- Null vector (no magnitude, but any direction)

  nothing to represent !

  Null vector is denoted by $\underline{0}$ (zero with vector sign)

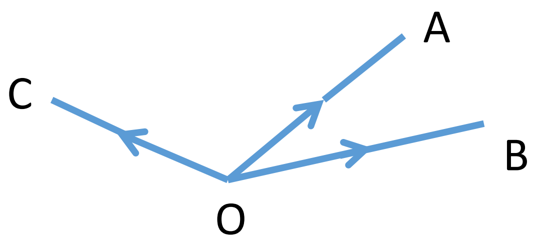
- Unit vector (magnitude is one(unit))



If $\underline{a}$ is a unit vector, then $|\underline{a}| = 1$

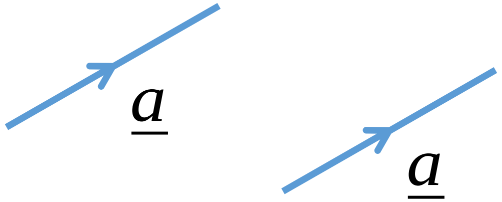
Types of vectors

- **Position vectors** : Vectors are drawn with one fixed reference point.



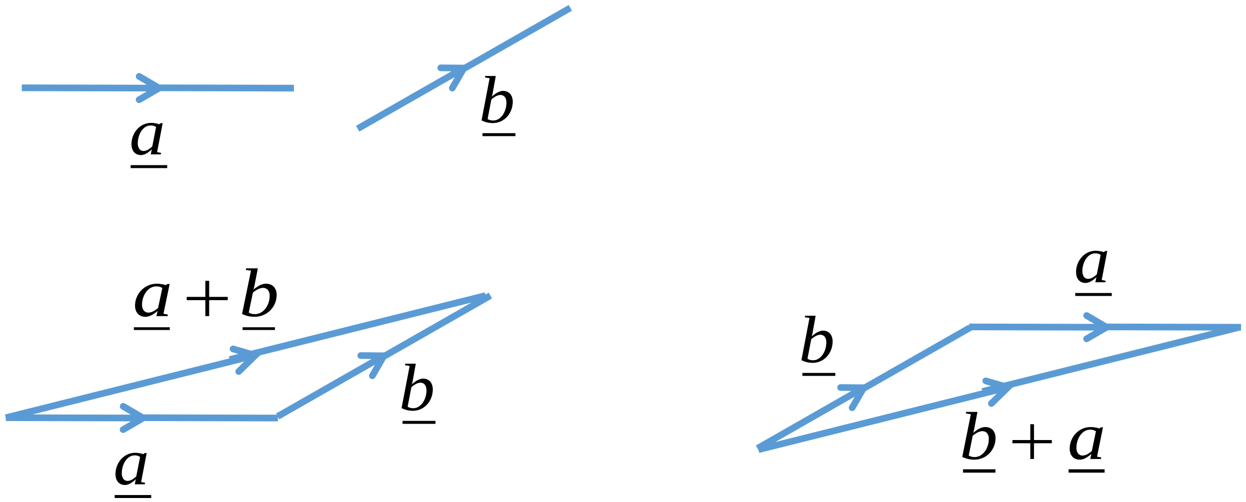
$\overline{\mathrm{OA}}$ denotes the position of A with respect to a fixed-point O.

- **Free vectors** : No restriction on position or line as in the previous cases.

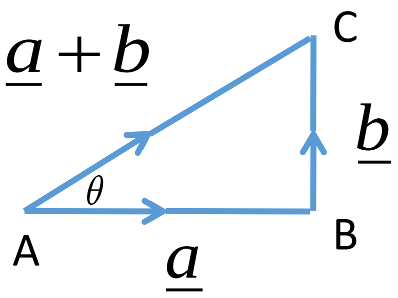


Vector addition(Triangular Law)

- Addition of two vectors $\underline{a}$ and $\underline{b}$, $\underline{a}+\underline{b}$
- Draw them as a chain starting the second where the first ends
- Resultant vector from the starting point to the end point of the chain.



Addition of vectors ctd.



$\underline{a}$ — force of 40N horizontally

$\underline{b}$ — force of 30N vertically

Resultant force : $\underline{a}+\underline{b}$ (or $\overline{\mathrm{AC}}$)

Magnitude $= \left|\overline{\mathrm{AC}}\right| = \mathrm{AC}$

$$= \sqrt{\mathrm{AB}^2 + \mathrm{BC}^2}$$

$$= \sqrt{\mathbf{40}^2 + \mathbf{30}^2}$$

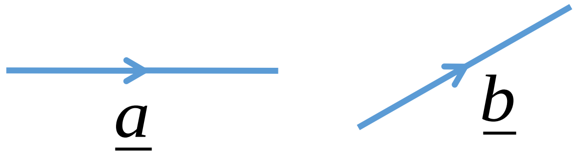
$$= \mathbf{50}\ N$$

Direction = Direction from A to C

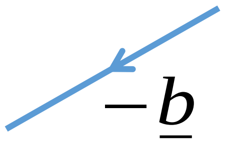
= θ upwards from horizontal direction where tanθ = ¾.

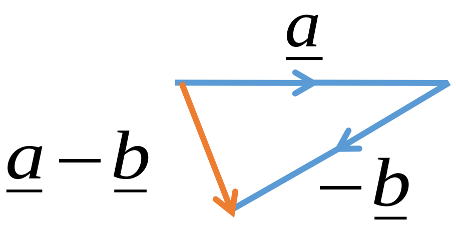
Addition of vectors ctd.

- About subtraction.... $\underline{a} - \underline{b}$



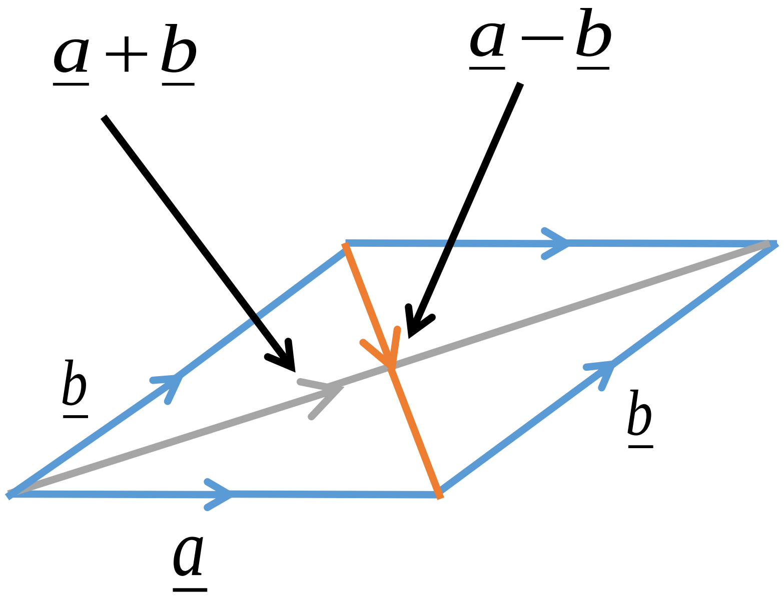
- Subtraction as an addition.... $\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$



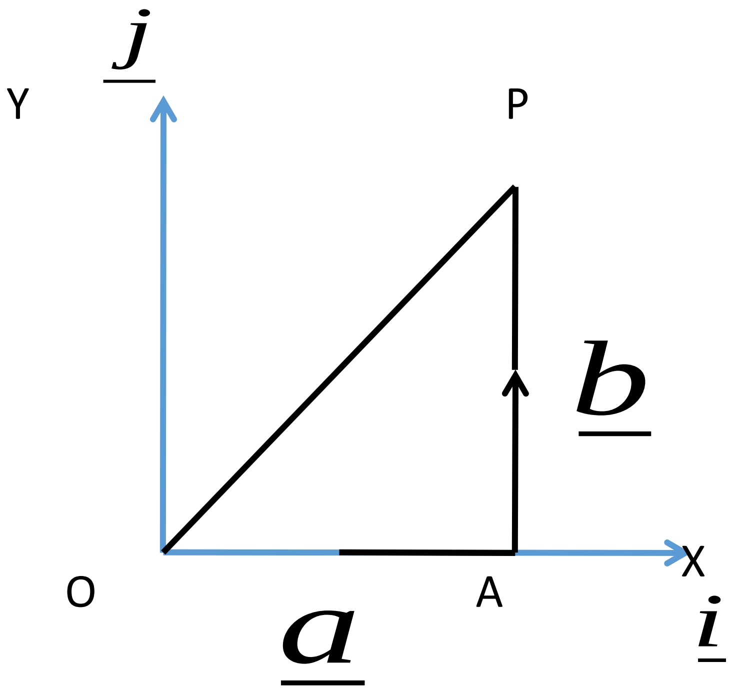


Addition of vectors (Parallelogram Law)

- Both $\underline{a}+\underline{b}$ and $\underline{a}-\underline{b}$ in the same diagram…



Components of a vector

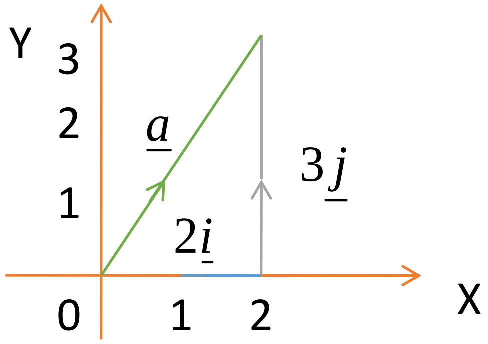


$$\underline{a} = a\underline{i}$$

$$\underline{b} = b\,\underline{j}$$

Then $\overline{OP} = a\underline{i} + b\,\underline{j}$ and $\left|\overline{OP}\right| = \sqrt{a^2 + b^2}$

$$Eg : \underline{a} = 2\underline{i} + 3\,\underline{j}$$



Example

Suppose $\underline{a} = -2\underline{i} + 3\underline{j}$, $\underline{b} = \underline{i} - 4\underline{j}$, $\underline{c} = 2\underline{j}$

a) Do the required simplification and display the resulting vector for the following cases.

b) Find the magnitude of each resulting vector.

$i.\ \underline{a} + \underline{b}$

$ii.\ 2\underline{a}$

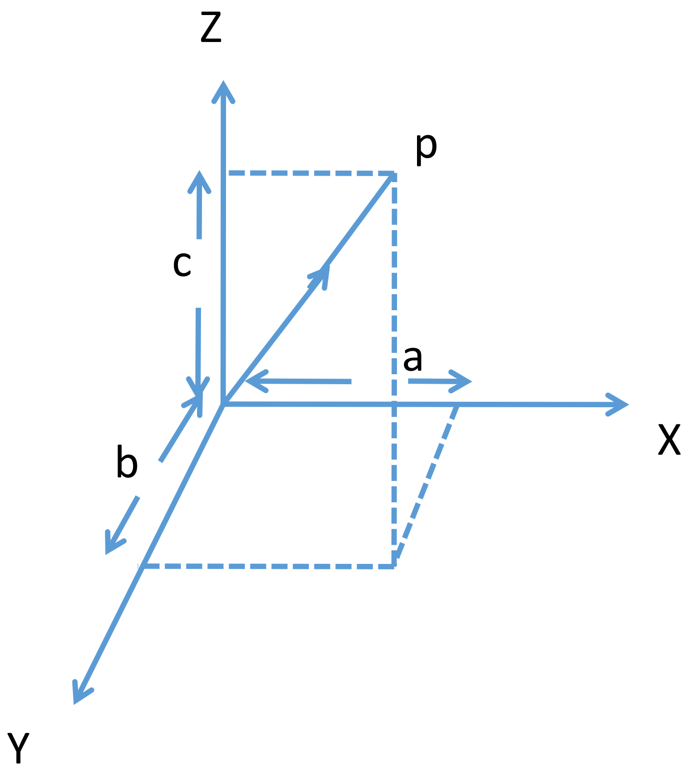
$iii.\ \underline{a} + \underline{b} + \underline{c}$

$iv.\ \underline{a} - \underline{b}$

$v.\ \underline{a} + \underline{b} - \underline{c}$

$vi.\ -2\underline{a} + 3\underline{b}$

3D Vectors



$a,\ b,\ c$ – lengths along OX, OY, OZ respectively.

Then, $\overline{OP} = a\underline{i} + b\underline{j} + c\underline{k}$

Hence, $\left|\overline{OP}\right| = \sqrt{a^2 + b^2 + c^2}$

Example

Ex : Suppose $\underline{a} = \underline{i} + \underline{j} + \underline{k}$, $\underline{b} = 2\underline{i} - 2\underline{j} - \underline{k}$, $\underline{c} = 2\underline{i} + \underline{k}$

a) Do the required simplification and display the resulting vector for the following cases.

b) Find the magnitude of each resulting vector.

$i.\ \ \underline{a} - \underline{b} + \underline{c}$

$ii.\ \ 2\underline{a} + 3\underline{c}$

$iii.\ \ \underline{a} - 2\underline{b} - 3\underline{c}$

Properties

For vectors $\underline{a},\ \underline{b},\ \underline{c}$ and scalars k, k'

$i.\ (\underline{a}+\underline{b})+\underline{c}=\underline{a}+(\underline{b}+\underline{c})$

$ii.\ \underline{a}+\underline{0}=\underline{a}$

$iii.\ \underline{a}+(-\underline{a})=\underline{0}$

$iv.\ \underline{a}+\underline{b}=\underline{b}+\underline{a}$

$v.\ k(\underline{a}+\underline{b})=k\underline{a}+k\underline{b}$

$vi.\ (k+k')\underline{a}=k\underline{a}+k'\underline{a}$

$vii.\ (kk')\underline{a}=k(k'\underline{a})$

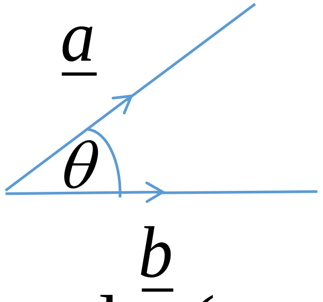
$viii.\ 1\underline{a}=\underline{a}$

Scalar Product (Dot Product)

Scalar product of two vectors $\underline{a}$ and $\underline{b}$ is defined by $|\underline{a}|\ |\underline{b}|\cos\theta$

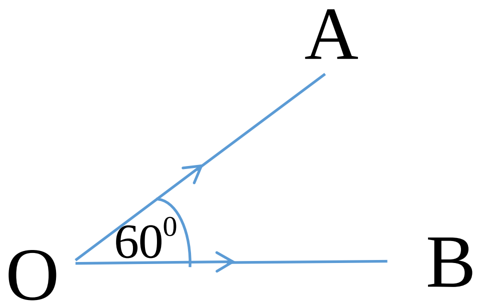
where θ is the angle between $\underline{a}$ and $\underline{b}$. It is denoted by $\underline{a}\cdot\underline{b}$.

ie. $\underline{a}\cdot\underline{b} = |\underline{a}|\ |\underline{b}|\cos\theta$



So, scalar product produces a scalar (number) as its result.

eg.



If $\left|\overline{OA}\right| = 4,\ \left|\overline{OB}\right| = 3,$ and $A\hat{O}B = 60^0$

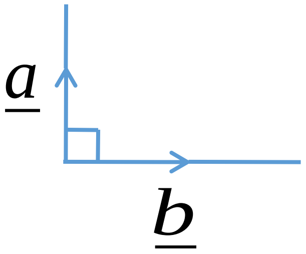
then $\overline{OA}\cdot\overline{OB} = \left|\overline{OA}\right|\left|\overline{OB}\right|\cos 60^0$

$$= 4 \times 3 \times \frac{1}{2}$$

$$= 6$$

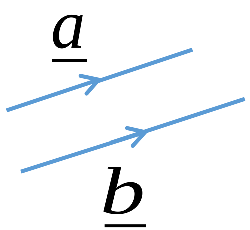
Key illustrations

1. Scalar product of two perpendicular vectors.



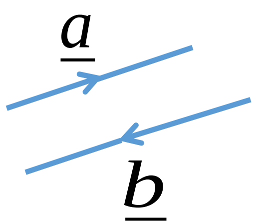
$$\underline{a}\cdot\underline{b} = |\underline{a}|\ |\underline{b}| \cos 90^0$$
$$= |\underline{a}|\ |\underline{b}|\ 0$$
$$= 0$$

2. Scalar product of two vectors in the same direction.



$$\underline{a}\cdot\underline{b} = |\underline{a}|\ |\underline{b}| \cos 0^0$$
$$= |\underline{a}|\ |\underline{b}|$$

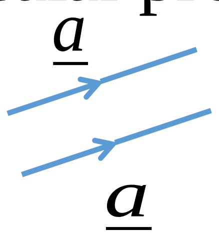
3. Scalar product of two vectors in the opposite direction.



$$\underline{a}\cdot\underline{b} = |\underline{a}|\ |\underline{b}| \cos 180^0$$
$$= |\underline{a}|\ |\underline{b}|\ (-1)$$
$$= -|\underline{a}||\underline{b}|$$

Key illustrations ctd.

4. Scalar product of two equal vectors.



$$\underline{a}\cdot\underline{a} = |\underline{a}|\,|\underline{a}|\cos 0^0$$
$$= |\underline{a}|^2$$

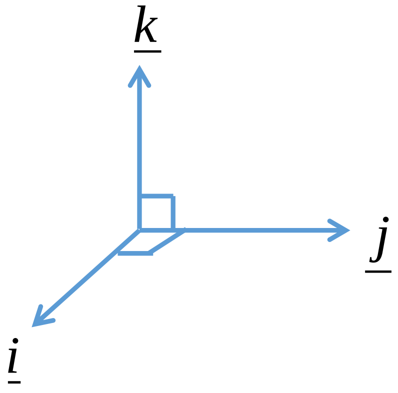
5. Scalar product of unit vectors.

$$\underline{i}\cdot\underline{i} = |\underline{i}|\,|\underline{i}|\,\cos 0^0 = 1\times 1\times 1 = 1$$

Similarlly, $\underline{j}\cdot\underline{j} = 1$ and $\underline{k}\cdot\underline{k} = 1$

$$\underline{i}\cdot\underline{j} = |\underline{i}|\,|\underline{j}|\,\cos 90^0 = 1\times 1\times 0 = 0$$

Similarlly, $\underline{i}\cdot\underline{k} = 0$ and $\underline{j}\cdot\underline{k} = 0$



6. Scalar product with null vectors.

$$\underline{a}\cdot\underline{0} = |\underline{a}|\,|\underline{0}|\cos\theta$$
$$= |\underline{a}|.0.\cos\theta = 0$$

Basic properties of scalar product

1. $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$
2. $\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}$
3. $(k\underline{a})\cdot\underline{b} = \underline{a}\cdot(k\underline{b}) = k(\underline{a}\cdot\underline{b}) \qquad k - \text{scalar}$

Example with unit vector representation.

If $\underline{a} = a_1\underline{i}+a_2\underline{j}+a_3\underline{k}$

$\underline{b} = b_1\underline{i}+b_2\underline{j}+b_3\underline{k}$

Then, $\underline{a}\cdot\underline{b} = (a_1\underline{i}+a_2\underline{j}+a_3\underline{k}).(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$

$$= \left(\begin{array}{l} a_1b_1\ \underline{i}\cdot\underline{i} + a_1b_2\ \underline{i}\cdot\underline{j} + a_1b_3\ \underline{i}\cdot\underline{k} + \\ a_2b_1\ \underline{j}\cdot\underline{i} + a_2b_2\ \underline{j}\cdot\underline{j} + a_2b_3\ \underline{j}\cdot\underline{k} + \\ a_3b_1\ \underline{k}\cdot\underline{i} + a_3b_2\ \underline{k}\cdot\underline{j} + a_3b_3\ \underline{k}\cdot\underline{k} \end{array} \right) = a_1b_1 + a_2b_2 + a_3b_3$$

Examples

1) If $\underline{a} = \underline{i} - 2\underline{j} + 3\underline{k}$ and $\underline{b} = 4\underline{i} + 6\underline{j} + \underline{k}$

Then, $\underline{a} \cdot \underline{b} = 4\ \underline{i} \cdot \underline{i} + (-12)\ \underline{j} \cdot \underline{j} + 3\ \underline{k} \cdot \underline{k}$

$= 4 - 12 + 3$

$= -5$

2) Find the angle between $\underline{a} = \underline{i} + \underline{j} + \underline{k}$ and $\underline{b} = 3\underline{i} - 2\underline{j} + 6\underline{k}$

using dot product. (Another way to find angle between vectors...!)

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$

$$7 = \sqrt{3}\, 7 \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Examples

3) Let $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{b} = -2\underline{i} + \underline{j} - 3\underline{k}$, $\underline{c} = \underline{i} - \underline{j}$

Evaluate the following dot products.

i. $\underline{a} \cdot \underline{b}$

ii. $\underline{a} \cdot \underline{c}$

iii. $\underline{a} \cdot \left(\underline{b} + \underline{c}\right)$

iv. $2\underline{a} \cdot 3\underline{b}$

v. $-\underline{a} \cdot \underline{c}$

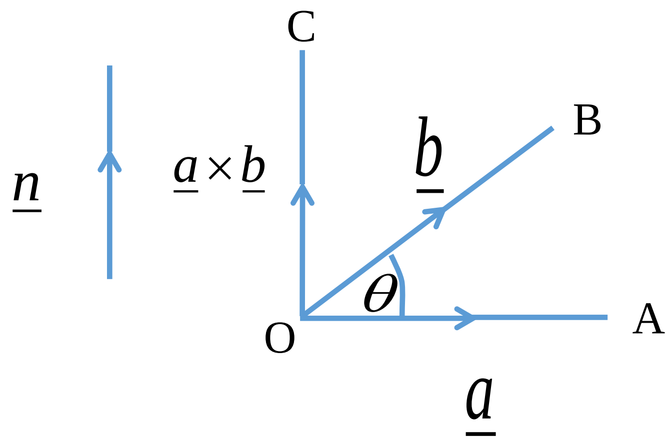
vi. $\left(\underline{a} + \underline{b}\right) \cdot \left(-\underline{b} + 2\underline{c}\right)$

Vector product(Cross product)

Vector product of two vectors $\underline{a}$ and $\underline{b}$ is defined by a vector with the magnitude $|\underline{a}|\ |\underline{b}|\sin\theta$, where θ $(0 \le \theta \le 180^0)$ is the angle between $\underline{a}$ and $\underline{b}$ and with the direction perpendicular to both $\underline{a}$ and $\underline{b}$.

$$\underline{a} \times \underline{b} = |\underline{a}|\ |\underline{b}|\sin\theta\ \overset{\wedge}{\underline{n}}$$

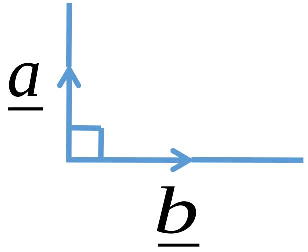
The vector $\overset{\wedge}{\underline{n}}$ is the unit vector directs towards the direction of a screw once it rotates from the direction of $\underline{a}$ to direction of $\underline{b}$.



Vector product $\underline{a}$ to $\underline{b}$ is denoted by $\underline{a} \times \underline{b}$ or $\underline{a} \wedge \underline{b}$.

Key illustrations on vector product

1. Vector product of two perpendicular vectors.

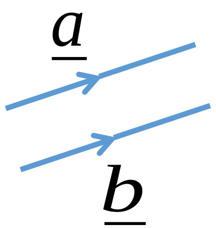


$$\underline{a} \times \underline{b} = |\underline{a}|\ |\underline{b}| \sin 90^0\ \underline{n}$$
$$= |\underline{a}|\ |\underline{b}|\ \underline{n}$$

Then, $|\underline{a} \times \underline{b}| = |\underline{a}|\ |\underline{b}|$

($\underline{n}$ has its original meaning........!)

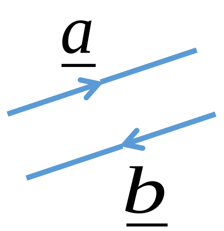
2. Vector product of two vectors in the same direction.



$$\underline{a} \times \underline{b} = |\underline{a}|\ |\underline{b}| \sin 0^0\ \underline{n}$$
$$= \underline{0} \ \text{(null vector)}$$

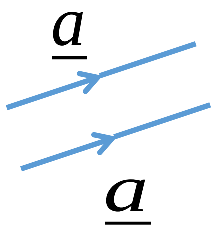
Key illustrations ctd.

3. Vector product of two vectors in the opposite direction.



$$\begin{aligned}\underline{a}\times\underline{b} &= |\underline{a}|\,|\underline{b}|\sin 180^0\ \underline{n}\\ &= |\underline{a}|\,|\underline{b}|\,.\,0\,.\,\underline{n}\\ &= \underline{0}\end{aligned}$$

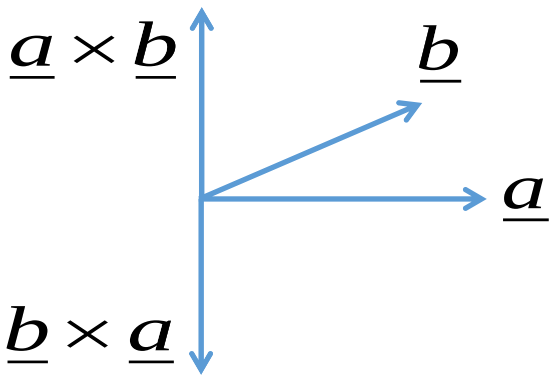
4. Vector product of two equal vectors.



$$\begin{aligned}\underline{a}\times\underline{a} &= |\underline{a}|\,|\underline{a}|\sin 0^0\ \underline{n}\\ &= \underline{0}\end{aligned}$$

Properties

1. $\underline{b}\times\underline{a} = -\left(\underline{a}\times\underline{b}\right)$



2. $\underline{a}\times\left(\underline{b}+\underline{c}\right) = \left(\underline{a}\times\underline{b}\right)+\left(\underline{a}\times\underline{c}\right)$

3. $\left(k\underline{a}\right)\times\underline{b} = \underline{a}\times\left(k\underline{b}\right) = k\left(\underline{a}\times\underline{b}\right) \qquad k-\text{scalar}$

Vector product using determinant

Consider the determinant

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- First row consists of unit vectors
- Second row consists of corresponding coefficients of $\underline{a}$
- Third row consists of corresponding coefficients of $\underline{b}$

Example

If $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 3\underline{i} + 4\underline{j} + \underline{k}$

i) Find $\underline{a} \times \underline{b}$.

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -1 \\ 3 & 4 & 1 \end{vmatrix} = \underline{i} \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 6\underline{i} - 4\underline{j} - 2\underline{k}$$

ii) Find the unit vector in the direction of $\underline{a} \times \underline{b}$.

$$\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{6\underline{i} - 4\underline{j} - 2\underline{k}}{\sqrt{6^2 + (-4)^2 + (-2)^2}} = \frac{6}{\sqrt{56}}\underline{i} - \frac{4}{\sqrt{56}}\underline{j} - \frac{2}{\sqrt{56}}\underline{k}$$

iii) Find the angle between $\underline{a}$ and $\underline{b}$.

(Another way to find angle between vectors...!)

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin\theta$$

$$\sin\theta = \frac{|6\underline{i} - 4\underline{j} - 2\underline{k}|}{|\underline{i} + 2\underline{j} - \underline{k}|\,|3\underline{i} + 4\underline{j} + \underline{k}|} = \frac{\sqrt{56}}{\sqrt{6}\sqrt{26}} = \sqrt{\frac{14}{39}}$$

$$\theta = \sin^{-1}\left(\sqrt{\frac{14}{39}}\right)$$

Example ctd.

iii) Find the angle between $\underline{a}$ and $\underline{b}$.

(Another way to find angle between vectors...!)

$$|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}|\, sin\,\theta$$

$$sin\,\theta = \frac{|6\,\underline{i} - 4\,\underline{j} - 2\underline{k}|}{|\underline{i} + 2\,\underline{j} - \underline{k}|\;|3\underline{i} + 4\,\underline{j} + \underline{k}|} = \frac{\sqrt{56}}{\sqrt{6}\,\sqrt{26}} = \sqrt{\frac{14}{39}}$$

$$\theta = sin^{-1}\left(\sqrt{\frac{14}{39}}\right)$$

Example

Let $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{b} = -2\underline{i} + \underline{j} - 2\underline{k}$, $\underline{c} = \underline{i} - \underline{j}$

Evaluate the followings.

i. $\underline{a} \times \underline{b}$

ii. $\underline{b} \times \underline{c}$

iii. $(\underline{b} \times \underline{a}) + (\underline{b} \times \underline{c})$

iv. $-2\underline{b} \times \underline{a}$

v. $(\underline{a} - \underline{b}) \times (-2\underline{a} + 3\underline{c})$

Triple product of vectors

For three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, two types of triple products are defined.

1. Scalar triple product

$$\underline{a} \, . \, (\underline{b} \times \underline{c}) \longrightarrow \text{resulting a scalar}$$

2. Vector triple product

$$\underline{a} \times (\underline{b} \times \underline{c}) \longrightarrow \text{resulting a vector}$$

Ex. Find the above two triple products for

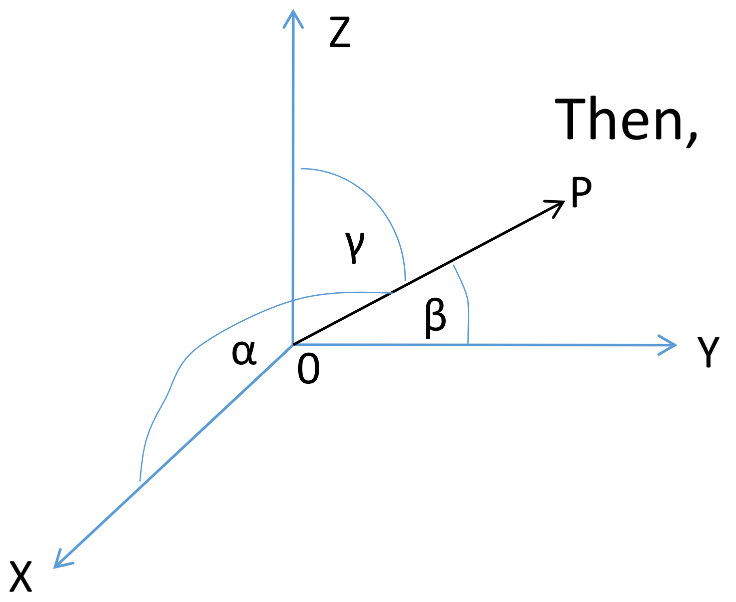
$$\underline{a} = 2\underline{i} + \underline{j} - 2\underline{k}, \ \underline{b} = -\underline{i} - 2\underline{j} + \underline{k} \ \text{and} \ \underline{c} = 3\underline{i} + \underline{j} - 2\underline{k} \ .$$

Direction cosines

The direction of a vector can be determined by the angles which the vector marks with the three axes of reference.

Let

$$\overline{OP} = \underline{r} = a\underline{i} + b\underline{j} + c\underline{k} \text{ and } |\underline{r}| = r$$



Then,

$$\frac{a}{r} = \cos\alpha \Rightarrow a = r\cos\alpha$$

$$\frac{b}{r} = \cos\beta \Rightarrow b = r\cos\beta$$

$$\frac{c}{r} = \cos\gamma \Rightarrow c = r\cos\gamma$$

Direction cosines ctd…

Once we denote $\cos\alpha$ as l, $\cos\beta$ as m and $\cos\gamma$ as n, $[l,m,n]$ written in square brackets are called direction cosines of vector $\underline{r}$.

Hence, we have an interesting relationship as follows.

Since $a^2+b^2+c^2=r^2$, we have

$$r^2\cos^2\alpha+r^2\cos^2\beta+r^2\cos^2\gamma=r^2$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

ie. $l^2+m^2+n^2=1$

Direction cosines ctd…

$eg.$ Find the direction cosines of the vector $\underline{i} - 2\underline{j} + 3\underline{k}$.

Let, $\underline{r} = \underline{i} - 2\underline{j} + 3\underline{k}$. With usual notation, we have

$$a = 1,\ b = -2,\ c = 3 \quad \text{and} \quad r = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

Then, direction cosines: $l = \dfrac{a}{r} = \dfrac{1}{\sqrt{14}}$, $m = \dfrac{b}{r} = \dfrac{-2}{\sqrt{14}}$, $n = \dfrac{c}{r} = \dfrac{3}{\sqrt{14}}$

In bracket notation : $\left[\dfrac{1}{\sqrt{14}}, \dfrac{-2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right]$

Example

Find the direction cosines of the following vectors in 3D space.

i. $\underline{i} + \underline{j} + \underline{k}$

ii. $3\underline{i} - 2\underline{j} + 6\underline{k}$

iii. $-2\underline{i} + \underline{j} - \underline{k}$

iv. $3\underline{i} + 4\underline{j}$