NEUTRON STAR OSCILLATIONS

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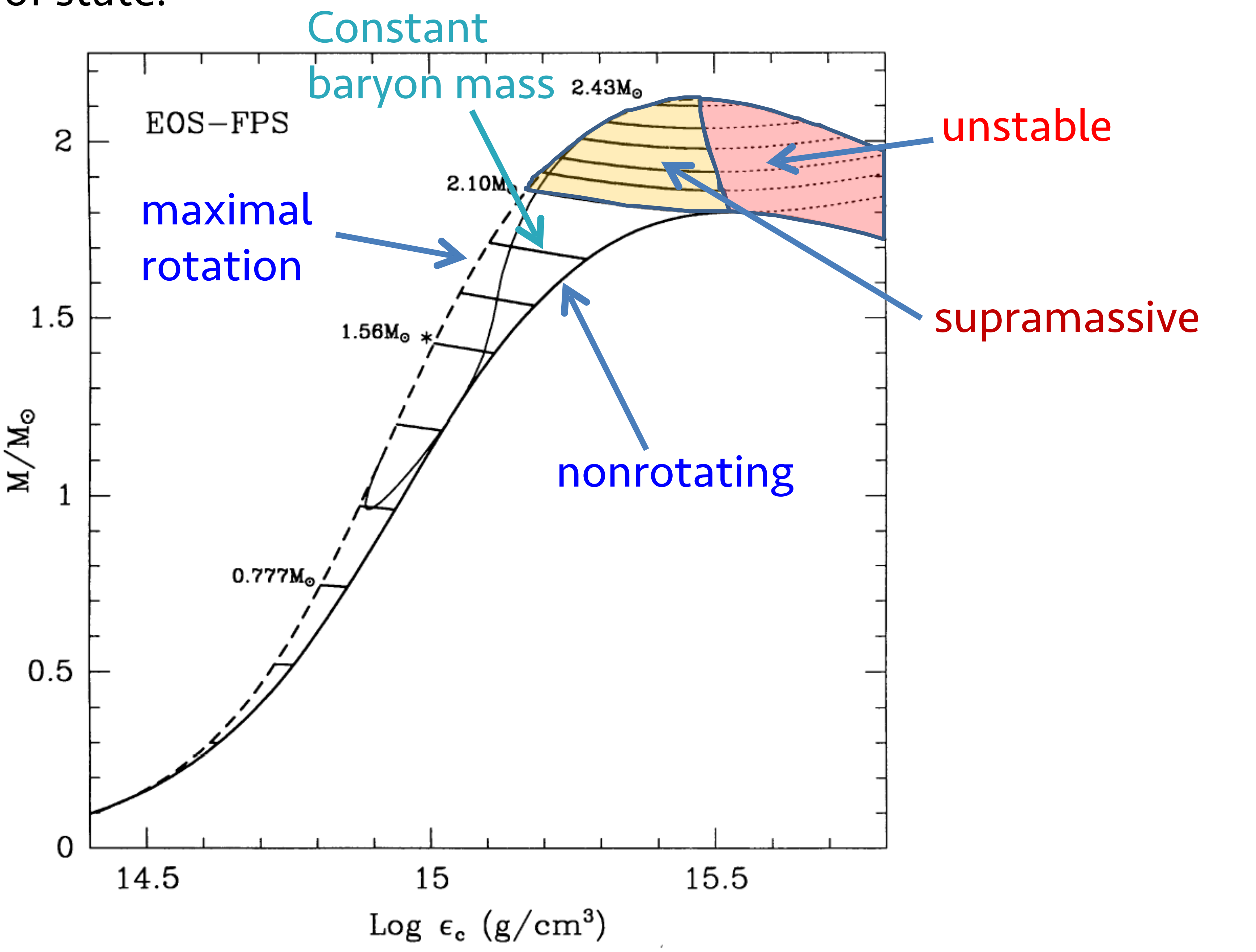
Equilibria of Rotating Stars

Uniformly rotating equilibrium models for a realistic neutron star equation of state.



Axisymmetric Instability to Collapse

Rotating stars are subject to a ***secular*** axisymmetric instability, if:

$$\left(\frac{\partial M}{\partial \epsilon_c}\right)_J < 0$$

(Friedman, Ipser & Sorkin, 1988).

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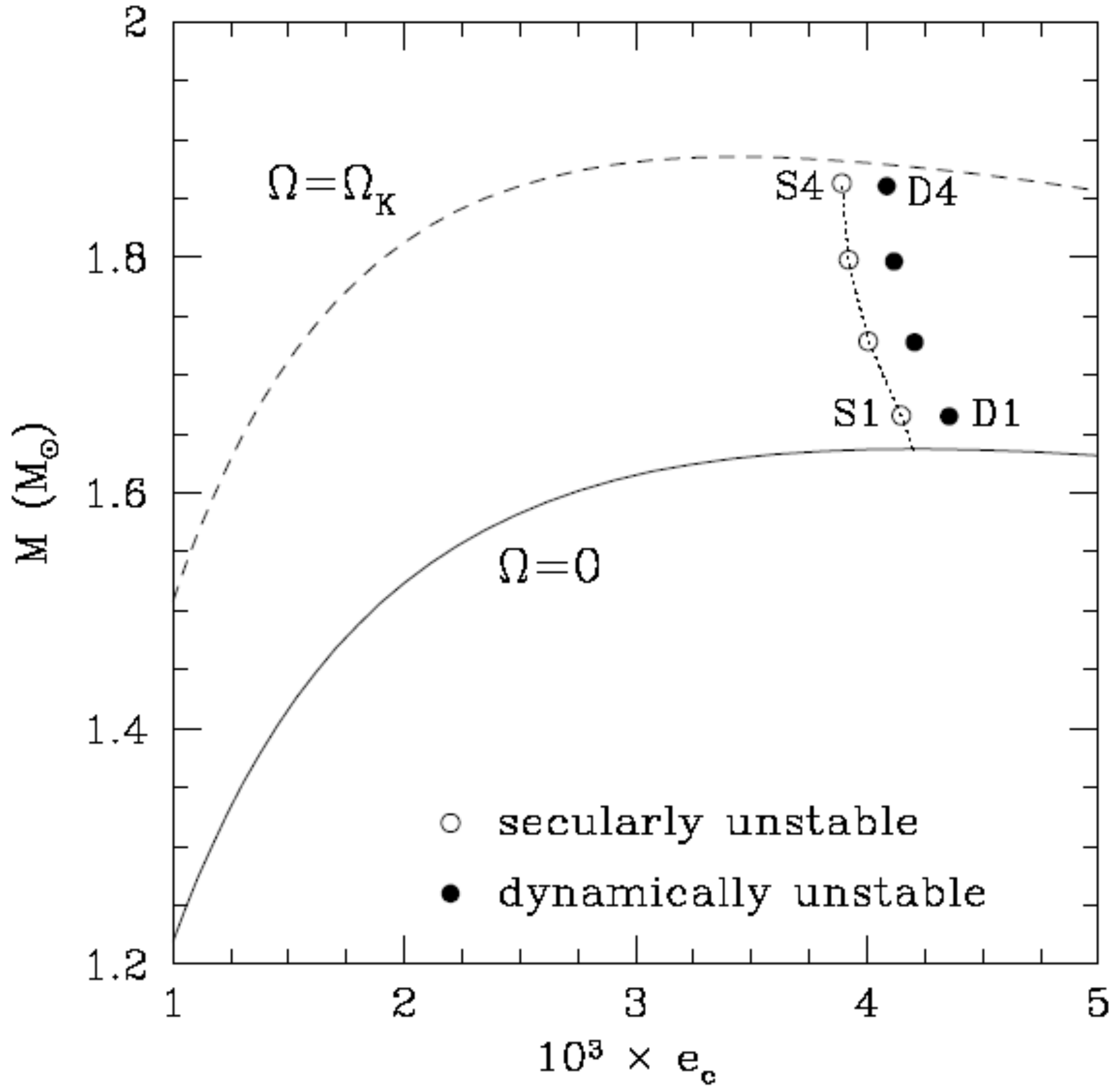
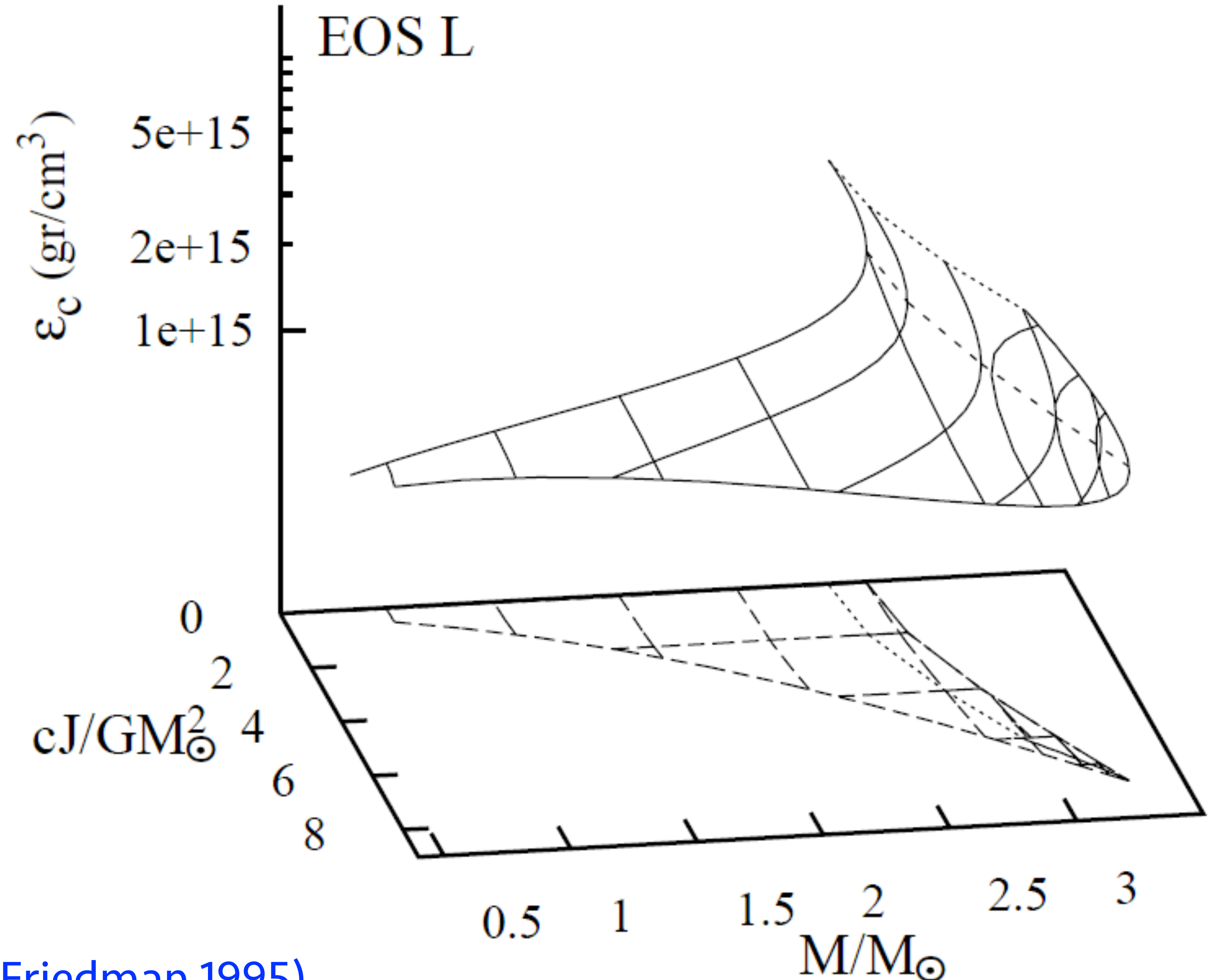


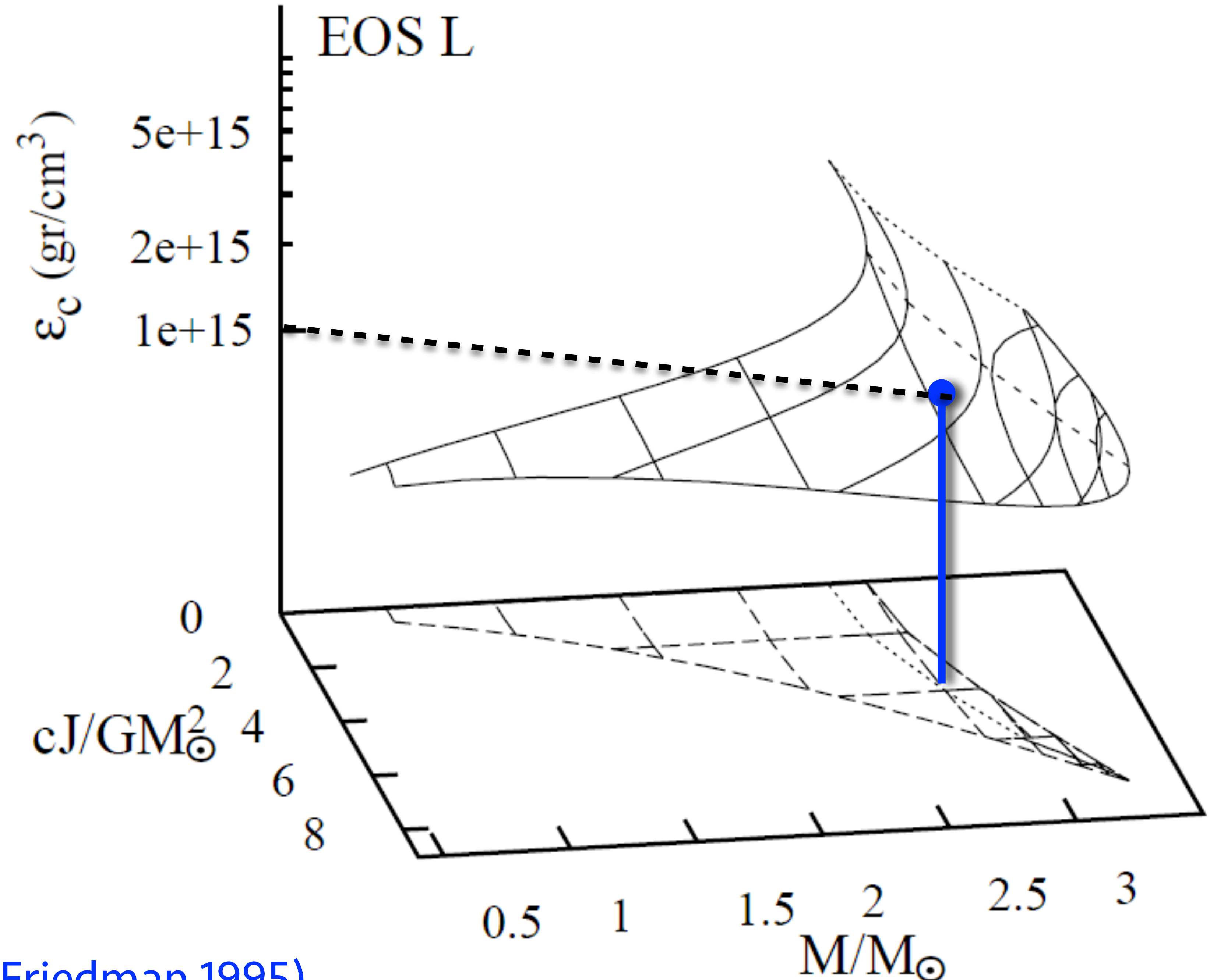
Dynamical instability soon after onset of secular instability.

Equilibrium Sequences



(NS & Friedman 1995)

Equilibrium Sequences



(NS & Friedman 1995)

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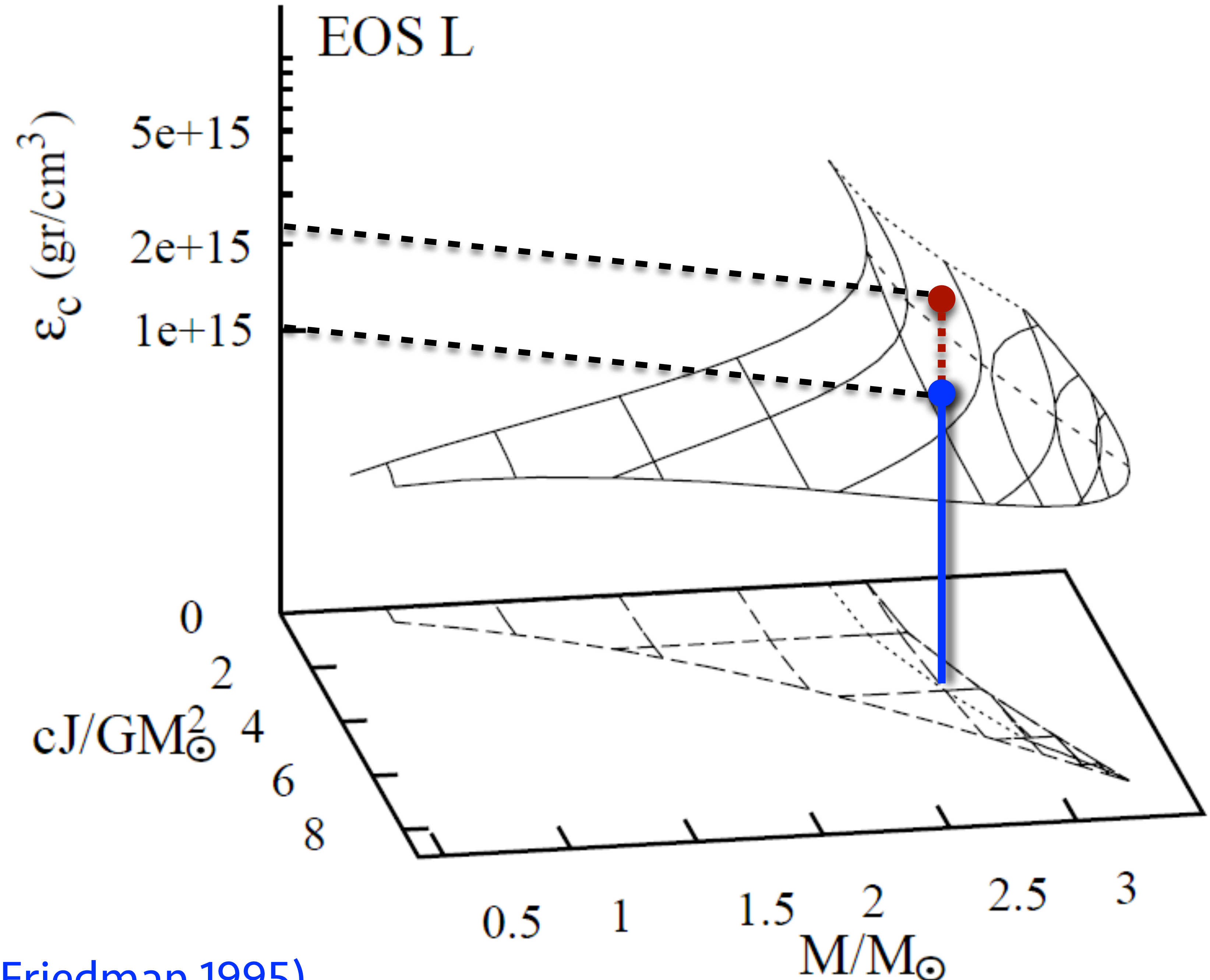
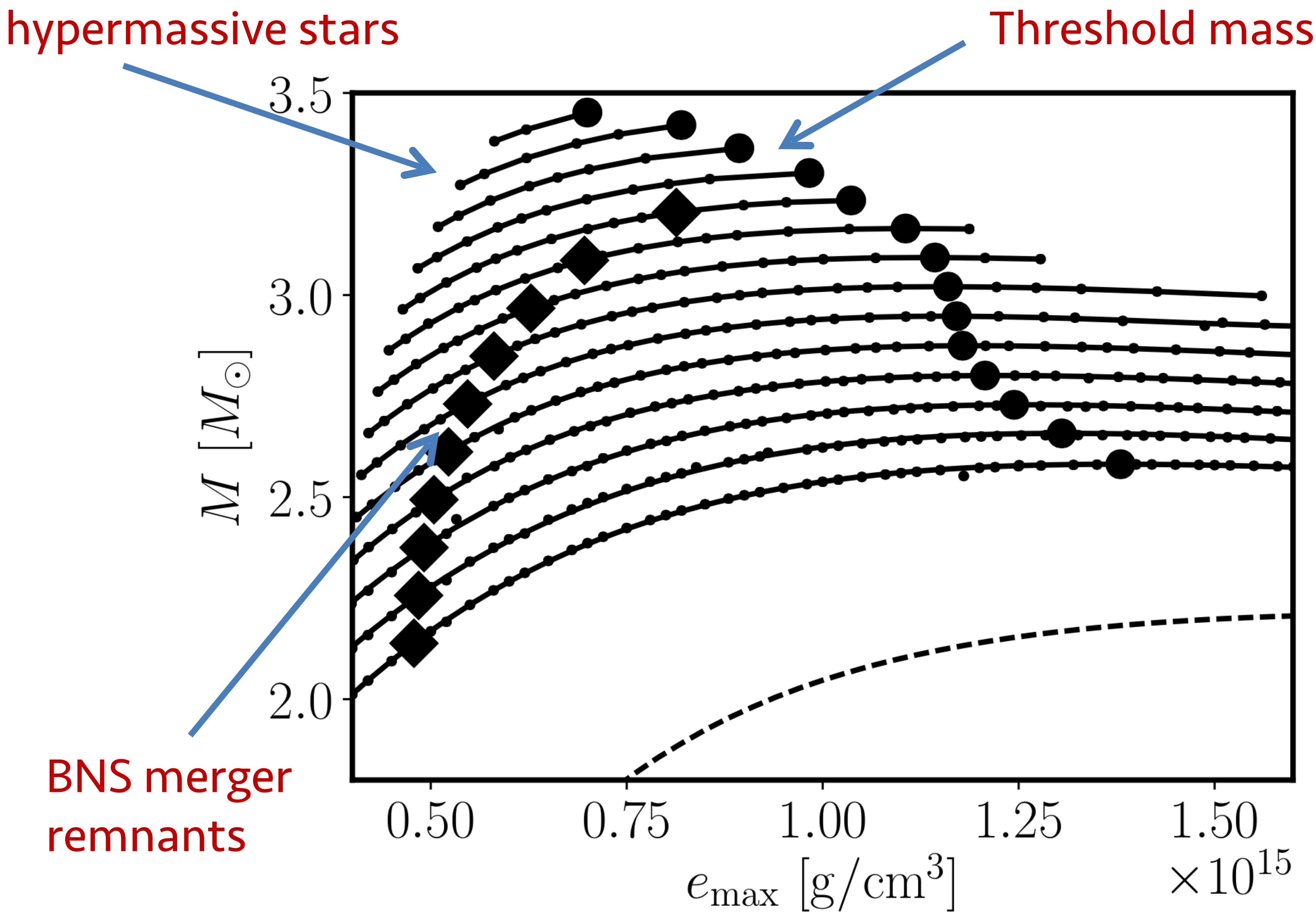


(NS & Friedman 1995)

Differentially Rotating Models

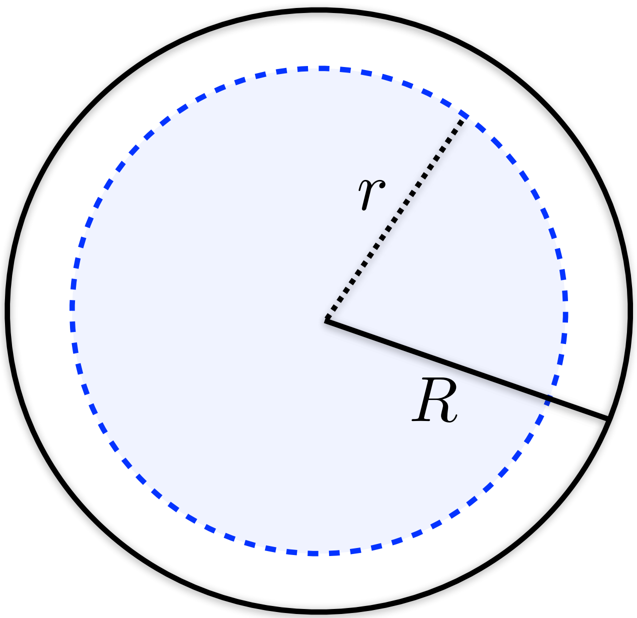


(Bauswein & NS 2017)

Radial Oscillations

Consider first a nonrotating Newtonian star, with radius *R*. The mass contained within an internal radius *r* is $\mathcal{M}_r$.

The conservation of mass (continuity) and momentum are



$$\frac{\partial \mathcal{M}_r}{\partial r} = 4\pi r^2 \rho$$

$$\ddot{r} = -4\pi r^2 \left(\frac{\partial P}{\partial \mathcal{M}_r}\right) - \frac{G\mathcal{M}_r}{r^2}$$

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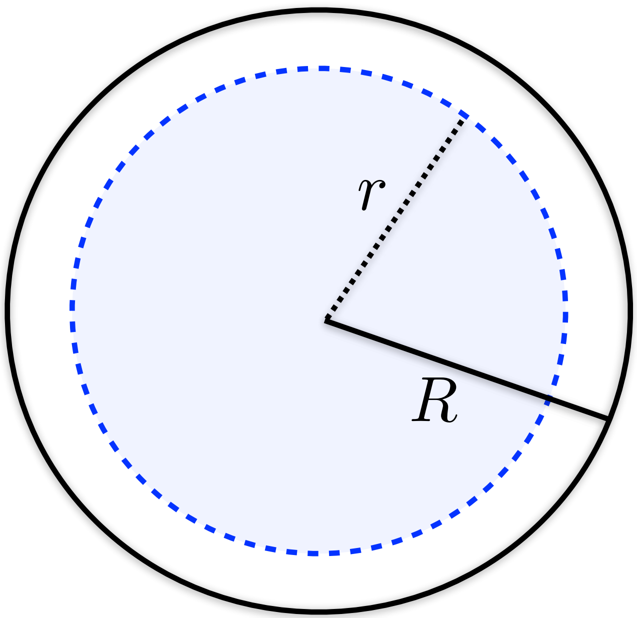

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Next, consider small *Lagrangian perturbations*

$$|\delta r/r_0| \ll 1 \text{ and } |\delta\rho/\rho_0| \ll 1$$

Then

$$\frac{\delta\rho}{\rho_0} = -3\,\frac{\delta r}{r_0} - r_0\,\frac{\partial\left(\delta r/r_0\right)}{\partial r_0}$$

$$\rho_0 r_0\,\frac{d^2\,\delta r/r_0}{dt^2} = -\left(4\,\frac{\delta r}{r_0} + \frac{\delta P}{P_0}\right)\frac{\partial P_0}{\partial r_0} - P_0\,\frac{\partial\left(\delta P/P_0\right)}{\partial r_0}$$

Radial Oscillations

If we assume a *harmonic time dependence*

$$\frac{\delta r(t, r_0)}{r_0} = \frac{\delta r(r_0)}{r_0}\, e^{i\sigma t} = \zeta(r_0)\, e^{i\sigma t}$$

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Then

$$\begin{aligned} \frac{d\zeta}{dr} &= -\frac{1}{r}\left(3\zeta + \frac{1}{\Gamma_1}\frac{\delta P}{P}\right) \\ \frac{d\,(\delta P/P)}{dr} &= -\frac{d\ln P}{dr}\left(4\zeta + \frac{\sigma^2 r^3}{G\mathcal{M}_r}\,\zeta + \frac{\delta P}{P}\right) \end{aligned}$$

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$$\frac{d\,(\delta P/P)}{dr} \;=\; -\frac{d\ln P}{dr}\left(4\zeta + \frac{\sigma^2 r^3}{G\mathcal{M}_r}\,\zeta + \frac{\delta P}{P}\right)$$

where $\Gamma_1 = \left(\dfrac{\partial \ln P}{\partial \ln \rho}\right)_{\rm ad}$ is the *adiabatic index* in equilibrium.

We have assumed *adiabatic oscillations*, so that

$$\frac{\delta P}{P_0} = \Gamma_1\,\frac{\delta\rho}{\rho_0}$$

Examples

- Assuming *uniform density*, the fundamental mode has ζ = const. and

$$\sigma_0^2 = \frac{4\pi}{3} G\rho(3\Gamma_1 - 4)$$

which reveals a fundamental scaling between frequency and density

$$\sigma \sim \sqrt{\rho} \sim \sqrt{M/R^3}$$

ξ / ξ_1

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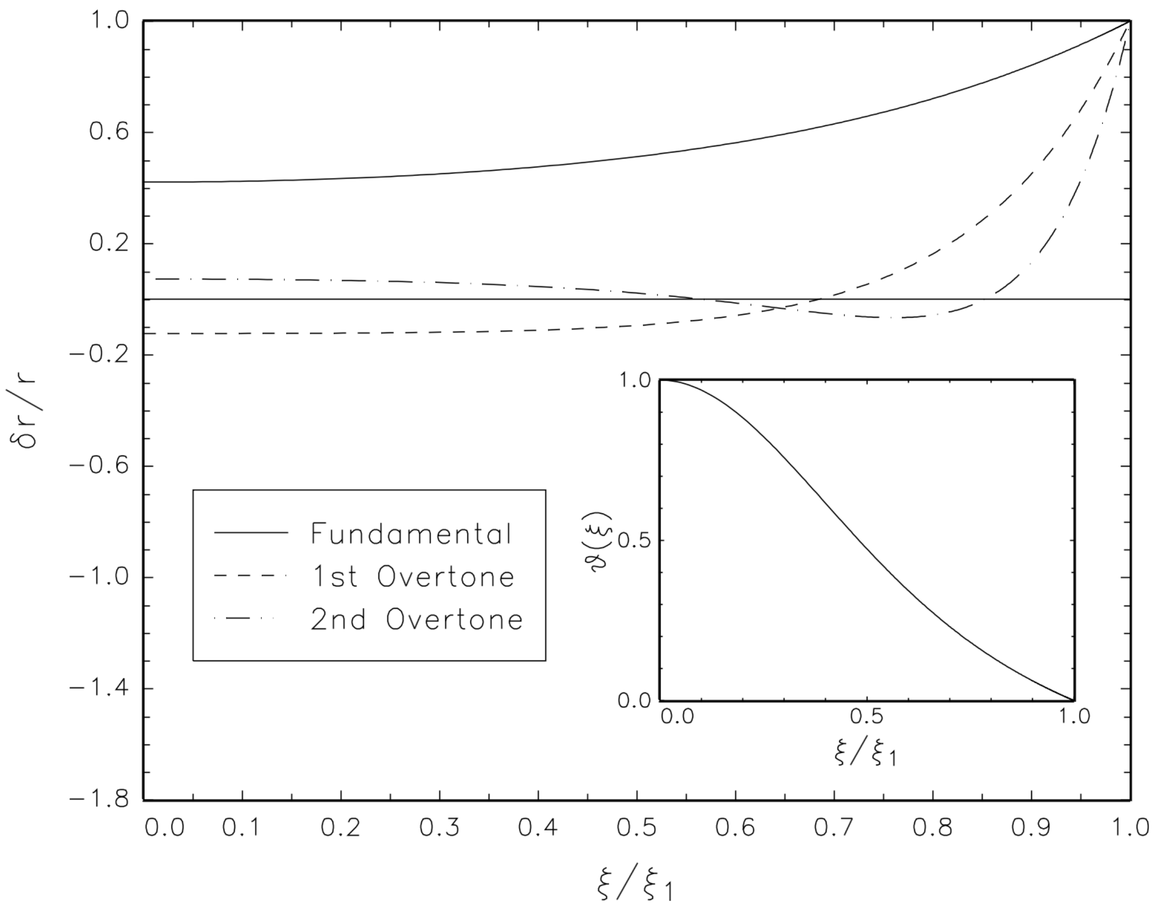
- For n=2 polytropes, the first three eigenfunctions are:



Nonradial Oscillations

For nonradial oscillations, we need to considered the general system of the Poisson equation plus energy and momentum conservation, without symmetries

$$
\begin{aligned}
\nabla^2 \Phi &= 4\pi G \rho \\
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \bullet (\rho \mathbf{v}) &= 0 \\
\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \bullet \boldsymbol{\nabla} \right) \mathbf{v} &= -\boldsymbol{\nabla} P - \rho \, \boldsymbol{\nabla} \Phi
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\end{aligned}
$$

The displacement vector with harmonic time dependence

$$
\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(\mathbf{r})\, e^{i\sigma t}
$$

is then decomposed in terms of spherical harmonics as

$$
\begin{aligned}
\boldsymbol{\xi}(r, \theta, \varphi) &= \xi_r(r, \theta, \varphi)\, \mathbf{e}_{\boldsymbol{r}} + \xi_\theta(r, \theta, \varphi)\, \mathbf{e}_{\boldsymbol{\theta}} + \xi_\varphi(r, \theta, \varphi)\, \mathbf{e}_{\boldsymbol{\varphi}} \\
&= \left[\xi_r(r)\, \mathbf{e}_{\boldsymbol{r}} + \xi_t(r)\, \mathbf{e}_{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} + \xi_t(r)\, \mathbf{e}_{\boldsymbol{\varphi}} \frac{1}{\sin\theta} \frac{\partial}{\partial \varphi} \right] Y_{\ell m}(\theta, \varphi)
\end{aligned}
$$

Nonradial Oscillations

Notice that

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi}$$

The Eulerian perturbation in the density is

$$\rho'(r,\theta,\phi,t) = \rho'(r)Y_{lm}(\theta,\phi)e^{i\sigma t}$$

The perturbed system of equations is of 4th order (because of the perturbation in *Φ*) and admits normal mode solutions that are characterized by the spherical harmonic indices *l*, *m* and the radial order *n*.

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A general perturbation in the density is then

$$\rho'(r,\theta,\phi,t) = \sum_l^\infty \sum_m^\infty \sum_n^\infty a_{lmn}\rho'_n(r)P_l^m(\theta,\phi)e^{i(\sigma_{lmn}t+m\phi)}$$

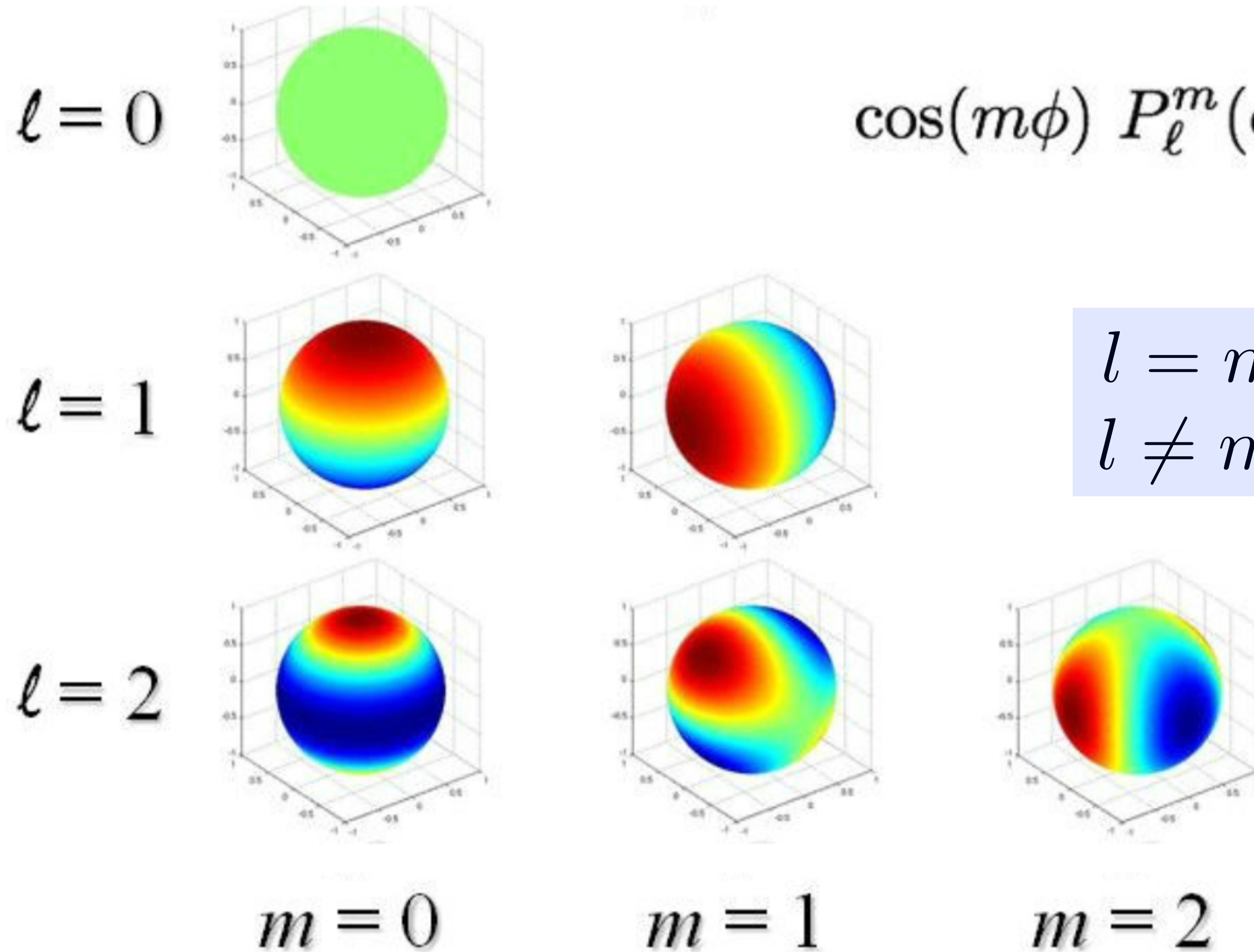
(but only the *m*=0 case is relevant for spherical stars).

Nonradial Oscillations

$$\cos(m\phi)\ P_\ell^m(\cos\theta)$$

$l = m$ (sectoral)

$l \neq m$ (tesseral)



(zonal)

Nonradial Oscillations of Neutron Stars

Main oscillation modes:

1. *f*-modes / *p*-modes

   fluid modes restored by pressure

2. *g*-modes

   restored by gravity/buoyancy in non-isentropic stars

3. inertial modes (*r*-modes)

   restored by the Coriolis force in rotating stars

4. *w*-modes

   spacetime modes (similar to black hole modes)

GW-detection: *f*-, *p*-, *g*-, *r*-modes : stable oscillations

instabilities

Mode Excitation

Various ways to excite modes:

1. Core collapse
2. Binary merger (if HMNS forms)
3. Star quakes (e.g. SGR activity)
4. Secular instabilities (f-modes, r-modes)
5. Dynamical instabilities (bar-mode, $T/W > 0.27$)
6. Low T/W instabilities

Frequency and Pattern Speed

For a nonrotating, uniform density star the *f*-mode frequency is

$$\omega^2 = \frac{2l(l-1)}{2l+1}\frac{GM}{R^3}$$

For *l* = 2, this is ~2 kHz for a typical neutron star.

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Since oscillations are proportional to $e^{i\omega t + m\phi}$, surfaces of constant phase correspond to

$$m\phi + \omega t = \text{const.} \quad \Rightarrow \quad \frac{d\phi}{dt} = -\frac{\omega}{m} := \omega_p$$

which defines the pattern speed, with which a mode revolves around the star (notice that the *m* = 0 modes are standing waves).

Nonradial Oscillations of Relativistic Stars

In GR, nonradial oscillations with $l > 1$ radiate gravitational waves and modes become quasi-normal, acquiring an imaginary part.

$$\omega = \sigma + \frac{i}{\tau_{\mathrm{GW}}}$$

The spacetime metric is perturbed as

$$\begin{aligned} ds^2 = & - e^{2\nu}\left(1 + r^l H_0^{lm} Y_{lm} e^{i\omega t}\right) dt^2 - 2i\omega r^{l+1} H_1^{lm} Y_{lm} e^{i\omega t} dt\, dr \\ & + e^{2\lambda}\left(1 - r^l H_0^{lm} Y_{lm} e^{i\omega t}\right) dr^2 + r^2\left(1 - r^l K^{lm} Y_{lm} e^{i\omega t}\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \end{aligned}$$

and the displacement vector is

$$\begin{aligned} \xi_r &= e^{\lambda} r^{l-1} W^{lm} Y_{lm} e^{i\omega t}, \\ \xi_\theta &= -r^l V^{lm} \partial_\theta Y_{lm} e^{i\omega t}, \\ \xi_\phi &= -r^l V^{lm} \partial_\phi Y_{lm} e^{i\omega t}, \end{aligned}$$

Quasi-normal Modes

The perturbation in the energy density is

$$\delta\varepsilon = r^l \delta\varepsilon^{lm} Y_{lm} e^{i\omega t}$$

A useful redefinition of variables is

$$X^{lm} = \omega^2(\varepsilon + p)e^{-\nu}V^{lm} - \frac{1}{r}\frac{dp}{dr}e^{\nu-\lambda}W^{lm} + \frac{e^{\nu}}{2}(\varepsilon + p)H_0^{lm}$$

One then arrives at a 4th-order system of equations that can be solved for arbitrary complex eigenfrequencies (mathing to BH perturbations in the exterior), which includes both incoming and outgoing gravitational waves.

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A discrete set of *quasi-normal modes* is obtained by requiring *purely outgoing* gravitational waves.

The Perturbed System for Spherical Stars

$$H_1^{lm\prime} = -\frac{1}{r}\left[l+1+\frac{2me^{2\lambda}}{r}+4\pi r^2 e^{2\lambda}(p-\epsilon)\right]H_1^{lm} + \frac{e^{2\lambda}}{r}\left[H_0^{lm}+K^{lm}-16\pi(\epsilon+p)V^{lm}\right],$$

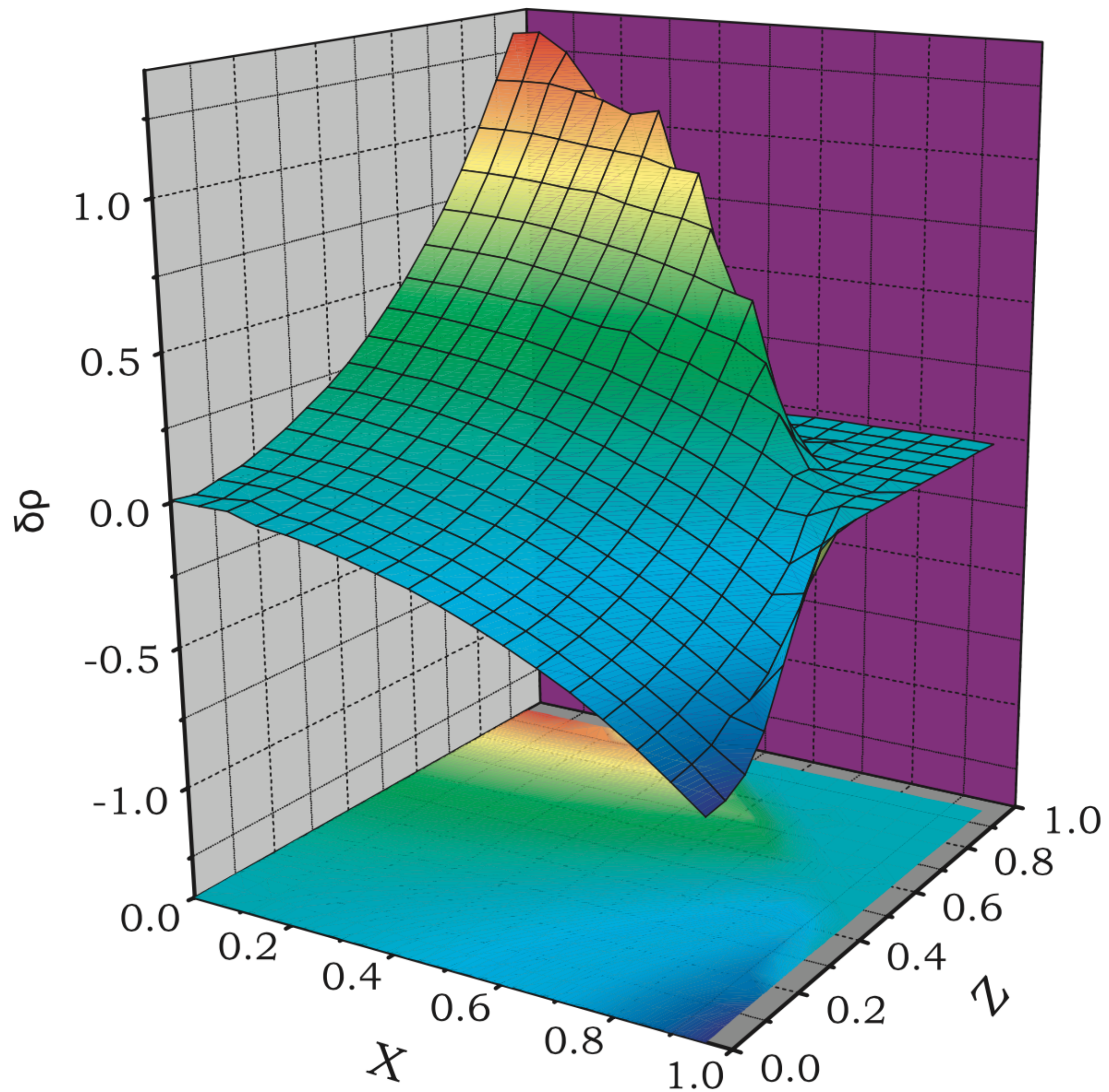
$$K^{lm\prime} = \frac{1}{r}H_0^{lm} + \frac{l(l+1)}{2r}H_1^{lm} - \left[\frac{l+1}{r}-\nu'\right]K^{lm} - 8\pi(\epsilon+p)\frac{e^{\lambda}}{r}W^{lm},$$

$$W^{lm\prime} = -\frac{l+1}{r}W^{lm} + re^{\lambda}\left[\frac{e^{-\nu}}{\upsilon_s^2(\epsilon+p)}X^{lm} - \frac{l(l+1)}{r^2}V^{lm} + \frac{1}{2}H_0^{lm} + K^{lm}\right],$$

$$\begin{aligned} X^{lm\prime} = -\frac{l}{r}X^{lm} + \frac{(\epsilon+p)e^{\nu}}{2}\Bigg\{&\left(\frac{1}{r}-\nu'\right)H_0^{lm} + \left(r\omega^2 e^{-2\nu} + \frac{l(l+1)}{2r}\right)H_1^{lm} \\ &+ \left(3\nu' - \frac{1}{r}\right)K^{lm} - \frac{2l(l+1)}{r^2}\nu' V^{lm} \\ &- \frac{2}{r}\left[4\pi(\epsilon+p)e^{\lambda} + \omega^2 e^{\lambda-2\nu} - \frac{r^2}{2}\left(\frac{2e^{-\lambda}}{r^2}\nu'\right)'\right]W^{lm}\Bigg\},\end{aligned}$$

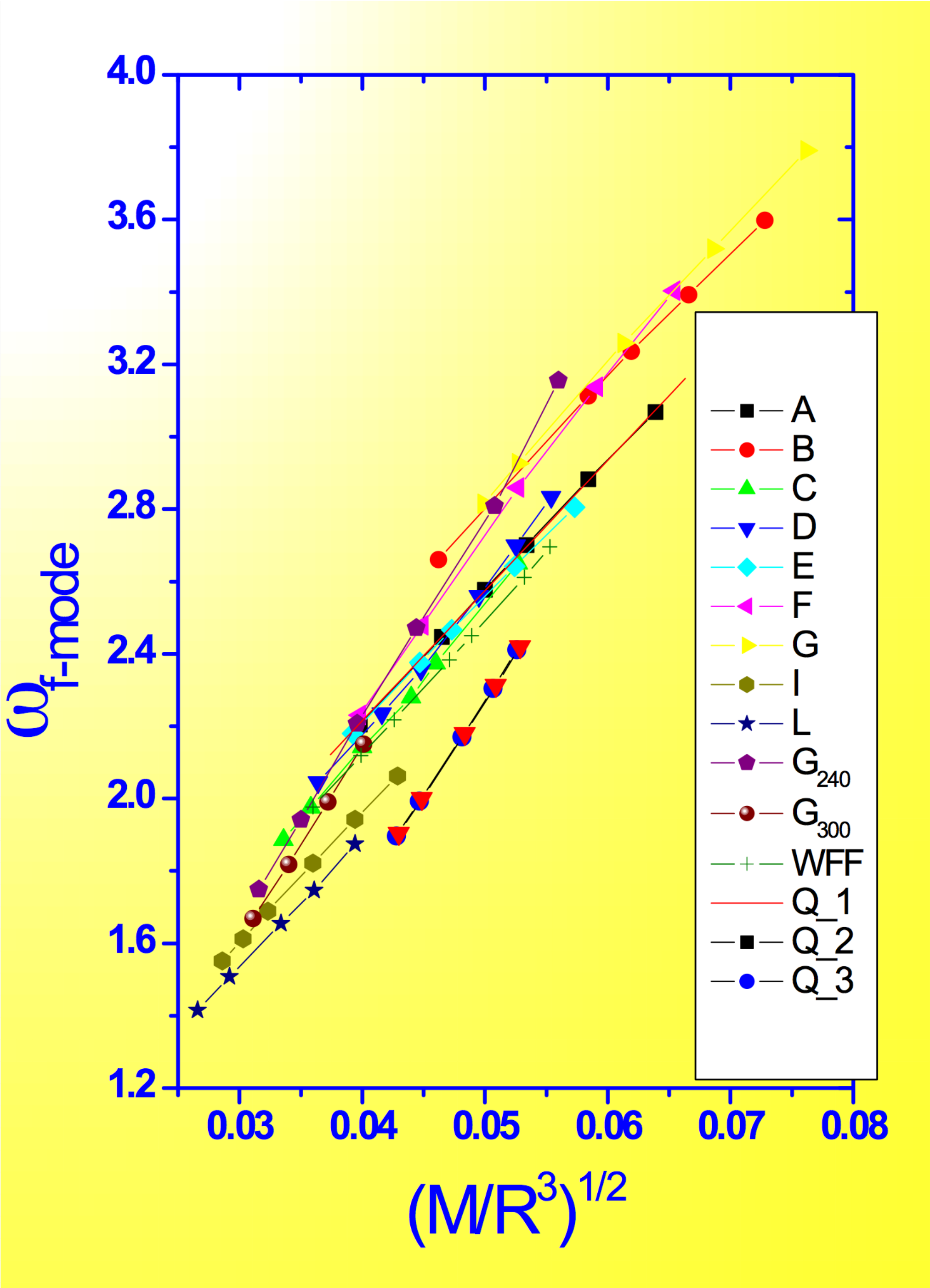
Eigenfunction of Quadrupole Oscillation

Typical example of the eigenfunction $\delta\rho(r, \theta)$ for the l = 2 mode:



NS, Apostolatos & Font (2004)

Quadrupole Frequencies for Nonrotating Stars



Empirical relations for GW asteroseismology:

$$\omega_f\,(kHz) \approx 0.78 + 1.637\left(\frac{M_{1.4}}{R_{10}{}^3}\right)^{1/2}$$

Andersson & Kokkotas (1998)

GW emission

The luminosity in gravitational waves of the quadrupole mode is:

$$\left\langle \frac{dE}{dt} \right\rangle_{\mathrm{GW}} = -\frac{4\pi}{75}\sigma^6 \left(\int_0^R r^4 \delta\rho(r) dr \right)^2$$

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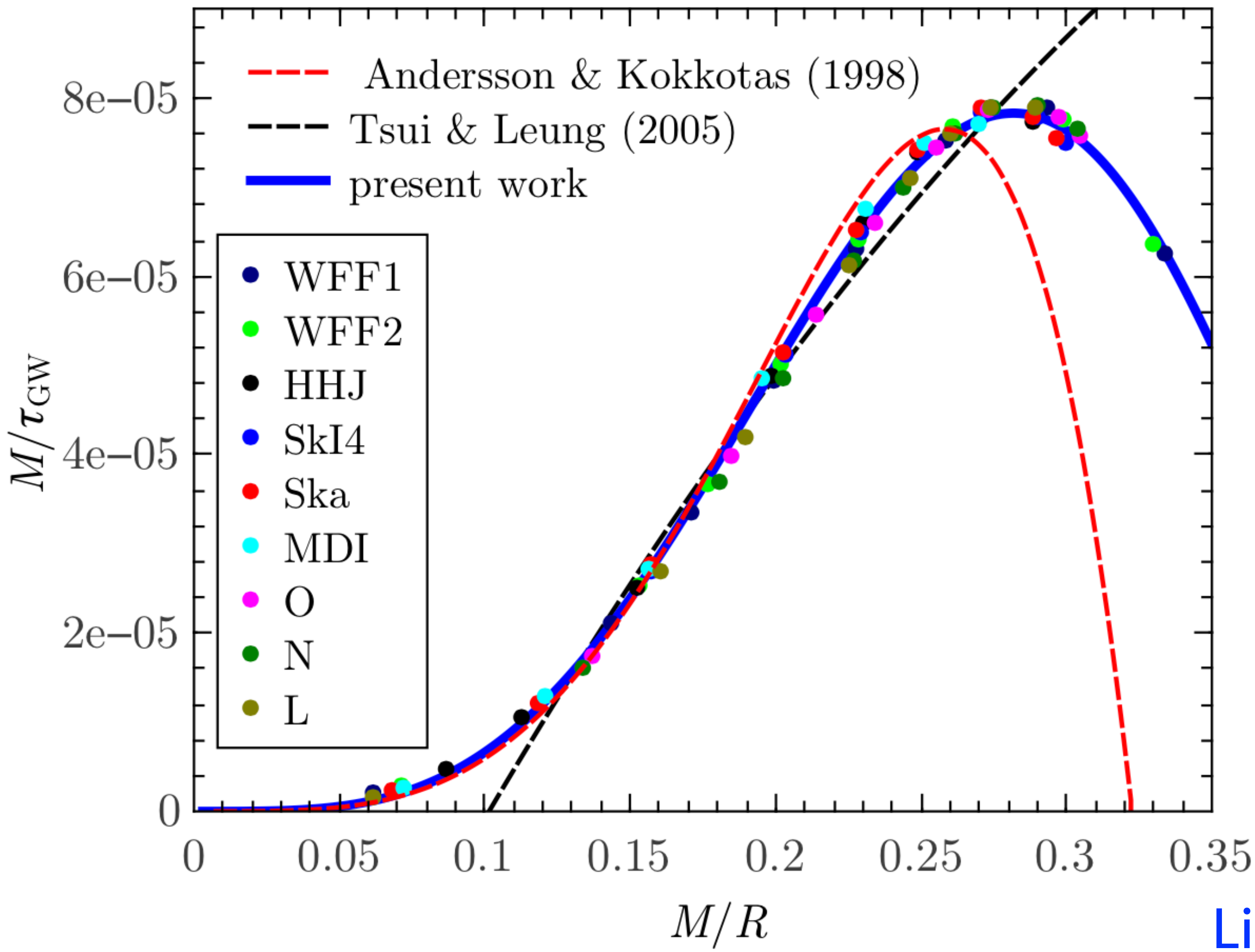
The gravitational-wave damping timescale also satisfies an empirical relation:

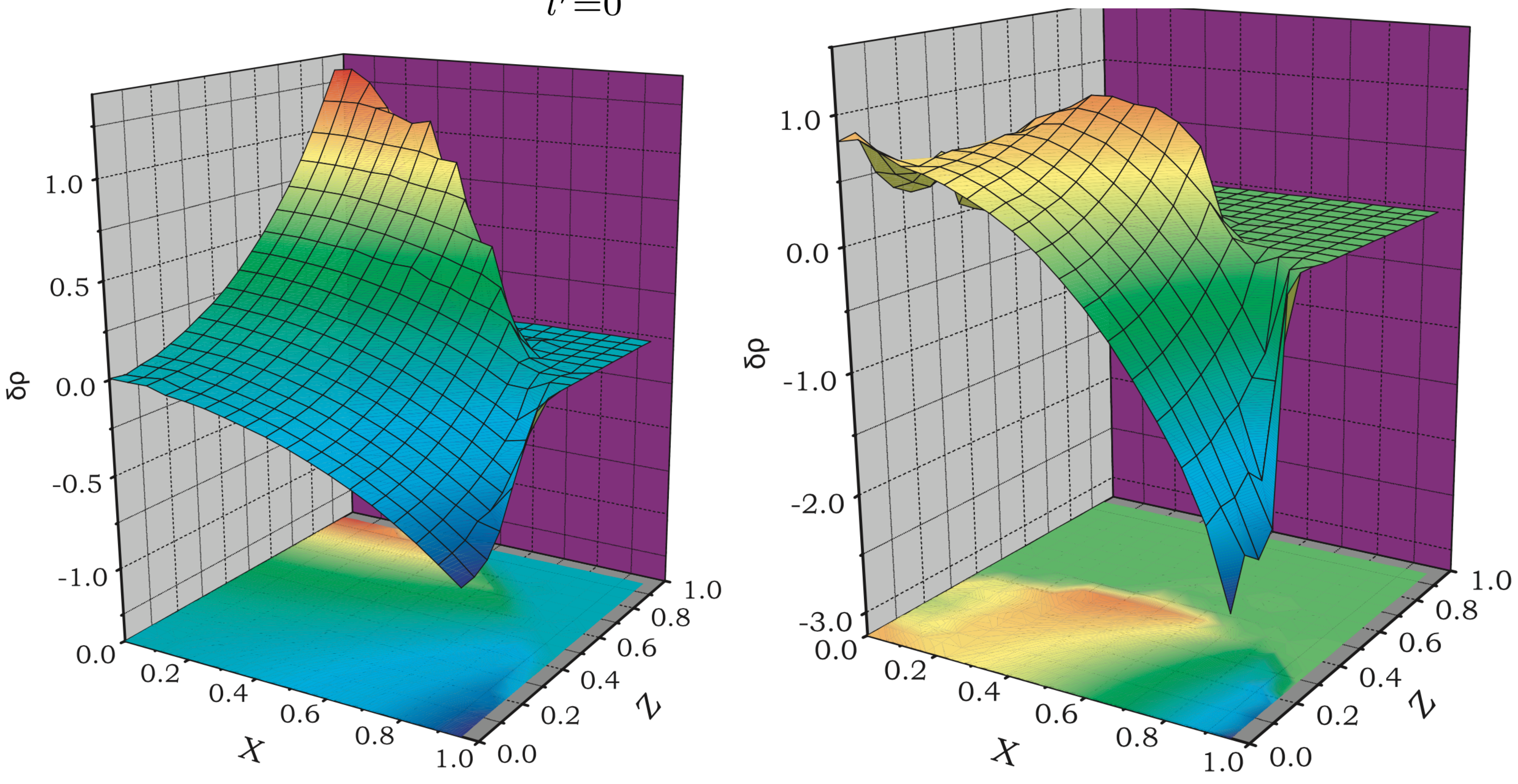


Lioutas & NS (2017)

Effects of Rotation on Axisymmetric Modes

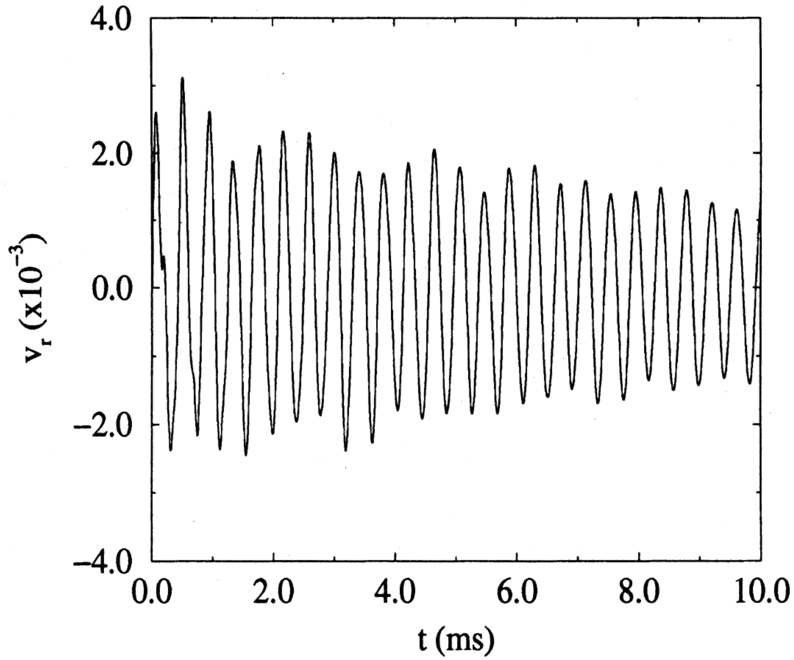
In rotating stars, the mode eigenfunctions are distorted and acquire higher-order terms and axial terms:

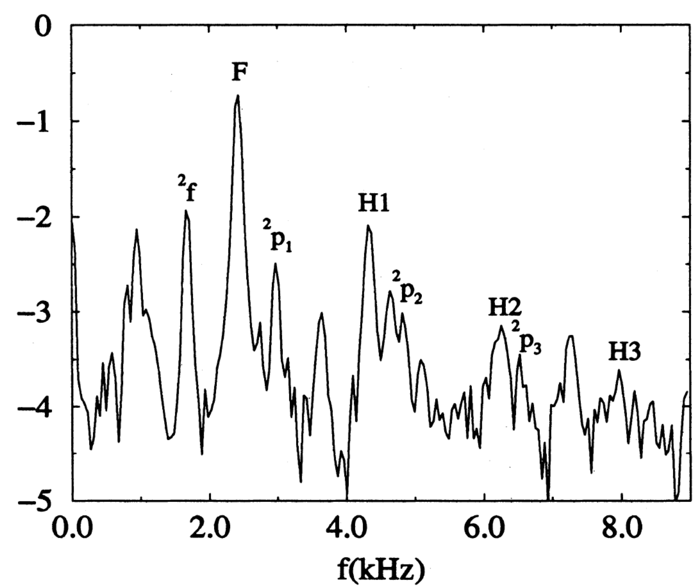
$$P_l^{\text{rot}} \sim \sum_{l'=0}^{\infty} (P_{l+2l'} + A_{l+2l'\pm 1})$$

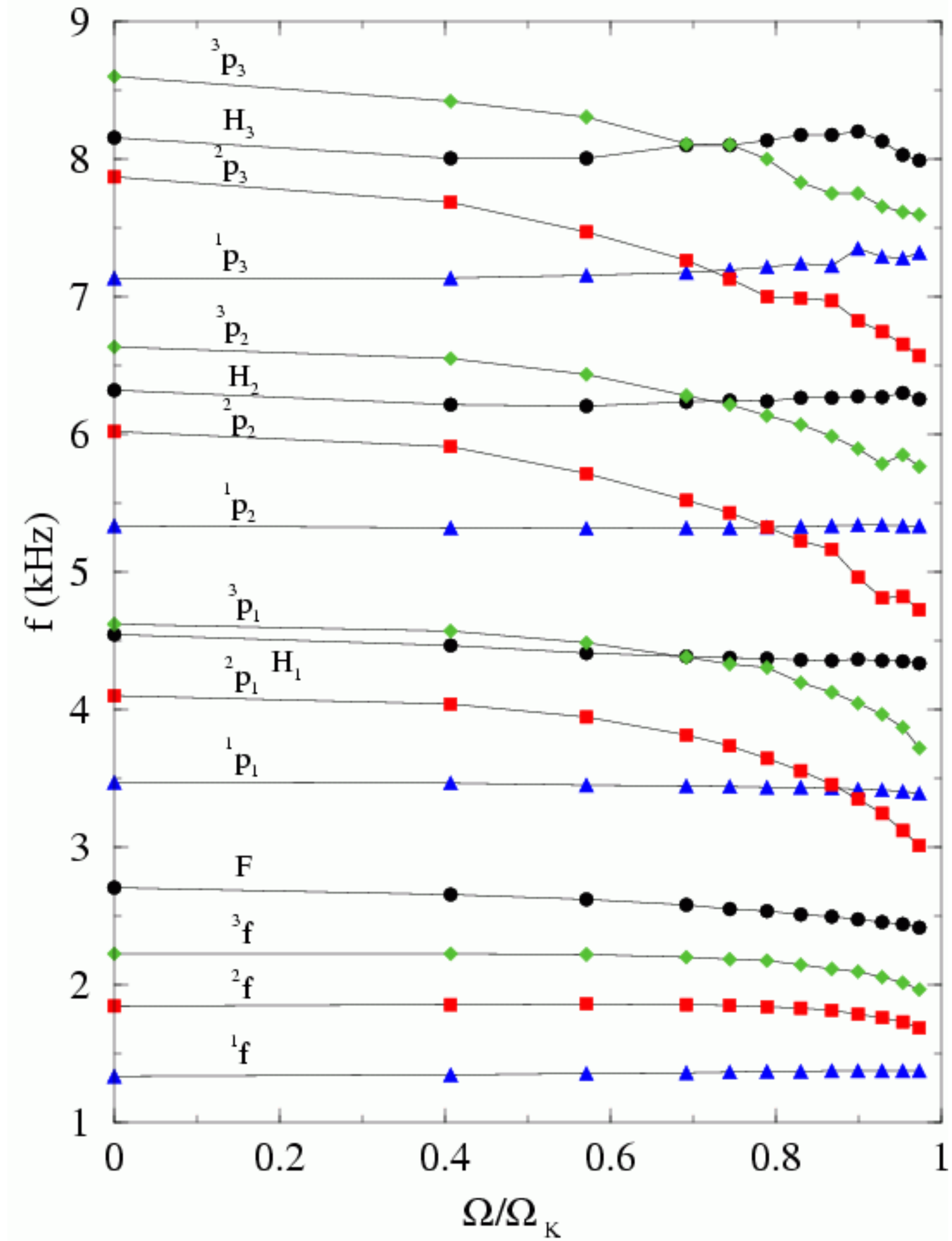


Axisymmetric Pulsations

Nonlinear simulations of N=1 polytropes in the Cowling approximation:

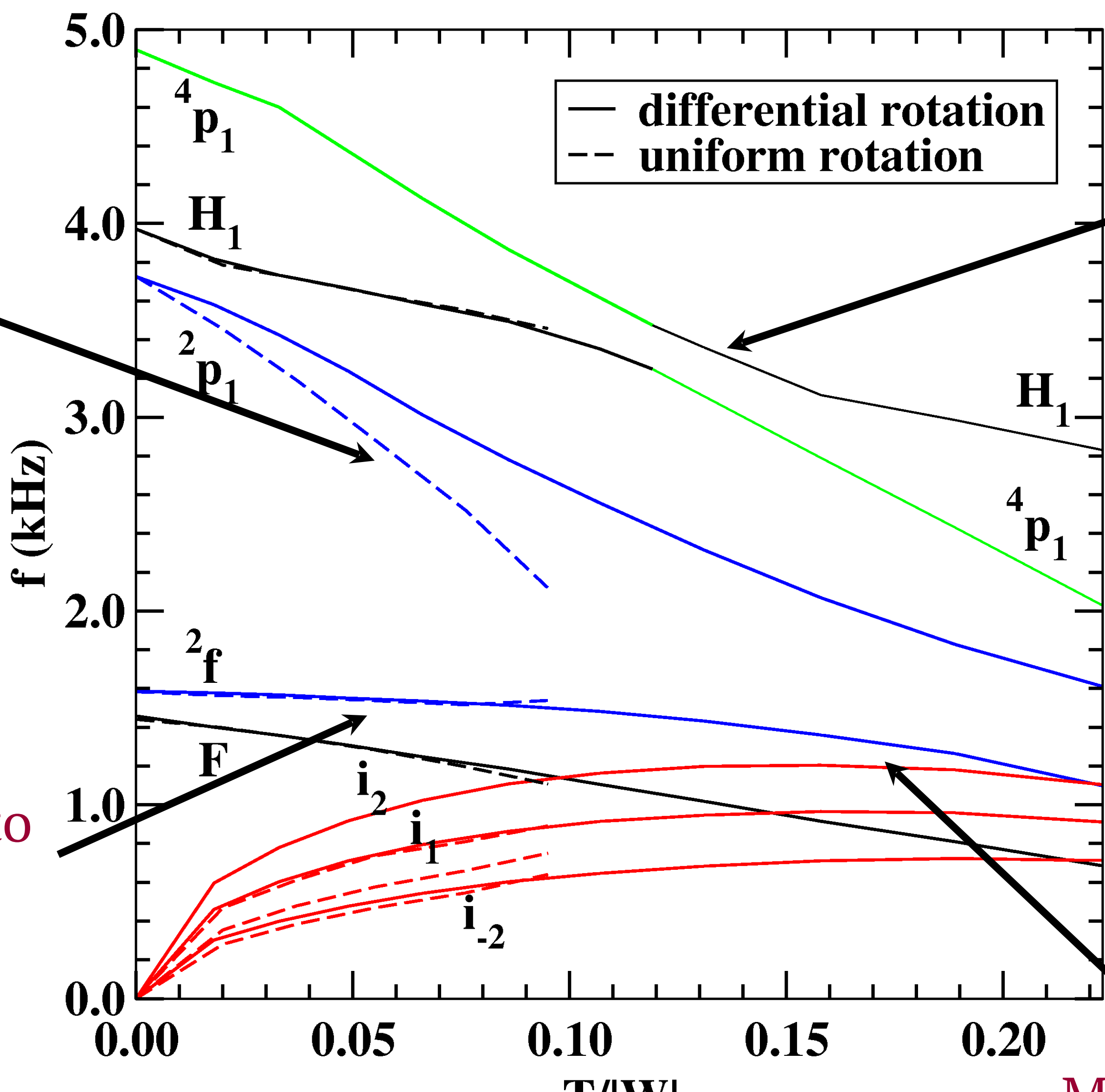






Axisymmetric Modes in CFC Approximation

Spacetime evolution with Spatially Conformally Flat Condition (CFC/IWM)



Sensitive to Differential Rotation

Not sensitive to Differential Rotation

Avoided Crossing

Frequencies Shifted to 1-2 kHz range

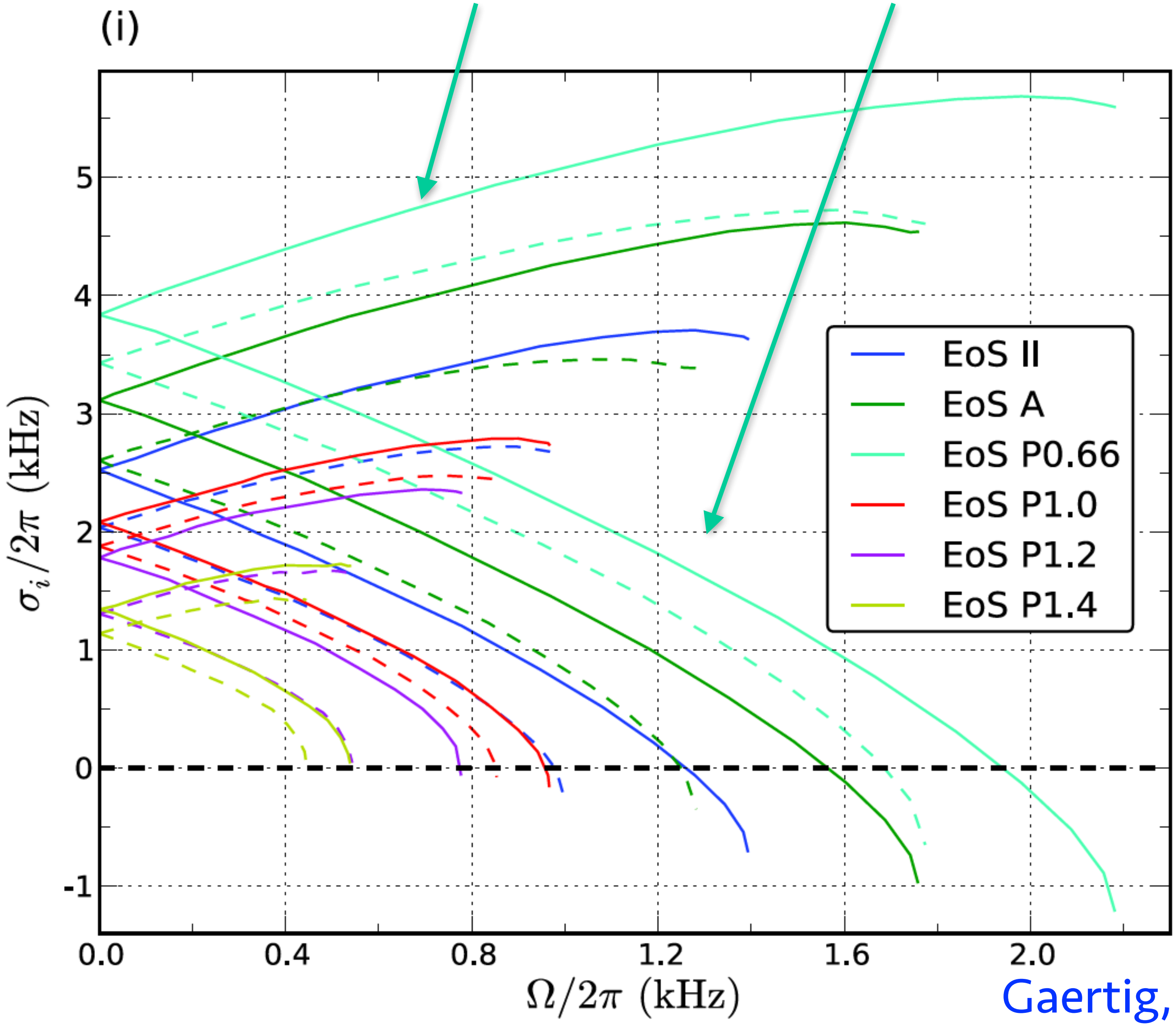
Maximum Frequency

Dimmelmeier, NS & Font (2006)

Effect of Rotation on Non-axisymmetric Modes

l = *m* = +/- 2 *f*-mode (in Cowling approximation),

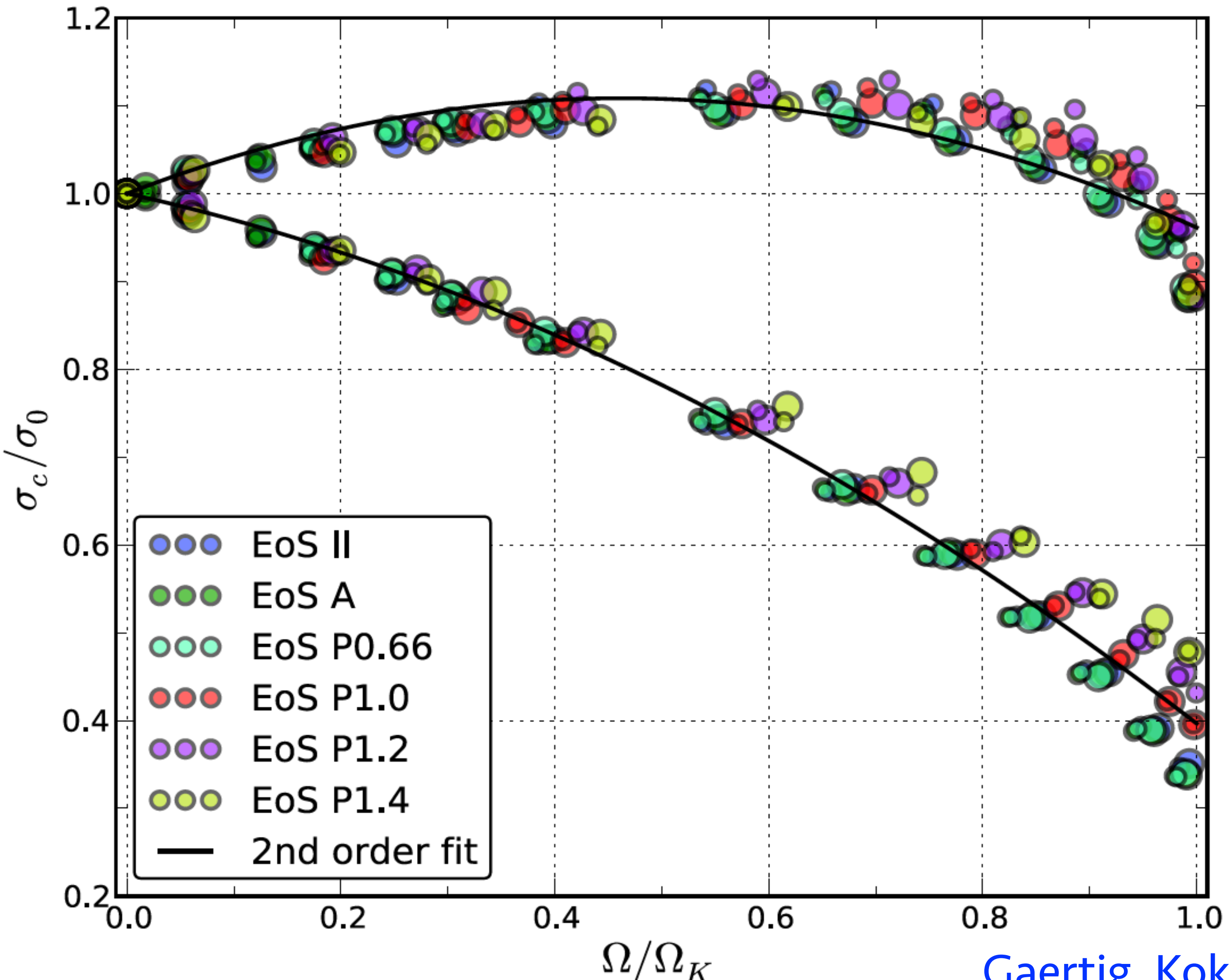
Frequency splitting into *co-rotating* and *counter-rotating* branches.



Gaertig, Kokkotas (2011)

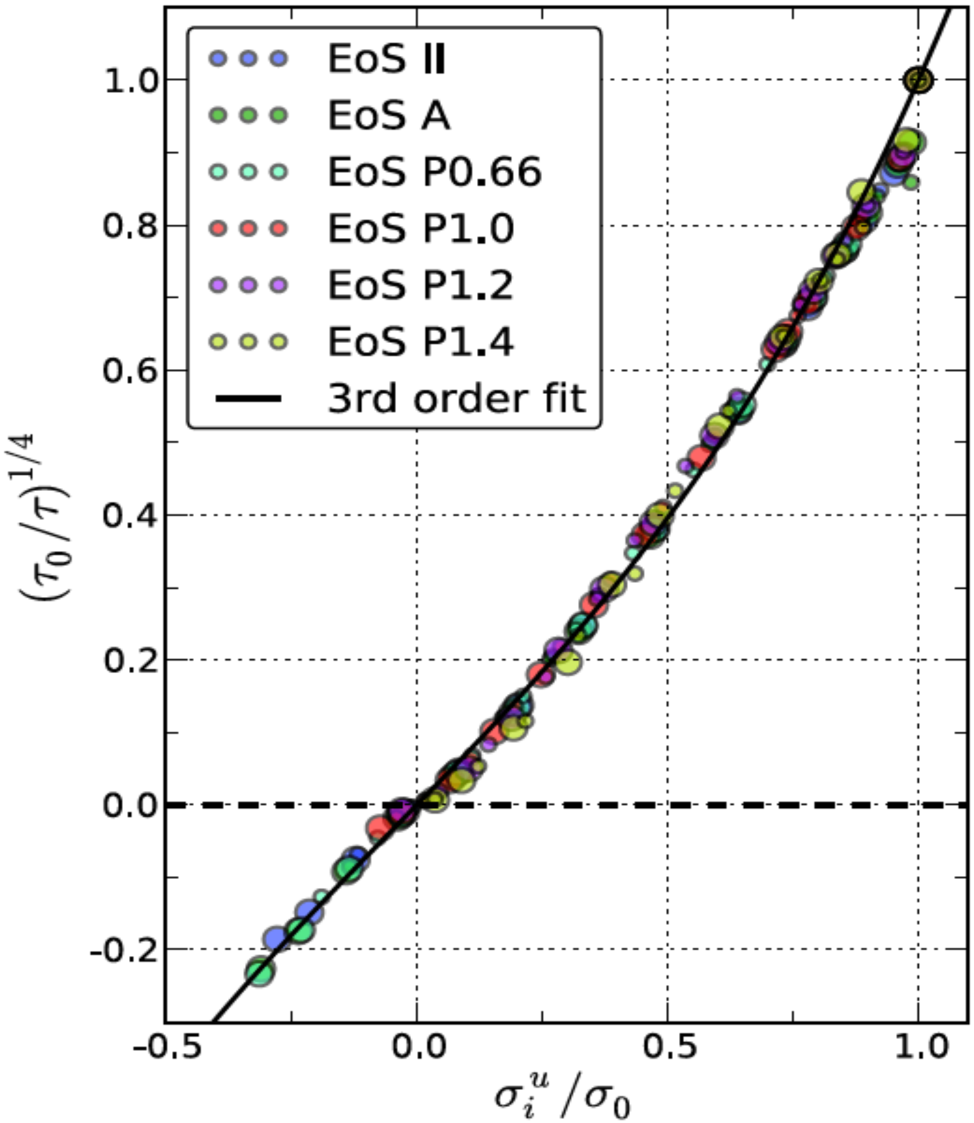
Effect of Rotation on Non-axisymmetric Modes

The effect on rotation is universal (independent of EOS)!

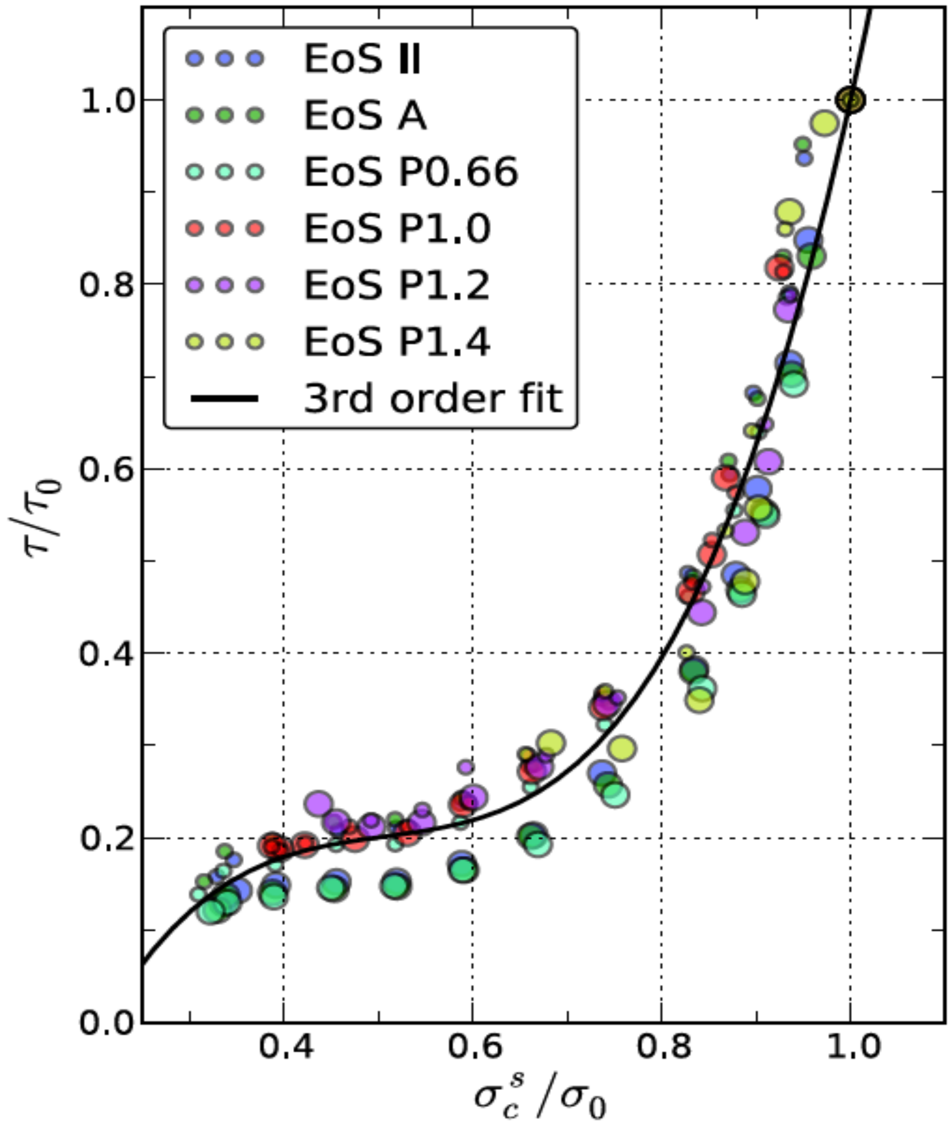


Gaertig, Kokkotas (2011)

Empirical Relations for Damping Timescale



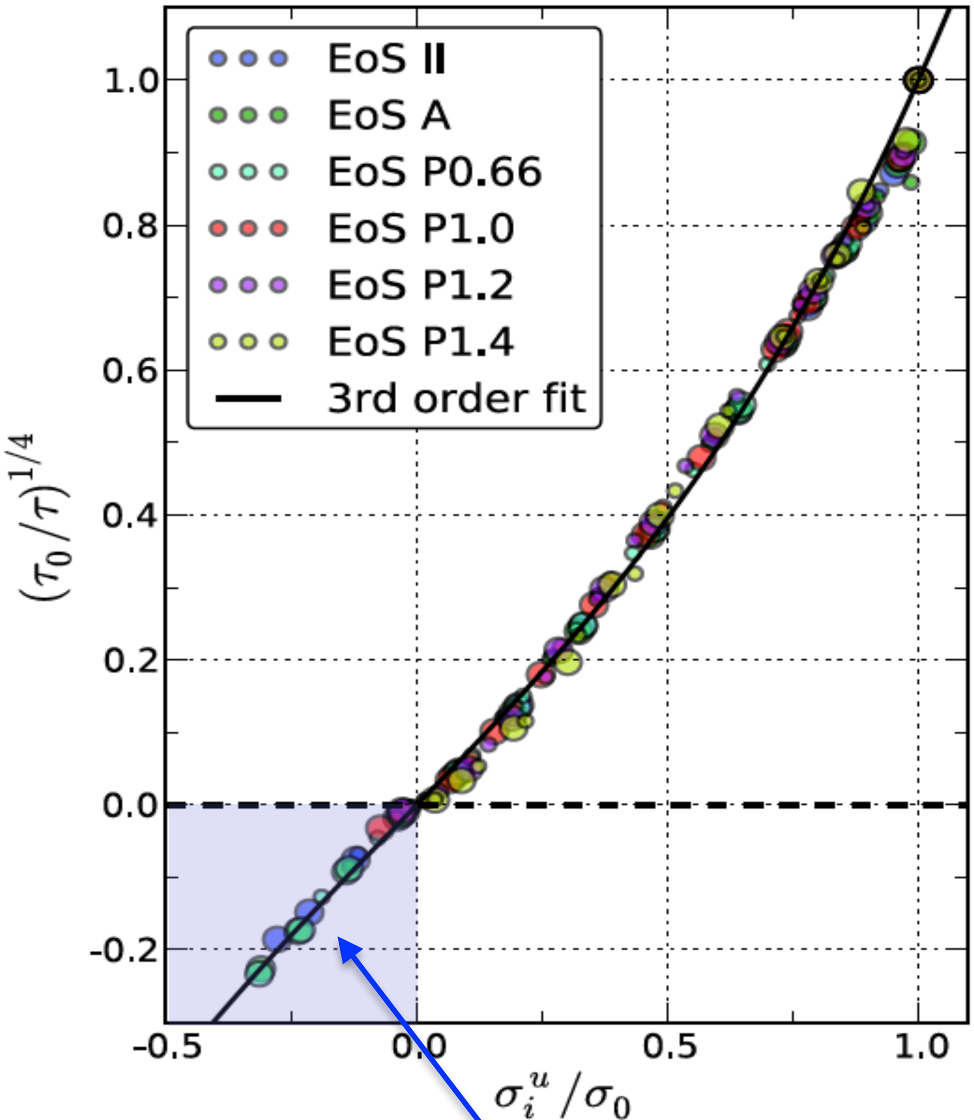
Counter-rotating (unstable) branch vs. *f* in *inertial* frame.

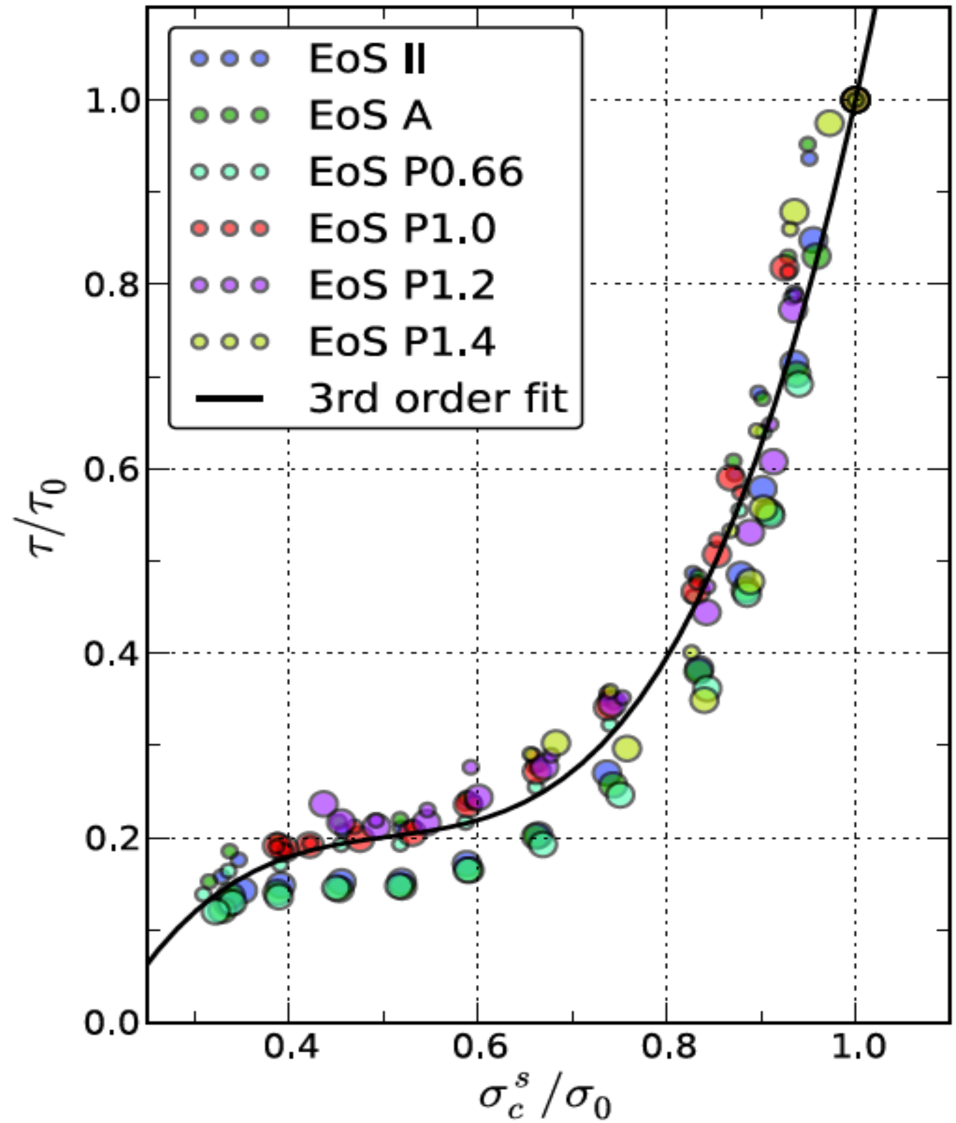


Corotating (stable) branch vs. *f* in *corotating* frame.

Gaertig, Kokkotas (2011)

Empirical Relations for Damping Timescale



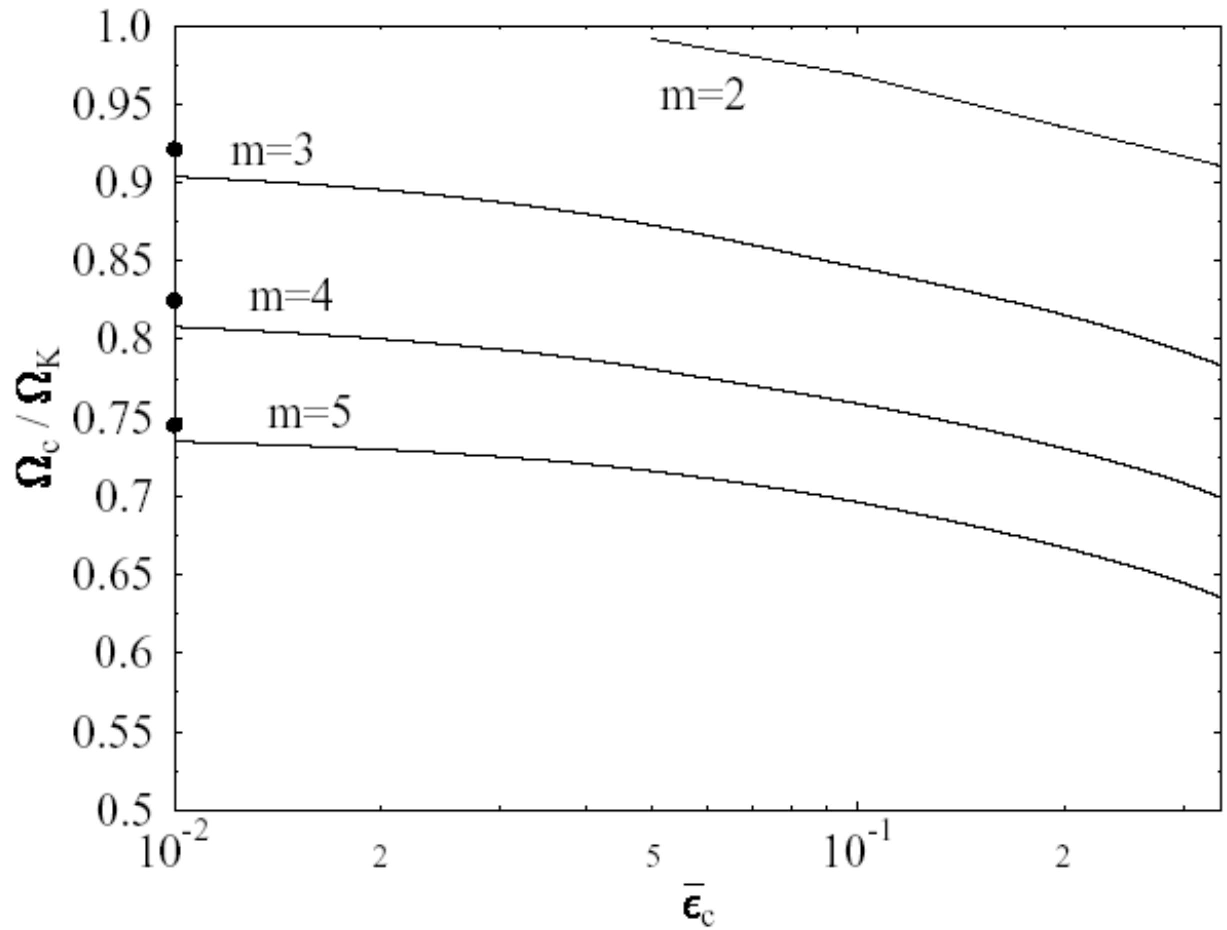


Counter-rotating (unstable) branch vs. f in *inertial* frame.

Corotating (stable) branch vs. f in *corotating* frame.

Chandrasekhar - Friedman - Schutz (CFS) instability

F-Mode Instability in Relativistic Stars

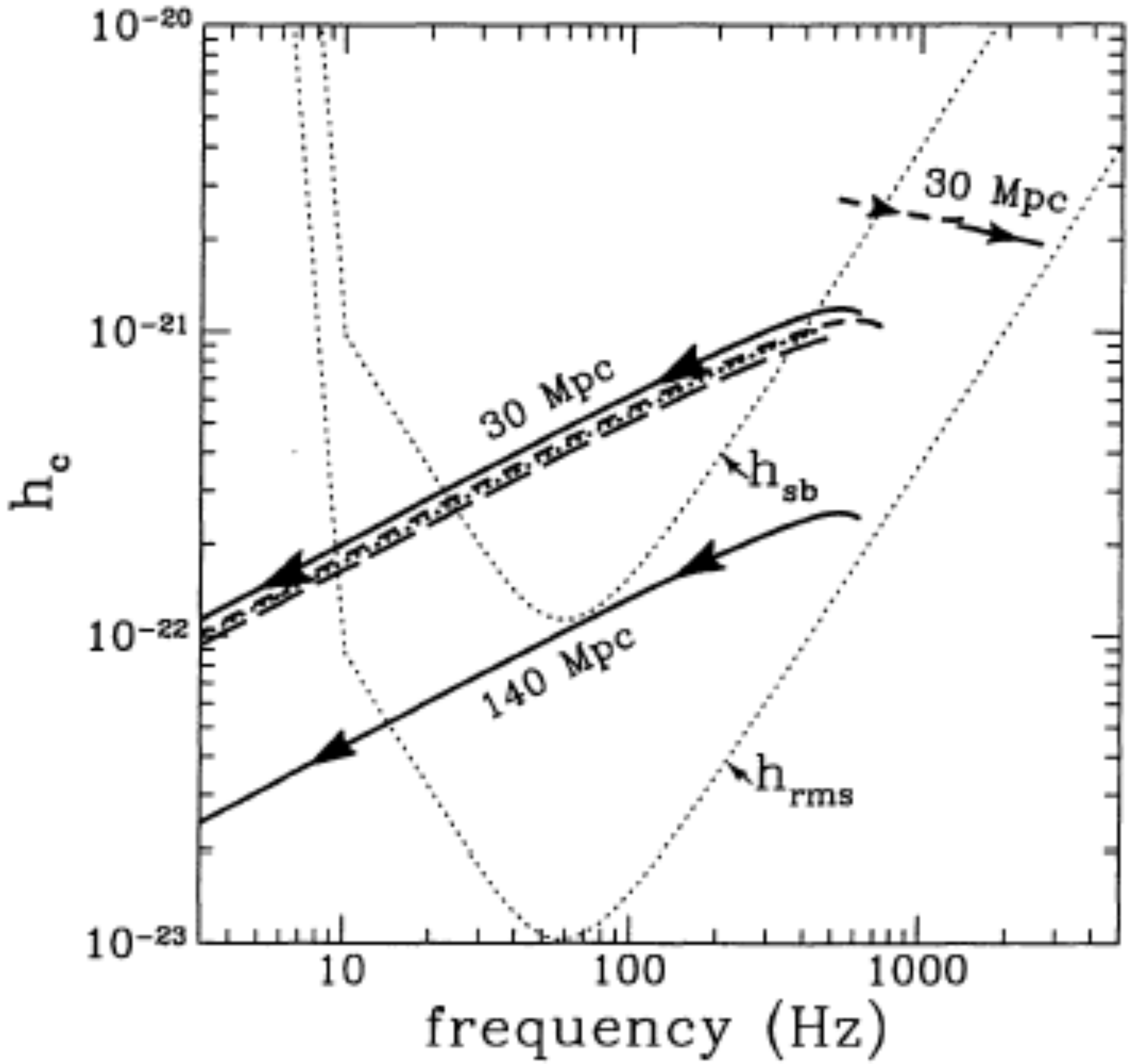


Onset of $l = m = 2$ instability for:

(Full GR)	(Newtonian)
$\Omega > 0.85\,\Omega_K$	$\Omega > 0.95\,\Omega_K$
$T/W > 0.07$	$T/W > 0.14$
$N < 1.3$	$N < 0.81$

(N.S & Friedman 1998)

Nonlinear Development of F-Mode Instability



Lai & Shapiro, 1995

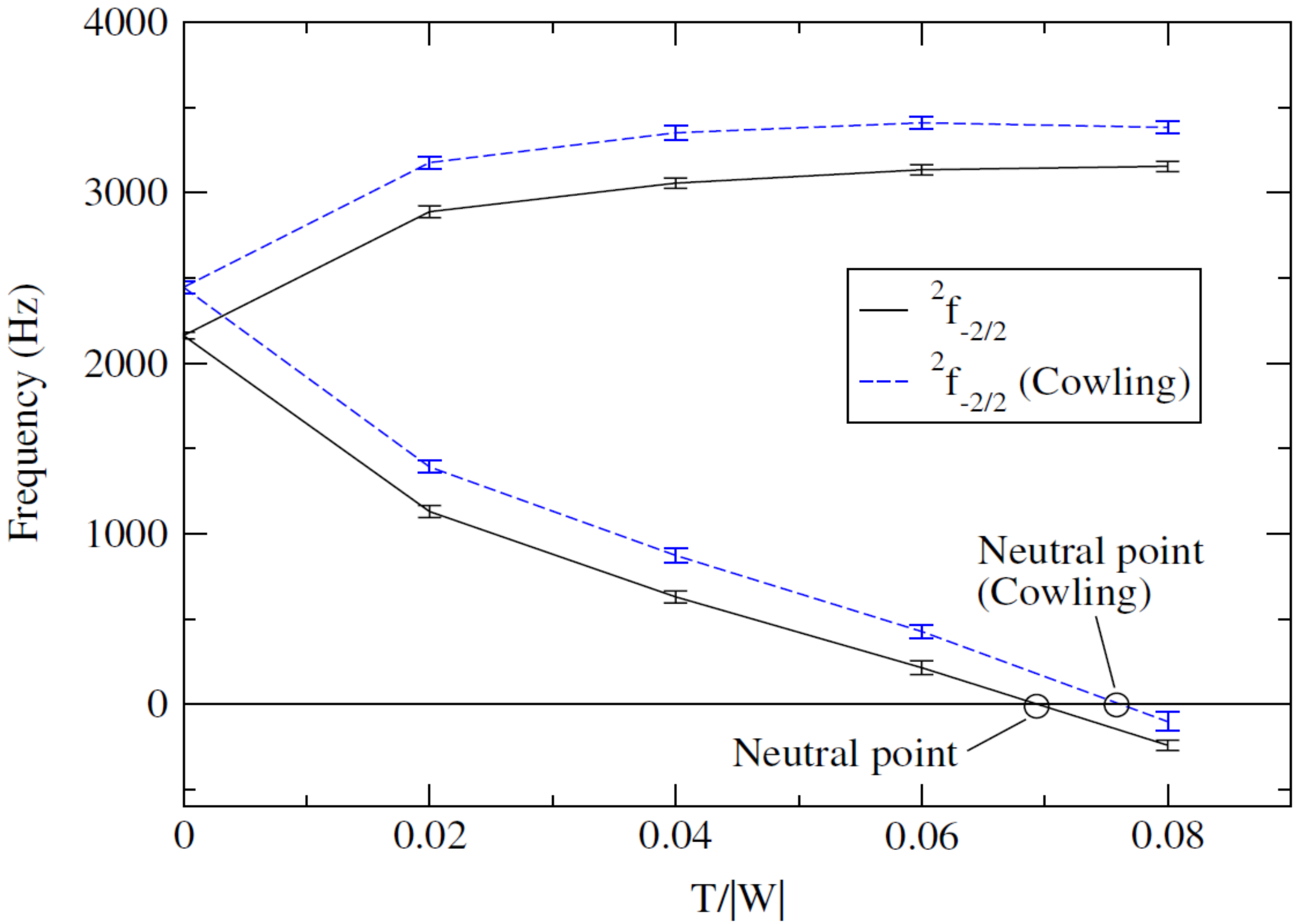
Main uncertainties:

1. Relativistic growth times
2. Nonlinear saturation
3. Initial rotation rates of protoneutron stars
4. Effect of magnetic fields

Rotational Instabilities

Zink, Korobkin, Schnetter, NS (201

Rapid Rotation, full GR, f-modes
(3-D THOR code)

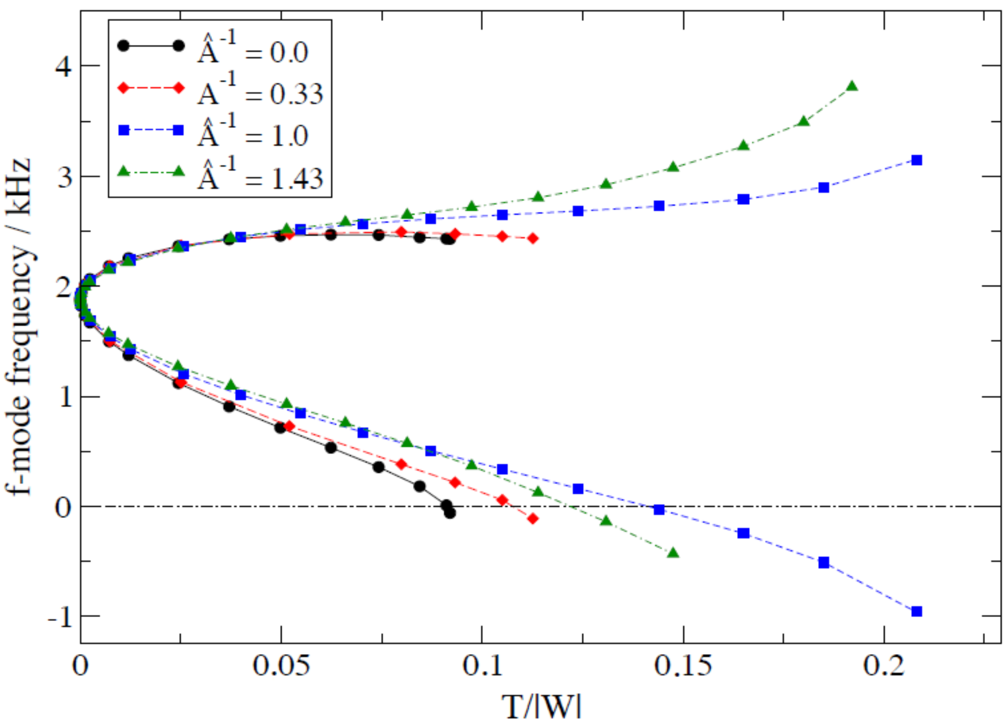


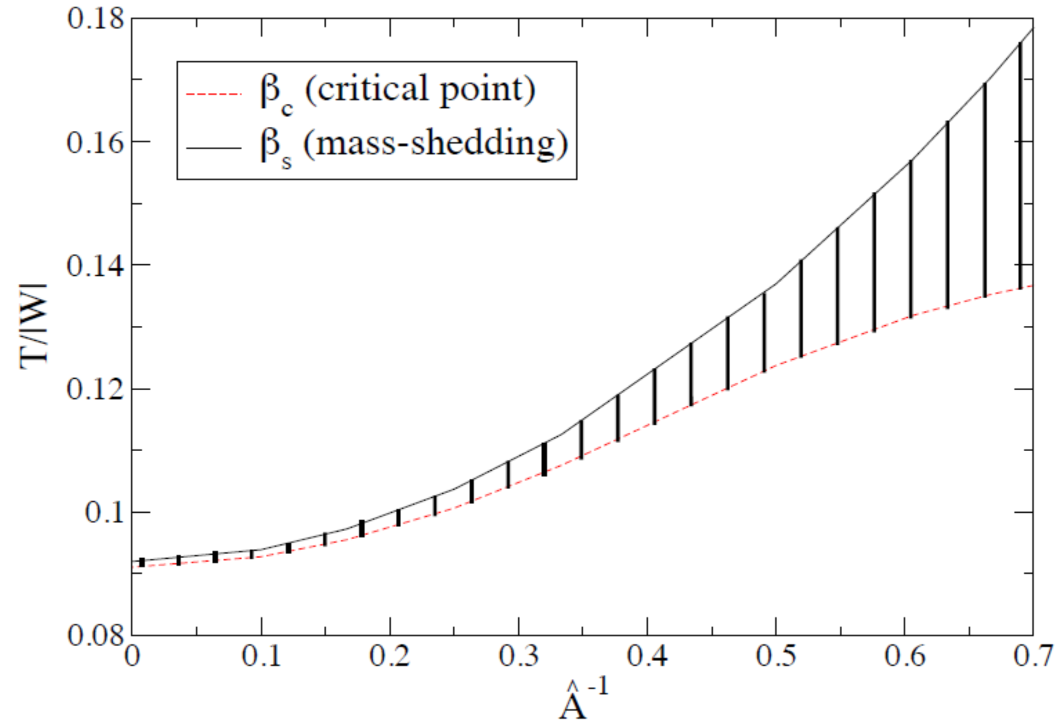
f-mode instability underestimated in Cowling approximation.
Neutral point near perturbative result by NS, Friedman (1998)

f-Mode Instability in Differentially Rotating Stars

Krueger, Gaertig, Kokkotas (20

Rapid <u>differential</u> rotation, Cowling approximation (linear time-evolution code)



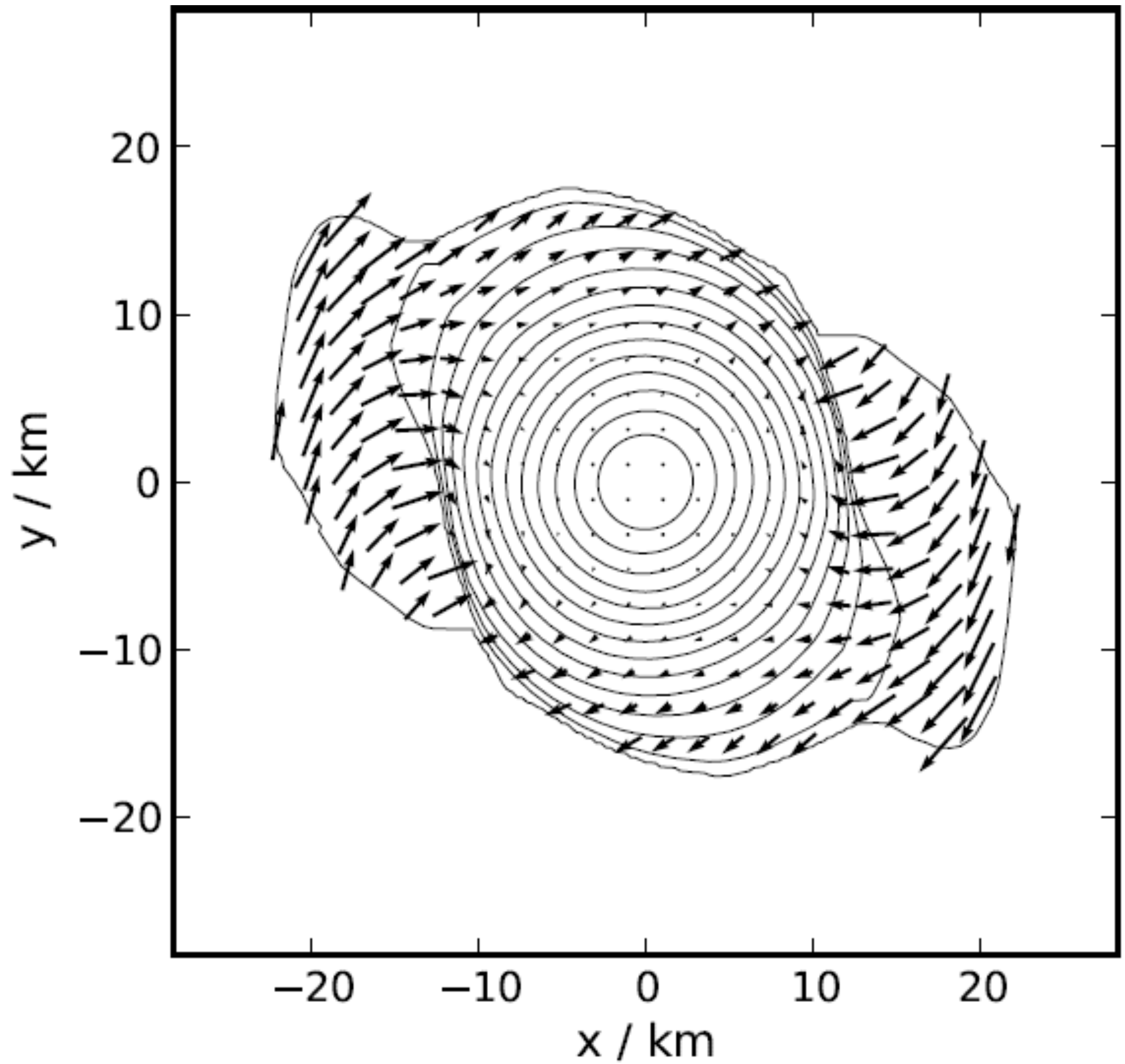


The critical absolute value of β=T/|W| for the instability increases.

At the same time, the value of β *at the mass-shedding limit* increases much more.

Nonlinear Saturation of l=m=2 f-Modes

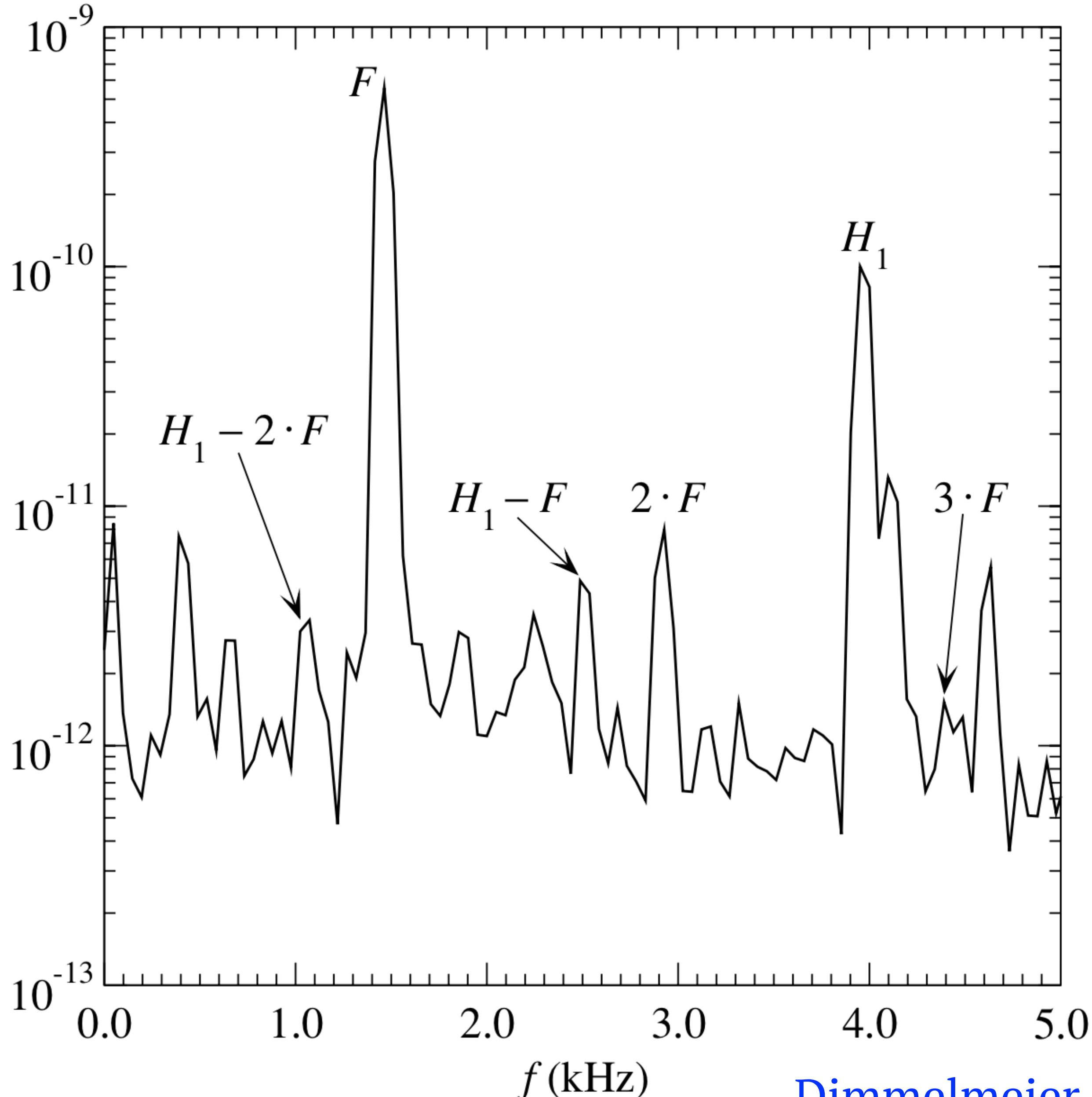
At amplitude of a few times 10^{-2} the f-mode is saturated by wave-breaking at the surface.



Kastaun, Willburger & Kokkotas (2010)

Quasi-linear Combination Frequencies

Linear sums and differences of linear mode frequencies:

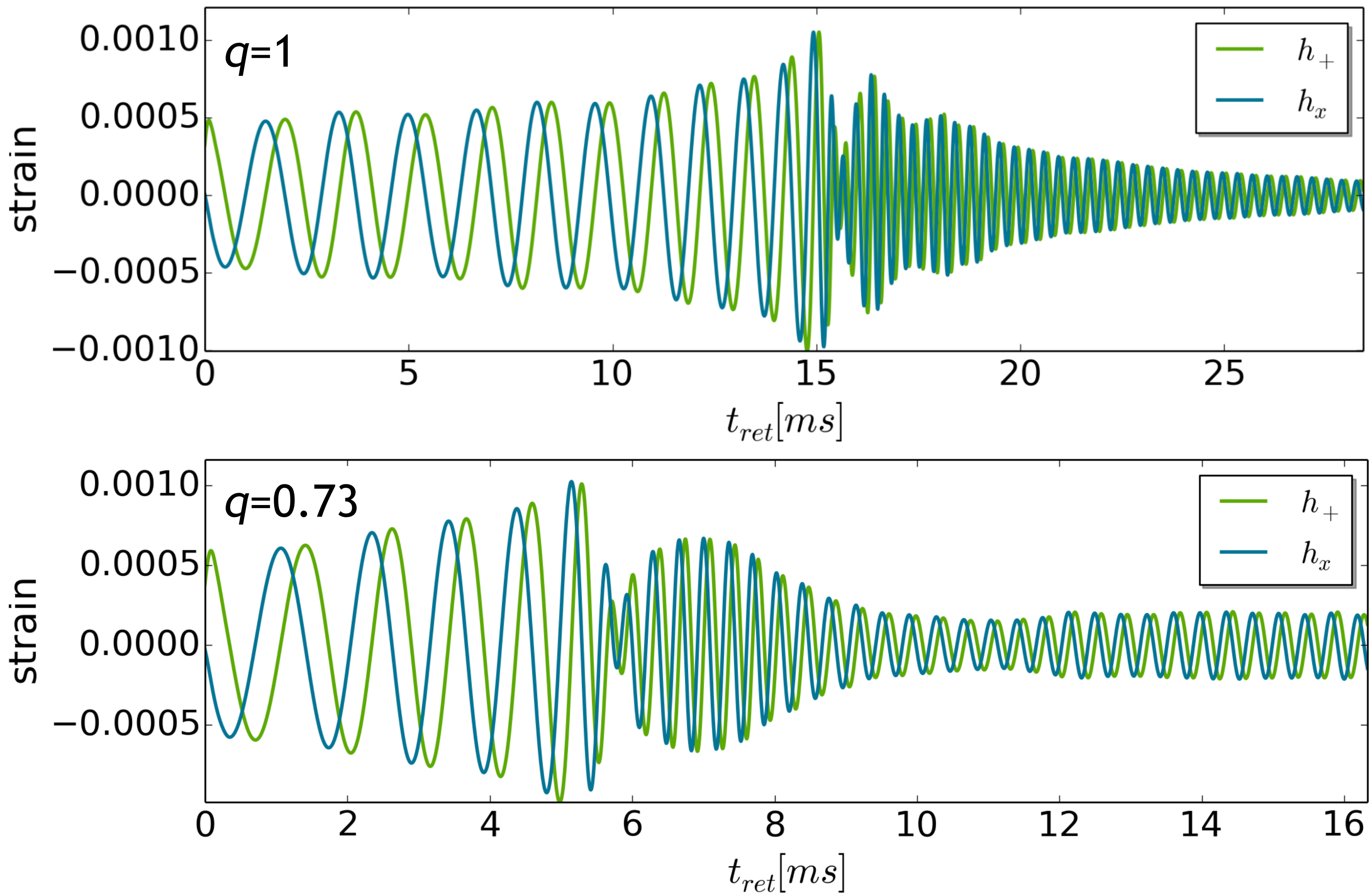


Dimmelmeier, NS & Font (2006)

Gravitational Waves

Soultanis & NS (2018)

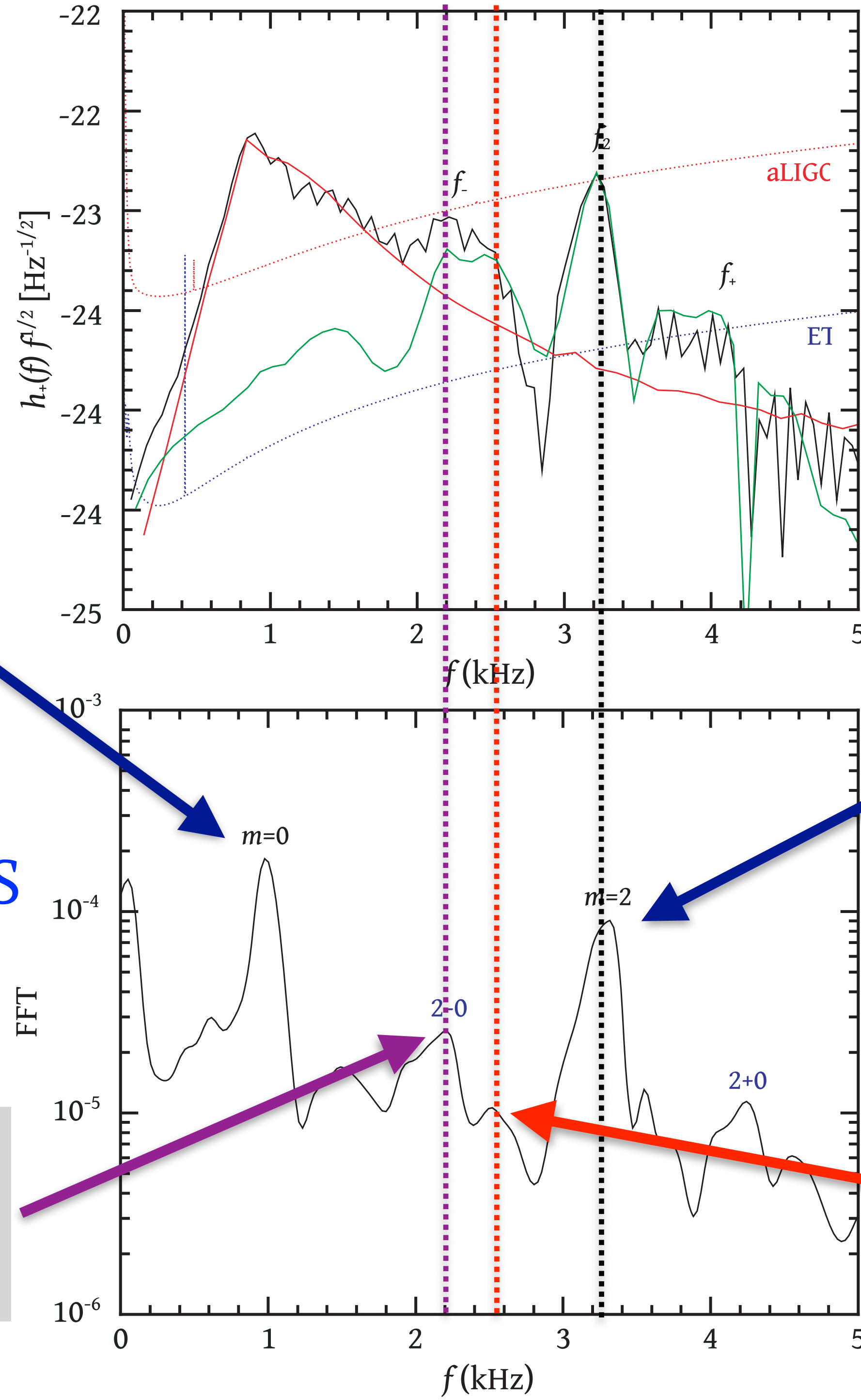
Numerical simulations using the MPA1 EOS and $M_{\rm chirp} = 1.186$
A hypermassive neutron star (HMNS) is formed, supported by differential rotation.



Post-merger Oscillations

NS, Bauswein, Zagkouris & Janka (2011)

GRAVITATIONAL WAVE SPECTRUM



l=m=0
linear quasi-radial mode

l=m=2
linear f-mode

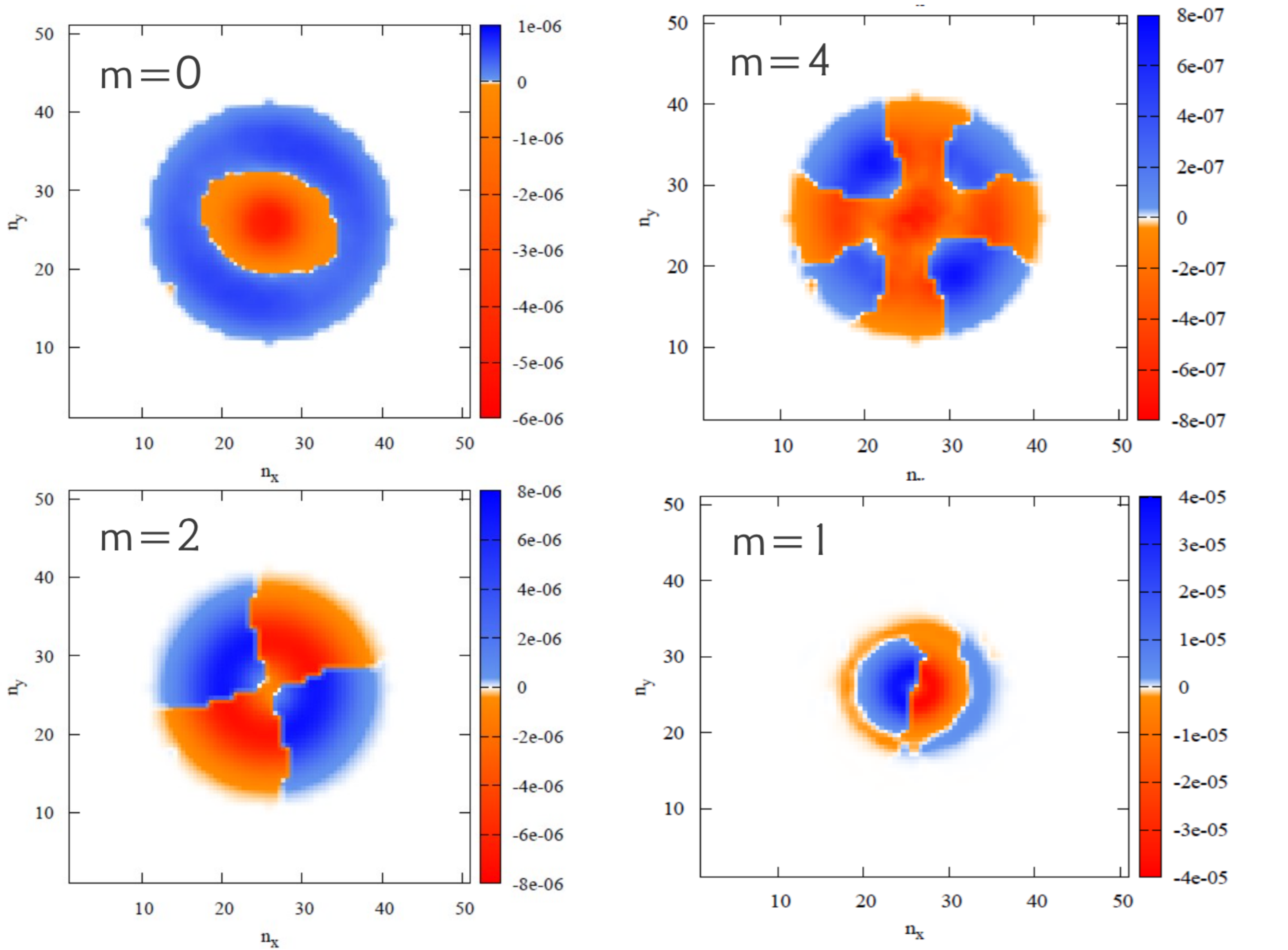
FFT OF HYDRODYNAMICS IN EQUATORIAL PLANE

"2-0" quasi-linear
combination frequency

nonlinear
spiral frequency

Eigenfunctions

m=0, 2, 4 (Shen $1.35M_{sun}+1.35M_{sun}$) m=1 (MIT60 $1.2M_{sun}+1.35M_{sun}$)

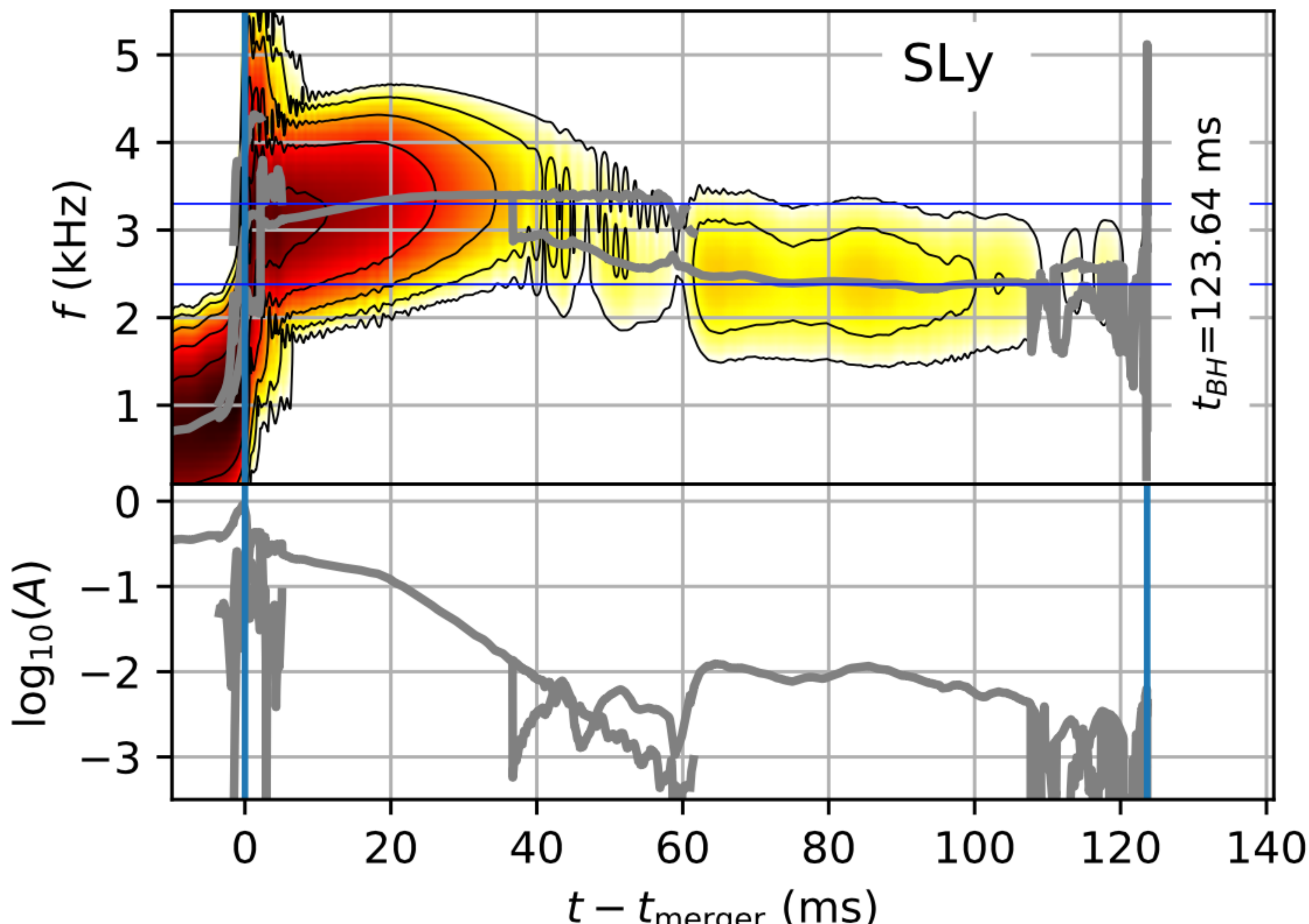


Inertial Modes!

(PRL, 2018)

Convective Excitation of Inertial Modes in Binary Neutron Star Mergers

Roberto De Pietri,[1,2] Alessandra Feo,[3,2] José A. Font,[4,5] Frank Löffler,[6,7] Francesco Maione,[1,2] Michele Pasquali,[1,2] and Nikolaos Stergioulas[8]



30 million core hours (PRACE)

Convective Instability

The local convective instability depends on the sign of the Schwarzschild discriminant

$$A_\alpha = \frac{1}{\varepsilon + p}\nabla_\alpha \varepsilon - \frac{1}{\Gamma_1 p}\nabla_\alpha p.$$

where

$$\Gamma_1 \; := \; (\varepsilon + p)/p(dp/d\varepsilon)_s \; = \; (d\ln p/d\ln \rho)_s$$

is the adiabatic index.

$A_\alpha < 0$ convective stability

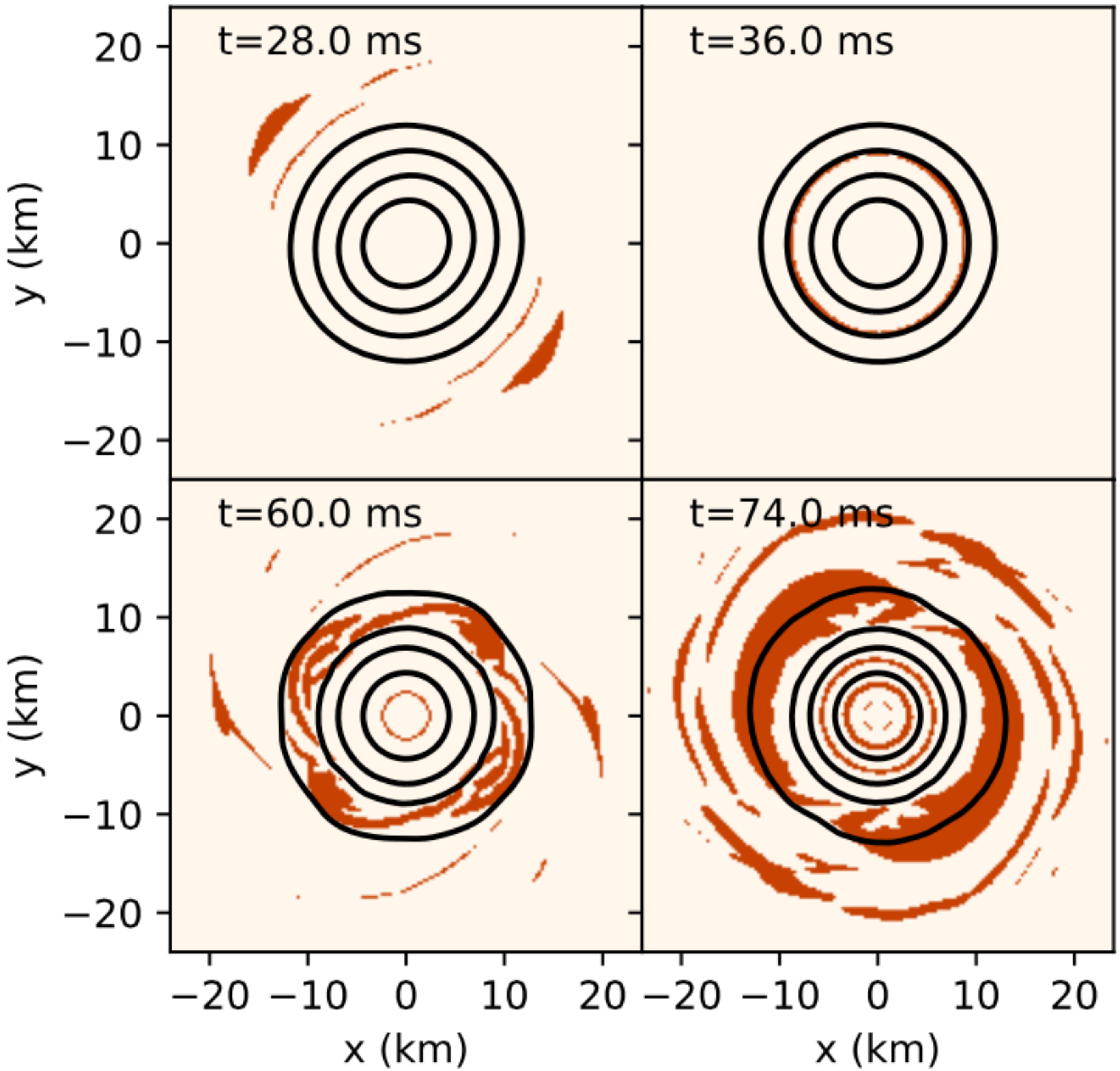
$A_\alpha > 0$ convective instability

For a piecewise-polytropic EOS with a thermal component, we find analytically:

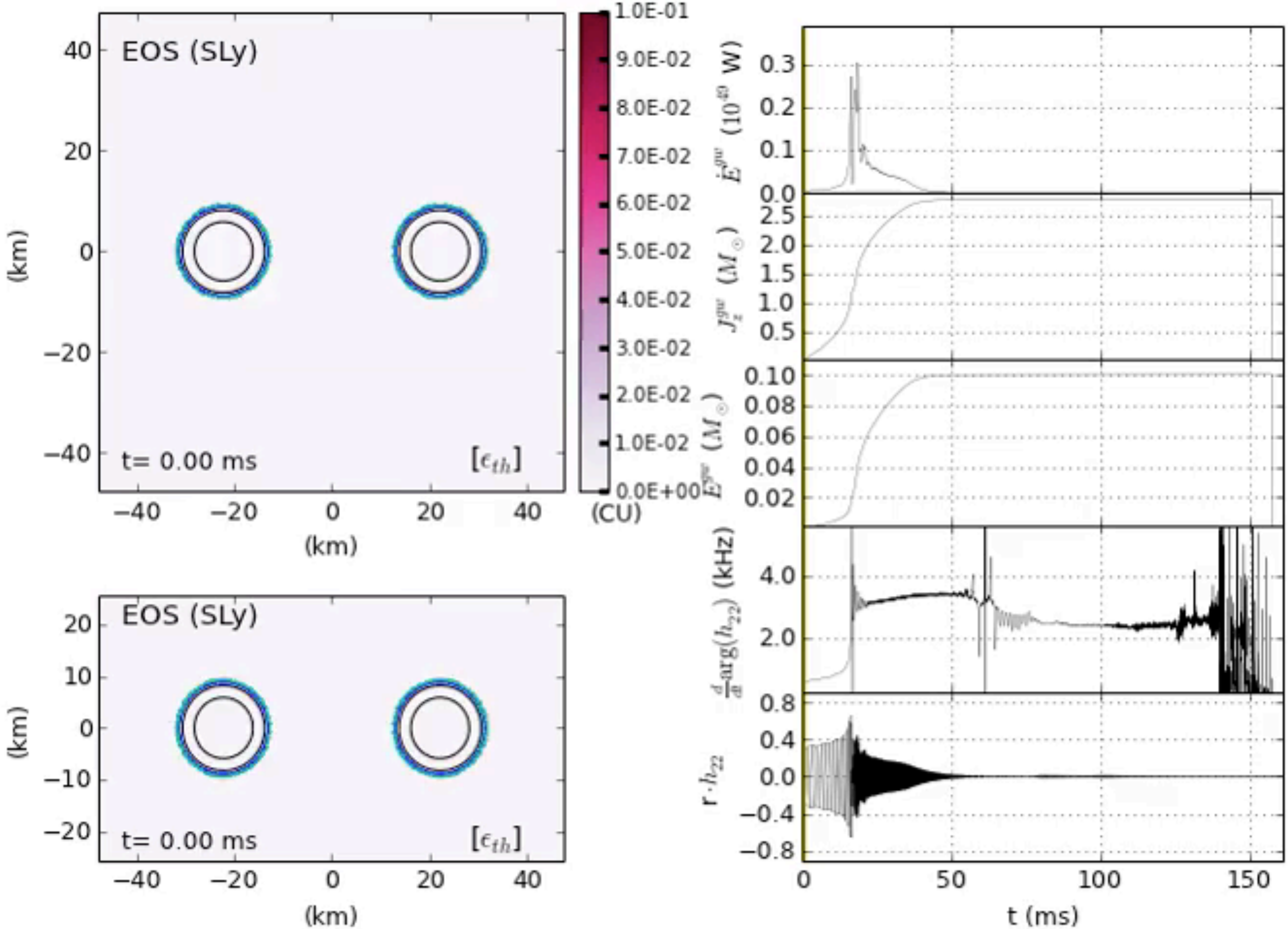
$$\Gamma_1 = \Gamma_{\rm th} + (\Gamma_i - \Gamma_{\rm th})\frac{K_i \rho^{\Gamma_i}}{p}.$$

Convective Instability

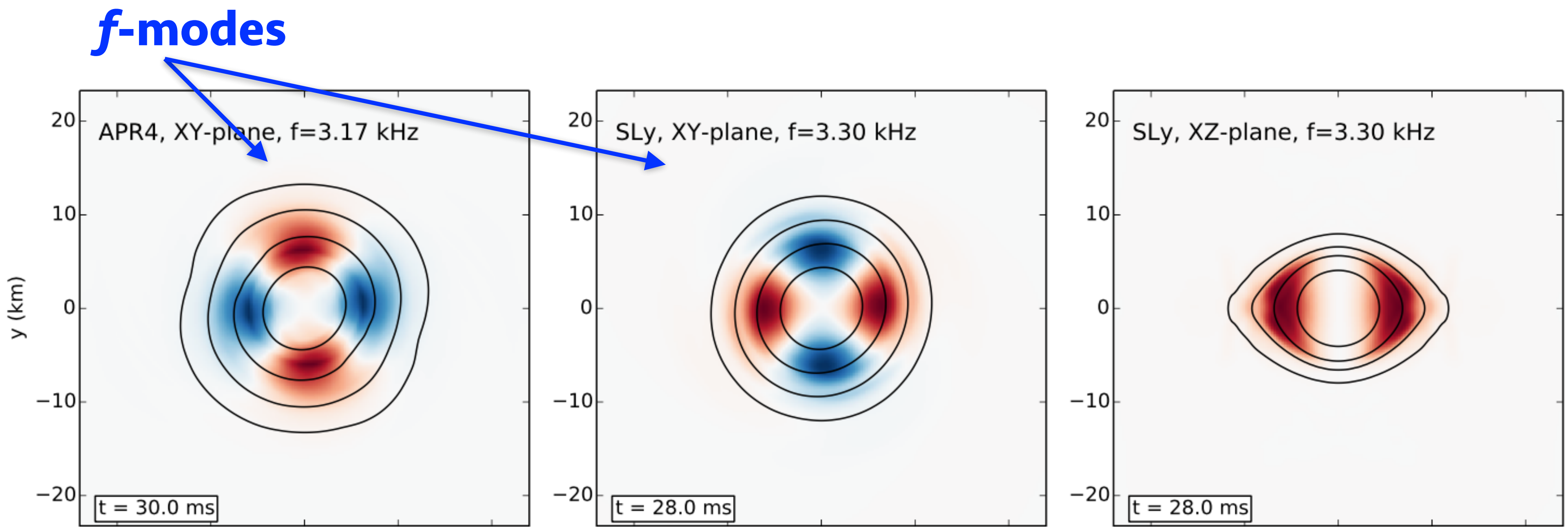
The sign of A_r in the equatorial plane:



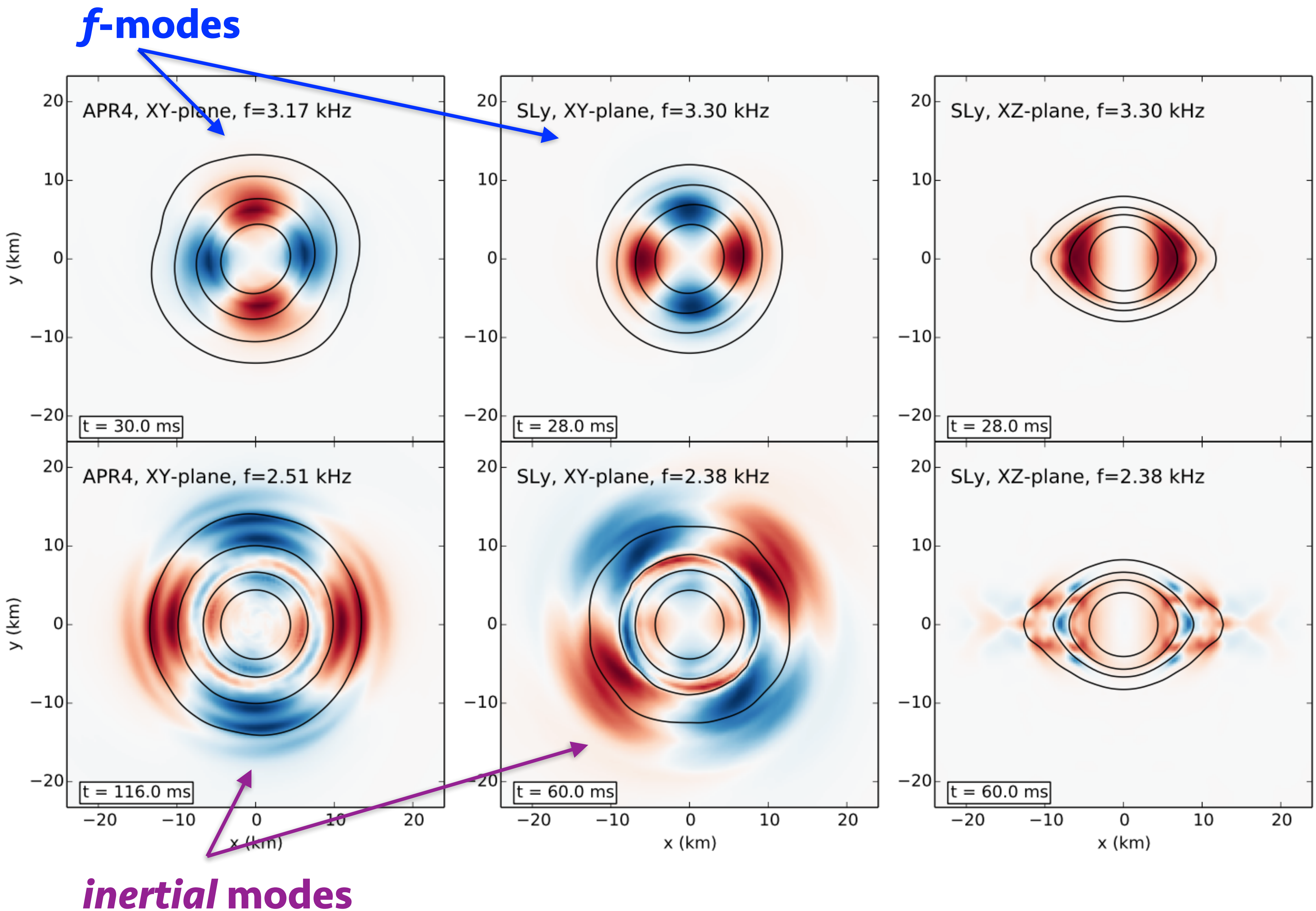
Thermal Evolution



Oscillations



Oscillations



Gravitational Wave Spectrum