1 Гармонический осциллятор

$$V = \frac{1}{2}m\omega^2 x^2,\ \dot{q}_k = \frac{p_k}{m},\ \dot{p}_k = -\left.\frac{\partial V}{\partial x}\right|_{q_k} = -m\omega^2 q_k$$

$$g_k = N\exp\left(\frac{1}{\hbar}\left(-\frac{1}{2}m\omega(x-q_k)^2 + ip_k(x-q_k)\right)\right) = \exp\left(\frac{1}{\hbar}\left(-\frac{1}{2}m\omega x^2 + \xi_k x + \eta_k\right)\right)$$

$$\xi_k = m\omega q_k + ip_k,\ \eta_k = \hbar\ln N - \frac{1}{2}m\omega q_k^2 - iq_k p_k$$

$$\dot{\xi}_k = m\omega\dot{q}_k + i\dot{p}_k = \omega p_k - im\omega^2 q_k = -i\omega(m\omega q_k + ip_k) = -i\omega\xi_k$$

$$\dot{\eta}_k = -m\omega q_k\dot{q}_k - i(\dot{q}_k p_k + q_k\dot{p}_k) = -\dot{q}_k(m\omega q_k + ip_k) - iq_k\dot{p}_k = -\frac{p_k\xi_k}{m} + im\omega^2 q_k^2$$

$$\mathbb{S}_{mk} = \langle g_m|g_k\rangle = \int\exp\left(\frac{1}{\hbar}\left(-m\omega x^2 + (\xi_m^* + \xi_k)x + \eta_m^* + \eta_k\right)\right)dx =$$

$$= \int\exp\left(\frac{1}{\hbar}\left(-m\omega\left[x - \frac{\xi_m^* + \xi_k}{2m\omega}\right]^2 + \frac{(\xi_m^* + \xi_k)^2}{4m\omega} + \eta_m^* + \eta_k\right)\right)dx =$$

$$= \exp\left(\frac{1}{\hbar}\left(\frac{(\xi_m^* + \xi_k)^2}{4m\omega} + \eta_m^* + \eta_k\right)\right)\int\exp\left(-\frac{m\omega}{\hbar}y^2\right)dy =$$

$$= \exp\left(\frac{1}{\hbar}\left(\frac{(\xi_m^* + \xi_k)^2}{4m\omega} + \eta_m^* + \eta_k\right)\right)\sqrt{\frac{\hbar\pi}{m\omega}}$$

$$\mathbb{S}_{kk} = N^2\sqrt{\frac{\hbar\pi}{m\omega}} = 1\ \Rightarrow\ N = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$$

$$\langle g_m\left|x\right|g_k\rangle = \int x\cdot\exp\left(\frac{1}{\hbar}\left(-m\omega x^2 + (\xi_m^* + \xi_k)x + \eta_m^* + \eta_k\right)\right)dx =$$

$$= \hbar\int\frac{\partial}{\partial(\xi_m^* + \xi_k)}\exp\left(\frac{1}{\hbar}\left(-m\omega x^2 + (\xi_m^* + \xi_k)x + \eta_m^* + \eta_k\right)\right)dx$$

$$= \hbar\frac{\partial\mathbb{S}_{mk}}{\partial(\xi_m^* + \xi_k)} = \frac{(\xi_m^* + \xi_k)}{2m\omega}\mathbb{S}_{mk}$$

$$\left\langle g_m\left|x^2\right|g_k\right\rangle = \int x^2\cdot\exp\left(\frac{1}{\hbar}\left(-m\omega x^2 + (\xi_m^* + \xi_k)x + \eta_m^* + \eta_k\right)\right)dx =$$

$$= \hbar^2\frac{\partial^2\mathbb{S}_{mk}}{\partial(\xi_m^* + \xi_k)^2} = \frac{\hbar}{2m\omega} + \left(\frac{\xi_m^* + \xi_k}{2m\omega}\right)^2$$

$$\tau_{mk} = \langle g_m|\dot{g}_k\rangle = \frac{1}{\hbar}\mathbb{S}_{mk}\left(\dot{\eta}_k + \dot{\xi}_k\frac{\xi_k + \xi_m^*}{2m\omega}\right) = \frac{1}{\hbar}\mathbb{S}_{mk}\left(im\omega^2 q_k^2 - \frac{p_k\xi_k}{m} - i\frac{\xi_k^2 + \xi_k\xi_m^*}{2m}\right) =$$

$$= \frac{1}{\hbar}\mathbb{S}_{mk}\left(im\omega^2 q_k^2 - \frac{i\xi_k\xi_m^*}{2m} - \frac{i\xi_k(\xi_k - 2ip_k)}{2m}\right) = \frac{1}{\hbar}\mathbb{S}_{mk}\left(im\omega^2 q_k^2 - \frac{i\xi_k\xi_m^*}{2m} - \frac{i|\xi_k|^2}{2m}\right) =$$

$$= \frac{1}{\hbar}\mathbb{S}_{mk}\left(im\omega^2 q_k^2 - \frac{i\xi_k\xi_m^*}{2m} - \frac{i(m^2\omega^2 q_k^2 + p_k^2)}{2m}\right) = \frac{1}{\hbar}\mathbb{S}_{mk}\left(\frac{i}{2}m\omega^2 q_k^2 - \frac{ip_k^2}{2m} - \frac{i\xi_k\xi_m^*}{2m}\right)$$

$$\mathbb{H}_{mk} = \left\langle g_m \left|\hat{T}\right| g_k\right\rangle + \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle = -\frac{\hbar^2}{2m}\left\langle g_m\left|\frac{\partial^2}{\partial x^2}\right| g_k\right\rangle + \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle =$$

$$= -\frac{\hbar^2}{2m}\left\langle g_m\left|\frac{1}{\hbar}\frac{\partial}{\partial x}(-m\omega x + \xi_k)\right| g_k\right\rangle + \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle =$$

$$= \frac{\hbar\omega}{2}\langle g_m|g_k\rangle - \frac{1}{2m}\left\langle g_m\left|(-m\omega x + \xi_k)^2\right| g_k\right\rangle + \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle =$$

$$= \left(\frac{\hbar\omega}{2} - \frac{\xi_k^2}{2m}\right)\langle g_m|g_k\rangle + \omega\xi_k\left\langle g_m\left|x\right| g_k\right\rangle - \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle + \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle =$$

$$= \left(\left(\frac{\hbar\omega}{2} - \frac{\xi_k^2}{2m}\right) + \omega\xi_k\frac{(\xi_m^* + \xi_k)}{2m\omega}\right)\langle g_m|g_k\rangle = \left(\frac{\hbar\omega}{2} + \frac{\xi_m^*\xi_k}{2m}\right)\langle g_m|g_k\rangle$$

$$\mathbb{H}_{mk} - i\hbar\boldsymbol{\tau}_{mk} = \left(\frac{\hbar\omega}{2} + \frac{\xi_m^*\xi_k}{2m} + \frac{1}{2}m\omega^2 q_k^2 - \frac{p_k^2}{2m} - \frac{\xi_m^*\xi_k}{2m}\right)\langle g_m|g_k\rangle =$$

$$= \left(\frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 q_k^2 - \frac{p_k^2}{2m}\right)\mathbb{S}_{mk}$$

$$\left(-i\mathbb{S}^{-1}\left(\mathbb{H} - i\hbar\boldsymbol{\tau}\right)\right)_{nk} = -i\sum_m \mathbb{S}^{-1}_{nm}\left(\mathbb{H}_{mk} - i\hbar\boldsymbol{\tau}_{mk}\right) =$$

$$= -i\left(\frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 q_k^2 - \frac{p_k^2}{2m}\right)\sum_m \mathbb{S}^{-1}_{nm}\mathbb{S}_{mk} = -i\left(\frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 q_k^2 - \frac{p_k^2}{2m}\right)\delta_{nk}$$

$$\dot{C}_n = -\frac{i}{\hbar}\sum_k\left(\frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 q_k^2 - \frac{p_k^2}{2m}\right)\delta_{nk}C_k = -\frac{i}{\hbar}\left(\frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 q_n^2 - \frac{p_n^2}{2m}\right)C_n$$

$$C_n(t) = C_n(0)\,\exp\left(-\frac{i}{\hbar}\int_0^t\left(\frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 q_n^2 - \frac{p_n^2}{2m}\right)dt'\right)$$

$$L_n = \frac{p_n^2}{2m} - \frac{1}{2}m\omega^2 q_n^2$$

$$L = \vec{C}^\dagger\left(\mathbb{T} - \mathbb{V}\right)\vec{C} = \sum_{m,k}C_m^*C_k\left(-\frac{\hbar^2}{2m}\left\langle g_m\left|\frac{\partial^2}{\partial x^2}\right| g_k\right\rangle - \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle\right) =$$

$$= \sum_{m,k}C_m^*C_k\left(\left(\frac{\hbar\omega}{2} - \frac{\xi_k^2}{2m}\right)\mathbb{S}_{mk} + \omega\xi_k\left\langle g_m\left|x\right| g_k\right\rangle -\right.$$

$$\left.-\frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle - \frac{1}{2}m\omega^2\left\langle g_m\left|x^2\right| g_k\right\rangle\right)$$

$$= \sum_{m,k} C_m^* C_k \left(\left(\frac{\hbar\omega}{2} - \frac{\xi_k^2}{2m} \right) \mathbb{S}_{mk} + \omega\xi_k \left\langle g_m \left| x \right| g_k \right\rangle - m\omega^2 \left\langle g_m \left| x^2 \right| g_k \right\rangle \right) =$$

$$= \sum_{m,k} C_m^* C_k \left(\frac{\hbar\omega}{2} - \frac{\xi_k^2}{2m} + \frac{\omega\xi_k(\xi_m^* + \xi_k)}{2m\omega} - m\omega^2 \left(\frac{\hbar}{2m\omega} + \left(\frac{\xi_m^* + \xi_k}{2m\omega} \right)^2 \right) \right) \mathbb{S}_{mk}$$

$$= \sum_{m,k} C_m^* C_k \mathbb{S}_{mk} \left(\frac{2\xi_k\xi_m^* - (\xi_m^* + \xi_k)^2}{4m} \right) = - \sum_{m,k} C_m^* C_k \mathbb{S}_{mk} \frac{(\xi_m^*)^2 + \xi_k^2}{4m} =$$

$$= -\frac{1}{4m} \left(\sum_{m,k} C_m^* C_k \mathbb{S}_{mk} \xi_k^2 + \sum_{m,k} C_m^* C_k \mathbb{S}_{mk} (\xi_m^2)^* \right) =$$

$$= -\frac{1}{4m} \left(\sum_{m,k} C_m^* C_k \mathbb{S}_{mk} \xi_k^2 + \sum_{m,k} (C_m C_k^* \mathbb{S}_{km} \xi_m^2)^* \right) =$$

$$= -\frac{1}{2m} \sum_{m,k} Re(C_m^* C_k \mathbb{S}_{mk} \xi_k^2) = -\frac{1}{2m} \sum_{m,k} \left(m^2\omega^2 q_k^2 - p_k^2 \right) Re(C_m^* \mathbb{S}_{mk} C_k) =$$

$$= \sum_{m,k} \left(\frac{p_k^2}{2m} - \frac{1}{2} m\omega^2 q_k^2 \right) Re(C_m^* \mathbb{S}_{mk} C_k) =$$

$$= \sum_{m,k} L_k Re \left(C_m^*(0) C_k(0) \exp \left(-\frac{i}{\hbar} \int_0^t (L_m - L_k)\, dt' \right) \mathbb{S}_{mk} \right) =$$

$$= \sum_{m,k} L_k \cos \left(\frac{1}{\hbar} \int_0^t (L_m - L_k)\, dt' \right) Re(C_m^*(0) \mathbb{S}_{mk} C_k(0))$$

$$C(t) = C(0) \exp \left(-\frac{i}{\hbar} \int_0^t \left(\frac{\hbar\omega}{2} - \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \right) dt \right)$$

$$\dot{q} = \frac{p}{m}$$

$$\dot{p} = -m\omega^2 q$$

$$\ddot{q} = \frac{\dot{p}}{m} = -\omega^2 q$$

$$q(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

$$q(0) = A_1 + A_2$$

$$\dot{q}(0) = \frac{p(0)}{m} = i\omega(A_1 - A_2)$$

$$A_1 = \frac{1}{2} \left(q(0) - \frac{ip(0)}{m\omega} \right)$$

$$A_2 = \frac{1}{2}\left(q(0) + \frac{ip(0)}{m\omega}\right)$$

$$q(t) = \frac{1}{2}\left(\left(q(0) - \frac{ip(0)}{m\omega}\right)e^{i\omega t} + \left(q(0) + \frac{ip(0)}{m\omega}\right)e^{-i\omega t}\right) =$$

$$= \frac{1}{2}\left(q(0)\left(e^{i\omega t} + e^{-i\omega t}\right) - \frac{ip(0)}{m\omega}\left(e^{i\omega t} - e^{-i\omega t}\right)\right) =$$

$$= q(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t)$$

$$p(t) = m\dot{q}(t) = -m\omega q(0)\sin(\omega t) + p(0)\cos(\omega t)$$

$$\int_0^t \frac{p(t')^2}{2m}\,dt' = \frac{1}{2m}\left(p(0)^2\int_0^t \cos^2(\omega t')\,dt' + m^2\omega^2 q(0)^2\int_0^t \sin^2(\omega t')\,dt' - \right.$$

$$\left. - m\omega q(0)p(0)\int_0^t 2\sin(\omega t)\cos(\omega t')\,dt'\right) =$$

$$= \frac{p(0)^2}{4m}\int_0^t (1+\cos(2\omega t'))\,dt' + \frac{m\omega^2 q(0)^2}{4}\int_0^t (1-\cos(2\omega t'))\,dt' -$$

$$- q(0)p(0)\int_0^t \sin(\omega t')\,d\sin(\omega t') =$$

$$= \frac{1}{2}\left(\frac{p(0)^2}{2m} + \frac{1}{2}m\omega^2 q(0)^2\right)t + \frac{1}{4\omega}\left(\frac{p(0)^2}{2m} - \frac{1}{2}m\omega^2 q(0)^2\right)\sin(2\omega t) -$$

$$- \frac{q(0)p(0)}{2}\sin^2(\omega t)$$

$$\frac{1}{2}m\omega^2\int_0^t q(t')^2\,dt' = \frac{1}{2}m\omega^2\left(q(0)^2\int_0^t \cos^2(\omega t')\,dt' + \frac{p(0)^2}{m^2\omega^2}\int_0^t \sin^2(\omega t')\,dt' + \right.$$

$$\left. + \frac{q(0)p(0)}{m\omega}\int_0^t 2\sin(\omega t')\cos(\omega t')\,dt'\right) =$$

$$= \frac{1}{4}m\omega^2 q(0)^2\int_0^t (1+\cos(2\omega t'))\,dt' + \frac{p(0)^2}{4m}\int_0^t (1-\cos(2\omega t'))\,dt' +$$

$$+ q(0)p(0)\int_0^t \sin(\omega t')\,d\sin(\omega t') =$$

$$= \frac{t}{2}\left(\frac{p(0)^2}{2m} + \frac{1}{2}m\omega^2 q(0)^2\right) - \frac{1}{4\omega}\left(\frac{p(0)^2}{2m} - \frac{1}{2}m\omega^2 q(0)^2\right)\sin(2\omega t) +$$

$$+ \frac{q(0)p(0)}{2}\sin^2(\omega t)$$

$$\int_0^t \left(\frac{p(t')^2}{2m} - \frac{1}{2}m\omega^2 q(t')^2\right)dt' = \left(\frac{p(0)^2}{2m} - \frac{1}{2}m\omega^2 q(0)^2\right)\frac{\sin(2\omega t)}{2\omega} - q(0)p(0)\sin^2(\omega t)$$

$$C(t)=C(0)\text{exp}\left(-\frac{i\omega t}{2}+\left(\frac{p(0)^2}{2m}-\frac{1}{2}m\omega^2q(0)^2\right)\frac{i\sin(2\omega t)}{2\hbar\omega}\right)$$

$$C(t)=C(0)\text{exp}\left(-\frac{i}{\hbar}\left(\frac{\hbar\omega t}{2}-L(0)\frac{\sin(2\omega t)}{2\omega}+q(0)p(0)\sin^2(\omega t)\right)\right) \qquad (1)$$

$$g_k=N\text{exp}\left(\frac{1}{\hbar}\left(-\frac{1}{2}m\omega(x-q)^2+ipx\right)\right)=\text{exp}\left(\frac{1}{\hbar}\left(-\frac{1}{2}m\omega x^2+\xi_k x+\eta_k\right)\right)$$

$$\xi_k=m\omega q_k+ip_k,\ \ \eta_k=-\frac{1}{2}m\omega q^2$$

$$\dot{\xi}_k=m\omega\dot{q}_k+i\dot{p}_k=\omega p_k-im\omega^2q_k=-\omega(im\omega q_k-p_k)=-i\omega(m\omega q_k+ip_k)=-i\omega\xi_k$$

$$\dot{\eta}_k=-m\omega q_k\dot{q}_k=-\omega q_k p_k$$

$$\tau_{mk}=\frac{1}{\hbar}\langle g_m|\dot{g}_k\rangle=\frac{1}{\hbar}\mathbb{S}_{mk}\left(\dot{\eta}_k+\dot{\xi}_k\frac{\xi_m^*+\xi_k}{2m\omega}\right)=\frac{1}{\hbar}\mathbb{S}_{mk}\left(-\omega q_k p_k-i\omega\xi_k\frac{\xi_m^*+\xi_k}{2m\omega}\right)=$$

$$=\frac{1}{\hbar}\mathbb{S}_{mk}\left(-\omega q_k p_k-\frac{i\xi_k^2}{2m}-\frac{i\xi_m^*\xi_k}{2m}\right)=$$

$$=\frac{1}{\hbar}\mathbb{S}_{mk}\left(-\omega q_k p_k-\frac{i}{2m}(m^2\omega^2q_k^2-p_k^2+2im\omega q_k p_k)-\frac{i\xi_m^*\xi_k}{2m}\right)=\frac{1}{\hbar}\mathbb{S}_{mk}\left(iL_k-\frac{i\xi_m^*\xi_k}{2m}\right)$$

$$\mathbb{H}_{mk}=\mathbb{S}_{mk}\left(\frac{\hbar\omega}{2}+\frac{\xi_m^*\xi_k}{2m}\right)$$

$$\mathbb{H}_{mk}-i\hbar\tau_{mk}=\mathbb{S}_{mk}\left(\frac{\hbar\omega}{2}+L_k\right)$$

$$\dot{C}_n=-\frac{i}{\hbar}\sum_{m,k}\mathbb{S}^{-1}_{nm}\mathbb{S}_{mk}\left(\frac{\hbar\omega}{2}+L_k\right)C_k=-\frac{i}{\hbar}\sum_{k}\left(\sum_{m}\mathbb{S}^{-1}_{nm}\mathbb{S}_{mk}\right)\left(\frac{\hbar\omega}{2}+L_k\right)C_k$$

$$\dot{C}_n=-\frac{i}{\hbar}\sum_{k}\left(\frac{\hbar\omega}{2}+L_k\right)\delta_{nk}C_k=-\frac{i}{\hbar}\left(\frac{\hbar\omega}{2}+L_n\right)C_n$$

$$C(t)=C(0)\text{exp}\left(-\frac{i}{\hbar}\left(\frac{\hbar\omega t}{2}+L(0)\frac{\sin(2\omega t)}{2\omega}-q(0)p(0)\sin^2(\omega t)\right)\right) \qquad (2)$$