

Gravitational Waves

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Generation of GWs

- Linearized field equations

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Solution

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}$$

where

$$G(x - x') = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \delta(x_{\text{ret}}^0 - x'^0)$$

is a Green's function, satisfying

$$\square_x G(x - x') = \delta^4(x - x')$$

and

$$t_{\text{ret}} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}$$

Generation of GWs

- The solution becomes

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right)$$

- Define the *spatial projector* normal to a direction $\hat{\mathbf{n}}$

$$P_{ij} := \delta_{ij} - n_i n_j$$

then

$$\begin{aligned} \Lambda_{ij,kl}(\hat{\mathbf{n}}) &= P_{ik}P_{jl} - \frac{1}{2}P_{ij} \\ &= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ &\quad + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l P_{kl} \end{aligned}$$

Transverse Traceless Gauge and Far-Field Approximation

- If $h_{\mu\nu}$ is in Lorentz gauge (in vacuum), then it is brought to the *TT gauge* via the projection

$$h_{ij}^{\text{TT}} = \Lambda_{ij,kl} h_{kl}$$

and the solution in vacuum is then

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right)$$

- Far from the source, we can expand (where d is the source size)

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + \mathcal{O}\left(\frac{d^2}{r}\right)$$

and obtain the *far-field approximation*

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}' \right)$$

Non-relativistic Sources

- Let us Fourier transform the stress-energy tensor:

$$T_{kl} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right) = \int \frac{d^4 k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega(t-r/c + \mathbf{x}' \cdot \hat{\mathbf{n}}) + i\mathbf{k} \cdot \mathbf{x}'}$$

If the source has a maximum frequency ω_s and is *non-relativistic* ($\omega_s d \ll c$) and because $|\mathbf{x}'| \lesssim d$, only frequencies for which

$$\frac{\omega}{c} \mathbf{x}' \cdot \hat{\mathbf{n}} \lesssim \frac{\omega_s d}{c} \ll 1$$

contribute. Then, expanding in terms of $\omega \mathbf{x}' \cdot \hat{\mathbf{n}}/c$

$$e^{-i\omega(t-r/c + \mathbf{x}' \cdot \hat{\mathbf{n}}/c) + i\mathbf{k} \cdot \mathbf{x}'} = e^{-i\omega(t-r/c)} \left[1 - i \frac{\omega}{c} x'^i n^i + \frac{1}{2} \left(-i \frac{\omega}{c} \right)^2 x^i x'^j n^i n^j + \dots \right]$$

or, in the time domain:

$$T_{kl} \left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}' \right) = T_{kl} \left(t - r/c, \mathbf{x}' \right) + \frac{x'^i n^i}{c} \partial_0 T_{kl} + \frac{1}{2c^2} x^i x'^j n^i n^j \partial_0^2 T_{kl} + \dots$$

Multipole Moments of the Stress-Energy Tensor

- The multipole moments of $T_{\mu\nu}$ are

$$\begin{aligned} S^{ij} &= \int d^3x T^{ij}(t, \mathbf{x}) \\ S^{ij,k} &= \int d^3x T^{ij}(t, \mathbf{x}) x^k \\ S^{ij,kl} &= \int d^3x T^{ij}(t, \mathbf{x}) x^k x^l \\ &\dots \end{aligned}$$

and the solution becomes

$$h_{ij}^{\text{TT}} = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \dots \right]_{\text{ret}}$$

Mass Density and Momentum Density Multipole Moments

- In terms of the mass density $(1/c^2) T^{00}$ one can define the moments

$$M = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x})$$

$$M^i = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i$$

$$M^{ij} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j$$

$$M^{ijk} = \frac{1}{c^2} \int d^3x T^{00}(t, \mathbf{x}) x^i x^j x^k, \quad \dots$$

and in terms of the momentum density $(1/c) T^{0i}$

$$P^i = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x})$$

$$P^{i,j} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j$$

$$P^{i,jk} = \frac{1}{c} \int d^3x T^{0i}(t, \mathbf{x}) x^j x^k, \quad \dots$$

Mass Quadrupole Radiation

- The quadrupole moment of $T_{\mu\nu}$ is written in terms of the mass-density quadrupole moment as

$$S^{ij} = \frac{1}{2} \ddot{M}^{ij}$$

and the solution becomes to leading order in v/c

$$[h_{ij}^{\text{TT}}(t, \mathbf{x})]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^2} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \ddot{M}^{kl}(t - r/c)$$

Define the *reduced (trace-free) quadrupole moment tensor*

$$Q^{ij} := M^{ij} - \frac{1}{3} \delta^{ij} M_{kk} \tag{1}$$

$$\simeq \int d^3x \rho(t, \mathbf{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right) \tag{2}$$

(to leading order in v/c it becomes the Newtonian expression) and

$$Q_{ij}^{\text{TT}} = \Lambda_{ij,kl}(\mathbf{n}) Q_{kl}$$

Quadrupole Approximation

- The *quadrupole formula* for GW radiation is

$$\boxed{[h_{ij}^{\text{TT}}(t, \mathbf{x})]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^4} \ddot{Q}_{ij}^{\text{TT}}(t - r/c)}$$

Notice that $\ddot{Q}_{ij}^{\text{TT}} = \Lambda_{ij,kl} \ddot{Q}_{ij} = \Lambda_{ij,kl} \ddot{M}_{ij}$ (the latter is preferred in calculations)

- EXAMPLE:** Emission along $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Then $P_{ij} = \delta_{ij} - n_i n_j$ becomes

$$P_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For any 3×3 matrix A_{ij}

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \left[P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right] A_{kl} \\ &= (PAP)_{ij} - \frac{1}{2} P_{ij} \text{Tr}(PA) \end{aligned}$$

Quadrupole Approximation

and

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while $\text{Tr}(PA) = A_{11} + A_{22}$. Then:

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} - \frac{A_{11} + A_{22}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \\ &= \begin{pmatrix} (A_{11} - A_{22})/2 & A_{12} & 0 \\ A_{21} & -(A_{11} - A_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \end{aligned}$$

Thus

$$\Lambda_{ij,kl} \ddot{M}_{kl} = \begin{pmatrix} (\ddot{M}_{11} - \ddot{M}_{22})/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -(\ddot{M}_{11} - \ddot{M}_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

Quadrupole Approximation

- Comparing to

$$h_{ij}^{\text{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

we immediately find

$$\begin{aligned} h_+ &= \frac{1}{r} \frac{G}{c^4} \left(\ddot{M}_{11} - \ddot{M}_{22} \right) \\ h_\times &= \frac{2}{r} \frac{G}{c^4} \ddot{M}_{12} \end{aligned}$$

(the r.h.s. is computed in the retarded time $t - r$).

Emission Along Arbitrary Direction

- Along an arbitrary direction \hat{n} , with components in a Cartesian system

$$n_i = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta)$$

the two polarizations are:

$$\begin{aligned} h_+(t; \theta, \phi) = \frac{1}{r} \frac{G}{c^4} \bigg[& \ddot{M}_{11} (\cos^2 \phi - \sin^2 \phi \cos^2 \theta) + \ddot{M}_{22} (\sin^2 \phi - \cos^2 \phi \cos^2 \theta) \\ & - \ddot{M}_{33} \sin^2 \theta - \ddot{M}_{12} \sin 2\phi (1 + \cos^2 \theta) \\ & + \ddot{M}_{13} \sin \phi \sin 2\theta + \ddot{M}_{23} \cos \phi \sin 2\theta \bigg] \end{aligned}$$

and

$$\begin{aligned} h_\times(t; \theta, \phi) = \frac{1}{r} \frac{G}{c^4} \bigg[& (\ddot{M}_{11} - \ddot{M}_{22}) \sin 2\phi \cos \theta + 2\ddot{M}_{12} \cos 2\phi \cos \theta \\ & - 2\ddot{M}_{13} \cos \phi \sin \theta + 2\ddot{M}_{23} \sin \phi \sin \theta \bigg] \end{aligned}$$

Emitted Energy and Linear Momentum of GWs

- Energy is emitted by GWs at a rate

$$\frac{dE_{\text{GW}}}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \rangle \quad (3)$$

$$= \frac{c^3 r^2}{16\pi G} \int d\Omega \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \quad (4)$$

$$\simeq \frac{G}{5c^5} \langle \ddot{Q}_{jk} \ddot{Q}^{jk} \rangle \quad (5)$$

$$= \frac{G}{5c^5} \left\langle \ddot{M}_{ij} \ddot{M}_{ij} - \frac{1}{3} \left(\ddot{M}_{kk} \right)^2 \right\rangle \quad (6)$$

- There is no loss of linear momentum in the quadrupole approximation

$$\frac{\partial P_{\text{GW}}^k}{dt} = -\frac{G}{8\pi c^5} \int d\Omega \ddot{Q}_{ij}^{\text{TT}} \partial^k \ddot{Q}_{ij}^{\text{TTT}} = 0 \quad (7)$$

because Q_{ij} is invariant and $\partial^i \rightarrow -\partial^i$ under a reflection $\mathbf{x} \rightarrow -\mathbf{x}$.

Angular Momentum Emitted by GWs

- The angular momentum carried away by GWs is

$$\frac{dJ^i}{dt} = \frac{c^3}{32\pi G} \int r^2 d\Omega \left\langle -\epsilon^{ikl} \dot{h}_{ab}^{\text{TT}} x^k \partial^\ell h_{ab}^{\text{TT}} + 2\epsilon^{ikl} \dot{h}_{al}^{\text{TT}} h_{ak}^{\text{TT}} \right\rangle$$

In the quadrupole approximation, this becomes

$$\left(\frac{dJ^i}{dt} \right)_{\text{quad}} = \frac{2G}{5c^5} \epsilon^{ikl} \left\langle \ddot{Q}_{ka} \ddot{Q}_{la} \right\rangle$$

GWs from a Binary System

- Consider a binary with circular orbits. The trajectories of the two stars are $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ and the relative coordinate is $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$. The center of mass is

$$\mathbf{x}_{\text{CM}} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}$$

For a nonrelativistic system, the mass quadrupole moment is

$$\begin{aligned} M^{ij} &= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j \\ &= m x_{\text{CM}}^i x_{\text{CM}}^j + \mu \left(x_{\text{CM}}^i x_0^j + x_{\text{CM}}^j x_0^i \right) + \mu x_0^i x_0^j \end{aligned}$$

where $m = m_1 + m_2$ and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass. If we choose $\mathbf{x}_{\text{CM}} = 0$ as the origin of our coordinate system, then the mass quadrupole moment is

$$M^{ij}(t) = \mu x_0^i(t) x_0^j(t)$$

GWs from a Binary System

- In the CM frame, the dynamics reduces to a one-body problem with reduced mass μ .
- Choose a circular orbit with angular frequency ω_s in the plane with $z_0 = 0$

$$\begin{aligned}x_0(t) &= R \cos \left(\omega_s t + \frac{\pi}{2} \right) \\y_0(t) &= R \sin \left(\omega_s t + \frac{\pi}{2} \right) \\z_0(t) &= 0\end{aligned}$$

Then

$$M_{11} = \mu R^2 \frac{1 - \cos 2\omega_s t}{2} \quad (8)$$

$$M_{22} = \mu R^2 \frac{1 + \cos 2\omega_s t}{2} \quad (9)$$

$$M_{12} = -\frac{1}{2} \mu R^2 \sin 2\omega_s t \quad (10)$$

(other components are zero).

GWs from a Binary System

- Taking two time-derivatives:

$$\begin{aligned}\ddot{M}_{11} &= -\ddot{M}_{22} = 2\mu R^2 \omega_s^2 \cos 2\omega_s t \\ \ddot{M}_{12} &= 2\mu R^2 \omega_s^2 \sin 2\omega_s t\end{aligned}$$

and

$$\begin{aligned}h_+(t; \theta, \phi) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left(\frac{1+\cos^2 \theta}{2} \right) \cos(2\omega_s t_{\text{ret}} + 2\phi) \\ h_\times(t; \theta, \phi) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos \theta \sin(2\omega_s t_{\text{ret}} + 2\phi)\end{aligned}$$

- If we can neglect the proper motion of the source, then the angle ϕ is fixed and by a change of the origin of time one can set it to zero.
- If we view the system from an *inclination* $\iota = \theta$, then

$$\begin{aligned}h_+(t) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left(\frac{1+\cos^2 \iota}{2} \right) \cos(2\omega_s t) \\ h_\times(t) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos \iota \sin(2\omega_s t)\end{aligned}$$

- For $\iota = 0 \Rightarrow$ circular polarization, for $\iota = 90^\circ \Rightarrow$ linear polarization, otherwise elliptic polarization. Measuring polarization, recovers ι .

- The two polarizations can be written as

$$h_+(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{2/3} \frac{1+\cos^2 \theta}{2} \cos(2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi)$$

$$h_\times(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{2/3} \cos \theta \sin(2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi)$$

where $\omega_{\text{gw}} = 2\omega_s$ and

$$f_{\text{gw}} = \omega_{\text{gw}} / (2\pi)$$

is the frequency of the GWs and

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

is the *chirp mass*.

- Kepler's law is

$$\omega_s^2 = \frac{Gm}{R^3}$$

Radiated Power

- The angular distribution of the radiated power is

$$\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$

or

$$\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{2}{\pi} \frac{c^5}{G} \left(\frac{GM_c \omega_{\text{gw}}}{2c^3}\right)^{10/3} g(\theta)$$

where

$$g(\theta) = \left(\frac{1 + \cos^2 \theta}{2}\right)^2 + \cos^2 \theta$$

which has an angular average of

$$\int \frac{d\Omega}{4\pi} g(\theta) = \frac{4}{5}$$

Radiate Power

- The radiated power is

$$P_{\text{quad}} = \frac{1}{10} \frac{G\mu^2}{c^5} R^4 \omega_{\text{gw}}^6$$

or

$$P_{\text{quad}} = \frac{32}{5} \frac{c^5}{G} \left(\frac{GM_c \omega_{\text{gw}}}{2c^3} \right)^{10/3}$$

- The energy radiated in one period $T = 2\pi/\omega_s$ is (with $v = \omega_s R$)

$$\langle E_{\text{quad}} \rangle_T = \frac{64\pi}{5} \frac{G\mu^2}{R} \left(\frac{v}{c} \right)^5$$

i.e. the energy scale $G\mu^2/R$ is suppressed by a factor $(v/c)^5$.

Frequency evolution

- The orbital energy is

$$\begin{aligned} E_{\text{orbit}} &= E_{\text{kin}} + E_{\text{pot}} \\ &= -\frac{Gm_1m_2}{2R} \\ &= -\left(G^2M_c^5\omega_{\text{gw}}^2/32\right)^{1/3} \end{aligned}$$

- Assume that

$$\left|\frac{dE_{\text{orbit}}}{dt}\right| = P_{\text{quad}}$$

Then

$$\dot{f}_{\text{gw}} = \frac{96}{5}\pi^{8/3} \left(\frac{GM_c}{c^3}\right)^{5/3} f_{\text{gw}}^{11/3}$$

- Integrating \dot{f}_{gw} , we see that it diverges at a finite time t_{coal} . The remaining time to coalescence is then

$$\tau = t_{\text{coal}} - t$$

and the frequency evolution is written as

$$f_{\text{gw}}(\tau) = \frac{1}{\pi} \left(\frac{5}{256} \frac{1}{\tau} \right)^{3/8} \left(\frac{GM_c}{c^3} \right)^{-5/8}$$

or

$$f_{\text{gw}}(\tau) \simeq 134\text{Hz} \left(\frac{1.21M_{\odot}}{M_c} \right)^{5/8} \left(\frac{1\text{s}}{\tau} \right)^{3/8}$$

- The time to coalescence is thus

$$\tau \simeq 2.18\text{s} \left(\frac{1.21M_{\odot}}{M_c} \right)^{5/3} \left(\frac{100\text{Hz}}{f_{\text{gw}}} \right)^{8/3}$$

Number of cycles

- When the period $T(t)$ is slowly varying, the number of cycles in a time interval dt is

$$d\mathcal{N}_{\text{cyc}} = \frac{dt}{T(t)} = f_{\text{gw}}(t)dt$$

and thus the number of cycles spent between frequencies f_{min} and f_{max} is

$$\begin{aligned}\mathcal{N}_{\text{cyc}} &= \int_{t_{\text{min}}}^{t_{\text{max}}} f_{\text{gw}}(t)dt \\ &= \int_{f_{\text{min}}}^{f_{\text{max}}} df_{\text{gw}} \frac{f_{\text{gw}}}{\dot{f}_{\text{gw}}}\end{aligned}$$

or

$$\mathcal{N}_{\text{cyc}} = \frac{1}{32\pi^{8/3}} \left(\frac{GM_c}{c^3} \right)^{-5/3} \left(f_{\text{min}}^{-5/3} - f_{\text{max}}^{-5/3} \right)$$

If $f_{\text{min}}^{-5/3} - f_{\text{max}}^{-5/3} \simeq f_{\text{min}}^{-5/3}$, then

$$\mathcal{N}_{\text{cyc}} \simeq 1.6 \times 10^4 \left(\frac{10\text{Hz}}{f_{\text{min}}} \right)^{5/3} \left(\frac{1.2M_{\odot}}{M_c} \right)^{5/3}$$

Orbital Evolution

- From Kepler's law and the equation for \dot{f}_{gw} we find that the radius of the orbit shrinks according to

$$\frac{\dot{R}}{R} = -\frac{2}{3} \frac{\dot{f}_{\text{gw}}}{f_{\text{gw}}} = -\frac{1}{4\tau}$$

If at $t = t_0$ the radius is $R = R_0$ and $\tau_0 = t_{\text{coal}} - t_0$, then integrating:

$$R(\tau) = R_0 \left(\frac{\tau}{\tau_0} \right)^{1/4}$$

- From Kepler's law and the equation for \dot{f}_{gw} we find

$$\tau_0 = \frac{5}{256} \frac{c^5 R_0^4}{G^3 m^2 \mu}$$

or

$$\tau_0 \simeq 9.829 \times 10^6 \text{yr} \left(\frac{T_0}{1\text{hr}} \right)^{8/3} \left(\frac{M_\odot}{m} \right)^{2/3} \left(\frac{M_\odot}{\mu} \right)$$

- Because $\omega_{\text{gw}} = d\Phi/dt$, the evolution of the phase is

$$\Phi(t) = \int_{t_0}^t dt' \omega_{\text{gw}}(t')$$

or, with $\Phi_0 = \Phi(\tau = 0)$

$$\Phi(\tau) = -2 \left(\frac{5GM_c}{c^3} \right)^{-5/8} \tau^{5/8} + \Phi_0$$

- The waveform is

$$h_+(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}(t_{\text{ret}})}{c} \right)^{2/3} \left(\frac{1+\cos^2 \iota}{2} \right) \cos [\Phi(t_{\text{ret}})]$$

$$h_x(t) = \frac{4}{r} \left(\frac{GM_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\text{gw}}(t_{\text{ret}})}{c} \right)^{2/3} \cos \iota \sin [\Phi(t_{\text{ret}})]$$

or

$$h_+(\tau) = \frac{1}{r} \left(\frac{GM_c}{c^2} \right)^{5/4} \left(\frac{5}{c\tau} \right)^{1/4} \left(\frac{1+\cos^2 \iota}{2} \right) \cos[\Phi(\tau)]$$

$$h_\times(\tau) = \frac{1}{r} \left(\frac{GM_c}{c^2} \right)^{5/4} \left(\frac{5}{c\tau} \right)^{1/4} \cos \iota \sin[\Phi(\tau)]$$