Lecture Notes on Relativistic Stars

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Preface

These lecture notes are distributed as part of the teaching material for the GR course in the MSc program at the Aristotle University of Thessaloniki.

Conventions

Gravitational units ($G = c = 1$) are used in equations, while numerical results are listed in appropriate units (cgs, km, $M_\odot$ etc.). The signature of the spacetime metric is $(- + ++)$. Abstract spacetime indices are Greek, α, $\beta, \ldots$, while spatial indices are Latin $a, b, \ldots$. Indices μ, ν, λ and i, j, k will be concrete, taking values $\mu = 0, 1, 2, 3$, $i = 1, 2, 3$ etc.

Numbers that rely on physical constants are based on the values $c = 2.9979 \times 10^{10}$ cm s^{-1}, $G = 6.670 \times 10^{-8}$g^{-1}cm^{3}s^{-2}, $\hbar = 1.0545 \times 10^{-27}$g cm^{2}s^{-1}, baryon mass $m_B = 1.659 \times 10^{-24}$g, and $M_\odot = 1.989 \times 10^{33}$g $= 1.477$ km.

1 Perfect fluids

The stress-energy tensor. In a perfect fluid one assumes that a mean velocity field u^α and a mean stress-energy tensor $T^{\alpha\beta}$ can be defined in fluid elements small compared to the macroscopic length scale but large compared to the mean free path. An observer moving with the average velocity u^α of the fluid will see the collisions randomly distribute the nearby particle velocities so that the particle distribution will appear locally isotropic. Therefore, the components of the fluid's energy momentum tensor in frame of a comoving observer must have no preferred direction and $T^{\alpha\beta}u_\beta$ must be invariant under rotations that fix u^α. It follows that the only nonzero parts of $T^{\alpha\beta}$ are the rotational scalars

$$\epsilon := T^{\alpha\beta}u_\alpha u_\beta, \tag{1}$$

and

$$p := \frac{1}{3}q_{\gamma\delta}T^{\gamma\delta}, \tag{2}$$

where $q_{\gamma\delta} = g_{\gamma\delta} + u_\gamma u_\delta$ is the projection tensor normal to the fluid. For such an isotropic, ideal fluid the stress-energy tensor takes the form

$$\boxed{T^{\alpha\beta} = \epsilon u^\alpha u^\beta + p q^{\alpha\beta}}. \tag{3}$$

The scalars ϵ and p are the *energy density* and the *pressure*, as measured by a *comoving* observer.

Thermodynamics. We denote by n the baryon number density. The rest-mass density (baryon-mass density) is then

$$\rho := m_B n, \tag{4}$$

where m_B is the mass per baryon. We restrict attention to the case of a perfect fluid with equilibrium composition, where the ϵ and p depend on ρ and the *specific entropy* (entropy per unit rest mass) s,

$$\epsilon = \epsilon(\rho, s), \qquad\qquad p = p(\rho, s), \tag{5}$$

(or, on equivalent sets of parameters). The thermodynamics of the fluid is described by the first law, which takes the form

$$\boxed{d\epsilon = \rho T ds + h d\rho}, \tag{6}$$

where T is *temperature* and h is the *specific enthalpy* (enthalpy per unit rest mass),

$$\boxed{h := \frac{\epsilon + p}{\rho}}. \tag{7}$$

Exercise 1.1: Derive (6) from its more common form in terms of extensive quantities

$$dE = TdS - p\,dV + \mu\,dN \equiv TdS - p\,dV + g\,dM_0, \tag{8}$$

by introducing the energy E, entropy S, volume V, baryon number N, and rest mass $M_0 = m_B N$ of a fluid element as measured by a comoving observer. Here $\mu = g m_B$, where $g = \frac{\epsilon+p}{\rho} - Ts$ is the *Gibbs free energy.*

The *specific internal energy* (internal energy per unit rest mass) e is defined by the relation

$$e = \frac{\epsilon}{\rho} - 1. \tag{9}$$

The *Newtonian expression for the specific enthalpy* is

$$h_{\text{Newtonian}} = h - 1 = e + p/\rho. \tag{10}$$

and differs from the relativistic enthalpy h because the relativistic energy density ϵ includes the rest-mass density ρ.

Baroclinic flow. From the definition (7) and using (6), one finds

$$\begin{aligned} dh &= \frac{d\epsilon}{\rho} + \frac{dp}{\rho} - \frac{\epsilon + p}{\rho^2} d\rho = Tds + h\frac{d\rho}{\rho} + \frac{dp}{\rho} - h\frac{d\rho}{\rho}, \\ &= Tds + \frac{dp}{\rho}, \\ \Rightarrow \quad & \boxed{d\ln h = \frac{T}{h} ds + \frac{dp}{\epsilon + p}}, \end{aligned} \tag{11a}$$

implying

$$\nabla_\alpha \ln h = \frac{T}{h}\nabla_\alpha s + \frac{1}{\epsilon + p}\nabla_\alpha p. \tag{12}$$

Exercise 1.2: Taking the curl of (12), show that in the presence of entropy gradients ($\nabla s \neq 0$), surfaces of constant energy density (*isopycnic surfaces*) do not, in general, coincide with surfaces of constant pressure (*isobaric surfaces*). Such a flow is called *baroclinic* and, for a rotating star, it implies the presence of *meridional circulation.*

Barotropic flow. Within a short time after formation, neutrino emission cools a newly born neutron star to $10^{10}K \simeq 1$ MeV, which is much smaller than the Fermi energy of the interior, $E_F(0.16\,\mathrm{fm}^{-3}) \approx 60\,\mathrm{MeV}$. A neutron star is in this sense cold, and, because nuclear reaction times are shorter than the cooling time, one can use a zero-temperature *equation of state* (EOS) to describe the matter:

$$\epsilon = \epsilon(\rho), \quad p = p(\rho), \tag{13}$$

or, equivalently,

$$\epsilon = \epsilon(p). \tag{14}$$

A one-parameter equation of state of the form (14) holds, also in more general situations, such as when the specific entropy is constant throughout the star ($\nabla s = 0$, *homentropic flow).* or even when $s = s(\rho)$, so that effectively the energy density still depends on one parameter only. In such cases, the isopycnic and isobaric surfaces coincide and the flow is *barotropic.*

In a homentropic star, the first law, (6), takes the form

$$d\epsilon = h d\rho, \tag{15}$$

and using (11a) the specific enthalpy is also given by

$$h = \exp\left(\int_0^p \frac{dp}{\epsilon + p}\right), \tag{16}$$

(notice that $\epsilon/\rho = 1$ at $p = 0$, since the gas is nonrelativistic at low densities).

Conservation of baryons. The baryon mass M_0 of a fluid element is conserved by the motion of the fluid. The proper volume of a fluid element is the volume V of a slice orthogonal to u^α through the history of the fluid element; and conservation of baryons can be written in the form $0 = \Delta M_0 = \Delta(\rho V)$. The fractional change in V in a proper time $\Delta\tau$ is given by the 3-dimensional divergence of the velocity, in the subspace orthogonal to u^α:

$$\frac{\Delta V}{V} = q^{\alpha\beta}\nabla_\alpha u_\beta \Delta\tau. \tag{17}$$

Because $u^\beta u_\beta = -1$, we have $u^\beta \nabla_\alpha u_\beta = \frac{1}{2}\nabla_\alpha(u_\beta u^\beta) = 0$, implying

$$q^{\alpha\beta}\nabla_\alpha u_\beta = \nabla_\beta u^\beta. \tag{18}$$

With $u^\alpha \nabla_\alpha \rho = \dfrac{d}{d\tau}\rho$, conservation of baryons takes the form

$$0 = \frac{\Delta(\rho V)}{V} = \Delta\rho + \rho\frac{\Delta V}{V} = (u^\alpha\nabla_\alpha\rho + \rho\nabla_\alpha u^\alpha)\Delta\tau, \tag{19}$$

or

$$\boxed{\nabla_\alpha(\rho u^\alpha) = 0}. \tag{20}$$

Conservation of stress-energy tensor. The Bianchi identities and the Einstein field equations imply that the stress-energy tensor is conserved

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{21}$$

By projecting this conservation along the fluid velocity and normal to it, one obtains separate equations for the conservation of energy and momentum.

Exercise 1.3: Show that the projection along the fluid velocity $u_\alpha \nabla_\beta T^{\alpha\beta} = 0$ leads to

$$\boxed{\nabla_\beta(\epsilon u^\beta) = -p\nabla_\beta u^\beta}. \tag{22}$$

(*conservation of energy*) and that the projection normal to the fluid velocity $q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} = 0$ leads to

$$\boxed{(\epsilon + p)u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta}\nabla_\beta p}, \tag{23}$$

(*conservation of momentum* or equations of motion).

Exercise 1.4: For a barotropic fluid with constant entropy (a homentropic fluid) write the relativistic Euler equation in the form

$$u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta}\nabla_\beta \ln h. \tag{24}$$

or, equivalently,

$$u^\beta \omega_{\alpha\beta} = 0, \tag{25}$$

where

$$\omega_{\alpha\beta} = \nabla_\alpha(h\, u_\beta) - \nabla_\beta(h\, u_\alpha), \tag{26}$$

is the *relativistic vorticity.*

Spacetime symmetries. A vector field ξ^α is a Killing vector if it Lie derives the metric,

$$\mathcal{L}_\xi g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0. \tag{27}$$

We will call ξ^α a *symmetry vector* of a perfect-fluid spacetime if ξ^α is a Killing vector that also Lie derives the fluid variables:

$$\mathcal{L}_\xi u^\alpha = 0, \quad \mathcal{L}_\xi \epsilon = 0, \quad \mathcal{L}_\xi p = 0. \tag{28}$$

Exercise 1.5: Show that $hu_\beta\, \xi^\beta$ is conserved along the spacetime trajectories of the fluid

$$\mathcal{L}_{\rm u}\left(h\, u_\beta \xi^\beta\right) = 0. \tag{29}$$

Stationary flow - Bernoulli's law. If a spacetime has an asymptotically timelike symmetry vector, t^α, the flow is stationary. The corresponding conservation law

$$\boxed{\mathcal{L}_{\rm u}(h\, u_t) = 0}, \tag{30}$$

is the relativistic form of Bernoulli's law, the conservation of *enthalpy per unit rest mass*, $-hu_t$, along the trajectories of a stationary flow.

Axisymmetric Flow. An axisymmetric flow is described by a spacetime with a rotational symmetry vector, ϕ^α, a spacelike vector field whose orbits are circles, except on an axis of symmetry, where $\phi^\alpha = 0$. The corresponding conservation law

$$\boxed{\mathcal{L}_{\rm u}(h\,u_\phi) = 0}, \tag{31}$$

expresses the conservation of a fluid element's *specific angular momentum*, $j := hu_\phi$, the angular momentum per unit rest mass about the axis of symmetry associated with ϕ^α.

Isentropic flow. In the absence of shocks, the flow of a perfect fluid remains *isentropic*, i.e. each fluid element conserves its specific entropy along its trajectory,

$$u^\alpha \nabla_\alpha s = 0. \tag{32}$$

Formally, the relation follows from conservation of baryons (20), conservation of energy (22), and from the first law (6).

Exercise 1.6: Show that in barotropic flows the *relativistic vorticity* $\omega_{\alpha\beta}$ is conserved along the fluid trajectories

$$\mathcal{L}_{\rm u}\omega_{\alpha\beta} = 0. \tag{33}$$

Equivalently, the *circulation* of the flow along a closed curve

$$\int_{c_\tau} h\,u_\alpha\, dl^\alpha = \int_{c_\tau} \frac{\epsilon + p}{\rho}\, u_\alpha\, dl^\alpha, \tag{34}$$

is independent of τ, conserved by the fluid flow.

2 The spacetime of a rotating star

A rotating star can be modeled by a stationary, axisymmetric, perfect-fluid spacetime, whose circular velocity field u^α can be written in terms of the two Killing vectors t^α and ϕ^α,

$$\boxed{u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha)}, \tag{35}$$

where

$$\boxed{\Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi}{dt}}, \tag{36}$$

is the angular velocity of the fluid as seen by an observer at rest at infinity. A star is called *uniformly rotating* (as seen from infinity) if Ω is constant.

Exercise 2.1: Defining the *shear tensor* as

$$\sigma_{\alpha\beta} := q_\alpha{}^\gamma q_\beta{}^\delta \nabla_{(\gamma} u_{\delta)} - \frac{1}{3} q_{\alpha\beta} \nabla_\gamma u^\gamma, \tag{37}$$

show that, locally, the flow is shear-free if and only if rotation is uniform. Thus, uniform rotation corresponds to constant Ω for observers at infinity and shear-free flow for local observers.

Geometry. In order to arrive at a metric suitable for describing a rotating star, one makes the following assumptions:

1. *The spacetime is asymptotically flat.*

2. *The spacetime is stationary and axisymmetric*: There exist an asymptotically timelike symmetry vector t^α and a rotational symmetry vector ϕ^α.

   The spacetime is said to be *strictly* stationary if t^α is everywhere timelike. (Some rapidly rotating stellar models, as well as black-hole spacetimes, have *ergospheres*, regions in which t^α is spacelike.)

3. *The Killing vectors commute*,
$$[t, \phi] \equiv \mathcal{L}_t \phi^\alpha = 0, \tag{38}$$
   and there is an isometry of the spacetime that simultaneously reverses the direction of t^α and ϕ^α,
$$t^\alpha \longrightarrow -t^\alpha, \quad \phi^\alpha \longrightarrow -\phi^\alpha. \tag{39}$$

For strictly stationary spacetimes, one does not need (38) as a separate assumption, since it follows from a theorem by Carter [14].

The Frobenius theorem now implies the existence of scalars t and ϕ [34, 13] for which

$$\begin{aligned} t^\alpha \nabla_\alpha t &= \phi^\alpha \nabla_\alpha \phi = 1, &(40)\\ t^\alpha \nabla_\alpha \phi &= \phi^\alpha \nabla_\alpha t = 0. &(41) \end{aligned}$$

and one can choose coordinates $x^0 = t$ and $x^3 = \phi$ so that

$$\begin{aligned} t^\alpha &= \boldsymbol{\partial}_t, &(42)\\ \phi^\alpha &= \boldsymbol{\partial}_\phi. &(43) \end{aligned}$$

The following metric components are formed by t^α and ϕ^α

$$\begin{aligned} t_\alpha t^\alpha &= g_{tt}, &(44a)\\ \phi_\alpha \phi^\alpha &= g_{\phi\phi}, &(44b)\\ t_\alpha \phi^\alpha &= g_{t\phi}. &(44c) \end{aligned}$$

Notice that t^α and ϕ^α are not orthogonal to each other and the lack of orthogonality implies a *dragging of inertial frames.* Also, the fluid is not invariant under $t \to -t$ reversal (a rotating fluid with circular flow is not static, but only stationary, there is invariance only under the simultaneous inversion $t \to -t, \phi \to -\phi$).

On the other hand, if the flow is not circular, be there exist meridional convective currents, then there is no invariance even under the simultaneous inversion $t \to -t, \phi \to -\phi$, because the the direction of the circulation changes. In this case the asymmetry means that there will be no family of surfaces orthogonal to t^α and ϕ^α, and the spacetime metric (47) will have additional off-diagonal components.

Quasi-isotropic coordinates. The surfaces of constant t and ϕ are a family of 2-surfaces orthogonal to t^α and ϕ^α that can be described by coordinates x^1 and x^2. A common choice for x^1 and x^2 are *quasi-isotropic coordinates*, for which

$$g_{r\theta} = 0, \qquad g_{\theta\theta} = r^2 g_{rr} \tag{45}$$

in spherical polar coordinates, or

$$g_{\varpi z} = 0, \qquad g_{zz} = r^2 g_{\varpi\varpi} \tag{46}$$

in cylindrical coordinates. Because of the orthogonality of the 2-surfaces to t^α and ϕ^α, four metric components vanish: $g_{tr} = 0$, $g_{t\theta} = 0$ and $g_{\phi r} = 0$, $g_{\phi\theta} = 0$ (equivalently, $g_{t\varpi} = 0$, $g_{tz} = 0$ and $g_{\phi\varpi} = 0$, $g_{\phi z} = 0$).

Spacetime metric. It follows that the metric of an axisymmetric rotating star with circular flow can be written in the form

$$\boxed{ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2 d\theta^2)}, \tag{47}$$

or

$$\boxed{ds^2 = -e^{2\nu}\, dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz)^2}, \tag{48}$$

where ν, ψ, ω and μ are four metric functions that depend on the coordinates r and θ (or ϖ and z) only. Notice that, in the exterior vacuum one can reduce the number of metric functions to three. It is convenient to write e^ψ in the form [5]

$$e^\psi = r \sin\theta B e^{-\nu}, \tag{49}$$

where B is again a function of r and θ only.

The three metric functions ν, ψ and ω are related to the norms of the Killing vectors, t^α and ϕ^α, and to their dot product by the relations

$$t_\alpha t^\alpha = g_{tt} = -e^{2\nu} + \omega^2 e^{2\psi}, \tag{50a}$$
$$\phi_\alpha \phi^\alpha = g_{\phi\phi} = e^{2\psi}, \tag{50b}$$
$$t_\alpha \phi^\alpha = g_{t\phi} = -\omega e^{2\psi}. \tag{50c}$$

The corresponding components of the contravariant metric are

$$g^{tt} = \nabla_\alpha t \nabla^\alpha = -e^{-2\nu}, \tag{51a}$$
$$g^{\phi\phi} = \nabla_\alpha \phi \nabla^\alpha \phi = e^{-2\psi} - \omega^2 e^{-2\nu}, \tag{51b}$$
$$g^{t\phi} = \nabla_\alpha t \nabla^\alpha \phi = -\omega e^{-2\nu}. \tag{51c}$$

The geometry of the orthogonal 2-surfaces is determined by the conformal factor $e^{2\mu}$.

Dragging of inertial frames. In the spacetime of a rotating star, particles (inertial observers) dropped from infinity with zero angular momentum acquire a *nonzero angular velocity* in the direction of the star's rotation. This relativistic effect is called *dragging of inertial frames*. A freely falling particle follows a geodesic, along which its angular momentum per unit rest mass $L = u_\alpha \phi^\alpha = u_\phi$ is conserved. For a particle with $L = 0$

$$u_\phi = 0 \quad \Rightarrow \quad e^{2\psi}(u^\phi - \omega u^t) = 0,$$

so that

$$\boxed{\frac{u^\phi}{u^t} = \omega}. \tag{52}$$

Thus, an radially infalling particle that starts out with zero angular momentum at infinity, will acquire a nonzero angular velocity

$$\Omega = \omega, \tag{53}$$

(as measured by an inertial observer at infinity) even though it maintains zero angular momentum along its trajectory.

ZAMOs. In describing the fluid, it is helpful to introduce a family of zero-angular-momentum-observers (ZAMOs) [4, 5], observers whose velocity has at each point the form

$$\boxed{u^\alpha_{\rm ZAMO} = u^t(t^\alpha + \omega\phi^\alpha)}. \tag{54}$$

Several properties of a fluid can be conveniently expressed with respect to ZAMOs (see, for example, the 3-velocity, defined below).

4-Velocity. The 4-velocity for a circular flow is written as in (35)

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha). \tag{55}$$

In the metric (48) the normalization $u_\alpha u^\alpha = -1$ determines u^t as

$$u^t = \frac{e^{-\nu}}{\sqrt{1-(\Omega-\omega)^2 e^{2(\psi-\nu)}}}. \tag{56}$$

Defining

$$v := (\Omega-\omega)e^{(\psi-\nu)}, \tag{57}$$

the contravariant and covariant components of the 4-velocity take the form

$$u^t = \frac{e^{-\nu}}{\sqrt{1-v^2}}, \qquad u^\phi = \Omega u^t, \tag{58}$$

$$u_t = -\frac{e^{\nu}}{\sqrt{1-v^2}}(1+e^{\psi-\nu}\omega v), \qquad u_\phi = \frac{e^{\psi}v}{\sqrt{1-v^2}}. \tag{59}$$

Written in this way, the denominator has the form of a Lorentz factor. Indeed, as will be shown below, v is identified as the 3-velocity measured in the frame of a ZAMO.

Notice that the 4-velocity of a ZAMO becomes $u^\alpha_{\rm ZAMO} = e^{-\nu}(t^\alpha + \omega\phi^\alpha)$.

3-Velocity. The spatial 3-velocity v does not have a covariant meaning, so one has to define it with respect to a chosen physical frame. One can construct an *orthonormal tetrad*, in which the metric has locally the form of the Minkowski metric

$$ds^2 = \eta_{\mu\nu}\boldsymbol{\omega}^{\hat\mu}\otimes\boldsymbol{\omega}^{\hat\nu}, \tag{60}$$

where $\boldsymbol{\omega}^{\hat{\mu}}$ are the basis covectors (the index denotes the different vectors, not components). A suitable example is the frame defined by the basis covectors

$$\boldsymbol{\omega}^{\hat{0}} = e^{\nu}\boldsymbol{dt}, \quad \boldsymbol{\omega}^{\hat{1}} = e^{\psi}(\boldsymbol{d\phi} - \omega\boldsymbol{dt}), \quad \boldsymbol{\omega}^{\hat{2}} = e^{\mu}\boldsymbol{d\varpi}, \quad \boldsymbol{\omega}^{\hat{3}} = e^{\mu}\boldsymbol{dz}, \tag{61}$$

with corresponding contravariant basis vectors

$$\boldsymbol{e}_{\hat{0}} = e^{-\nu}(\boldsymbol{\partial}_t + \omega\boldsymbol{\partial}_\phi), \quad \boldsymbol{e}_{\hat{1}} = e^{-\psi}\boldsymbol{\partial}_\phi, \quad \boldsymbol{e}_{\hat{2}} = e^{-\mu}\boldsymbol{\partial}_\varpi, \quad \boldsymbol{e}_{\hat{3}} = e^{-\mu}\boldsymbol{\partial}_z. \tag{62}$$

Along these frame vectors, the nonzero components of the four velocity $u^{\hat{\mu}}$ are written in terms of a fluid 3-velocity v as in Minkowski spacetime

$$u^{\hat{0}} = \frac{1}{\sqrt{1-v^2}}, \qquad u^{\hat{1}} = \frac{v}{\sqrt{1-v^2}}. \tag{63}$$

Exercise 2.2: Transform the above 4-velocity components to the coordinate frame, via $u^\alpha = u^{\hat{\mu}} e^\alpha_{\hat{\mu}}$, and show that one obtains the components (58) only if

$$\boxed{v = (\Omega - \omega)e^{\psi-\nu}}, \tag{64}$$

as in (57).

Since $v = 0$ for $\Omega = \omega$ (for the ZAMO), the 3-velocity v is defined with respect to this observer.

Time dilation. From Eq. (54) it follows that $e^{-\nu}$ is the *time dilation factor* relating the proper time of the local ZAMO to coordinate time t (proper time at infinity).

Redshift. A zero-angular momentum photon sent to infinity by a ZAMO from a point P suffers a redshift that is given by

$$\frac{\omega_{ZAMO}}{\omega_\infty} = \frac{k_\alpha u^\alpha_{ZAMO}}{k_\beta t^\beta} = e^{-\nu}\Big|_P = 1 + z. \tag{65}$$

Circumferential radius. The *proper circumference* of a circle around the axis of symmetry is

$$\int_0^{2\pi} \sqrt{g_{\phi\phi}}d\phi = 2\pi e^{\psi}. \tag{66}$$

Therefore the *proper circumferential radius* is

$$\boxed{R := e^{\psi}}. \tag{67}$$

Ergospheres. In highly relativistic models, rapid rotation can lead to frame dragging extreme enough that all physical particles are dragged forward relative to an observer at infinity, or, equivalently, relative to the Killing vector t^α. A region in which this is true is called an *ergosphere*, whose definition (mentioned above), is a region in which the asymptotically timelike Killing vector t^α is spacelike. Because physical particles move along timelike or null lines, the definition implies that no physical particle can remain at rest relative to t^α.

At any point in the spacetime, the angular velocity of a physical particle is restricted in a way that looks asymmetric relative to infinity, but *symmetric relative to a ZAMO*. In Minkowski space, a particle can have a timelike or null trajectory only if $\varpi\Omega < 1$, implying $-1/\varpi < \Omega < 1/\varpi$. In the spacetime of a rotating star, a particle can have arbitrary 4-velocity

$$u^\alpha = u^t(t^\alpha + v^\alpha + \Omega\phi^\alpha),$$

where $v^\alpha \perp t^\alpha,\ \phi^\alpha$, which is timelike if $u_\alpha u^\alpha = -1 < 0$. This implies that

$$t^\alpha t_\alpha + 2\Omega t^\alpha \phi_\alpha + \Omega^2 \phi^\alpha \phi_\alpha + v^\alpha v_\alpha \leq 0.$$

Then $v^\alpha v_\alpha \geq 0$ implies $\Omega_- \leq \Omega \leq \Omega_+$, where

$$\Omega_\pm = -\frac{\boldsymbol{t}\cdot\boldsymbol{\phi}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} \pm \left[\left(\frac{\boldsymbol{t}\cdot\boldsymbol{\phi}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}\right)^2 - \frac{\boldsymbol{t}\cdot\boldsymbol{t}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}\right]^{1/2}$$

or

$$\Omega_\pm = \omega \pm \left(\omega^2 - \frac{\boldsymbol{t}\cdot\boldsymbol{t}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}\right)^{1/2}. \tag{68}$$

For the metric (48)

$$\Omega_\pm = \omega \pm e^{\nu-\psi}. \tag{69}$$

These extrema are reached when $v^\alpha = 0$, that is, for circular motion in the equatorial plane. One obtains the same expression (69) when requiring that $|v| < 1$, for circular motion in the equatorial plane. At the boundary of the ergosphere (called the *stationary limit*), where $t^\alpha t_\alpha = 0$, we have $\Omega_- = 0$. Within the ergosphere, both Ω_+ and Ω_- are positive, implying that, seen from infinity, all particles, must move in the direction of the star's rotation. In stellar models, ergospheres are toroidal and typically enclose the star's equator. In principle, energy could be extracted from the ergosphere by the Penrose process.

Asymptotic Behavior. The lowest-order asymptotic behavior of the metric functions ν and ω is

$$\nu \sim -\frac{M}{r} + \frac{Q}{r^3}P_2(\cos\theta), \tag{70}$$

$$\omega \sim \frac{2J}{r^3}, \tag{71}$$

where M, J and Q are the gravitational mass, angular momentum and quadrupole moment of the source of the gravitational field. The asymptotic expansion of the dragging potential ω shows that it decays rapidly far from the star, so that its effect will be significant mainly in the vicinity of the star.

In addition, the asymptotic relations

$$e^\psi = \varpi(e^{-\nu} + O(r^{-2})), \qquad e^\mu = e^{-\nu} + O(r^{-2}), \tag{72}$$

hold, because any stationary, asymptotically flat spacetime agrees with the Schwarzschild geometry to order r^{-1}. If, following Bardeen and Wagoner (1971), we write

$$\beta := \psi + \nu, \qquad \zeta := \mu + \nu, \qquad B := \frac{1}{\varpi}\, e^{\beta}, \tag{73}$$

then, asymptotically, β (or B) deviates by $O(r^{-2})$ from its value in the isotropic Schwarzschild metric; and ζ, which vanishes for isotropic Schwarzschild, is itself of order r^{-2}.

Nonrotating Limit. In the non-rotating limit, the metric (48) reduces to the metric of a spherical relativistic star in *isotropic coordinates*, so that, in the exterior vacuum region, the following relations hold

$$e^{\nu} = \frac{1 - M/2r}{1 + M/2r}, \quad e^{\psi} = \varpi(1 + M/2r)^2, \quad e^{\mu} = (1 + M/2r)^2, \tag{74}$$

where M is the gravitational mass of the star (see [53]).

3 Einstein's field equation

When an equation of state has been specified and if an equilibrium solution exists, the structure of the star is determined by solving four components of Einstein's gravitational field equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, or

$$R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right), \tag{75}$$

(where $R_{\alpha\beta}$ is the Ricci tensor and $T = T_{\alpha}{}^{\alpha}$), together with the equation of hydrostationary equilibrium (see next section). One approach for deriving the necessary equations is to select four components of the Einstein field equation, expressed in the tetrad frame of the ZAMO. In this frame, the stress-energy tensor becomes

$$T^{\hat{0}\hat{0}} = \frac{\epsilon + pv^2}{1 - v^2}, \qquad T^{\hat{0}\hat{1}} = \epsilon + p \frac{v}{1 - v^2}, \tag{76}$$

$$T^{\hat{1}\hat{1}} = \frac{\epsilon v^2 + p}{1 - v^2}, \qquad T^{\hat{2}\hat{2}} = T^{\hat{3}\hat{3}} = p. \tag{77}$$

With $\zeta = \mu + \nu$, one common choice for the components of the gravitational field equation is [5, 12]

$$\begin{aligned}
\nabla \cdot (B \nabla \nu) &= \frac{1}{2} r^2 \sin^2 \theta B^3 e^{-4\nu} \nabla\omega \cdot \nabla\omega \\
&\quad + 4\pi B e^{2\zeta - 2\nu} \left[\frac{(\epsilon + p)(1 + v^2)}{1 - v^2} + 2p \right], && (78a) \\
\nabla \cdot (r^2 \sin^2 \theta B^3 e^{-4\nu} \nabla\omega) &= -16\pi r \sin\theta B^2 e^{2\zeta - 4\nu} \frac{(\epsilon + p) v}{1 - v^2}, && (78b) \\
\nabla \cdot (r \sin\theta \nabla B) &= 16\pi r \sin\theta B e^{2\zeta - 2\nu} p, && (78c)
\end{aligned}$$

(these are, respectively the $R_{\hat{0}\hat{0}}, R_{\hat{0}\hat{3}}$, and $R_{\hat{0}\hat{0}} - R_{\hat{3}\hat{3}}$ field equations), supplemented by a first-order differential equation for ζ (see [12]), which comes from $e^{-\beta+2\mu}(G^{\hat{3}\hat{3}} - G^{\hat{2}\hat{2}}) = e^{-\beta}(G_{zz} -$

$G_{\varpi\varpi}) = 0$:

$$\frac{1}{\varpi}\zeta_{,\varpi} + \frac{1}{B}(B_{,\varpi}\zeta_{,\varpi} - B_{,z}\zeta_{,z}) = \frac{1}{2\varpi^2 B}(\varpi^2 B_{,\varpi})_{,\varpi} - \frac{1}{2B}B_{,zz} + (\nu_{,\varpi})^2 - (\nu_{,z})^2 - \frac{1}{4}\varpi^2 B^2 e^{-4\nu}\left[(\omega_{,\varpi})^2 - (\omega_{,z})^2\right]. \tag{79}$$

In the first three equations above, $\boldsymbol{\nabla}$ is the ordinary *flat* three-dimensional derivative operator, the derivative operator of the flat 3-metric

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = d\varpi^2 + dz^2 + \varpi^2 d\phi^2.$$

Thus, three of the four components of the field equation are of elliptic type, while the fourth is a first-order partial-differential equation, relating only metric functions. The remaining non-zero components of the gravitational field equation yield two more elliptic equations and one first-order partial-differential equation, which are consistent with the above set of four equations.

4 Hydrostationary equilibrium equation

For a stationary, axisymmetric perfect fluid star, the equation of hydrostationary equilibrium can be written in several equivalent forms. Because scalars and components of vectors (in t, r, θ, ϕ coordinates) depend only on r and θ, the equation has only r and θ components: The t and ϕ components of each term vanish identically, and $q_\alpha{}^\beta\nabla_\beta p = \nabla_\alpha p$. The relativistic Euler equation (23) for a stationary, axisymmetric star then takes the form

$$\frac{\nabla_\alpha p}{(\epsilon + p)} = -u^\beta\nabla_\beta u_\alpha. \tag{80}$$

We note first that the equation can be written in terms of the scalars $u^t = u^\alpha\nabla_\alpha t$ and $u_\phi = u_\alpha\phi^\alpha$ as

$$\frac{\nabla_\alpha p}{(\epsilon + p)} = \nabla_\alpha \ln u^t - u^t u_\phi \nabla_\alpha\Omega. \tag{81}$$

To obtain (81), one uses the fact that, for a circular flow, the 4-velocity is a linear combination of two Killing vectors, $u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha)$. For uniform rotation, $k^\alpha := u^\alpha/u^t$ is also a Killing vector, satisfying $\nabla_{(\alpha}k_{\beta)} = 0$. More generally, for differential rotation

$$\nabla_{(\alpha}k_{\beta)} = \phi_{(\alpha}\nabla_{\beta)}\Omega, \tag{82}$$

which leads directly to

$$u^\beta\nabla_\beta u_\alpha = -\nabla_\alpha \ln u^t + u^t u_\phi\nabla_\alpha\Omega, \tag{83}$$

where we have used $u^\beta\nabla_\beta \ln u^t = 0 = u^\beta\nabla_\beta\Omega$ (from axisymmetry and stationarity) and $u^\beta\nabla_\alpha u_\beta = 0$ (from the normalization $u^\alpha u_\alpha = -1$).

Next, replacing u^t and u_ϕ in Eq. (81) by their expressions in Eqs. (58), (59), and using Eq. (64) for v, we have

$$\begin{aligned}\frac{\nabla p}{(\epsilon + p)} &= \nabla \ln \frac{e^{-\nu}}{\sqrt{1 - v^2}} - \frac{e^{\psi-\nu}v}{1 - v^2}\nabla\Omega,\\ &= -\frac{1}{1 - v^2}\left(\nabla\nu - v^2\nabla\psi + e^{\psi-\nu}v\nabla\omega\right),\end{aligned} \tag{84}$$

Exercise 4.1: Derive the following equivalent forms of the hydrostationary equilibrium equation:

$$\frac{\nabla p}{(\epsilon + p)} = \nabla \ln u^t - u^t u_\phi \nabla\Omega, \tag{85a}$$

$$= \nabla \ln u^t - \frac{l}{1-\Omega l}\nabla\Omega, \tag{85b}$$

$$= -\nabla \ln(-u_t) + \frac{\Omega}{1-\Omega l}\nabla l, \tag{85c}$$

$$= -\nabla\nu + \frac{1}{1-v^2}\left(v\nabla v - \frac{v^2\nabla\Omega}{\Omega-\omega}\right), \tag{85d}$$

where $l := -u_\phi/u_t$ is conserved along fluid trajectories (since hu_t and hu_ϕ are conserved, so is their ratio and l is the angular momentum per unit energy).

Note that (84) is explicitly independent of $\nabla\Omega$, with the $\nabla\Omega$ term in each equation of (85) canceling a term involving $\nabla\Omega$ in $\nabla \ln u^t$, $\nabla \ln(-u_t)$, or ∇v.

5 The Poincaré-Wavre theorem

For barotropes, one can prove a number of important properties. Since $\epsilon = \epsilon(p)$, one can define a function

$$H(p) := \int_0^p \frac{dp'}{\epsilon(p') + p'}, \tag{86}$$

satisfying $\nabla H = \nabla \ln h - \frac{T}{h}\nabla s$, so that (81) becomes

$$\nabla(H - \ln u^t) = -F\nabla\Omega, \tag{87}$$

where we have set $F := u^t u_\phi$. For homentropic stars, $H = \ln h$, and the equation of hydrostationary equilibrium takes the form

$$\nabla\left(\ln\frac{h}{u^t}\right) = -F\nabla\Omega. \tag{88}$$

Because scalars are independent of t and ϕ, we can regard ∇ in Eqs. (81) and (84) as the two-dimensional gradient in the $r-\theta$ subspace. With A, B indices in that subspace, we have

$$\nabla_A(H - \ln u^t) = -F\nabla_A\Omega, \tag{89}$$

The curl of (89) has the form $\nabla_{[A}F\ \nabla_{B]}\Omega = 0$, implying either

$$\Omega = \text{constant}, \tag{90}$$

(*uniform rotation*), or

$$F = F(\Omega), \tag{91}$$

in the case of *differential rotation*. In the latter case, (89) becomes
$H - \ln u^t + \displaystyle\int_{\Omega_0}^{\Omega} F(\Omega')d\Omega' = \text{constant}$, or

$$H - \ln u^t + \int_{\Omega_{\rm pole}}^{\Omega} F(\Omega')d\Omega' = \nu|_{\rm pole}, \tag{92}$$

where the lower limit, Ω_0 is chosen as the value of Ω at the pole, where H and v vanish. The above *global* first integral of the hydrostationary equilibrium equations is useful in constructing numerical models of rotating stars[1].

For a uniformly rotating star, (92) can be written as

$$H - \ln u^t = \nu|_{\rm pole}, \tag{93}$$

which, in the case of a homentropic star, becomes

$$\frac{h}{u^t} = \mathcal{E}, \tag{94}$$

with $\mathcal{E} = e^\nu\,|_{\rm pole}$ constant over the star. The constancy of $\mathcal{E}$ follows from the fact that an equilibrium configuration is an extremum of the mass for perturbations that move baryons from one place to another, with angular momentum and entropy fixed.

Another consequence of (92) is that *the effective gravity can be derived from a potential*, $\Phi_{\rm eff}$, as is clear from

$$\frac{\nabla_\alpha p}{\epsilon + p} = \nabla_\alpha \Phi_{\rm eff} := \nabla_\alpha\left(\ln u^t - \int_{\Omega_0}^{\Omega} F(\Omega')d\Omega'\right). \tag{95}$$

Using (92), one finds

$$\nabla_\alpha \Phi_{\rm eff} = \nabla_\alpha H, \tag{96}$$

and the surfaces of constant effective gravity (*level surfaces*) coincide with the surfaces of constant energy density ϵ (isopycnic surfaces).

In the Newtonian limit, because $e^\psi = \varpi + O(\lambda^2), \quad e^\nu = 1 + O(\lambda^2)$, we have, to Newtonian order,

$$u^t u_\phi = v\varpi = \varpi^2\Omega, \tag{97}$$

and the functional dependence of Ω implied by Eq. (91) becomes the familiar requirement that, for a barotropic equation of state, Ω be stratified on cylinders,

$$\Omega = \Omega(\varpi). \tag{98}$$

The Newtonian limit of the integral of motion (92) is

$$h_{\rm Newtonian} - \frac{1}{2}v^2 + \Phi = \text{constant}. \tag{99}$$

With the assumption that the topology of the star's surface is either spherical or toroidal, Abramowicz [1] shows in the relativistic context that the surfaces of constant Ω are topological cylinders.

From Eq. (87) and our subsequent discussion, the relativistic version of the classical Poincaré-Wavre theorem [49] follows: Consider a model of a rotating star, a stationary axisymmetric spacetime with a bounded connected perfect fluid having 4-velocity along $k^\alpha = t^\alpha + \Omega\phi^\alpha$. *Any one of the following statements implies the other three:*

[1] The global first integral (92) and its special case (94), are sometimes mistakenly referred to as Bernoulli's law. In Bernoulli's law (30), however, the conserved quantity is hu_t, and it is conserved only along each fluid trajectory; in the equation of hydrostationary equilibrium, the constant quantity is h/u^t (for a uniformly rotating, homentropic star), and it is constant throughout the star. The confusion may arise from the fact that, for a uniformly rotating star, the Newtonian form of the conserved quantity appearing in Bernoulli's law, $h_{\rm Newtonian} + \frac{1}{2}v^2 + \Phi$, differs from the corresponding Newtonian first integral (99) only in the sign of the v^2 term.

1. *$F := u^t u_\phi$ is a function of Ω only.*

2. *The effective gravity can be derived from a potential.*

3. *The effective gravity is normal to the surfaces of constant ϵ.*

4. *The surfaces of constant p and ϵ (isobaric and isopycnic surfaces) coincide.*

6 Equation of state

Relativistic Polytropes. In a nonrelativistic Fermi gas, the Fermi momentum is $p_F \sim \hbar n^{1/3}$ corresponding to a pressure $p \sim p_F v n = \dfrac{p_F^2}{m} n \sim \dfrac{\hbar^2}{m} n^{5/3}$ (with m the particle mass); In the relativistic case, v approaches 1, implying a degeneracy pressure of order $p \sim p_F n \sim \hbar n^{4/3}$. Old neutron stars have temperatures much smaller than the Fermi energy of their constituent particles, so one can ignore entropy gradients. If one also neglects the slow change in composition, then the increase in pressure and density toward the star's center is adiabatic and the first law (6), with $ds = 0$, becomes

$$d\epsilon = \frac{\epsilon + p}{\rho} d\rho, \tag{100}$$

with p given in terms of ρ by

$$\frac{\rho}{p}\frac{dp}{d\rho} = \frac{\epsilon + p}{p}\frac{dp}{d\epsilon} = \Gamma_1. \tag{101}$$

Here Γ_1 is the *adiabatic index*, the fractional change in pressure per fractional change in comoving volume, at constant entropy and composition:

$$\Gamma_1 := \frac{\partial \log p(\rho, s, Y_1, \ldots, Y_N)}{\partial \log \rho} = \frac{\epsilon + p}{p}\frac{\partial p(\epsilon, s, Y_1, \ldots, Y_N)}{\partial \epsilon}, \tag{102}$$

with Y_k the fractional number density of the kth species of constituent particle ($Y_k = n_k/n$, with n_k the number density of the kth species and n the total number density of baryons). In an ideal degenerate Fermi gas, in the nonrelativistic and ultrarelativistic regimes, Γ_1 has the constant values 5/3 and 4/3, respectively. Except in the outer crust, neutron-star matter is far from an ideal Fermi gas, but models often assume a constant effective adiabatic index, chosen to match an average stellar compressibility. An equation of state of the form

$$p = K\rho^\Gamma, \tag{103}$$

with K and Γ constants, is called *polytropic*; K and Γ are the *polytropic constant* and *polytropic exponent*, respectively. The corresponding relation between ϵ and p follows from Eq. (100), rewritten in the form $d\dfrac{\epsilon}{\rho} = \dfrac{p}{\rho^2} d\rho$. Requiring $\lim\limits_{p \to 0} \dfrac{\epsilon}{\rho} = 1$, we have

$$\frac{\epsilon}{\rho} = 1 + \int_0^\rho K\rho^{\Gamma-2} d\rho = 1 + K\frac{\rho^{\Gamma-1}}{\Gamma - 1},$$

or

$$\epsilon = \rho + \frac{p}{\Gamma - 1}. \tag{104}$$

The polytropic exponent Γ is commonly replaced by a *polytropic index* N, given by

$$\Gamma = 1 + \frac{1}{N}. \tag{105}$$

For the above polytropic EOS, the quantity $c^{(\Gamma-2)/(\Gamma-1)}\sqrt{K^{1/(\Gamma-1)}/G}$ has units of length. In gravitational units ($c = G = 1$), one can thus use $K^{N/2}$ as a fundamental length scale to define dimensionless quantities. Equilibrium models are then characterized by the polytropic index N and their dimensionless central energy density. Equilibrium properties can be scaled to different dimensional values, using appropriate values for K. For $N < 1.0$ ($N > 1.0$) one obtains stiff (soft) models, while for $N \sim 0.5 - 1.0$, one obtains models whose masses and radii are roughly consistent with observed neutron-star masses and with the weak constraints on radius imposed by present observations and by candidate equations of state.

The definition (103), (104) of the relativistic polytropic EOS was introduced by Tooper [51], to allow a polytropic exponent Γ that coincides with the adiabatic index of a relativistic fluid with constant entropy per baryon (a homentropic fluid). Unfortunately, Tooper had previously used the form $p = K\epsilon^\Gamma$ [50]; because this equation of state does not satisfy Eq. (100), it is not consistent with the first law of thermodynamics for a fluid with uniform entropy. The literature, however, has not universally accepted one convention. In particular, two of the best introductions to the equation of state of compact objects, by Shapiro and Teukolsky [45] and by Glendenning [24], use the two different definitions, with Shapiro-Teukolsky adopting the choice, $p = K\rho^\Gamma$, that we regard as more natural and will use here. (Note, however, that Glendenning's use of the equation of state $p = K\epsilon^\Gamma$ is restricted to two cases where the definitions agree: nonrelativistic matter, for which $\epsilon = \rho$, and the ultrarelativistic Fermi gas, for which $\epsilon = 3p$.)

Hadronic Equations of State. (TBD)

Strange Quark Matter

Before a density of about $6\rho_0$ is reached, lattice QCD calculations indicate a phase transition from quarks confined to nucleons (or hyperons) to a collection of free quarks (and gluons); and heavy ion collisions at CERN and RHIC show evidence of the formation of such a quark-gluon plasma. A density for the phase transition higher than that needed for strange quarks in hyperons is similarly high enough to give a mixture of up, down and strange quarks in quark matter, and the expected strangeness per unit baryon number is $\simeq -1$. If densities high enough for a phase transition to quark matter are reached, neutron-star cores may contain a transition region with a mixed phase of quark droplets in neutron matter [24].

Bodmer [8] and later Witten [54] pointed out that experimental data does not rule out the possibility that the ground state of matter at zero pressure and large baryon number is not iron but strange quark matter. If this is the case, all "neutron stars" may be strange quark stars, a lower density version of the quark-gluon plasma, again with roughly equal numbers of up, down and strange quarks, together with electrons to give overall charge neutrality [8, 20]. The first extensive study of strange quark star properties is due to Witten [54] (but, see also [30, 11]), while hybrid stars that have a mixed-phase region of quark and hadronic matter, have also been studied extensively (see, for example, Glendenning's review [24]).

The strange quark matter equation of state can be represented by the following linear relation between pressure and energy density

$$p = a(\epsilon - \epsilon_0), \tag{106}$$

where ϵ_0 is the energy density at the surface of a bare strange star (neglecting a possible thin crust of normal matter). The MIT bag model of strange quark matter involves three parameters, the bag constant, $\mathcal{B} = \epsilon_0/4$, the mass of the strange quark, m_s, and the QCD coupling constant, α_c. The constant a in (106) is equal to 1/3 if one neglects the mass of the strange quark, while it takes the value of $a = 0.289$ for $m_s = 250$ MeV. When measured in units of $\mathcal{B}_{60} = \mathcal{B}/(60 \text{ MeV fm}^{-3})$, the constant $\mathcal{B}$ is restricted to be in the range

$$0.9821 < \mathcal{B}_{60} < 1.525, \tag{107}$$

assuming $m_s = 0$. The lower limit is set by the requirement of stability of neutrons with respect to a spontaneous fusion into strangelets, while the upper limit is determined by the energy per baryon of ^{56}Fe at zero pressure (930.4 MeV). For other values of m_s the above limits are modified somewhat (see also [19, 25] for other attempts to describe deconfined strange quark matter).

7 Rotation law

At birth, neutron stars are likely to be differentially rotating, with a rotation law that can depend on the rotation of their progenitors and on the way they are formed – the collapse of stellar cores [56, 37, 2], the accretion-induced collapse of white dwarfs [38], the merger of degenerate cores in double-degenerate systems, or the merger of neutron stars [42, 47, 41, 48].

Uniform angular velocity minimizes the total mass-energy of a configuration for a given baryon number and total angular momentum [10, 28]. As a result, apart from slight differential rotation following glitches, neutron stars, soon after birth, are expected to rotate uniformly. As the neutron star cools, several mechanisms act to enforce uniform rotation. Magnetic braking of differential rotation by Alfvén waves [44, 16, 39] could be the most effective damping mechanism, acting on a timescale of minutes or shorter. In contrast, kinematical shear viscosity acts on a timescale [21, 22, 18]

$$\tau \sim 18 \times \left(\frac{\rho}{10^{15}\text{g cm}^{-3}}\right)^{-5/4} \left(\frac{T}{10^9\text{K}}\right)^2 \left(\frac{R}{10^6\text{cm}}\right) \quad \text{yr}, \tag{108}$$

where ρ, T and R are the central density, temperature and radius of the star. It has also been suggested that convective and turbulent motions may enforce uniform rotation on a timescale of the order of days [29].

If the cooling to degeneracy, and hence to a nearly barotropic configuration, is faster than the time to establish uniform rotation, condition (91) is likely to govern the rotation law of nascent neutron stars. A simple choice of a differential rotation law that satisfies this condition is

$$F(\Omega) = A^2(\Omega_\mathrm{c} - \Omega) = \frac{(\Omega - \omega)e^{2(\psi-\nu)}}{1 - (\Omega - \omega)^2 e^{2(\psi-\nu)}}, \tag{109}$$

where A is a constant with units of length that determines the length scale over which the angular velocity changes within the star [32, 33]. When $A \to \infty$, the above rotation law reduces to the uniform rotation case. In the Newtonian limit and when $A \to 0$, the rotation law becomes a so-called j–constant rotation law (where $j := hu_\phi$ is the specific angular momentum), a law that satisfies the Rayleigh criterion for local dynamical stability against axisymmetric disturbances: j should not decrease outwards, $dj/d\Omega < 0$. The same criterion is also satisfied in the relativistic case (see [33]). It should be noted that differentially rotating stars may also be subject to a shear instability that tends to suppress differential rotation [55]. The above rotation law is a simple choice and more physically plausible choices must be obtained through detailed numerical simulations of the formation of rotating relativistic stars.

8 Equilibrium quantities

The mass and angular momentum of a stationary axisymmetric spacetime can each be defined as an integral at spatial infinity (their Komar form [31]), namely

$$M = -\frac{1}{4\pi}\int_{S_\infty} \nabla^\alpha t^\beta dS_{\alpha\beta}, \tag{110}$$

$$J = \frac{1}{8\pi}\int_{S_\infty} \nabla^\alpha \phi^\beta dS_{\alpha\beta}, \tag{111}$$

where $\displaystyle\int_{S_\infty} := \lim_{r\to\infty}\int_{S_r}$, and S_r is a sphere of constant radial coordinate r appropriate to the asymptotically flat metric. Stokes's theorem relates the surface integrals for M and J to integrals over the matter and implies that the surface integrals are independent of surface for any surface enclosing the matter. To see this, one uses the Killing-vector identity

$$\nabla_\beta \nabla^\alpha \xi^\beta = R^\alpha{}_\beta \xi^\beta, \tag{112}$$

together with the field equation in the form

$$R^\alpha{}_\beta = 8\pi(T^\alpha{}_\beta - \delta^\alpha_\beta T), \tag{113}$$

to write

$$M = -2\int (T_\alpha{}^\beta - \frac{1}{2}\delta^\beta_\alpha T)\, t^\alpha dS_\beta, \tag{114}$$

$$J = \int T_\alpha{}^\beta \phi^\alpha dS_\beta. \tag{115}$$

In a time-dependent spacetime, the asymptotic forms above can be used to define the total mass and angular momentum associated with timelike and rotational symmetry vectors of a flat asymptotic metric, but they can no longer be written as integrals over the matter.

Associated with the differential baryon (and lepton) conservation law (20) is a conserved rest mass. That is, the integral of $\nabla_\alpha(\rho u^\alpha)$ over a 4-volume ${}^4V$ bounded by 3-surfaces V_1 and V_2 yields

$$\int_{{}^4V} \nabla_\alpha(\rho u^\alpha) d^4V = \left(\int_{V_2} - \int_{V_1}\right) \rho u^\alpha dS_\alpha,$$

implying that the total rest mass,

$$M_0 = \int_V \rho u^\alpha dS_\alpha, \tag{116}$$

is independent of the 3-surface V, for any 3-surface containing the fluid.

We can now make the connection between the specific angular momentum, $j = hu_\alpha\phi^\alpha$, introduced in Eq. (31), and the total angular momentum: If we denote by $dM_0 = \rho u^\alpha dS_\alpha$ the integrand in Eq. (116), then Eq. (115) for the total angular momentum has, when evaluated on an axisymmetric hypersurface, the form

$$J = \int_V j dM_0. \tag{117}$$

This follows from the fact that ϕ^α is tangent to an axisymmetric hypersurface, implying $\phi^\alpha dS_\alpha = 0$. Then

$$T^\alpha{}_\beta \phi^\beta dS_\alpha = (\epsilon + p) u_\beta \phi^\beta u^\alpha dS_\alpha = \frac{\epsilon + p}{\rho} u_\beta \phi^\beta \rho u^\alpha dS_\alpha = j dM_0.$$

The rotational kinetic energy is defined by

$$T = \frac{1}{2}\int \Omega dJ, \quad \text{where} \quad dJ = T_\alpha{}^\beta \phi^\alpha dS_\beta, \tag{118}$$

and it takes the form $T = \frac{1}{2} J\Omega$ for uniform rotation. A natural definition of the scalar moment of inertia for a uniformly rotating star is similarly

$$I = J/\Omega. \tag{119}$$

Although J can be computed as an integral over the source, the scalar moment of inertia does not have an independent integral definition. As a result, it has no natural definition as a local integral for a star with arbitrary velocity field. [2]

The internal energy density, given by

$$u = \epsilon - \rho = \rho e, \tag{120}$$

can be used to define a total internal energy $U = \int u\, u^\alpha dS_\alpha$. Finally, with mass, rest mass, kinetic energy, and internal energy defined, one can define a gravitational binding energy W as the difference

$$W = M - M_0 - T - U. \tag{121}$$

(A *proper mass*, $M_P = M_0 + U$, is also commonly used, allowing one to write $W = M - M_P - T$.)

A summary of integral quantities that characterize stationary, axisymmetric models, together with the equations that define them, is displayed in Table 1.

The *mass-shedding* or *Kepler limit* along a sequence of rotating stellar models is reached when, at the equator, the angular velocity of the fluid, $\Omega(r_e, \pi/2)$, reaches the angular velocity of a free particle in circular orbit, Ω_K. The latter has the form

$$\Omega_K = \omega + \frac{\omega'}{2\psi'} + \left[e^{2\nu - 2\psi}\frac{\nu'}{\psi'} + \left(\frac{\omega'}{2\psi'}\right)^2 \right]^{1/2}, \tag{122}$$

[2] Multipole moments of the mass and current distributions are defined in terms of the asymptotic metric at spatial infinity, but these are symmetric, tracefree tensors and do not define I.

Table 1: Equilibrium properties

gravitational mass	$M = -2\int(T_\alpha{}^\beta - \frac{1}{2}\delta_\alpha^\beta T)\, t^\alpha n_\beta dV$
rest mass	$M_0 = \int \rho u^\beta n_\beta dV$
internal energy	$U = \int u\ u^\beta n_\beta dV$
gravitational binding energy	$W = M - M_0 - T - U$
angular momentum	$J = \int T_\alpha{}^\beta \phi^\alpha n_\beta\, dV \equiv \int dJ$
kinetic energy	$T = \frac{1}{2}\int \Omega dJ$
moment of inertia (uniform rotation)	$I = J/\Omega$

where ($'$) means ∂_r. Eq. (122) is obtained from the geodesic equation, $u^\beta \nabla_\beta u^\alpha = 0$, for a satellite in circular orbit, with $u^\alpha = u^t(t^\alpha + \Omega_K \phi^\alpha)$. As in Eq. (64), the normalization $u^\alpha u_\alpha = -1$ implies $u^t = \dfrac{e^{-\nu}}{\sqrt{1-v_K^2}}$, where $v_K = e^{\psi-\nu}(\Omega_K - \omega)$. Then, the geodesic equation gives

$$0 = (1 - v_K^2)\partial_r \ln u^t = -\nu' + v_K^2 \psi' - v_K e^{\psi-\nu}\omega'. \tag{123}$$

Note that this relation agrees with (and could be deduced from) the hydrostationary equilibrium equation, (84), when $\nabla p = 0$: At maximum rotation (when $\Omega = \Omega_K$), no pressure gradient contributes to the support of a ring of fluid at the equator and the balance of forces on a fluid element is set by the equality of the gravitational[3] and centrifugal forces. A further increase in uniform rotation would lead to mass shedding at the equator. Solving the quadratic equation (123) for v_K, we have

$$v_K = e^{\psi-\nu}\frac{\omega'}{2\psi'} + \left[\frac{\nu'}{\psi'} + e^{2\psi-2\nu}\left(\frac{\omega'}{2\psi'}\right)^2\right]^{1/2}$$

and Eq. (122) immediately follows.

For a spherical star, Eq. (122) takes the Newtonian form

$$\Omega_K = \left(\frac{M}{R}\right)^{1/2}, \tag{124}$$

with R the Schwarzschild coordinate of the equator. As a spherical star spins up, it becomes oblate, and the Kepler frequency of a particle at larger equatorial radius is correspondingly smaller. By the time the star itself rotates at the Kepler frequency, Ω_K has typically fallen to about 60% of its value for the spherical configuration with the same baryon number.

Finally, we note that in the quasi-isotropic coordinates, $e^\psi = \sqrt{\phi^\alpha \phi_\alpha}$ is the radial coordinate for which $2\pi e^\psi$ is the circumference of the circular orbits of the symmetry vector ϕ^α. We will write $r_c(r) = e^\psi(r, \theta = \pi/2)$, noting that, for spherical stars, r_c coincides with the usual Schwarzschild radial coordinate. At the equator, r_c is the *circumferential equatorial radius* of the star, also denoted as R (coinciding with the Schwarzschild radial coordinate at the equator for spherical stars).

[3] more precisely, by the curvature of spacetime, including the effect of frame dragging.

9 The 3+1 split

Detailed accounts on the historical development of the subject can be found e.g. in [3, 9, 7, 26, 46].

If the spacetime is *globally hyperbolic*, the initial-value problem is well-defined and the spacetime can be foliated by non-intersecting *spacelike hypersurfaces* (Cauchy surfaces) Σ_t, which are parametrized by a suitably chosen global "time" coordinate t. Then, the 1-form $\nabla_\alpha t$ is *normal* to the t = constant spacelike surfaces Σ_t and is everywhere *timelike*. In a chart $\{t, x^i\}$, its components are

$$\nabla_\mu t = \delta^t_\mu = (1,0,0,0). \tag{125}$$

The future pointing contravariant *unit vector normal to each slice* Σ_t is thus

$$\boxed{n^\alpha := -\frac{\nabla^\alpha t}{\sqrt{-\nabla_\beta t \nabla^\beta t}}}, \tag{126}$$

(the future is the direction in which t increases and we assume a $- + ++$ metric signature).

For a given spacetime metric $g_{\alpha\beta}$ and a unit normal vector n^α to each Σ_t, the tensor

$$\boxed{\gamma_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta}, \tag{127}$$

is *orthogonal* to the normal n^α, since $\gamma_{\alpha\beta} n^\alpha = 0$. Then,

$$\gamma^\alpha{}_\beta := \delta^\alpha_\beta + n^\alpha n_\beta, \tag{128}$$

is called the *projection tensor* onto each Σ_t, with $\gamma^\alpha{}_\beta n^\beta = 0$. If a vector is decomposed into parts that are tangent and orthogonal to Σ_t, e.g. in the form $v^\alpha = v^\alpha_\| + v_\perp n^\alpha$, then $\gamma^\alpha{}_\beta v^\beta = v^\alpha_\|$.

The coordinate lines connecting fixed points x^i on different hypersurfaces Σ_t are *tangent to the vector field*

$$\boxed{t^\alpha := (\partial/\partial t)^\alpha}, \tag{129}$$

with components $t^\mu = \delta^\mu_t = (1,0,0,0)$ (only time t changes along these lines). In a spacetime where the coordinates are time-independent (there is no shift in the position of points with same coordinates x^i between adjacent hypersurfaces) $t^\alpha || n^\alpha$ (an example is the Schwarzschild metric).

There is a *freedom in foliating the spacetime* so that proper time advances along n^α in a chosen, non-uniform way (for the same dt, the proper time $d\tau$ is position-dependent). Then $t^\alpha = \alpha n^\alpha$, where the *lapse function* α encodes the chosen foliation and $d\tau = \alpha\, dt$ is the *proper time* elapsed in a normal direction between coordinate times t and $t + dt$.

But, there is also a *freedom in moving the origin of spatial coordinates* when progressing from one hypersurface Σ_t to a nearby Σ_{t+dt} (an example is the spacetime of a rotating star, where the nonzero $g_{t\phi}$ metric component implies a position-dependent angular velocity $\omega = d\phi/dt$, as seen from infinity, for ZAMOs). This change in the origin of spatial coordinates is described by a *shift vector* β^α, tangent to Σ_t, in the sense that

$$\beta^\alpha n_\alpha = 0 \Rightarrow \beta^\alpha = (0, \beta^i). \tag{130}$$

Then, $\beta^\alpha\, dt$ is the *spatial shift in position* if one advances along t^α as opposed to advancing along the normal n^α, between two hypersurfaces Σ_t and Σ_{t+dt}.

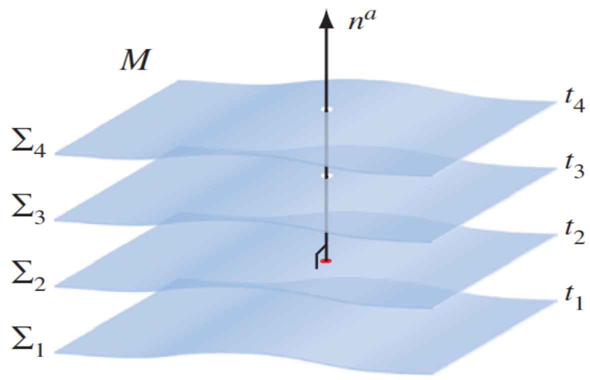

Figure 1: The spacetime M is foliated by the spacelike hypersurfaces Σ_t. At each point on a spacelike hypersurface, a unit normal vector n^α is defined. Figure from [7].

In general, the timelike vector t^α can thus have components both along n^α and tangent to Σ_t

$$\boxed{t^\alpha = \alpha n^\alpha + \beta^\alpha}. \tag{131}$$

From the above relation, the lapse function is obtained as

$$\boxed{\alpha \equiv -t^\alpha n_\alpha}, \tag{132}$$

and the shift vector is obtained as

$$\boxed{\beta^\alpha \equiv \gamma^\alpha_\beta t^\beta}. \tag{133}$$

Exercise 9.1: Derive the following explicit components for t^α, n^α, β^α and γ_{ij}

$$\begin{aligned}
t^\mu &= (1, 0, 0, 0), && (134)\\
t_\mu &= (-\alpha^2 + \beta_i\beta^i, \beta_i), && (135)\\
n^\mu &= \frac{1}{\alpha} \cdot (1, -\beta^i), && (136)\\
n_\mu &= -\alpha \cdot (1, 0, 0, 0), && (137)\\
\beta^t &= 0, && (138)\\
\beta_t &= \beta_i\beta^i, && (139)\\
\beta_i &= \gamma_{ij}\beta^j, && (140)\\
\gamma_{tt} &= g_{tt} + \alpha^2, && (141)\\
\gamma_{ti} &= g_{ti}, && (142)\\
\gamma_{ij} &= g_{ij}, && (143)
\end{aligned}$$

and show that

$$\sqrt{-\nabla_\beta t \nabla^\beta t} = \alpha^{-1}. \tag{144}$$

Notice that γ_{ij} is simply the *spatial part* of $g_{\mu\nu}$. Furthermore, it follows that

$$g_{tt} = t_\alpha t^\alpha = -\alpha^2 + \beta_i\beta^i. \tag{145}$$

and thus

$$\gamma_{tt} = \beta_i\beta^i. \tag{146}$$

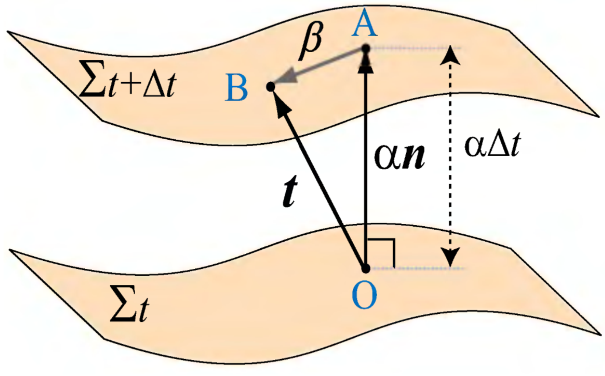

Figure 2: The timelike vector field t^α (connecting points of fixed coordinates x^i on different hypersurfaces) is, in general, decomposed into αn^α (normal to Σ_t) and β^α (tangent to Σ_t). The proper time elapsed between the two hypersurfaces is $\alpha \Delta t$. Figure from [46].

and by the definition (133), the covariant component of the shift vector is

$$\beta_i = \gamma_{ti}. \tag{147}$$

Combining the above results, it follows immediately that the metric element is written as

$$\boxed{ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)}. \tag{148}$$

corresponding to the mentric tensor

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}. \tag{149}$$

The inverse metric tensor is

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}. \tag{150}$$

10 The induced metric

The *pullback* of $\gamma_{\alpha\beta}$ to Σ_t is the 3-dimensional tensor

$$\boxed{\gamma_{ab} = \gamma^\alpha{}_a \gamma^\beta{}_b \gamma_{\alpha\beta}}, \tag{151}$$

which is the Riemannian *3-metric*, or *induced metric* on Σ_t. In the charts $\{x^i\}$ for Σ_t and $\{t, x^i\}$ for spacetime M, $\gamma^\alpha{}_a$ has components

$$\gamma^\mu{}_i = \delta^\mu_i, \tag{152}$$

because $n_i = 0$. The spatial components of γ_{ab}, $\gamma_{\alpha\beta}$ and $g_{\alpha\beta}$ all coincide:

$$\gamma_{ij} = g_{ij}. \tag{153}$$

Similarly, the pullback of β_α on Σ_t is

$$\boxed{\beta_a = \gamma^\alpha{}_a \beta_\alpha}, \tag{154}$$

which is a 3-dimensional covariant vector on Σ_t. Defining the determinant of the 3-metric as $\gamma = \det(\gamma_{ab})$, the volume element is

$$\sqrt{|g|} = \alpha\sqrt{\gamma}. \tag{155}$$

The choice of a 3+1 decomposition $\mathbb{R} \times \Sigma$ of the spacetime allows one to identify each hypersurface $\Sigma_t = \{t\} \times \Sigma$ with the fixed space Σ. We can then regard $\alpha(t)$, $\beta^a(t)$, $\gamma_{ab}(t)$, and the 3-dimensional projections of the fluid variables as time-dependent quantities on Σ.

The correspondence between fields $\alpha(t), \beta^a(t)$, and $\gamma_{ab}(t)$ on Σ and α, β^α, and $\gamma_{\alpha\beta}$ on M extends to a correspondence between the time derivatives $\dot{\alpha} = \partial_t \alpha$, $\dot{\beta}^a = \partial_t \beta^a$, $\dot{\gamma}_{ab} = \partial_t \gamma_{ab}$ on Σ and the Lie derivatives $\dot{\alpha} = \mathcal{L}_t \alpha$, $\dot{\beta}^\alpha = \mathcal{L}_t \beta^\alpha$, $\dot{\gamma}_{\alpha\beta} = \mathcal{L}_t \gamma_{\alpha\beta}$ on M. For example,

$$\dot{\gamma}_{ab} := \partial_t \gamma_{ab} = \gamma^\alpha{}_a \gamma^\beta{}_b \mathcal{L}_t \gamma_{\alpha\beta} = \gamma^\alpha{}_a \gamma^\beta{}_b \dot{\gamma}_{\alpha\beta}. \tag{156}$$

This follows immediate from the fact that the spatial components of corresponding tensors coincide.

The pullback of the covariant derivative operator ∇_α on Σ is

$$D_a = \gamma^\alpha{}_a \nabla_\alpha. \tag{157}$$

and the *extrinsic curvature* K_{ab} of Σ is defined as[4]

$$\boxed{K_{ab} := -\frac{1}{2}\gamma^\alpha{}_a \gamma^\beta{}_b \mathcal{L}_{\mathrm{n}} \gamma_{\alpha\beta}}, \tag{158}$$

which can be regarded (up to a factor of -1/2) as a "time-derivative" along the normal n^α. The contraction K of the extrinsic curvature is

$$K = K_a{}^a = \gamma^{ab} K_{ab}. \tag{159}$$

Exercise 10.1: Using Eqs. (158) and (131), show that

$$\begin{aligned} K_{ab} &= -\frac{1}{2}\gamma^\alpha{}_a \gamma^\beta{}_b \mathcal{L}_{\mathrm{n}} \gamma_{\alpha\beta} = -\frac{1}{2\alpha}[\partial_t \gamma_{ab} - \gamma^\alpha{}_a \gamma^\beta{}_b (\nabla_\alpha \beta_\beta + \nabla_\beta \beta_\alpha)] \\ &= -\frac{1}{2\alpha}(\partial_t \gamma_{ab} - D_a \beta_b - D_b \beta_a). \end{aligned} \tag{160}$$

Eq. (160), in the form

$$\boxed{\partial_t \gamma_{ab} = -2\alpha K_{ab} + D_a \beta_b + D_b \beta_a}, \tag{161}$$

serves as an evolution equation for the 3-metric γ_{ab}.

The energy density ρ_{E}, the momentum density j_a, and the stress tensor S_{ab} *as measured by an observer whose 4-velocity is* n^α are

$$\rho_E := n^\alpha n^\beta T_{\alpha\beta} = (\epsilon + p)(\alpha u^t)^2 - p, \tag{162a}$$

$$j_a := -\gamma_a{}^\alpha n^\beta T_{\alpha\beta} = (\epsilon + p)\alpha u^t u_a, \tag{162b}$$

$$S_{ab} := \gamma_a{}^\alpha \gamma_b{}^\beta T_{\alpha\beta} = (\epsilon + p) u_a u_b + p\gamma_{ab}, \tag{162c}$$

[4]There is no consistent convention in the literature for the sign of the extrinsic curvature of a spacelike hypersurface. Our convention agrees with that of MTW [40] and disagrees with Wald's [52].

where $u_a = \gamma^\alpha{}_a u_\alpha$. We denote by R_{ab} the *3-dimensional Ricci tensor of the 3-metric* γ_{ab} and by

$$R = \gamma^{ab} R_{ab}, \tag{163}$$

the corresponding *3-dimensional Ricci scalar.* The 4-dimensional Ricci scalar will be written ${}^4R$.

The field equation $E^{\alpha\beta} := G^{\alpha\beta} - 8\pi T^{\alpha\beta} = 0$ can be decomposed in terms of components that are either entirely tangent to Σ_t, or entirely normal to Σ_t as well as in terms of mixed components. The projection that is entirely tangent to Σ_t, which is $E_{\parallel}^{\alpha\beta} \equiv \gamma^\alpha{}_\gamma \gamma^\beta{}_\delta E^{\gamma\delta} = 0$, has the form

$$\boxed{\partial_t K_{ab} = -D_a D_b \alpha + \alpha(R_{ab} + K K_{ab} - 2K_{ac}K^c{}_b) + \mathcal{L}_{\boldsymbol{\beta}} K_{ab} - 8\pi\alpha \left[S_{ab} - \frac{\gamma_{ab}}{2}(S_c{}^c - \rho_{\rm E})\right]}, \tag{164}$$

which is as evolution equation for K_{ab}.

The projection that is entirely normal to Σ_t, which is $E_{\perp\perp} \equiv E^{\alpha\beta} n_\alpha n_\beta = 0$, is the *Hamiltonian constraint*

$$\boxed{R + K^2 - K_{ab}K^{ab} - 16\pi\rho_{\rm E} = 0}, \tag{165}$$

whereas the mixed projection, $E_{\parallel\perp}^a \equiv E^{\alpha\beta}\gamma^a{}_\alpha n_\beta = 0$, is the *momentum constraint*

$$\boxed{D_b(K^{ab} - \gamma^{ab}K) - 8\pi j^a = 0}. \tag{166}$$

The system of equations (161), (164) describes the time evolution of the three-dimensional tensor fields $\gamma_{ab}(t)$ and $K_{ab}(t)$ on Σ. Since these equations do not contain time derivatives of the lapse function α or of the shift vector β^a, these metric functions are not dynamical variables. One can regard the four degrees of gauge freedom associated with the choice of coordinates (t, x^i) as the freedom to choose α and β^a. Once α and β^a are prescribed, one needs initial data $\gamma_{ab}(0)$ and $K_{ab}(0)$ satisfying the constraint equations (165,166).

Next, we consider the 3+1 decomposition of the conservation equations governing the fluid. Baryon conservation (20), has the form $0 = \nabla_\alpha(\rho u^\alpha\sqrt{|g|}) = \partial_\mu(\rho u^\mu\sqrt{|g|})$. The upper and lower components are related by $u_i = \gamma_{i\mu}u^\mu = \gamma_{ij}u^j + \beta_i u^t$, or

$$u^i = \gamma^{ij}u_j - \beta^i u^t. \tag{167}$$

The difference is related to the relation between decompositions with n^α and with $t^\alpha = \alpha n^\alpha + \beta^\alpha$ as the timelike vector. To write this relation, we introduce the 3-velocity v_a measured by an observer with velocity n^α. Using the correspondence $v_a = \gamma_a{}^\alpha v_\alpha$ between v_a and a 4-vector v_α for which $v_\alpha n^\alpha = 0$, we have

$$u_\alpha = W(n_\alpha + v_\alpha), \tag{168}$$

where $u^\alpha u_\alpha = -1$ implies that W is the Lorentz factor

$$W = \frac{1}{\sqrt{1 - v^2}}, \tag{169}$$

with $v^2 = v^\alpha v_\alpha = \gamma_{ab}v^a v^b$. Using Eqs. (??) and (??), we can write the components of Eq. (168) as

$$W = \alpha u^t, \quad u^i = u^t\left(\alpha v^i - \beta^i\right), \tag{170}$$

and the decomposition of u^α with respect to t^α is

$$u^\alpha = u^t(t^\alpha + \alpha v^\alpha - \beta^\alpha). \tag{171}$$

The evolution of a perfect fluid is governed by conservation of baryons (the continuity equation), (20), the Euler equation, (23), and conservation of energy, (22), together with an equation of state. These first three equations have the 3+1 form

$$\begin{aligned}
\partial_t(\rho\, u^t\alpha\sqrt{\gamma}) &= -D_b[\rho u^t(\alpha v^b - \beta^b)\sqrt{\gamma}], &(172)\\
\partial_t j^a &= \mathcal{L}_\beta j^a + \alpha(2K^a{}_b + \delta^a_b K)j^b - D_b(\alpha S^{ab}) - \rho_E D^a\alpha. &(173)\\
\partial_t\rho_E &= \mathcal{L}_\beta\rho_E + \alpha K\rho_E - \frac{1}{\alpha}D_b(\alpha^2 j^b) + \alpha K_{ab}S^{ab}. &(174)
\end{aligned}$$

For a one-parameter equation of state, $\epsilon = \epsilon(\rho)$, $p = p(\rho)$, satisfying the zero-entropy first law of thermodynamics, (100), the energy conservation equation, (174), is redundant, implied by baryon conservation and the equation of state. The evolution of the fluid is then given by Eqs. (172), (173), and the equation of state. Initial data for the fluid are the values of baryon mass density ρ and fluid 3-velocity u^a, with initial values of ϵ, p and u^t determined by the equation of state and the normalization $u_\alpha u^\alpha = -1$.

A solution $\gamma_{ab}(t), K_{ab}(t), \alpha(t), \beta^a(t), \epsilon(t), p(t), u^a(t)$ to the Einstein-Euler system, in the decomposed form (161), (164)-(173), then yields the four-dimensional metric (148) whose source is a perfect fluid having 4-velocity $u^\alpha = u^t(t^\alpha + \gamma^\alpha{}_a u^a)$. In solving the system numerically, one specifies initial data satisfying the constraint equations at $t = 0$; and one essentially solves only the evolution equations, (161), (164), for the metric and (172), (173) for the fluid. The resulting evolution then preserves the constraints: Eqs. (165) and (166) are automatically satisfied. Preservation of the constraints is a consequence of the Bianchi identity, $\nabla_\beta G^{\alpha\beta} = 0$, and the fluid equation $\nabla_\beta T^{\alpha\beta} = 0$, because the equation $\nabla_\beta(G^{\alpha\beta} - 8\pi T^{\alpha\beta}) = 0$ expresses the time derivative of the constraint $(G^{\alpha\beta} - 8\pi T^{\alpha\beta})n_\alpha$ at a time t in terms of the field equation and its spatial derivatives at t.

The time evolution of a fluid with smooth initial data is, in general, smooth for a finite time (see below), after which shock waves appear. Within a perfect-fluid description, shocks are solutions in which the fluid variables are discontinuous on a characteristic hypersurface – the history of a two-surface that moves at the speed of sound. Formally maintaining a one-parameter equation of state can be appropriate for solutions without shocks or with small shocks; but it is not appropriate for strong shocks, because it ignores the heat generated by the shock. There is, however, a common approximation that roughly accounts for the heat generated without introducing the complications of neutrino and radiative transport. One adopts a two-parameter EOS of the form (5), and uses the continuity and energy conservation equations, (172) and (174), as independent evolution equations for ρ and ϵ. With no shocks, as we have just seen, this evolution preserves the thermodynamic relation (1st law) between ϵ and ρ that maintains a constant-entropy equation of state, $\epsilon = \epsilon(\rho)$. When there are shocks, the constant-entropy first law is violated by the time evolution. The evolution equations again determine ϵ and ρ, but ϵ is no longer a one-parameter function $\epsilon(\rho)$.

For both numerical evolution and for proving existence of solutions, it is important to note that the system as described is not strongly hyperbolic. To turn it into a strongly hyperbolic system one can add to the dynamical equations linear combinations of the constraint equations or use harmonic coordinates, coordinates for which $\partial_\nu(g^{\mu\nu}\sqrt{|g|}) = 0$ (see also Sect. **??** for generalized version). The harmonic gauge condition, in effect, replaces spatial derivatives in the constraint equations by time derivatives and leaves each component of the Ricci tensor in the manifestly hyperbolic form $R_{\mu\nu} = -\frac{1}{2}g^{\sigma\tau}\partial_\sigma\partial_\tau g_{\mu\nu} + F_{\mu\nu}(\partial g, g)$. With this form, local existence of solutions to the vacuum Einstein equation for analytic data is immediate from the Cauchy-Kovalevskaya theorem (see [17]) and was first proved for smooth data by Choquet-Bruhat [23]: For any smooth initial data $(\gamma_{ab}, \partial_t\gamma_{ab})$ on a hypersurface Σ there is a solution $g_{\alpha\beta}$ to the vacuum Einstein equation in a neighborhood of $\{0\} \times \Sigma$ in $M = \mathbb{R} \times \Sigma$, and the solution is unique up to isometry. Local existence for perfect fluids without boundary was proved by Lichnerowicz [36], but for local evolution of stars, modeled as perfect fluids with boundary in an asymptotically flat spacetime, there is still no proof available (see [15, 43] for reviews of the Cauchy problem). Proof of existence of solutions with shocks (weak solutions to the Einstein-Euler system in which the fluid variables are discontinuous) is so far restricted to a few equations of state and to a few symmetric spacetimes [27, 35, 6].

11 The CFC approximation

Following Wilson et al. (1996) (see also Flanagan 1999; Mathews & Wilson 2000) we approximate the general metric $g_{\mu\nu}$ by replacing its spatial three-metric γ_{ij} with the conformally flat (CF) three-metric (conformal flatness condition ? CFC hereafter):

$$\gamma_{ij} = \phi^4 \hat{\gamma}_{ij} \tag{175}$$

where $\hat{\gamma}_{ij}$ is the flat metric ($\hat{\gamma}^{ij} = \delta_{ij}$ in Cartesian coordinates). In general, the conformal factor ϕ depends on the coordinates x^i. Therefore, at all times during a numerical simulation we assume that all off-diagonal components of the three-metric are zero, and the diagonal elements have the common factor ϕ^4. Note that all metric quantities with a hat are defined with respect to the flat three-metric $\hat{\gamma}_{ij}$.

Within this approximation the ADM equations reduce to a set of five coupled elliptic (Poisson-like) equations for the metric components,

$$\begin{aligned}
\hat{\Delta}\phi &= -2\pi\phi^5\left(\rho h W^2 - P + \frac{K_{ij}K^{ij}}{16\pi}\right), &(176)\\
\hat{\Delta}(\alpha\phi) &= 2\pi\alpha\phi^5\left(\rho h\left(3W^2 - 2\right) + 5P + \frac{7K_{ij}K^{ij}}{16\pi}\right), &(177)\\
\hat{\Delta}\beta^i &= 16\pi\alpha\phi^4 S^i + 2\hat{K}^{ij}\hat{\nabla}_j\left(\frac{\alpha}{\phi^6}\right) - \frac{1}{3}\hat{\nabla}^i\hat{\nabla}_k\beta^k, &(178)
\end{aligned}$$

where $\hat{\nabla}$ and $\hat{\Delta}$ are the flat space Nabla and Laplace operator, respectively. The transformation behavior between the extrinsic curvature defined on γ_{ij} and $\hat{\gamma}_{ij}$ is as follows:

$$K_{ij} = \phi^{-2}\hat{K}_{ij}, \quad K^{ij} = \phi^{-10}\hat{K}^{ij}. \tag{179}$$

The metric Eqs. (7)?(9) couple to each other via their right hand sides, and in case of the equations for β^i via the operator $\hat{\Delta}$ acting on the vector β^i . The equations are dominated by the source terms involving the hydrodynamic quantities ρ, P and v^i, whereas the nonlinear coupling through the other, purely metric, source terms becomes only important for strong gravity. On each time slice the metric is solely determined by the instantaneous hydrodynamic state, i.e. the distribution of matter in space.

12 Linear Perturbations and Gravitational Waves

Assume a metric $g_{\mu\nu}$, that differs from Minkowski metric $\eta_{\mu\nu}$ as

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}, \tag{180}$$

with $|h_{\mu\nu}| \ll 1$. Using the flat metric tensor, one can define

$$h_\mu{}^\nu \equiv \eta^{\nu\beta} h_{\mu\beta}, \tag{181}$$

$$h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}, \tag{182}$$

$$h \equiv h^\alpha{}_\alpha = \eta^{\mu\alpha} h_{\mu\alpha}. \tag{183}$$

The metric must satisfy

$$g_{\mu\nu} g^{\nu\beta} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\beta} + \delta g^{\nu\beta}) = \delta_\mu^\beta, \tag{184}$$

which leads to

$$\delta g^{\mu\nu} = -h^{\mu\nu}. \tag{185}$$

In order to derived the linearized Riemann tensor, we first need to linearize the Christoffel symbols. In a chosen coordinate system, the Christoffel symbols can be expressed as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu\alpha,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right). \tag{186}$$

The perturbation of the partial derivatives of the metric are, e.g.

$$(\delta g_{\nu\alpha})_{,\beta} = \eta_{\nu\alpha,\beta} + h_{\nu\alpha,\beta} = h_{\nu\alpha,\beta}.$$

The linearized Christoffel symbols are then

$$\begin{aligned} \delta\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\eta^{\mu\nu} \left(h_{\nu\alpha,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu} \right), \\ &= \frac{1}{2} \left(h_\alpha{}^\mu{}_{,\beta} + h_\beta{}^\mu{}_{,\alpha} - h_{\alpha\beta}{}^{,\mu} \right). \end{aligned}$$

The linearized Riemann tensor is simply

$$\begin{aligned} \delta R_{\alpha\mu\beta\nu} &\equiv \frac{1}{2}\delta(g_{\alpha\nu,\mu\beta} - g_{\alpha\beta,\mu\nu} + g_{\mu\beta,\alpha\nu} - g_{\mu\nu,\alpha\beta}), \\ &= \frac{1}{2}(h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu} + h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta}), \end{aligned} \tag{187}$$

the linearized Ricci tensor is

$$\begin{aligned} \delta R_{\mu\nu} = \eta^{\alpha\beta}\delta R_{\alpha\mu\beta\nu} &= \frac{1}{2}\eta^{\alpha\beta}(h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu} + h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta}), \\ &= \frac{1}{2}(h_\nu{}^\alpha{}_{,\mu\alpha} - h_{,\mu\nu} + h_\mu{}^\alpha{}_{,\alpha\nu} - h_{\mu\nu,\alpha}{}^\alpha), \end{aligned} \tag{188}$$

and the linearized Ricci scalar is

$$\delta R = \eta^{\mu\nu}\delta R_{\mu\nu}. \tag{189}$$

Then, the linearized field equations become

$$h_{\mu\alpha,}{}^{\alpha}{}_{\nu} + h_{\mu\alpha,}{}^{\alpha}{}_{\mu} - h_{\mu\nu,}{}^{\alpha}{}_{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta,}{}^{\alpha\beta} - h_{,\alpha}{}^{\alpha}\right) = 16\pi\delta T_{\mu\nu}$$

The wave-like character of the linearized field equations is more clearly revealed if one works with the *trace-reversed* perturbation

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h,$$

for which $\bar{h} = -h$. Then, the linearized field equations become

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 16\pi\delta T_{\mu\nu}.$$

The first term can be written using the short-hand notation for the d'Alembertian (wave) operator

$$\begin{aligned} \Box &= \eta^{\alpha\beta}\partial_\alpha\partial_\beta \\ &= \partial_\alpha\partial^\alpha \\ &= -\frac{\partial^2}{\partial t^2} + \nabla^2, \end{aligned}$$

so that

$$\boxed{-\Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 16\pi\delta T_{\mu\nu}}.$$

which is more compact than the original equation. As we will see next, the second and third terms on the left side can be eliminated by an infinitesimal coordinate transformation.

12.1 Gauge freedom for infinitesimal perturbations

Linear perturbations of a spacetime metric have a gauge freedom to infinitesimal transformations of the coordinates, of the form

$$x^{\alpha'} = x^\alpha + \xi^\alpha, \tag{190}$$

where ξ^α is an infinitesimal displacement. In the new coordinate system, the metric becomes

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x). \tag{191}$$

Since

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha_\beta - \xi^\alpha{}_{,\beta}, \tag{192}$$

it follows that

$$\begin{aligned} g'_{\mu\nu} &= (\delta^\alpha_\mu - \xi^\alpha{}_{,\mu})(\delta^\beta_\nu - \xi^\beta{}_{,\nu})(\eta_{\alpha\beta} + h_{\alpha\beta}), \\ &= \eta_{\mu\nu} + h_{\mu\nu} - \delta^\alpha_\nu\eta_{\alpha\beta}\xi^\beta{}_{,\nu} - \delta^\beta_\nu\eta_{\alpha\beta}\xi^\alpha{}_{,\mu} + \mathscr{O}[\xi^2, \xi\cdot h], \\ &= \eta_{\mu\nu} + h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}, \end{aligned} \tag{193}$$

and the two metric perturbations are related as

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}. \tag{194}$$

The infinitesimal displacement vector can then be suitably chosen to enforce a certain gauge condition. Choosing the four components of ξ^α is equivalent to specifying four conditions on the components of $\bar{h}_{\mu\nu}$.

In terms of the traceless metric perturbation $\bar{h}^{\mu\nu}$, Eq. (194) becomes

$$\bar{h'}^{\mu\nu} = \bar{h}^{\mu\nu} - \xi^{\mu,\nu} - \xi^{\nu,\mu} + \eta^{\mu\nu}\xi^{\alpha}{}_{,\alpha}, \tag{195}$$

and taking a derivative, one obtains

$$\bar{h'}^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \Box\xi^\mu. \tag{196}$$

12.2 Lorentz Gauge

The linearized field equations simplify considerably, if one transforms to a new coordinate system x', in which

$$\overline{h}'^{\mu\nu}{}_{,\nu} = 0, \tag{197}$$

$$\Rightarrow \quad \Box\xi^\mu = \overline{h}^{\mu\nu}{}_{,\nu} \equiv f^\mu(x). \tag{198}$$

The latter is an inhomogeneous wave equation, with source $f^\mu(x)$. The d'Alembertian operator is invertible, having a Green's function $G(x - x')$, i.e.

$$\Box_x G(x - x') = \delta^4(x - x'), \tag{199}$$

with corresponding solution

$$\xi^\mu(x) = \int d^4x' G(x - x') f^\mu(x'). \tag{200}$$

One can thus always find an infinitesimal displacement to enforce the condition Eq. (197). By analogy with electromagnetism, this condition is called the Lorentz gauge (the correct name would be Lorenz gauge). It is the weak-field limit of the harmonic (or De-Donder) gauge in strong fields.

But, notice that there exist trivial displacements, satisfying $\Box\xi^\mu = 0$, that preserve the Lorentz gauge, since then

$$\bar{h'}^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} = 0. \tag{201}$$

A trivial displacement is thus a solution of $\Box\xi^\mu = 0$, of the form

$$\xi^\alpha = C^\alpha e^{ik_\mu x^\mu}, \tag{202}$$

where k^μ is a wave 4-vector and C^α is the amplitude 4-vector.

In vacuum, one can choose four arbitraty component of ξ^α and construct the tensor

$$\xi_{\mu\nu} \equiv \xi_{\nu,\mu} + \xi_{\mu,\nu} - \eta_{\mu\nu}\xi^{\rho}{}_{,\rho}, \tag{203}$$

which also satisfies $\Box\xi_{\mu\nu} = 0$, when $\Box\xi_\mu = 0$. By subtracting $\xi_{\mu\nu}$ from $\bar{h}_{\mu\nu}$, one can impose four arbitrary conditions on the latter.

The initial 10 components of the perturbed metric $h_{\mu\nu}$ are reduced to just $10 - 4 - 4 = 2$ independent components, because of the general Lorentz gauge Eq. (197) and of the freedom to specify the components of the 4-vector C^α in Eq. (202).

In the Lorentz gauge the linearized field equations become a simple wave equation

$$\Box\overline{h}_{\mu\nu} = 16\pi T_{\mu\nu}. \tag{204}$$

In vacuum, one can show that $\Box\bar{h}_{\mu\nu} = \Box h_{\mu\nu} = 0$. Therefore, in vacuum the metric perturbations propagate as waves.

12.3 The Transverse-Traceless Gauge

In vacuum, one can choose ξ^t so that

$$\boxed{\bar{h} = h = 0}, \tag{205}$$

$$\Rightarrow \ \bar{h}_{\mu\nu} = h_{\mu\nu}, \tag{206}$$

which makes the perturbed metric *traceless*. Next, one can choose the three spatial components ξ^i in order to set

$$\boxed{\bar{h}^{ti} = 0}. \tag{207}$$

With the above choices, the t-component of the Lorentz gauge Eq. (197) becomes

$$h^{tt}{}_{,t} = 0, \tag{208}$$

which means that h_{tt} is constant in time (this corresponds to the static part of the gravitational interaction). The gravitational wave itself corresponds to a time-dependent perturbation, so that for gravitational waves:

$$\boxed{h^{tt} = 0}, \tag{209}$$

which also implies that since the perturbed metric is traceless:

$$\boxed{h_i{}^i = 0}. \tag{210}$$

Finally, with the above choices, the spatial components of the Lorentz gauge Eq. (197) become

$$\boxed{h^{ij}{}_{,j} = 0}. \tag{211}$$

The perturbed metric in this transverse-traceless gauge is denoted as h_{ij}^{TT} and contains only two remaining degrees of freedom.

Notice that within the source, one can still choose a trivial displacement satisfying $\Box\xi^\mu = 0$ and construct a tensor satisfying $\Box\xi_{\mu\nu} = 0$, but now the wave equation has a source term, $\Box\bar{h}_{\mu\nu} = 16\pi T_{\mu\nu}$, so that by subtracting $\xi_{\mu\nu}$ from $\bar{h}_{\mu\nu}$, one cannot set to zero any further components.

12.4 Plane waves

The wave equation $\Box\bar{h}_{\mu\nu} = 0$ has plane wave solutions of the form

$$\boxed{h_{ij}^{\mathrm{TT}}(x^\mu) = e_{ij}(k^i)e^{ik_\alpha x^\alpha}}, \tag{212}$$

where is a null wave 4-vector ($k_\mu k^\mu = 0$) with components $k^\mu = (\omega/c, k^i)$ and $\omega/c = \sqrt{k_i k^i}$. The tensor $e_{ij}(k^i)$ is the *polarization tensor*. For propagation along the z−axis, the polarization tensor in the TT gauge can be written as

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)], \tag{213}$$

where h_+ and $h_\times$ are the amplitudes of the "plus" and "cross" polarizations of the plane wave, respectively. Notice that the perturbation is nonzero only in the *transverse* direction, with respect to the direction of propagation.

Appendix

Derivatives and integrals

The covariant derivative operator of the spacetime metric $g_{\alpha\beta}$ will be written ∇_α, and the partial derivative of a scalar f with respect to one of the coordinates – say r – will be written $\partial_r f$ or $f_{,r}$. Lie derivatives along a vector u^α will be denoted by $\mathcal{L}_{\mathrm{u}}$. The Lie derivative of an arbitrary tensor $T^{a\cdots b}{}_{c\cdots d}$ is

$$\begin{aligned}\mathcal{L}_{\mathrm{u}} T^{a\cdots b}{}_{c\cdots d} &= u^e \nabla_e T^{a\cdots b}{}_{c\cdots d} - T^{e\cdots b}{}_{c\cdots d} \nabla_e u^a - \cdots - T^{a\cdots e}{}_{c\cdots d} \nabla_e u^b \\ &\quad + T^{a\cdots b}{}_{e\cdots d} \nabla_c u^e + \cdots + T^{a\cdots b}{}_{c\cdots e} \nabla_d u^e. \end{aligned} \tag{214}$$

Our notation for integrals is as follows. We denote by d^4V the spacetime volume element. In a chart $\{x^0, x^1, x^2, x^3\}$, the notation means,

$$d^4V = \epsilon_{0123} dx^0 dx^1 dx^2 dx^3 = \sqrt{|g|}\, d^4x, \tag{215}$$

where g is the determinant of the matrix $\|g_{\mu\nu}\|$. Gauss's theorem (presented in Sect. **??** of the Appendix) has the form

$$\int_\Omega \nabla_\alpha A^\alpha d^4V = \int_{\partial\Omega} A^\alpha dS_\alpha, \tag{216}$$

with $\partial\Omega$ the boundary of the region Ω. In a chart (u, x^1, x^2, x^3) for which V is a surface of constant u, $dS_\alpha = \sqrt{|g|}\ \nabla_\alpha u d^3x$, and

$$\int_V A^\alpha dS_\alpha = \int_V A^u \sqrt{|g|} d^3x. \tag{217}$$

If V is nowhere null, one can define a unit normal,

$$\widehat{n}_\alpha = \frac{\nabla_\alpha u}{\left|\nabla_\beta u \nabla^\beta u\right|^{1/2}}, \tag{218}$$

and write

$$dS_\alpha = \widehat{n}_\alpha dV, \tag{219}$$

where

$$dV = \sqrt{|^3g|}\ d^3x, \tag{220}$$

where ^3g is the determinant of the 3-metric induced on the surface V. But Gauss's theorem has the form (216) for any 3-surface S, bounding a 4-dimensional region $\mathcal{R}$, regardless of whether S is timelike, spacelike or null.[5]

Similarly, if $F^{\alpha\beta}$ is an antisymmetric tensor, its integral over a 2-surface S of constant coordinates u and v is written

$$\int_S F^{\alpha\beta} dS_{\alpha\beta} = \int_S F^{uv} \sqrt{|g|} d^2x, \tag{221}$$

[5]Note that in the text, n_α denotes the *future* pointing unit normal to a t = constant hypersurface, $n_\alpha = -\nabla_\alpha t/|\nabla_\beta t \nabla^\beta t|^{1/2}$. In order that, for example, $\int \rho u^\alpha dS_\alpha$, be positive on a t = constant surface, one must use $dS_\alpha = \nabla_\alpha t \sqrt{|g|} d^3x = \widehat{n}_\alpha dV = -n_\alpha dV$.

and a corresponding generalized Gauss's theorem has the form

$$\int_V \nabla_\beta F^{\alpha\beta} dS_\alpha = \int_{\partial V} F^{\alpha\beta} dS_{\alpha\beta}. \tag{222}$$

If n_α and $\widetilde{n}_\alpha$ are orthogonal unit normals to the surface S, for which $(\mathrm{n}, \widetilde{\mathrm{n}}, \boldsymbol{\partial}_2, \boldsymbol{\partial}_3)$ is positively oriented, then $dS_{\alpha\beta} = n_{[\alpha}\widetilde{n}_{\beta]}\sqrt{|^2g|}\, d^2x$.

Asymptotic notation: O and o

We will use the symbols $O(x)$ and $o(x)$ to describe asymptotic behavior of functions. For a function $f(x)$, $f = O(x)$ if there is a constant C for which $|f/x| < C$, for sufficiently small $|x|$; and $f = o(x)$ if $\lim_{x\to 0}|f/x| = 0$. For example, if A is constant, $A/r = O(r^{-1})$, and $A/r^{3/2} = o(r^{-1})$.

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