ADAPTIVE PREDICTION THEORY COMBINING OFFLINE AND ONLINE LEARNING *

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Abstract. Real-world intelligence systems usually operate by combining offline learning and online adaptation with highly correlated and non-stationary system data or signals, which, however, has rarely been investigated theoretically in the literature. This paper initiates a theoretical investigation on the prediction performance of a two-stage learning framework combining offline and online algorithms for a class of nonlinear stochastic dynamical systems. For the offline-learning phase, we establish an upper bound on the generalization error for approximate nonlinear-least-squares estimation under general datasets with strong correlation and distribution shift, leveraging the Kullback-Leibler divergence to quantify the distributional discrepancies. For the online-adaptation phase, we address, on the basis of the offline-trained model, the possible uncertain parameter drift in real-world target systems by proposing a meta-LMS prediction algorithm. This two-stage framework, integrating offline learning with online adaptation, demonstrates superior prediction performances compared with either purely offline or online methods. Both theoretical guarantees and empirical studies are provided.

Key words. System identification, nonlinear-least-squares, Kullback-Leibler divergence, offline learning, multiple models, least-mean-squares algorithm.

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1. Introduction.

1.1. Background. The development of big data and large models, particularly deep neural network-based learning methods, have provided many new approaches for the prediction and control of stochastic dynamical systems with uncertainties. Among these, the integration of offline learning and online adaptation is the most widely applied(e.g., [12, 21, 20, 27]). Offline learning leverages pre-collected historical datasets for model training through batch processing. By consolidating all computationally intensive tasks into a single upfront training phase, this strategy enables efficient and rapid predictions while preserving the capacity to capture essential system dynamics(e.g., [20, 21]). Furthermore, offline learning from multiple historical trajectories or from multiple environments, unlike single trajectory approaches, enables the acquisition of generalized behavioral patterns across broader operating regimes(e.g., [31, 21]). However, this approach often requires excessive storage space and struggles to adapt to dynamically changing environments or unfamiliar scenarios that often arise during real-world deployment. Achieving guaranteed generalization performance may require the smallness of either the parameter sensitivity function or the parameter estimation error, which is an outcome that depends on strong data or model structure conditions such as persistent excitation and expansiveness condition(e.g., [25, 32, 38]). When discrepancies exist between real and training environments, the prediction performance often degrades significantly and not robust. In contrast, online learning processes data, typically acquired from a single system trajectory, in a streaming fashion, requiring minimal storage and computational resources. Its lightweight, streaming processing enables flexibility and scalability. Fortunately, many convergence results have been established for online learning algorithms, particularly when applied to linear and saturated linear stochastic systems in the area of adaptive systems theory(e.g., [7, 34, 36]). However, online adaptation methods typically face several limitations. They generally require random initialization, which may induce large

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transient errors(e.g., [35]). Moreover, the theoretical analysis of pure online adaptation for general nonlinear dynamical systems may lead to estimating all the unknown parameters which may otherwise be partly estimated via offline learning. In other words, real-time model online updates usually necessitate full parameter optimization, demanding substantial computational resources that may be challenging to implement in embedded systems or real-time application scenarios. Extensive empirical evidence indicates that full parameter updates often underperform compared to carefully designed partial update strategies, such as linear probing(e.g., [13]).

Given their complementary strengths and weaknesses above, integrating offline and online approaches presents a promising path towards performance advancement. In the era of big data and large models, the increasing complexity of engineering systems, from industrial process control to autonomous navigation and robotic manipulation, has made such offline-online framework not just appealing, but necessary. The existing representative methods usually follow a two-stage approach: first, they learn either basis functions for unknown system dynamics(e.g., [21]) or control policies via deep reinforcement learning(e.g., [12]) in idealized environments. Then, during real-time processing, the methods apply online adaptation techniques such as Kalman-filtering or recursive-least-squares to refine model parameters in real time. This two-stage framework begins with extensive offline training on historical or simulated data to develop high-capacity foundation models, which serve as a solid baseline. It is then followed by online linear-probing in the target system, combining the strengths of both learning approaches to ensure robust and reliable performance under real-world uncertainties(e.g., [20, 12, 21, 27]).

Despite compelling empirical evidence across multiple domains, there has been limited theoretical investigation on the integration of offline-learning and online-adaptation, particularly in quantitatively characterizing their complementary effects and the resulting overall prediction performance. The state-of-the-art applications still rely primarily on heuristic combination offline and online components, with limited unifying theory offering end-to-end performance guarantees for the complete learning pipeline. Most existing frameworks fundamentally assume that offline learning can sufficiently capture the unknown system dynamics(e.g., [20, 21]). The core theoretical challenges appear in two lines of sight. First, differences between offline training environments and online deployment conditions cause distribution shift (e.g., [3]), while parameter drift between source and target systems further violates key assumptions like identical distribution and stationarity that underpin traditional statistical learning theory (e.g., [25, 38, 37]). The offline datasets often originate from previous system executions, thus carrying both temporal dependencies that cannot satisfy classical i.i.d. requirements. Apart from this, real-world systems often undergo parameter drift, where system parameters evolve over time due to factors like aging or environmental changes. When combined with distribution shift, this drift results in a substantial mismatch between offline-trained models and their behavior during processing. Existing theoretical frameworks are ill-equipped to capture such coupled dynamics, making it difficult to provide reliable performance guarantees. Thus, it finally renders the learning methods inapplicable to the safety-critical applications like the autonomous driving, weather systems, medical devices, and industrial control fields where traceable and strong performance is mandated.

In this paper, we will provide formal prediction performance guarantees for the complete two-stage learning pipeline and our theoretical development will address the coupled challenges of distribution shift and parameter drift, while providing practical guidance for implementing two-stage learning systems in safety-critical applications.

1.2. Contributions. This paper addresses the theoretical foundation of a two-stage offline and online learning framework by making contributions in three aspects: **(i)Offline-learning phase.** Based on historical trajectories generated by the source system, we estimate the unknown parameter through approximate nonlinear-least-squares and establish a novel upper bound on the generalization error. This bound explicitly addresses non-i.i.d. data with strong correlations and distribution shift, leveraging Kullback-Leibler(KL) divergence to quantify the discrepancy between the training and new data under mild assumptions.

This result not only provides theoretical guarantees for the offline-learning generalization in nonlinear dynamical systems with bounded nonlinear mappings(encompassing a class of deep neural networks) but also establishes a foundation for the subsequent online-adaptation phase. **(ii)Online-adaptation phase.** On the basis of the offline-learning phase, we further propose a meta-LMS prediction algorithm that integrates a multi-model meta-algorithm with a projected least-mean-squares(LMS) algorithm. It is designed to circumvent the parameter drift in the target system and mitigate the excessive transient error inherent in single-model approaches. We then provide a rigorous theoretical analysis of the prediction performance of this two-stage offline and online learning framework with a special focus on cross-system drift in the target system. Our analysis demonstrates that, under certain typical conditions, the framework can achieve near-optimal or even optimal prediction performance. **(iii)Comparison with purely offline/online methods.** Finally, we provide a rigorous comparative analysis demonstrating the advantages of the proposed two-stage learning framework over traditional purely offline or online methods. Specifically, we systematically articulate the necessity of online adaptation and the fundamental role of offline learning through both theoretical analysis and experimental validation.

The remainder of this paper is organized as follows. Section 2 introduces the notations and presents the two-stage algorithm, detailing both the offline-learning and online-adaptation phases. Section 3 establishes a Generalization Lemma and states the main theorems on the prediction performance, while also examining advantages of our framework relative to conventional purely offline and purely online methods. Proofs of the main theorems are provided in Section 4. Numerical simulations demonstrating effectiveness of the two-stage algorithm are provided in Section 5. Concluding remarks are given in Section 6.

2. Problem Formulation.

2.1. Notation. In this paper, the symbol $x \in \mathbb{X}^p \subseteq \mathbb{R}^p$ represents a p-dimensional column vector in $\mathbb{X}^p$, and x^τ, $\|x\|$, $\|x\|_1$ and $\|x\|_\infty$ denote its transpose, Euclidean norm, 1-norm and infinity-norm, respectively. The inner product is $\langle x, y \rangle \triangleq x^\tau y$ for $x, y \in \mathbb{X}^p$. The radius of a compact set $\mathcal{M}$ is defined as $\frac{1}{2} \max_{x,x' \in \mathcal{M}} \|x - x'\|$ and is denoted by $R_\mathcal{M}$. The minimum(maximum) eigenvalue of a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is denoted by $\lambda_{\min}(A)(\lambda_{\max}(A))$. The spectral radius $\rho(B)$ of a matrix B is defined as the maximum modulus among all its eigenvalues. For a matrix $\Gamma \in \mathbb{R}^{n \times m}$, $\|\Gamma\|$ and $\|\Gamma\|_F$ denote the operator norm induced by the Euclidean norm and Frobenius norm, respectively. The operator $vec(\Gamma)$ denotes the vectorization of Γ, obtained by stacking its columns into a single column vector. In a probability space $(\Omega, \mathcal{F}, P)$, Ω is the sample space, the collection of events is referred to as the σ-algebra $\mathcal{F}$ on Ω and P is a probability measure on $(\Omega, \mathcal{F})$. For an $\mathcal{F}$-measurable set $\mathcal{A}$, its complement $\mathcal{A}^c$ is defined by $\mathcal{A}^c = \Omega - \mathcal{A}$. The event $\mathcal{A}$ is said to happen almost surely(a.s.) if $P(\mathcal{A}) = 1$. Expectation and probability with respect to all the randomness of the underlying probability space are denoted by $\mathbb{E}$ and P, respectively. Expectation with respect to a random variable X is denoted as $\mathbb{E}_X$ or $\mathbb{E}_{P_X}$, which is defined as :$\mathbb{E}_X f(X, Y) = \int f(X, Y) dP_X$, and P_X denotes the marginal distribution of X. $N(\mu, \sigma^2)$ denotes the Gaussian distribution with mean μ and variance σ^2. Let $\{Z_t\}_{t=0}^{T-1}$ be a stochastic process with joint distribution P_Z. For each pair (i, j), let $P_{Z_{i:j}}$ denote the joint distribution of $\{Z_t\}_{t=i}^{j}$ and $\mathcal{Z}_{ij} := \sigma\{Z_i, \ldots, Z_j\}$ the σ-algebra generated by $\{Z_t\}_{t=i}^{j}$. $D(P\|Q)$ is the Kullback–Leibler(KL) divergence between two distributions P and Q and is defined as: $D(P\|Q) = \int \log(\frac{dP}{dQ}) dP$, which is nonnegative and can be ∞.

We need the following definitions in our analysis :

DEFINITION 2.1. *The dependency matrix of* $\{Z_t\}_{t=0}^{T-1}$ *is the matrix* $\Gamma_{\mathrm{dep}}(P_T) \triangleq \{\Gamma_{ij}\}_{i,j=0}^{T-1}$ *where:*

$$\Gamma_{ij} = \sqrt{2 \sup_{\substack{A \in \mathcal{Z}_{0:i} \\ B \in \mathcal{Z}_{j:T-1}}} |P_{Z_{j:T-1}}(B|A) - P_{Z_{j:T-1}}(B)|},$$

for $i < j$, $\Gamma_{ii} = 1$, *and* $\Gamma_{ij} = 0$, *for* $i > j$.

DEFINITION 2.2. *For a given compact set $D \subseteq \mathbb{R}^m$, the projection operator $\Pi_D(\cdot)$ is defined as*

$$\Pi_D(\beta) = \arg \min_{\omega \in D} \|\beta - \omega\|, \quad \forall \beta \in \mathbb{R}^m.$$

2.2. Offline-learning phase. At this stage, we attempt to estimate the unknown parameter α^* from datasets generated by certain source system through least squares.

Consider the following source system with N_1 trajectories:

$$y_{t+1,i} = f_t(\alpha^*, \beta_0(t), x_{t,i}) + v_{t+1,i}, \quad i = 1, 2, \cdots, N_1,$$

where $\alpha^* \in \mathcal{M}$ is unknown and can be regarded as an "internal" parameter, $\mathcal{M}$ is a known convex and compact set in $\mathbb{R}^n$ with radius $R_{\mathcal{M}}$; the form of f_t is known; the time-varying parameter $\beta_0(t)$ is known and deterministic in the source system, which can be regarded as an "environmental" parameter; $x_{t,i} \in \mathbb{X}^p$ is the regression vector, $y_{t+1,i} \in \mathbb{Y}^q$ is the output and $v_{t+1,i}$ is the noise. Here $\mathbb{X}$ can be a compact set or $\mathbb{R}$.

In this section, denote $\{x_{0,i}, x_{1,i}, \cdots, x_{T-1,i}\} \triangleq x_{0:T-1,i}$ and $\{y_{1,i}, y_{2,i}, \cdots, y_{T,i}\} \triangleq y_{1:T,i}$. Given datasets $\{\hat{x}_{0:T-1,i}, \hat{y}_{1:T,i}\}_{i=1}^{N_1}$ generated by the trajectories described above, we aim at using least-squares to estimate the true parameters α^* and then derive a generalization error bound with respect to a new dataset $x_{0:T-1}$, defined as:

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{x_{0:T-1}} \left\| f_t(\alpha^*, \beta_0(t), x_t) - f_t(\hat{\alpha}, \beta_0(t), x_t) \right\|^2.$$

Now we list the assumptions underlying our theoretical analysis, each accompanied by some remarks that will be used in the subsequent analysis:

ASSUMPTION 2.1. *The N_1 different regression vector sets $\hat{x}_{0:T-1,i}$, $i = 1, 2 \cdots, N_1$, are independent and identically distributed, that is, $\hat{x}_{0:T-1,i} \stackrel{i.i.d.}{\sim} P_T$ for $i = 1, 2, \cdots, N_1$. Moreover, the N_1 noise sets $v_{1:T,i}$, $i = 1, 2, \cdots, N_1$, are mutually independent.*

Remark 2.3. It should be emphasized that Assumption 2.1 here pertains to datasets of different trajectories, not necessarily requires that the data within any given trajectory to be independent. For example, if $x_{t,i} = (y_{t,i}, y_{t-1,i}, \cdots, y_{t-p+1,i}, u_{t,i}, u_{t-1,i}, \cdots, u_{t-q+1,i})^\tau$, then Assumption 2.1 holds provided that the initial conditions, input sets $u_{0:T-1,i}$, $i = 1, 2, \cdots, N_1$, and noise sequences of different system trajectories are i.i.d, but clearly, for any given i, $\{x_{t,i}, t = 0, 1, \cdots, T-1\}$ itself is a dependent sequence.

ASSUMPTION 2.2. *There exists $b_1 > 0, b_1' > 0$, $b_2, b_2' \in [0, 1)$, such that for any $i = 1, 2, \cdots, N_1$,*

$$\|\Gamma_{dep}(P_{T,i})\|^2 \le b_1 T^{b_2}, \quad \|\Gamma_{dep}(P_T')\|^2 \le b_1' T^{b_2'},$$

where $\Gamma_{dep}(\cdot)$ is the dependency matrix of the regression vectors defined in Definition 2.1 and $\Gamma_{dep}(P_T')$ denotes the dependency matrix of the new data sequence $x_{0:T-1}$ defined on the same probability space as the training one and $x_{0:T-1} \in \mathbb{X}^{pT}$.

Remark 2.4. This assumption characterizes the degree of dependence among the regression vectors along a single system trajectory either for training data or new data. In cases where the regression vectors are independent, m-dependent or ϕ-mixing with $\sum_{i=1}^\infty \sqrt{\phi(i)} < \infty$, it follows that $b_2 = 0$; see [25, 22, 38] for a more detailed discussion.

Remark 2.5. We now show that for a general class of nonlinear time series models, one may also have $b_2 = 0$. Let $x_t = (y_t, y_{t-1}, \cdots, y_{t-p+1})^\tau$. Suppose that $|f_t(\alpha^*, \beta_0(t), x)|$ is bounded by a constant $M_f < \infty$ for all $x \in \mathbb{R}^p, t \ge 0$, and that the noise v_{t+1} is $\stackrel{i.i.d.}{\sim} N(0, 1)$ and independent of y_0, then the regression vector sequence $\{x_t\}$ is a time-inhomogeneous Markov chain. The aforementioned source system can be formalized as:

$$x_{t+1} = F_t(x_t, x_{t-1}, \cdots, x_{t-p+1}) + V_{t+1},$$

where $V_{t+1} = (v_{t+1}, v_t, \cdots, v_{t-p+2})^\tau$ and $F_t(x_t, x_{t-1}, \cdots, x_{t-p+1}) = (f_t, f_{t-1}, \cdots, f_{t-p+1})^\tau$, $f_s \triangleq f_s(\alpha^*, \beta_0(s), x_s)$, $t - p + 1 \leq s \leq t$.

Since F_t is bounded by $\sqrt{p} M_f$, for any $n \geq 0$, $x_0 \in \mathbb{R}^p$, $A \in \mathcal{B}(\mathbb{R}^p)$, we have

$$P_{n,n+p}(x_0, A) = P(x_{n+p} \in A | x_n = x_0)$$
$$\geq \int_A (\frac{1}{\sqrt{2\pi}})^p \exp\{\frac{-\|y - F_{n+p-1}\|^2}{2}\} dy_1 dy_2 \cdots dy_p$$
$$\geq \delta v(A),$$

where $\delta = (\frac{1}{\sqrt{2}})^p \exp\{-p M_f^2\}$ and $v(\cdot)$ is a probability measure induced by p-dimensional normal distribution $N(0, \frac{1}{2} I_p)$, thus $\delta v(\cdot)$ is a nonnegative measure with non-zero mass δ. By Remark 1 of Theorem 1 in Chapter 2.4 of [5], this time-inhomogeneous Markov chain $\{x_t\}$ is uniformly geometrically ϕ-mixing, thus $b_2 = 0$.

Moreover, if $x_t = (y_t, y_{t-1}, ..., y_{t-p+1}, u_t, u_{t-1}, ..., u_{t-q+1})^\tau$, and the inputs $\{u_t\}$ is independent and identically distributed standard normal, a similar argument also shows that $b_2 = 0$. For a class of closed-loop systems with feedback control that satisfies Assumption 2.2, see [18, 2, 19].

ASSUMPTION 2.3. *There exists a constant $L > 0$, such that for any $\alpha', \alpha'' \in \mathcal{M}, x \in \mathbb{X}^p$ and $t \geq 0$,*
$$\|f_t(\alpha', \beta_0(t), x) - f_t(\alpha'', \beta_0(t), x)\| \leq L \|\alpha' - \alpha''\|.$$

Remark 2.6. This assumption is fairly mild when $\mathbb{X}$ is compact and $\beta_0(t)$ is bounded, and it is also possible to derive a specific value of L, which can be interpreted as an upper bound on the sensitivity function of f_t with respect to its unknown parameter α, as discussed in [16]. For example, consider a deep neural network f of width H and depth D, i.e., $f(\alpha, x) = W_D^\tau \sigma(W_{D-1}^\tau \sigma(\cdots \sigma(W_1^\tau x) \cdots))$, where the activation function σ satisfies $\|\sigma(x) - \sigma(y)\| \leq l \|x - y\|$ for any $x, y \in \mathbb{R}^H$ (e.g., ReLu, tanh, sigmoid, etc.) and $\alpha = (vec^\tau(W_D), vec^\tau(W_{D-1}), \cdots, vec^\tau(W_1))^\tau$. Denote $h_i \triangleq \sigma(W_i^\tau \sigma(\cdots \sigma(W_1^\tau x)) \cdots)$. We have
$$\|h_1 - h_1'\| \leq l \|W_1 - W_1'\| \|x\|,$$

and for $i = 2, 3, \cdots, D - 1$,

$$\|h_i - h_i'\| \leq l \|W_i^\tau h_{i-1} - W_i'^\tau h_{i-1}'\|$$
$$\leq l \|h_{i-1}\| \|W_i - W_i'\| + l \|W_i'\| \|h_{i-1} - h_{i-1}'\|,$$

and,

$$\|f(\alpha, x) - f(\alpha', x)\| = \|W_D^\tau h_{D-1} - W_D'^\tau h_{D-1}'\|$$
$$\leq \|W_D - W_D'\| \|h_{D-1}\| + l \|W_D'\| \|h_{D-1} - h_{D-1}'\|.$$

Besides, by Lipschitz property of σ, we have $\|\sigma(x)\| \leq l \|x\| + \|\sigma(0)\|$, $\|h_1\| \leq l \|W_1\| \|x\| + \|\sigma(0)\|$, $\|h_i\| \leq l \|W_i\| \|h_{i-1}\| + \|\sigma(0)\|$. Together with $\|W_i - W_i'\| \leq \|W_i - W_i'\|_F \leq \|\alpha - \alpha'\|$, we know that f satisfies this assumption and the Lipschitz constant L is bounded by a known polynomial of $\|x\|$. Moreover, we emphasize that if in Assumption 2.3, a time t belongs to $[0, T-1]$ and L depends on T, then, the corresponding conclusion remains valid under certain conditions, see Corollary 3.8.

Let $\mathcal{F}_{t,i} \triangleq \sigma\{x_{0,i}, x_{1,i}, ..., x_{t,i}, v_{1,i}, v_{2,i}, \cdots, v_{t,i}\}$ be the σ-algebra generated by the past regressors and noises up to time t of the *i*th trajectory.

ASSUMPTION 2.4. *There exists a constant $\sigma_v^2 > 0$, such that for all $\lambda \in \mathbb{R}, \|u\| = 1, i = 1, 2, \cdots, N_1$, the following inequality holds: $\mathbb{E}[e^{\lambda \langle v_{t+1,i}, u \rangle} | \mathcal{F}_{t,i}] \leq e^{\frac{\lambda^2 \sigma_v^2}{2}}$.*

Remark 2.7. It can be concluded that the noise sequence must be a martingale difference sequence by Jensen inequality. Moreover, consider any bounded noise $v_{t+1,i}$ for which the conditional expectation $\mathbb{E}[v_{t+1,i} | \mathcal{F}_{t,i}] = \mu_t$ is known. We can then apply a transformation by

defining $y'_{t+1} = y_{t+1} - \mu_t, v'_{t+1,i} = v_{t+1,i} - \mu_t$. According to Example 2.4 in Chapter 2 of [28], Assumption 2.4 still holds under this transformation. Besides, taking a constant $c > 2\sigma_v^2$ and defining $A_{t,i} \triangleq \{v_{t,i} : \|v_{t,i}\|^2 \geq c \log t\}$, by Markov inequality, we have $P(A_{t,i}|\mathcal{F}_{t-1,i}) \leq \inf_{s>0} \exp\{\frac{s^2\sigma_v^2}{2} - s\sqrt{c\log t}\} = \exp\{-\frac{c\log t}{2\sigma_v^2}\}$, thus $\|v_t\|^2 = O(\log t)$ by Borel-Cantelli Lemma [9].

ASSUMPTION 2.5. *Given a fixed $\varepsilon^* \geq 0$ and N_1 datasets $\{\hat{x}_{0:T-1,i}, \hat{y}_{1:T,i}\}_{i=1}^{N_1}$ generated by these system trajectories, a sub-optimal estimator $\hat{\alpha} \in \mathcal{M}$ can be obtained in the following sense:*

$$\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \left\|\hat{y}_{t+1,i} - f_t(\hat{\alpha}, \beta_0(t), \hat{x}_{t,i})\right\|^2$$

$$\leq \min_{\alpha \in \mathcal{M}} \frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \left\|\hat{y}_{t+1,i} - f_t(\alpha, \beta_0(t), \hat{x}_{t,i})\right\|^2 + \varepsilon^*.$$

Remark 2.8. Notice that the nonlinear-least-squares estimate over $\mathcal{M}$ satisfies this inequality with $\varepsilon^* = 0$. For any given $\varepsilon^* > 0$, an estimate satisfying this inequality can be obtained using random search method similar to those proposed in [8, 29]. Besides, we emphasize that no statistical assumptions on the error ε^* are required. According to Remark 2.6, a deep neural network with a suitable class of activation functions satisfies Assumption 2.3. Therefore, when the form of f_t is unknown, the Universal Approximation Theorem for deep neural networks(e.g., [4]) allows us to reformulate the problem as one of learning the network parameters and ε^* can be decomposed as $\varepsilon^* = \varepsilon_{app} + \varepsilon_{alg}$, where ε_{app} denotes the approximation error inherent to network's capacity, and ε_{alg} represents the error introduced by certain learning algorithms.

2.3. Online-adaptation phase. Unlike the previously studied source system, some parameters of the actual system may be unknown and subject to drift and the new sequence of regression vectors $x_{0:T-1}$ may exhibit distribution shift compared with those during the offline training. We now give a detailed description.

Let us consider the scalar output case, i.e., $q = 1$, for convenience. Consider the following real-world target system:

$$y_{t+1} = f_t(\alpha^*, \beta(t), x_t) + w_{t+1},$$

where compared with the source system in the offline-learning phase, the "environmental" parameter sequence may differ from that of the source system and thus $\beta(t) \neq \beta_0(t)$, which is now assumed to be unknown; w_{t+1} is either a stochastic or a deterministic bounded perturbation satisfying $|w_{t+1}| \leq W_{\max} < \infty$ for any $t \geq 0$, which may not satisfy Assumption 2.4. In addition, since most real-world systems are subject to fundamental physical laws that inherently guarantee their boundedness, we assume that $|f_t(\alpha^*, \beta(t), x_t)| \leq M_f < \infty$ for any $t \geq 0$, even though $f_t(\alpha^*, \beta(t), x_t)$ may not satisfy Assumption 2.3.

The constants M_f and $W_{\max}$ are both known a prior, and for simplicity, the new sequence of regression vectors $x_{0:T-1}$ is considered to be independent of the training datasets $\{\hat{x}_{0:T-1,i}, y_{1:T,i}\}_{i=1}^{N_1}$ originating from the source system(see Corollaries 3.7, 3.8 for the non-independent case).

In this section, the parameter estimate $\hat{\alpha}$ obtained from the offline-learning phase remains fixed, while online adaptation is employed to adapt the unknown, time-varying parameter $\beta(t)$ only. Our focus will be put on the prediction performance of the target system output in an average sense:

$$J_T \triangleq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(y_{t+1} - y_{t+1}^{\text{pred}})^2,$$

where $y_{t+1}^{\text{pred}} \in \sigma\{\hat{\alpha}, x_0, x_1, \cdots, x_t, x_{t+1}, y_1, y_2, \cdots, y_t\}$ is a predicted value of the unseen y_{t+1}, and the expectation is taken over all randomness.

In the following, we consider a typical case where $f_t(\alpha, \beta, x)$ can be linearly parameterized or approximated by a separable model of the form $\sum_{i=1}^{m} \lambda_i(t)\varphi_i(\beta, t)\psi_i(\alpha, x, t)$, which is hold for quite general classes of nonlinear functions, see, e.g., the related mathematical theorems in [23, 24, 11, 26]. Thus we may assume that for all $\alpha \in \mathcal{M}$ and $t \geq 0$,

$$(2.1) \qquad f_t(\alpha, \beta(t), x_t) = \beta_t^\tau \phi_t(\alpha, x_t),$$

where β_t is an unknown m-dimensional vector whose components are the weighting coefficients $\lambda_i(t)\varphi_i(\beta(t), t)$ of the "basis" functions $\psi_i(\alpha, x_t, t)$ represented by the components of the vector ϕ_t. In the presence of approximation error, it can in fact be treated as part of the perturbation, thereby yielding analogous theoretical results.

Therefore, we consider the following real-world target system:

$$(2.2) \qquad y_{t+1} = \beta_t^\tau \phi_t(\alpha^*, x_t) + w_{t+1}.$$

Additionally, we impose the following assumption:

ASSUMPTION 2.6. *There exists known constants A, B, such that for the $\hat{\alpha} \in \mathcal{M}$ obtained from the offline-learning phase, $\|\phi_t(\hat{\alpha}, x_t)\| \leq A$, $\|\beta_t\| \leq B$, for any $t \geq 0$.*

The original target system (2.2) can be rewritten as:

$$(2.3) \qquad y_{t+1} = \beta_t^\tau \phi_t(\hat{\alpha}, x_t) + \varepsilon_t + w_{t+1},$$

where $\varepsilon_t \triangleq \beta_t^\tau \phi(\alpha^*, x_t) - \beta_t^\tau \phi(\hat{\alpha}, x_t)$ represents the error caused by the discrepancy between the parameter $\hat{\alpha}$ learned through offline training and the true system parameters α^*. Under Assumption 2.6, β_t is time-varying and bounded, $\phi_t(\hat{\alpha}, x_t)$ is bounded.

Similar to (2.1), we can also write $f_t(\alpha, \beta_0(t), x_t) = \beta_t^{0\tau} \phi_t(\alpha, x_t)$, where β_t^0 is known when $\beta_0(t)$ is known as assumed in the offline-learning phase. Furthermore, the error ε_t is straightforwardly bounded by $M_f + AB$, thus there always exist a constant $L_1 \geq 0$, and a bounded sequence $L_0(t) \geq 0$, such that for all $t \geq 0$:

$$(2.4) \qquad \mathbb{E}\varepsilon_t^2 \leq L_1 \mathbb{E}\left(\beta_t^{0\tau}\phi_t(\alpha^*, x_t) - \beta_t^{0\tau}\phi_t(\hat{\alpha}, x_t)\right)^2 + L_0(t).$$

This means that the generalization error at time t on the target system is bounded by that on the source system plus an offset term $L_0(t)$. Moreover, L_1 and $L_0(t)$ serve to characterize the parameter drift from the source to the target system, and their specific choices satisfying (2.4) are clearly not unique as detailed in Appendix A, and will influence the output prediction error bound as discussed in Remark 3.11. As remarked in Case 2 of Appendix A, the term $L_0(t)$ can reflect the squared norm of the inner product of the generalization error $\phi_t(\alpha^*, x_t) - \phi_t(\hat{\alpha}, x_t)$ and the projection residual of β_t to the linear space spanned by β_t^0.

Next, to predict the output at the next step, following previous studies [6, 27, 1, 33, 35], we propose a meta-LMS prediction Algorithm 2.1, aiming at enhancing the transient performance through a multi-model framework with prediction-error-based weighting, by employing multiple initializations under a single hyperparameter d, a mechanism analogous to the switching strategy in [35]. Unlike the multiple hyperparameters in [33], our focus is to generate the predicted outputs rather than the estimated parameters.

Similarly, we derive the following lemma to highlight the theoretical advantage of multi-models over a single-model baseline: Algorithm 2.1 has a performance floor guaranteed by the worst model and asymptotically achieves the performance of the best model:

LEMMA 2.9. *If the learning rate $\lambda < \frac{1}{2(M_f+AB+W_{\max})^2}$, then the average prediction error produced by meta-LMS prediction Algorithm 2.1 satisfies:*

$$(2.5) \qquad \frac{1}{T}\sum_{t=0}^{T-1}(y_{t+1} - y_{t+1}^{pred})^2 \leq \max_{1 \leq i \leq N_2}\left\{\frac{1}{T}\sum_{t=0}^{T-1}(y_{t+1} - y_{t+1,i}^{pred})^2\right\}, \quad a.s.,$$

Algorithm 2.1 Meta-LMS Prediction

Require: Learning rate $\lambda > 0$, $\gamma \in (0,1)$, $d > \frac{A^2}{(1-\gamma)^2}$.

1: $w_{0,i} > 0$ and $\sum_{i=1}^{N_2} w_{0,i} = 1$, $D \triangleq \left\{ \beta \in \mathbb{R}^m \middle| \|\beta\| \leq B \right\}$, $\|\phi_0(\hat{\alpha}, x_0)\| \leq m_0 \leq \frac{A}{1-\gamma}$,

and the initial values $\hat{\beta}_{0,i}$ of these N_2 models are chosen arbitrarily in D and mutually different.

2: **for** $t = 0, 1, \ldots, T-1$ **do**

3:     Obtain prediction $y_{t+1,i}^{\text{pred}} = \hat{\beta}_{t,i}^{\tau} \phi_t(\hat{\alpha}, x_t)$ from each model i.

4:     Aggregate prediction $y_{t+1}^{\text{pred}} = \sum_{i=1}^{N_2} w_{t,i} y_{t+1,i}^{\text{pred}}$.

5:     Incur loss $l_t(y_{t+1,i}^{\text{pred}}) \triangleq \left(y_{t+1} - y_{t+1,i}^{\text{pred}} \right)^2$ for each model i.

6:     Update the weights:

$$w_{t+1,i} = \frac{w_{t,i} \exp(-\lambda l_t(y_{t+1,i}^{\text{pred}}))}{\sum_{i=1}^{N_2} w_{t,i} \exp(-\lambda l_t(y_{t+1,i}^{\text{pred}}))}.$$

7:     Update the estimate of β_{t+1} in parallel using projected LMS algorithm:

$$\hat{\beta}_{t+1,i} = \Pi_D \left\{ \hat{\beta}_{t,i} + \frac{\phi_t(\hat{\alpha}, x_t)}{d + m_t^2} \left(y_{t+1} - \hat{\beta}_{t,i}^{\tau} \phi_t(\hat{\alpha}, x_t) \right) \right\}, \quad i = 1, 2, \cdots, N_2.$$

8:     Update m_{t+1}:

$$m_{t+1} = \gamma m_t + \|\phi_{t+1}(\hat{\alpha}, x_{t+1})\|.$$

9: **end for**

---

and,

$$(2.6) \qquad J_T = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(y_{t+1} - y_{t+1}^{pred})^2 \leq \min_{1 \leq i \leq N_2} \left\{ \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(y_{t+1} - y_{t+1,i}^{pred})^2 + \frac{\log \frac{1}{w_{0,i}}}{\lambda T} \right\}.$$

3. Main Results.

3.1. Generalization Lemma. In Section 2.2, an estimate $\hat{\alpha}$ of the true parameter α^* was obtained by employing an approximate nonlinear-least-squares approach on datasets originating from the source system. We first give the following Generalization Lemma on the bound of the generalization error due to distribution shift between the training data and new data. The proof is supplied in Appendix B.

LEMMA 3.1. *Under Assumptions 2.1-2.5, there exists $N_0 > 0$ depending on $n, R_{\mathcal{M}}, L, \sigma_v$ and p such that when $N_1 \cdot T > N_0$, the generalization error based on the estimate $\hat{\alpha}$ obtained in the offline-learning phase can be bounded as follows:*

(3.1)

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{x_{0:T-1}} \left\| f_t(\alpha^*, \beta_0(t), x_t) - f_t(\hat{\alpha}, \beta_0(t), x_t) \right\|^2$$

$$\leq C_1 \frac{\log(N_1 T)}{N_1 T^{1-b_2}} + 8 \sup_{\alpha \in \mathcal{M}} M_{T,N_1}(\alpha) + 8L^2 R_{\mathcal{M}}^2 b_1' \frac{D(P_T \| P_T')}{T^{1-b_2'}} + 16\varepsilon^*, \quad a.s.,$$

and the second term on the right-hand side is a measurable function of the training datasets

$\{\hat{x}_{0:T-1,i}, \hat{y}_{1:T,i}\}_{i=1}^{N_1}$ *satisfying:*

$$(3.2) \qquad \mathbb{E}\left[\sup_{\alpha \in \mathcal{M}} M_{T,N_1}(\alpha)\right] \leq C_0 \sigma_v^2 \frac{\log(N_1 T)}{N_1 T},$$

where the constant C_1 depends on $n, R_\mathcal{M}, b_1, L, p$; C_0 depends on $n, R_\mathcal{M}, L, p$. The constants N_0, C_1 and C_0 can be obtained according to the proof in Appendix B. P'_T denotes the distribution of the new data $x_{0:T-1}$ and P_T denotes the distribution of the training data $\hat{x}_{0:T-1,i}$.

Remark 3.2. Inspired by [25], this lemma derives an upper bound on the generalization error across multiple source system trajectories, while allowing for distribution shift between the training and new data without assuming Assumptions 4 and 5 from [25] which require a kind of persistence of excitation condition on the data and a quadratic identifiability condition. The third term on the right-hand side of (3.1) captures the error due to distribution shift which depends on the data marginal distributions, and the fourth term corresponds to the possible optimization errors in Assumption 2.5.

Remark 3.3. For many dynamical systems, the distribution shift may arise from possible parameter drift in system dynamics, calling for a more refined analysis of (3.1) in Generalization Lemma 3.1, see, e.g., Corollaries 3.6, 3.7 and 3.8 in the subsequent section. As is well known, the KL divergence can sometimes be infinite. To address this issue, alternative measures, such as Total Variation distance, Rényi divergence, and concentration coefficient, have been proposed to quantify the difference between distributions(e.g., [3, 30]). However, these alternatives often require stronger assumptions and may be less effective than the KL divergence under some typical situations, e.g., Corollaries 3.6, 3.7 and 3.8.

3.1.1. Corollaries of Generalization Lemma 3.1. We note that, under the i.i.d. assumption, i.e., $\hat{x}_t \overset{i.i.d.}{\sim} P_0$ and $x_t \overset{i.i.d.}{\sim} P'_0$ for all $t \geq 0$, the KL divergence term simplifies to $\frac{D(P_T \| P'_T)}{T^{1-b'_2}} = \frac{D(P_T \| P'_T)}{T} = D(P_0 \| P'_0)$, and thus may not vanish, as is also noted in Remark 2 of [30]. Nevertheless, under Assumption 2.3, the expected generalization error can be further bounded by $L^2 \mathbb{E}\|\alpha^* - \hat{\alpha}\|^2$, which vanishes as $\hat{\alpha} \to \alpha^*$ in the mean square sense.

Thus, the convergence of the parameter estimators plays an important role in characterizing the robustness of the generalization error. As a trade-off, it may require certain "excitation" conditions on the data and/or model structure. We now establish two sufficient conditions for the mean square convergence of the nonlinear-least-squares(NLS) parameter estimation based on Generalization Lemma 3.1 with proof given in Appendix C.

COROLLARY 3.4. *("Excitation" conditions for estimator convergence) Suppose that Assumptions 2.3 and 2.4 hold. Then, under either of the following two "excitation" conditions:*

$$(3.3) \qquad T^{b_2} \log T = o\left(\mathbb{E} \min_{\alpha' \in \mathcal{M}} \lambda_{\min}\left(\sum_{t=0}^{T-1} F_t(\alpha^*, \alpha') F_t^\tau(\alpha^*, \alpha')\right)\right),$$

or,

$$(3.4) \qquad \log T = o\left(\min_{\alpha' \in \mathcal{M}} \lambda_{\min}\left(\sum_{t=0}^{T-1} F_t(\alpha^*, \alpha') F_t^\tau(\alpha^*, \alpha')\right)\right) \quad a.s.,$$

where $F_t(\alpha^*, \alpha') \triangleq \int_0^1 \nabla_\alpha f_t(\alpha' + s(\alpha^* - \alpha'), \beta_0(t), \hat{x}_{t,1}) ds$, the NLS estimator $\hat{\alpha}$ satisfies:

$$\mathbb{E}\|\hat{\alpha} - \alpha^*\|^2 \to 0, \quad as \ T \to \infty.$$

Remark 3.5. In fact, the almost sure convergence of $\hat{\alpha}$ to the true parameter α^* can be obtained under Condition (3.4), and the proof is supplied in Appendix C. For either linear stable model or nonlinear dynamical model with bounded nonlinear mapping, it is known that $b_2 = 0$(e.g., [25, 38] and Remark 2.5). Conditions (3.3) and (3.4) are reminiscent of the weakest possible convergence condition for the standard linear-least-squares algorithm, see, [14].

We now provide three corollaries that illustrate the application of Generalization Lemma 3.1 to three typical classes of nonlinear dynamical systems. For convenience, consider the case of a single trajectory and scalar outputs and let the optimization error $\varepsilon^* = 0$, while the general case can be considered similarly. The related proofs are provided in Appendix C.

COROLLARY 3.6. *(Generalizability for isomorphic systems) Consider the following two systems, and it is the second system that generates the new data:*

$$y_{t+1} = f_t(\alpha^*, \beta_0(t), \phi_t, u_t) + w_{t+1},$$

$$y'_{t+1} = f_t(\alpha^*, \beta_0(t), \phi'_t, u'_t) + w'_{t+1},$$

where

$$\phi_t = (y_t, y_{t-1}, \cdots, y_{t-p+1}, u_{t-1}, \cdots, u_{t-q+1})^\tau,$$

$$\phi'_t = (y'_t, y'_{t-1}, \cdots, y'_{t-p+1}, u'_{t-1}, \cdots, u'_{t-q+1})^\tau.$$

Assume that:

(i) $\|f_t(\alpha, \beta_0(t), \phi, u)\|$ is bounded by $M_f < \infty$ for all $t \geq 0$, $\alpha \in \mathcal{M}$ and $(\phi, u) \in \mathbb{R}^{p+q}$; and for any compact set $B \subset \mathbb{R}^{p+q}$, there exists a constant $C_B < \infty$, such that $\|\nabla_\alpha f_t(\alpha, \beta_0(t), \phi, u)\| \leq C_B$, for all $t \geq 0$, $\alpha \in \mathcal{M}$ and $(\phi, u) \in B$, e.g., a deep neural network in Remark 2.6 with bounded activation functions.

(ii) These two systems are independent of each other (initial values, inputs and noises) and both systems share the same feedback input structure, i.e., $u_t = \pi_t(\phi_t) + v_t$, $u'_t = \pi_t(\phi'_t) + v'_t$, and $|\pi_t(\phi)| \leq M_f < \infty$, for all $\phi \in \mathbb{R}^{p+q-1}, t \geq 0$.

(iii) $\{w_{t+1}, v_t\}$ and $\{w'_{t+1}, v'_t\}$ are both independent and identically distributed; w_{t+1} and w'_{t+1} are independent of $\sigma\{v_{0:t}, \phi_0\}$ and $\sigma\{v'_{0:t}, \phi'_0\}$, respectively; v_t and v'_t are independent of $\sigma\{w_{1:t}, \phi_0\}$ and $\sigma\{w'_{1:t}, \phi'_0\}$, respectively. Besides, they are zero-mean, bounded by $W > 2M_f$ and satisfy $\inf_{v \in [-W,W]} p(v) > 0$, $\inf_{w \in [-W,W]} p(w) > 0$, where $p(\cdot)$ is denoted as the probability density function.

(iv) The KL-divergence of the initial values $x_0 \triangleq (\phi_0^\tau, u_0)^\tau, x'_0 \triangleq (\phi_0'^\tau, u'_0)^\tau$ satisfies $D(D_0 \| D'_0) < \infty$, where $x_0 \sim D_0$ and $x'_0 \sim D'_0$.

Let $\hat{\alpha}$ be the nonlinear-least-squares estimate of α^*, i.e.,

$$\hat{\alpha} \triangleq \arg\min_{\alpha \in \mathcal{M}} \frac{1}{T} \sum_{t=0}^{T-1} \left(y_{t+1} - f_t(\alpha, \beta_0(t), \phi_t, u_t) \right)^2,$$

where the training dataset $\{\phi_{0:T-1}, u_{0:T-1}, y_{1:T}\}$ comes from the first system.

Then, the generalization error has the following convergence rate:

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left(f_t(\alpha^*, \beta_0(t), \phi'_t, u'_t) - f_t(\hat{\alpha}, \beta_0(t), \phi'_t, u'_t) \right)^2 = O\left(\frac{\log T}{T} \right),$$

where the expectation is taken over all randomness.

COROLLARY 3.7. *(Generalizability for time series models) Consider the following system with l-dimensional outputs:*

$$Y_{t+1} = F(\alpha^*, Y_t) + W_{t+1},$$

where the $F : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^l$ and the noise process $\{W_{t+1}\}$ satisfy assumptions (i) and (iii), respectively, in Corollary 3.6. Besides, assume that the initial value Y_0 is supported on $B \triangleq B_l(0, W - M_f)$, i.e., a ball of radius $W - M_f$ centered at the origin, and has a density $p_0(x)$ satisfying $\int_{\{x|p_0(x)>1\}} p_0(x) \log p_0(x) dx < \infty$.

Let $\hat{\alpha}$ be the NLS estimate based on the system's outputs $Y_{0:T}$. Then, the generalization error of the next trajectory of length T starting from time T has the following convergence rate:

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{Y_{T:2T-1}} \left\| F(\alpha^*, Y_{t+T}) - F(\hat{\alpha}, Y_{t+T}) \right\|^2 = O\left(\frac{\log T}{T} \right) \quad a.s.$$

COROLLARY 3.8. *(Generalizability under feedback control with learned parameter) Consider the following two systems:*

$$y_{t+1} = f_t(\beta_0(t), y_t, u_t) + v_{t+1}, \quad u_t = \pi_t(\alpha^*, y_t);$$

$$y'_{t+1} = f_t(\beta_0(t), y'_t, u'_t) + v'_{t+1}, \quad u'_t = \pi_t(\hat{\alpha}, y'_t);$$

where $\hat{\alpha}$ is the NLS estimate of α^* obtained from the first system.

Suppose that:

(i) f_t is bounded and $\|\nabla_\alpha f_t(\beta_0(t), y, \pi_t(\alpha, y))\| \leq a|y|^b + c$ for some constants a, b, c, for all $\alpha \in \mathcal{M}, y \in \mathbb{R}, t \geq 0$, and $b \geq 0$ is known(e.g., a deep neural network in Remark 2.6).

(ii) The noise v_t is independent of $\sigma\{v_{0:t-1}, y_0\}$, and v'_t is independent of $\sigma\{v'_{0:t-1}, y'_0\}$ and both $\{v_t\}$ and $\{v'_t\} \overset{i.i.d.}{\sim} N(0, \sigma^2)$.

(iii) The KL-divergence between the distributions of the initial values y_0, y'_0 is finite.

Then, given the estimator $\hat{\alpha}$ produced by the control policy $\pi_t(\alpha^*, \cdot)$, the generalization error on the second system with control policy $\pi_t(\hat{\alpha}, \cdot)$ has the following convergence rate:

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{y'_{0:T-1}} \left(f_t(\beta_0(t), y'_t, \pi_t(\hat{\alpha}, y'_t)) - f_t(\beta_0(t), y'_t, \pi_t(\alpha^*, y'_t)) \right)^2 = O\left(\frac{\log^c T}{T} \right) \quad a.s.,$$

where $c \geq 1$ is a constant depending on n, b.

3.2. Theorems on the prediction ability. Next, by combining the above Generalization Lemma 3.1 with the meta-LMS prediction Algorithm 2.1 for online adaptation of fine-tuning the drifting parameters in the target system, we can obtain the following theorems.

Denote:

$$\sigma_T^2 \triangleq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} w_{t+1}^2 \leq W_{\max}^2, \quad L_{0,T} \triangleq \frac{1}{T} \sum_{t=0}^{T-1} L_0(t),$$

$$\delta_T^2 \triangleq \frac{1}{T} \sum_{t=0}^{T-1} \|\beta_{t+1} - \beta_t\|^2 \geq \left(\frac{1}{T} \sum_{t=0}^{T-1} \|\beta_{t+1} - \beta_t\| \right)^2.$$

THEOREM 3.9. *Let the assumptions of the Generalization Lemma 3.1 as well as Assumption 2.6 holds. If we take the design parameters $d > \frac{A^2}{(1-\gamma)^2}$ and $\lambda < \frac{1}{2(M_f + AB + W_{\max})^2}$ in Algorithm 2.1, then the average prediction error for the combined two-stage process of offline learning and online adaptation is bounded by:*

$$(3.5) \qquad\qquad J_T \leq J_{mis} + J_{opt} + J_{est},$$

where

$$J_{mis} \triangleq \mathcal{N}_d \left(L_1 C \frac{D(P_T \| P'_T)}{T^{1-b'_2}} + L_{0,T} \right),$$

$$J_{opt} \triangleq \mathcal{N}_d L_1 \varepsilon^*,$$

$$J_{est} \triangleq \mathcal{N}_d \left(L_1 \frac{C \log(N_1 T)}{N_1 T^{1-b_2}} + B\delta_T + \delta_T^2 + \frac{B^2}{T} \right) + \left(\sqrt{C_d} + 1 + \frac{1}{d} \right)^2 \sigma_T^2,$$

and where $C_d \triangleq \left(1 + \frac{A^2}{d} \right)^2 \left(1 + \frac{A^2}{(1-\gamma)^2 d} \right) \left(1 + \frac{1}{d} \right) \geq 1$, the constant $\mathcal{N}_d$ depends on the design parameter d only and $\mathcal{N}_d = O(d)$; the constant C depends on $n, R_{\mathcal{M}}, b_1, b'_1, L, p, \sigma_v$ and the constants $\mathcal{N}_d, C$ can be obtained according to the proofs in Section 4.1 and Appendix B.

Remark 3.10. As can be seen from (3.5), the bound of the prediction error J_T contains three error terms: **(i)** $J_{\mathbf{mis}}$**:** the model mismatch containing both the distribution shift characterized by $D(P_T \| P'_T)$ and the parameter drift between the source and target systems

characterized by $L_1, L_{0,T}$ in Inequality (2.4). Note that $L_0(t)$ also depends on the generalization error on ϕ_t, which is a key element linking offline learning in the source system to cross-system drift in the target system; **(ii)**$J_{\mathbf{opt}}$**:** the optimization error ε^* multiplied by $\mathcal{N}_d, L_1$; and **(iii)**$J_{\mathbf{est}}$**:** the estimation errors determined by the number of training trajectories N_1 and the data-size T, by the parameter variation δ_T in the target system, and by the "variance" of the random/deterministic disturbances described by σ_T^2. Of course, all the three error terms depend on the design parameter d in Algorithm 2.1.

Remark 3.11. We remark that various options are available for the selection of L_1 and $L_0(t)$ satisfying (2.4), and these choices directly influence the prediction error bound. For guidance on selecting suitable L_1 and $L_0(t)$ to further refine the prediction error bound in different scenarios, see Appendix A for details. Furthermore, when multiple source systems with different outer parameters $\beta_t^1, \beta_t^2, \cdots, \beta_t^{N_1}$ are considered, the offset term $L_0(t)$ vanishes under condition that the target parameter β_t lies within the span of the set $\{\beta_t^1, \beta_t^2, \cdots, \beta_t^{N_1}\}$, and L_1 can be taken as the maximum over $t = 0$ to $T - 1$ of the sum of the squared magnitudes of these N_1 representation coefficients at each time t, as guaranteed by Cauchy-Schwarz inequality similar to Case 2 in Appendix A. This holds regardless of whether the offline-learned parameter $\hat{\alpha}$ converges to the true parameter α^* and provides theoretical validation that the multi-environment learning strategy, as used in [21], indeed enhances the generalization ability of the offline-learning phase.

Remark 3.12. Theorem 3.9 is established based on (2.5) in Lemma 2.9. Analogous result can be derived from (2.6) with initial weights $w_{0,i} = \frac{1}{N_2}$, which takes the form of a modified version of (3.9) wherein the term $\frac{B^2}{T}$ in J_{est} is replaced by $\frac{\min_{1 \leq i \leq N_2} \mathbb{E}\|\hat{\beta}_{0,i} - \beta_0\|^2}{4T} (\leq \frac{B^2}{T})$, together with the inclusion of an additional term $\frac{\log N_2}{\lambda T}$. As shown in FIG. 1 of Section 5, employing multiple models significantly enhances the transient response of the system without compromising prediction accuracy. And, for regression vectors of the form $x_t = (y_t, \cdots, y_{t-p+1}, u_t, \cdots, u_{t-q+1})^\tau$, analogous conclusions apply *mutatis mutandis* to cases with model-specific regression vectors $x_{t,i}$, including scenarios where the feedback input u_t depends on current parameter estimates, as discussed in [6], and the proof in Section 4.1 remains valid when substituting x_t with $x_{t,i}$, e.g., [35]. We further emphasize that Algorithm 2.1 and its analysis can be generalized to settings with random β_t, as established in [6]. Moreover, as is pointed out in Remark 3.3, for many dynamical systems, the KL divergence term in J_{mis}, i.e., $\frac{D(P_T\|P_T')}{T^{1-b_2'}}$, which quantifies the distribution shift, can be further characterized by parameter drift terms including the source-to-target parameter drift terms $L_1, L_{0,T}$, and the time-varying parameter-induced drift term δ_T, and thus can become particularly small in certain cases.

Remark 3.13. Under conditions of no parameter drift or disturbance, i.e., $\delta_T = 0, \varepsilon_t = 0$ and $w_{t+1} = 0$, our projected LMS algorithm guarantees bounded regret, as shown in (4.1) in the proof of Theorem 3.9. In contrast with the related result in Theorem 1 of [33], under the same conditions, [33] yields a dynamic regret of order $O(T)$ for any fixed hyperparameter. The fundamental reason for this difference lies in the distinct Lyapunov functions used in the analysis. Moreover, following the ideas in [33], we can mitigate the trade-off arising from the coefficient N_d and the noise term coefficient as functions of d. By incorporating multiple values of design parameters d_i in the meta-algorithm, a tighter upper bound on the prediction error J_T can be derived by using the Lyapunov function in the proof of our Theorem 3.9 in Section 4.1.

We now show that as $T \to \infty$, the average prediction error will be closed to the optimal value under some typical situations including the case where the disturbance is a martingale difference sequence.

THEOREM 3.14. *Under the assumptions of Theorem 3.9, suppose further that the noise w_{t+1} is a bounded martingale difference sequence with respect to $\mathcal{F}_t'$ and that L_1 is known. If $\delta_T^2, L_{0,T} \to 0$, and if either $L_1 = 0$ or $L_1 > 0$ with $\frac{D(P_T\|P_T')}{T^{1-b_2'}} \to 0$ as $T \to \infty$, then for any*

$\varepsilon > 0$, by choosing $\varepsilon^* < \frac{\varepsilon}{320L_1}$ (or $\varepsilon^* < \infty$ when $L_1 = 0$) and $d > \max\{\frac{A^2}{(1-\gamma)^2}, \frac{8A^2W_{\max}^2}{\varepsilon}\}$, the asymptotic prediction error is bounded as follows:

$$(3.6) \qquad\qquad \sigma_{\inf}^2 \leq \liminf_{T \to \infty} J_T \leq \limsup_{T \to \infty} J_T \leq \sigma_{\sup}^2 + \varepsilon,$$

where $\mathcal{F}_t' \triangleq \sigma\{\hat{\alpha}, x_0, x_1, ..., x_t, w_1, ..., w_t\}$, $\sigma_{\inf}^2 \triangleq \liminf_{T \to \infty} \sigma_T^2$ and $\sigma_{\sup}^2 \triangleq \limsup_{T \to \infty} \sigma_T^2$.

Remark 3.15. Following the same approach as in the proof of Theorem 3.14 in Section 4.2, if we additionally assume that either $L_1 = 0$ or $L_1 > 0$ with $\varepsilon^* = 0$ and δ_t decays at a known rate, then the optimal prediction performance(i.e., $\varepsilon = 0$) can be guaranteed by designing a time-varying hyperparameter $d_t = o(\min\{\frac{1}{\delta_t}, t\})$ and $d_t \to \infty$ instead of a fixed d.

3.3. Further discussions. Based on the above theorems, let us explain why this two-stage learning method outperforms either purely offline or purely online methods. The simple reason is that this two-stage combined method shares the advantages of both offline and online learning phases. To the best of the authors' knowledge, this natural idea has rarely been explored and elaborated mathematically in theory before.

(i) Why the online-adaptation phase is necessary? Relying solely on offline-learning without online-adaptation typically fails to achieve good prediction performance in real-world systems. More precisely, compared with the relatively ideal source system in the offline-learning phase, the real-world target system may contain unknown changed and changing parameters, besides the negative influence due to the possible non-convergence of the $\hat{\alpha}$ learned in the offline phase. Luckily, as is pointed out in Chapter 6 of [9], we are capable of addressing these uncertainties using the projected LMS. In contrast, if we do not adaptively adjust the $\hat{\beta}_t$, and instead directly use the parameters β_t^0 in the offline-learning phase, the errors caused by the model mismatch and disturbance w_t in the system will accumulate(see, FIG. 1-Fixed Parameters). Along with the simulation result, we next provide a theoretical example where the average prediction error J_T becomes unbounded without the online-adaptation phase. Suppose that all the vectors are 1-dimensional for convenience, and that $\beta_t^0 \equiv \beta_0 \equiv \beta > 0$, $\beta_{t+1} = \beta_t + \frac{1}{t+1}$, and $0 < \phi_0 \leq \mathbb{E}\phi_t^2(\alpha^*, x_t) \leq \phi_1 < \infty$ for all $t \geq 0$. Additionally, we assume $\hat{\alpha}$ is the NLS estimator of α^*(i.e., $\varepsilon^* = 0$), the training and new regressor sequences are identically distributed, and Assumption 2.2 holds. In this case, without online-adaptation, even if the true value α^* is known and no external disturbances are present, the prediction error J_T can be lower-bounded as: $\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(\beta_t\phi_t(\alpha^*, x_t) - \beta\phi_t(\alpha^*, x_t))^2 \geq \frac{1}{T}\sum_{t=1}^{T-1}(1 + \frac{1}{2} + \cdots + \frac{1}{t})^2\phi_0 \to \infty$, as $T \to \infty$. In contrast, if online-adaptation is performed in this case, it is straightforward to verify that $\delta_T^2 = O(\frac{1}{T})$ and $B = O(\log T)$. And, since $\beta_t = (1 + \frac{1 + \frac{1}{2} + \cdots + \frac{1}{t}}{\beta})\beta$, by Case 2 in Appendix A, we have $L_1 = O(\log T), L_0(t) \equiv 0$. Therefore, Theorem 3.9 implies that as $T \to \infty$, J_T will be bounded by $O(\sigma_T^2)$ and the "O" constant depends on d only.

(ii) Why the offline-learning phase is important? The offline-learning phase enables a reduction in model uncertainty and an improvement in computational efficiency by using historical datasets. More precisely: **a).** A well-estimated $\hat{\alpha}$ obtained offline encapsulates valuable prior knowledge that, when integrated into the multi-model meta-algorithm, may improve transient performance particularly when $\hat{\beta}_{0,i}^\tau \phi_0(\hat{\alpha}, x_0) \approx \beta_0^\tau \phi_0(\alpha^*, x_0)$. This stands in contrast to most purely online adaptation methods that typically suffer from large transient errors due to arbitrary initialization(e.g., [35]), as confirmed in FIG. 1-Single Model. **b).** When the dimension of the unknown parameter α^* is quite large, the importance of its offline-learning becomes more apparent, since it can help to reduce the computational burden and increase the flexibility of adaptation in the online-adaptation phase. **c).** Once we can obtain historical datasets generated from multiple system trajectories/environments, offline learning overcomes the known limitations of single-trajectory system identification (e.g., [31]) by leveraging multi-trajectory datasets. By ensuring a sufficiently large number of experimental trials N_1, this approach, as formalized in Generalization Lemma 3.1, is designed to

handle cases where the training data $\hat{x}_{0,T-1,i}$ from any trajectory violate Assumption 2.2, i.e., $b_2 > 1$.

4. Proofs of the main theorems. In this section, we present proofs of our main theorems(Theorems 3.9 and 3.14) which are based on [6, 9]. We first prove Lemma 2.9 based on the ideas of [27, 1, 33].

Proof of Lemma 2.9. By boundedness of β_t, $\phi(\hat{\alpha}, x_t)$, $\beta_t^\tau \phi(\alpha^*, x_t)$ and w_{t+1} and Lemma 2 in [27], it follows that $(y_{t+1} - y_{t+1}^{\text{pred}})^2 \le (M_f + AB + W_{\max})^2$ is bounded and is λ-exp-concave for any $0 < \lambda < \frac{1}{2(M_f + AB + W_{\max})^2}$. Therefore, by the definition $y_{t+1}^{\text{pred}} = \sum_{i=1}^{N_2} w_{t,i} y_{t+1,i}^{\text{pred}}$ in Algorithm 2.1 and Jensen inequality,

$$\exp(-\lambda S_T) \ge \exp(-\lambda S_{T-1}) \sum_{i=1}^{N_2} w_{T-1,i} \exp(-\lambda s_{T-1,i})$$

$$\ge \exp(-\lambda S_{T-2}) \sum_{i=1}^{N_2} w_{T-2,i} \exp[-\lambda(s_{T-1,i} + s_{T-2,i})]$$

$$\ge \cdots \ge \sum_{i=1}^{N_2} w_{0,i} \exp\left(-\lambda \sum_{t=0}^{T-1} s_{t,i}\right),$$

where $S_t \triangleq \sum_{j=0}^{t-1}(y_{j+1} - y_{j+1}^{\text{pred}})^2$, $s_{t,i} \triangleq (y_{t+1} - y_{t+1,i}^{\text{pred}})^2$ and $S_{T,i} \triangleq \sum_{t=0}^{T-1} s_{t,i}$, and the above inequalities hold due to the definition of the weights for $j = 1, 2, \cdots, T-1$:

$$w_{T-j,i} = \frac{w_{T-j-1,i} \exp(-\lambda s_{T-j-1})}{\sum_{i=1}^{N_2} w_{T-j-1,i} \exp(-\lambda s_{T-j-1})}.$$

Furthermore, since $\sum_{i=1}^{N_2} w_{0,i} \exp(-\lambda S_{T,i}) \ge \min_{1 \le i \le N_2} \{\exp(-\lambda S_{T,i})\}$, it is not difficult to see that:

$$\frac{1}{T} \sum_{t=0}^{T-1}(y_{t+1} - y_{t+1}^{\text{pred}})^2 \le \max_{1 \le i \le N_2} \left\{\frac{1}{T} \sum_{t=0}^{T-1}(y_{t+1} - y_{t+1,i}^{\text{pred}})^2\right\}.$$

Similarly, since $\sum_{i=1}^{N_2} w_{0,i} \exp(-\lambda S_{T,i}) \ge \max_{1 \le i \le N_2} \{w_{0,i} \exp(-\lambda S_{T,i})\}$, we have:

$$\frac{1}{T} \sum_{t=0}^{T-1}(y_{t+1} - y_{t+1}^{\text{pred}})^2 \le \min_{1 \le i \le N_2} \left\{\frac{1}{T} \sum_{t=0}^{T-1}(y_{t+1} - y_{t+1,i}^{\text{pred}})^2 + \frac{1}{\lambda T} \log \frac{1}{w_{0,i}}\right\}.$$

Taking expectations yields (2.6) immediately from the above inequality.

4.1. Proof of Theorem 3.9. For $i = 1, 2, \cdots, N_2$ and $k \ge 0$, define the Lyapunov function as:

$$V_{k,i} = \|\beta_k - \hat{\beta}_{k,i}\|^2 \triangleq \|\tilde{\beta}_{k,i}\|^2 \le 4B^2,$$

and denote $\varphi_k \triangleq \phi_k(\hat{\alpha}, x_k)$, $\delta_k \triangleq \|\beta_{k+1} - \beta_k\|$.

Since $\|\varphi_k\| \le A$ and $m_k = \gamma m_{k-1} + \|\varphi_k\|$, we have $\|\varphi_k\|^2 \le m_k^2 \le \frac{A^2}{(1-\gamma)^2}$.

By boundedness of $\beta_k, \hat{\beta}_{k+1,i}$,

$$V_{k+1,i} = \|\beta_{k+1} - \hat{\beta}_{k+1,i}\|^2$$

$$= \|\beta_k + (\beta_{k+1} - \beta_k) - \hat{\beta}_{k+1,i}\|^2$$

$$\le \|\beta_k - \hat{\beta}_{k+1,i}\|^2 + 4B\delta_k + \delta_k^2,$$

furthermore, since $\beta_k \in D$, by properties of projection,

$$\|\beta_k - \hat{\beta}_{k+1,i}\|^2$$

$$\leq \left\| \left(I - \frac{\varphi_k \varphi_k^\tau}{d + m_k^2} \right) \tilde{\beta}_{k,i} - \frac{\varphi_k w_{k+1} + \varphi_k \varepsilon_k}{d + m_k^2} \right\|^2$$

$$\leq \|\tilde{\beta}_{k,i}\|^2 - \left(2 - \frac{A^2}{d + A^2} \right) \frac{(\tilde{\beta}_{k,i}^\tau \varphi_k)^2}{d + m_k^2} + 2 \frac{|\tilde{\beta}_{k,i}^\tau \varphi_k| \cdot |w_{k+1} + \varepsilon_k|}{d + m_k^2} + \frac{A^2 (w_{k+1} + \varepsilon_k)^2}{d^2}$$

$$\leq \|\tilde{\beta}_{k,i}\|^2 - \left(1 - \frac{A^2}{d + A^2} \right) \frac{(\tilde{\beta}_{k,i}^\tau \varphi_k)^2}{d + m_k^2} + \left(\frac{A^2}{d} + 1 \right) \frac{(w_{k+1} + \varepsilon_k)^2}{d}$$

$$= V_{k,i} - \left(1 - \frac{A^2}{d + A^2} \right) \frac{(\tilde{\beta}_{k,i}^\tau \varphi_k)^2}{d + m_k^2} + \left(1 + \frac{A^2}{d} \right) \frac{(w_{k+1} + \varepsilon_k)^2}{d},$$

where the last inequality holds due to $2|x\|y| \leq x^2 + y^2$.

Therefore,

$$\left(1 - \frac{A^2}{d + A^2} \right) \frac{(\tilde{\beta}_{k,i}^\tau \varphi_k)^2}{d + m_k^2} \leq V_{k,i} - V_{k+1,i} + \left(1 + \frac{A^2}{d} \right) \frac{(w_{k+1} + \varepsilon_k)^2}{d} + 4B\delta_k + \delta_k^2.$$

Since $V_{0,i} = \|\beta_0 - \hat{\beta}_{0,i}\|^2 \leq 4B^2$, by summing up from 0 to $T - 1$, we can conclude that

$$V_{T,i} + \frac{d}{d + A^2} \sum_{t=0}^{T-1} \frac{(\tilde{\beta}_{t,i}^\tau \varphi_t)^2}{d + m_t^2} \leq \frac{A^2 + d}{d^2} \sum_{t=0}^{T-1} (w_{t+1}^2 + 2\varepsilon_t w_{t+1} + \varepsilon_t^2) + 4B \sum_{t=0}^{T-1} \delta_t + \sum_{t=0}^{T-1} \delta_t^2 + 4B^2.$$

Take $d > \frac{A^2}{(1-\gamma)^2}$ to be sufficiently large. It follows that

$$(4.1) \quad \sum_{t=0}^{T-1} (\tilde{\beta}_{t,i}^\tau \varphi_t)^2 \leq \left(1 + \frac{A^2}{d} \right)^2 \left(1 + \frac{A^2}{(1-\gamma)^2 d} \right) \sum_{t=0}^{T-1} \left(\left(1 + \frac{1}{d} \right) w_{t+1}^2 + (1+d)\varepsilon_t^2 \right)$$
$$+ 16dBT\delta_T + 4dT\delta_T^2 + 16dB^2.$$

Note that the above inequality holds for all $i = 1, 2, \cdots, N_2$. By (2.5) in Lemma 2.9, the average prediction error can be upper bounded as follows:

$$\frac{1}{T} \sum_{t=0}^{T-1} (y_{t+1} - y_{t+1}^{\text{pred}})^2$$

$$\leq \max_{1 \leq i \leq N_2} \left\{ \frac{1}{T} \sum_{t=0}^{T-1} \left(\beta_t^\tau \phi_t(\hat{\alpha}, x_t) + \varepsilon_t + w_{t+1} - \hat{\beta}_{t,i}^\tau \phi_t(\hat{\alpha}, x_t) \right)^2 \right\}$$

$$\leq \frac{\sqrt{C_d} d + 1 + d}{T} \max_{1 \leq i \leq N_2} \left\{ \sum_{t=0}^{T-1} \left(\frac{1}{\sqrt{C_d} d} (\tilde{\beta}_{t,i}^\tau \varphi_t)^2 + \varepsilon_t^2 + \frac{1}{d} w_{t+1}^2 \right) \right\},$$

where $C_d \triangleq \left(1 + \frac{A^2}{d} \right)^2 \left(1 + \frac{A^2}{(1-\gamma)^2 d} \right) \left(1 + \frac{1}{d} \right) \geq 1$.

Substituting (4.1) into the above inequality and taking the expectation with respect to all randomness yields:

$$J_T \leq C_d'(B\delta_T + \delta_T^2 + \frac{B^2}{T}) + C_d'' \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\varepsilon_t^2 + \left(\sqrt{C_d} + 1 + \frac{1}{d} \right)^2 \sigma_T^2$$

$$\leq C_d'(B\delta_T + \delta_T^2 + \frac{B^2}{T}) + \left(\sqrt{C_d} + 1 + \frac{1}{d} \right)^2 \sigma_T^2$$

$$+ C_d'' L_1 \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(\beta_0^\tau(t)(\phi_t(\alpha^*, x_t) - \phi_t(\hat{\alpha}, x_t))^2 + C_d'' L_0(t).$$

Finally, by Inequality (2.4) and Generalization Lemma 3.1, we can get the desired result by direct calculation.

4.2. Proof of Theorem 3.14. Define the Lyapunov function as $V_{k,i} = \mathbb{E}\|\beta_k - \hat{\beta}_{k,i}\|^2$. Since $\varepsilon_k, m_k^2, \varphi_k \in \mathcal{F}_k'$, similar to the proof of Theorem 3.9, we have

$$V_{k+1,i}$$
$$\leq \mathbb{E}\|\beta_k - \hat{\beta}_{k+1,i}\|^2 + 4B\delta_k + \delta_k^2$$
$$\leq \mathbb{E}\left\|\left(I - \frac{\varphi_k\varphi_k^\tau}{d + m_k^2}\right)\tilde{\beta}_{k,i} - \frac{\varphi_k(w_{k+1} + \varepsilon_k)}{d + m_k^2}\right\|^2 + 4B\delta_k + \delta_k^2$$
$$\leq \mathbb{E}\|\tilde{\beta}_{k,i}\|^2 - (2 - \frac{A^2}{d + A^2})\mathbb{E}\frac{(\tilde{\beta}_{k,i}^\tau\varphi_k)^2}{d + m_k^2} + 2\mathbb{E}\frac{|\tilde{\beta}_{k,i}^\tau\varphi_k| \cdot |\varepsilon_k|}{d + m_k^2} + \mathbb{E}\frac{A^2(w_{k+1}^2 + \varepsilon_k^2)}{d^2} + 4B\delta_k + \delta_k^2$$
$$\leq V_{k,i} - (1 - \frac{A^2}{d + A^2})\mathbb{E}\frac{(\tilde{\beta}_{k,i}^\tau\varphi_k)^2}{d + m_k^2} + \frac{d + A^2}{d^2}\mathbb{E}\varepsilon_k^2 + 4B\delta_k + \delta_k^2 + \frac{A^2}{d^2}\mathbb{E}w_{k+1}^2,$$

where the third inequality holds due to the martingale difference property of w_{k+1}.

Summing up from $k = 0$ to $T - 1$ yields:

$$\frac{d}{d + A^2}\mathbb{E}\sum_{t=0}^{T-1}\frac{(\tilde{\beta}_{t,i}^\tau\varphi_t)^2}{d + m_t^2} \leq \frac{d + A^2}{d^2}\mathbb{E}\sum_{t=0}^{T-1}\varepsilon_t^2 + 4BT\delta_T + T\delta_T^2 + \frac{A^2}{d^2}TW_{\max}^2 + 4B^2.$$

Since $d > \frac{A^2}{(1-\gamma)^2} \geq m_t^2$, we have

(4.2)
$$\sum_{t=0}^{T-1}\mathbb{E}(\tilde{\beta}_{t,i}^\tau\varphi_t)^2 \leq 4\sum_{t=0}^{T-1}\mathbb{E}\varepsilon_t^2 + 8(d + A^2)BT\delta_T + 2(d + A^2)T\delta_T^2 + 16dB^2$$
$$+ \frac{4A^2}{d}TW_{\max}^2.$$

By (2.6) in Lemma 2.9, the average prediction error can be upper bounded as follows:

$$J_T \leq \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(\beta_t^\tau\phi_t(\hat{\alpha}, x_t) + \varepsilon_t + w_{t+1} - \hat{\beta}_{t,i}^\tau\phi_t(\hat{\alpha}, x_t))^2 + \frac{\log\frac{1}{w_{0,i}}}{\lambda T}$$
$$= \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(\tilde{\beta}_{t,i}^\tau\phi_t(\hat{\alpha}, x_t) + \varepsilon_t)^2 + \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}w_{t+1}^2 + \frac{\log\frac{1}{w_{0,i}}}{\lambda T}$$
$$\leq \frac{2}{T}\sum_{t=0}^{T-1}\mathbb{E}(\tilde{\beta}_{t,i}^\tau\phi_t(\hat{\alpha}, x_t))^2 + \frac{2}{T}\sum_{t=0}^{T-1}\mathbb{E}\varepsilon_t^2 + \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}w_{t+1}^2 + \frac{\log\frac{1}{w_{0,i}}}{\lambda T}.$$

Since $\varepsilon^* \leq \frac{\varepsilon}{320L_1}$ and $d > \max\{\frac{A^2}{(1-\gamma)^2}, \frac{8A^2W_{\max}^2}{\varepsilon}\}$, by (4.2) and Generalization Lemma 3.1, it follows that Theorem 3.14 holds.

5. Numerical Simulation. In this section, simulation experiments are conducted to demonstrate that adopting a meta prediction strategy during the online-adaptation phase is beneficial for reducing the prediction error. Furthermore, we demonstrate that, the proposed two-stage offline-online prediction approach maintains satisfactory performance even when the parameter $\hat{\alpha}$ identified during the initial offline stage deviates considerably from the true value α^*.

Consider the real-world target system:

$$y_{t+1} = a_t\sigma_t(bx_t + c) + d_t + w_{t+1},$$

where $\sigma_t(x) = \frac{1}{t + e^{-x}}$ and $W_t \overset{\text{i.i.d.}}{\sim} N(0,1)$. $a_t = -50\sigma_t(t) + \frac{1}{t^2}$, $d_t = 15\sigma_t(t) + \frac{2}{\log(t+1)}$, $b = 1.5$, $c = -0.5$ and a_t, b, c, d_t are unknown beforehand.

The regression vector sequence $\{x_t\}$ is defined as follows:

$$x_{t+1} = 100\sigma_t(x_t) + w_{t+1}', \quad x_0 \sim N(10, 1),$$

where w'_{t+1} is independent of w_{t+1} and is standard normal.

For simplicity, let $N_1 = 1$. Parameter estimates of (b, c) are first obtained via offline learning from a dataset generated by the source system. Then, these estimated parameters are fixed in the target system, where Algorithm 2.1 performs continual adaptation of the remaining unknown drifting parameters in an online manner. The perform index of our algorithm in this simulation is the average prediction error:

$$(5.1) \qquad J_T \triangleq \frac{1}{T} \sum_{t=0}^{T-1} (y_{t+1} - y_{t+1}^{\mathrm{pred}})^2.$$

Offline-learning Phase: Consider the source system in the offline-learning phase as:

$$y_{t+1}^{off} = a_t^{off} \sigma_t(b x_t^{off} + c) + d_t^{off} + w_{t+1}^{off},$$

where W_t^{off} is independent and identical distributed of W_t, $a_t^{off} \equiv 20$, $b_t^{off} \equiv -10$, and the training dataset $\{x_t^{off}\}$ is defined as follows:

$$x_{t+1}^{off} = 100 \sigma_t(x_t^{off}) + w_{t+1}^{off'}, \quad x_0^{off} \sim N(0, 1),$$

where $w_{t+1}^{off'}$ is independent of other random variables and is standard normal.

We then use grid-search in Matlab to get an estimate of (b, c) through least squares: the search domain $\mathcal{M}$ is $[-3.5, 6.5] \times [-5.5, 4.5]$, and each dimension is uniformly divided into 50 segments.

Online-adaptation Phase: Fixing the parameters obtained via offline learning and setting $N_2 = 500, \lambda = 10^{-3}, d = 10^3$ and $D = \{(a, d) | a^2 + d^2 \le 10^7\}$, we initialize $w_{0,i} = \frac{1}{500}$, $\hat{\beta}_{0,1} = [10, -10]^\tau$ and $\hat{\beta}_{0,i} \sim N(-3, 1)$, for $i \ge 2$. Algorithm 2.1 is subsequently employed to fine-tune the time-varying parameters.
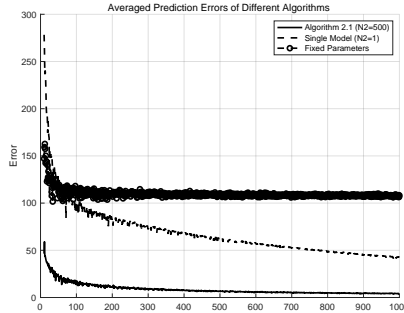

Fig. 1. *Average prediction error*

The experimental results demonstrate that the meta approach in Algorithm 2.1 outperforms the single-model method and reduces prediction error(see, FIG. 1), even when the parameter estimate $(\hat{b}, \hat{c})$ fails to converge(see, FIG. 2). Moreover, although a poor initialization leads to a large error in the initial transient response, applying online-adaptation in the second step eventually yields better performance than without adjustment(see, FIG. 1-Single Model and Fixed Parameters).

6. Concluding Remarks. In this paper, we have investigated the prediction performance of a two-stage learning framework combining offline learning and online adaptation for a class of nonlinear stochastic dynamical systems. For the offline phase, we derived a generalization error bound for approximate nonlinear-least-squares estimation under data scenarios involving strong correlation and distribution shift, showing that the generalization error can be quite small under some typical situations. For the online phase, we addressed
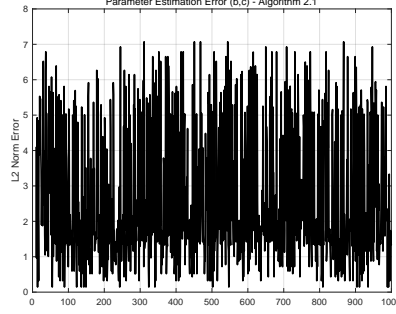
FIG. 2. *Parameter estimation error of (b, c) in Algorithm 2.1*

potential parameter drift in the real-world system by proposing a meta-LMS prediction algorithm that builds upon the offline-trained model. To the best of the authors' knowledge, this paper appears to be the first to demonstrate in theory that this framework can synthesize the complementary strengths of both offline and online learning, resulting in superior prediction performance compared with purely offline or purely online methods. Notably, our results apply under the more general condition of a potentially biased offline estimator. This study confirms that in safety-critical dynamic environments, the integration of offline learning and online adaptation is of paramount importance. It establishes a key theoretical foundation for building reliable learning systems that can withstand various uncertainties, including parameter drift, distribution shift, and external disturbances, etc. Fundamental challenges remain, including developing weaker assumptions on data dependencies and integrating offline reinforcement learning policies into online adaptive control for enhanced performance.

Appendix A. Remarks on Inequality (2.4). To further refine the upper bound in Theorem 3.9 by selecting appropriate L_1 and $L_0(t)$, let us discuss three cases below:

Case 1: $\|\beta_t \beta_t^\tau - K_t \beta_t^0 \beta_t^{0\tau}\| \le B_t$, K_t is scalar, and $\mathbb{E}\|\phi_t(\alpha^*, x_t) - \phi_t(\hat{\alpha}, x_t)\|^2 \le M_t$, for all $t \ge 0$. In this case, $\lambda_{\max}(\beta_t \beta_t^\tau - K_t \beta_t^0 \beta_t^{0\tau}) \le \|\beta_t \beta_t^\tau - K_t \beta_t^0 \beta_t^{0\tau}\| \le B_t$, thus, by triangle inequality, Inequality (2.4) holds provided that $L_1 \ge \max_{0 \le t \le T-1} |K_t|$, $L_0(t) \ge B_t M_t$.

Case 2: $\|\beta_t - V_t \beta_t^0\| \le B_t$, where $V_t \in \mathbb{R}^{m \times m}$ and $\mathbb{E}\|\phi_t(\alpha^*, x_t) - \phi_t(\hat{\alpha}, x_t)\|^2 \le M_t$, for all $t \ge 0$. Denote $V_t'(k) \triangleq V_t \beta_t^0 \beta_t^{0\tau} V_t^\tau - k \beta_t^0 \beta_t^{0\tau} \in \mathbb{R}^{m \times m}$ and its spectral radius $\rho_t(k) \triangleq \rho(V_t'(k))$ for $k \ge 0$. Since $\lambda_{max}(\beta_t \beta_t^\tau - V_t \beta_t^0 \beta_t^{0\tau} V_t^\tau) \le \|\beta_t \beta_t^\tau - V_t \beta_t^0 \beta_t^{0\tau} V_t^\tau\| \le B_t(\|\beta_t\| + \|V_t \beta_t^0\|)$, similar to Case 1, by triangle inequality, a feasible choice satisfying Inequality (2.4) is $L_1 = k$, $L_0(t) = [\rho_t(k) + B_t(\|\beta_t\| + \|V_t \beta_t^0\|)]M_t$.

Since the offset term $L_0(t)$ can not be eliminated from the finial prediction error(see Theorem 3.9), we then attempt to determine an k such that $\rho_t(k) \ge 0$ is the smallest:

(i) β_t^0 is an eigenvector of V_t corresponding to an eigenvalue λ_t. In this case, $\rho_t(|\lambda_t|^2) = 0$ is minimal. Thus $L_1 = \max_{0 \le t \le T-1} |\lambda_t|^2$, $L_0(t) = B_t M_t(\|\beta_t\| + \|V_t \beta_t^0\|)$.

Furthermore, as a special case of (i), if $\beta_{t,i} = \beta_{t,i}^0$, $i \ne i_0$, $|\beta_{t,i_0} - \beta_{t,i_0}^0| \le b_t$ and $|\phi_t(\alpha^*, x_t)_{i_0} - \phi_t(\hat{\alpha}, x_t)_{i_0}|^2 \le m_t, a.s.$, for all $t \ge 0$, then by triangle inequality we can conclude that $|\beta_t^\tau(\phi_t(\alpha^*, x_t) - \phi_t(\hat{\alpha}, x_t))| \le |\beta_t^{0\tau}(\phi_t(\alpha^*, x_t) - \phi_t(\hat{\alpha}, x_t))| + b_t m_t$. Thus by Cauchy-Schwarz inequality, it follows that $L_1 = 2, L_0(t) = 2b_t^2 m_t$ and it suffices to fine-tune parameter β_{t,i_0} without adjusting the entire m elements of β_t by refining $y'_{t+1} \triangleq y_{t+1} - \sum_{i \ne i_0} \beta_{t,i}^0 \phi_i(\hat{\alpha}, x_t)$ in the online-adaptation phase.

(ii) $V_t \beta_t^0$ and β_t^0 are linearly independent, thus $\rho_t(k) > 0$ for all $k \ge 0$. In this case, the spectral radius of $V_t'(k)$ coincides with that of its restriction to $S_t \triangleq span\{\beta_t^0, V_t \beta_t^0\}$. Let $e_1 = \frac{\beta_t^0}{\|\beta_t^0\|}, e_2 = \frac{V_t \beta_t^0 - (\beta_t^{0\tau} V_t^\tau e_1) e_1}{\|V_t \beta_t^0 - (\beta_t^{0\tau} V_t^\tau e_1) e_1\|}$ form an orthonormal basis of S_t. Then the matrix representation of $V_t'(k)$ with respect to $\{e_1, e_2\}$ is $\begin{pmatrix} a_t^2 r^2 - k b_t^2 & a_t^2 r_t \sqrt{1 - r_t^2} \\ a_t^2 r_t \sqrt{1 - r_t^2} & a_t^2(1 - r_t^2) \end{pmatrix}$, where

$a_t \triangleq \|V_t \beta_t^0\|, b_t \triangleq \|\beta_t^0\|$ and $r_t \triangleq \frac{\beta_t^{0\tau} V_t \beta_t^0}{ab} < 1$. A direct calculation shows that the spectral radius of this 2×2 matrix, when minimized over $k \geq 0$, achieves its minimum at the value $\frac{a_t^2}{2}\left(1 - r_t^2 + \sqrt{(1 - r_t^2)(1 + 3r_t^2)}\right)$ at $k = \frac{a_t^2 r_t^2}{b_t^2} \geq 0$.

Case 3: It follows that under additional assumptions, such as the "excitation" conditions in Corollary 3.4, the nonlinear-least-squares estimate $\hat{\alpha}$ obtained from the offline phase will converge to the true parameter α^* in the mean square sense as $T \to \infty$. Assume that $\beta_t^\tau \phi_t(\alpha, x)$ is uniformly continues with respect to α. By boundedness of $\phi_t(\hat{\alpha}, x_t), \beta_t$ and $\beta_t^\tau \phi_t(\alpha^*, x_t)$, we can conclude that $\mathbb{E}\left(\beta_t^\tau\left(\phi_t(\alpha^*, x_t) - \phi_t(\hat{\alpha}, x_t)\right)\right)^2 \to 0$, i.e., $L_1 = 0, L_0(t) \to 0$.

Appendix B. Proof of Generalization Lemma 3.1.

Following the approaches in [25, 37, 15], we first prove Generalization Lemma 3.1. And the proof begins with two key lemmas as follows:

LEMMA B.1 (see [30, 37]). *Let P and Q be two probability measures defined on a common measurable space $(\Omega, \mathcal{F})$, with $P \ll Q$(i.e., P is absolutely continuous with respect to Q) and let $F : \Omega \to \mathbb{R}$ be a measurable function such that $\mathbb{E}_Q[e^F] < \infty$.*
Then, the following inequality holds:

$$(B.1) \qquad \mathbb{E}_P[F] - \log \mathbb{E}_Q[e^F] \leq D(P\|Q).$$

LEMMA B.2 (see [25, 22, 38]). *For each t, let $g_t : \mathcal{M} \times \mathbb{X}^p \to \mathbb{R}$ be such that $0 \leq g_t(\alpha, x) \leq C$, for all $(\alpha, x) \in \mathcal{M} \times \mathbb{X}^p$, for some $C > 0$. Then, for any $\lambda > 0$ and $\alpha \in \mathcal{M}$ we have:*

$$(B.2)$$
$$\mathbb{E}\exp\left(-\lambda \sum_{t=0}^{T-1} g_t(\alpha, x_t)\right) \leq \exp\left(-\lambda \sum_{t=0}^{T-1} \mathbb{E}g_t(\alpha, x_t) + \frac{\lambda^2 \|\Gamma_{dep}(P_T)\|^2}{2} \sum_{t=0}^{T-1} \mathbb{E}g_t^2(\alpha, x_t)\right).$$

Proof of Generalization Lemma 3.1: For the new data $x_{0:T-1}$, denote

$$0 \leq g_t(\alpha, x_t) \triangleq \|f_t(\alpha, \beta_0(t), x_t) - f_t(\alpha^*, \beta_0(t), x_t)\|^2 \leq 4L^2 R_{\mathcal{M}}^2.$$

Take $F = -\lambda \sum_{t=0}^{T-1} g_t(\alpha, x_t)$ with $\lambda > 0$ to be determined later. By Lemma B.1, we have:

$$(B.3) \qquad \int -\lambda \sum_{t=0}^{T-1} g_t(\alpha, x_t) dP_T - \log \int e^{-\lambda \sum_{t=0}^{T-1} g_t(\alpha, x_t)} dP_T' \leq D(P_T \| P_T').$$

By Lemma B.2, for the second term on the LHS of (B.3), we have:

$$(B.4)$$
$$\log \mathbb{E}_{P_T'} \exp\left(-\lambda \sum_{t=0}^{T-1} g_t(\alpha, x_t)\right)$$
$$\leq \log\left(\exp\left(-\lambda \sum_{t=0}^{T-1} \mathbb{E}_{P_T'} g_t(\alpha, x_t) + \frac{\lambda^2 \|\Gamma_{dep}(P_T')\|^2}{2} \sum_{t=0}^{T-1} \mathbb{E}_{P_T'} g_t^2(\alpha, x_t)\right)\right)$$
$$\leq -\lambda \mathbb{E}_{P_T'} \sum_{t=0}^{T-1} g_t(\alpha, x_t) + \frac{\lambda^2 b_1' T^{b_2'}}{2} \cdot 4L^2 R_{\mathcal{M}}^2 \mathbb{E}_{P_T'} \sum_{t=0}^{T-1} g_t(\alpha, x_t).$$

Substituting (B.4) into (B.3) and dividing both sides by $\lambda > 0$ yields:

$$\mathbb{E}_{P_T'} \sum_{t=0}^{T-1} g_t(\alpha, x_t) \leq \frac{D(P_T \| P_T')}{\lambda} + 2\lambda b_1' T^{b_2'} L^2 R_{\mathcal{M}}^2 \mathbb{E}_{P_T'} \sum_{t=0}^{T-1} g_t(\alpha, x_t) + \mathbb{E}_{P_T} \sum_{t=0}^{T-1} g_t(\alpha, x_t).$$

Without loss of generality, assume $\mathbb{E}_{P'_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t) > 0$, as the conclusion holds trivially otherwise.

If $D(P_T \| P'_T) > 0$, by setting $\lambda = \sqrt{\frac{D(P_T \| P'_T)}{2b'_1 T^{b'_2} L^2 R_{\mathcal{M}}^2 \mathbb{E}_{P'_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t)}} > 0$, we then have:

$$
\begin{aligned}
& \mathbb{E}_{P'_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t) \\
\text{(B.5)} \quad & \leq \left(\sqrt{2b'_1 T^{b'_2} L^2 R_{\mathcal{M}}^2 D(P_T \| P'_T)} + \sqrt{\mathbb{E}_{P_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t) + 2b'_1 T^{b'_2} L^2 R_{\mathcal{M}}^2 D(P_T \| P'_T)} \right)^2 \\
& \leq 2\mathbb{E}_{P_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t) + 8b'_1 T^{b'_2} L^2 R_{\mathcal{M}}^2 D(P_T \| P'_T);
\end{aligned}
$$

else, $D(P_T \| P'_T) = 0$, we have $\mathbb{E}_{P_{T'}} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t) = \mathbb{E}_{P_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t)$, which demonstrates that (B.5) still holds.

Dividing both sides of (B.5) by T yields:

$$
\text{(B.6)} \qquad \frac{1}{T} \mathbb{E}_{P'_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t) \leq \frac{2}{T} \mathbb{E}_{P_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t) + \frac{8b'_1 L^2 R_{\mathcal{M}}^2 D(P_T \| P'_T)}{T^{1-b'_2}}.
$$

Next, based on the ideas of [25], we analyze the term $\frac{1}{T} \mathbb{E}_{P_T} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, x_t)$ in (B.6), where $x_{0:T-1}$ is independent and identically distributed to the training data $\hat{x}_{0:T-1,i}$.

By Lemma B.2, we have for any $\lambda > 0, \alpha \in \mathcal{M}$:

$$
\begin{aligned}
\mathbb{E} \exp\left(-\lambda \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\alpha, \hat{x}_{t,i}) \right) &= \prod_{i=1}^{N_1} \mathbb{E} \exp\left(-\lambda \sum_{t=0}^{T-1} g_t(\alpha, \hat{x}_{t,i}) \right) \\
&\leq \exp\left(-\lambda \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\alpha, \hat{x}_{t,i}) + \frac{\lambda^2 b_1 T^{b_2}}{2} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t^2(\alpha, \hat{x}_{t,i}) \right),
\end{aligned}
$$

thus, by Markov inequality,

$$
\begin{aligned}
& P\left(\frac{1}{T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\alpha, \hat{x}_{t,i}) \leq \frac{1}{2T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\alpha, \hat{x}_{t,i}) \right) \\
\text{(B.7)} \quad & \leq \inf_{\lambda > 0} P\left(\exp\left(-\lambda \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\alpha, \hat{x}_{t,i}) \right) \geq \exp\left(-\frac{\lambda}{2} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\alpha, \hat{x}_{t,i}) \right) \right) \\
& \leq \inf_{\lambda > 0} \frac{\mathbb{E}_{P_T} \exp\left(-\lambda \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\alpha, \hat{x}_{t,i}) \right)}{\exp\left(-\frac{\lambda}{2} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\alpha, \hat{x}_{t,i}) \right)} \leq \exp\left(-\frac{N_1 \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\alpha, \hat{x}_{t,1})}{32 b_1 T^{b_2} L^2 R_{\mathcal{M}}^2} \right).
\end{aligned}
$$

Denote $h_t(\alpha, \hat{x}_{t,i}) \triangleq f_t(\alpha, \beta_0(t), \hat{x}_{t,i}) - f_t(\alpha^*, \beta_0(t), \hat{x}_{t,i})$ and $h(\alpha, \hat{x}_{1,i}, \hat{x}_{2,i}, \cdots, \hat{x}_{T-1,i}) \triangleq (h_0(\alpha, \hat{x}_{0,i}), h_1(\alpha, \hat{x}_{1,i}), \cdots, h_{T-1}(\alpha, \hat{x}_{T-1,i}))$. Then $\|h_t\|^2 = g_t$.

Define the following function spaces:

$$
H^* = \{ \gamma h(\alpha, X) \mid \gamma \in [0,1], \alpha \in \mathcal{M}, X = (\hat{x}_{0,i}, \hat{x}_{1,i}, \cdots, \hat{x}_{T-1,i}) \in \mathbb{X}^{pT} \},
$$

$$
H_r^* = \{ \bar{h} \in H^* \mid \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \| \bar{h}_t(\alpha, \hat{x}_{t,i}) \|^2 \leq r^2 \},
$$

$$
\partial H_r^* = \{ \bar{h} \in H^* \mid \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \| \bar{h}_t(\alpha, \hat{x}_{t,i}) \|^2 = r^2 \}, \quad r \text{ to be determined later.}
$$

For any $\overline{h} \in H^*$, by (B.7), we have:

$$P\left(\frac{1}{T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|\overline{h}_t(\alpha, \hat{x}_{t,i})\|^2 \le \frac{1}{2T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|\overline{h}_t(\alpha, \hat{x}_{t,i})\|^2 \right)$$

$$= P\left(\frac{1}{T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|h_t(\alpha, \hat{x}_{t,i})\|^2 \le \frac{1}{2T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,i})\|^2 \right)$$

$$\le \exp\left(-\frac{N_1 \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,1})\|^2}{32 b_1 T^{b_2} L^2 R_{\mathcal{M}}^2} \right)$$

$$\le \exp\left(-\frac{N_1 \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|\overline{h}_t(\alpha, \hat{x}_{t,1})\|^2}{32 b_1 T^{b_2} L^2 R_{\mathcal{M}}^2} \right).$$

Let N_r be the $\frac{r}{\sqrt{8}}$ - covering set of ∂H_r^* in L^∞ sense with minimal cardinality. Specifically, a $\frac{r}{\sqrt{8}}$ - covering set is a set such that for any $\overline{h} \in \partial H_r^*$, there exists $\overline{g} \in N_r \subset \partial H_r^*$, satisfying $\|\overline{h} - \overline{g}\|_\infty = \max_{0 \le t \le T-1} \|\overline{h}_t - \overline{g}_t\|_\infty \le \frac{r}{\sqrt{8}}$. Moreover, among all sets possessing this covering property, N_r has minimal cardinality.

By definition of H^*, we have for any $\overline{h} \in N_r$, there exists $\overline{\gamma} \in [0,1], \overline{\alpha} \in \mathcal{M}$, s.t. $\overline{h} = \overline{\gamma} h(\overline{\alpha}, X)$.

Denote the event:

$$\epsilon = \bigcup_{\overline{\gamma} h(\overline{\alpha}, X) \in N_r} \left\{ \frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \overline{\gamma}^2 \|h_t(\overline{\alpha}, \hat{x}_{t,i})\|^2 \le \frac{1}{2T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} (\overline{\gamma}^2 \|h_t(\overline{\alpha}, \hat{x}_{t,1})\|^2) \right\}.$$

We then have:

$$P(\epsilon) \le |N_r| \exp\left(-\frac{N_1 r^2 T^{1-b_2}}{32 b_1 L^2 R_{\mathcal{M}}^2} \right).$$

Similar to (40) of [25], $|N_r|$ is bounded when f_t is L-Lipschitz with respect to α as follows:

$$|N_r| \le N(\frac{r}{\sqrt{8}}, H^*, \|\cdot\|_\infty) \le \frac{8\sqrt{8} L R_{\mathcal{M}}}{r} \left(\frac{6\sqrt{8} L R_{\mathcal{M}}}{r} \right)^n \triangleq C(L, n, R_{\mathcal{M}}) \frac{1}{r^{n+1}}$$

Therefore,

$$P(\epsilon) \le C(L, n, R_{\mathcal{M}}) \frac{1}{r^{n+1}} \exp\left(-\frac{N_1 r^2 T^{1-b_2}}{32 b_1 L^2 R_{\mathcal{M}}^2} \right).$$

By the definition of N_r, for all $\overline{h} \in \partial H_r^*$, there exists $\overline{h}' \in N_r$, s.t. $\|\overline{h}(X) - \overline{h}'(X)\|_\infty \le \frac{r}{\sqrt{8}}$, for any $X \in \mathbb{X}^{pT}$, thus, by the above inequality and $2|x||y| \ge -2x^2 - \frac{1}{2} y^2$, we have $\|\overline{h}_t(\hat{x}_{t,i})\|^2 \ge \frac{\|\overline{h}_t'^2(\hat{x}_{t,i})\|^2}{2} - \frac{r^2}{8}$ for any $t = 0, 1, \cdots, T-1, i = 1, 2, \cdots, N_1$. Summing over i and t from 1 to N_1 and 0 to $T-1$, respectively, and then averaging, yields

$$\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|\overline{h}_t(\hat{x}_{t,i})\|^2 \ge \frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \frac{\|\overline{h}'(\hat{x}_{t,i})\|^2}{2} - \frac{r^2}{8}.$$

Furthermore, on ϵ^c, from the above inequality, we can conclude that:

$$\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|\overline{h}_t(\hat{x}_{t,i})\|^2 \ge \frac{r^2}{4} - \frac{r^2}{8} = \frac{r^2}{8}.$$

Therefore:

$$P\left(\inf_{\overline{h} \in \partial H_r^*} \left(\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|\overline{h}_t(\hat{x}_{t,i})\|^2 - \frac{r^2}{8} \right) \le 0 \right) \le P(\epsilon) \le C \frac{1}{r^{n+1}} \exp\left(-\frac{N_1 r^2 T^{1-b_2}}{32 b_1 L^2 R_{\mathcal{M}}^2} \right).$$

Furthermore, for all $\overline{h} \in H^* \setminus H_r^*$,

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|\overline{h}_t(\hat{x}_{t,1})\|^2 = r'^2 > r^2,$$

this yields $\frac{r}{r'} < 1$, thus $\frac{r}{r'} \overline{h}(X) \in \partial H_r^*$.
Hence,

$$P\left(\inf_{\overline{h} \in H^* \setminus H_r^*} \left(\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|\overline{h}_t(\hat{x}_{t,i})\|^2 - \frac{1}{8T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|\overline{h}_t(\hat{x}_{t,1})\|^2\right) \leq 0\right)$$
$$\leq C \frac{1}{r^{n+1}} \exp\left(-\frac{N_1 r^2 T^{1-b_2}}{32 b_1 L^2 R_{\mathcal{M}}^2}\right).$$

Denote $B_r \triangleq \left\{\alpha \in \mathcal{M} | \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,1})\|^2 \leq r^2\right\}$. For all $\alpha \in \mathcal{M} \setminus B_r$, we have

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,1})\|^2 > r^2,$$

hence,

$$h(\alpha, X) \in H^* \setminus H_r^*.$$

Therefore,

$$P\left(\inf_{\alpha \in \mathcal{M} \setminus B_r} \left(\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|h_t(\alpha, \hat{x}_{t,i})\|^2 - \frac{1}{8T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,1})\|^2\right) \leq 0\right)$$
$$\leq C \frac{1}{r^{n+1}} \exp\left(-\frac{N_1 r^2 T^{1-b_2}}{32 b_1 L^2 R_{\mathcal{M}}^2}\right).$$

Define the above event as:

$$\mathcal{A} = \left\{\inf_{\alpha \in \mathcal{M} \setminus B_r} \left(\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|h_t(\alpha, \hat{x}_{t,i})\|^2 - \frac{1}{8T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,1})\|^2\right) \leq 0\right\}.$$

On $\mathcal{A}^c$, we have

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,1})\|^2 \leq \frac{8}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|h_t(\alpha, \hat{x}_{t,i})\|^2 + r^2,$$

on $\mathcal{A}$,

$$\frac{1}{T} \sum_{t=0}^{T-1} \|h_t(\alpha, \hat{x}_{t,1})\|^2 \leq 4L^2 R_{\mathcal{M}}^2.$$

Therefore,

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \left\|f_t(\alpha, \beta_0(t), \hat{x}_{t,1}) - f_t(\alpha^*, \beta_0(t), \hat{x}_{t,1})\right\|^2 = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} \|h_t(\alpha, \hat{x}_{t,1})\|^2$$

$$\leq 4L^2 R_{\mathcal{M}}^2 P(A) + \frac{8}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|h_t(\alpha, \hat{x}_{t,i})\|^2 + r^2.$$

Given that the new data $x_{0:T-1}$ and the training data $\hat{x}_{0:T-1,1}$ are identically distributed, we have

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\alpha, x_t) = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\alpha, \hat{x}_{t,1}) \leq 4L^2 R_{\mathcal{M}}^2 P(A) + \frac{8}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\alpha, \hat{x}_{t,i}) + r^2.$$

Note that the above holds for all $\alpha \in \mathcal{M}$, specifically, let $\alpha = \hat{\alpha}$ which is learned from the training datasets. It yields

$$(B.8) \qquad \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{P_T} g_t(\hat{\alpha}, x_t) \leq 4L^2 R_{\mathcal{M}}^2 P(A) + \frac{8}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, \hat{x}_{t,i}) + r^2.$$

Furthermore, by Assumption 2.5:

$$\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|\hat{y}_{t+1,i} - f_t(\hat{\alpha}, \beta_0(t), \hat{x}_{t,i})\|^2 \leq \frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \|\hat{y}_{t+1,i} - f_t(\alpha^*, \beta_0(t), \hat{x}_{t,i})\|^2 + \varepsilon^*.$$

By direct computation, we have
(B.9)

$$\frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, \hat{x}_{t,i}) \leq \frac{4}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \langle v_{t+1,i}, h_t(\hat{\alpha}, \hat{x}_{t,i}) \rangle - \frac{1}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\hat{\alpha}, \hat{x}_{t,i}) + 2\varepsilon^*$$

$$\triangleq M_{T,N_1}(\hat{\alpha}) + 2\varepsilon^*$$

$$\leq M_{T,N_1}(\mathcal{M}) + 2\varepsilon^*,$$

where $M_{T,N_1}(\mathcal{M}) \triangleq \sup_{\alpha \in \mathcal{M}} M_{T,N_1}(\alpha)$ is the martingale offset term(see, e.g., [17, 38, 25, 37])
and is a measurable function of $\{\hat{x}_{0:T-1,i}, \hat{y}_{1:T,i}\}_{i=1}^{N_1}$.

Let N_γ be the γ-covering set of $\{h(\alpha, X) | \alpha \in \mathcal{M}, X \in \mathbb{X}^{pT}\} \subset H^*$ with minimal cardinality. Since h_t is bounded, by Assumptions 2.3,2.4, similar to (23) of [15], we have for any $\gamma > 0$:

$$\mathbb{E}[M_{T,N_1}(\mathcal{M})]$$

$$\leq 4\gamma \sigma_v p + \gamma^2 + \frac{1}{N_1 T} \mathbb{E}\left(\sup_{h \in N_\gamma} \frac{4}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \langle v_{t+1,i}, h_t(\hat{x}_{t,i}) \rangle - \frac{1}{2N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\hat{x}_{t,i}) \right).$$

For the last term of the expression above, similar to Lemmas 2,3 in [37], we have for any $\lambda \in [0, \frac{1}{4\sigma_v^2}]$,

$$\mathbb{E} \sup_{h \in N_\gamma} \left(\frac{4\lambda}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \langle v_{t+1,i}, h_t(\hat{x}_{t,i}) \rangle - \frac{\lambda}{2N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\hat{x}_{t,i}) \right)$$

$$\leq \log \mathbb{E} \exp \left\{ \sup_{h \in N_\gamma} \frac{4\lambda}{N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} \langle v_{t+1,i}, h_t(\hat{x}_{t,i}) \rangle - \frac{\lambda}{2N_1 T} \sum_{i=1}^{N_1} \sum_{t=0}^{T-1} g_t(\hat{x}_{t,i}) \right\}$$

$$\leq \log |N_\gamma| \leq (n+1) \log \left(C(L, n, R_{\mathcal{M}}) \frac{1}{\gamma} \right).$$

Taking $\lambda = \frac{1}{4\sigma_v^2}$ and combining the above two inequalities yields:

$$(B.10) \qquad \mathbb{E}[M_{T,N_1}(\mathcal{M})] \leq 4\gamma \sigma_v p + \gamma^2 + \frac{4\sigma_v^2(n+1)}{N_1 T} \log(\frac{C}{\gamma}).$$

Finally, let $\gamma = \frac{1}{N_1 T}, r^2 = \frac{\log(N_1 T)^{\gamma_0}}{N_1 T^{1-b_2}}$, where $\gamma_0 > 16b_1 L^2 R_{\mathcal{M}}^2(n+3)$ is fixed. By (B.6) and (B.8)-(B.10), we can get the desired result by direct computation when $N_1 \cdot T$ is sufficiently large.

Appendix C. Proofs of Corollaries 3.4, 3.6, 3.7 and 3.8.

C.1. Proof of Corollary 3.4. (i) Under Condition (3.3): By Lemma 2 in [10], without loss of generality, assume that the NLS estimate $\hat{\alpha}$ is a measurable function of the training dataset $\{\hat{x}_{0:T-1,1}, \hat{y}_{1:T,1}\}$. Let the new data $x_{0:T-1}$ be identically distributed to the training data $\{\hat{x}_{0:T-1,1}\}$ and independent of the NLS estimate $\hat{\alpha}$. By Generalization Lemma 3.1 and the integral form of Mean Value Theorem,

$$
O\left(\frac{\log T}{T^{1-b_2}}\right) = \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left\|f_t(\alpha^*, \beta_0(t), x_t) - f_t(\hat{\alpha}, \beta_0(t), x_t)\right\|^2
$$

$$
= \mathbb{E}(\alpha^* - \hat{\alpha})^\tau\left(\frac{1}{T}\sum_{t=0}^{T-1}F_t(\alpha^*, \hat{\alpha})F_t^\tau(\alpha^*, \hat{\alpha})\right)(\alpha^* - \hat{\alpha})
$$

$$
\geq \mathbb{E}\min_{\alpha'\in\mathcal{M}}\lambda_{\min}\left(\frac{1}{T}\sum_{t=0}^{T-1}F_t(\alpha^*, \alpha')F_t^\tau(\alpha^*, \alpha')\right)\|\alpha^* - \hat{\alpha}\|^2
$$

$$
= \mathbb{E}\left(\min_{\alpha'\in\mathcal{M}}\lambda_{\min}\left(\frac{1}{T}\sum_{t=0}^{T-1}F_t(\alpha^*, \alpha')F_t^\tau(\alpha^*, \alpha')\right)\right)\mathbb{E}\|\alpha^* - \hat{\alpha}\|^2,
$$

where the last equality holds due to the independence of $x_{0:T-1}$ and $\hat{\alpha}$. Thus we can conclude that $\hat{\alpha}$ converges to α^* in the mean square sense.

(ii) Under Condition (3.4): By (92) in [25] and Borel-Cantelli Lemma, we can conclude that the martingale offset term satisfies: $\sup_{\alpha\in\mathcal{M}} M_{T,1}(\alpha) = O(\frac{\log T}{T}), a.s.$, thus, by (B.9) in the proof of Generalization Lemma 3.1, it follows that

$$
\frac{1}{T}\sum_{t=0}^{T-1}\left\|f_t(\alpha^*, \beta_0(t), \hat{x}_{t,1}) - f_t(\hat{\alpha}, \beta_0(t), \hat{x}_{t,1})\right\|^2 = O\left(\frac{\log T}{T}\right)\quad a.s.,
$$

similar to (i), by the integral form of Mean Value Theorem, under Condition (3.4), we can conclude that $\hat{\alpha}$ converges to α^* almost surely.

C.2. Proof of Corollary 3.6. Denote:

$$
x_t \triangleq (\phi_t^\tau, u_t)^\tau = \left(y_t, y_{t-1}, \cdots, y_{t-p+1}, u_{t-1}, \cdots, u_{t-q+1}, \pi_t(\phi_t) + v_t\right)^\tau,
$$

$$
x_t' \triangleq (\phi_t'^\tau, u_t')^\tau = \left(y_t', y_{t-1}', \cdots, y_{t-p+1}', u_{t-1}', \cdots, u_{t-q+1}', \pi_t(\phi_t') + v_t'\right)^\tau.
$$

We first verify that $\{x_t\}$ and $\{x_t'\}$ satisfy Assumptions 2.2, 2.3 and 2.4.

By boundedness of f_t, π_t, $\{w, v\}$ and $\{w', v'\}$, denote $s = \max\{p, q-1\}$, we conclude that $\{x_t\}$ and $\{x_t'\}$ are both bounded by $\sqrt{p+q}(W + M_f)$ for $t \geq s$. Hence $\mathbb{X}$ is a compact set in $\mathbb{R}$, and f_t satisfies Assumption 2.3.

Besides, since $\phi_{t+1} = (y_{t+1}, y_t, \cdots, y_{t-p+2}, u_t, u_{t-1}, \cdots, u_{t-q+2})^\tau \triangleq F_t'(x_t, w_{t+1})$, by the definition of x_t, it yields:

$$
x_{t+1} = \begin{pmatrix} f_t(\alpha^*, \beta_0(t), x_t) \\ f_t(\alpha^*, \beta_0(t-1), x_{t-1}) \\ \vdots \\ f_t(\alpha^*, \beta_0(t-p+1), x_{t-p+1}) \\ \pi_t(\phi_t) \\ \vdots \\ \pi_{t-q+2}(\phi_{t-q+2}) \\ \pi_{t+1}(F_t'(x_t, w_{t+1})) \end{pmatrix} + \begin{pmatrix} w_{t+1} \\ w_t \\ \vdots \\ w_{t-p+2} \\ v_t \\ \vdots \\ v_{t-q+2} \\ v_{t+1} \end{pmatrix},
$$

by boundedness of f_t and π_t, similar to Remark 2.5, we have that, if $k > s$, for both systems:

$$
P_{n,n+k}(x, A) \geq \delta v(A),
$$

where $\delta \triangleq \inf_{w \in [-W,W]^{p+q}} p(w) > 0$ and $p(w)$ is the density function of the noise, which is a $(p+q)$-dimensional random vector, and $v(\cdot)$ is the Lebesgue measure restricted to $[-W + M_f, W - M_f]^{p+q}$. Thus Assumption 2.2 holds with $b_2 = b_2' = 0$.

Furthermore, since $\{w_{t+1}\}, \{w_{t+1}'\}$ are independent, zero-mean and bounded random variables, Assumption 2.4 holds.

Finally, since $\{x_t'\}$ is a measurable function of $\{w_{t+1}', v_{t+1}', x_0'\}$ and $\{x_t, y_{t+1}\}$ is a measurable function of $\{W_{t+1}, v_{t+1}, x_0\}$, thus $\{x_t'\}$ is independent of the training dataset.

Next, we address $D(P_T \| P_T')$. Since

$$x_{t+1} = G_t(x_t, w_{t+1}, v_{t+1}), \quad x_{t+1}' = G_t(x_t', w_{t+1}', v_{t+1}'),$$

where

$$G_t(x, w, v) \triangleq \left(f_t(\alpha^*, \beta_t, x) + w, x_1, x_2, \cdots, x_{p-1}, x_{p+1} \cdots, x_{p+q-1}, \pi_{t+1}(F_t'(x, w)) + v \right)^\tau,$$

therefore, these two independent systems have identical transition kernels, i.e.,

$$P_T = P(x_0, x_1, \cdots, x_{T-1}) = D_0(x_0) \prod_{i=1}^{T-1} P_i(x_i|x_{i-1}),$$

$$P_T' = P'(x_0, x_1, \cdots, x_{T-1}) = D_0'(x_0) \prod_{i=1}^{T-1} P_i(x_i|x_{i-1}).$$

By the chain rule of KL divergence, we have

$$D(P_T \| P_T') = \sum_{i=0}^{T-1} \mathbb{E}_{x_{0:i-1}} D\left(P(x_i|x_{0:i-1}) \middle\| P'(x_i|x_{0:i-1}) \right)$$

$$= D(D_0 \| D_0') + \sum_{i=1}^{T-1} \mathbb{E}_{x_{i-1}} D\left(P_i(x_i|x_{i-1}) \middle\| P_i(x_i|x_{i-1}) \right)$$

$$= D(D_0 \| D_0') < \infty.$$

Consequently, by Generalization Lemma 3.1, the total expectation of the generalization error diminishes to zero at a rate $\frac{\log T}{T}$.

C.3. Proof of Corollary 3.7. It is obvious that x_t is not independent of the previous sequence, and the training dataset $\{\hat{x}_{0:T-1}, \hat{y}_{1:T}\}$ is $\{y_{0:T-1}, y_{1:T}\}$ and the new data is $y_{T:2T-1}$.

Similar to the proof of Corollary 3.6, we can conclude that $p(y|x) = p_W(y - F(\alpha^*, x))$, thus for any $y \in B$, by boundedness of F, $\|y - F(\alpha^*, x)\| \le W$. Thus, there exists $\delta_0 > 0$, such that the density function of the transition kernel satisfies $q(y|x) \ge \delta_0 I\{y \in B\}$.

Thus, for $y \in B$, $p_T(y) = \int_X p_{T-1}(x) q(y|x) dx \ge \delta_0$.

Therefore, the KL divergence of x_0, x_T can be bounded above as follows:

$$D(P(x_0) \| P(x_T)) = \int_B p_0(x) \log \frac{p_0(x)}{p_T(x)} dx \le \int_{\{x|p_0(x)>1\}} p_0(x) \log p_0(x) dx + \log \frac{1}{\delta_0} < \infty.$$

Note that this upper bound is independent of T. By the proof of Corollary 3.4 in Appendix B, we know that the martingale offset term satisfies $M_{T,1}(\mathcal{M}) = O(\frac{\log T}{T})$ a.s., similar to the proof of Corollary 3.6, we can conclude that the generalization error over the new data is of order $O(\frac{\log T}{T})$.

C.4. Proof of Corollary 3.8. By Remark 2.7, we have $v_t^2 = O(\log t), v_t'^2 = O(\log t)$, thus the radius of $\mathbb{X}$ is of order $\sqrt{\log T}$, $L = O(\log T^{\frac{b}{2}})$.

Similar to the proof of Corollary 3.6, we can conclude that $b_2 = b_2' = 0$.

Besides, for two Gaussian distributions, it yields $D(N(a, \sigma^2)\|N(b, \sigma^2)) = \frac{2(a-b)^2}{\sigma^2}$. Note that $D(P_T\|P_T')$ is random since P_T' depends on the random variable $\hat{\alpha}$. Therefore, by the chain rule of KL divergence,

$$
\begin{aligned}
D(P_T\|P_T') &= \sum_{i=0}^{T-1} \mathbb{E}_{y_{0:i-1}} D\bigg(P(y_i|y_{0:i-1}) \bigg\| P'(y_i|y_{0:i-1}) \bigg) \\
&= D(D_0\|D_0') + \sum_{i=1}^{T-1} \mathbb{E}_{y_{i-1}} D\bigg(P_i(y_i|y_{i-1}) \bigg\| P_i'(y_i|y_{i-1}) \bigg) \\
&\leq D(D_0\|D_0') + \sum_{t=0}^{T-2} \mathbb{E}_{y_t} \bigg(\frac{2(f_t(\beta_0(t), y_t, \pi_t(\alpha^*, y_t)) - f_t(\beta_0(t), y_t, \pi_t(\hat{\alpha}, y_t))^2}{\sigma^2} \bigg) \\
&= D(D_0\|D_0') + O(\log^c T) \quad a.s.,
\end{aligned}
$$

where the inequality holds due to $b_2 = b_2' = \varepsilon^* = D(P_T\|P_T) = 0$ and (3.1) in Generalization Lemma 3.1.

Similar to the proof of Corollary 3.6, the generalization error on the new data $\{y_t'\}$ will converge to 0 as $T \to \infty$.

REFERENCES

[1] N. CESA-BIANCHI AND G. LUGOSI, *Prediction, Learning, and Games*, Cambridge University Press, 2006.
[2] H. CHEN AND L. GUO, *Convergence of adaptive mpc for linear stochastic systems*, Science China Information Sciences, 66 (2023).
[3] C. CORTES, Y. MANSOUR, AND M. MOHRI, *Learning bounds for importance weighting*, in Advances in Neural Information Processing Systems, vol. 23, Curran Associates, Inc., 2010.
[4] G. V. CYBENKO, *Approximation by superpositions of a sigmoidal function*, Mathematics of Control, Signals and Systems, 2 (1989), pp. 303–314.
[5] P. DOUKHAN, *Mixing: Properties and Examples*, Springer-Verlag, New York, 1994.
[6] L. GUO, *On adaptive stabilization of time-varying stochastic systems*, SIAM Journal on Control and Optimization, 28 (1990), pp. 1432–1451.
[7] L. GUO, *Convergence and logarithm laws of self-tuning regulators*, Automatica, 31 (1995), pp. 435–450.
[8] L. GUO, *Self-convergence of weighted least-squares with applications to stochastic adaptive control*, IEEE Transactions on Automatic Control, 41 (1996), pp. 79–89.
[9] L. GUO, *Time-Varying Stochastic Systems: Stability and Adaptive Theory*, Science Press, Beijing, 2nd ed., 2020.
[10] R. I. JENNRICH, *Asymptotic properties of non-linear least squares estimators*, The Annals of Mathematical Statistics, 40 (1969), pp. 633–643.
[11] S. JEONG AND A. TOWNSEND, *Extending mercer's expansion to indefinite and asymmetric kernels*, in The Thirteenth International Conference on Learning Representations, 2025.
[12] E. KAUFMANN, L. BAUERSFELD, A. LOQUERCIO, M. MÜLLER, V. KOLTUN, AND D. SCARAMUZZA, *Champion-level drone racing using deep reinforcement learning*, Nature, 620 (2023), pp. 982–987.
[13] A. KUMAR, A. RAGHUNATHAN, R. M. JONES, T. MA, AND P. LIANG, *Fine-tuning can distort pretrained features and underperform out-of-distribution*, in International Conference on Learning Representations, 2022.
[14] T. L. LAI AND C. Z. WEI, *Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems*, The Annals of Statistics, 10 (1982), pp. 154–166.
[15] B. D. LEE, I. ZIEMANN, G. J. PAPPAS, AND N. MATNI, *Active learning for control-oriented identification of nonlinear systems*, in 2024 IEEE 63rd Conference on Decision and Control (CDC), 2024, pp. 3011–3018.
[16] C. LI AND L. GUO, *On feedback capability in a class of nonlinearly parameterized uncertain systems*, IEEE Transactions on Automatic Control, 56 (2011), pp. 2946–2951.
[17] T. LIANG, A. RAKHLIN, AND K. SRIDHARAN, *Learning with square loss: Localization through offset rademacher complexity*, in Proceedings of The 28th Conference on Learning Theory, vol. 40, PMLR, 03–06 Jul 2015, pp. 1260–1285.

[18] S. Meyn and L. Guo, *Geometric ergodicity of a doubly stochastic time series model*, Journal of Time Series Analysis, 14 (2008), pp. 93 – 108.

[19] S. Meyn, R. L. Tweedie, and P. W. Glynn, *Markov Chains and Stochastic Stability*, Cambridge Mathematical Library, Cambridge University Press, 2 ed., 2009.

[20] P. K. Mishra, M. V. Gasparino, and G. Chowdhary, *Deep model predictive control with stability guarantees*, IEEE Transactions on Automatic Control, 70 (2025), pp. 5460–5467.

[21] M. O'Connell, G. Shi, X. Shi, K. Azizzadenesheli, A. Anandkumar, Y. Yue, and S.-J. Chung, *Neural-fly enables rapid learning for agile flight in strong winds*, Science Robotics, 7 (2022), p. eabm6597.

[22] P.-M. Samson, *Concentration of measure inequalities for Markov chains and Φ-mixing processes*, The Annals of Probability, 28 (2000), pp. 416 – 461.

[23] J. Song, *High-dimensional functions and manifolds in their optimal representation in low-dimensional visual space*, Chinese Science Bulletin, (2001), pp. 977–984.

[24] J. Song, *Remarks on representation of multivariate functions or cloud data in visualized low-dimensional spaces*, Journal of Systems Science and Mathematical Sciences, 45 (2025), 1, pp. 1–4.

[25] C. Stamouli, I. Ziemann, and G. J. Pappas, *Rate-optimal non-asymptotics for the quadratic prediction error method*, in 2024 IEEE 63rd Conference on Decision and Control (CDC), 2024, pp. 5723–5730.

[26] R. Takhanov, *On the speed of uniform convergence in mercer's theorem*, Journal of Mathematical Analysis and Applications, 518 (2023), p. 126718.

[27] A. Vlachos, A. Tsiamis, A. Karapetyan, E. C. Balta, and J. Lygeros, *Online residual learning from offline experts for pedestrian tracking*, in 2024 IEEE 63rd Conference on Decision and Control (CDC), 2024, pp. 3390–3395.

[28] M. J. Wainwright, *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2019.

[29] X. Wang and L. Guo, *A new convergent algorithm for online empirical risk minimization*, in 2017 36th Chinese Control Conference (CCC), 2017, pp. 11172–11176.

[30] X. Wu, J. H. Manton, U. Aickelin, and J. Zhu, *On the generalization for transfer learning: An information-theoretic analysis*, IEEE Transactions on Information Theory, 70 (2024), pp. 7089–7124.

[31] L. Xin, B. She, Q. Dou, G. T.-C. Chiu, and S. Sundaram, *Learning linearized models from nonlinear systems under initialization constraints with finite data*, Automatica, 180 (2025), p. 112478.

[32] L. Xin, L. Ye, G. Chiu, and S. Sundaram, *Learning dynamical systems by leveraging data from similar systems*, IEEE Transactions on Automatic Control, 70 (2025), pp. 4833–4840.

[33] L. Zhang, S. Lu, and Z.-H. Zhou, *Adaptive online learning in dynamic environments*, in Proceedings of the 32nd International Conference on Neural Information Processing Systems, Curran Associates Inc., 2018, p. 1330–1340.

[34] L. Zhang, Y. Zhao, and L. Guo, *Identification and adaptation with binary-valued observations under non-persistent excitation condition*, Automatica, 138 (2022), p. 110158.

[35] Y. Zhang and L. Guo, *Stochastic adaptive switching control based on multiple models*, J. Systems Science and Complexity, 15 (2002), pp. 18–34.

[36] X. Zheng and L. Guo, *l_1-based adaptive identification with saturated observations*, IEEE Transactions on Automatic Control, 70 (2025), pp. 5836–5847.

[37] I. Ziemann, H. Sandberg, and N. Matni, *Single trajectory nonparametric learning of nonlinear dynamics*, arXiv preprint arXiv:2202.08311, (2022).

[38] I. Ziemann and S. Tu, *Learning with little mixing*, arXiv preprint arXiv:2206.08269, (2024).