Gravitational Waves

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The Two Polarizations in the TT gauge

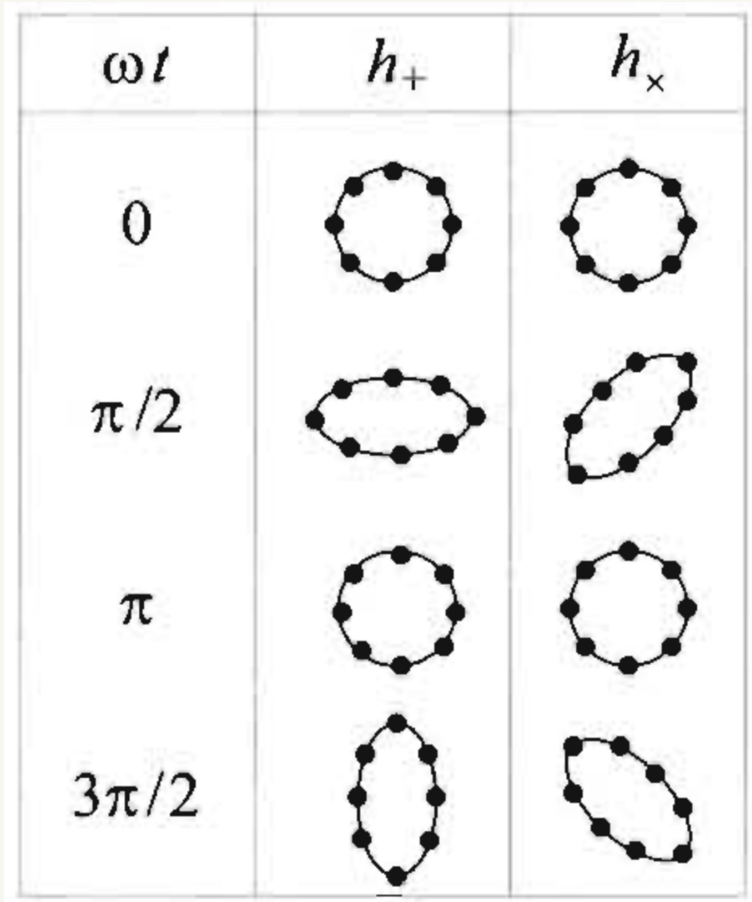

Figure 1: The effect of the two polarizations on a circle. Figure from [1].

Generation of GWs

- Linearized field equations

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Solution

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' G\left(x - x'\right) T_{\mu\nu}$$

where

$$\boxed{G\left(x - x'\right) = -\frac{1}{4\pi \left|\mathbf{x} - \mathbf{x}'\right|} \delta\left(x^0_{\text{ret}} - x'^0\right)}$$

is a Green's function, satisfying

$$\Box_x G\left(x - x'\right) = \delta^4\left(x - x'\right)$$

and

$$t_{\text{ret}} = t - \frac{\left|\mathbf{x} - \mathbf{x}'\right|}{c}$$

Generation of GWs

- The solution becomes

$$\boxed{\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right)}$$

- Define the *spatial projector* normal to a direction $\hat{\mathbf{n}}$

$$P_{ij} := \delta_{ij} - n_i n_j$$

then

$$\begin{aligned}
\Lambda_{ij,kl}(\hat{\mathbf{n}}) &= P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \\
&= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\
&\quad + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l
\end{aligned}$$

Transverse Traceless Gauge and Far-Field Approximation

- If $h_{\mu\nu}$ is in Lorentz gauge (in vacuum), then it is brought to the *TT gauge* via the projection

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} h_{kl}$$

and the solution in vacuum is then

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right)$$

- Far from the source, we can expand (where d is the source size)

$$\left| \mathbf{x} - \mathbf{x}' \right| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + \mathcal{O}\left(\frac{d^2}{r} \right)$$

and obtain the *far-field approximation*

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}' \right)$$

Non-relativistic Sources

- Let us Fourier transform the stress-energy tensor:

$$T_{kl}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega(t-r/c+\mathbf{x}'\hat{\mathbf{n}})+i\mathbf{k}\cdot\mathbf{x}'}$$

If the source has a maximum frequency ω_s and is *non-relativistic* $(\omega_s d \ll c)$ and because $|\mathbf{x}'| \lesssim d$, only frequencies for which

$$\frac{\omega}{c}\mathbf{x}'\cdot\hat{\mathbf{n}} \lesssim \frac{\omega_s d}{c} \ll 1$$

contribute. Then, expanding in terms of $\omega\mathbf{x}'\cdot\hat{\mathbf{n}}/c$

$$e^{-i\omega(t-r/c+\mathbf{x}'\hat{\mathbf{n}}/c)+i\mathbf{k}\cdot\mathbf{x}'}=e^{-i\omega(t-r/c)}\left[1-i\frac{\omega}{c}x'^i n^i+\frac{1}{2}\left(-i\frac{\omega}{c}\right)^2 x^i x'^j n^i n^j+\ldots\right.$$

or, in the time domain:

$$T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c},\mathbf{x}'\right)=T_{kl}\left(t-r/c,\mathbf{x}'\right)+\frac{x'^i n^i}{c}\partial_0 T_{kl}+\frac{1}{2c^2}x^i x'^j n^i n^j\partial_0^2 T_{kl}+$$

Multipole Moments of the Stress-Energy Tensor

- The multipole moments of $T_{\mu\nu}$ are

$$
\begin{aligned}
S^{ij} &= \int d^3x T^{ij}(t, \mathbf{x}) \\
S^{ij,k} &= \int d^3x T^{ij}(t, \mathbf{x}) x^k \\
S^{ij,kl} &= \int d^3x T^{ij}(t, \mathbf{x}) x^k x^l \\
&\dots
\end{aligned}
$$

and the solution becomes

$$
h_{ij}^{\mathrm{TT}} = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \dots \right]_{\mathrm{ret}}
$$

Mass Density and Momentum Density Multipole Moments

- In terms of the mass density $\left(1/c^2\right) T^{00}$ one can define the moments

$$M = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x})$$
$$M^i = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i$$
$$M^{ij} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j$$
$$M^{ijk} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j x^k, \quad \cdots$$

and in terms of the momentum density $(1/c)T^{0i}$

$$P^i = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x})$$
$$P^{i,j} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j$$
$$P^{i,jk} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j x^k, \quad \cdots$$

Mass Quadrupole Radiation

- The quadrupole moment of $T_{\mu\nu}$ is written in terms of the mass-density qudrupole moment as

$$\boxed{S^{ij} = \frac{1}{2}\ddot{M}^{ij}}$$

and the solution becomes to leading order in v/c

$$\boxed{\left[h_{ij}^{\mathrm{TT}}(t,\mathbf{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^2}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}^{kl}(t-r/c)}$$

Define the *reduced (trace-free) quadrupole moment tensor*

$$Q^{ij} := M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} \tag{1}$$

$$\simeq \int d^3x\rho(t,\mathbf{x})\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right) \tag{2}$$

(to leading order in v/c it becomes the Newtonian expression) and

$$Q_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}(\mathbf{n})Q_{ij}$$

Quadrupole Approximation

- The *quadrupole formula* for GW radiation is

$$\boxed{\left[h_{ij}^{\mathrm{TT}}(t,\mathbf{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{\mathrm{TT}}(t-r/c)}$$

  Notice that $\ddot{Q}_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}\ddot{Q}_{ij} = \Lambda_{ij,kl}\ddot{M}_{ij}$ (the latter is preferred in calculations)

- **EXAMPLE**: Emission along $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Then $P_{ij} = \delta_{ij} - n_i n_j$ becomes

$$P_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  For any 3×3 matrix A_{ij}

$$\begin{aligned} \Lambda_{ij,kl}A_{kl} &= \left[P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right]A_{kl} \\ &= (PAP)_{ij} - \frac{1}{2}P_{ij}\,\mathrm{Tr}(PA) \end{aligned}$$

Quadrupole Approximation

and

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while $\mathrm{Tr}(PA) = A_{11} + A_{22}$. Then:

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} = & \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} - \frac{A_{11} + A_{22}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \\ = & \begin{pmatrix} \left(A_{11} - A_{22}\right)/2 & A_{12} & 0 \\ A_{21} & -\left(A_{11} - A_{22}\right)/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \end{aligned}$$

Thus

$$\Lambda_{ij,kl} \ddot{M}_{kl} = \begin{pmatrix} \left(\ddot{M}_{11} - \ddot{M}_{22}\right)/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -\left(\ddot{M}_{11} - \ddot{M}_{22}\right)/2 & 0 \\ 0 & 0 & \end{pmatrix}_{ij}$$

Quadrupole Approximation

- Comparing to

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

we immediately find

$$\begin{aligned} h_+ &= \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) \\ h_\times &= \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12} \end{aligned}$$

(the r.h.s. is computed in the retarded time $t - r$).

Emission Along Arbitrary Direction

- Along an arbitrary direction $\hat{\mathbf{n}}$, with components in a Cartesian system

$$n_i = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$$

the two polarizations are:

$$\begin{aligned} h_+(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\Big[&\ddot{M}_{11}\left(\cos^2\phi - \sin^2\phi\cos^2\theta\right) + \ddot{M}_{22}\left(\sin^2\phi - \cos^2\phi\cos^2\theta\right) \\ &- \ddot{M}_{33}\sin^2\theta - \dot{M}_{12}\sin 2\phi\left(1+\cos^2\theta\right) \\ &+ \ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta\Big] \end{aligned}$$

and

$$\begin{aligned} h_\times(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\Big[&\left(\ddot{M}_{11} - \ddot{M}_{22}\right)\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta \\ &- 2\ddot{M}_{13}\cos\phi\sin\theta + 2\ddot{M}_{23}\sin\phi\sin\theta\Big] \end{aligned}$$

Emitted Energy and Linear Momentum of GWs

- Energy is emitted by GWs at a rate

$$\begin{aligned} \frac{dE_{\mathrm{GW}}}{dt} &= \frac{c^3 r^2}{32\pi G} \int d\Omega \left\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \right\rangle && (3) \\ &= \frac{c^3 r^2}{16\pi G} \int d\Omega \left\langle \dot{h}_{+}^{2} + \dot{h}_{\times} \right\rangle && (4) \\ &\simeq \frac{G}{5c^5} \left\langle \dddot{Q}_{jk} \dddot{Q}^{jk} \right\rangle && (5) \\ &= \frac{G}{5c^5} \left\langle \ddot{M}_{ij} \ddot{M}_{ij} - \frac{1}{3} \left(\ddot{M}_{kk} \right)^2 \right\rangle && (6) \end{aligned}$$

- There is no loss of linear momentum in the quadrupole approximation

$$\frac{\partial P_{\mathrm{GW}}^{k}}{dt} = -\frac{G}{8\pi c^5} \int d\Omega \ddot{Q}_{ij}^{\mathrm{TT}} \partial^k \ddot{Q}_{ij}^{\mathrm{TTT}} = 0 \qquad (7)$$

because Q_{ij} is invariant and $\partial^i \to -\partial^i$ under a reflection $\mathbf{x} \to -\mathbf{x}$.

Angular Momentum Emitted by GWs

- The angular momentum carried away by GWs is

$$\frac{dJ^i}{dt} = \frac{c^3}{32\pi G} \int r^2 d\Omega \left\langle -\epsilon^{ikl} \dot{h}_{ab}^{\mathrm{TT}} x^k \partial^\ell h_{ab}^{\mathrm{TT}} + 2\epsilon^{ikl} \dot{h}_{al}^{\mathrm{TT}} h_{ak}^{\mathrm{TT}} \right\rangle$$

In the quadrupole approximation, this becomes

$$\left(\frac{dJ^i}{dt}\right)_{\text{quad}} = \frac{2G}{5c^5} \epsilon^{ikl} \left\langle \ddot{Q}_{ka} \dddot{Q}_{la} \right\rangle$$

GWs from a Binary System

- Consider a binary with circular orbits. The trajectories of the two stars are $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ and the relative coordinate is $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$. The center of mass is

$$\mathbf{x}_{\mathrm{CM}} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}$$

For a nonrelativistic system, the mass quadrupole moment is

$$\begin{aligned} M^{ij} &= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j \\ &= m x_{\mathrm{CM}}^i x_{\mathrm{CM}}^j + \mu \left(x_{\mathrm{CM}}^i x_0^j + x_{\mathrm{CM}}^j x_0^i \right) + \mu x_0^i x_0^j \end{aligned}$$

where $m = m_1 + m_2$ and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass. If we choose $\mathbf{x}_{\mathrm{CM}} = 0$ as the origin of our coordinate system, then the mass quadrupole moment is

$$\boxed{M^{ij}(t) = \mu x_0^i(t) x_0^j(t)}$$

GWs from a Binary System

- In the CM frame, the dynamics reduces to a one-body problem with reduced mass μ.
- Choose a circular orbit with angular frequency ω_s in the plane with $z_0 = 0$

$$x_0(t) = R\cos\left(\omega_s t + \tfrac{\pi}{2}\right)$$
$$y_0(t) = R\sin\left(\omega_s t + \tfrac{\pi}{2}\right)$$
$$z_0(t) = 0$$

Then

$$M_{11} = \mu R^2 \frac{1 - \cos 2\omega_s t}{2} \tag{8}$$
$$M_{22} = \mu R^2 \frac{1 + \cos 2\omega_s t}{2} \tag{9}$$
$$M_{12} = -\frac{1}{2}\mu R^2 \sin 2\omega_s t \tag{10}$$

(other components are zero).

GWs from a Binary System

- Taking two time-derivatives:

$$\ddot{M}_{11} = -\ddot{M}_{22} = 2\mu R^2 \omega_s^2 \cos 2\omega_s t$$
$$\ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin 2\omega_s t$$

and

$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\omega_s t_{\rm ret} + 2\phi\right)$$
$$h_\times(t;\theta,\phi)) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta \sin\left(2\omega_s t_{\rm ret} + 2\phi\right)$$

- If we can neglect the proper motion of the source, then the angle ϕ is fixed and by a change of the origin of time one can set it to zero.
- If we view the system from an *inclination* $\iota = \theta$, then

$$h_+(t) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left(2\omega_s t\right)$$
$$h_\times(t) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\iota \sin\left(2\omega_s t\right)$$

- For $\iota = 0 \Rightarrow$ circular polarization, for $\iota = 90^o \Rightarrow$ linear polarization, otherwise elliptic polarization. Measuring polarization, recovers ι.

- The two polarizations can be written as

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos\left(2\pi f_{\rm gw}t_{\rm ret}+2\phi\right)$$
$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}}{c}\right)^{2/3}\cos\theta\sin\left(2\pi f_{\rm gw}t_{\rm ret}+2\phi\right)$$

  where $\omega_{\rm gw} = 2\omega_s$ and

$$f_{\rm gw} = \omega_{\rm gw}/(2\pi)$$

  is the frequency of the GWs and

$$M_c = \mu^{3/5}m^{2/5} = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

  is the *chirp mass*.
- Kepler’s law is

$$\omega_s^2 = \frac{Gm}{R^3}$$

Radiated Power

- The angular distribution of the radiated power is

$$\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$

or

$$\boxed{\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{2}{\pi}\frac{c^5}{G}\left(\frac{GM_c\omega_{\text{gw}}}{2c^3}\right)^{10/3} g(\theta)}$$

where

$$g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta$$

which has an angular average of

$$\int \frac{d\Omega}{4\pi} g(\theta) = \frac{4}{5}$$

Radiate Power

- The radiated power is

$$P_{\mathrm{quad}} = \frac{1}{10}\frac{G\mu^2}{c^5}R^4\omega_{\mathrm{gw}}^6$$

or

$$\boxed{P_{\mathrm{quad}} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{\mathrm{gw}}}{2c^3}\right)^{10/3}}$$

- The *energy radiated in one period* $T = 2\pi/\omega_s$ is (with $v = \omega_s R$)

$$\langle E_{\mathrm{quad}}\rangle_T = \frac{64\pi}{5}\frac{G\mu^2}{R}\left(\frac{v}{c}\right)^5$$

i.e. the energy scale $G\mu^2/R$ is suppressed by a factor $(v/c)^5$.

Frequency evolution

- The orbital energy is

$$
\begin{aligned}
E_{\text{orbit}} &= E_{\text{kin}} + E_{\text{pot}} \\
&= -\frac{G m_1 m_2}{2R} \\
&= -\left(G^2 M_c^5 \omega_{\rm gw}^2 / 32\right)^{1/3}
\end{aligned}
$$

- Assume that

$$
\left|\frac{dE_{\text{orbit}}}{dt}\right| = P_{\rm quad}
$$

Then

$$
\boxed{\dot{f}_{\rm gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{\rm gw}^{11/3}}
$$

- Integrating $\dot{f}_{\mathrm{gw}}$, we see that it *diverges* at a finite time t_{coal}. The remaining time to coalescence is then

$$\boxed{\tau = t_{\mathrm{coal}} - t}$$

and the frequency evolution is written as

$$f_{\mathrm{gw}}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

or

$$\boxed{f_{\mathrm{gw}}(\tau) \simeq 134\mathrm{Hz}\left(\frac{1.21M_\odot}{M_c}\right)^{5/8}\left(\frac{1s}{\tau}\right)^{3/8}}$$

- The time to coalescence is thus

$$\boxed{\tau \simeq 2.18\mathrm{s}\left(\frac{1.21M_\odot}{M_c}\right)^{5/3}\left(\frac{100\mathrm{Hz}}{f_{\mathrm{gw}}}\right)^{8/3}}$$

Number of cycles

- When the period $T(t)$ is slowly varying, the number of cycles in a time interval dt is
$$d\mathcal{N}_{\rm cyc} = \frac{dt}{T(t)} = f_{\rm gw}(t)dt$$
and thus the number of cycles spent between frequencies $f_{\rm min}$ and $f_{\rm max}$ is
$$\mathcal{N}_{\rm cyc} = \int_{t_{\rm min}}^{t_{\rm max}} f_{\rm gw}(t)dt$$
$$= \int_{f_{\rm min}}^{f_{\rm max}} df_{\rm gw} \frac{f_{\rm gw}}{\dot{f}_{\rm gw}}$$
or
$$\mathcal{N}_{\rm cyc} = \frac{1}{32\pi^{8/3}} \left(\frac{GM_c}{c^3}\right)^{-5/3} \left(f_{\rm min}^{-5/3} - f_{\rm max}^{-5/3}\right)$$
If $f_{\rm min}^{-5/3} - f_{\rm max}^{-5/3} \simeq f_{\rm min}^{-5/3}$, then
$$\boxed{\mathcal{N}_{\rm cyc} = \simeq 1.6 \times 10^4 \left(\frac{10\rm Hz}{f_{\rm min}}\right)^{5/3} \left(\frac{1.2M_\odot}{M_c}\right)^{5/3}}$$

Orbital Evolution

- From Kepler's law and the equation for $\dot{f}_{\mathrm{gw}}$ we find that the radius of the orbit shrinks according to

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{f}_{\mathrm{gw}}}{f_{\mathrm{gw}}} = -\frac{1}{4\tau}$$

If at $t = t_0$ the radius is $R = R_0$ and $\tau_0 = t_{\mathrm{coal}} - t_0$, then integrating:

$$\boxed{R(\tau) = R_0 \left(\frac{\tau}{\tau_0}\right)^{1/4}}$$

- From Kepler's law and the equation for f_{gw} we find

$$\tau_0 = \frac{5}{256}\frac{c^5 R_0^4}{G^3 m^2 \mu}$$

or

$$\boxed{\tau_0 \simeq 9.83 \times 10^6 \mathrm{yr} \left(\frac{T_0}{1\mathrm{hr}}\right)^{8/3} \left(\frac{M_\odot}{m}\right)^{2/3} \left(\frac{M_\odot}{\mu}\right)}$$

Phase Evolution

- Because $\omega_{\rm gw} = d\Phi/dt$, the evolution of the phase is

$$\Phi(t) = \int_{t_0}^{t} dt' \omega_{\rm gw}\left(t'\right)$$

or, with $\Phi_0 = \Phi(\tau = 0)$

$$\boxed{\Phi(\tau) = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8} \tau^{5/8} + \Phi_0}$$

- The waveform is

$$h_+(t) = \tfrac{4}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t_{\textsf{ret}})}{c}\right)^{2/3}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left[\Phi\left(t_{\textsf{ret}}\right)\right]$$
$$h_x(t) = \tfrac{4}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t_{\textsf{ret}})}{c}\right)^{2/3}\cos\iota\sin\left[\Phi\left(t_{\textsf{ret}}\right)\right]$$

or

$$h_+(\tau) = \tfrac{1}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/4}\left(\tfrac{5}{c\tau}\right)^{1/4}\left(\frac{1+\cos^2\iota}{2}\right)\cos[\Phi(\tau)]$$
$$h_\times(\tau) = \tfrac{1}{r}\left(\tfrac{GM_c}{c^2}\right)^{5/4}\left(\tfrac{5}{c\tau}\right)^{1/4}\cos\iota\sin[\Phi(\tau)]$$

Higher-order Multipoles

- For a binary system in circular orbit and assuming a *flat background*, the *mass octupole* and the *current quadrupole* emit GWs at both frequencies ω_s and $3\omega_s$. The power emitted (compared to the mass quadrupole) is

$$P_{\mathrm{oct+cq}}(\omega_s) = \frac{19}{672}\left(\frac{v}{c}\right)^2 P_{\mathrm{quad}}(2\omega_s)$$
$$P_{\mathrm{oct+cq}}(3\omega_s) = \frac{135}{224}\left(\frac{v}{c}\right)^2 P_{\mathrm{quad}}(2\omega_s)$$

  so it is suppressed by a factor of $(v/c)^2$.

- Notice, however, that the orbit is also affected at order $(v/c)^2$ by relativistic effects, so that the above calculation is not consistent to this order, but only indicates the order of magnitude.

ISCO

- The inspiral phase terminates when the orbit becomes unstable. For a Schwarzschild spacetime of mass $m = m_1 + m_2$, the ISCO radius is

$$r_{1\mathrm{SCO}} = \frac{6Gm}{c^2}$$

The orbital frequency at the ISCO is

$$(f_s)_{\mathrm{ISCO}} = \frac{1}{6\sqrt{6}(2\pi)} \frac{c^3}{Gm}$$

or

$$\boxed{(f_s)_{\mathrm{ISCO}} \simeq 2.2\mathrm{kHz} \left(\frac{M_\odot}{m} \right)}$$

General TT Plane Wave Solution

- The general solution of the wave equation in the TT-gauge $\Box h_{ij}^{\mathrm{TT}} = 0$ can be written as

$$h_{ij}^{\mathrm{TT}}(x) = \int \frac{d^3k}{(2\pi)^3} \left(\mathcal{A}_{ij}(\mathbf{k}) e^{ik^\mu x_\mu} + \mathcal{A}_{ij}^*(\mathbf{k}) e^{-ik^\mu x_\mu} \right)$$

where $k^\mu = (\omega/c, \mathbf{k})$ with $\mathbf{k}/|\mathbf{k}| = \hat{\mathbf{n}}$ and $|\mathbf{k}| = \omega/c = (2\pi f)/c$. Therefore

$$d^3k = |\mathbf{k}|^2 d|\mathbf{k}| d\Omega = (2\pi/c)^3 f^2 df\, d\Omega$$

with $f > 0$. Setting $d\cos\theta d\phi := d^2\hat{\mathbf{n}}$, the solution is written as

$$h_{ij}^{\mathrm{TT}}(x) = \frac{1}{c^3} \int_0^\infty df\, f^2 \int d^2\hat{\mathbf{n}} \left[\mathcal{A}_{ij}(f, \hat{\mathbf{n}}) e^{-2\pi i f (t - \hat{\mathbf{n}} \cdot \mathbf{x}/c)} + \mathrm{c.c.} \right]$$

(notice that both terms in the parenthesis correspond to waves traveling in the $+\hat{\mathbf{n}}$ direction and only physical frequencies $f > 0$ appear in the expansion).

Plane Wave from Specific Direction

- For a plane wave coming from a specific direction $\hat{\mathbf{n}}_0$

$$\boxed{\mathcal{A}_{ij}(f,\hat{\mathbf{n}}) := A_{ij}(f)\delta^{(2)}\left(\hat{\mathbf{n}} - \hat{\mathbf{n}}_0\right)}$$

  (this does not apply to stochastic backgrounds that arrive from different directions).

- In the TT-gauge $k^i\mathcal{A}_{ij}(\mathbf{k}) = 0 \Rightarrow n^i\mathcal{A}_{ij}(f,\hat{\mathbf{n}}) = 0$ and therefore the plane wave is described by only the indices $a, b = 1, 2$ in the transverse direction. We can thus drop the TT label and write h_{ab} for the wave and $\tilde{h}_{ab}$ for its Fourier transform. Also in this gauge

$$\tilde{h}_{ab}(f) = \begin{pmatrix} \tilde{h}_+(f) & \tilde{h}_\times(f) \\ \tilde{h}_\times(f) & -\tilde{h}_+(f) \end{pmatrix}_{ab}$$

Plane Wave at Detector

- The plane wave solution arriving at a detector from a specific direction $\hat{\mathbf{n}}_0$ can thus be written as

$$h_{ab}(t,\mathbf{x}) = \int_0^\infty df \left[\tilde{h}_{ab}(f,\mathbf{x})e^{-2\pi i f t} + \tilde{h}^*_{ab}(f,\mathbf{x})e^{2\pi i f t}\right]$$

where

$$\begin{aligned}\tilde{h}_{ab}(f,\mathbf{x}) &= \frac{f^2}{c^3}\int d^2\hat{\mathbf{n}}\mathcal{A}_{ab}(f,\hat{\mathbf{n}})e^{2\pi i f \hat{\mathbf{n}}\cdot\mathbf{x}/c} \\ &= \frac{f^2}{c^3}A_{ab}(f)e^{2\pi i f\hat{\mathbf{n}}_0\cdot\mathbf{x}/c}\end{aligned}$$

- For the ground-based detectors, the length of each arm is much smaller than the reduced wavelength $\lambda/(2\pi)$ of detectable GWs and taking the detector as the center of the coordinate system we have $\exp\{2\pi i f\hat{\mathbf{n}}\cdot\mathbf{x}/c\} = \exp\{2\pi i\hat{\mathbf{n}}\cdot\mathbf{x}/\lambda\} \simeq 1$. In this case

$$\boxed{h_{ab}(t) \simeq \int_0^\infty df \left[\tilde{h}_{ab}(f)e^{-2\pi i f t} + \tilde{h}^*_{ab}(f)e^{2\pi i f t}\right]}$$

where $\tilde{h}_{ab}(f) = \tilde{h}_{ab}(f,\mathbf{x}=0) = (f^2/c^3)A_{ab}(f)$.

Fourier Transform

- If we extend the definition of $\tilde{h}_{ab}(f)$ to negative frequencies as

  $$\tilde{h}_{ab}(-f) = \tilde{h}^*_{ab}(f)$$

  then we can write the plane wave solution at the detector as

  $$\boxed{h_{ab}(t) = \int_{-\infty}^{\infty} df\, \tilde{h}_{ab}(f) e^{-2\pi i f t}}$$

  which means that $\tilde{h}_{ab}(f)$ is the *Fourier transform* of $h_{ab}(t)$

  $$\boxed{\tilde{h}_{ab}(f) = \int_{-\infty}^{\infty} dt\, h_{ab}(t) e^{2\pi i f t}}$$

Fourier Transform during Inspiral Phase

- For the inspiral phase ($-\infty < t < t_{\rm coal}$) the Fourier transform for a circular binary inspiral is

$$\tilde{h}_+(f) = A e^{i\Psi_+(f)} \frac{c}{r} \left(\frac{GM_c}{c^3}\right)^{5/6} \frac{1}{f^{7/6}} \left(\frac{1+\cos^2\iota}{2}\right) \quad (11)$$

$$\tilde{h}_\times(f) = A e^{i\Psi_\times(f)} \frac{c}{r} \left(\frac{GM_c}{c^3}\right)^{5/6} \frac{1}{f^{7/6}\cos\iota} \quad (12)$$

where

$$A = \frac{1}{\pi^{2/3}} \left(\frac{5}{24}\right)^{1/2}$$

and

$$\Psi_+(f) = 2\pi f\,(t_c + r/c) - \Phi_0 - \frac{\pi}{4} + \frac{3}{4}\left(\frac{GM_c}{c^3} 8\pi f\right)^{-5/3}$$

$$\Psi_\times = \Psi_+ + (\pi/2)$$

with $\Phi_0 = \Phi(\tau = 0)$. For accurate matched filtering, post-Newtonian corrections to the phase $\Psi_{+,\times}(f)$ must be included.

GW Energy Spectrum during Inspiral Phase

- We have seen that the energy emitted in GWs is

$$\left(\frac{dE}{dt}\right)_{\mathrm{GW}} = \frac{c^3 r^2}{16\pi G} \int d\Omega \left\langle \dot{h}_+^2 + \dot{h}_\times \right\rangle$$

The total energy flowing through solid angle $d\Omega$ is thus

$$\left(\frac{dE}{d\Omega}\right)_{\mathrm{GW}} = \frac{c^3 r^2}{16\pi G} \int_{-\infty}^{\infty} dt \left(\dot{h}_+^2 + \dot{h}_\times\right)$$

(because we integrate over all times, the average $\langle\rangle$ over a few periods is not required). Inserting the plane wave solution for a signal with $\lambda >> L_{\mathrm{detector}}$ and restricting to positive frequencies only, we obtain

$$\boxed{\left(\frac{dE}{d\Omega}\right)_{\mathrm{GW}} = \frac{\pi c^3}{2G} \int_0^{\infty} df\, f^2 \left(\left|\tilde{h}_+(f)\right|^2 + \left|\tilde{h}_\times(f)\right|^2\right)}$$

GW Energy Spectrum during Inspiral Phase

- Integrating over a sphere surround the source, the energy spectrum of GWs is

$$\boxed{\frac{dE}{df} = \frac{\pi c^3}{2G} f^2 r^2 \int d\Omega \left(\left| \tilde{h}_+(f) \right|^2 + \left| \tilde{h}_\times(f) \right|^2 \right)}$$

- For a circular binary inspiral, this becomes

$$\boxed{\frac{dE}{df} = \frac{\pi^{2/3}}{3G} (GM_c)^{5/3} f^{-1/3}}$$

Total Energy Emitted During Inspiral

- The total energy emitted in the inspiral phase (up to a maximum frequency $f_{\max}$ is

$$\Delta E_{\rm rad} \sim \frac{\pi^{2/3}}{2G} (GM_c)^{5/3} f_{\max}^{2/3}$$

or

$$\Delta E_{\rm rad} \sim 4.2 \times 10^{-2} M_\odot c^2 \left(\frac{M_c}{1.21 M_\odot}\right)^{5/3} \left(\frac{f_{\max}}{1\rm kHz}\right)^{2/3}$$

If we take $f_{\max} \simeq 2\,(f_s)_{\rm ISCO}$, then

$$\boxed{\Delta E_{\rm rad} \sim 8 \times 10^{-2} \mu c^2}$$

which depends only on the reduced mass of the system.

- A better estimate is obtained considering the binding energy of the binary system at the ISCO

$$E_{\text{binding}} = (1 - \sqrt{8/9})\mu c^2 \simeq 5.7 \times 10^{-2} \mu c^2$$

Elliptic Orbits

- The elliptic orbit is described by polar coordinates (r, ψ) with origin at the center of mass.

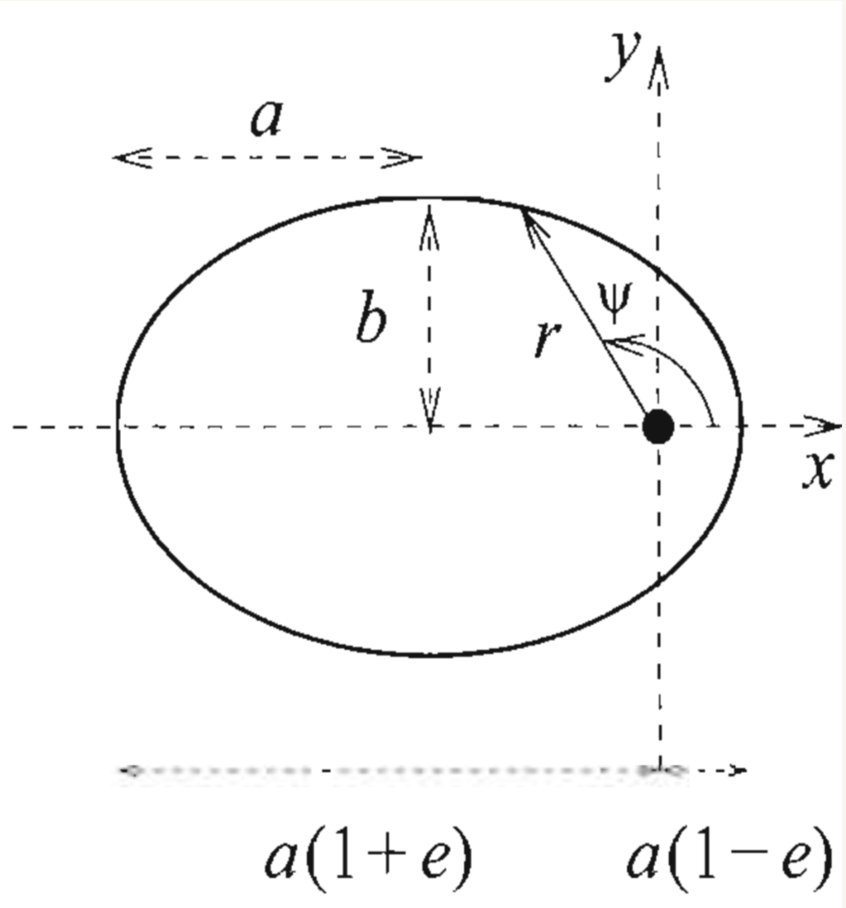

Figure 2: The function $f(e)$. Figure from [1].

Elliptic Orbits

- The motion is equivalent to an effective one-body problem with mass μ and angular momentum $L = \mu r^2 \dot{\psi}$. The total orbital energy is

$$\begin{aligned} E &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\psi}^2\right) - \frac{G\mu m}{r} && (13) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{G\mu m}{r} && (14) \end{aligned}$$

($E < 0$). Integrating $dr/d\psi = \dot{r}/\dot{\psi}$, the equation of the orbit is

$$\frac{1}{r} = \frac{1}{R}(1 + e\cos\psi)$$

where the eccentricity

$$e = \sqrt{1 + \frac{2EL^2}{G^2 m^2 \mu^3}}$$

and the length scale

$$R = \frac{L^2}{Gm\mu^2}$$

are constants of motion.

Elliptic Orbits

- The two semi-axes of the ellipse are

$$a = \frac{R}{1-e^2} = \frac{Gm\mu}{2|E|}$$

$$b = \frac{R}{\left(1-e^2\right)^{1/2}}$$

- In terms of a and e the equation for the orbit is written as

$$\boxed{r = \frac{a\left(1-e^2\right)}{1+e\cos\psi}}$$

and Kepler's law is

$$\omega_0^2 = \frac{Gm}{a^3}$$

with

$$T = \frac{2\pi}{\omega_0} \tag{15}$$

being the period of the orbit.

Elliptic Orbits

- Integrating $\dot{r}$ and $\dot{\psi}$ the time-dependent orbit $r(t)$, $\psi(t)$ is given in parametric form as

$$\begin{aligned} r &= a[1 - e\cos u] \\ \cos\psi &= \frac{\cos u - e}{1 - e\cos u} \end{aligned} \tag{16}$$

where the time parameter u (the *eccentric anomaly*) is related to t through

$$\boxed{\beta \equiv u - e\sin u = \omega_0 t} \tag{17}$$

With $\psi(t=0) = 0$ we can also write

$$\psi(u) = A_e(u) \equiv 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{u}{2}\right] \tag{18}$$

where $A_e(u)$ is called the *true anomaly*.

- Notice that for $e = 0 \Rightarrow \psi = u$.

The True Anomaly

- $-\pi \leq \psi \leq \pi$ and $-\pi \leq u \leq \pi$

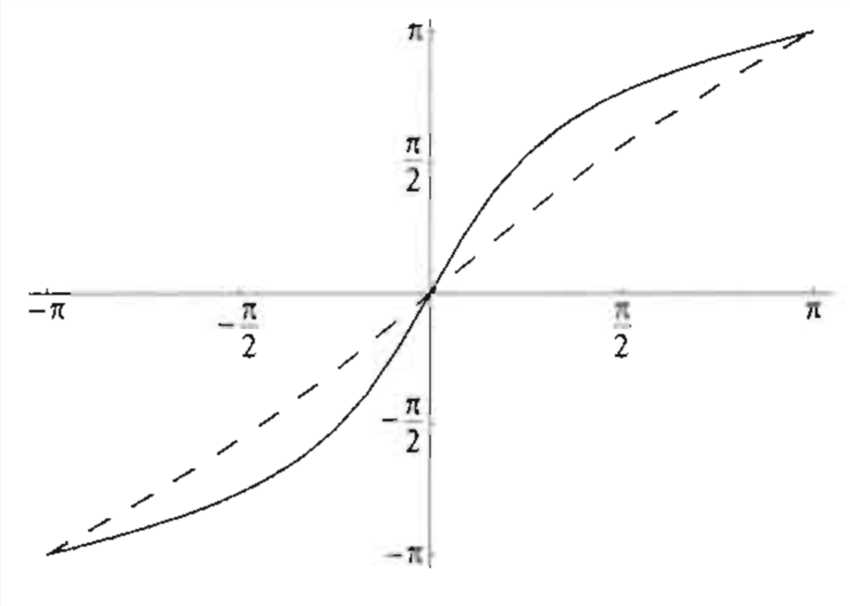

Figure 3: The function $\psi(u)$ for $e = 0.2$ (dashed line) and $e = 0.75$ (solid line). Figure from [1].

- In Cartesian coordinates, the orbit is

$$\begin{aligned} x(t) &= r\cos\psi = a[\cos u(t) - e] \\ y(t) &= r\sin\psi = b\sin u(t) \end{aligned} \tag{19}$$

Elliptic Orbits

- For an elliptic orbit of eccentricity e and semi-major axis a, the power emitted in gravitational waves is

$$\boxed{P = \left(\frac{dE}{dt}\right)_{\mathrm{GW}} = \frac{32G^4\mu^2m^3}{5c^5a^5}f(e)}$$

where

$$f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)$$

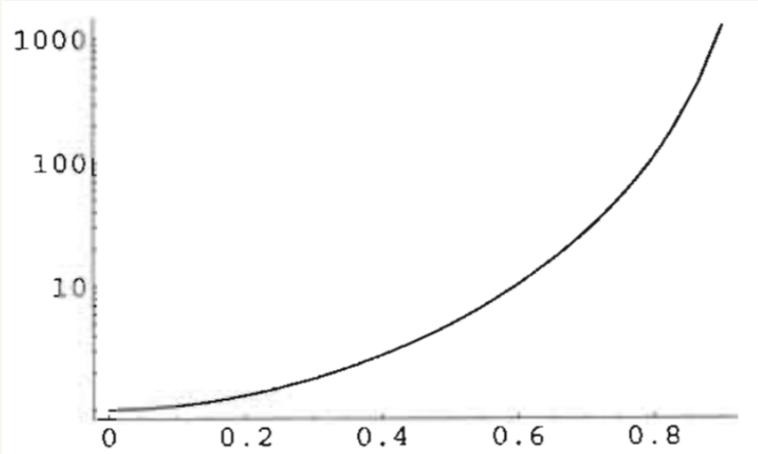

Figure 4: The function $f(e)$. Figure from [1].

- Rewriting Kepler's law, we see that $T \propto (-E)^{-3/2}$, so that

$$\frac{\dot{T}}{T} = -\frac{3}{2}\frac{\dot{E}}{E} \tag{20}$$

and substituting $dE/dt = -(dE/dt)_{\mathrm{GW}}$ we find that the period changes according to

$$\frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^3 \mu m^2}{c^5 a^4} f(e) \tag{21}$$

or

$$\boxed{\frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^{5/3} \mu m^{2/3}}{c^5}\left(\frac{T}{2\pi}\right)^{-8/3} f(e)} \tag{22}$$

(this equation was used to compare the observations of the Hulse-Taylor pulsar, which has $e = 0.617$, to the theoretical prediction of a decreasing period due to GW emission).

Fourier Transform of the Orbit

- The orbit $x(t)$, $y(t)$ is a periodic function of u or β with period 2π. Restricting β to $-\pi \leqslant \beta \leqslant \pi$ we write the *discrete Fourier transform*

$$x(\beta) = \sum_{n=-\infty}^{\infty} \tilde{x}_n e^{-in\beta}$$

$$y(\beta) = \sum_{n=-\infty}^{\infty} \tilde{y}_n e^{-in\beta}$$

  with $\tilde{x}_n = \tilde{x}^*_{-n}$ and $\tilde{y}_n = \tilde{y}^*_{-n}$ (since $x(\beta)$ and $y(\beta)$ are real functions).

- Choosing the origin of time such as $y(t=0)=0$

$$x(\beta) = \sum_{n=0}^{\infty} a_n \cos(n\beta) \quad (23)$$

$$y(\beta) = \sum_{n=1}^{\infty} b_n \sin(n\beta) \quad (24)$$

  with $a_0 = \tilde{x}_0$ and $a_n = 2\tilde{x}_n$ and $b_n = -2i\tilde{y}_n$ for $n \geq 1$.

- With $\beta = \omega_0 t$ and $\omega_n = n\omega_0$

$$x(t) = \sum_{n=0}^{\infty} a_n \cos \omega_n t \tag{25}$$

$$y(t) = \sum_{n=1}^{\infty} b_n \sin \omega_n t \tag{26}$$

with

$$a_0 = \frac{1}{\pi} \int_0^{\pi} d\beta x(\beta) = -(3/2)ae$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} d\beta x(\beta) \cos(n\beta) = \frac{a}{n} \left[J_{n-1}(ne) - J_{n+1}(ne) \right] \tag{27}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} d\beta y(\beta) \sin(n\beta) = \frac{b}{n} \left[J_{n-1}(ne) + J_{n+1}(ne) \right] \tag{28}$$

where $J(x)$ are Bessel functions.

- To compute the GW spectrum, we need the Fourier decomposition of $x^2(t)$, $y^2(t)$ and $x(t)y(t)$, which are

$$x^2(t) = \sum_{n=0}^{\infty} A_n \cos \omega_n t$$

$$y^2(t) = \sum_{n=0}^{\infty} B_n \cos \omega_n t$$

$$x(t)y(t) = \sum_{n=1}^{\infty} C_n \sin \omega_n t$$

where

$$A_n = \frac{a^2}{n} \left[J_{n-2}(ne) - J_{n+2}(ne) - 2e \left(J_{n-1}(ne) - J_{n+1}(ne) \right) \right]$$
$$B_n = \frac{b^2}{n} \left[J_{n+2}(ne) - J_{n-2}(ne) \right]$$
$$C_n = \frac{ab}{n} \left[J_{n+2}(ne) + J_{n-2}(ne) - e \left(J_{n+1}(ne) + J_{n-1}(ne) \right) \right]$$

- Then, the radiated power is a sum of harmonics

$$P = \sum_{n=1}^{\infty} P_n$$

where

$$P_n = \frac{G\mu^2\omega_0^6}{15c^5} n^6 \left(A_n^2 + B_n^2 + 3C_n^2 - A_n B_n\right)$$

This can be written as

$$\boxed{P_n = \frac{32G^4\mu^2 m^3}{5c^5 a^5} g(n,e)}$$

where

$$\boxed{g(n,e) = \frac{n^6}{96a^4}\left[A_n^2(e) + B_n^2(e) + 3C_n^2(e) - A_n(e)B_n(e)\right]}$$

Power of Harmonics for Elliptical Orbits

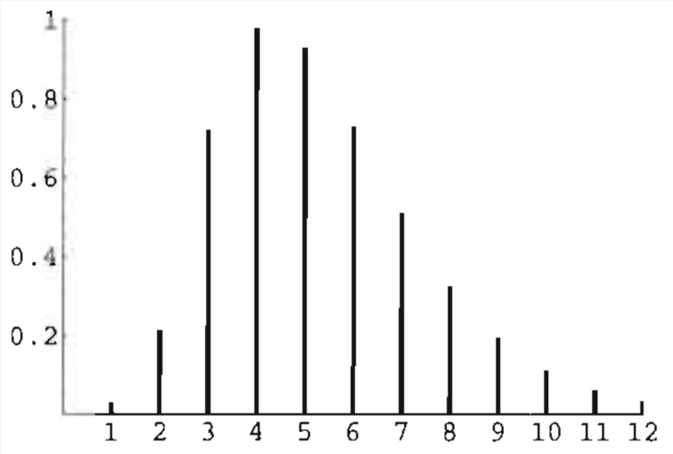

Figure 5: The power P_n as function of n for $e = 0.5$. Figure from [1].

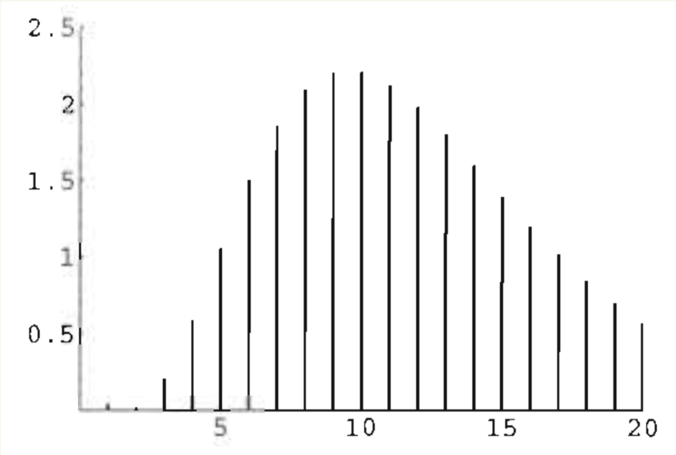

Figure 6: The power P_n as function of n for $e = 0.7$. Figure from [1].

Evolution of Orbital Parameters

- The energy and angular momentum of the orbit evolve as

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4\mu^2 m^3}{c^5 a^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$
$$\frac{dL}{dt} = -\frac{32}{5}\frac{G^{7/2}\mu^2 m^{5/2}}{c^5 a^{7/2}}\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$

which can be written as evolution equations for a and e

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^3\mu m^2}{c^5 a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$
$$\frac{de}{dt} = -\frac{304}{15}\frac{G^3\mu m^2}{c^5 a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

Notice that for $e > 0 \Rightarrow de/dt < 0$ (elliptic orbits circularize due to emission of GWs) and that for $e = 0 \Rightarrow de/dt = 0$ (circular orbits remain circular).

Evolution of Orbital Parameters

- Numerically it is challenging to compute $a(t)$ and $e(t)$ over large timescales, but $a(e)$ can be determined analytically, by solving the equation

$$\frac{da}{de} = \frac{12}{19} a \frac{1 + (73/24)e^2 + (37/96)e^4}{e\,(1-e^2)\,[1 + (121/304)e^2]}$$

  We find

$$\boxed{a(e) = c_0 \frac{e^{12/19}}{1-e^2}\left(1 + \frac{121}{304}e^2\right)^{870/2299}}$$

  where c_0 is determined by the initial condition $a = a_0$ when $e = e_0$.

- The Hulse-Taylor binary pulsar has $a_0 = 2 \times 10^9$m and $e = 0.617$ today. By the time the separation becomes $a \simeq 1000$km (~ 100 neutron star radii) the eccentricity will have become $e \simeq 6 \times 10^{-6}$, practically circular.

Evolution of Orbital Parameters

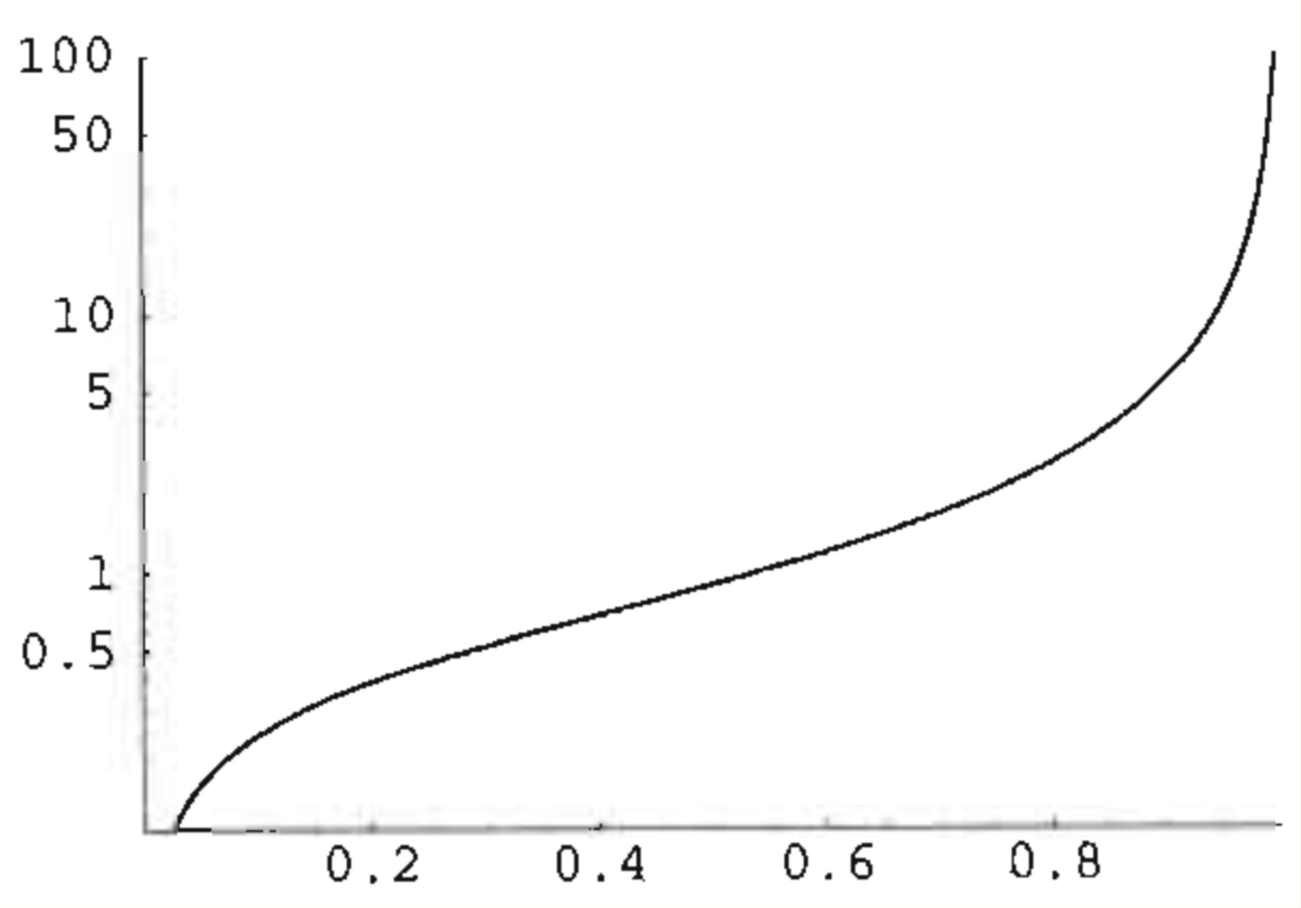

Figure 7: The scaled semi-major axis $a(e)/c_0$ as a function of e. Figure from [1].

Time to Coalescence

- The time to coalescence for an elliptical orbit with initial a_0 and e_0 is

$$\boxed{\tau_0\left(a_0, e_0\right) \simeq 9.83 \times 10^6 \mathrm{yr} \left(\frac{T_0}{1\mathrm{hr}}\right)^{8/3} \left(\frac{M_\odot}{m}\right)^{2/3} \left(\frac{M_\odot}{\mu}\right) F\left(e_0\right)}$$

where

$$F\left(e_0\right) = \frac{48}{19} \frac{1}{g^4\left(e_0\right)} \int_0^{e_0} de \frac{g^4(e)\left(1-e^2\right)^{5/2}}{e\left(1+\frac{121}{304}e^2\right)}$$

where

$$g(e) = \frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{870/2299}$$

- For the Hulse-Taylor binary pulsar, $T_0 = 7.75\,$h, $e_0 = 0.617$ and $m_1 = m_2 \simeq 1.4 M_\odot$ and we find a time to coalescence of $\simeq 300\,$Myr.

References

[1] M. Maggiore. *Gravitational Waves: Volume 1: Theory and Experiments*. Oxford University Press, 2008.