Lecture Notes on Relativistic Stars

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Preface

These lecture notes are distributed as part of the teaching material for the GR course in the MSc program at the Aristotle University of Thessaloniki.

Conventions

Gravitational units ($G = c = 1$) are used in equations, while numerical results are listed in appropriate units (cgs, km, $M_\odot$ etc.). The signature of the spacetime metric is $(- + ++)$. Abstract spacetime indices are Greek, α, $\beta, \ldots$, while spatial indices are Latin $a, b, \ldots$. Indices μ, ν, λ and i, j, k will be concrete, taking values $\mu = 0, 1, 2, 3$, $i = 1, 2, 3$ etc.

Numbers that rely on physical constants are based on the values $c = 2.9979 \times 10^{10}$ cm s^{-1}, $G = 6.670 \times 10^{-8}\mathrm{g}^{-1}\mathrm{cm}^3\mathrm{s}^{-2}$, $\hbar = 1.0545 \times 10^{-27}\mathrm{g\ cm}^2\mathrm{s}^{-1}$, baryon mass $m_B = 1.659 \times 10^{-24}$g, and $M_\odot = 1.989 \times 10^{33}\mathrm{g} = 1.477$ km.

1 Perfect fluids

The stress-energy tensor. In a perfect fluid one assumes that a mean velocity field u^α and a mean stress-energy tensor $T^{\alpha\beta}$ can be defined in fluid elements small compared to the macroscopic length scale but large compared to the mean free path. An observer moving with the average velocity u^α of the fluid will see the collisions randomly distribute the nearby particle velocities so that the particle distribution will appear locally isotropic. Therefore, the components of the fluid's energy momentum tensor in frame of a comoving observer must have no preferred direction and $T^{\alpha\beta}u_\beta$ must be invariant under rotations that fix u^α. It follows that the only nonzero parts of $T^{\alpha\beta}$ are the rotational scalars

$$\epsilon := T^{\alpha\beta}u_\alpha u_\beta, \tag{1}$$

and

$$p := \frac{1}{3}q_{\gamma\delta}T^{\gamma\delta}, \tag{2}$$

where $q_{\gamma\delta} = g_{\gamma\delta} + u_\gamma u_\delta$ is the projection tensor normal to the fluid. For such an isotropic, ideal fluid the stress-energy tensor takes the form

$$\boxed{T^{\alpha\beta} = \epsilon u^\alpha u^\beta + pq^{\alpha\beta}}. \tag{3}$$

The scalars ϵ and p are the *energy density* and the *pressure*, as measured by a *comoving* observer.

Thermodynamics. We denote by n the baryon number density. The rest-mass density (baryon-mass density) is then

$$\rho := m_B n, \tag{4}$$

where m_B is the mass per baryon. We restrict attention to the case of a perfect fluid with equilibrium composition, where the ϵ and p depend on ρ and the *specific entropy* (entropy per unit rest mass) s,

$$\epsilon = \epsilon(\rho, s), \qquad\qquad p = p(\rho, s), \tag{5}$$

(or, on equivalent sets of parameters). The thermodynamics of the fluid is described by the first law, which takes the form

$$\boxed{d\epsilon = \rho T ds + h d\rho}, \tag{6}$$

where T is *temperature* and h is the *specific enthalpy* (enthalpy per unit rest mass),

$$\boxed{h := \frac{\epsilon + p}{\rho}}. \tag{7}$$

Exercise 1.1: Derive (6) from its more common form in terms of extensive quantities

$$dE = TdS - p\,dV + \mu\,dN \equiv TdS - p\,dV + g\,dM_0, \tag{8}$$

by introducing the energy E, entropy S, volume V, baryon number N, and rest mass $M_0 = m_B N$ of a fluid element as measured by a comoving observer. Here $\mu = g m_B$, where $g = \frac{\epsilon+p}{\rho} - Ts$ is the *Gibbs free energy.*

The *specific internal energy* (internal energy per unit rest mass) e is defined by the relation

$$e = \frac{\epsilon}{\rho} - 1. \tag{9}$$

The *Newtonian expression for the specific enthalpy* is

$$h_{\text{Newtonian}} = h - 1 = e + p/\rho. \tag{10}$$

and differs from the relativistic enthalpy h because the relativistic energy density ϵ includes the rest-mass density ρ.

Baroclinic flow. From the definition (7) and using (6), one finds

$$\begin{aligned} dh &= \frac{d\epsilon}{\rho} + \frac{dp}{\rho} - \frac{\epsilon + p}{\rho^2} d\rho = T ds + h\frac{d\rho}{\rho} + \frac{dp}{\rho} - h\frac{d\rho}{\rho}, \\ &= T ds + \frac{dp}{\rho}, \\ \Rightarrow \quad & \boxed{d \ln h = \frac{T}{h} ds + \frac{dp}{\epsilon + p}}, \end{aligned} \tag{11a}$$

implying

$$\nabla_\alpha \ln h = \frac{T}{h} \nabla_\alpha s + \frac{1}{\epsilon + p} \nabla_\alpha p. \tag{12}$$

Exercise 1.2: Taking the curl of (12), show that in the presence of entropy gradients ($\nabla s \neq 0$), surfaces of constant energy density (*isopycnic surfaces*) do not, in general, coincide with surfaces of constant pressure (*isobaric surfaces*). Such a flow is called *baroclinic* and, for a rotating star, it implies the presence of *meridional circulation.*

Barotropic flow. Within a short time after formation, neutrino emission cools a newly born neutron star to $10^{10}K \simeq 1$ MeV, which is much smaller than the Fermi energy of the interior, $E_F(0.16\,\mathrm{fm}^{-3}) \approx 60\,\mathrm{MeV}$. A neutron star is in this sense cold, and, because nuclear reaction times are shorter than the cooling time, one can use a zero-temperature *equation of state* (EOS) to describe the matter:

$$\epsilon = \epsilon(\rho), \quad p = p(\rho), \tag{13}$$

or, equivalently,

$$\epsilon = \epsilon(p). \tag{14}$$

A one-parameter equation of state of the form (14) holds, also in more general situations, such as when the specific entropy is constant throughout the star ($\nabla s = 0$, *homentropic flow).* or even when $s = s(\rho)$, so that effectively the energy density still depends on one parameter only. In such cases, the isopycnic and isobaric surfaces coincide and the flow is *barotropic.*

In a homentropic star, the first law, (6), takes the form

$$d\epsilon = h d\rho, \tag{15}$$

and using (11a) the specific enthalpy is also given by

$$h = \exp\left(\int_0^p \frac{dp}{\epsilon + p}\right), \tag{16}$$

(notice that $\epsilon/\rho = 1$ at $p = 0$, since the gas is nonrelativistic at low densities).

Conservation of baryons. The baryon mass M_0 of a fluid element is conserved by the motion of the fluid. The proper volume of a fluid element is the volume V of a slice orthogonal to u^α through the history of the fluid element; and conservation of baryons can be written in the form $0 = \Delta M_0 = \Delta(\rho V)$. The fractional change in V in a proper time $\Delta\tau$ is given by the 3-dimensional divergence of the velocity, in the subspace orthogonal to u^α:

$$\frac{\Delta V}{V} = q^{\alpha\beta}\nabla_\alpha u_\beta \Delta\tau. \tag{17}$$

Because $u^\beta u_\beta = -1$, we have $u^\beta \nabla_\alpha u_\beta = \frac{1}{2}\nabla_\alpha(u_\beta u^\beta) = 0$, implying

$$q^{\alpha\beta}\nabla_\alpha u_\beta = \nabla_\beta u^\beta. \tag{18}$$

With $u^\alpha \nabla_\alpha \rho = \dfrac{d}{d\tau}\rho$, conservation of baryons takes the form

$$0 = \frac{\Delta(\rho V)}{V} = \Delta\rho + \rho\frac{\Delta V}{V} = (u^\alpha\nabla_\alpha\rho + \rho\nabla_\alpha u^\alpha)\Delta\tau, \tag{19}$$

or

$$\boxed{\nabla_\alpha(\rho u^\alpha) = 0}. \tag{20}$$

Conservation of stress-energy tensor. The Bianchi identities and the Einstein field equations imply that the stress-energy tensor is conserved

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{21}$$

By projecting this conservation along the fluid velocity and normal to it, one obtains separate equations for the conservation of energy and momentum.

Exercise 1.3: Show that the projection along the fluid velocity $u_\alpha \nabla_\beta T^{\alpha\beta} = 0$ leads to

$$\boxed{\nabla_\beta(\epsilon u^\beta) = -p\nabla_\beta u^\beta}. \tag{22}$$

(*conservation of energy*) and that the projection normal to the fluid velocity $q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} = 0$ leads to

$$\boxed{(\epsilon + p)u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta}\nabla_\beta p}, \tag{23}$$

(*conservation of momentum* or equations of motion).

Exercise 1.4: For a barotropic fluid with constant entropy (a homentropic fluid) write the relativistic Euler equation in the form

$$u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta}\nabla_\beta \ln h. \tag{24}$$

or, equivalently,

$$u^\beta \omega_{\alpha\beta} = 0, \tag{25}$$

where

$$\omega_{\alpha\beta} = \nabla_\alpha(h\, u_\beta) - \nabla_\beta(h\, u_\alpha), \tag{26}$$

is the *relativistic vorticity*.

Spacetime symmetries. A vector field ξ^α is a Killing vector if it Lie derives the metric,

$$\mathcal{L}_\xi g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0. \tag{27}$$

We will call ξ^α a *symmetry vector* of a perfect-fluid spacetime if ξ^α is a Killing vector that also Lie derives the fluid variables:

$$\mathcal{L}_\xi u^\alpha = 0, \quad \mathcal{L}_\xi \epsilon = 0, \quad \mathcal{L}_\xi p = 0. \tag{28}$$

Exercise 1.5: Show that $hu_\beta\, \xi^\beta$ is conserved along the spacetime trajectories of the fluid

$$\mathcal{L}_{\rm u}\left(h\, u_\beta \xi^\beta\right) = 0. \tag{29}$$

Stationary flow - Bernoulli's law. If a spacetime has an asymptotically timelike symmetry vector, t^α, the flow is stationary. The corresponding conservation law

$$\boxed{\mathcal{L}_{\rm u}(h\, u_t) = 0}, \tag{30}$$

is the relativistic form of Bernoulli's law, the conservation of *enthalpy per unit rest mass*, $-hu_t$, along the trajectories of a stationary flow.

Axisymmetric Flow. An axisymmetric flow is described by a spacetime with a rotational symmetry vector, ϕ^α, a spacelike vector field whose orbits are circles, except on an axis of symmetry, where $\phi^\alpha = 0$. The corresponding conservation law

$$\boxed{\mathcal{L}_{\mathrm{u}}(h\, u_\phi) = 0}, \tag{31}$$

expresses the conservation of a fluid element's *specific angular momentum*, $j := hu_\phi$, the angular momentum per unit rest mass about the axis of symmetry associated with ϕ^α.

Isentropic flow. In the absence of shocks, the flow of a perfect fluid remains *isentropic*, i.e. each fluid element conserves its specific entropy along its trajectory,

$$u^\alpha \nabla_\alpha s = 0. \tag{32}$$

Formally, the relation follows from conservation of baryons (20), conservation of energy (22), and from the first law (6).

Exercise 1.6: Show that in barotropic flows the *relativistic vorticity* $\omega_{\alpha\beta}$ is conserved along the fluid trajectories

$$\mathcal{L}_{\mathrm{u}} \omega_{\alpha\beta} = 0. \tag{33}$$

Equivalently, the *circulation* of the flow along a closed curve

$$\int_{c_\tau} h\, u_\alpha\, dl^\alpha = \int_{c_\tau} \frac{\epsilon + p}{\rho}\, u_\alpha\, dl^\alpha, \tag{34}$$

is independent of τ, conserved by the fluid flow.

2 The spacetime of a rotating star

A rotating star can be modeled by a stationary, axisymmetric, perfect-fluid spacetime, whose circular velocity field u^α can be written in terms of the two Killing vectors t^α and ϕ^α,

$$\boxed{u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha)}, \tag{35}$$

where

$$\boxed{\Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi}{dt}}, \tag{36}$$

is the angular velocity of the fluid as seen by an observer at rest at infinity. A star is called *uniformly rotating* (as seen from infinity) if Ω is constant.

Exercise 2.1: Defining the *shear tensor* as

$$\sigma_{\alpha\beta} := q_\alpha{}^\gamma q_\beta{}^\delta \nabla_{(\gamma} u_{\delta)} - \frac{1}{3} q_{\alpha\beta} \nabla_\gamma u^\gamma, \tag{37}$$

show that, locally, the flow is shear-free if and only if rotation is uniform. Thus, uniform rotation corresponds to constant Ω for observers at infinity and shear-free flow for local observers.

Geometry. In order to arrive at a metric suitable for describing a rotating star, one makes the following assumptions:

1. *The spacetime is asymptotically flat.*

2. *The spacetime is stationary and axisymmetric*: There exist an asymptotically timelike symmetry vector t^α and a rotational symmetry vector ϕ^α.

   The spacetime is said to be *strictly* stationary if t^α is everywhere timelike. (Some rapidly rotating stellar models, as well as black-hole spacetimes, have *ergospheres*, regions in which t^α is spacelike.)

3. *The Killing vectors commute*,
$$[t, \phi] \equiv \mathcal{L}_t \phi^\alpha = 0, \tag{38}$$
   and there is an isometry of the spacetime that simultaneously reverses the direction of t^α and ϕ^α,
$$t^\alpha \longrightarrow -t^\alpha, \quad \phi^\alpha \longrightarrow -\phi^\alpha. \tag{39}$$

For strictly stationary spacetimes, one does not need (38) as a separate assumption, since it follows from a theorem by Carter [6].

The Frobenius theorem now implies the existence of scalars t and ϕ [7, 5] for which

$$t^\alpha \nabla_\alpha t = \phi^\alpha \nabla_\alpha \phi = 1, \tag{40}$$
$$t^\alpha \nabla_\alpha \phi = \phi^\alpha \nabla_\alpha t = 0. \tag{41}$$

and one can choose coordinates $x^0 = t$ and $x^3 = \phi$ so that

$$t^\alpha = \boldsymbol{\partial}_t, \tag{42}$$
$$\phi^\alpha = \boldsymbol{\partial}_\phi. \tag{43}$$

The following metric components are formed by t^α and ϕ^α

$$t_\alpha t^\alpha = g_{tt}, \tag{44a}$$
$$\phi_\alpha \phi^\alpha = g_{\phi\phi}, \tag{44b}$$
$$t_\alpha \phi^\alpha = g_{t\phi}. \tag{44c}$$

Notice that t^α and ϕ^α are not orthogonal to each other and the lack of orthogonality implies a *dragging of inertial frames.* Also, the fluid is not invariant under $t \to -t$ reversal (a rotating fluid with circular flow is not static, but only stationary, there is invariance only under the simultaneous inversion $t \to -t, \phi \to -\phi$).

On the other hand, if the flow is not circular, be there exist meridional convective currents, then there is no invariance even under the simultaneous inversion $t \to -t, \phi \to -\phi$, because the the direction of the circulation changes. In this case the asymmetry means that there will be no family of surfaces orthogonal to t^α and ϕ^α, and the spacetime metric (47) will have additional off-diagonal components.

Quasi-isotropic coordinates. The surfaces of constant t and ϕ are a family of 2-surfaces orthogonal to t^α and ϕ^α that can be described by coordinates x^1 and x^2. A common choice for x^1 and x^2 are *quasi-isotropic coordinates*, for which

$$g_{r\theta} = 0, \qquad g_{\theta\theta} = r^2 g_{rr} \tag{45}$$

in spherical polar coordinates, or

$$g_{\varpi z} = 0, \qquad g_{zz} = r^2 g_{\varpi\varpi} \tag{46}$$

in cylindrical coordinates. Because of the orthogonality of the 2-surfaces to t^α and ϕ^α, four metric components vanish: $g_{tr} = 0$, $g_{t\theta} = 0$ and $g_{\phi r} = 0$, $g_{\phi\theta} = 0$ (equivalently, $g_{t\varpi} = 0$, $g_{tz} = 0$ and $g_{\phi\varpi} = 0$, $g_{\phi z} = 0$).

Spacetime metric. It follows that the metric of an axisymmetric rotating star with circular flow can be written in the form

$$\boxed{ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2 d\theta^2)}, \tag{47}$$

or

$$\boxed{ds^2 = -e^{2\nu}\, dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz)^2}, \tag{48}$$

where ν, ψ, ω and μ are four metric functions that depend on the coordinates r and θ (or ϖ and z) only. Notice that, in the exterior vacuum one can reduce the number of metric functions to three. It is convenient to write e^ψ in the form [3]

$$e^\psi = r \sin\theta B e^{-\nu}, \tag{49}$$

where B is again a function of r and θ only.

The three metric functions ν, ψ and ω are related to the norms of the Killing vectors, t^α and ϕ^α, and to their dot product by the relations

$$t_\alpha t^\alpha = g_{tt} = -e^{2\nu} + \omega^2 e^{2\psi}, \tag{50a}$$
$$\phi_\alpha \phi^\alpha = g_{\phi\phi} = e^{2\psi}, \tag{50b}$$
$$t_\alpha \phi^\alpha = g_{t\phi} = -\omega e^{2\psi}. \tag{50c}$$

The corresponding components of the contravariant metric are

$$g^{tt} = \nabla_\alpha t \nabla^\alpha = -e^{-2\nu}, \tag{51a}$$
$$g^{\phi\phi} = \nabla_\alpha \phi \nabla^\alpha \phi = e^{-2\psi} - \omega^2 e^{-2\nu}, \tag{51b}$$
$$g^{t\phi} = \nabla_\alpha t \nabla^\alpha \phi = -\omega e^{-2\nu}. \tag{51c}$$

The geometry of the orthogonal 2-surfaces is determined by the conformal factor $e^{2\mu}$.

Dragging of inertial frames. In the spacetime of a rotating star, particles (inertial observers) dropped from infinity with zero angular momentum acquire a *nonzero angular velocity* in the direction of the star's rotation. This relativistic effect is called *dragging of inertial frames*. A freely falling particle follows a geodesic, along which its angular momentum per unit rest mass $L = u_\alpha \phi^\alpha = u_\phi$ is conserved. For a particle with $L = 0$

$$u_\phi = 0 \quad \Rightarrow \quad e^{2\psi}(u^\phi - \omega u^t) = 0,$$

so that

$$\boxed{\frac{u^\phi}{u^t} = \omega}. \tag{52}$$

Thus, an radially infalling particle that starts out with zero angular momentum at infinity, will acquire a nonzero angular velocity

$$\Omega = \omega, \tag{53}$$

(as measured by an inertial observer at infinity) even though it maintains zero angular momentum along its trajectory.

ZAMOs. In describing the fluid, it is helpful to introduce a family of zero-angular-momentum-observers (ZAMOs) [2, 3], observers whose velocity has at each point the form

$$\boxed{u^\alpha_{\mathrm{ZAMO}} = u^t(t^\alpha + \omega\phi^\alpha)}. \tag{54}$$

Several properties of a fluid can be conveniently expressed with respect to ZAMOs (see, for example, the 3-velocity, defined below).

4-Velocity. The 4-velocity for a circular flow is written as in (35)

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha). \tag{55}$$

In the metric (48) the normalization $u_\alpha u^\alpha = -1$ determines u^t as

$$u^t = \frac{e^{-\nu}}{\sqrt{1-(\Omega-\omega)^2 e^{2(\psi-\nu)}}}. \tag{56}$$

Defining

$$v := (\Omega - \omega)e^{(\psi-\nu)}, \tag{57}$$

the contravariant and covariant components of the 4-velocity take the form

$$u^t = \frac{e^{-\nu}}{\sqrt{1-v^2}}, \qquad u^\phi = \Omega u^t, \tag{58}$$

$$u_t = -\frac{e^{\nu}}{\sqrt{1-v^2}}(1 + e^{\psi-\nu}\omega v), \qquad u_\phi = \frac{e^{\psi} v}{\sqrt{1-v^2}}. \tag{59}$$

Written in this way, the denominator has the form of a Lorentz factor. Indeed, as will be shown below, v is identified as the 3-velocity measured in the frame of a ZAMO.

Notice that the 4-velocity of a ZAMO becomes $u^\alpha_{\mathrm{ZAMO}} = e^{-\nu}(t^\alpha + \omega\phi^\alpha)$.

3-Velocity. The spatial 3-velocity v does not have a covariant meaning, so one has to define it with respect to a chosen physical frame. One can construct an *orthonormal tetrad*, in which the metric has locally the form of the Minkowski metric

$$ds^2 = \eta_{\mu\nu}\boldsymbol{\omega}^{\hat{\mu}} \otimes \boldsymbol{\omega}^{\hat{\nu}}, \tag{60}$$

where $\boldsymbol{\omega}^{\hat{\mu}}$ are the basis covectors (the index denotes the different vectors, not components). A suitable example is the frame defined by the basis covectors

$$\boldsymbol{\omega}^{\hat{0}} = e^{\nu}\boldsymbol{d}t, \quad \boldsymbol{\omega}^{\hat{1}} = e^{\psi}(\boldsymbol{d}\phi - \omega\boldsymbol{d}t), \quad \boldsymbol{\omega}^{\hat{2}} = e^{\mu}\boldsymbol{d}\varpi, \quad \boldsymbol{\omega}^{\hat{3}} = e^{\mu}\boldsymbol{d}z, \tag{61}$$

with corresponding contravariant basis vectors

$$\boldsymbol{e}_{\hat{0}} = e^{-\nu}(\boldsymbol{\partial}_t + \omega\boldsymbol{\partial}_\phi), \quad \boldsymbol{e}_{\hat{1}} = e^{-\psi}\boldsymbol{\partial}_\phi, \quad \boldsymbol{e}_{\hat{2}} = e^{-\mu}\boldsymbol{\partial}_\varpi, \quad \boldsymbol{e}_{\hat{3}} = e^{-\mu}\boldsymbol{\partial}_z. \tag{62}$$

Along these frame vectors, the nonzero components of the four velocity $u^{\hat{\mu}}$ are written in terms of a fluid 3-velocity v as in Minkowski spacetime

$$u^{\hat{0}} = \frac{1}{\sqrt{1-v^2}}, \qquad u^{\hat{1}} = \frac{v}{\sqrt{1-v^2}}. \tag{63}$$

Exercise 2.2: Transform the above 4-velocity components to the coordinate frame, via $u^\alpha = u^{\hat{\mu}} e^\alpha_{\hat{\mu}}$, and show that one obtains the components (58) only if

$$\boxed{v = (\Omega - \omega)e^{\psi-\nu}}, \tag{64}$$

as in (57).

Since $v = 0$ for $\Omega = \omega$ (for the ZAMO), the 3-velocity v is defined with respect to this observer.

Time dilation. From Eq. (54) it follows that $e^{-\nu}$ is the *time dilation factor* relating the proper time of the local ZAMO to coordinate time t (proper time at infinity).

Redshift. A zero-angular momentum photon sent to infinity by a ZAMO from a point P suffers a redshift that is given by

$$\frac{\omega_{ZAMO}}{\omega_\infty} = \frac{k_\alpha u^\alpha_{ZAMO}}{k_\beta t^\beta} = e^{-\nu}\Big|_P = 1 + z. \tag{65}$$

Circumferential radius. The *proper circumference* of a circle around the axis of symmetry is

$$\int_0^{2\pi} \sqrt{g_{\phi\phi}}d\phi = 2\pi e^{\psi}. \tag{66}$$

Therefore the *proper circumferential radius* is

$$\boxed{R := e^{\psi}}. \tag{67}$$

Ergospheres. In highly relativistic models, rapid rotation can lead to frame dragging extreme enough that all physical particles are dragged forward relative to an observer at infinity, or, equivalently, relative to the Killing vector t^α. A region in which this is true is called an *ergosphere*, whose definition (mentioned above), is a region in which the asymptotically timelike Killing vector t^α is spacelike. Because physical particles move along timelike or null lines, the definition implies that no physical particle can remain at rest relative to t^α.

At any point in the spacetime, the angular velocity of a physical particle is restricted in a way that looks asymmetric relative to infinity, but *symmetric relative to a ZAMO*. In Minkowski space, a particle can have a timelike or null trajectory only if $\varpi\Omega < 1$, implying $-1/\varpi < \Omega < 1/\varpi$. In the spacetime of a rotating star, a particle can have arbitrary 4-velocity

$$u^\alpha = u^t(t^\alpha + v^\alpha + \Omega\phi^\alpha),$$

where $v^\alpha \perp t^\alpha,\ \phi^\alpha$, which is timelike if $u_\alpha u^\alpha = -1 < 0$. This implies that

$$t^\alpha t_\alpha + 2\Omega t^\alpha \phi_\alpha + \Omega^2 \phi^\alpha \phi_\alpha + v^\alpha v_\alpha \leq 0.$$

Then $v^\alpha v_\alpha \geq 0$ implies $\Omega_- \leq \Omega \leq \Omega_+$, where

$$\Omega_\pm = -\frac{\boldsymbol{t}\cdot\boldsymbol{\phi}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} \pm \left[\left(\frac{\boldsymbol{t}\cdot\boldsymbol{\phi}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}\right)^2 - \frac{\boldsymbol{t}\cdot\boldsymbol{t}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}\right]^{1/2}$$

or

$$\Omega_\pm = \omega \pm \left(\omega^2 - \frac{\boldsymbol{t}\cdot\boldsymbol{t}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}\right)^{1/2}. \tag{68}$$

For the metric (48)

$$\Omega_\pm = \omega \pm e^{\nu-\psi}. \tag{69}$$

These extrema are reached when $v^\alpha = 0$, that is, for circular motion in the equatorial plane. One obtains the same expression (69) when requiring that $|v| < 1$, for circular motion in the equatorial plane. At the boundary of the ergosphere (called the *stationary limit*), where $t^\alpha t_\alpha = 0$, we have $\Omega_- = 0$. Within the ergosphere, both Ω_+ and Ω_- are positive, implying that, seen from infinity, all particles, must move in the direction of the star's rotation. In stellar models, ergospheres are toroidal and typically enclose the star's equator. In principle, energy could be extracted from the ergosphere by the Penrose process.

Asymptotic Behavior. The lowest-order asymptotic behavior of the metric functions ν and ω is

$$\nu \sim -\frac{M}{r} + \frac{Q}{r^3}P_2(\cos\theta), \tag{70}$$

$$\omega \sim \frac{2J}{r^3}, \tag{71}$$

where M, J and Q are the gravitational mass, angular momentum and quadrupole moment of the source of the gravitational field. The asymptotic expansion of the dragging potential ω shows that it decays rapidly far from the star, so that its effect will be significant mainly in the vicinity of the star.

In addition, the asymptotic relations

$$e^\psi = \varpi(e^{-\nu} + O(r^{-2})), \qquad e^\mu = e^{-\nu} + O(r^{-2}), \tag{72}$$

hold, because any stationary, asymptotically flat spacetime agrees with the Schwarzschild geometry to order r^{-1}. If, following Bardeen and Wagoner (1971), we write

$$\beta := \psi + \nu, \qquad \zeta := \mu + \nu, \qquad B := \frac{1}{\varpi}\, e^{\beta}, \tag{73}$$

then, asymptotically, β (or B) deviates by $O(r^{-2})$ from its value in the isotropic Schwarzschild metric; and ζ, which vanishes for isotropic Schwarzschild, is itself of order r^{-2}.

Nonrotating Limit. In the non-rotating limit, the metric (48) reduces to the metric of a spherical relativistic star in *isotropic coordinates*, so that, in the exterior vacuum region, the following relations hold

$$e^{\nu} = \frac{1 - M/2r}{1 + M/2r}, \quad e^{\psi} = \varpi(1 + M/2r)^2, \quad e^{\mu} = (1 + M/2r)^2, \tag{74}$$

where M is the gravitational mass of the star (see [9]).

3 Einstein's field equation

When an equation of state has been specified and if an equilibrium solution exists, the structure of the star is determined by solving four components of Einstein's gravitational field equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, or

$$R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right), \tag{75}$$

(where $R_{\alpha\beta}$ is the Ricci tensor and $T = T_{\alpha}{}^{\alpha}$), together with the equation of hydrostationary equilibrium (see next section). One approach for deriving the necessary equations is to select four components of the Einstein field equation, expressed in the tetrad frame of the ZAMO. In this frame, the stress-energy tensor becomes

$$T^{\hat{0}\hat{0}} = \frac{\epsilon + p v^2}{1 - v^2}, \qquad T^{\hat{0}\hat{1}} = \epsilon + p \frac{v}{1 - v^2}, \tag{76}$$

$$T^{\hat{1}\hat{1}} = \frac{\epsilon v^2 + p}{1 - v^2}, \qquad T^{\hat{2}\hat{2}} = T^{\hat{3}\hat{3}} = p. \tag{77}$$

With $\zeta = \mu + \nu$, one common choice for the components of the gravitational field equation is [3, 4]

$$\begin{aligned}
\nabla \cdot (B \nabla \nu) &= \frac{1}{2} r^2 \sin^2 \theta B^3 e^{-4\nu} \nabla\omega \cdot \nabla\omega \\
&\quad + 4\pi B e^{2\zeta - 2\nu} \left[\frac{(\epsilon + p)(1 + v^2)}{1 - v^2} + 2p \right], && (78a) \\
\nabla \cdot (r^2 \sin^2 \theta B^3 e^{-4\nu} \nabla\omega) &= -16\pi r \sin\theta B^2 e^{2\zeta - 4\nu} \frac{(\epsilon + p) v}{1 - v^2}, && (78b) \\
\nabla \cdot (r \sin\theta \nabla B) &= 16\pi r \sin\theta B e^{2\zeta - 2\nu} p, && (78c)
\end{aligned}$$

(these are, respectively the $R_{\hat{0}\hat{0}}, R_{\hat{0}\hat{3}}$, and $R_{\hat{0}\hat{0}} - R_{\hat{3}\hat{3}}$ field equations), supplemented by a first-order differential equation for ζ (see [4]), which comes from $e^{-\beta + 2\mu}(G^{\hat{3}\hat{3}} - G^{\hat{2}\hat{2}}) = e^{-\beta}(G_{zz} - G_{\varpi\varpi}) =$

0:

$$\frac{1}{\varpi}\zeta_{,\varpi}+\frac{1}{B}(B_{,\varpi}\zeta_{,\varpi}-B_{,z}\zeta_{,z}) = \frac{1}{2\varpi^2 B}(\varpi^2 B_{,\varpi})_{,\varpi}-\frac{1}{2B}B_{,zz}+(\nu_{,\varpi})^2 -(\nu_{,z})^2-\frac{1}{4}\varpi^2 B^2 e^{-4\nu}\left[(\omega_{,\varpi})^2-(\omega_{,z})^2\right]. \tag{79}$$

In the first three equations above, $\boldsymbol{\nabla}$ is the ordinary *flat* three-dimensional derivative operator, the derivative operator of the flat 3-metric

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = d\varpi^2 + dz^2 + \varpi^2 d\phi^2.$$

Thus, three of the four components of the field equation are of elliptic type, while the fourth is a first-order partial-differential equation, relating only metric functions. The remaining non-zero components of the gravitational field equation yield two more elliptic equations and one first-order partial-differential equation, which are consistent with the above set of four equations.

4 Hydrostationary equilibrium equation

For a stationary, axisymmetric perfect fluid star, the equation of hydrostationary equilibrium can be written in several equivalent forms. Because scalars and components of vectors (in t, r, θ, ϕ coordinates) depend only on r and θ, the equation has only r and θ components: The t and ϕ components of each term vanish identically, and $q_\alpha{}^\beta\nabla_\beta p = \nabla_\alpha p$. The relativistic Euler equation (23) for a stationary, axisymmetric star then takes the form

$$\frac{\nabla_\alpha p}{(\epsilon+p)} = -u^\beta\nabla_\beta u_\alpha. \tag{80}$$

We note first that the equation can be written in terms of the scalars $u^t = u^\alpha\nabla_\alpha t$ and $u_\phi = u_\alpha\phi^\alpha$ as

$$\frac{\nabla_\alpha p}{(\epsilon+p)} = \nabla_\alpha \ln u^t - u^t u_\phi \nabla_\alpha\Omega. \tag{81}$$

To obtain (81), one uses the fact that, for a circular flow, the 4-velocity is a linear combination of two Killing vectors, $u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha)$. For uniform rotation, $k^\alpha := u^\alpha/u^t$ is also a Killing vector, satisfying $\nabla_{(\alpha}k_{\beta)} = 0$. More generally, for differential rotation

$$\nabla_{(\alpha}k_{\beta)} = \phi_{(\alpha}\nabla_{\beta)}\Omega, \tag{82}$$

which leads directly to

$$u^\beta\nabla_\beta u_\alpha = -\nabla_\alpha \ln u^t + u^t u_\phi\nabla_\alpha\Omega, \tag{83}$$

where we have used $u^\beta\nabla_\beta \ln u^t = 0 = u^\beta\nabla_\beta\Omega$ (from axisymmetry and stationarity) and $u^\beta\nabla_\alpha u_\beta = 0$ (from the normalization $u^\alpha u_\alpha = -1$).

Next, replacing u^t and u_ϕ in Eq. (81) by their expressions in Eqs. (58), (59), and using Eq. (64) for v, we have

$$\begin{aligned}\frac{\nabla p}{(\epsilon+p)} &= \nabla \ln\frac{e^{-\nu}}{\sqrt{1-v^2}} - \frac{e^{\psi-\nu}v}{1-v^2}\nabla\Omega, \\ &= -\frac{1}{1-v^2}\left(\nabla\nu - v^2\nabla\psi + e^{\psi-\nu}v\nabla\omega\right),\end{aligned} \tag{84}$$

Exercise 4.1: Derive the following equivalent forms of the hydrostationary equilibrium equation:

$$\frac{\nabla p}{(\epsilon + p)} = \nabla \ln u^t - u^t u_\phi \nabla\Omega, \tag{85a}$$

$$= \nabla \ln u^t - \frac{l}{1-\Omega l}\nabla\Omega, \tag{85b}$$

$$= -\nabla \ln(-u_t) + \frac{\Omega}{1-\Omega l}\nabla l, \tag{85c}$$

$$= -\nabla \nu + \frac{1}{1-v^2}\left(v\nabla v - \frac{v^2\nabla\Omega}{\Omega - \omega}\right), \tag{85d}$$

where $l := -u_\phi/u_t$ is conserved along fluid trajectories (since hu_t and hu_ϕ are conserved, so is their ratio and l is the angular momentum per unit energy).

Note that (84) is explicitly independent of $\nabla\Omega$, with the $\nabla\Omega$ term in each equation of (85) canceling a term involving $\nabla\Omega$ in $\nabla \ln u^t$, $\nabla \ln(-u_t)$, or ∇v.

5 The Poincaré-Wavre theorem

For barotropes, one can prove a number of important properties. Since $\epsilon = \epsilon(p)$, one can define a function

$$H(p) := \int_0^p \frac{dp'}{\epsilon(p') + p'}, \tag{86}$$

satisfying $\nabla H = \nabla \ln h - \frac{T}{h}\nabla s$, so that (81) becomes

$$\nabla(H - \ln u^t) = -F\nabla\Omega, \tag{87}$$

where we have set $F := u^t u_\phi$. For homentropic stars, $H = \ln h$, and the equation of hydrostationary equilibrium takes the form

$$\nabla\left(\ln \frac{h}{u^t}\right) = -F\nabla\Omega. \tag{88}$$

Because scalars are independent of t and ϕ, we can regard ∇ in Eqs. (81) and (84) as the two-dimensional gradient in the $r-\theta$ subspace. With A, B indices in that subspace, we have

$$\nabla_A(H - \ln u^t) = -F\nabla_A\Omega, \tag{89}$$

The curl of (89) has the form $\nabla_{[A}F\ \nabla_{B]}\Omega = 0$, implying either

$$\Omega = \text{constant}, \tag{90}$$

(*uniform rotation*), or

$$F = F(\Omega), \tag{91}$$

in the case of *differential rotation.* In the latter case, (89) becomes $H - \ln u^t + \displaystyle\int_{\Omega_0}^{\Omega} F(\Omega')d\Omega' = \text{constant}$, or

$$H - \ln u^t + \int_{\Omega_{\text{pole}}}^{\Omega} F(\Omega')d\Omega' = \nu|_{\text{pole}}, \tag{92}$$

where the lower limit, Ω_0 is chosen as the value of Ω at the pole, where H and v vanish. The above *global* first integral of the hydrostationary equilibrium equations is useful in constructing numerical models of rotating stars[1].

For a uniformly rotating star, (92) can be written as

$$H - \ln u^t = \nu|_{\text{pole}}, \tag{93}$$

which, in the case of a homentropic star, becomes

$$\frac{h}{u^t} = \mathcal{E}, \tag{94}$$

with $\mathcal{E} = e^{\nu}\,|_{\text{pole}}$ constant over the star. The constancy of $\mathcal{E}$ follows from the fact that an equilibrium configuration is an extremum of the mass for perturbations that move baryons from one place to another, with angular momentum and entropy fixed.

Another consequence of (92) is that *the effective gravity can be derived from a potential*, Φ_{eff}, as is clear from

$$\frac{\nabla_\alpha p}{\epsilon + p} = \nabla_\alpha \Phi_{\text{eff}} := \nabla_\alpha \left(\ln u^t - \int_{\Omega_0}^{\Omega} F(\Omega')d\Omega' \right). \tag{95}$$

Using (92), one finds

$$\nabla_\alpha \Phi_{\text{eff}} = \nabla_\alpha H, \tag{96}$$

and the surfaces of constant effective gravity (*level surfaces*) coincide with the surfaces of constant energy density ϵ (isopycnic surfaces).

In the Newtonian limit, because $e^{\psi} = \varpi + O(\lambda^2), \quad e^{\nu} = 1 + O(\lambda^2)$, we have, to Newtonian order,

$$u^t u_\phi = v\varpi = \varpi^2\Omega, \tag{97}$$

and the functional dependence of Ω implied by Eq. (91) becomes the familiar requirement that, for a barotropic equation of state, Ω be stratified on cylinders,

$$\Omega = \Omega(\varpi). \tag{98}$$

The Newtonian limit of the integral of motion (92) is

$$h_{\text{Newtonian}} - \frac{1}{2}v^2 + \Phi = \text{constant}. \tag{99}$$

With the assumption that the topology of the star's surface is either spherical or toroidal, Abramowicz [1] shows in the relativistic context that the surfaces of constant Ω are topological cylinders.

From Eq. (87) and our subsequent discussion, the relativistic version of the classical Poincaré-Wavre theorem [8] follows: Consider a model of a rotating star, a stationary axisymmetric spacetime with a bounded connected perfect fluid having 4-velocity along $k^\alpha = t^\alpha + \Omega\phi^\alpha$. *Any one of the following statements implies the other three:*

[1]The global first integral (92) and its special case (94), are sometimes mistakenly referred to as Bernoulli's law. In Bernoulli's law (30), however, the conserved quantity is hu_t, and it is conserved only along each fluid trajectory; in the equation of hydrostationary equilibrium, the constant quantity is h/u^t (for a uniformly rotating, homentropic star), and it is constant throughout the star. The confusion may arise from the fact that, for a uniformly rotating star, the Newtonian form of the conserved quantity appearing in Bernoulli's law, $h_{\text{Newtonian}} + \frac{1}{2}v^2 + \Phi$, differs from the corresponding Newtonian first integral (99) only in the sign of the v^2 term.

1. $F := u^t u_\phi$ is a function of Ω only.

2. The effective gravity can be derived from a potential.

3. The effective gravity is normal to the surfaces of constant ϵ.

4. The surfaces of constant p and ϵ (isobaric and isopycnic surfaces) coincide.

Appendix

Derivatives and integrals

The covariant derivative operator of the spacetime metric $g_{\alpha\beta}$ will be written ∇_α, and the partial derivative of a scalar f with respect to one of the coordinates – say r – will be written $\partial_r f$ or $f_{,r}$. Lie derivatives along a vector u^α will be denoted by $\mathcal{L}_{\mathrm{u}}$. The Lie derivative of an arbitrary tensor $T^{a\cdots b}{}_{c\cdots d}$ is

$$\begin{aligned}\mathcal{L}_{\mathrm{u}} T^{a\cdots b}{}_{c\cdots d} &= u^e \nabla_e T^{a\cdots b}{}_{c\cdots d} - T^{e\cdots b}{}_{c\cdots d} \nabla_e u^a - \cdots - T^{a\cdots e}{}_{c\cdots d} \nabla_e u^b \\ &\quad + T^{a\cdots b}{}_{e\cdots d} \nabla_c u^e + \cdots + T^{a\cdots b}{}_{c\cdots e} \nabla_d u^e. \end{aligned} \tag{100}$$

Our notation for integrals is as follows. We denote by d^4V the spacetime volume element. In a chart $\{x^0, x^1, x^2, x^3\}$, the notation means,

$$d^4V = \epsilon_{0123} dx^0 dx^1 dx^2 dx^3 = \sqrt{|g|}\, d^4x, \tag{101}$$

where g is the determinant of the matrix $\|g_{\mu\nu}\|$. Gauss's theorem (presented in Sect. **??** of the Appendix) has the form

$$\int_\Omega \nabla_\alpha A^\alpha d^4V = \int_{\partial\Omega} A^\alpha dS_\alpha, \tag{102}$$

with $\partial\Omega$ the boundary of the region Ω. In a chart (u, x^1, x^2, x^3) for which V is a surface of constant u, $dS_\alpha = \sqrt{|g|}\, \nabla_\alpha u d^3x$, and

$$\int_V A^\alpha dS_\alpha = \int_V A^u \sqrt{|g|} d^3x. \tag{103}$$

If V is nowhere null, one can define a unit normal,

$$\widehat{n}_\alpha = \frac{\nabla_\alpha u}{\left|\nabla_\beta u \nabla^\beta u\right|^{1/2}}\ , \tag{104}$$

and write

$$dS_\alpha = \widehat{n}_\alpha dV, \tag{105}$$

where

$$dV = \sqrt{|^3g|}\, d^3x, \tag{106}$$

where ^3g is the determinant of the 3-metric induced on the surface V. But Gauss's theorem has the form (102) for any 3-surface S, bounding a 4-dimensional region $\mathcal{R}$, regardless of whether S is timelike, spacelike or null.[2]

Similarly, if $F^{\alpha\beta}$ is an antisymmetric tensor, its integral over a 2-surface S of constant coordinates u and v is written

$$\int_S F^{\alpha\beta} dS_{\alpha\beta} = \int_S F^{uv} \sqrt{|g|} d^2x, \tag{107}$$

and a corresponding generalized Gauss's theorem has the form

$$\int_V \nabla_\beta F^{\alpha\beta} dS_\alpha = \int_{\partial V} F^{\alpha\beta} dS_{\alpha\beta}. \tag{108}$$

If n_α and $\widetilde{n}_\alpha$ are orthogonal unit normals to the surface S, for which $(\mathrm{n}, \widetilde{\mathrm{n}}, \boldsymbol{\partial}_2, \boldsymbol{\partial}_3)$ is positively oriented, then $dS_{\alpha\beta} = n_{[\alpha} \widetilde{n}_{\beta]} \sqrt{|^2g|}\, d^2x$.

[2]Note that in the text, n_α denotes the *future* pointing unit normal to a t = constant hypersurface, $n_\alpha = -\nabla_\alpha t / |\nabla_\beta t \nabla^\beta t|^{1/2}$. In order that, for example, $\int \rho u^\alpha dS_\alpha$, be positive on a t = constant surface, one must use $dS_\alpha = \nabla_\alpha t \sqrt{|g|} d^3x = \widehat{n}_\alpha dV = -n_\alpha dV$.

Asymptotic notation: O and o

We will use the symbols $O(x)$ and $o(x)$ to describe asymptotic behavior of functions. For a function $f(x)$, $f = O(x)$ if there is a constant C for which $|f/x| < C$, for sufficiently small $|x|$; and $f = o(x)$ if $\lim_{x\to 0} |f/x| = 0$. For example, if A is constant, $A/r = O(r^{-1})$, and $A/r^{3/2} = o(r^{-1})$.

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