CM 1606 Computational Mathematics

Functions

Week 3| Ganesha Thondilege

Learning Outcomes

- Covers LO1 for CM1606
- On completion of this lecture, students are expected to be able to:
    - identify the rule of a function.
    - determine domain, codomain and range of a function.
    - Identify single valued functions.
    - construct compositions of functions.
    - construct the inverse of a function.
    - recognize special features of several types of functions.
    - Critical value analysis for quadratic functions

CONTENT

- Components of a function
- Range of a function
- Graph of a function
- Vertical line test – single valued functions
- Addition and subtraction of functions
- Composition of functions
- Inverse of a function
- Graph of the inverse rule
- Several categories of functions
- Critical values of quadratic functions

Components of a function

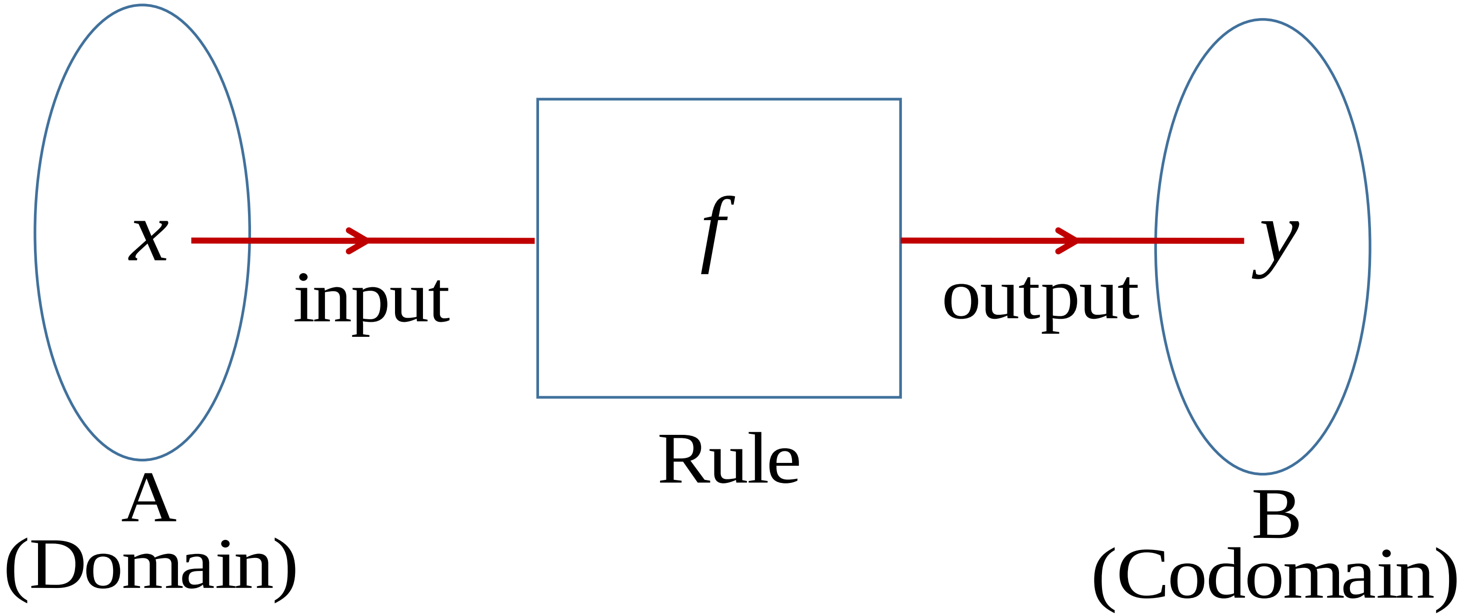
- Let A and B be non-empty sets.

"Each element in A is assigned by a certain rule to a

unique (Single valued) element in B"

- This assignment is called a function (mapping) from A to B.
- Then, the set A is called the domain
- The set B is called the codomain.
- The rule that assigns elements from A to B is considered as the rule of the function.

Components of a function ctd.



Notation :

$f : A \to B$ or $A \xrightarrow{f} B$

eg. Rule of raising to the power 2 (or squaring)

Components of a function ctd.

So, rule can be expressed as $f(x) = x^2$, where generally,

$f(x)$ is for "function of x ".

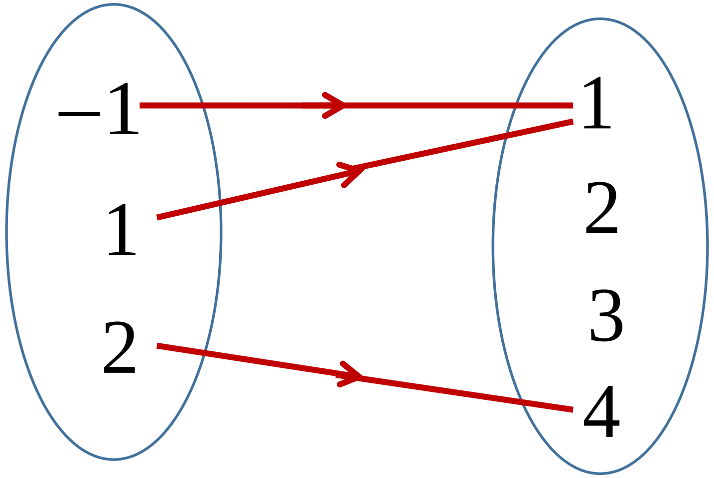
Alternatively it can be denoted by

$$x \mapsto x^2 \text{ as well.}$$

eg. Let $A = \{-1, 1, 2\}$, $B = \{1, 2, 3, 4\}$

$$f : A \to B$$

$$f(x) = x^2$$



Range of a function

Let f be a function with f:A $\to$ B. Suppose $a \in$ A is assigned to $b \in$ B by f

(ie. $f(a) = b$).

Then b is called the image of a.

Set of all images of the elements in the domain is called the range

of the function.

Usually, it is denoted by f (A) .

Note that $f\,(\mathrm{A}) \subseteq \mathrm{B}$.

Range of a function ctd.

$$\text{eg. Let } A = \{-1,\ 1,\ 2\}\ ,\ B = \{1,\ 2,\ 3,\ 4\}$$

$$f\ :\ A \to B$$

$$f(x) = x^2$$

$$\text{Image of } -1 \text{ is } f(-1) = 1$$

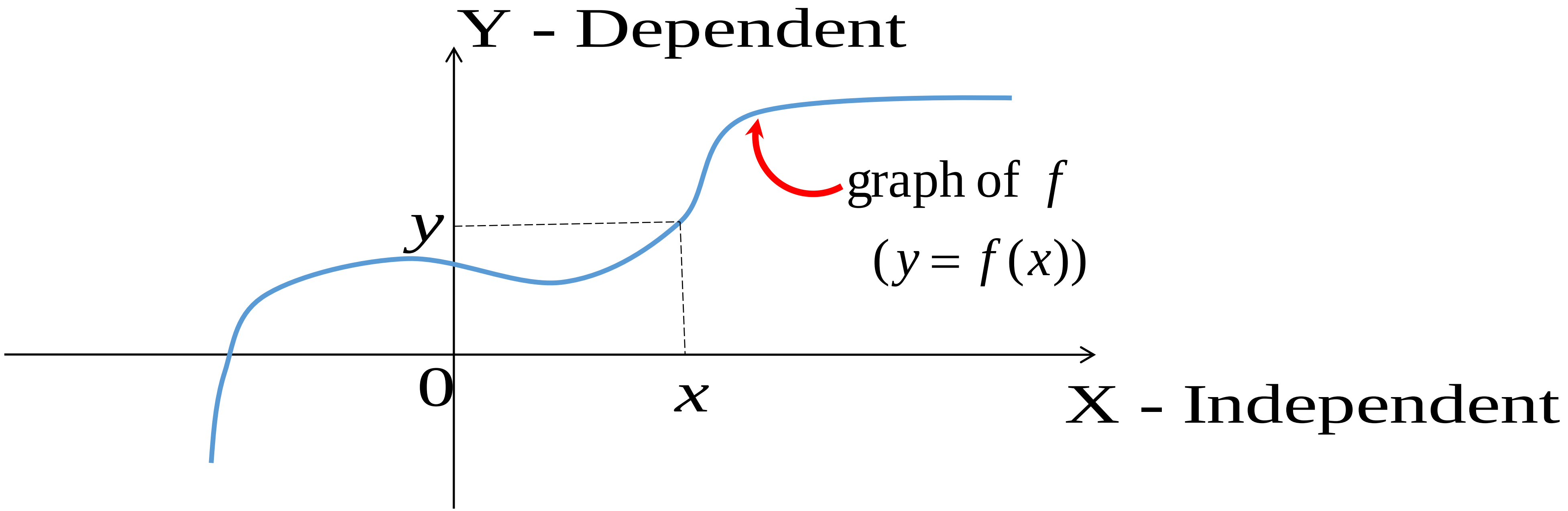
$$\text{Image of } 1 \text{ is } f(1) = 1$$

$$\text{Image of } 2 \text{ is } f(2) = 4$$

$$\text{So, Range of } f = \{1, 4\}$$

Graph of a function

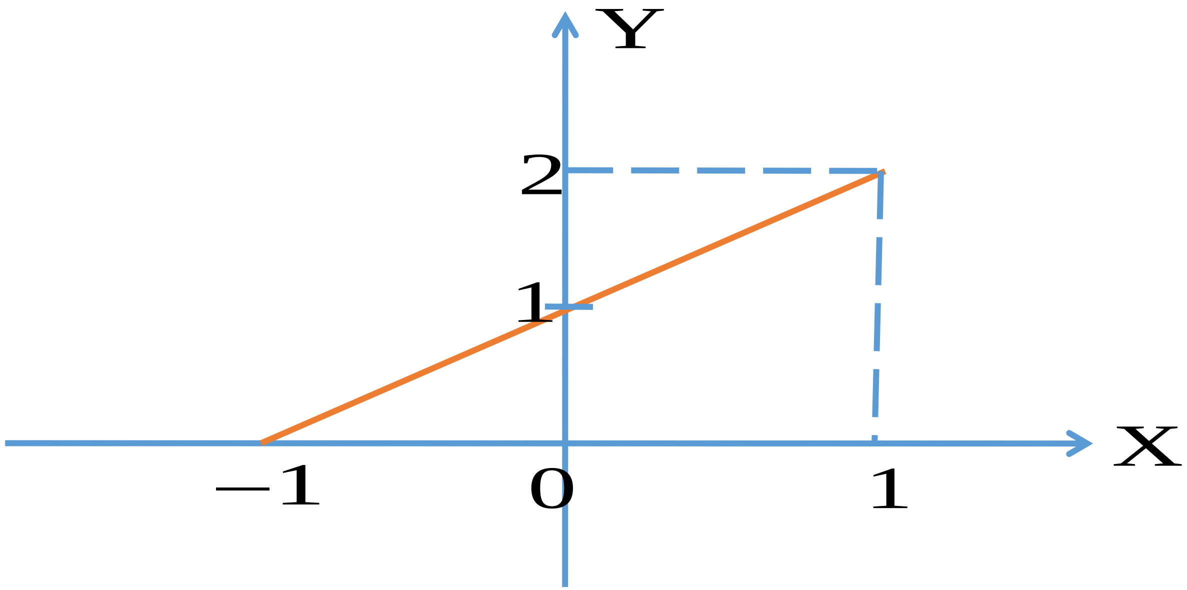
- Let the rule of a function denoted as $y=f(x)$
- The graph or the curve of f is given by the points (x,y)
- Domain in x *axis* and range in y *axis*



Example

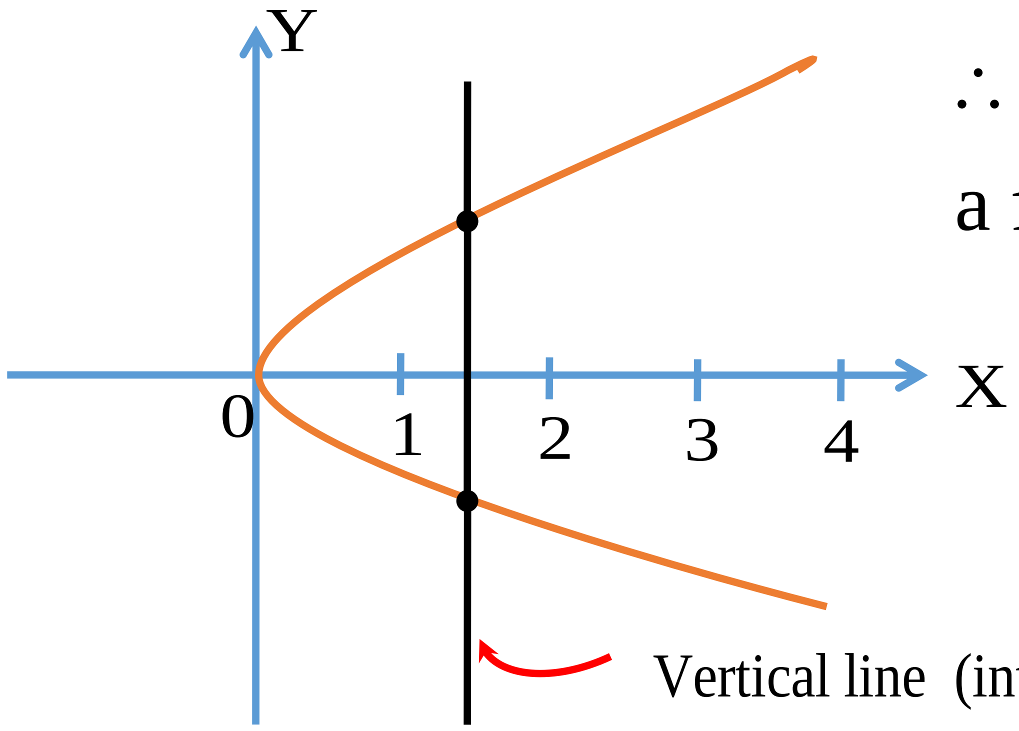
What is the rule of the function plotted in the following figure ?

Determine the domain and the range.



Vertical line test – Single valued feature

Consider the graph of $y = \pm\sqrt{x}$



$\therefore \quad y = \pm\sqrt{x}$ cannot be a rule of a function.

Vertical line (intersects the graph more than once)

Addition and subtraction of functions

- Consider two functions $f(x) = x^2 + 5$ and $g(x) = -2x^2 - 3$

  - $\lambda f(x) \pm \mu g(x) = (\lambda f \pm \mu g)(x)$

$i)(f+g)(x) = -x^2 + 2$

$ii)(f-g)(x) = 3x^2 + 8$

$iii)(f+g)(0) = 2$

$iv)(f-g)(1) = 11$

Composition of functions

- Chain of functions can be built up where,

  output of one function forms the input to the next function

$$x \longrightarrow \boxed{\text{function } f} \longrightarrow f(x) \longrightarrow \boxed{\text{function } g} \longrightarrow g(f(x))$$

$$\Uparrow$$

output of f

becomes input for g

Composition of functions ctd.

For two functions $f : A \to B$ and $g : B \to C$, the composition of f and g, denoted by $g \circ f$ is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(x) \equiv g(f(x)).$$

We read $g \circ f$ as "g circle f" or "g of f".

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Example

1) If the rules are given by $f(x) = x+1$ and $g(x) = x^2$,

then find the composed rules for $g_\circ f$ and $f_\circ g$.

2) Find composed rules with $f(x) = \dfrac{1}{x}$ and $g(x) = x^2$

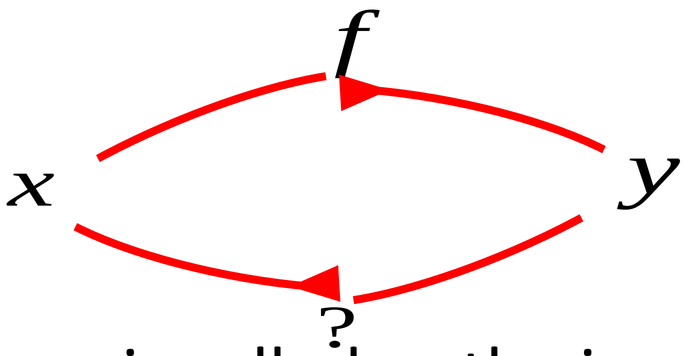
in both ways.

Result : Let f:A $\rightarrow$ B , g:B $\rightarrow$ C , h:C $\rightarrow$ D .

Then $h_\circ(g_\circ f) = (h_\circ g)_\circ f$

Inverse of a function

- For the function $y=f(x)$, what is the rule from y to x?



- This reverse process is called as the inversion
- The required rule is called as the inverse function
- Denoted as $f^{-1}(x)$

Example

1) What is the rule of the inverse for $f(x) = x + 2$?

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & x+2 \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$$f^{-1}(x) = x - 2$$

2) For $f(x) = 2x - 1 \quad \Rightarrow \quad y = 2x - 1$

$$2x = y + 1$$

$$x = \frac{y+1}{2}$$

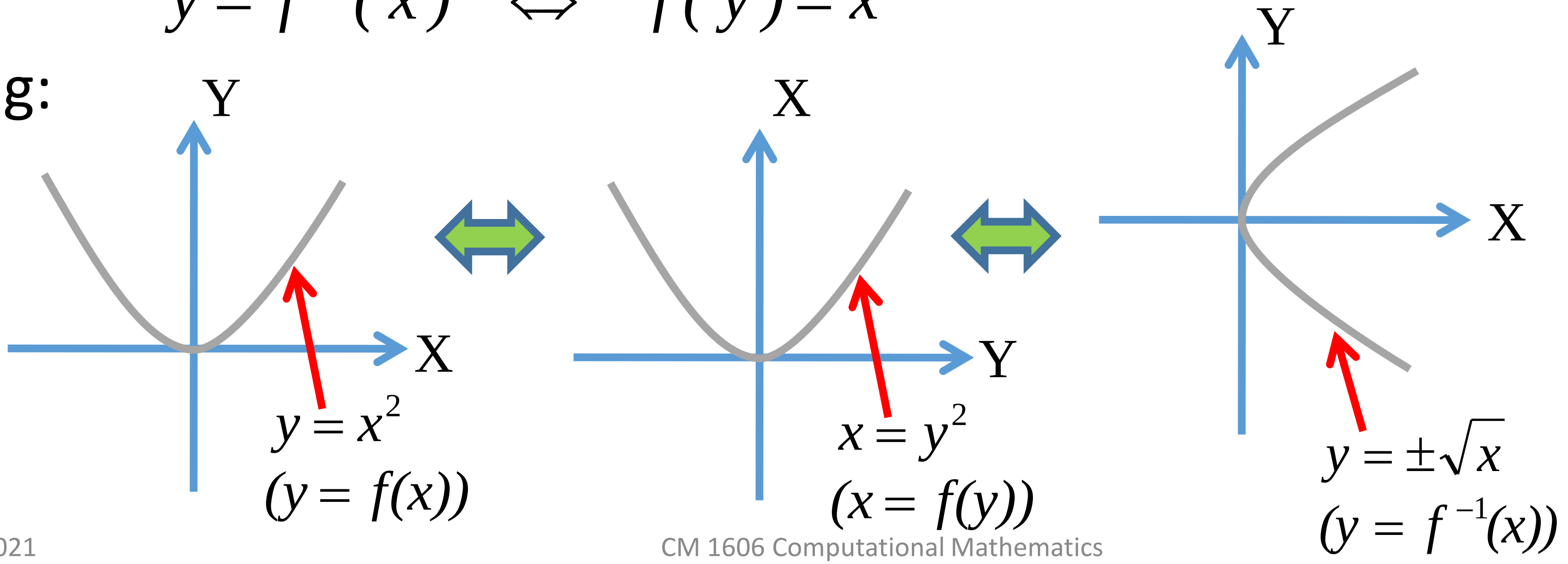
Then, $f^{-1}(y) = \dfrac{y+1}{2}$ or identically we have $f^{-1}(x) = \dfrac{x+1}{2}$.

Graph of the inverse rule

- Rule of $f^{-1}(x)$ can be plotted using the graph of $f(x)$ by interchanging X and Y axes appropreately.
- That interchange is based on the following implication.

$$y = f^{-1}(x) \Leftrightarrow f(y) = x$$

- Eg:



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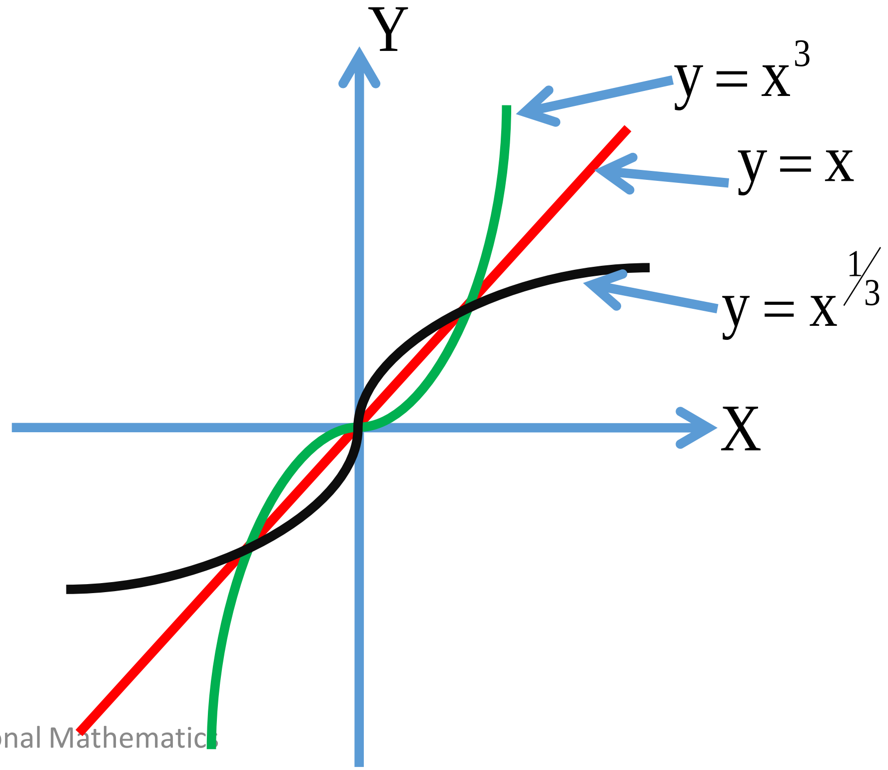
Graph of the inverse rule ctd.

- Alternatively, graph of $f^{-1}(x)$ can be taken as the reflection (mirror image) of the graph of $f(x)$ over the line $y = x$.

eg. $f(x) = x^3$

$$f^{-1}(x) = x^{1/3}$$



Polynomial functions

- n^{th} degree polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \quad ; \quad a_n \neq 0$$

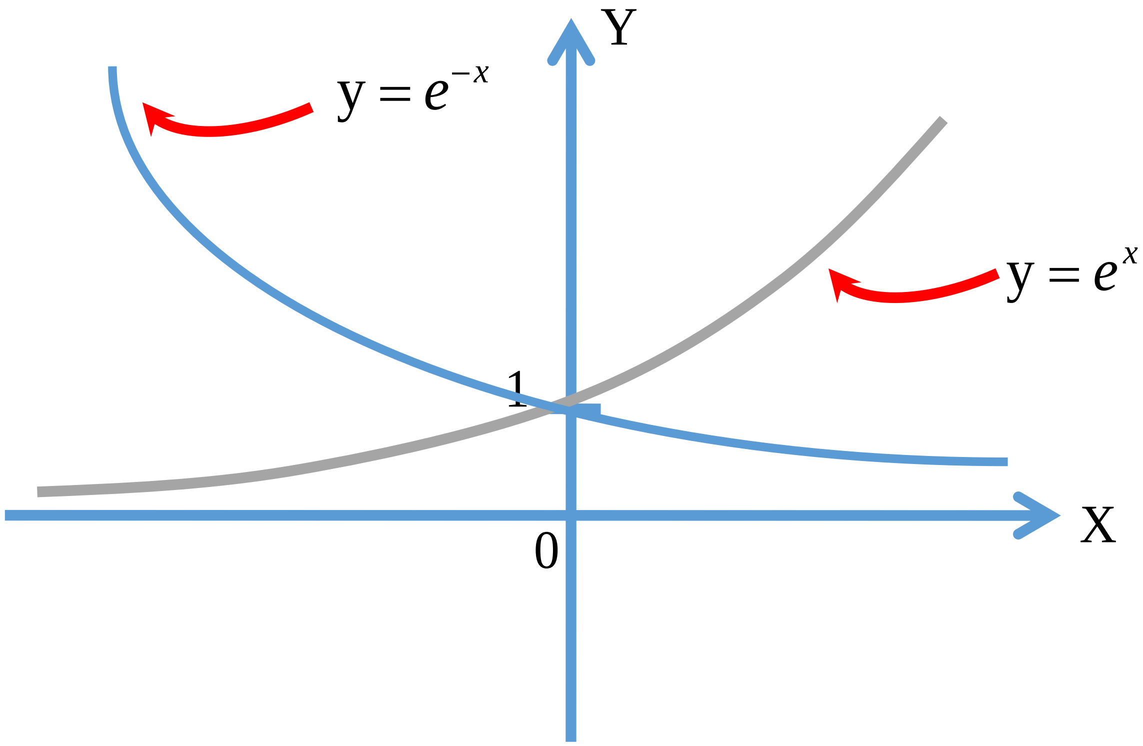
n is a non-negative integer and all a_i's are constants

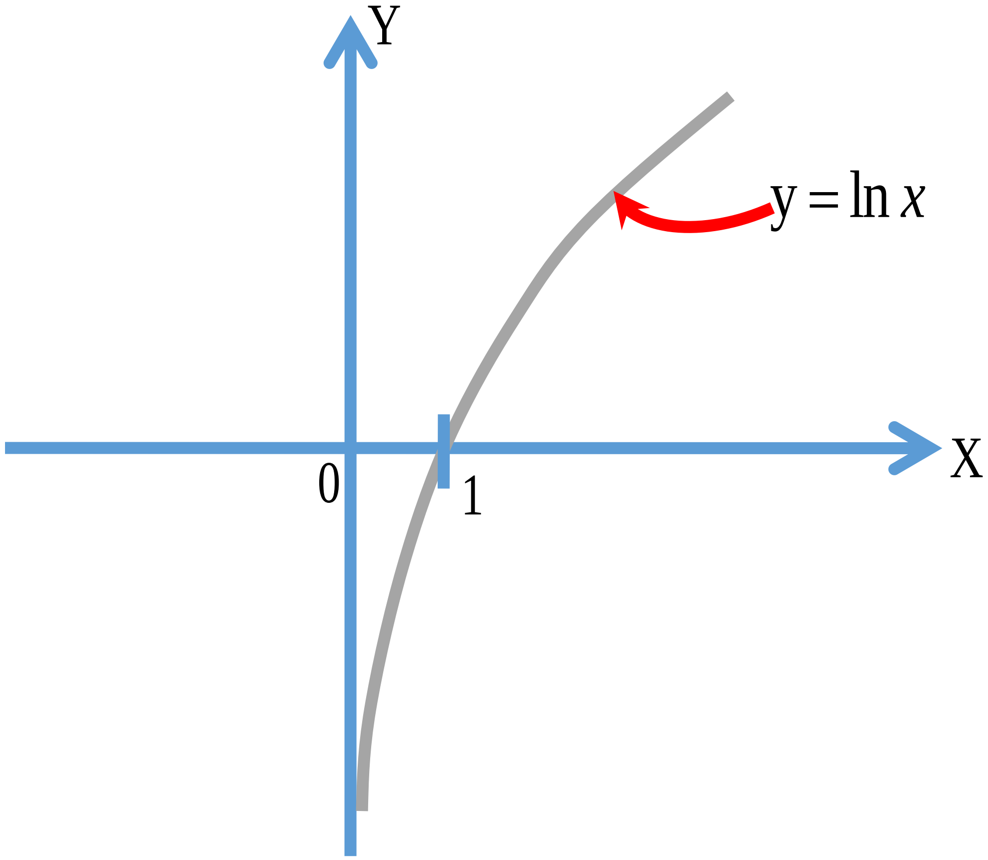
- Trigonometric functions
- Exponential and logarithmic functions

$$y = e^x \quad \Leftrightarrow \quad x = \log_e y \quad (\text{or } \ln y)$$

Thus exponential and logarithmic functions are inverse of each other (mutual inverses).

Graph of $f(x) = e^x$ and ln x





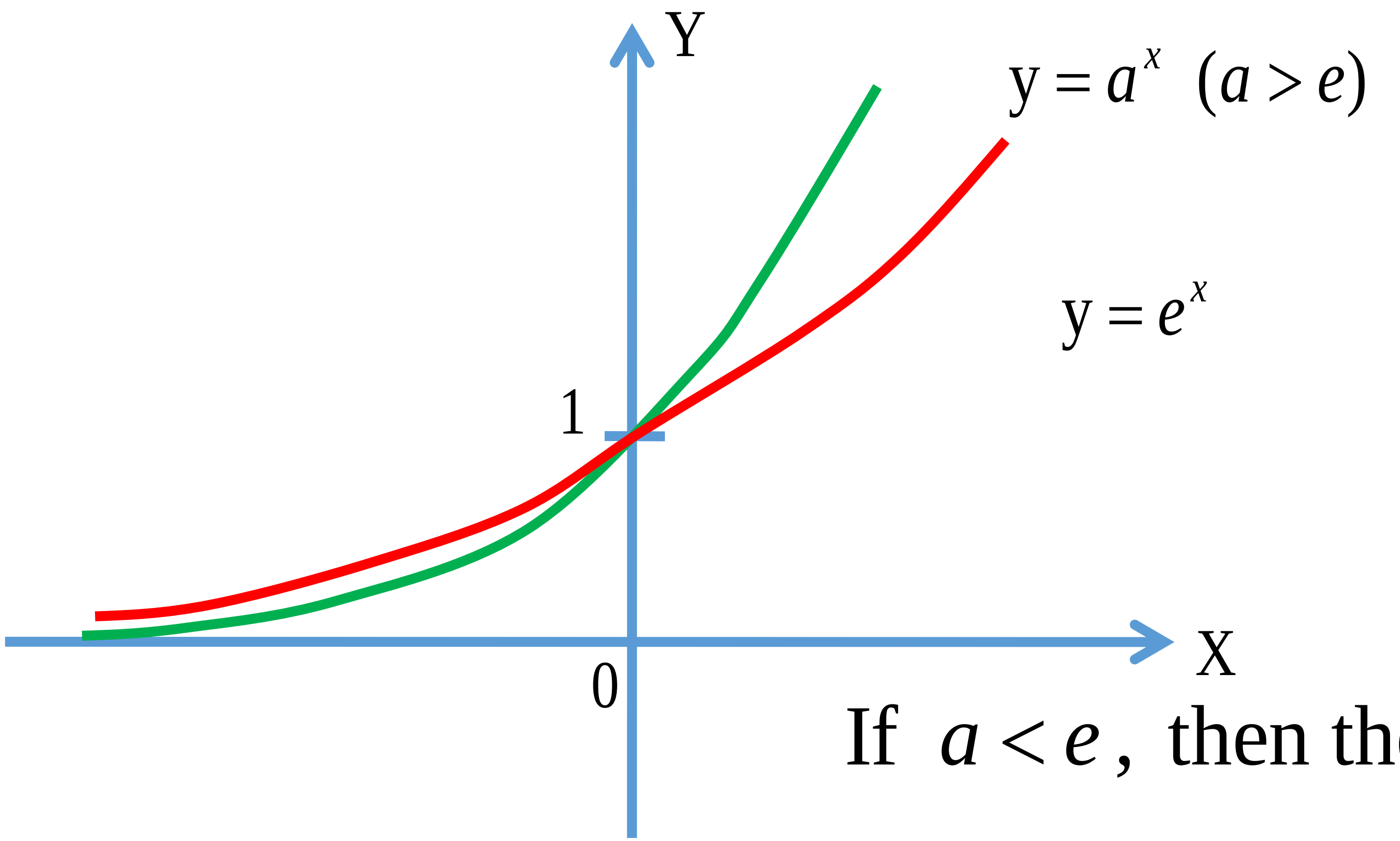
Logistic function

- Basic form: $y = \frac{1}{1+\mathrm{e}^{-x}}$

Ex:

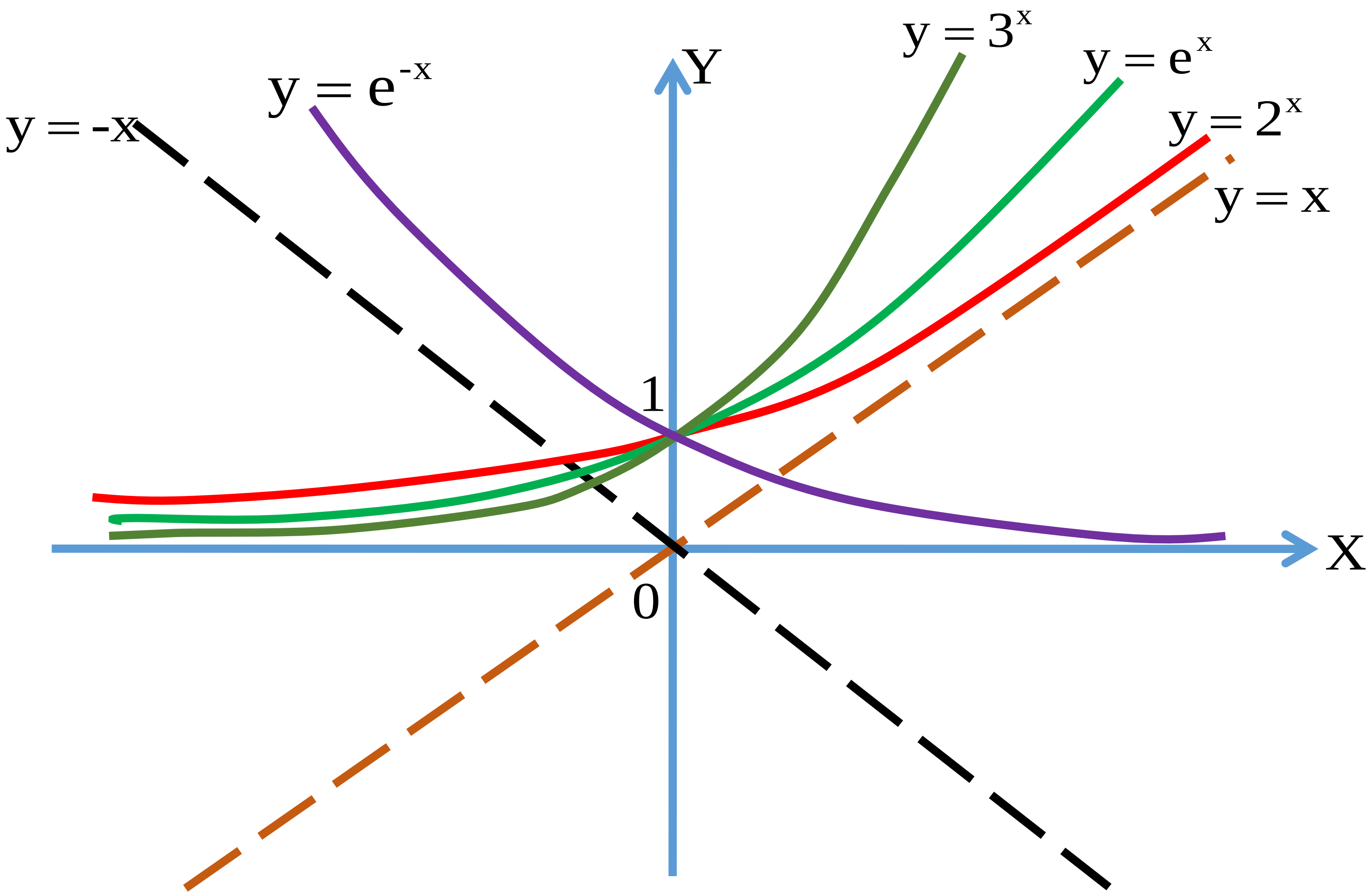
Find the maximum and minimum value for this function and sketch the function (Hint: Refer $\mathrm{y} = e^{-x}$)

Special features



If $a < e$, then the graph of a^x increases less quickly than the graph of e^x.

Special features



Critical Values of Quadratic functions

- Consider the function

$$f(x) = x^2 + 2x + 1$$

- By the method of completing squares

$$f(x) = x^2 + 2x + 1 = (x+1)^2$$

$Note:$

$$f(x) = x^2 \pm bx + c = \left(x \pm \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = \left(x \pm \frac{b}{2}\right)^2 \pm d$$

Critical Values of Quadratic functions ctd.

$$Note:$$

$$f(x)=x^2 \pm bx + c = \left(x \pm \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = \left(x \pm \frac{b}{2}\right)^2 \pm d$$

Eg: Find the minimum value of the function $f(x)=x^2+6x+15; x \in \Re$

$$f(x)=x^2+6x+15$$

$$=(x+3)^2+6$$

So $\min f(x)=6$

$when\ x = \text{-}3$