Gravitational Waves

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Generation of GWs

- Linearized field equations

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Solution

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' G\left(x - x'\right) T_{\mu\nu}$$

where

$$\boxed{G\left(x - x'\right) = -\frac{1}{4\pi\left|\mathbf{x} - \mathbf{x}'\right|} \delta\left(x^0_{\mathrm{ret}} - x'^0\right)}$$

is a Green's function, satisfying

$$\Box_x G\left(x - x'\right) = \delta^4\left(x - x'\right)$$

and

$$t_{\mathrm{ret}} = t - \frac{\left|\mathbf{x} - \mathbf{x}'\right|}{c}$$

Generation of GWs

- The solution becomes

$$\boxed{\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right)}$$

- Define the *spatial projector* normal to a direction $\hat{\mathbf{n}}$

$$P_{ij} := \delta_{ij} - n_i n_j$$

then

$$\begin{aligned}
\Lambda_{ij,kl}(\hat{\mathbf{n}}) &= P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \\
&= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\
&\quad + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l
\end{aligned}$$

Transverse Traceless Gauge and Far-Field Approximation

- If $h_{\mu\nu}$ is in Lorentz gauge (in vacuum), then it is brought to the *TT gauge* via the projection

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} h_{kl}$$

and the solution in vacuum is then

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right)$$

- Far from the source, we can expand (where d is the source size)

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + \mathcal{O}\left(\frac{d^2}{r} \right)$$

and obtain the *far-field approximation*

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}' \right)$$

Non-relativistic Sources

- Let us Fourier transform the stress-energy tensor:

$$T_{kl}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega(t-r/c+\mathbf{x}'\hat{\mathbf{n}})+i\mathbf{k}\cdot\mathbf{x}'}$$

If the source has a maximum frequency ω_s and is *non-relativistic* ($\omega_s d \ll c$) and because $|\mathbf{x}'| \lesssim d$, only frequencies for which

$$\frac{\omega}{c}\mathbf{x}'\cdot\hat{\mathbf{n}}\lesssim\frac{\omega_s d}{c}\ll 1$$

contribute. Then, expanding in terms of $\omega\mathbf{x}'\cdot\hat{\mathbf{n}}/c$

$$e^{-i\omega(t-r/c+\mathbf{x}'\hat{\mathbf{n}}/c)+i\mathbf{k}\cdot\mathbf{x}'}=e^{-i\omega(t-r/c)}\left[1-i\frac{\omega}{c}x'^in^i+\frac{1}{2}\left(-i\frac{\omega}{c}\right)^2x^ix'^jn^in^j+\ldots\right.$$

or, in the time domain:

$$T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c},\mathbf{x}'\right)=T_{kl}\left(t-r/c,\mathbf{x}'\right)+\frac{x'^in^i}{c}\partial_0T_{kl}+\frac{1}{2c^2}x^ix'^jn^in^j\partial_0^2T_{kl}+$$

Multipole Moments of the Stress-Energy Tensor

- The multipole moments of $T_{\mu\nu}$ are

$$S^{ij} = \int d^3x T^{ij}(t, \mathbf{x})$$
$$S^{ij,k} = \int d^3x T^{ij}(t, \mathbf{x}) x^k$$
$$S^{ij,kl} = \int d^3x T^{ij}(t, \mathbf{x}) x^k x^l$$
$$\dots$$

and the solution becomes

$$h_{ij}^{\mathrm{TT}} = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\left[S^{kl} + \frac{1}{c}n_m\dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p \ddot{S}^{kl,mp} + \dots\right]_{\mathrm{ret}}$$

Mass Density and Momentum Density Multipole Moments

- In terms of the mass density $\left(1/c^2\right) T^{00}$ one can define the moments

$$M = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x})$$
$$M^i = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i$$
$$M^{ij} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j$$
$$M^{ijk} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j x^k, \quad \cdots$$

and in terms of the momentum density $(1/c)T^{0i}$

$$P^i = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x})$$
$$P^{i,j} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j$$
$$P^{i,jk} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j x^k, \quad \cdots$$

Mass Quadrupole Radiation

- The quadrupole moment of $T_{\mu\nu}$ is written in terms of the mass-density qudrupole moment as

  $$\boxed{S^{ij} = \frac{1}{2}\ddot{M}^{ij}}$$

  and the solution becomes to leading order in v/c

  $$\boxed{\left[h_{ij}^{\mathrm{TT}}(t,\mathbf{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^2}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}^{kl}(t-r/c)}$$

  Define the *reduced (trace-free) quadrupole moment tensor*

  $$Q^{ij} := M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} \quad (1)$$

  $$\simeq \int d^3x\rho(t,\mathbf{x})\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right) \quad (2)$$

  (to leading order in v/c it becomes the Newtonian expression) and

  $$Q_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}(\mathbf{n})Q_{ij}$$

Quadrupole Approximation

- The *quadrupole formula* for GW radiation is

$$\boxed{\left[h_{ij}^{\mathrm{TT}}(t,\mathbf{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{\mathrm{TT}}(t-r/c)}$$

  Notice that $\ddot{Q}_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}\ddot{Q}_{ij} = \Lambda_{ij,kl}\ddot{M}_{ij}$ (the latter is preferred in calculations)

- **EXAMPLE**: Emission along $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Then $P_{ij} = \delta_{ij} - n_i n_j$ becomes

$$P_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  For any 3×3 matrix A_{ij}

$$\begin{aligned} \Lambda_{ij,kl}A_{kl} &= \left[P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right]A_{kl} \\ &= (PAP)_{ij} - \frac{1}{2}P_{ij}\,\mathrm{Tr}(PA) \end{aligned}$$

Quadrupole Approximation

and

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while $\mathrm{Tr}(PA) = A_{11} + A_{22}$. Then:

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} - \frac{A_{11} + A_{22}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \\ &= \begin{pmatrix} \left(A_{11} - A_{22}\right)/2 & A_{12} & 0 \\ A_{21} & -\left(A_{11} - A_{22}\right)/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \end{aligned}$$

Thus

$$\Lambda_{ij,kl} \ddot{M}_{kl} = \begin{pmatrix} \left(\ddot{M}_{11} - \ddot{M}_{22}\right)/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -\left(\ddot{M}_{11} - \ddot{M}_{22}\right)/2 & 0 \\ 0 & 0 & \end{pmatrix}_{ij}$$

Quadrupole Approximation

- Comparing to

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

we immediately find

$$\begin{aligned} h_+ &= \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) \\ h_\times &= \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12} \end{aligned}$$

(the r.h.s. is computed in the retarded time $t - r$).

Emission Along Arbitrary Direction

- Along an arbitrary direction $\hat{\mathbf{n}}$, with components in a Cartesian system

$$n_i = (\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\theta)$$

the two polarizations are:

$$\begin{aligned} h_+(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\Big[&\ddot{M}_{11}\left(\cos^2\phi - \sin^2\phi\cos^2\theta\right) + \ddot{M}_{22}\left(\sin^2\phi - \cos^2\phi\cos^2\theta\right) \\ &- \ddot{M}_{33}\sin^2\theta - \dot{M}_{12}\sin 2\phi\left(1+\cos^2\theta\right) \\ &+ \ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta\Big] \end{aligned}$$

and

$$\begin{aligned} h_\times(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\Big[&\left(\ddot{M}_{11} - \ddot{M}_{22}\right)\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta \\ &- 2\ddot{M}_{13}\cos\phi\sin\theta + 2\ddot{M}_{23}\sin\phi\sin\theta\Big] \end{aligned}$$

Emitted Energy and Linear Momentum of GWs

- Energy is emitted by GWs at a rate

$$\begin{aligned}\frac{dE_{\mathrm{GW}}}{dt} &= \frac{c^3 r^2}{32\pi G}\int d\Omega \left\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \right\rangle && (3)\\ &= \frac{c^3 r^2}{16\pi G}\int d\Omega \left\langle \dot{h}_{+}^{2} + \dot{h}_{\times} \right\rangle && (4)\\ &\simeq \frac{G}{5c^5}\left\langle \dddot{Q}_{jk} \dddot{Q}^{jk} \right\rangle && (5)\\ &= \frac{G}{5c^5}\left\langle \ddot{M}_{ij}\ddot{M}_{ij} - \frac{1}{3}\left(\ddot{M}_{kk}\right)^2 \right\rangle && (6)\end{aligned}$$

- There is no loss of linear momentum in the quadrupole approximation

$$\frac{\partial P^{k}_{\mathrm{GW}}}{dt} = -\frac{G}{8\pi c^5}\int d\Omega \ddot{Q}^{\mathrm{TT}}_{ij} \partial^k \ddot{Q}^{\mathrm{TTT}}_{ij} = 0 \qquad (7)$$

because Q_{ij} is invariant and $\partial^i \to -\partial^i$ under a reflection $\mathbf{x} \to -\mathbf{x}$.

Angular Momentum Emitted by GWs

- The angular momentum carried away by GWs is

$$\frac{dJ^i}{dt} = \frac{c^3}{32\pi G}\int r^2 d\Omega \left\langle -\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ab} x^k \partial^\ell h^{\mathrm{TT}}_{ab} + 2\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{al} h^{\mathrm{TT}}_{ak}\right\rangle$$

  In the quadrupole approximation, this becomes

$$\left(\frac{dJ^i}{dt}\right)_{\text{quad}} = \frac{2G}{5c^5}\epsilon^{ikl}\left\langle \ddot{Q}_{ka}\dddot{Q}_{la}\right\rangle$$

GWs from a Binary System

- Consider a binary with circular orbits. The trajectories of the two stars are $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ and the relative coordinate is $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$. The center of mass is

$$\mathbf{x}_{\rm CM} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}$$

For a nonrelativistic system, the mass quadrupole moment is

$$\begin{aligned} M^{ij} &= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j \\ &= m x_{\rm CM}^i x_{\rm CM}^j + \mu\left(x_{\rm CM}^i x_0^j + x_{\rm CM}^j x_0^i\right) + \mu x_0^i x_0^j \end{aligned}$$

where $m = m_1 + m_2$ and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass. If we choose $\mathbf{x}_{\rm CM} = 0$ as the origin of our coordinate system, then the mass quadrupole moment is

$$\boxed{M^{ij}(t) = \mu x_0^i(t) x_0^j(t)}$$

GWs from a Binary System

- In the CM frame, the dynamics reduces to a one-body problem with reduced mass μ.

- Choose a circular orbit with angular frequency ω_s in the plane with $z_0 = 0$

$$x_0(t) = R\cos\left(\omega_s t + \tfrac{\pi}{2}\right)$$
$$y_0(t) = R\sin\left(\omega_s t + \tfrac{\pi}{2}\right)$$
$$z_0(t) = 0$$

Then

$$M_{11} = \mu R^2 \frac{1-\cos 2\omega_s t}{2} \quad (8)$$
$$M_{22} = \mu R^2 \frac{1+\cos 2\omega_s t}{2} \quad (9)$$
$$M_{12} = -\frac{1}{2}\mu R^2 \sin 2\omega_s t \quad (10)$$

(other components are zero).

GWs from a Binary System

- Taking two time-derivatives:

$$\ddot{M}_{11} = -\ddot{M}_{22} = 2\mu R^2\omega_s^2 \cos 2\omega_s t$$
$$\ddot{M}_{12} = 2\mu R^2\omega_s^2 \sin 2\omega_s t$$

and

$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\omega_s t_{\rm ret} + 2\phi\right)$$
$$h_\times(t;\theta,\phi)) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta\sin\left(2\omega_s t_{\rm ret} + 2\phi\right)$$

- If we can neglect the proper motion of the source, then the angle ϕ is fixed and by a change of the origin of time one can set it to zero.
- If we view the system from an *inclination* $\iota = \theta$, then

$$h_+(t) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left(2\omega_s t\right)$$
$$h_\times(t) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\iota\sin\left(2\omega_s t\right)$$

- For $\iota = 0 \Rightarrow$ circular polarization, for $\iota = 90^o \Rightarrow$ linear polarization, otherwise elliptic polarization. Measuring polarization, recovers ι.

- The two polarizations can be written as

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos\left(2\pi f_{\rm gw}t_{\rm ret}+2\phi\right)$$
$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}}{c}\right)^{2/3}\cos\theta\sin\left(2\pi f_{\rm gw}t_{\rm ret}+2\phi\right)$$

  where $\omega_{\rm gw} = 2\omega_s$ and

$$f_{\rm gw} = \omega_{\rm gw}/(2\pi)$$

  is the frequency of the GWs and

$$M_c = \mu^{3/5}m^{2/5} = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

  is the *chirp mass*.
- Kepler’s law is

$$\omega_s^2 = \frac{Gm}{R^3}$$

Radiated Power

- The angular distribution of the radiated power is

$$\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$

or

$$\boxed{\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{2}{\pi}\frac{c^5}{G}\left(\frac{GM_c\omega_{\text{gw}}}{2c^3}\right)^{10/3} g(\theta)}$$

where

$$g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta$$

which has an angular average of

$$\int \frac{d\Omega}{4\pi} g(\theta) = \frac{4}{5}$$

Radiate Power

- The radiated power is

$$P_{\mathrm{quad}} = \frac{1}{10}\frac{G\mu^2}{c^5}R^4\omega_{\mathrm{gw}}^6$$

or

$$\boxed{P_{\mathrm{quad}} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{\mathrm{gw}}}{2c^3}\right)^{10/3}}$$

- The energy radiated in one period $T = 2\pi/\omega_s$ is (with $v = \omega_s R$)

$$\langle E_{\mathrm{quad}}\rangle_T = \frac{64\pi}{5}\frac{G\mu^2}{R}\left(\frac{v}{c}\right)^5$$

i.e. the energy scale $G\mu^2/R$ is suppressed by a factor $(v/c)^5$.

Frequency evolution

- The orbital energy is

$$\begin{aligned} E_{\text{orbit}} &= E_{\text{kin}} + E_{\text{pot}} \\ &= -\frac{G m_1 m_2}{2R} \\ &= -\left(G^2 M_c^5 \omega_{\rm gw}^2 / 32\right)^{1/3} \end{aligned}$$

- Assume that

$$\left| \frac{dE_{\text{orbit}}}{dt} \right| = P_{\rm quad}$$

  Then

$$\boxed{\dot{f}_{\rm gw} = \frac{96}{5} \pi^{8/3} \left(\frac{G M_c}{c^3} \right)^{5/3} f_{\rm gw}^{11/3}}$$

- Integrating $\dot{f}_{\rm gw}$, we see that it diverges at a finite time $t_{\rm coal}$. The remaining time to coalescence is then

$$\boxed{\tau = t_{\rm coal} - t}$$

and the frequency evolution is written as

$$f_{\rm gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

or

$$\boxed{f_{\rm gw}(\tau) \simeq 134{\rm Hz}\left(\frac{1.21M_\odot}{M_c}\right)^{5/8}\left(\frac{1s}{\tau}\right)^{3/8}}$$

- The time to coalescence is thus

$$\boxed{\tau \simeq 2.18{\rm s}\left(\frac{1.21M_\odot}{M_c}\right)^{5/3}\left(\frac{100{\rm Hz}}{f_{\rm gw}}\right)^{8/3}}$$

Number of cycles

- When the period $T(t)$ is slowly varying, the number of cycles in a time interval dt is

$$d\mathcal{N}_{\rm cyc} = \frac{dt}{T(t)} = f_{\rm gw}(t)dt$$

and thus the number of cycles spent between frequencies $f_{\rm min}$ and $f_{\rm max}$ is

$$\mathcal{N}_{\rm cyc} = \int_{t_{\rm min}}^{t_{\rm max}} f_{\rm gw}(t)dt = \int_{f_{\rm min}}^{f_{\rm max}} df_{\rm gw}\frac{f_{\rm gw}}{\dot{f}_{\rm gw}}$$

or

$$\mathcal{N}_{\rm cyc} = \frac{1}{32\pi^{8/3}}\left(\frac{GM_c}{c^3}\right)^{-5/3}\left(f_{\rm min}^{-5/3} - f_{\rm max}^{-5/3}\right)$$

If $f_{\rm min}^{-5/3} - f_{\rm max}^{-5/3} \simeq f_{\rm min}^{-5/3}$, then

$$\boxed{\mathcal{N}_{\rm cyc} =\simeq 1.6\times 10^4\left(\frac{10{\rm Hz}}{f_{\rm min}}\right)^{5/3}\left(\frac{1.2M_\odot}{M_c}\right)^{5/3}}$$

Orbital Evolution

- From Kepler's law and the equation for $\dot{f}_{\rm gw}$ we find that the radius of the orbit shrinks according to

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{f}_{\rm gw}}{f_{\rm gw}} = -\frac{1}{4\tau}$$

If at $t = t_0$ the radius is $R = R_0$ and $\tau_0 = t_{\rm coal} - t_0$, then integrating:

$$\boxed{R(\tau) = R_0 \left(\frac{\tau}{\tau_0}\right)^{1/4}}$$

- From Kepler's law and the equation for $f_{\rm gw}$ we find

$$\tau_0 = \frac{5}{256}\frac{c^5 R_0^4}{G^3 m^2 \mu}$$

or

$$\boxed{\tau_0 \simeq 9.829 \times 10^6 \mathrm{yr} \left(\frac{T_0}{1\mathrm{hr}}\right)^{8/3} \left(\frac{M_\odot}{m}\right)^{2/3} \left(\frac{M_\odot}{\mu}\right)}$$

- Because $\omega_{\rm gw} = d\Phi/dt$, the evolution of the phase is

$$\Phi(t) = \int_{t_0}^{t} dt' \omega_{\rm gw}\left(t'\right)$$

or, with $\Phi_0 = \Phi(\tau = 0)$

$$\boxed{\Phi(\tau) = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8} \tau^{5/8} + \Phi_0}$$

- The waveform is

$$h_+(t) = \tfrac{4}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t_{\rm ret})}{c}\right)^{2/3}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left[\Phi\left(t_{\rm ret}\right)\right]$$

$$h_x(t) = \tfrac{4}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t_{\rm ret})}{c}\right)^{2/3}\cos\iota\sin\left[\Phi\left(t_{\rm ret}\right)\right]$$

or

$$h_+(\tau) = \tfrac{1}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/4}\left(\tfrac{5}{c\tau}\right)^{1/4}\left(\frac{1+\cos^2\iota}{2}\right)\cos[\Phi(\tau)]$$

$$h_\times(\tau) = \tfrac{1}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/4}\left(\tfrac{5}{c\tau}\right)^{1/4}\cos\iota\sin[\Phi(\tau)]$$