

Gravitational Waves

Nikolaos Stergioulas

February 5, 2020

The Two Polarizations in the TT gauge

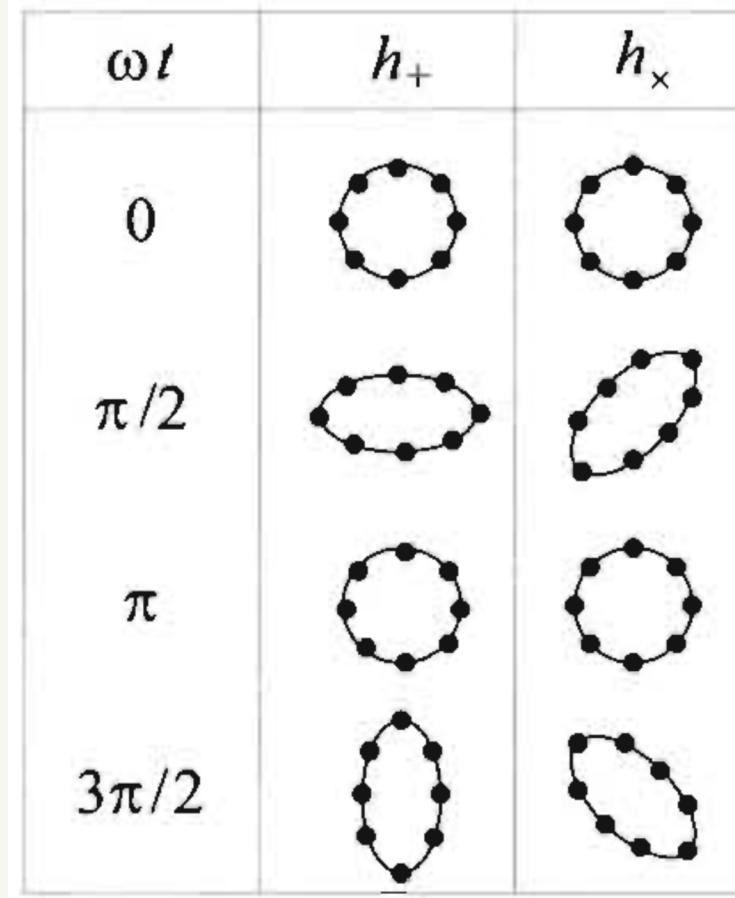


Figure 1: The effect of the two polarizations on a circle. Figure from [2].

Generation of GWs

- Linearized field equations

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Solution

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}$$

where

$$G(x - x') = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \delta(x_{\text{ret}}^0 - x'^0)$$

is a Green's function, satisfying

$$\square_x G(x - x') = \delta^4(x - x')$$

and

$$t_{\text{ret}} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}$$

Generation of GWs

- The solution becomes

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right)$$

- Define the *spatial projector* normal to a direction $\hat{\mathbf{n}}$

$$P_{ij} := \delta_{ij} - n_i n_j$$

then

$$\begin{aligned}\Lambda_{ij,kl}(\hat{\mathbf{n}}) &= P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \\ &= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ &\quad + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l\end{aligned}$$

Transverse Traceless Gauge and Far-Field Approximation

- If $h_{\mu\nu}$ is in Lorentz gauge (in vacuum), then it is brought to the *TT gauge* via the projection

$$h_{ij}^{\text{TT}} = \Lambda_{ij,kl} h_{kl}$$

and the solution in vacuum is then

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right)$$

- Far from the source, we can expand (where d is the source size)

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + \mathcal{O}\left(\frac{d^2}{r}\right)$$

and obtain the *far-field approximation*

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}' \right)$$

Non-relativistic Sources

- Let us Fourier transform the stress-energy tensor:

$$T_{kl} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right) = \int \frac{d^4 k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega(t-r/c+\mathbf{x}' \cdot \hat{\mathbf{n}}) + i\mathbf{k} \cdot \mathbf{x}'}$$

If the source has a maximum frequency ω_s and is *non-relativistic* ($\omega_s d \ll c$) and because $|\mathbf{x}'| \lesssim d$, only frequencies for which

$$\frac{\omega}{c} \mathbf{x}' \cdot \hat{\mathbf{n}} \lesssim \frac{\omega_s d}{c} \ll 1$$

contribute. Then, expanding in terms of $\omega \mathbf{x}' \cdot \hat{\mathbf{n}} / c$

$$e^{-i\omega(t-r/c+\mathbf{x}' \cdot \hat{\mathbf{n}}/c) + i\mathbf{k} \cdot \mathbf{x}'} = e^{-i\omega(t-r/c)} \left[1 - i\frac{\omega}{c} x'^i n^i + \frac{1}{2} \left(-i\frac{\omega}{c} \right)^2 x^i x'^j n^i n^j + \dots \right]$$

or, in the time domain:

$$T_{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}' \right) = T_{kl} (t - r/c, \mathbf{x}') + \frac{x'^i n^i}{c} \partial_0 T_{kl} + \frac{1}{2c^2} x^i x'^j n^i n^j \partial_0^2 T_{kl} +$$

Multipole Moments of the Stress-Energy Tensor

- The multipole moments of $T_{\mu\nu}$ are

$$S^{ij} = \int d^3x T^{ij}(t, \mathbf{x})$$

$$S^{ij,k} = \int d^3x T^{ij}(t, \mathbf{x}) x^k$$

$$S^{ij,kl} = \int d^3x T^{ij}(t, \mathbf{x}) x^k x^l$$

...

and the solution becomes

$$h_{ij}^{\text{TT}} = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \dots \right]_{\text{ret}}$$

Mass Density and Momentum Density Multipole Moments

- In terms of the mass density $(1/c^2) T^{00}$ one can define the moments

$$M = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x})$$

$$M^i = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i$$

$$M^{ij} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j$$

$$M^{ijk} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j x^k, \quad \dots$$

and in terms of the momentum density $(1/c) T^{0i}$

$$P^i = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x})$$

$$P^{i,j} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j$$

$$P^{i,jk} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j x^k, \quad \dots$$

Mass Quadrupole Radiation

- The quadrupole moment of $T_{\mu\nu}$ is written in terms of the mass-density quadrupole moment as

$$S^{ij} = \frac{1}{2} \ddot{M}^{ij}$$

and the solution becomes to leading order in v/c

$$[h_{ij}^{\text{TT}}(t, \mathbf{x})]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^2} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \ddot{M}^{kl}(t - r/c)$$

Define the *reduced (trace-free) quadrupole moment tensor*

$$Q^{ij} := M^{ij} - \frac{1}{3} \delta^{ij} M_{kk} \tag{1}$$

$$\simeq \int d^3x \rho(t, \mathbf{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right) \tag{2}$$

(to leading order in v/c it becomes the Newtonian expression) and

$$Q_{ij}^{\text{TT}} = \Lambda_{ij,kl}(\mathbf{n}) Q_{ij}$$

Quadrupole Approximation

- The *quadrupole formula* for GW radiation is

$$[h_{ij}^{\text{TT}}(t, \mathbf{x})]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^4} \ddot{Q}_{ij}^{\text{TT}}(t - r/c)$$

Notice that $\ddot{Q}_{ij}^{\text{TT}} = \Lambda_{ij,kl} \ddot{Q}_{ij} = \Lambda_{ij,kl} \ddot{M}_{ij}$ (the latter is preferred in calculations)

- EXAMPLE:** Emission along $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Then $P_{ij} = \delta_{ij} - n_i n_j$ becomes

$$P_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For any 3×3 matrix A_{ij}

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \left[P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right] A_{kl} \\ &= (PAP)_{ij} - \frac{1}{2} P_{ij} \text{Tr}(PA) \end{aligned}$$

Quadrupole Approximation

and

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while $\text{Tr}(PA) = A_{11} + A_{22}$. Then:

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \left(\begin{array}{ccc} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{array} \right)_{ij} - \frac{A_{11} + A_{22}}{2} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)_{ij} \\ &= \left(\begin{array}{ccc} (A_{11} - A_{22})/2 & A_{12} & 0 \\ A_{21} & -(A_{11} - A_{22})/2 & 0 \\ 0 & 0 & 0 \end{array} \right)_{ij} \end{aligned}$$

Thus

$$\Lambda_{ij,kl} \ddot{M}_{kl} = \left(\begin{array}{ccc} (\ddot{M}_{11} - \ddot{M}_{22})/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -(\ddot{M}_{11} - \ddot{M}_{22})/2 & 0 \\ 0 & 0 & 0 \end{array} \right)_{ij}$$

Quadrupole Approximation

- Comparing to

$$h_{ij}^{\text{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

we immediately find

$$\begin{aligned} h_+ &= \frac{1}{r} \frac{G}{c^4} \left(\ddot{M}_{11} - \ddot{M}_{22} \right) \\ h_\times &= \frac{2}{r} \frac{G}{c^4} \ddot{M}_{12} \end{aligned}$$

(the r.h.s. is computed in the retarded time $t - r$).

Emission Along Arbitrary Direction

- Along an arbitrary direction \hat{n} , with components in a Cartesian system

$$n_i = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta)$$

the two polarizations are:

$$\begin{aligned} h_+(t; \theta, \phi) = & \frac{1}{r} \frac{G}{c^4} \left[\ddot{M}_{11} (\cos^2 \phi - \sin^2 \phi \cos^2 \theta) + \ddot{M}_{22} (\sin^2 \phi - \cos^2 \phi \cos^2 \theta) \right. \\ & - \ddot{M}_{33} \sin^2 \theta - \dot{M}_{12} \sin 2\phi (1 + \cos^2 \theta) \\ & \left. + \ddot{M}_{13} \sin \phi \sin 2\theta + \ddot{M}_{23} \cos \phi \sin 2\theta \right] \end{aligned}$$

and

$$\begin{aligned} h_\times(t; \theta, \phi) = & \frac{1}{r} \frac{G}{c^4} \left[\left(\ddot{M}_{11} - \ddot{M}_{22} \right) \sin 2\phi \cos \theta + 2\ddot{M}_{12} \cos 2\phi \cos \theta \right. \\ & - 2\ddot{M}_{13} \cos \phi \sin \theta + 2\ddot{M}_{23} \sin \phi \sin \theta \left. \right] \end{aligned}$$

Emitted Energy and Linear Momentum of GWs

- Energy is emitted by GWs at a rate

$$\frac{dE_{\text{GW}}}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \left\langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \right\rangle \quad (3)$$

$$= \frac{c^3 r^2}{16\pi G} \int d\Omega \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \quad (4)$$

$$\approx \frac{G}{5c^5} \left\langle \ddot{Q}_{jk} \ddot{Q}^{jk} \right\rangle \quad (5)$$

$$= \frac{G}{5c^5} \left\langle \ddot{M}_{ij} \ddot{M}_{ij} - \frac{1}{3} \left(\ddot{M}_{kk} \right)^2 \right\rangle \quad (6)$$

- There is no loss of linear momentum in the quadrupole approximation

$$\frac{\partial P_{\text{GW}}^k}{dt} = -\frac{G}{8\pi c^5} \int d\Omega \ddot{Q}_{ij}^{\text{TT}} \partial^k \ddot{Q}_{ij}^{\text{TTT}} = 0 \quad (7)$$

because Q_{ij} is invariant and $\partial^i \rightarrow -\partial^i$ under a reflection $\mathbf{x} \rightarrow -\mathbf{x}$.

Angular Momentum Emitted by GWs

- The angular momentum carried away by GWs is

$$\frac{dJ^i}{dt} = \frac{c^3}{32\pi G} \int r^2 d\Omega \left\langle -\epsilon^{ikl} \dot{h}_{ab}^{\text{TT}} x^k \partial^\ell h_{ab}^{\text{TT}} + 2\epsilon^{ikl} \dot{h}_{al}^{\text{TT}} h_{ak}^{\text{TT}} \right\rangle$$

In the quadrupole approximation, this becomes

$$\left(\frac{dJ^i}{dt} \right)_{\text{quad}} = \frac{2G}{5c^5} \epsilon^{ikl} \left\langle \ddot{Q}_{ka} \ddot{Q}_{la} \right\rangle$$

GWs from a Binary System

- Consider a binary with circular orbits. The trajectories of the two stars are $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ and the relative coordinate is $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$. The center of mass is

$$\mathbf{x}_{\text{CM}} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}$$

For a nonrelativistic system, the mass quadrupole moment is

$$\begin{aligned} M^{ij} &= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j \\ &= m x_{\text{CM}}^i x_{\text{CM}}^j + \mu \left(x_{\text{CM}}^i x_0^j + x_{\text{CM}}^j x_0^i \right) + \mu x_0^i x_0^j \end{aligned}$$

where $m = m_1 + m_2$ and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass. If we choose $\mathbf{x}_{\text{CM}} = 0$ as the origin of our coordinate system, then the mass quadrupole moment is

$$M^{ij}(t) = \mu x_0^i(t) x_0^j(t)$$

GWs from a Binary System

- In the CM frame, the dynamics reduces to a one-body problem with reduced mass μ .
- Choose a circular orbit with angular frequency ω_s in the plane with $z_0 = 0$

$$\begin{aligned}x_0(t) &= R \cos\left(\omega_s t + \frac{\pi}{2}\right) \\y_0(t) &= R \sin\left(\omega_s t + \frac{\pi}{2}\right) \\z_0(t) &= 0\end{aligned}$$

Then

$$M_{11} = \mu R^2 \frac{1 - \cos 2\omega_s t}{2} \quad (8)$$

$$M_{22} = \mu R^2 \frac{1 + \cos 2\omega_s t}{2} \quad (9)$$

$$M_{12} = -\frac{1}{2} \mu R^2 \sin 2\omega_s t \quad (10)$$

(other components are zero).

GWs from a Binary System

- Taking two time-derivatives:

$$\begin{aligned}\ddot{M}_{11} &= -\ddot{M}_{22} = 2\mu R^2 \omega_s^2 \cos 2\omega_s t \\ \ddot{M}_{12} &= 2\mu R^2 \omega_s^2 \sin 2\omega_s t\end{aligned}$$

and

$$\begin{aligned}h_+(t; \theta, \phi) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left(\frac{1+\cos^2 \theta}{2} \right) \cos (2\omega_s t_{\text{ret}} + 2\phi) \\ h_\times(t; \theta, \phi) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos \theta \sin (2\omega_s t_{\text{ret}} + 2\phi)\end{aligned}$$

- If we can neglect the proper motion of the source, then the angle ϕ is fixed and by a change of the origin of time one can set it to zero.
- If we view the system from an *inclination* $\iota = \theta$, then

$$\begin{aligned}h_+(t) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left(\frac{1+\cos^2 \iota}{2} \right) \cos (2\omega_s t) \\ h_\times(t) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos \iota \sin (2\omega_s t)\end{aligned}$$

- For $\iota = 0 \Rightarrow$ circular polarization, for $\iota = 90^\circ \Rightarrow$ linear polarization, otherwise elliptic polarization. Measuring polarization, recovers ι .

- The two polarizations can be written as

$$h_+(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{2/3} \frac{1+\cos^2 \theta}{2} \cos (2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi)$$

$$h_\times(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{2/3} \cos \theta \sin (2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi)$$

where $\omega_{\text{gw}} = 2\omega_s$ and

$$f_{\text{gw}} = \omega_{\text{gw}} / (2\pi)$$

is the frequency of the GWs and

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

is the *chirp mass*.

- Kepler's law is

$$\omega_s^2 = \frac{Gm}{R^3}$$

Radiated Power

- The angular distribution of the radiated power is

$$\left(\frac{dP}{d\Omega} \right)_{\text{quad}} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$

or

$$\left(\frac{dP}{d\Omega} \right)_{\text{quad}} = \frac{2c^5}{\pi G} \left(\frac{GM_c \omega_{\text{gw}}}{2c^3} \right)^{10/3} g(\theta)$$

where

$$g(\theta) = \left(\frac{1 + \cos^2 \theta}{2} \right)^2 + \cos^2 \theta$$

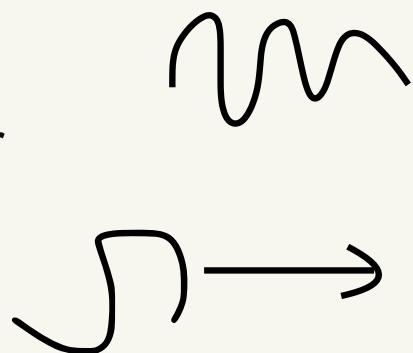
which has an angular average of

$$\int \frac{d\Omega}{4\pi} g(\theta) = \frac{4}{5}$$

Radiate Power

- The radiated power is

or



$$\underline{P_{\text{quad}} = \frac{1}{10} \frac{G\mu^2}{c^5} R^4 \omega_{\text{gw}}^6}$$

$$\boxed{\underline{P_{\text{quad}} = \frac{32}{5} \frac{c^5}{G} \left(\frac{GM_c \omega_{\text{gw}}}{2c^3} \right)^{10/3}}}$$

$$\int_a^b \frac{dx}{dt}$$

$$\int_a^b \frac{dx}{dt}$$

- The energy radiated in one period $T = 2\pi/\omega_s$ is (with $v = \omega_s R$)



$$\langle E_{\text{quad}} \rangle_T = \frac{64\pi}{5} \frac{G\mu^2}{R} \left(\frac{v}{c} \right)^5$$



i.e. the energy scale $G\mu^2/R$ is suppressed by a factor $(v/c)^5$.



Frequency evolution

- The orbital energy is

$$\begin{aligned} E_{\text{orbit}} &= E_{\text{kin}} + E_{\text{pot}} \\ &= -\frac{Gm_1m_2}{2R} \\ &= -\left(G^2 M_c^5 \omega_{\text{gw}}^2 / 32\right)^{1/3} \end{aligned}$$

- Assume that

$$\left| \frac{dE_{\text{orbit}}}{dt} \right| = P_{\text{quad}}$$

Then

$$\dot{f}_{\text{gw}} = \frac{96}{5} \pi^{8/3} \left(\frac{GM_c}{c^3} \right)^{5/3} f_{\text{gw}}^{11/3}$$

- Integrating \dot{f}_{gw} , we see that it *diverges* at a finite time t_{coal} . The remaining time to coalescence is then

$$\tau = t_{\text{coal}} - t$$

and the frequency evolution is written as

$$f_{\text{gw}}(\tau) = \frac{1}{\pi} \left(\frac{5}{256} \frac{1}{\tau} \right)^{3/8} \left(\frac{GM_c}{c^3} \right)^{-5/8}$$

or

$$f_{\text{gw}}(\tau) \simeq 134 \text{Hz} \left(\frac{1.21 M_{\odot}}{M_c} \right)^{5/8} \left(\frac{1 \text{s}}{\tau} \right)^{3/8}$$

- The time to coalescence is thus

$$\tau \simeq 2.18 \text{s} \left(\frac{1.21 M_{\odot}}{M_c} \right)^{5/3} \left(\frac{100 \text{Hz}}{f_{\text{gw}}} \right)^{8/3}$$

Number of cycles

- When the period $T(t)$ is slowly varying, the number of cycles in a time interval dt is

$$d\mathcal{N}_{\text{cyc}} = \frac{dt}{T(t)} = f_{\text{gw}}(t)dt$$

and thus the number of cycles spent between frequencies f_{\min} and f_{\max} is

$$\begin{aligned}\mathcal{N}_{\text{cyc}} &= \int_{t_{\min}}^{t_{\max}} f_{\text{gw}}(t)dt \\ &= \int_{f_{\min}}^{f_{\max}} df_{\text{gw}} \frac{f_{\text{gw}}}{\dot{f}_{\text{gw}}}\end{aligned}$$

or

$$\mathcal{N}_{\text{cyc}} = \frac{1}{32\pi^{8/3}} \left(\frac{GM_c}{c^3} \right)^{-5/3} \left(f_{\min}^{-5/3} - f_{\max}^{-5/3} \right)$$

If $f_{\min}^{-5/3} - f_{\max}^{-5/3} \simeq f_{\min}^{-5/3}$, then

$$\mathcal{N}_{\text{cyc}} = \simeq 1.6 \times 10^4 \left(\frac{10\text{Hz}}{f_{\min}} \right)^{5/3} \left(\frac{1.2M_{\odot}}{M_c} \right)^{5/3}$$

Orbital Evolution

- From Kepler's law and the equation for \dot{f}_{gw} we find that the radius of the orbit shrinks according to

$$\frac{\dot{R}}{R} = -\frac{2}{3} \frac{\dot{f}_{\text{gw}}}{f_{\text{gw}}} = -\frac{1}{4\tau}$$

If at $t = t_0$ the radius is $R = R_0$ and $\tau_0 = t_{\text{coal}} - t_0$, then integrating:

$$R(\tau) = R_0 \left(\frac{\tau}{\tau_0} \right)^{1/4}$$

- From Kepler's law and the equation for f_{gw} we find

$$\tau_0 = \frac{5}{256} \frac{c^5 R_0^4}{G^3 m^2 \mu}$$

or

$$\tau_0 \simeq 9.83 \times 10^6 \text{yr} \left(\frac{T_0}{1\text{hr}} \right)^{8/3} \left(\frac{M_\odot}{m} \right)^{2/3} \left(\frac{M_\odot}{\mu} \right)$$

Phase Evolution

- Because $\omega_{\text{gw}} = d\Phi/dt$, the evolution of the phase is

$$\Phi(t) = \int_{t_0}^t dt' \omega_{\text{gw}}(t')$$

or, with $\Phi_0 = \Phi(\tau = 0)$

$$\boxed{\Phi(\tau) = -2 \left(\frac{5GM_c}{c^3} \right)^{-5/8} \tau^{5/8} + \Phi_0}$$

- The waveform is

$$h_+(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}(t_{\text{ret}})}{c} \right)^{2/3} \left(\frac{1+\cos^2 \iota}{2} \right) \cos [\Phi(t_{\text{ret}})]$$
$$h_x(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}(t_{\text{ret}})}{c} \right)^{2/3} \cos \iota \sin [\Phi(t_{\text{ret}})]$$

or

$$h_+(\tau) = \frac{1}{r} \left(\frac{GM_c}{c^2} \right)^{5/4} \left(\frac{5}{c\tau} \right)^{1/4} \left(\frac{1+\cos^2 \iota}{2} \right) \cos [\Phi(\tau)]$$
$$h_\times(\tau) = \frac{1}{r} \left(\frac{GM_c}{c^2} \right)^{5/4} \left(\frac{5}{c\tau} \right)^{1/4} \cos \iota \sin [\Phi(\tau)]$$

Higher-order Multipoles

- For a binary system in circular orbit and assuming a *flat background*, the *mass octupole* and the *current quadrupole* emit GWs at both frequencies ω_s and $3\omega_s$. The power emitted (compared to the mass quadrupole) is

$$P_{\text{oct+cq}}(\omega_s) = \frac{19}{672} \left(\frac{v}{c}\right)^2 P_{\text{quad}}(2\omega_s)$$

$$P_{\text{oct+cq}}(3\omega_s) = \frac{135}{224} \left(\frac{v}{c}\right)^2 P_{\text{quad}}(2\omega_s)$$

so it is suppressed by a factor of $(v/c)^2$.

- Notice, however, that the orbit is also affected at order $(v/c)^2$ by relativistic effects, so that the above calculation is not consistent to this order, but only indicates the order of magnitude.

ISCO

- The inspiral phase terminates when the orbit becomes unstable. For a Schwarzschild spacetime of mass $m = m_1 + m_2$, the ISCO radius is

$$r_{\text{ISCO}} = \frac{6Gm}{c^2}$$

The orbital frequency at the ISCO is

$$(f_s)_{\text{ISCO}} = \frac{1}{6\sqrt{6}(2\pi)} \frac{c^3}{Gm}$$

or

$$(f_s)_{\text{ISCO}} \simeq 2.2 \text{kHz} \left(\frac{M_\odot}{m} \right)$$

General TT Plane Wave Solution

- The general solution of the wave equation in the TT -gauge $\square h_{ij}^{\text{TT}} = 0$ can be written as

$$h_{ij}^{\text{TT}}(x) = \int \frac{d^3k}{(2\pi)^3} \left(\mathcal{A}_{ij}(\mathbf{k}) e^{ik^\mu x_\mu} + \mathcal{A}_{ij}^*(\mathbf{k}) e^{-ik^\mu x_\mu} \right)$$

where $k^\mu = (\omega/c, \mathbf{k})$ with $\mathbf{k}/|\mathbf{k}| = \hat{\mathbf{n}}$ and $|\mathbf{k}| = \omega/c = (2\pi f)/c$.
Therefore

$$d^3k = |\mathbf{k}|^2 d|\mathbf{k}| d\Omega = (2\pi/c)^3 f^2 df d\Omega$$

with $f > 0$. Setting $d \cos \theta d\phi := d^2 \hat{\mathbf{n}}$, the solution is written as

$$h_{ij}^{\text{TT}}(x) = \frac{1}{c^3} \int_0^\infty df f^2 \int d^2 \hat{\mathbf{n}} \left[\mathcal{A}_{ij}(f, \hat{\mathbf{n}}) e^{-2\pi i f(t - \hat{\mathbf{n}} \cdot \mathbf{x}/c)} + \text{c.c.} \right]$$

(notice that both terms in the parenthesis correspond to waves traveling in the $+\hat{\mathbf{n}}$ direction and only physical frequencies $f > 0$ appear in the expansion).

Plane Wave from Specific Direction

- For a plane wave coming from a specific direction $\hat{\mathbf{n}}_0$

$$\mathcal{A}_{ij}(f, \hat{\mathbf{n}}) := A_{ij}(f) \delta^{(2)}(\hat{\mathbf{n}} - \hat{\mathbf{n}}_0)$$

(this does not apply to stochastic backgrounds that arrive from different directions).

- In the TT -gauge $k^i \mathcal{A}_{ij}(\mathbf{k}) = 0 \Rightarrow n^i \mathcal{A}_{ij}(f, \hat{\mathbf{n}}) = 0$ and therefore the plane wave is described by only the indices $a, b = 1, 2$ in the transverse direction. We can thus drop the TT label and write h_{ab} for the wave and \tilde{h}_{ab} for its Fourier transform. Also in this gauge

$$\tilde{h}_{ab}(f) = \begin{pmatrix} \tilde{h}_+(f) & \tilde{h}_\times(f) \\ \tilde{h}_\times(f) & -\tilde{h}_+(f) \end{pmatrix}_{ab}$$

Plane Wave at Detector

- The plane wave solution arriving at a detector from a specific direction $\hat{\mathbf{n}}_0$ can thus be written as

$$h_{ab}(t, \mathbf{x}) = \int_0^\infty df \left[\tilde{h}_{ab}(f, \mathbf{x}) e^{-2\pi ift} + \tilde{h}_{ab}^*(f, \mathbf{x}) e^{2\pi ift} \right]$$

where

$$\begin{aligned} \tilde{h}_{ab}(f, \mathbf{x}) &= \frac{f^2}{c^3} \int d^2 \hat{\mathbf{n}} \mathcal{A}_{ab}(f, \hat{\mathbf{n}}) e^{2\pi if \hat{\mathbf{n}} \cdot \mathbf{x}/c} \\ &= \frac{f^2}{c^3} A_{ab}(f) e^{2\pi if \hat{\mathbf{n}}_0 \cdot \mathbf{x}/c} \end{aligned}$$

- For the ground-based detectors, the length of each arm is much smaller than the reduced wavelength $\lambda/(2\pi)$ of detectable GWs and taking the detector as the center of the coordinate system we have $\exp\{2\pi if \hat{\mathbf{n}} \cdot \mathbf{x}/c\} = \exp\{2\pi i \hat{\mathbf{n}} \cdot \mathbf{x}/\lambda\} \simeq 1$. In this case

$$h_{ab}(t) \simeq \int_0^\infty df \left[\tilde{h}_{ab}(f) e^{-2\pi ift} + \tilde{h}_{ab}^*(f) e^{2\pi ift} \right]$$

where $\tilde{h}_{ab}(f) = \tilde{h}_{ab}(f, \mathbf{x} = 0) = (f^2/c^3) A_{ab}(f)$.

Fourier Transform

- If we extend the definition of $\tilde{h}_{ab}(f)$ to negative frequencies as

$$\tilde{h}_{ab}(-f) = \tilde{h}_{ab}^*(f)$$

then we can write the plane wave solution at the detector as

$$h_{ab}(t) = \int_{-\infty}^{\infty} df \tilde{h}_{ab}(f) e^{-2\pi i f t}$$

which means that $\tilde{h}_{ab}(f)$ is the *Fourier transform* of $h_{ab}(t)$

$$\tilde{h}_{ab}(f) = \int_{-\infty}^{\infty} dt h_{ab}(t) e^{2\pi i f t}$$

Fourier Transform during Inspiral Phase

- For the inspiral phase ($-\infty < t < t_{\text{coal}}$) the Fourier transform for a circular binary inspiral is

$$\tilde{h}_+(f) = A e^{i\Psi_+(f)} \frac{c}{r} \left(\frac{GM_c}{c^3} \right)^{5/6} \frac{1}{f^{7/6}} \left(\frac{1 + \cos^2 \iota}{2} \right) \quad (11)$$

$$\tilde{h}_\times(f) = A e^{i\Psi_\times(f)} \frac{c}{r} \left(\frac{GM_c}{c^3} \right)^{5/6} \frac{1}{f^{7/6} \cos \iota} \quad (12)$$

where

$$A = \frac{1}{\pi^{2/3}} \left(\frac{5}{24} \right)^{1/2}$$

and

$$\Psi_+(f) = 2\pi f (t_c + r/c) - \Phi_0 - \frac{\pi}{4} + \frac{3}{4} \left(\frac{GM_c}{c^3} 8\pi f \right)^{-5/3}$$

$$\Psi_\times = \Psi_+ + (\pi/2)$$

with $\Phi_0 = \Phi(\tau = 0)$. For accurate matched filtering, post-Newtonian corrections to the phase $\Psi_{+,\times}(f)$ must be included.

GW Energy Spectrum during Inspiral Phase

- We have seen that the energy emitted in GWs is

$$\left(\frac{dE}{dt} \right)_{\text{GW}} = \frac{c^3 r^2}{16\pi G} \int d\Omega \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle$$

The total energy flowing through solid angle $d\Omega$ is thus

$$\left(\frac{dE}{d\Omega} \right)_{\text{GW}} = \frac{c^3 r^2}{16\pi G} \int_{-\infty}^{\infty} dt \left(\dot{h}_+^2 + \dot{h}_\times^2 \right)$$

(because we integrate over all times, the average $\langle \rangle$ over a few periods is not required). Inserting the plane wave solution for a signal with $\lambda \gg L_{\text{detector}}$ and restricting to positive frequencies only, we obtain

$$\boxed{\left(\frac{dE}{d\Omega} \right)_{\text{GW}} = \frac{\pi c^3}{2G} \int_0^{\infty} df f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)}$$

GW Energy Spectrum during Inspiral Phase

- Integrating over a sphere surround the source, the energy spectrum of GWs is

$$\frac{dE}{df} = \frac{\pi c^3}{2G} f^2 r^2 \int d\Omega \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)$$

- For a circular binary inspiral, this becomes

$$\frac{dE}{df} = \frac{\pi^{2/3}}{3G} (GM_c)^{5/3} f^{-1/3}$$

Total Energy Emitted During Inspiral

- The total energy emitted in the inspiral phase (up to a maximum frequency f_{\max}) is

$$\Delta E_{\text{rad}} \sim \frac{\pi^{2/3}}{2G} (GM_c)^{5/3} f_{\max}^{2/3}$$

or

$$\Delta E_{\text{rad}} \sim 4.2 \times 10^{-2} M_{\odot} c^2 \left(\frac{M_c}{1.21 M_{\odot}} \right)^{5/3} \left(\frac{f_{\max}}{1 \text{kHz}} \right)^{2/3}$$

If we take $f_{\max} \simeq 2 (f_s)_{\text{ISCO}}$, then

$$\boxed{\Delta E_{\text{rad}} \sim 8 \times 10^{-2} \mu c^2}$$

which depends only on the reduced mass of the system.

- A better estimate is obtained considering the binding energy of the binary system at the ISCO

$$E_{\text{binding}} = (1 - \sqrt{8/9}) \mu c^2 \simeq 5.7 \times 10^{-2} \mu c^2$$

Elliptic Orbits

- The elliptic orbit is described by polar coordinates (r, ψ) with origin at the center of mass.

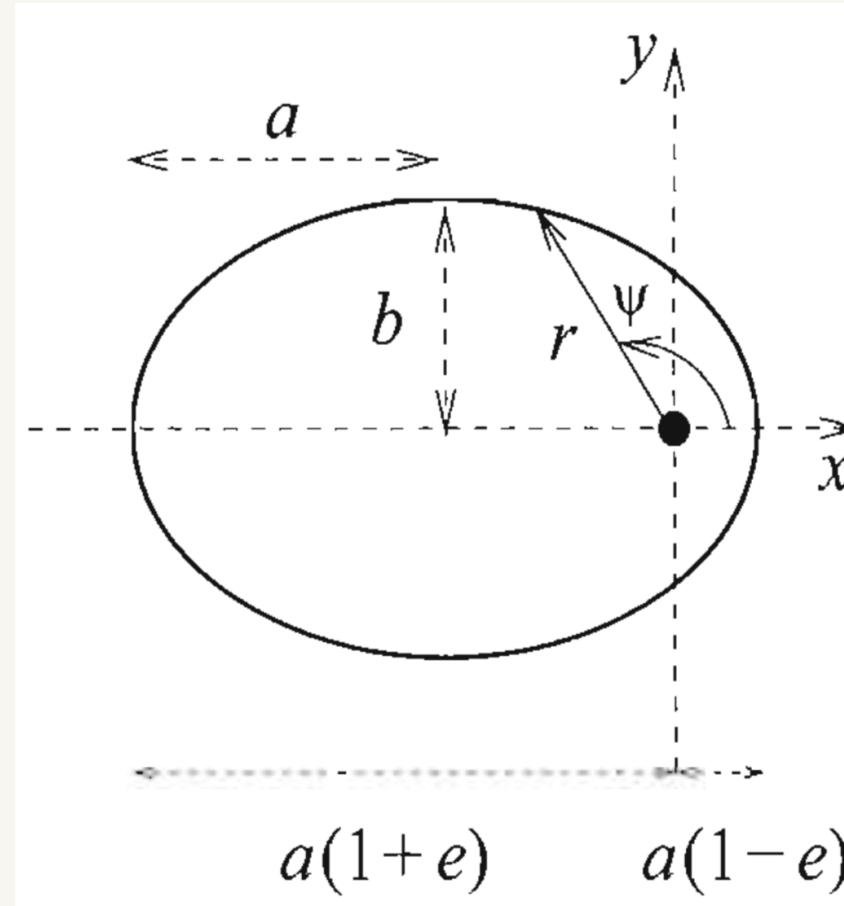


Figure 2: The function $f(e)$. Figure from [2].

Elliptic Orbits

- The motion is equivalent to an effective one-body problem with mass μ and angular momentum $L = \mu r^2 \dot{\psi}$. The total orbital energy is

$$E = \frac{1}{2} \mu \left(\dot{r}^2 + r^2 \dot{\psi}^2 \right) - \frac{G \mu m}{r} \quad (13)$$

$$= \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2 \mu r^2} - \frac{G \mu m}{r} \quad (14)$$

($E < 0$). Integrating $dr/d\psi = \dot{r}/\dot{\psi}$, the equation of the orbit is

$$\frac{1}{r} = \frac{1}{R} (1 + e \cos \psi)$$

where the eccentricity

$$e = \sqrt{1 + \frac{2 E L^2}{G^2 m^2 \mu^3}}$$

and the length scale

$$R = \frac{L^2}{G m \mu^2}$$

are constants of motion.

Elliptic Orbits

- The two semi-axes of the ellipse are

$$a = \frac{R}{1 - e^2} = \frac{Gm\mu}{2|E|}$$
$$b = \frac{R}{(1 - e^2)^{1/2}}$$

- In terms of a and e the equation for the orbit is written as

$$r = \frac{a(1 - e^2)}{1 + e \cos \psi}$$

and Kepler's law is

$$\omega_0^2 = \frac{Gm}{a^3}$$

with

$$T = \frac{2\pi}{\omega_0} \tag{15}$$

being the period of the orbit.

Elliptic Orbits

- Integrating \dot{r} and $\dot{\psi}$ the time-dependent orbit $r(t)$, $\psi(t)$ is given in parametric form as

$$\begin{aligned} r &= a[1 - e \cos u] \\ \cos \psi &= \frac{\cos u - e}{1 - e \cos u} \end{aligned} \tag{16}$$

where the time parameter u (the *eccentric anomaly*) is related to t through

$$\beta \equiv u - e \sin u = \omega_0 t \tag{17}$$

With $\psi(t = 0) = 0$ we can also write

$$\psi(u) = A_e(u) \equiv 2 \arctan \left[\left(\frac{1+e}{1-e} \right)^{1/2} \tan \frac{u}{2} \right] \tag{18}$$

where $A_e(u)$ is called the *true anomaly*.

- Notice that for $e = 0 \Rightarrow \psi = u$.

The True Anomaly

- $-\pi \leq \psi \leq \pi$ and $-\pi \leq u \leq \pi$

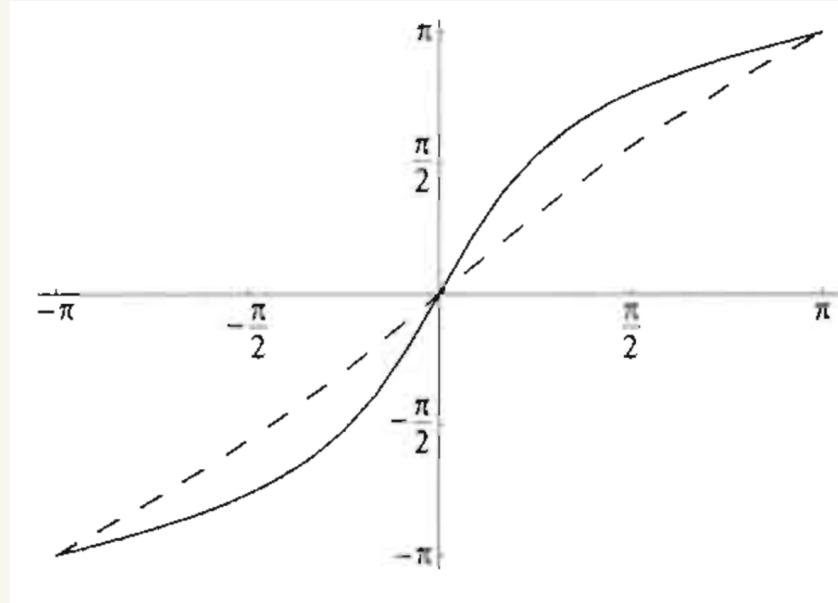


Figure 3: The function $\psi(u)$ for $e = 0.2$ (dashed line) and $e = 0.75$ (solid line). Figure from [2].

- In Cartesian coordinates, the orbit is

$$\begin{aligned}x(t) &= r \cos \psi = a[\cos u(t) - e] \\y(t) &= r \sin \psi = b \sin u(t)\end{aligned}\tag{19}$$

Elliptic Orbits

- For an elliptic orbit of eccentricity e and semi-major axis a , the power emitted in gravitational waves is

$$P = \left(\frac{dE}{dt} \right)_{\text{GW}} = \frac{32G^4\mu^2m^3}{5c^5a^5} f(e)$$

where

$$f(e) = \frac{1}{(1 - e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right)$$

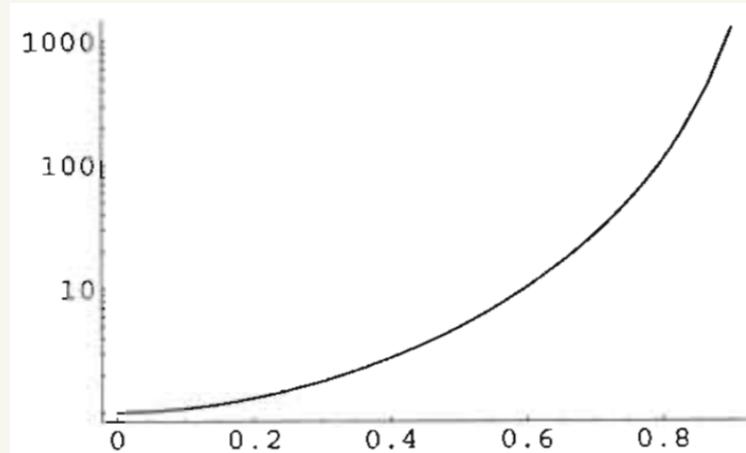


Figure 4: The function $f(e)$. Figure from [2].

- Rewriting Kepler's law, we see that $T \propto (-E)^{-3/2}$, so that

$$\frac{\dot{T}}{T} = -\frac{3}{2} \frac{\dot{E}}{E} \quad (20)$$

and substituting $dE/dt = -(dE/dt)_{\text{GW}}$ we find that the period changes according to

$$\frac{\dot{T}}{T} = -\frac{96}{5} \frac{G^3 \mu m^2}{c^5 a^4} f(e) \quad (21)$$

or

$$\boxed{\frac{\dot{T}}{T} = -\frac{96}{5} \frac{G^{5/3} \mu m^{2/3}}{c^5} \left(\frac{T}{2\pi}\right)^{-8/3} f(e)} \quad (22)$$

(this equation was used to compare the observations of the Hulse-Taylor pulsar, which has $e = 0.617$, to the theoretical prediction of a decreasing period due to GW emission).

Fourier Transform of the Orbit

- The orbit $x(t), y(t)$ is a periodic function of u or β with period 2π . Restricting β to $-\pi \leq \beta \leq \pi$ we write the *discrete Fourier transform*

$$x(\beta) = \sum_{n=-\infty}^{\infty} \tilde{x}_n e^{-in\beta}$$

$$y(\beta) = \sum_{n=-\infty}^{\infty} \tilde{y}_n e^{-in\beta}$$

with $\tilde{x}_n = \tilde{x}_{-n}^*$ and $\tilde{y}_n = \tilde{y}_{-n}^*$ (since $x(\beta)$ and $y(\beta)$ are real functions).

- Choosing the origin of time such as $y(t=0) = 0$

$$x(\beta) = \sum_{n=0}^{\infty} a_n \cos(n\beta) \tag{23}$$

$$y(\beta) = \sum_{n=1}^{\infty} b_n \sin(n\beta) \tag{24}$$

with $a_0 = \tilde{x}_0$ and $a_n = 2\tilde{x}_n$ and $b_n = -2i\tilde{y}_n$ for $n \geq 1$.

- With $\beta = \omega_0 t$ and $\omega_n = n\omega_0$

$$x(t) = \sum_{n=0}^{\infty} a_n \cos \omega_n t \quad (25)$$

$$y(t) = \sum_{n=1}^{\infty} b_n \sin \omega_n t \quad (26)$$

with

$$a_0 = \frac{1}{\pi} \int_0^\pi d\beta x(\beta) = -(3/2)ae$$

and

$$a_n = \frac{2}{\pi} \int_0^\pi d\beta x(\beta) \cos(n\beta) = \frac{a}{n} [J_{n-1}(ne) - J_{n+1}(ne)] \quad (27)$$

$$b_n = \frac{2}{\pi} \int_0^\pi d\beta y(\beta) \sin(n\beta) = \frac{b}{n} [J_{n-1}(ne) + J_{n+1}(ne)] \quad (28)$$

where $J(x)$ are Bessel functions.

- To compute the GW spectrum, we need the Fourier decomposition of $x^2(t)$, $y^2(t)$ and $x(t)y(t)$, which are

$$x^2(t) = \sum_{n=0}^{\infty} A_n \cos \omega_n t$$

$$y^2(t) = \sum_{n=0}^{\infty} B_n \cos \omega_n t$$

$$x(t)y(t) = \sum_{n=1}^{\infty} C_n \sin \omega_n t$$

where

$$A_n = \frac{a^2}{n} [J_{n-2}(ne) - J_{n+2}(ne) - 2e(J_{n-1}(ne) - J_{n+1}(ne))]$$

$$B_n = \frac{b^2}{n} [J_{n+2}(ne) - J_{n-2}(ne)]$$

$$C_n = \frac{ab}{n} [J_{n+2}(ne) + J_{n-2}(ne) - e(J_{n+1}(ne) + J_{n-1}(ne))]$$

- Then, the radiated power is a sum of harmonics

$$P = \sum_{n=1}^{\infty} P_n$$

where

$$P_n = \frac{G\mu^2\omega_0^6}{15c^5} n^6 (A_n^2 + B_n^2 + 3C_n^2 - A_n B_n)$$

This can be written as

$$P_n = \frac{32G^4\mu^2m^3}{5c^5a^5} g(n, e)$$

where

$$g(n, e) = \frac{n^6}{96a^4} [A_n^2(e) + B_n^2(e) + 3C_n^2(e) - A_n(e)B_n(e)]$$

Power of Harmonics for Elliptical Orbits

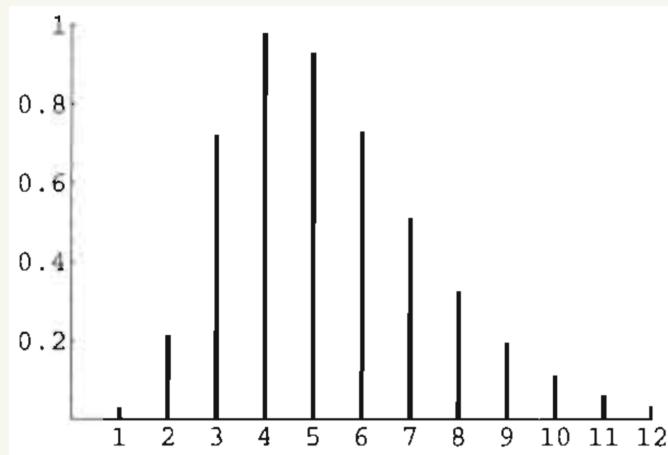


Figure 5: The power P_n as function of n for $e = 0.5$. Figure from [2].

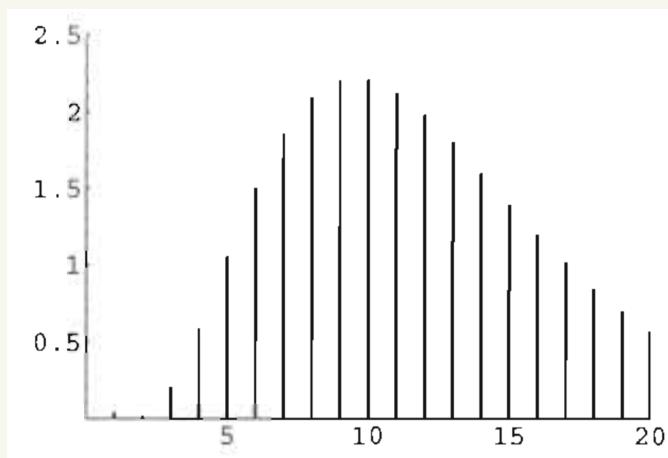


Figure 6: The power P_n as function of n for $e = 0.7$. Figure from [2].

Evolution of Orbital Parameters

- The energy and angular momentum of the orbit evolve as

$$\begin{aligned}\frac{dE}{dt} &= -\frac{32}{5} \frac{G^4 \mu^2 m^3}{c^5 a^5} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \\ \frac{dL}{dt} &= -\frac{32}{5} \frac{G^{7/2} \mu^2 m^{5/2}}{c^5 a^{7/2}} \frac{1}{(1-e^2)^2} \left(1 + \frac{7}{8} e^2\right)\end{aligned}$$

which can be written as evolution equations for a and e

$$\begin{aligned}\frac{da}{dt} &= -\frac{64}{5} \frac{G^3 \mu m^2}{c^5 a^3} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \\ \frac{de}{dt} &= -\frac{304}{15} \frac{G^3 \mu m^2}{c^5 a^4} \frac{e}{(1-e^2)^{5/2}} \left(1 + \frac{121}{304} e^2\right)\end{aligned}$$

Notice that for $e > 0 \Rightarrow de/dt < 0$ (elliptic orbits circularize due to emission of GWs) and that for $e = 0 \Rightarrow de/dt = 0$ (circular orbits remain circular).

Evolution of Orbital Parameters

- Numerically it is challenging to compute $a(t)$ and $e(t)$ over large timescales, but $a(e)$ can be determined analytically, by solving the equation

$$\frac{da}{de} = \frac{12}{19}a \frac{1 + (73/24)e^2 + (37/96)e^4}{e(1 - e^2)[1 + (121/304)e^2]}$$

We find

$$a(e) = c_0 \frac{e^{12/19}}{1 - e^2} \left(1 + \frac{121}{304}e^2\right)^{870/2299}$$

where c_0 is determined by the initial condition $a = a_0$ when $e = e_0$.

- The Hulse-Taylor binary pulsar has $a_0 = 2 \times 10^9$ m and $e = 0.617$ today. By the time the separation becomes $a \simeq 1000$ km (~ 100 neutron star radii) the eccentricity will have become $e \simeq 6 \times 10^{-6}$, practically circular.

Evolution of Orbital Parameters

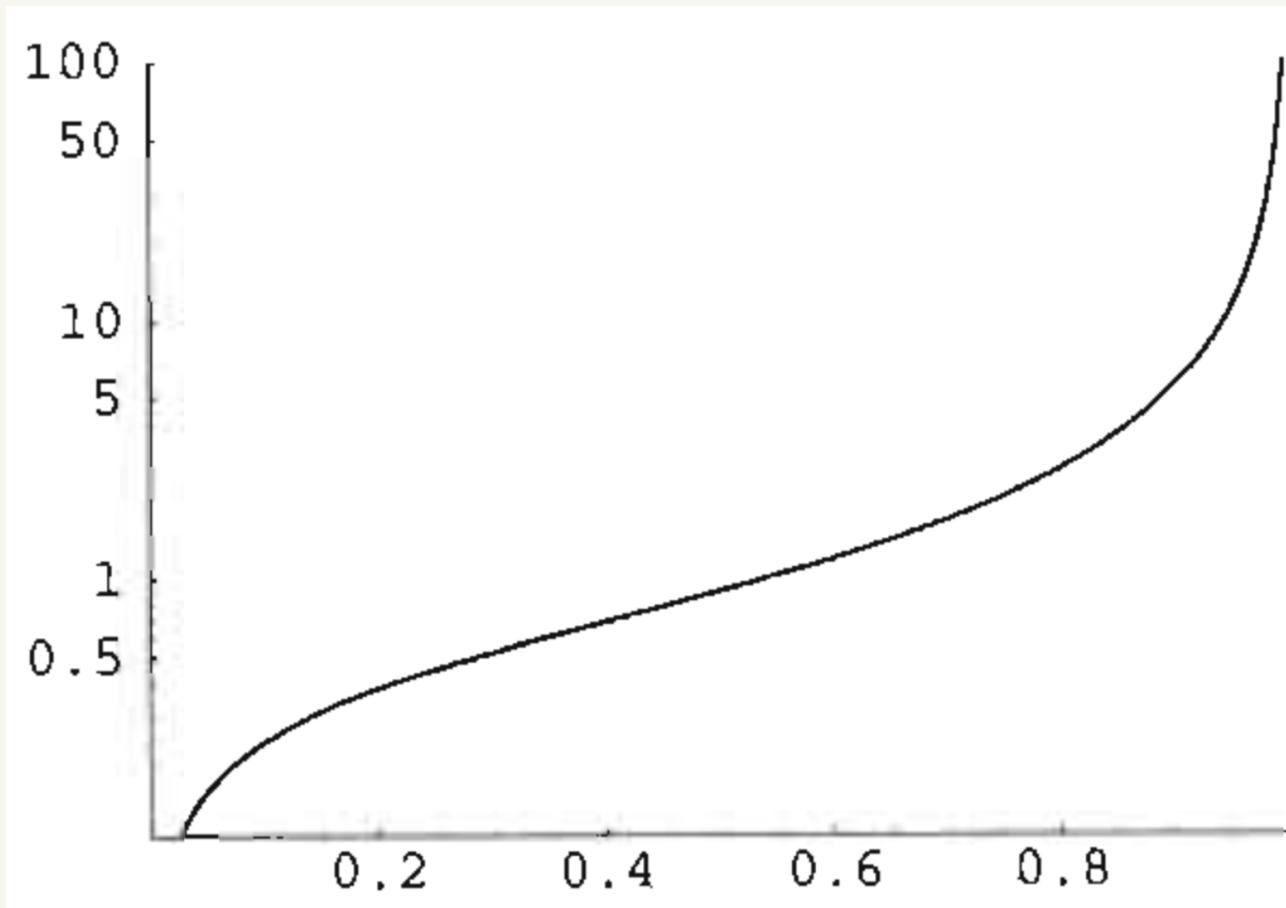


Figure 7: The scaled semi-major axis $a(e)/c_0$ as a function of e . Figure from [2].

Time to Coalescence

- The time to coalescence for an elliptical orbit with initial a_0 and e_0 is

$$\tau_0(a_0, e_0) \simeq 9.83 \times 10^6 \text{ yr} \left(\frac{T_0}{1 \text{ hr}} \right)^{8/3} \left(\frac{M_\odot}{m} \right)^{2/3} \left(\frac{M_\odot}{\mu} \right) F(e_0)$$

where

$$F(e_0) = \frac{48}{19} \frac{1}{g^4(e_0)} \int_0^{e_0} de \frac{g^4(e) (1 - e^2)^{5/2}}{e (1 + \frac{121}{304} e^2)}$$

where

$$g(e) = \frac{e^{12/19}}{1 - e^2} \left(1 + \frac{121}{304} e^2 \right)^{870/2299}$$

- For the Hulse-Taylor binary pulsar, $T_0 = 7.75 \text{ h}$, $e_0 = 0.617$ and $m_1 = m_2 \simeq 1.4 M_\odot$ and we find a time to coalescence of $\simeq 300 \text{ Myr}$.

Binaries at Cosmological Distances

- Advanced LIGO can detect binary BH mergers out to a few Gpc ($z \sim 0.25 - 0.5$, while LISA will reach $z \sim 5 - 10$).
- The metric in an FRW cosmological model is

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

where $a(t)$ is a scale factor and $k = 0$ for a spatially flat universe or $k = \pm 1$ for a spatially closed or open universe. The coordinates t, r, θ, ϕ are comoving coordinates (galaxies remain at fixed coordinates as the universe expands by the scale factor $a(t)$).

- Two galaxies that differ by coordinate distance $dr = r_2 - r_1$, differ by physical distance

$$r_{\text{phys}}(t) = a(t) \int_{r_1}^{r_2} \frac{dr}{(1 - kr^2)^{1/2}}$$

Binaries at Cosmological Distances

- Light signals travel along the light cone ($ds^2 = 0$). For a signal emitted at $r = r_2$ at time $t = t_{\text{emis}}$ and received at $r = r_1$ at time $t = t_{\text{obs}}$

$$\int_{t_{\text{emis}}}^{t_{\text{obs}}} \frac{cdt}{a(t)} = \int_{r_1}^{r_2} \frac{dr}{(1 - kr^2)^{1/2}}$$

A second signal is emitted at time $t = t_{\text{emis}} + \Delta t_{\text{emis}}$ and observed at time $t = t_{\text{obs}} + \Delta t_{\text{obs}}$. Then

$$\int_{t_{\text{emis}} + \Delta t_{\text{emis}}}^{t_{\text{obs}} + \Delta t_{\text{obs}}} \frac{cdt}{a(t)} = \int_{r_1}^{r_2} \frac{dr}{(1 - kr^2)^{1/2}}$$

The right side is the same and for $\Delta t_{\text{emis,obs}} \ll (t_{\text{obs}} - t_{\text{emis}})$ we find

$$\Delta t_{\text{obs}} = \frac{a(t_{\text{obs}})}{a(t_{\text{emis}})} \Delta t_{\text{emis}}$$

Redshift

- The redshift z of the source is defined by

$$1 + z = \frac{a(t_{\text{obs}})}{a(t_{\text{emis}})}$$

Then, the observed time interval is thus larger by a factor of $1 + z$

$$\Delta t_{\text{obs}} = (1 + z) \Delta t_{\text{emis}}$$

The observed wavelength is

$$\lambda_{\text{obs}} = (1 + z) \lambda_{\text{emis}}$$

and the observed frequency is

$$f_{\text{obs}} = \frac{f_{\text{emis}}}{1 + z}$$

and the observed energy is

$$E_{\text{obs}} = \frac{E_{\text{emis}}}{1 + z}$$

- If the emitted luminosity is

$$\mathcal{L} = \frac{dE_{\text{emis}}}{dt_{\text{emis}}}$$

then the observed luminosity is

$$\frac{dE_{\text{obs}}}{dt_{\text{obs}}} = \frac{1}{(1+z)^2} \frac{dE_{\text{emis}}}{dt_{\text{emis}}} = \frac{\mathcal{L}}{(1+z)^2}$$

- The spherical area at a coordinate distance r from a source is

$$A = 4\pi a^2(t) r^2$$

- The flux that the observer receives is

$$\mathcal{F} = \frac{1}{A} \frac{dE_{\text{obs}}}{dt_{\text{obs}}} = \frac{\mathcal{L}}{4\pi a^2(t_{\text{obs}}) r^2 (1+z)^2} = \frac{\mathcal{L}}{4\pi d_L^2}$$

where

$$d_L = (1+z)a(t_{\text{obs}})r$$

is the *luminosity distance*, which can be calculated if \mathcal{L} and \mathcal{F} are known.

Hubble Parameter

- Taylor expanding $a(t)$ around the present time, we can write

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + \dots$$

where the Hubble constant is

$$H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)}$$

and the deceleration parameter is

$$\begin{aligned} q_0 &\equiv -\frac{\ddot{a}(t_0)}{a(t_0)} \frac{1}{H_0^2} \\ &= -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}^2(t_0)} \end{aligned}$$

Since $a(t_0)/a(t) = 1 + z$, we can invert the expansion as

$$\boxed{\frac{H_0 d_L(z)}{c} = z + \frac{1}{2}(1 - q_0)z^2 + \dots}$$

The first term is Hubble's law: $cz \simeq H_0 d_L$, valid for small redshifts only.

Hubble Parameter

- More generally,

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}$$

and since $a(t)$ is a function of z , so is the Hubble parameter $H = H(z)$.

- For example, for a flat universe ($k = 0$) we find

$$\boxed{\frac{c}{H(z)} = \frac{d}{dz} \left(\frac{d_L(z)}{1+z} \right)}$$

An observational determination of $d_L(z)$ will allow us to calculate $H(z)$, thus $d_L(z)$ encodes the whole expansion history of the universe.

Gravitational Waves from Cosmological Distances

- The time to coalesce in the observer's frame is $\tau_{\text{obs}} = (1 + z)\tau_s$. The two polarizations are then

$$h_+(\tau_{\text{obs}}) = h_c(\tau_{\text{obs}}) \frac{1+\cos^2\iota}{2} \cos [\Phi(\tau_{\text{obs}})]$$
$$h_\times(\tau_{\text{obs}}) = h_c(\tau_{\text{obs}}) \cos\iota \sin [\Phi(\tau_{\text{obs}})]$$

where

$$\Phi(\tau_{\text{obs}}) = -2 \left(\frac{5G\mathcal{M}_c(z)}{c^3} \right)^{-5/8} \tau_{\text{obs}}^{5/8} + \Phi_0$$

and

$$h_c(\tau_{\text{obs}}) = \frac{4}{d_L(z)} \left(\frac{G\mathcal{M}_c(z)}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}^{(\text{obs})}(\tau_{\text{obs}})}{c} \right)^{2/3}$$

where the observed frequency is

$$f_{\text{gw}}^{(\text{obs})}(\tau_{\text{obs}}) = \frac{1}{\pi} \left(\frac{5}{256} \frac{1}{\tau_{\text{obs}}} \right)^{3/8} \left(\frac{G\mathcal{M}_c(z)}{c^3} \right)^{-5/8}$$

and we defined the redshifted chirp mass

$$\mathcal{M}_c = (1 + z)M_c$$

Detector Response

- The plane wave solution at the detector is

$$h_{ij}(t) = \int_{-\infty}^{\infty} df \tilde{h}_{ij}(f) e^{-2\pi ift}$$

If the wave is traveling along $\hat{\mathbf{n}}$ and we denote as $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ the two unit vectors orthogonal to $\hat{\mathbf{n}}$ and to each other, then:

$$\begin{aligned} h_{ij}(t) &= e_{ij}^+(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} df \tilde{h}_+(f) e^{-2\pi ift} + e_{ij}^\times(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} df \tilde{h}_\times(f) e^{-2\pi ift} \\ &= \sum_{A=+, \times} e_{ij}^A(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} df \tilde{h}_A(f) e^{-2\pi ift} \\ &= \sum_{A=+, x} e_{ij}^A(\hat{\mathbf{n}}) h_A(t) \end{aligned}$$

where the two *polarization tensors* are

$$e_{ij}^+(\hat{\mathbf{n}}) = \hat{\mathbf{u}}_i \hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i \hat{\mathbf{v}}_j$$

$$e_{ij}^\times(\hat{\mathbf{n}}) = \hat{\mathbf{u}}_i \hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i \hat{\mathbf{u}}_j$$

Detector Response

- If we choose $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $\hat{\mathbf{u}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{v}} = \hat{\mathbf{y}}$, then

$$e_{ij}^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ij}$$

$$e_{ij}^\times = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ij}$$

- The detector acts as a *linear system*. The effect of the detector on the signal is described by the *detector tensor* D^{ij} and the *detector input* is

$$h(t) = D^{ij} h_{ij}(t)$$

For a linear system, the Fourier transform of the *detector output* $h_{\text{out}}(t)$ is related to the Fourier transform of the detector input through

$$\tilde{h}_{\text{out}}(f) = T(f) \tilde{h}(f)$$

where $T(f)$ is the *transfer function* of the system.

Detector Noise

- The output of the detector will include noise $n_{\text{out}}(t)$, so that the total signal in the output is

$$s_{\text{out}}(t) = h_{\text{out}}(t) + n_{\text{out}}(t)$$

We can define the *input noise* $n(t)$ by

$$\tilde{n}(f) = T^{-1}(f)\tilde{n}_{\text{out}}(f)$$

It is a fictitious noise that if it were injected at the detector input, without any other noise present in the system, it would produce $n_{\text{out}}(t)$ at the output.

- We define the total signal at the input as

$$s(t) = h(t) + n(t)$$

so that we can compare the input signal to the input noise.

Detector Noise

- The *auto-correlation function* of the noise is

$$R(\tau) \equiv \langle n(t + \tau)n(t) \rangle$$

where $\langle \rangle$ is a time average. As τ increases, the noise at time $t + \tau$ becomes more and more uncorrelated from the noise at time t . For white noise, $R(\tau) \sim \delta(\tau)$, otherwise $R(\tau) \sim \exp\{-|\tau|/\tau_c\}$ where τ_c is a characteristic timescale. Since $R(\tau)$ goes to 0 very fast as $t \rightarrow \pm\infty$ it can be Fourier transformed:

$$R(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} df S_n(f) e^{-i2\pi f \tau}$$

and then

$$\langle n^2(t) \rangle = \int_0^{\infty} df S_n(f)$$

The factor 1/2 is used by convention, so that $S_n(f)$ is the *one-sided noise spectral density or one-sided power spectral density (PSD)*

$$S_n(f) \equiv 2 \int_{-\infty}^{\infty} d\tau R(\tau) e^{i2\pi f \tau}$$

Characteristic Strain and Amplitude Spectral Density

- For a signal $h(t)$, we define the *characteristic strain* as

$$[h_c(f)]^2 = 4f^2|\tilde{h}(f)|^2$$

with $h_c(f) = \sqrt{N_{\text{cycles}}}|\tilde{h}(f)|$ and for a detector with PSD $S_n(f)$, we define the *characteristic noise* as

$$[h_n(f)]^2 = fS_n(f)$$

Both $h_c(f)$ and $h_n(f)$ are dimensionless.

- From the last equation, we obtain the *amplitude spectral density* of the noise

$$\sqrt{S_n(f)} = h_n(f)f^{-1/2}$$

and by analogy, we define an equivalent function for the signal

$$\sqrt{S_h(f)} = h_c(f)f^{-1/2} = 2f^{1/2}|\tilde{h}(f)|$$

These have units of $\text{Hz}^{-1/2}$ and are the most commonly used definitions for the sensitivity curves.

Detector Noise

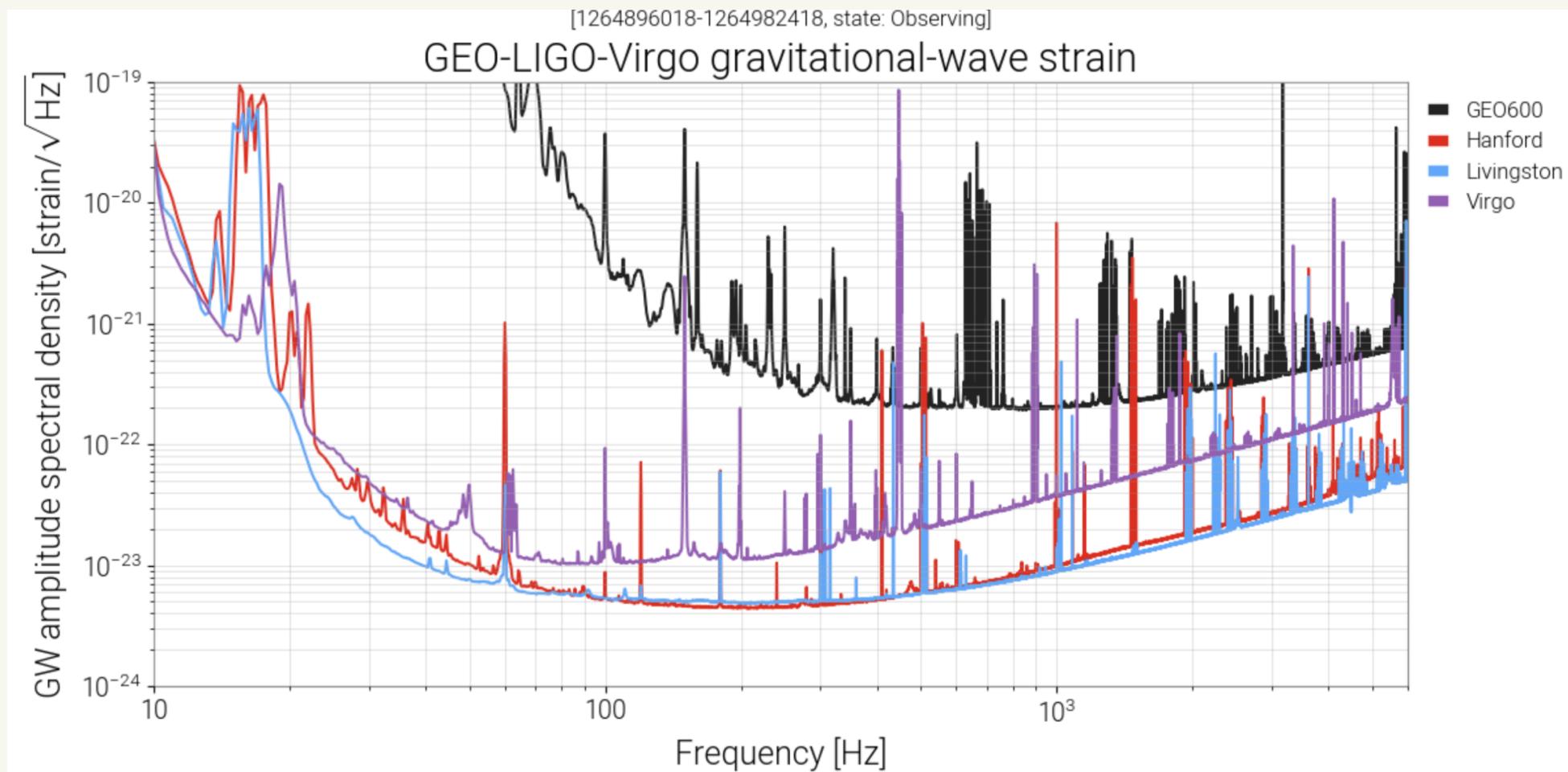


Figure 8: Sensitivity curves for 4 different detectors operating on Feb 4, 2020.

Matched Filtering

- When $s(t) = h(t) + n(t)$ we can calculate for an observation time T

$$\frac{1}{T} \int_0^T dt s(t)h(t) = \frac{1}{T} \int_0^T dt h^2(t) + \frac{1}{T} \int_0^T dt n(t)h(t)$$

Because $h(t) \sim h_0 \cos(\omega t)$, the first term on the right side becomes for large T

$$\frac{1}{T} \int_0^T dt h^2(t) \sim h_0^2$$

But, the second integral over the arbitrarily oscillating quantity $n(t)h(t)$ grows only as $T^{1/2}$ (typical for random walk) so that

$$\frac{1}{T} \int_0^T dt n(t)h(t) \sim \left(\frac{\tau_0}{T}\right)^{1/2} n_0 h_0$$

where n_0 is the characteristic amplitude of the noise and τ_0 a characteristic time (e.g. the period of the wave $h(t)$). In the limit $T \rightarrow \infty$, the second term averages to zero (*the noise is filtered out*).

Optimal Signal-to-Noise Ratio

- The optimal S/N for a signal $h(t)$ and a detector with one-sided PSD $S_n(f)$ is

$$\left(\frac{S}{N}\right)^2 = 4 \int_0^\infty df \frac{|\tilde{h}(f)|^2}{S_n(f)} = \int_{-\infty}^\infty d(\log f) \left[\frac{h_c(f)}{h_n(f)} \right]^2$$

- For a coalescing binary this becomes

$$\left(\frac{S}{N}\right)^2 = \frac{5}{6} \frac{1}{\pi^{4/3}} \frac{c^2}{r^2} \left(\frac{GM_c}{c^3}\right)^{5/3} |Q(\theta, \phi; \iota)|^2 \int_{f_{\min}}^{f_{\max}} df \frac{f^{-7/3}}{S_n(f)}$$

where $Q(\theta, \phi; \iota)$ is a geometric factor. When averaged over all angles and inclinations, this factor becomes

$$\langle |Q(\theta, \phi; \iota)|^2 \rangle^{1/2} = \frac{2}{5}$$

- A detection is claimed only when $S/N > 5$.

Signal Templates

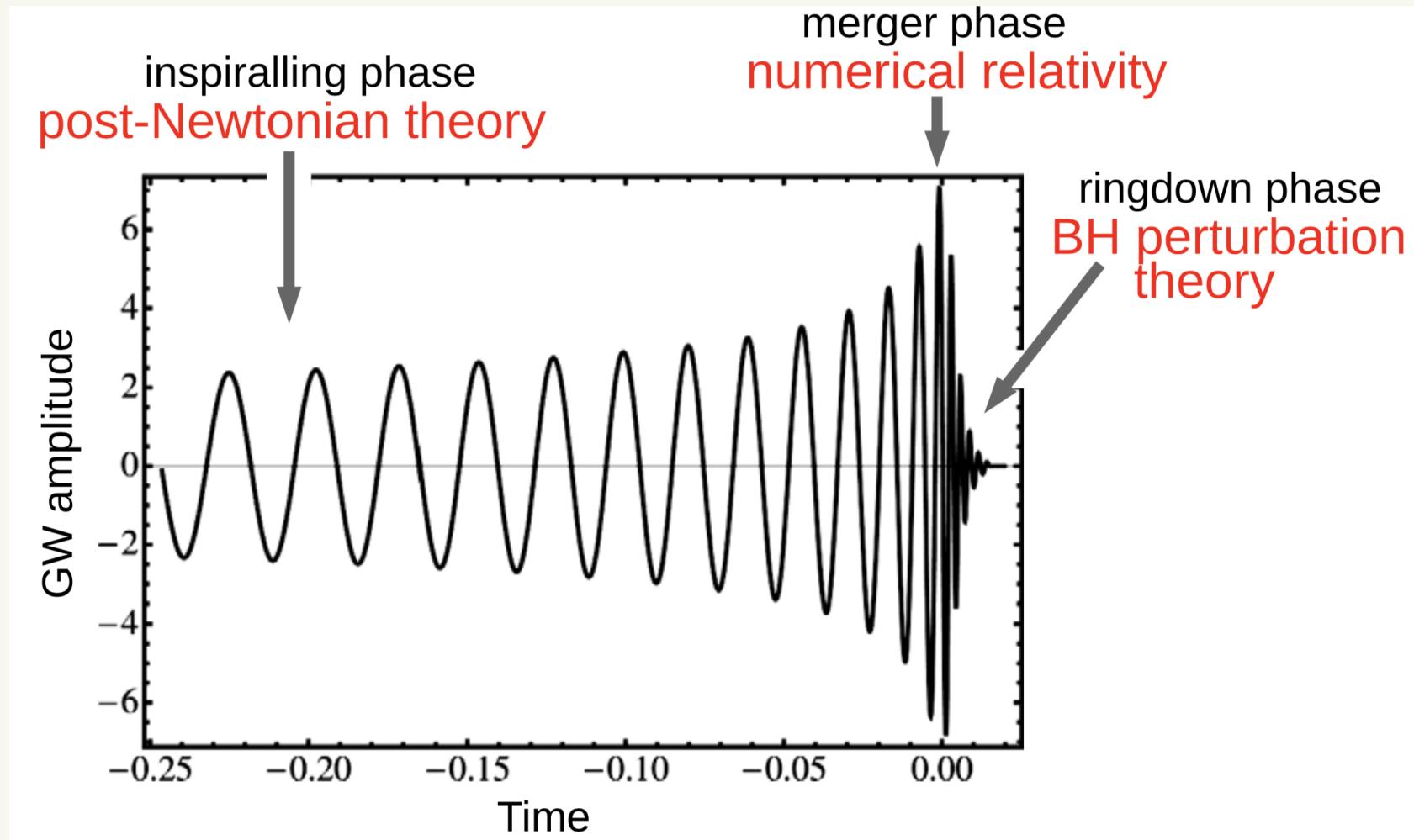


Figure 9: Construction of analytic templates for BBH signals. Figure from [1].

Detection of GW150914

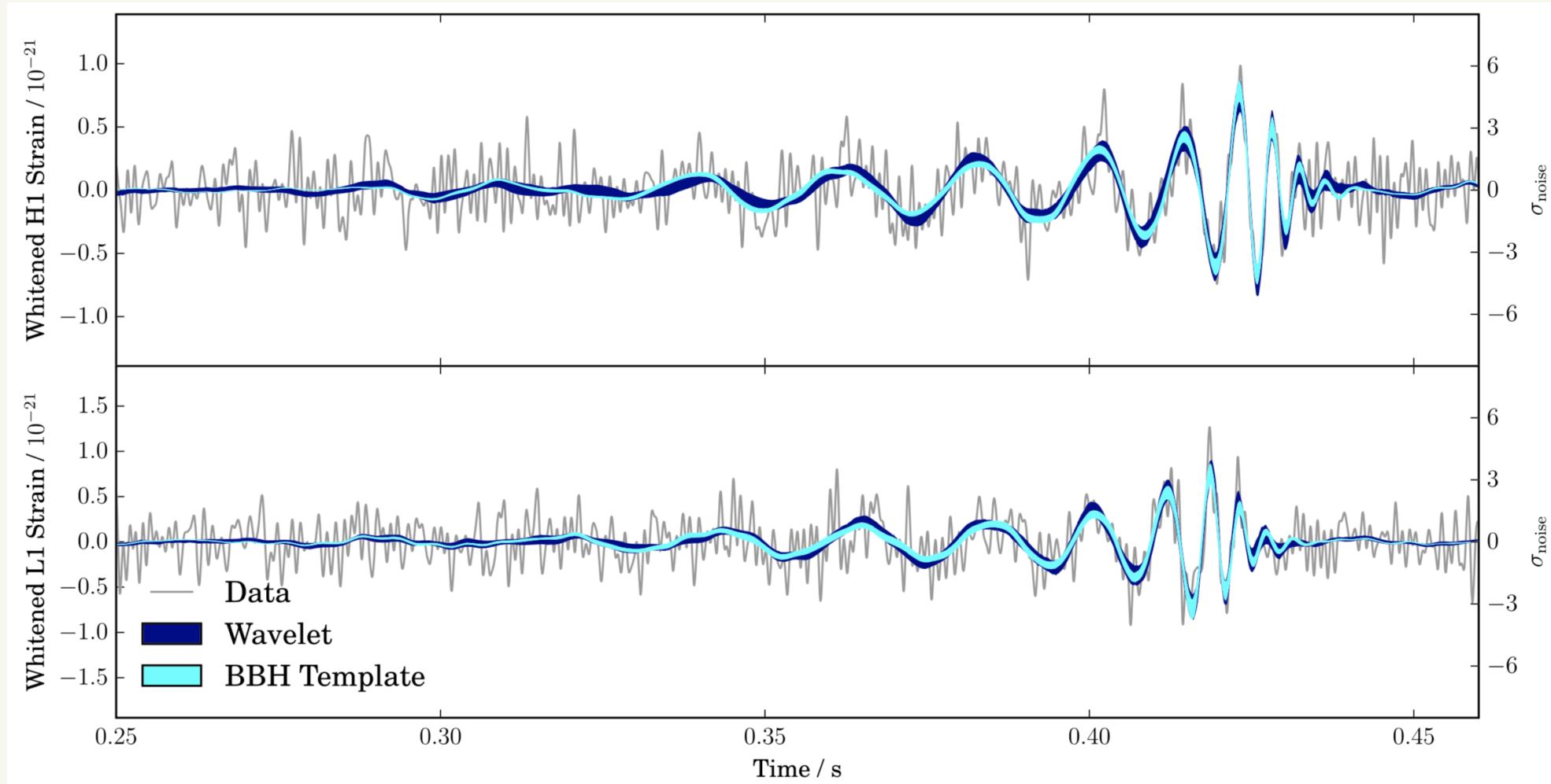


Figure 10: Comparison between data and analytic templates for BBH signal. Figure from [1].

Detector Range

- Inverting the previous relation, one can define the detector range, i.e. the distance to which a binary system can be detected with certain S/N using a detector that has one-sided noise spectral density $S_n(f)$

$$d_{\text{range}} = \frac{2}{5} \left(\frac{5}{6} \right)^{1/2} \frac{c}{\pi^{2/3}} \left(\frac{GM_c}{c^3} \right)^{5/6} \left[\int_{f_{\min}}^{f_{\max}} df \frac{f^{-7/3}}{S_n(f)} \right]^{1/2} (S/N)^{-1}$$

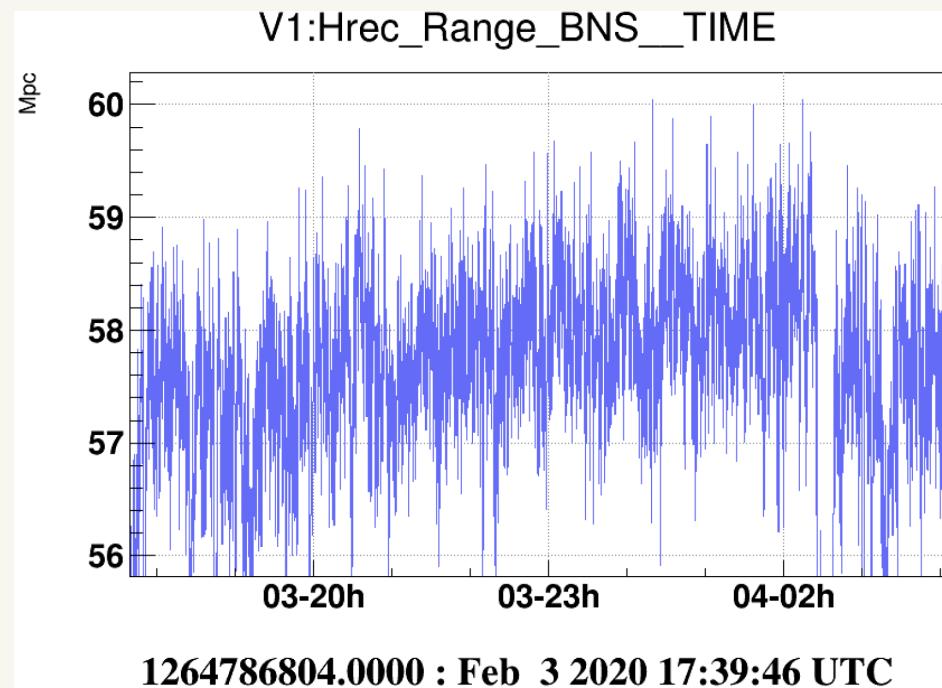


Figure 11: VIRGO detector range for a typical BNS detection.

Detector Range for BNS Inspiral

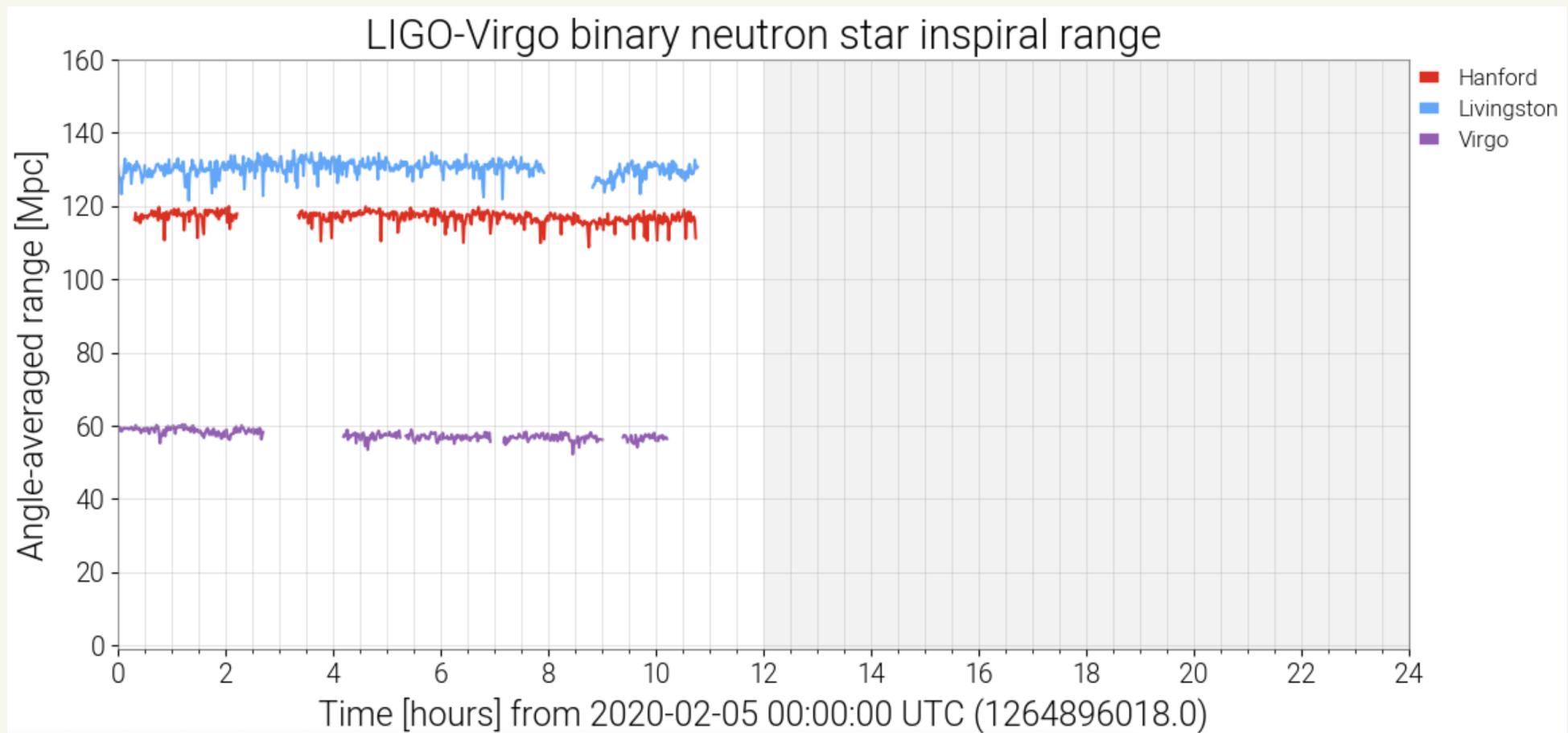


Figure 12: LIGO-VIRGO detector range for a typical BNS inspiral.

References

- [1] Luc Blanchet. Analyzing gravitational waves with general relativity. *Comptes Rendus Physique*, 20(6):507–520, Sep 2019.
- [2] M. Maggiore. *Gravitational Waves: Volume 1: Theory and Experiments*. Oxford University Press, 2008.