$$\min_x \tfrac{1}{2}\|x-z\|_2^2 \quad \text{s.t.} \quad e^T x \le B \quad x \ge 0$$

KKT. $0 \le (x-z) + \lambda e \quad \perp \quad x \ge 0$

$$x_i = 0 \Rightarrow -z_i - \lambda \ge 0 \Rightarrow z_i + \lambda \le 0$$
$$x_i > 0 \Rightarrow x_i - z_i - \lambda = 0 \Rightarrow x_i = z_i + \lambda$$

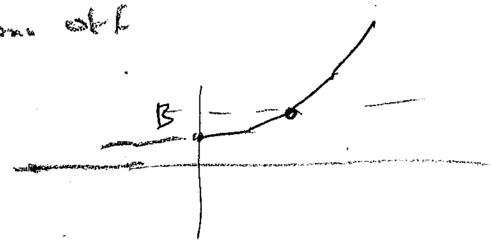
$$x_i = \max(z_i + \lambda, 0) \qquad i = 1 \dots n$$

$$t(\lambda) = \sum x_i = \sum \max(z_i + \lambda, 0).$$

want to find λ s.t. $t(\lambda) = B$

$t(\lambda)$ is increasing function of λ

$$t(0) = \sum_{i=1}^{n} \max(z_i, 0)$$



$$\lim_{\lambda \uparrow \infty} t(\lambda) = +\infty$$
$$\lim_{\lambda \downarrow -\infty} t(\lambda) = 0$$

Assume wlog. that $z_1 \ge z_2 \ge \dots \ge z_n$

$$t(\lambda) = \sum_{i \mid z_i + \lambda \ge 0} z_i + \lambda.$$

try $\lambda = -z_1$
$-z_2$
$\vdots$
$-z_n$

$\lambda < -z_1 \Rightarrow z_i + \lambda < 0 \quad \forall i = 1 \dots n$

$\lambda \in [-z_1, -z_2] \Rightarrow z_1 + \lambda \ge 0$
$z_i + \lambda \le z_i - z_2 \le 0 \qquad i = 2, \dots n$

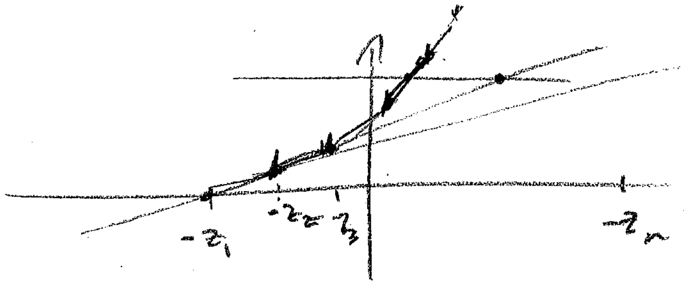
$\vdots$

$\lambda \in [-z_j, -z_{j+1}] \Rightarrow z_i + \lambda \ge 0 \qquad i = 1 \dots j$
$z_i + \lambda \le 0 \qquad i = j+1 \dots n$

so for $\lambda \in [-z_j, -z_{j+1}]$, $t(\lambda) = \sum_{i=1}^{j} (z_i + \lambda) = \left(\sum_{i=1}^{j} z_i\right) + j\lambda$

$= (\mathrm{zcum})_j + j\lambda$

$$\left.\begin{aligned} \lambda < -z_1 &\Rightarrow f(\lambda) = 0 \\ \lambda \in [-z_j, -z_{j+1}] &\Rightarrow f(\lambda) = (wcum)_j + j\lambda \\ \lambda > -z_n &\Rightarrow f(\lambda) = (wcum)_n + n\lambda \end{aligned}\right) \quad j = 1 \ldots n-1$$



$$f(-z_j) = wcum_j + j(-z_j)$$
$$= wcum_{j-1} + (j-1)(-z_j) \checkmark$$

$$f(\lambda) = B$$
$$\Rightarrow wcum_j + j\lambda = B$$
$$\Rightarrow \lambda = \frac{B - wcum_j}{j}$$

min $\frac{1}{2}\|x-z\|_2^2$ s.t. $B - e^T x \ge 0$ $\quad x \ge 0$

$$0 \le x - z + \lambda e \perp x \ge 0$$

$x_i = 0 \Rightarrow \lambda - z_i \ge 0 \Rightarrow z_i - \lambda \le 0$ $\qquad x_i = \max(z_i - \lambda, 0)$

$x_i > 0 \Rightarrow x_i = z_i - \lambda > 0$

$0 \le \lambda \perp B - e^T x \ge 0$

check: $e^T z \le B$, $z \ge 0 \Rightarrow \lambda = 0$, $x = z$

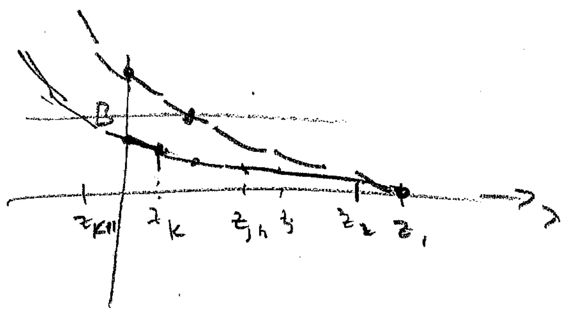
t! $z_1 \ge z_2 \ge \dots \ge z_n$

$$t(\lambda) = \sum_{i:\, z_i - \lambda \ge 0} z_i - \lambda$$

$\lim_{\lambda \downarrow -\infty} t(\lambda) = \infty$

$\lim_{\lambda \uparrow \infty} t(\lambda) = 0$

$t(0) = \sum_{z_i \ge 0} z_i \ge 0$.



$\lambda \in [z_{j+1}, z_j] \Rightarrow$ $z_i - \lambda \ge 0$ if $i \le j$

$z_i - \lambda \le 0$ if $i \ge j+1$

$$t(\lambda) = \sum_{i=1}^{j} (z_i - \lambda) = (zcum)_j - j\lambda$$

$$t(z_2) = (zcum)_1 - z_2 = z_1 - z_2$$

Breakpoints at $\underline{z_1 \ge z_2 \ge \dots \ge z_k} \ge 0 \ge z_{k+1}$

if $t(0) \le B \Rightarrow \lambda = 0$; $x = \max(z, 0)$

if $t(0) > B \Rightarrow$ examine $(0, z_k)$

$[z_k, z_{k-1}]$

$[z_{k-1}, z_{k-2}]$

$\vdots$

$[z_2, z_1]$

$t(z_k), t(z_{k-1}) \dots t(z_2), t(z_1) = 0$.