CM 1606 Computational Mathematics

Probability

Week 10| Ganesha Thondilege

ROBERT GORDON UNIVERSITY ABERDEEN

INFORMATICS INSTITUTE OF TECHNOLOGY

Learning Outcomes

Covers LO1 for Module

On completion of this lecture, students are expected to be able to:

*define any event on a sample space defined for an experiment

*understand classical probability

*definitions and axioms of probability

*probabilities of intersection/union of events, and disjoint events

*Techniques of counting

*understand how they can apply these in real world problems

CONTENT

- Random Experiments, Sample space and events
- Set operations on events
- Disjoint events
- The concept of probability
- Classical and frequency approach
- Axioms on probability
- Exhaustive events
- Theorems
- Conditional probability
- Techniques of counting

Random Experiments

We are familiar with the importance of experiments

Perform experiments under very nearly identical conditions

- able to control the values of variables

In some experiments

- unable to control the values of certain variables
- Results will vary from one performance to another
- even though most the conditions are same
  - These experiments aka **Random experiments**

Examples

- Tossing a fair coin
- Tossing a fair six-sided die
- Tossing a fair coin twice
- Checking the condition of bolts made by a machine

  Some are defectives , some are non defectives
- Measuring lifetime(L in hours) of bulbs produced by a certain company

  Assumption: no bulbs lasts more than 3500 hours

$$0 \le L \le 3500$$

Sample space

- Denoted as S
- Consists of all possible outcomes of a random experiment
- Each outcome is called a sample point (an element of S)
- Try to identify the sample space for each experiment above.

Sample space

- Finite sample space
- Countably infinite sample space (Set of natural numbers)
- Uncountable infinite sample space (all real numbers within $0<x<2$)
- Discrete sample space (Finite or countably infinite)

Events

A subset of the sample space S (A, B, C,....)

A set of possible outcomes

If the outcome of an experiment is an element of an event

Say A has occurred

If |A|=1, Simple or elementary event

Events ctd.

Sure, or certain events

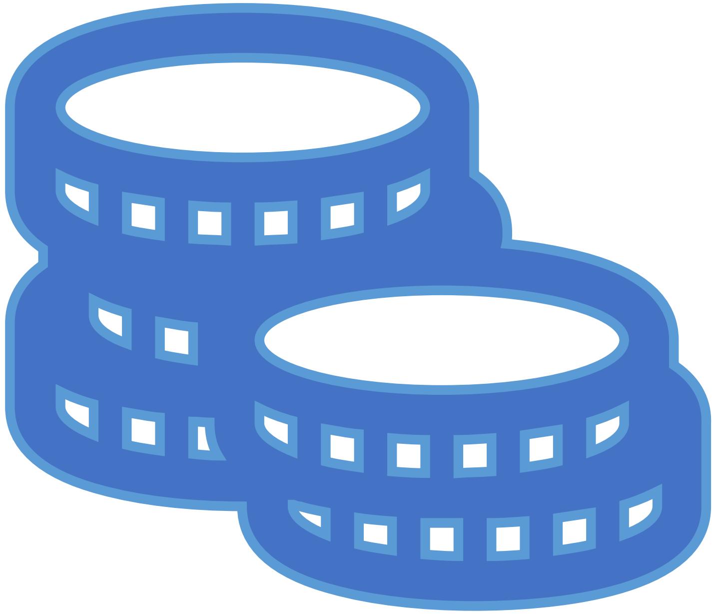
An event A is same as the sample space S – Exactly happening

Impossible events

For an event A,

- |A|=0 or A={ } → cannot occur

Example



1) Consider the experiment of tossing a fair coin twice

    a) Identify the sample space

    b) Identify the events A, B, C, D and E as

A be the event of getting only one head

B be the event of getting two tails

C be the event of getting at lease one head

D be the event of getting at most two tails

E be the event of getting head and tail in the same toss

    c) Identify impossible events, sure events, elementary events

Set operations on events

- Can obtain other events in S
- $A \cup B$ - either A or B occurred(or both)
- $A \cap B$ - both A and B
- A' - not A
- $A - B$ - A but not B

$$A' = S - A$$

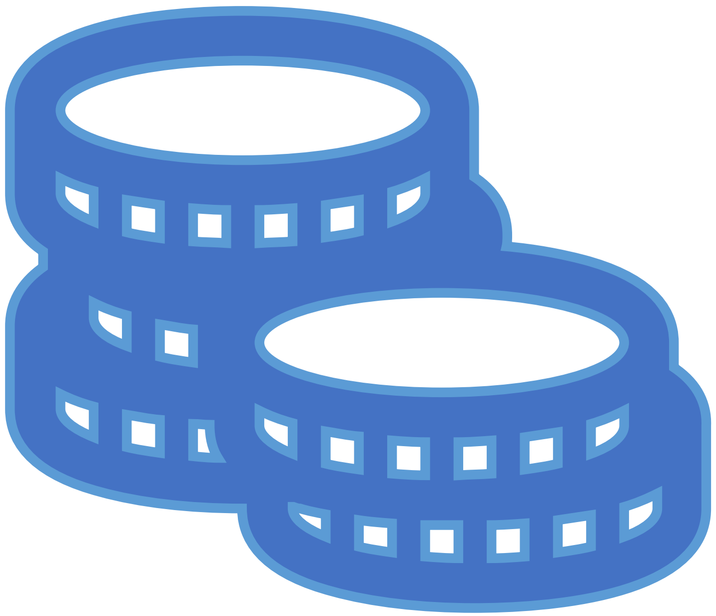
Disjoint events (Mutually exclusive)

If the sets corresponding to events A and B

i.e. $A \cap B = \phi$

We say A and B are mutually exclusive events.

- They cannot both occur.
- A collection of events, $A_1, A_2, \ldots\ldots\ldots, A_n$ is mutually exclusive if every pair in the collection is mutually exclusive.



Example

- Experiment: Tossing a fair coin twice
- Events A: Getting only one head

  B: Getting no heads

  C: Getting at least one head
- Identify the sample points of A, B and C
- Identify mutually exclusive events
- Also find

$$A \cup B, A', B \cup C, C - A$$

The concept of probability

As a measure of the *chance*

Assign a number between 0 and 1

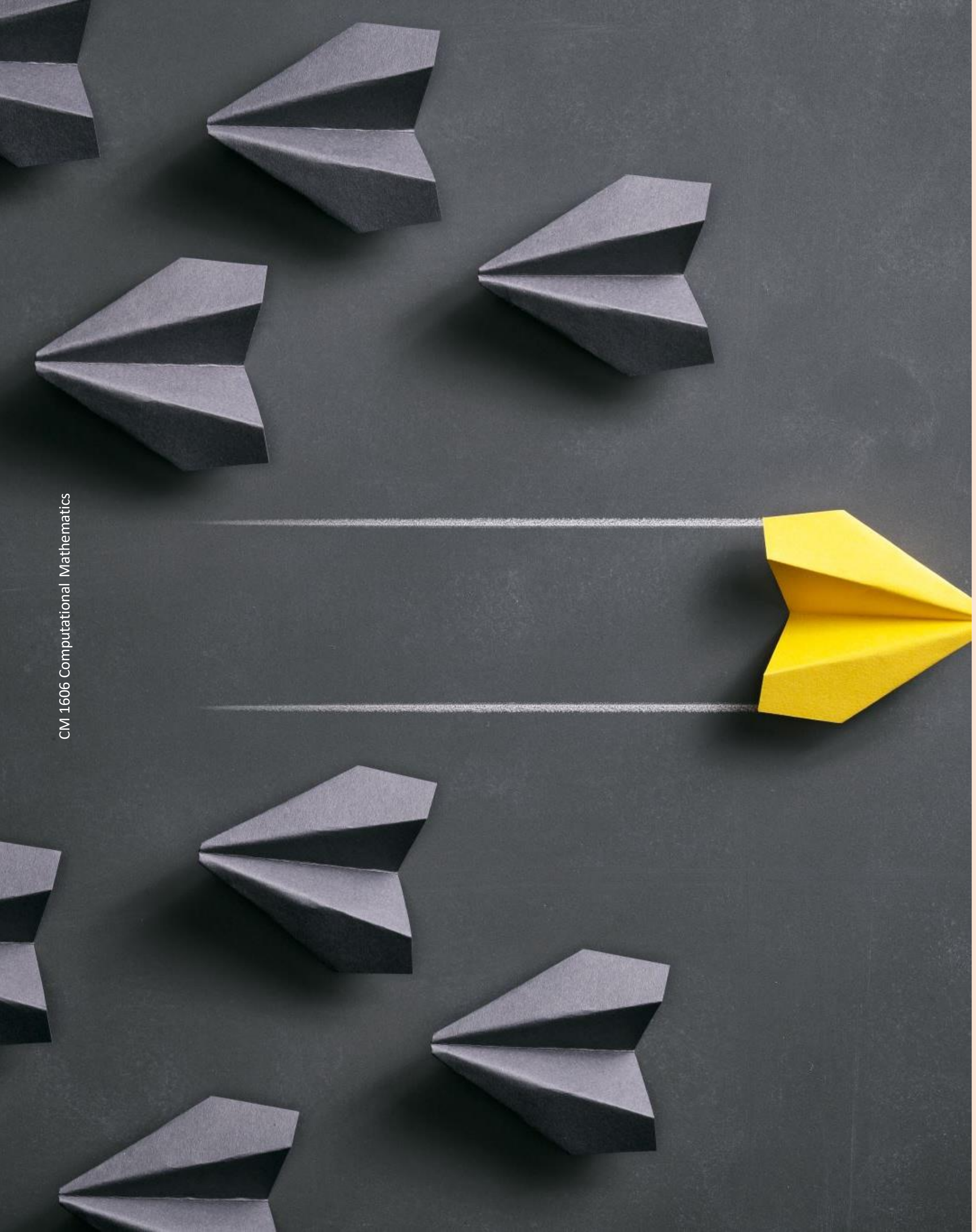
- For sure events → probability is 1 (or 100%)

- For impossible events →probability is 0 (or 0%)

Ex. If probability is ¾ for an event A,

there is a 75% chance it will occur

and 25% chance that it will not occur



Classical approach

If an event can occur in n different ways out of a total number of N possible ways, all of which are equally likely, then the

probability of the event is n/N.

Example

How we can find the probability of getting head in a single toss of a fair coin?

- There are two equally likely ways (N=2, Heads and Tails)
- Head can arise once (n=1)
- Probability of getting head = ½ (n/N)

ROBERT GORDON UNIVERSITY ABERDEEN
INFORMATICS INSTITUTE OF TECHNOLOGY

Frequency approach

If after N repetitions of an experiment, where N is very large, an event is observed to occur in n of these, then the

probability of the event is n/N

- Aka the empirical probability of the event

Example

- If we toss a coin 1000 times and found heads comes up 547 times.

In this case we can estimate the probability of getting head for the coin to be 547/1000=0.547

Drawbacks of both approaches:

- Need to be equally likely – Classical approach
- Sample space size need to be a large number – Frequency approach

Axioms of probability

- S – Sample space
- Then for each event A on S, we associate a real number $P(A)$
- P is called a probability function
- $P(A)$ is the probability of the event *A*,

if the following axioms are satisfied.

Axioms of probability ctd.

- For any event A $P(A) \geq 0$

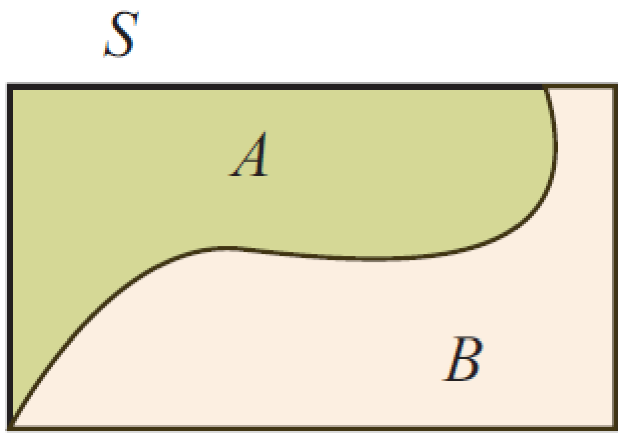
- For the sure or certain event S, $P(S) = 1$
-
- For any number of mutually exclusive events $A_1, A_2, \ldots\ldots, A_n$

$$P(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$$

Exhaustive events

The sequence of events $A_1, A_2, \ldots\ldots\ldots, A_n$ is said to be exhaustive, if the union of all events is the sample space s

$$P(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) = S$$



A and B are mutually exclusive and exhaustive events

Theorems on probability

Theorem I

If $A \subseteq B$, then $P(A) \leq P(B)$ and $P(B-A) = P(B) - P(A)$

Theorem II

For any event A, $0 \leq P(A) \leq 1$

Theorem III

If A’ is the complement of A, then $P(A') = 1 - P(A)$

Theorems on probability ctd.

Theorem IV

If $A = A_1 \cup A_2 \cup ......\cup A_n$, where $A_1, A_2, ..., A_n$ are mutually exclusive events, then

$$P(A) = P(A_1 \cup A_2 \cup ......\cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$$

if A=S

$$P(A_1) + P(A_2) + ... + P(A_n) = 1$$

Theorems on probability ctd.

Theorem v

For any two events A and B, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Theorems on probability ctd.

Theorem VI

For any event A and B

$$P(A) = P(A \cap B) + P(A \cap B')$$

Theorem VII

If an event A must result in the occurrences of one of the mutually

exclusive events, $A_1, A_2, ..., A_n$ then

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + ... + P(A \cap A_n)$$

Example

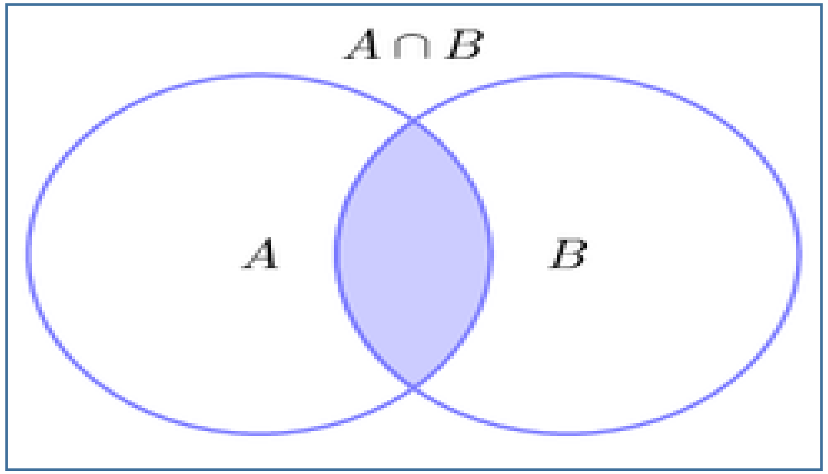
1) A single fair six-sided die is rolled once. Find the probability of a 2 or 4 turning up.

2) In a single throw of two dice, find the probability of getting a total of 7 or 12.

Conditional probability

- Let A and B be two events such that $P(A) > 0$.
- $P(B \mid A) > 0$ - The probability of B given that A has occurred
- New sample space A not S

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$



Alternatively,

$$P(A \cap B) = P(B \mid A).P(A)$$

Example

1) Find the probability that a single toss of a die will result in a number less than 4 if,

   a) no other information is given

   b) it is given that the toss resulted in an odd number

Theorems on Conditional probability

Theorem VII

For any three events *A, B* and *C,* then

$$P(A \cap B \cap C) = P(A).P(B \mid A).P(C \mid A \cap B)$$

Theorem VIII

If an event A must result in each one of the mutually exclusive events $A_1, A_2, \ldots\ldots\ldots, A_n$ then

$$P(A) = P(A_1)P(A \mid A_1) + P(A_2)P(A \mid A_2) + \ldots\ldots + P(A_n)P(A \mid A_n)$$

Independent events

- If $P(B \mid A) = P(B)$ → A and B independent
- Probability of B occurring is not affected by the occurrences or non occurrences of A

$$P(A \cap B) = P(B).P(A)$$

Furthermore

$$P(A \cap B \cap C) = P(B).P(A).P(C)$$

Bayes' Theorem

Suppose that $A_1, A_2, \ldots\ldots\ldots, A_n$ are mutually exclusive and exhaustive events, for any event *B,* Bayes' rule is

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$

`

Example

1) Assume that a certain school enrolled equal number of male and female students. 6% of male population is football players. Find the probability that a randomly selected student is a football player male.

Techniques of counting

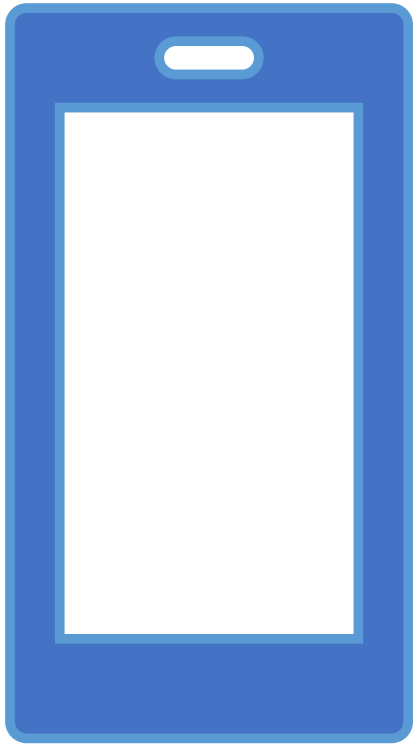
- Sample points in S is not very large
- Direction enumeration or counting needed
- Sometimes direct counting becomes a practical difficulty
- Combinatorial analysis make counting easy
- Aka *sophisticated way of counting*

Fundamental principle of counting

Tree diagrams → If one thing can be accomplished in n1 different ways

a second thing can be accomplished in n2 different ways → and………………..kth thing can be accomplished in nk different ways,

then all k things can be accomplished in the specific order → n1.n2.n3…………..nk different ways



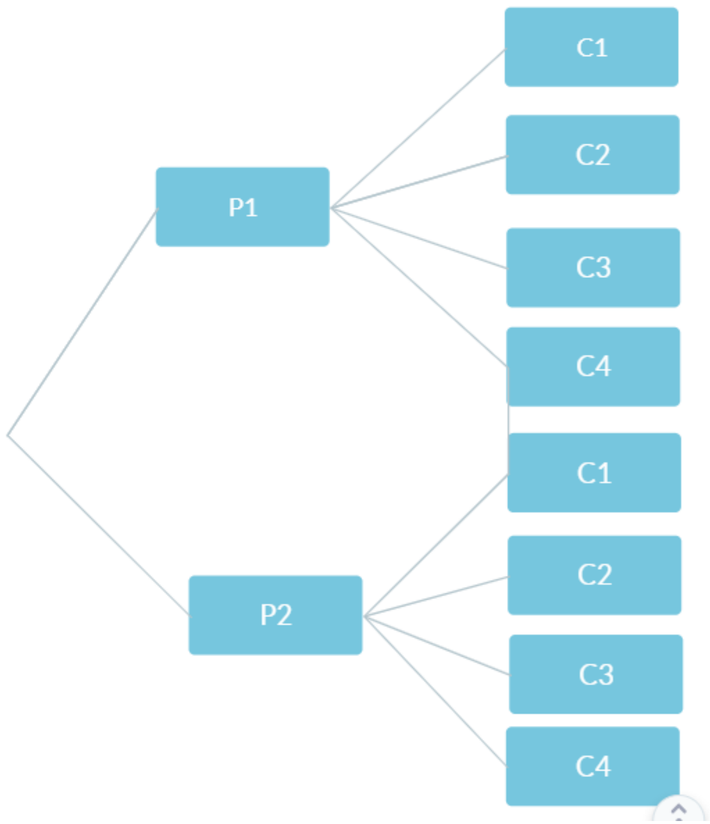
Example

1) If you have 2 phones (Same model) and 4 back covers in different colors , how many ways do you have to choose a phone and then a back cover?

Tree diagram

- first selection - one phone out of 2 phones
- Second selection - a back cover out of 4 covers for each selection above (or vise versa)
- How many combined branches do you have?

phone selection cover selection



Permutations

We have n distinct objects → Wish to arrange r of them in a line → n ways to select 1st object → (n-1) ways to select the 2nd → (n-2) ways to select the 3rd ………………

→ (n-(r-1)) = (n-r+1)ways to select the rth object → Number of different arrangements or permutations → n(n-1)(n-2)……….(n-r+1)→ Permutations of n objects taken r at a time

Permutations (All distinct objects)

- Selecting r from n object

$$^{n}P_{r} = n(n-1)(n-2)..........(n-r+1)$$

- Particularly , When r=n

$$^{n}P_{n} = n(n-1)(n-2)..........1 = n!$$

Factorial n

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$^{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \quad ; \ 0!=1$$

Permutations (Some same type objects)

- Arranging n number of objects in a line

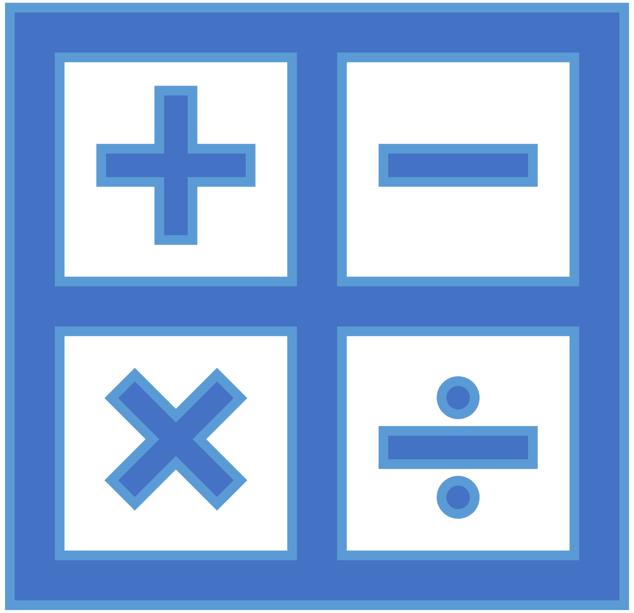
$$n = n_1 + n_2 + .. + n_k$$

- Number of different arrangements can be made by using these all-n objects

Total number of diffrent arrangements of n distinct objects=n!

If $n = n_1 + n_2 + n_3 + ... + n_k$

$$\text{Then total number of arrangements} = \frac{\text{n!}}{n_1!n_2!n_3!......n_k!}$$



Example

1) Find the number of different arrangements (permutations) can be made by using 5 letters of the word MATHS.

2) Find the total number of different arrangements can be made from the 11 letters of the word MISSISSIPPI.

Combinations

- Permutations – Order matters

E.g. abc and bca are two different permutations

- But the same combination – a, b, and c
- If interested on choosing/selecting objects only w/o regard to order

This type of selections are known as combinations

E.g. abc and bca are the same combination

Combinations

The combinations of n things taken r at a time is denoted as

$$^{n}C_{r} \text{ or } \binom{n}{r}$$

Where,

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Result:

$$^{n}C_{r} = {}^{n}C_{n-r}$$

Example

1) Evaluate ${}^{7}C_3, {}^{7}C_4, {}^{3}C_3$ and ${}^{10}C_7$

2) In how many ways can a committee of 4 people be chosen out of 7 people.

Example

3) A box contains 8 red, 3 white and 9 blue balls. If 3 balls are drawn at random without replacement, determine the probability that

a) all three are red

b) all three are white

c) 2 are red and 1 is white

d) at least one is white