NEUTRON STAR STRUCTURE

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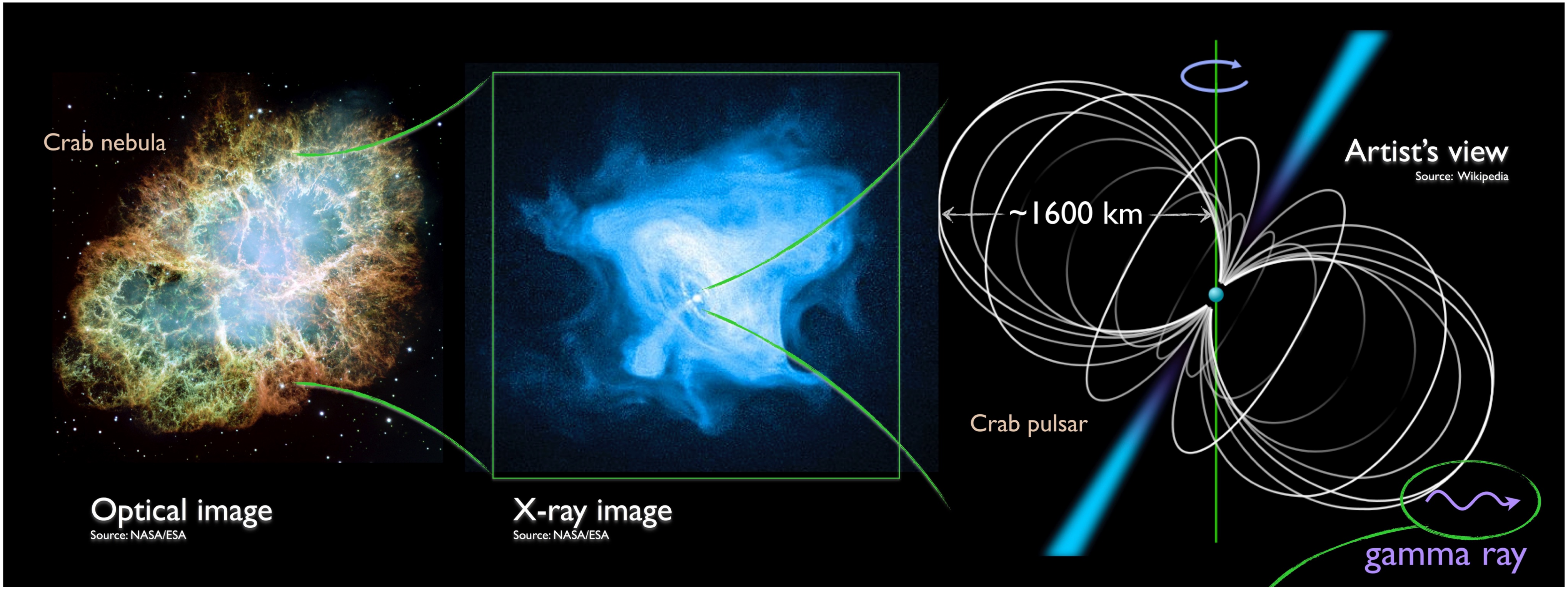
Neutron Stars



First neutron star detected almost 50 years ago.

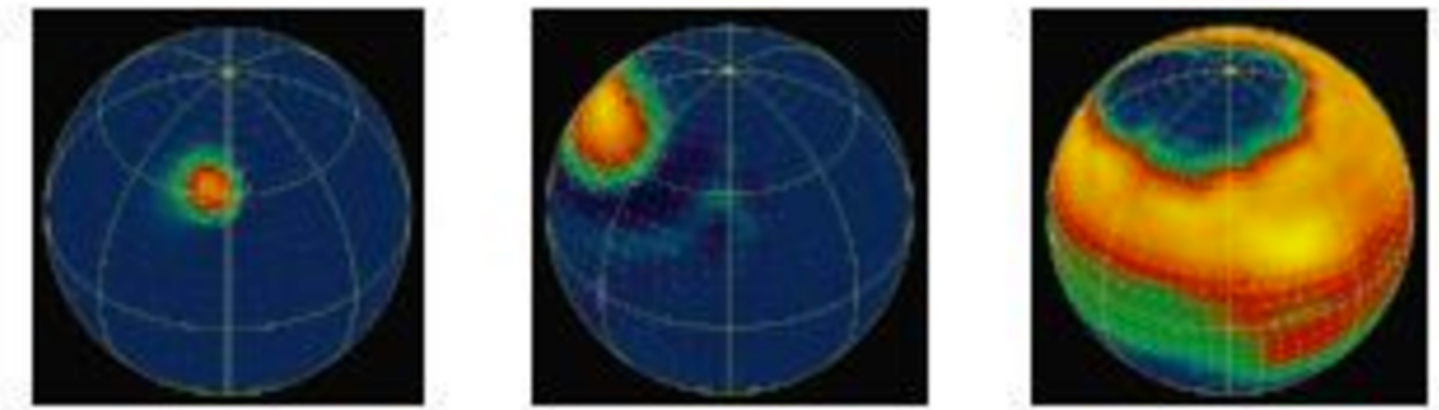
Properties of matter at the core?

Until today: No accurate radius determination.

Constraints on Neutron Star Radii or M/R

Main methods in EM spectrum:

- Thermonuclear X-ray bursts (*photospheric radius expansion*)



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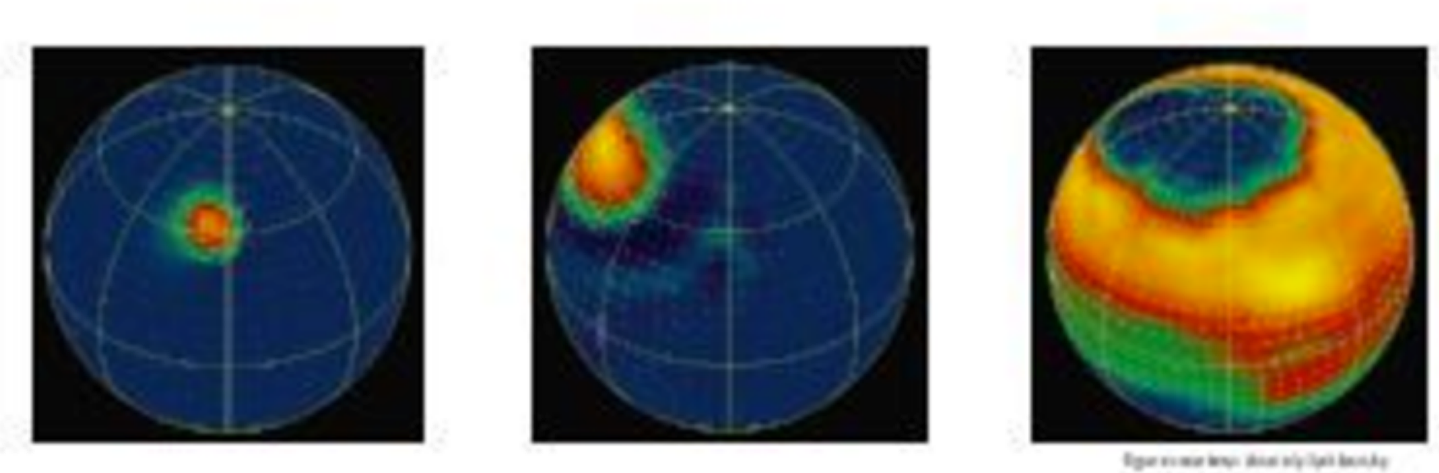
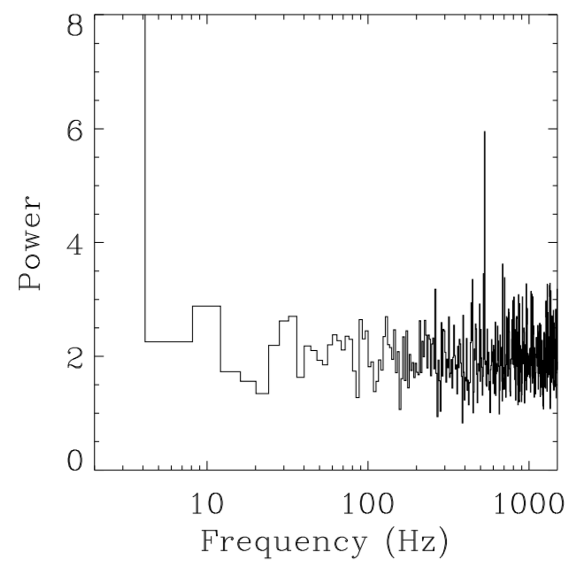


- Burst oscillations (*rotationally modulated waveform*)



Bhattacharyya et al. (2006)

Constraints on Neutron Star Radii or M/R

- Fits of continuum spectra of X-ray bursts

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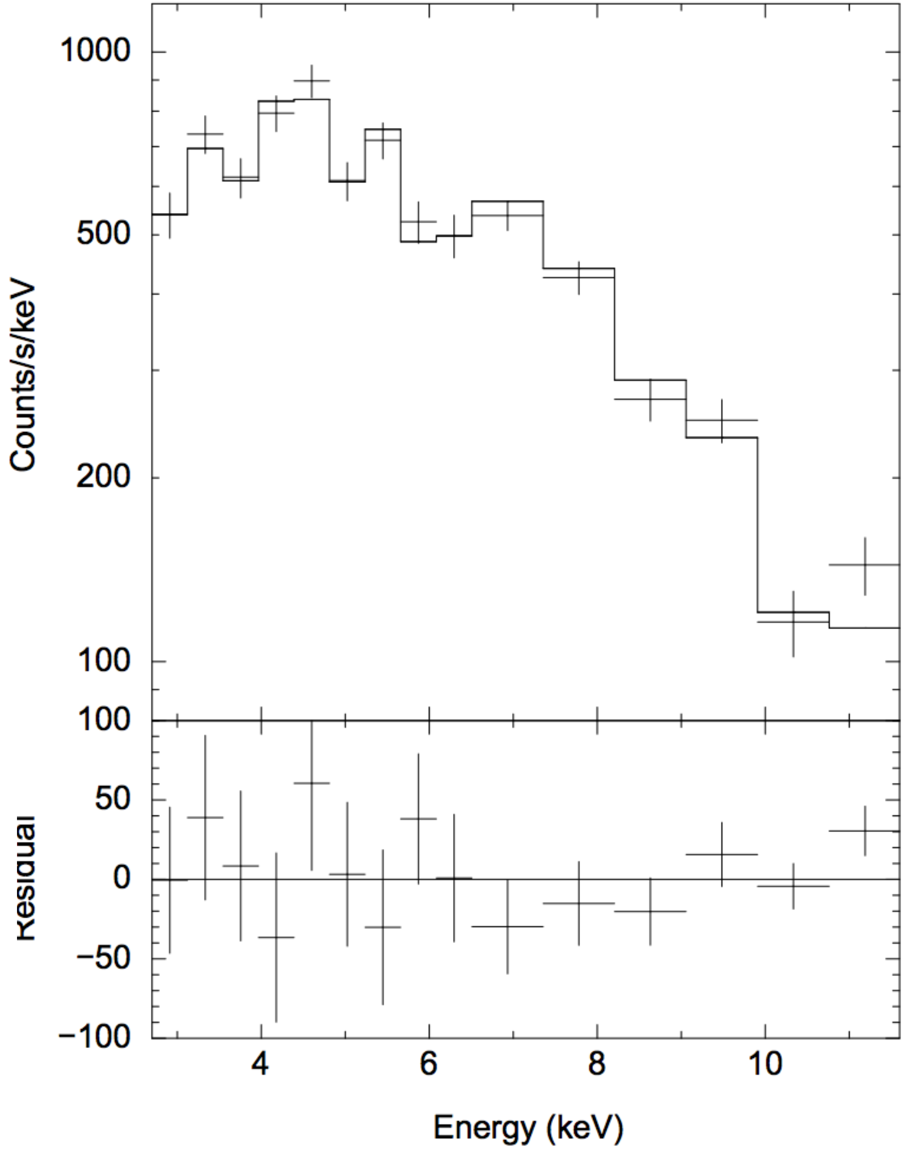

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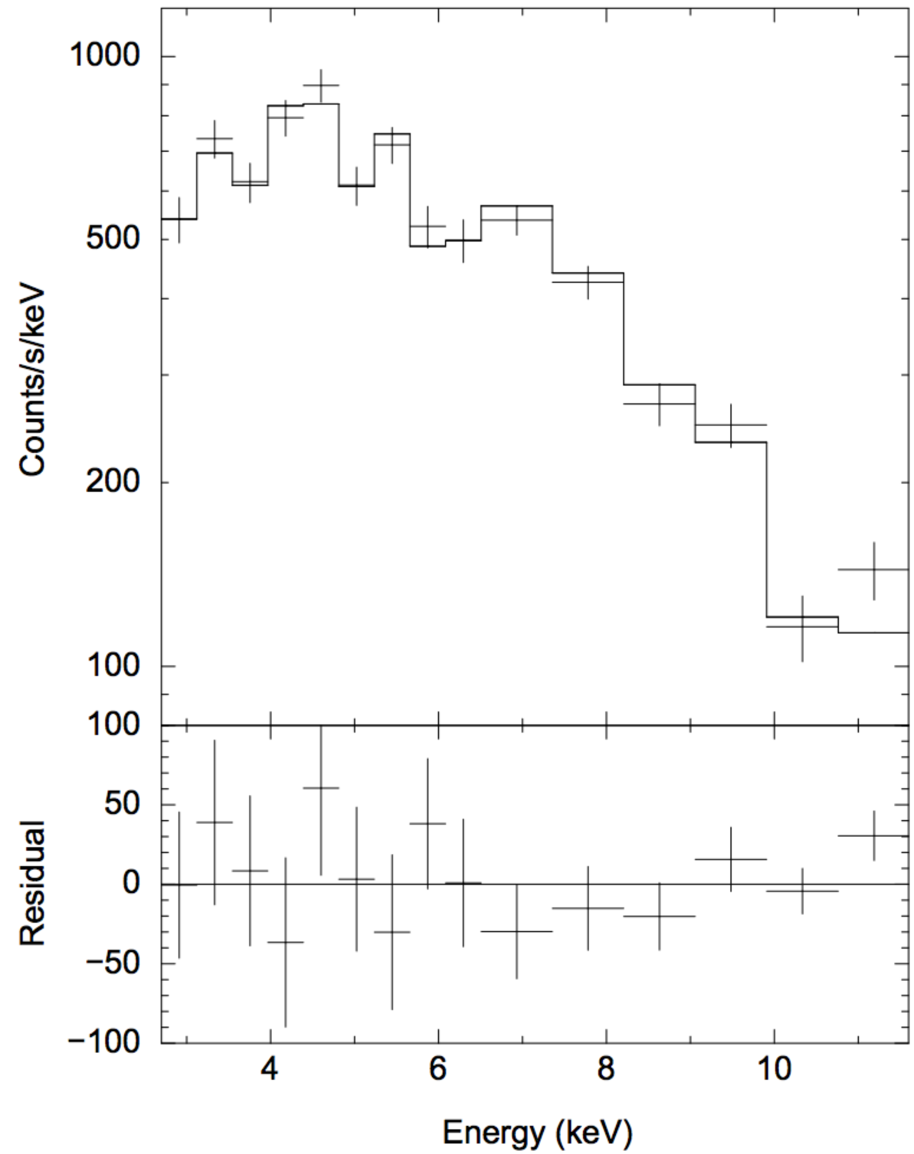
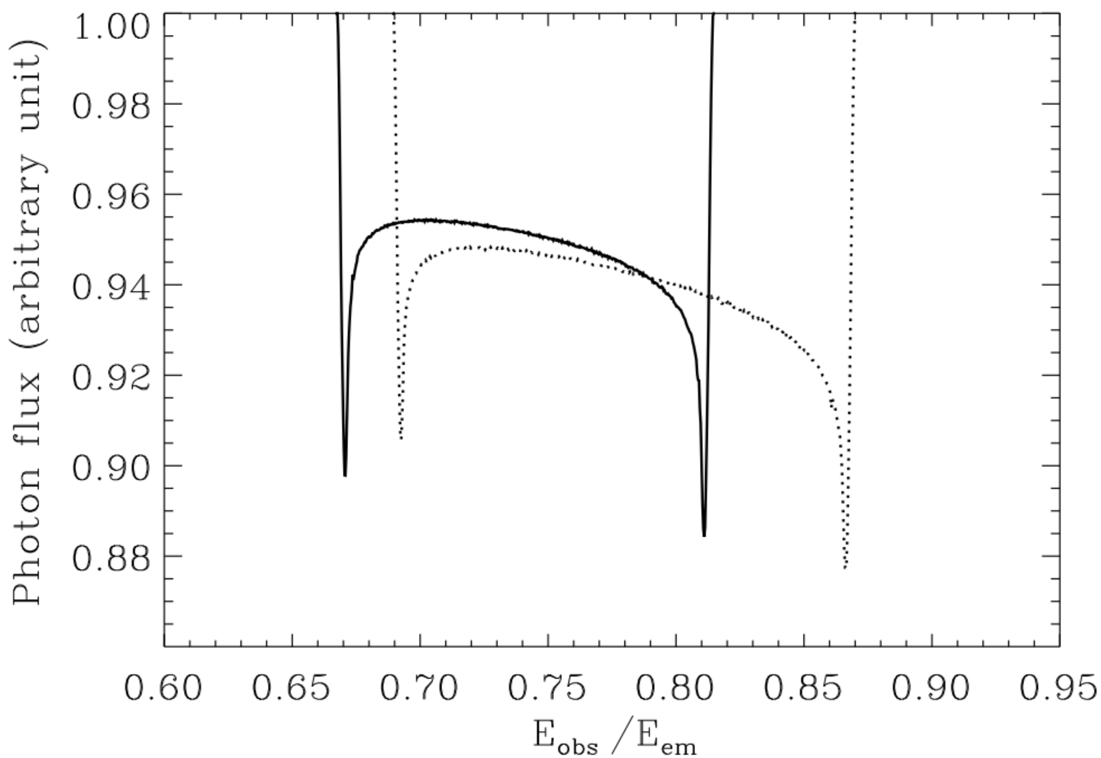


- Relativistic spectral line shifts



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Constraints on Neutron Star Radii or M/R

- Quasi-periodic oscillations in accretion disks around neutron stars

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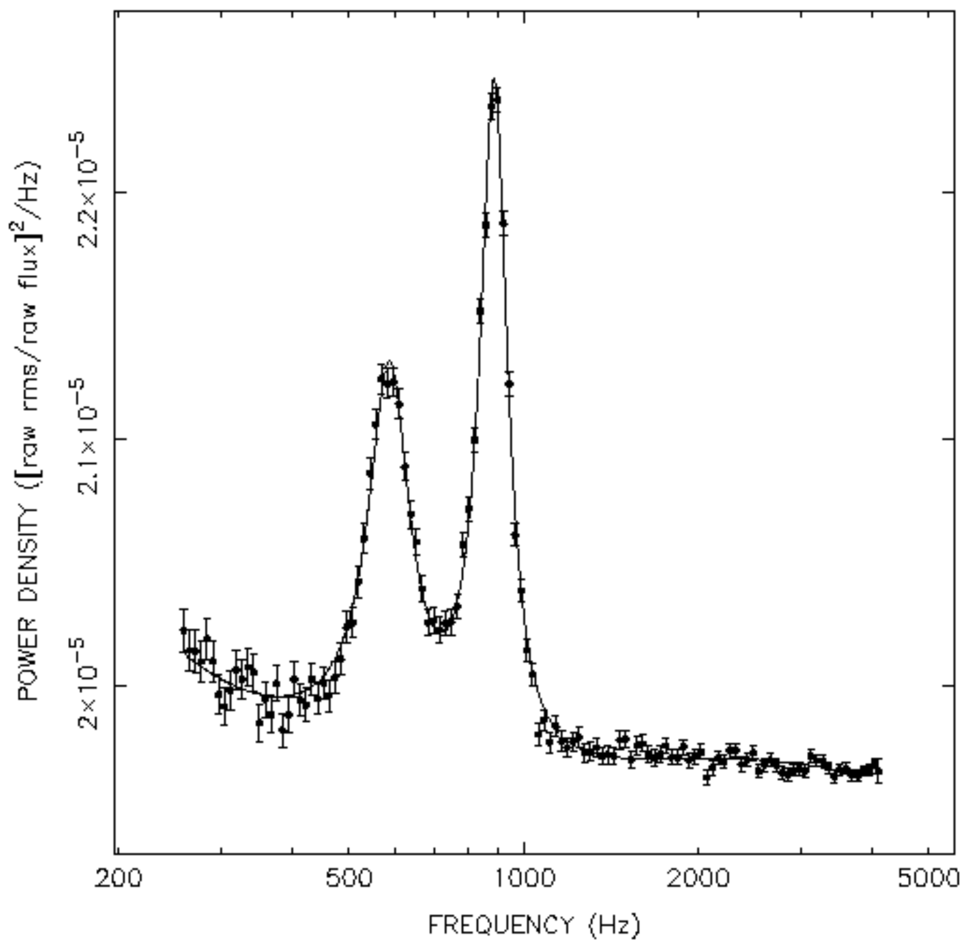

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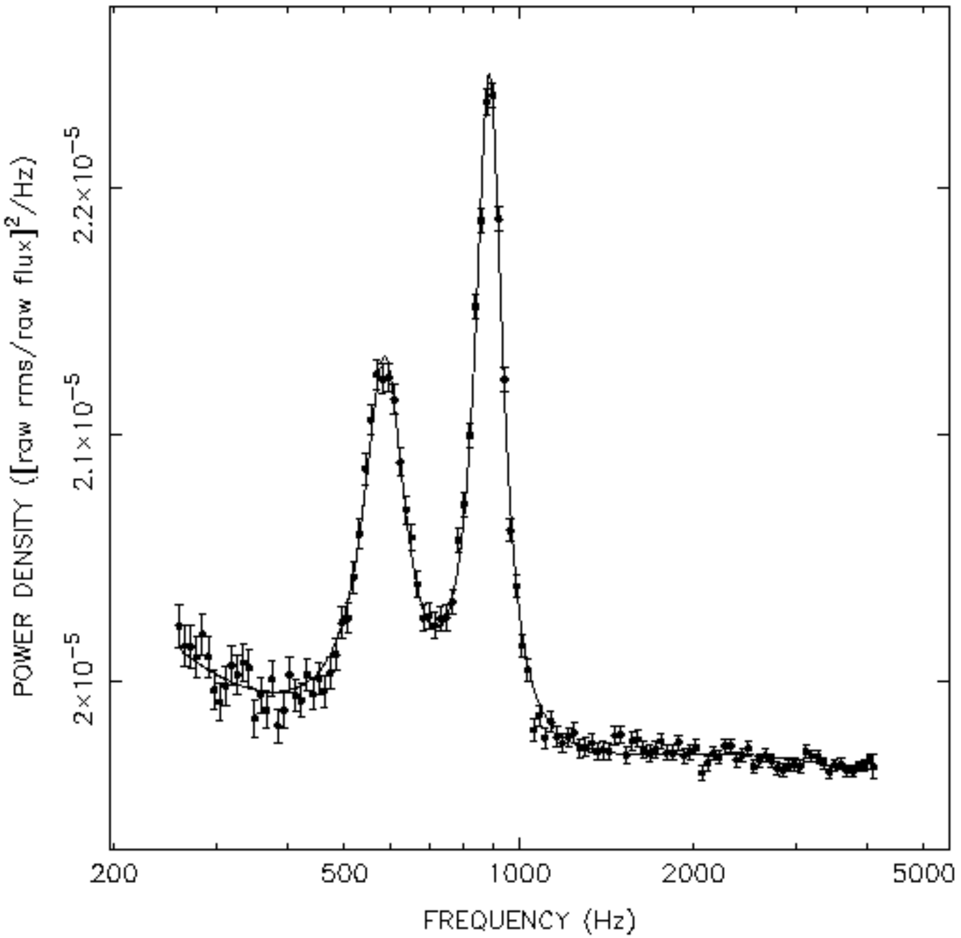
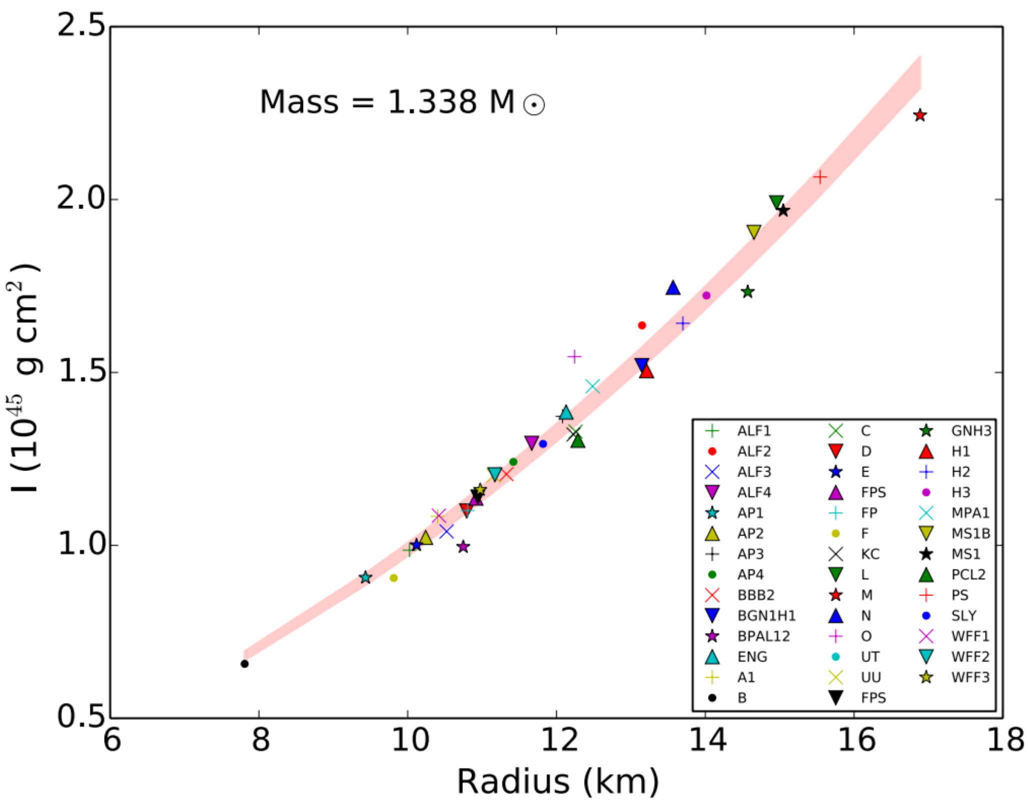


- Pericenter precession in relativistic binaries (*measuring moment of inertia in double pulsar J0737*)



(Raithel, Ozel, Psaltis 2016)

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Through *GRAVITATIONAL WAVES*:

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-> **Chatziioannou's** lecture)

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(promises radii with accuracy <4% in the future -> **Bauswein's** lecture)

Description of the fluid

For a perfect fluid, we define the following *intensive properties*, all measured by an observer *comoving* with the fluid:

n	*baryon number density*
ρ	*baryon mass density*
ε	*energy density*
p	*isotropic pressure*
h	*specific enthalpy*
s	*specific entropy*

where

$$h = \frac{\epsilon + p}{\rho}$$

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The *equation of state* (**EOS**) can be assumed to be of the form

$p=p(\rho)$, $\varepsilon=\varepsilon(\rho)$ *for barotropic (cold stars)*

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Note: if $s=s(\rho)$ the star is *pseudo-barotropic* and an equilibrium exists.

Governing Equations

- Field equations of general relativity (GR)

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

- Perfect fluid

$$T_{\alpha\beta} = (\epsilon + p)u_\alpha u_\beta + p g_{\alpha\beta}$$

- Conservation of stress-energy tensor

$$\nabla_\alpha T^{\alpha\beta} = 0$$

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- Conservation of stress-energy tensor

$$\nabla_\alpha T^{\alpha\beta} = 0$$

- Conservation of baryons

$$\nabla_\alpha (\rho u^a) = 0$$

- 1st law of thermodynamics

$$d\epsilon = \rho T ds + h d\rho$$

Structure of Nonrotating Neutron Stars

Metric tensor:

$$ds^2 = -e^{2\Phi} c^2 dt^2 + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

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- Static equilibrium (Tolman-Oppenheimer-Volkoff, TOV)

$$\frac{dm}{dr} = 4\pi r^2\epsilon/c^2$$

$$\frac{dp}{dr} = -\frac{G}{c^2}\frac{m\epsilon}{r^2}\left(1+\frac{p}{\epsilon}\right)\left(1+\frac{4\pi r^3 p}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1}$$

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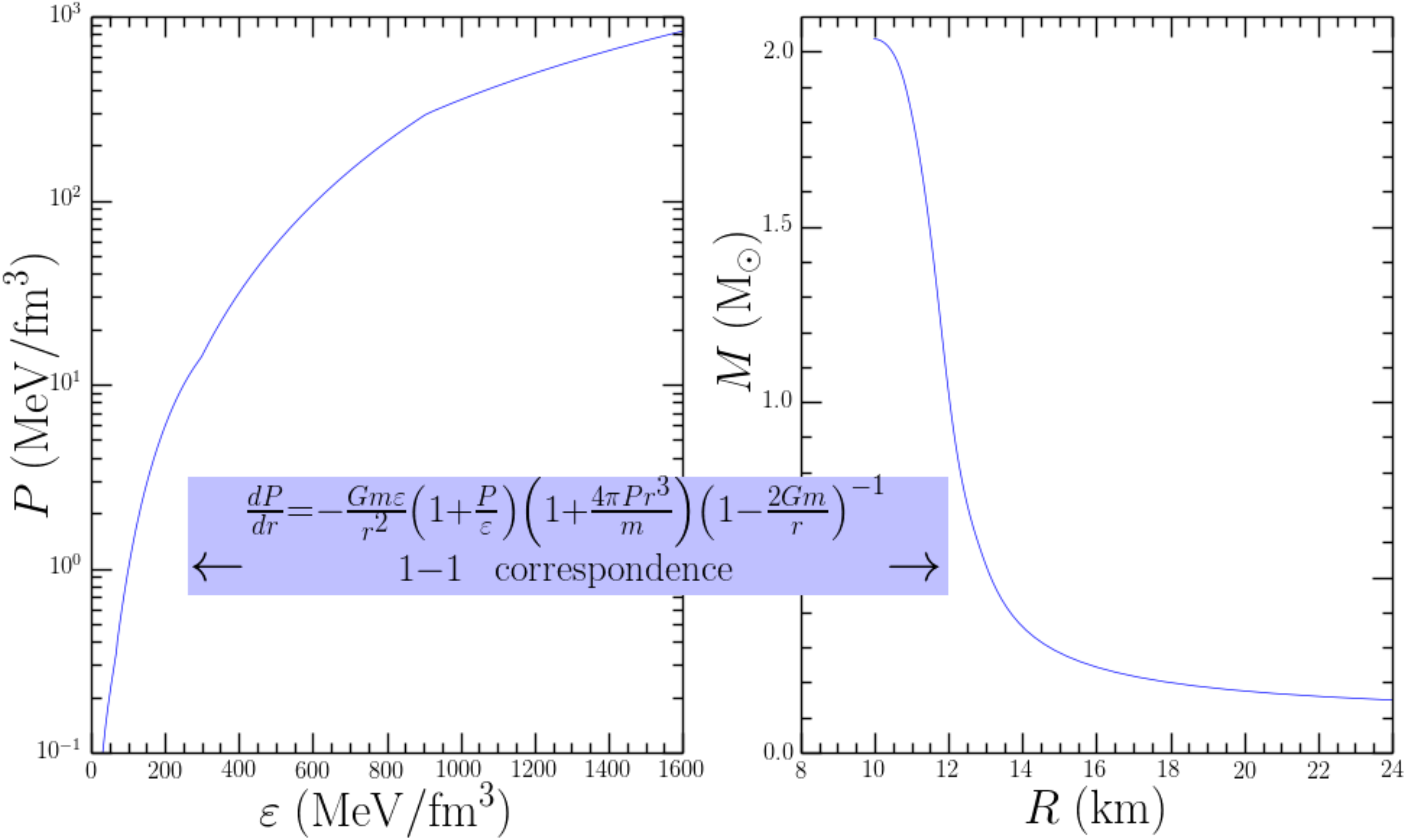
- Radius and gravitational mass

$$R = r(p = 0)$$

$$M = \int_0^R 4\pi r^2 \frac{\epsilon}{c^2} dr = m(r = R)$$

EOS and M-R Correspondence



(A. Steiner)

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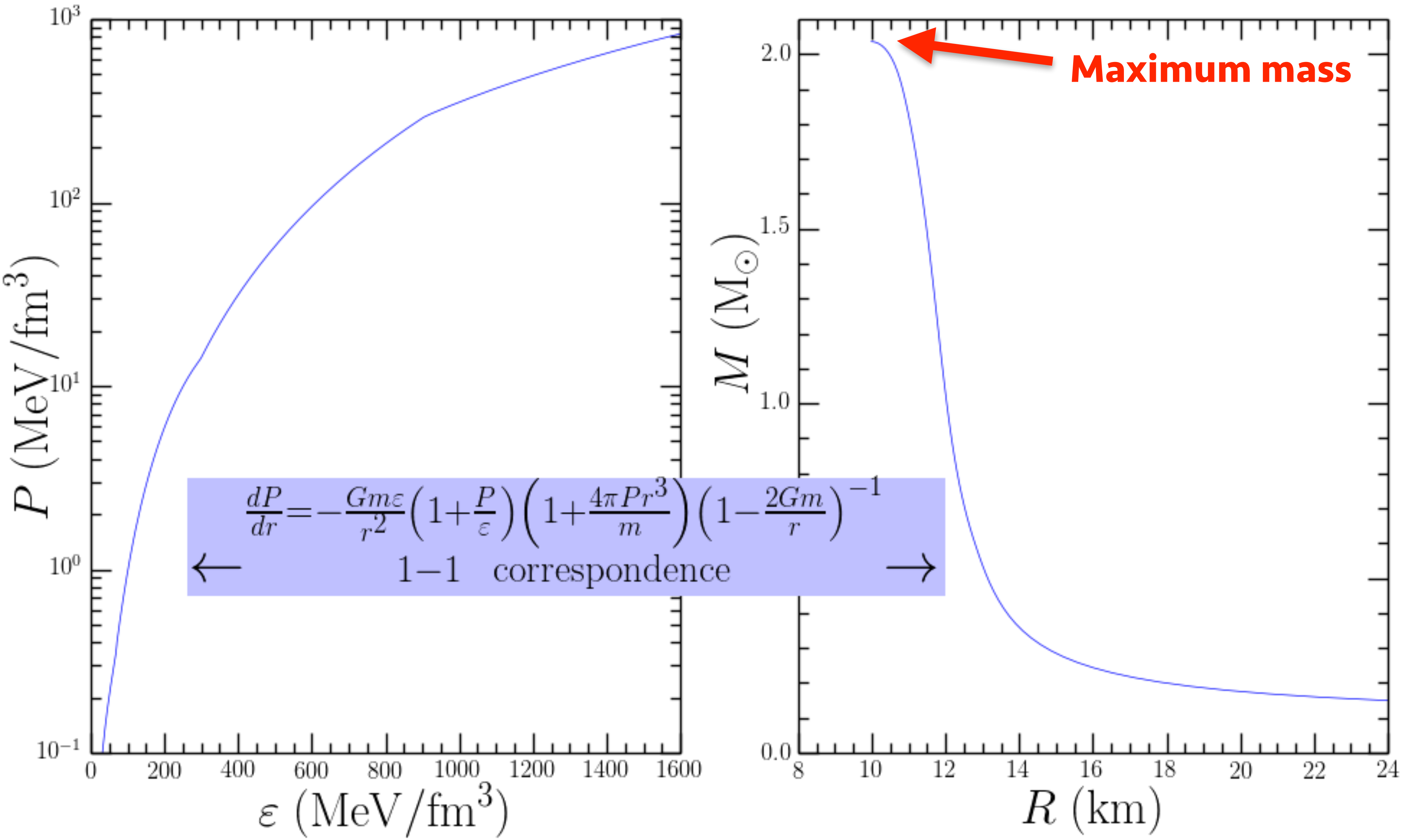
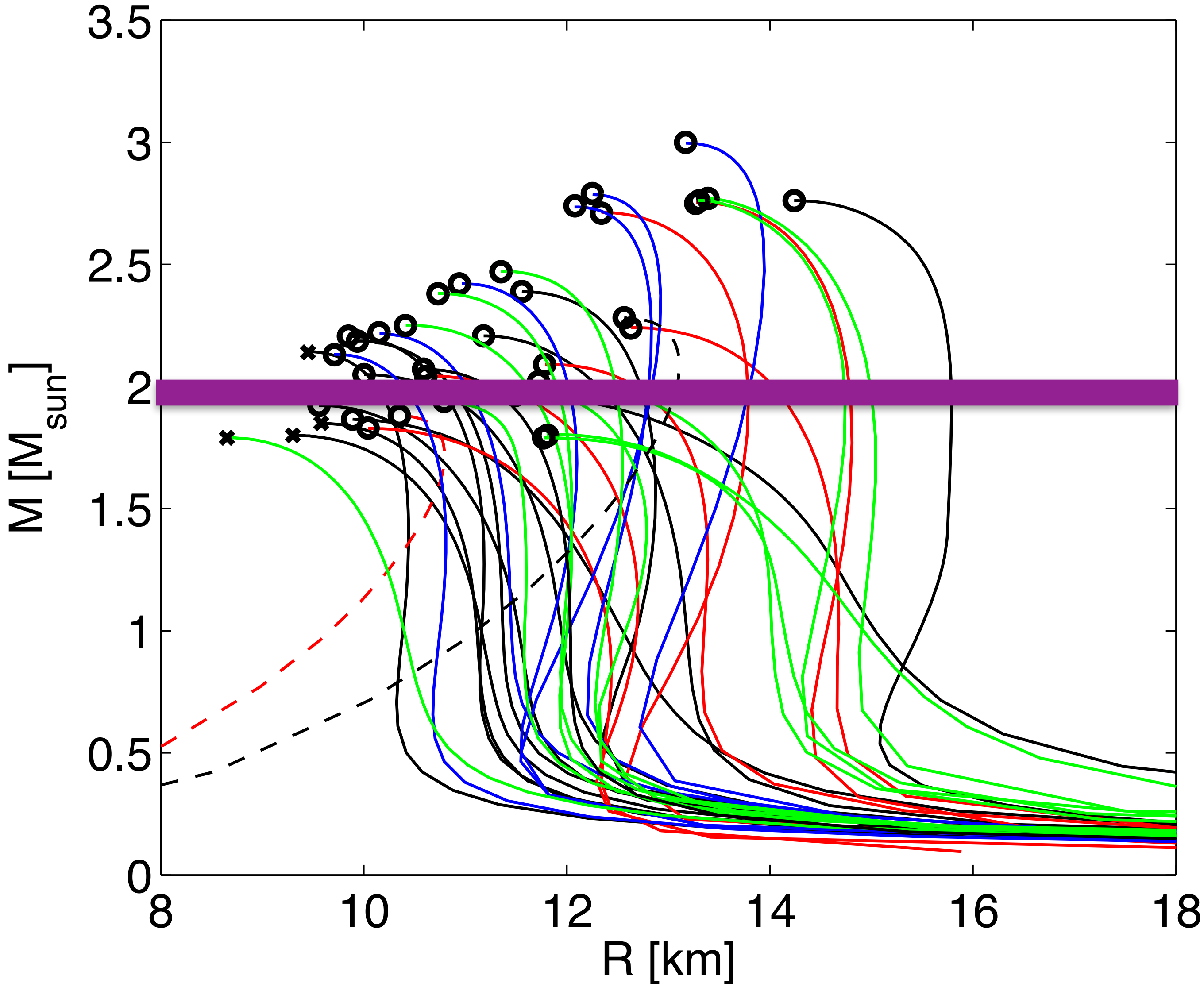


(A. Steiner)

Sample of M-R relations for different EOSs

A large number of theoretical extrapolations leads to very different mass-radius relations:



Hybrid Stars

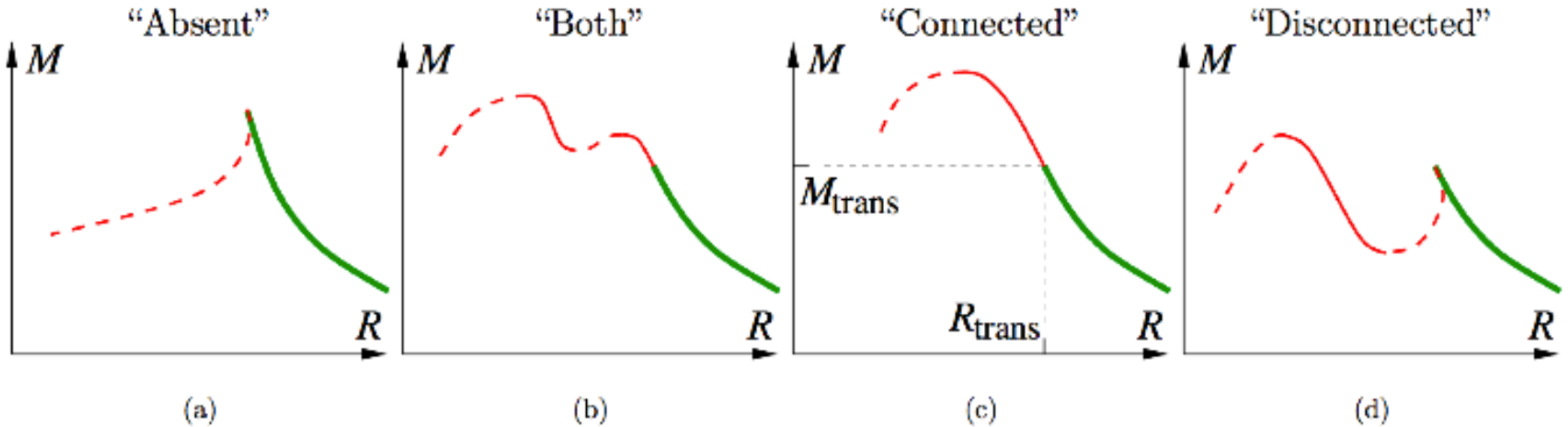


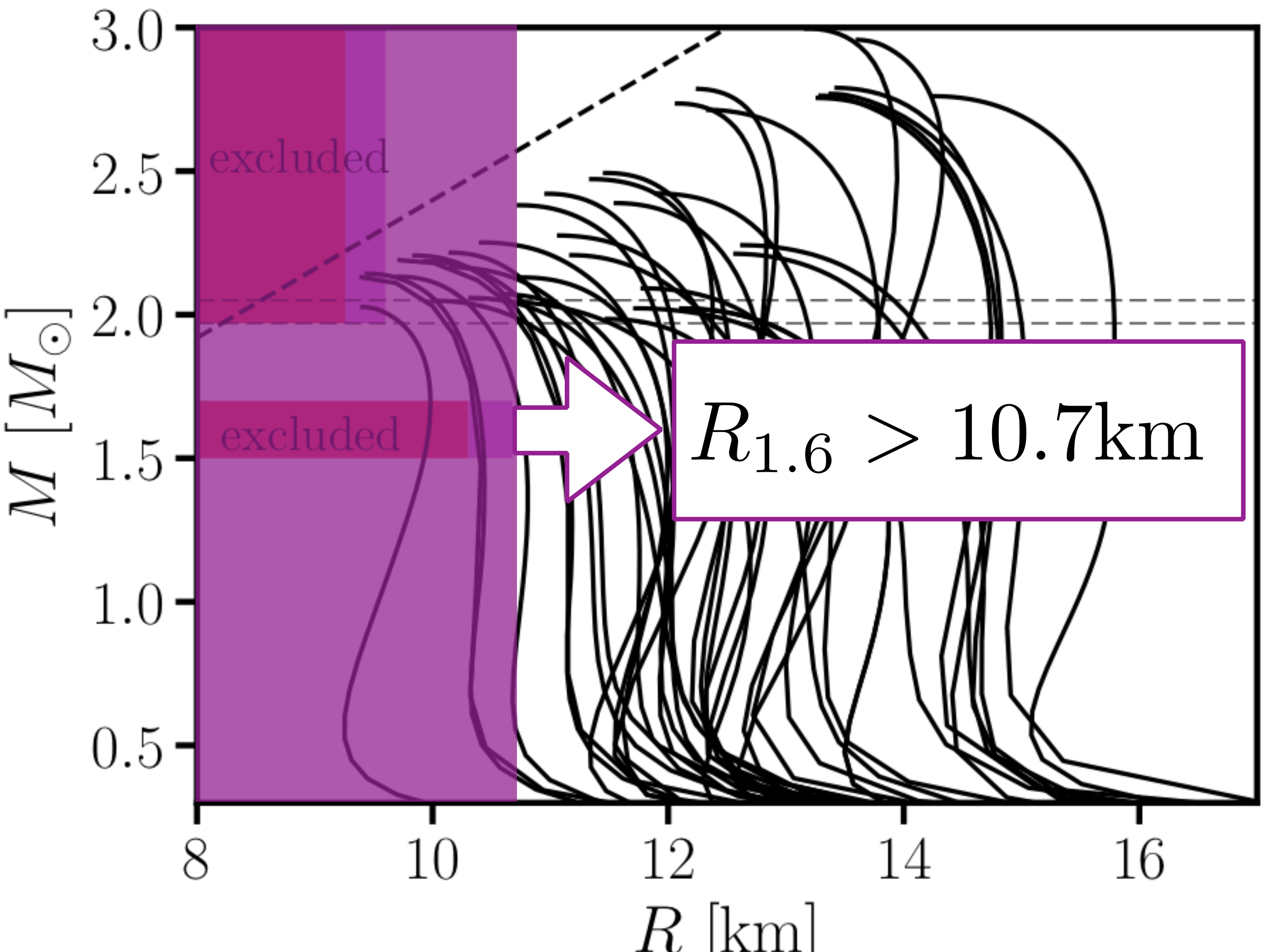
FIG. 2: Four possible topologies of the mass-radius relation for hybrid stars. The thick (green) line is the hadronic branch. Thin solid (red) lines are stable hybrid stars; thin dashed (red) lines are unstable hybrid stars. In (a) the hybrid branch is absent. In (c) there is a connected branch. In (d) there is a disconnected branch. In (b) there are both types of branch. In realistic neutron star $M(R)$ curves, the cusp that occurs in cases (a) and (d) is much smaller and harder to see [13, 14]

Alford, Han, and Prakash (2013)

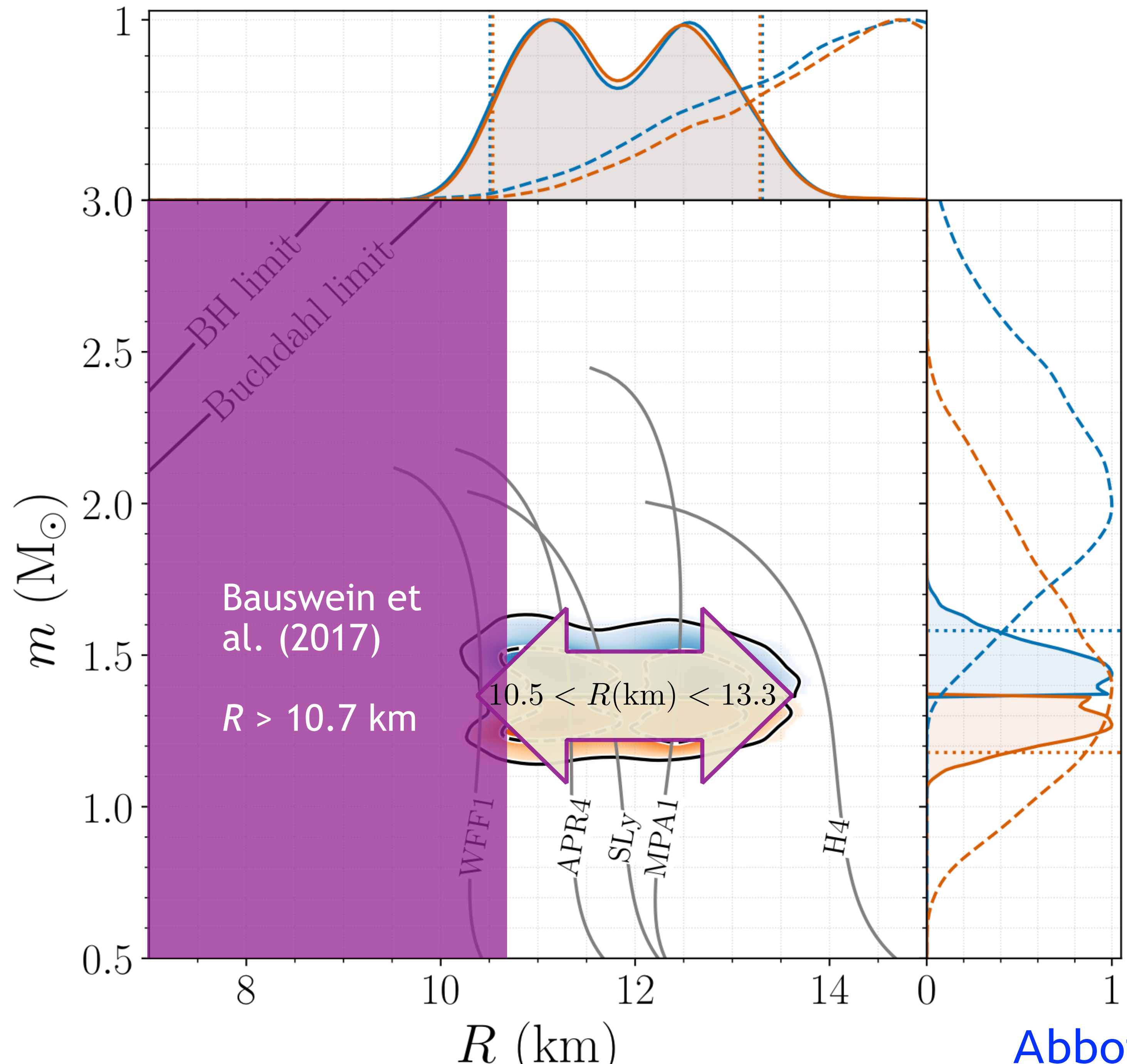
FIRST RADIUS CONSTRAINTS FROM GW's

Bauswein, Just, Janka & NS (2017)

Minimal assumptions: *no prompt collapse in GW170817 + causality*



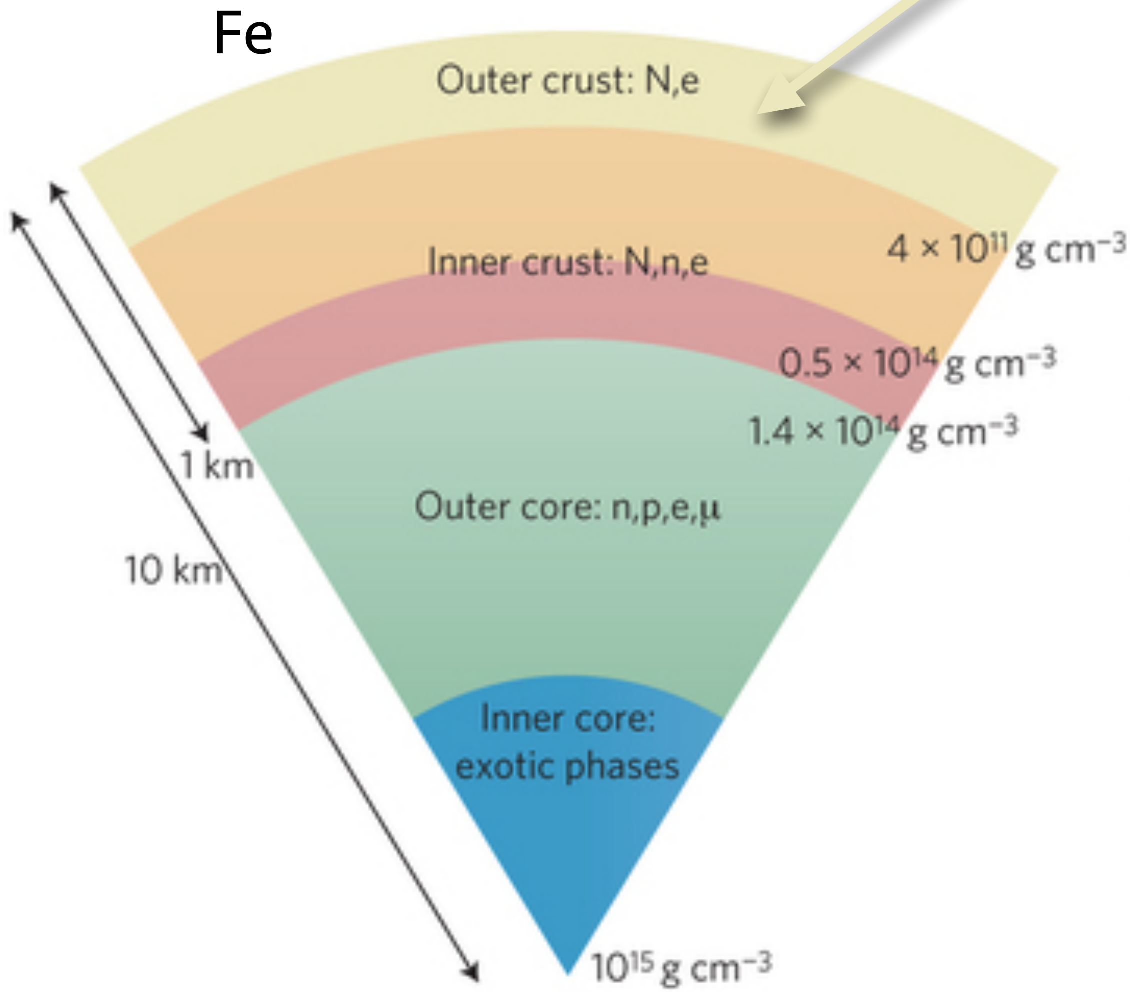
Constraints from the *inspiral* of GW170817



Abbot et al. (2018)

Interior Structure

Heavy nuclei in a solid state



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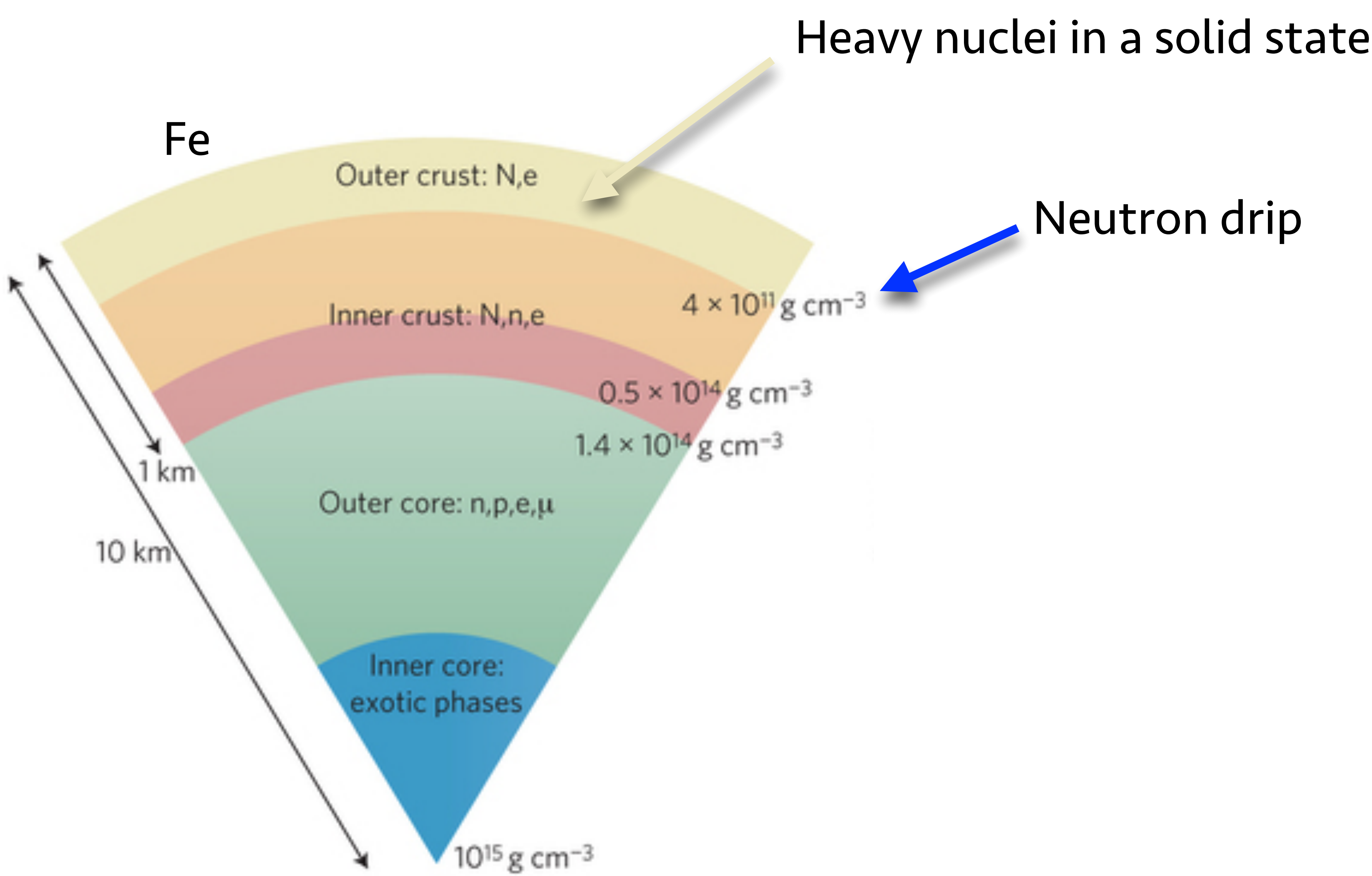

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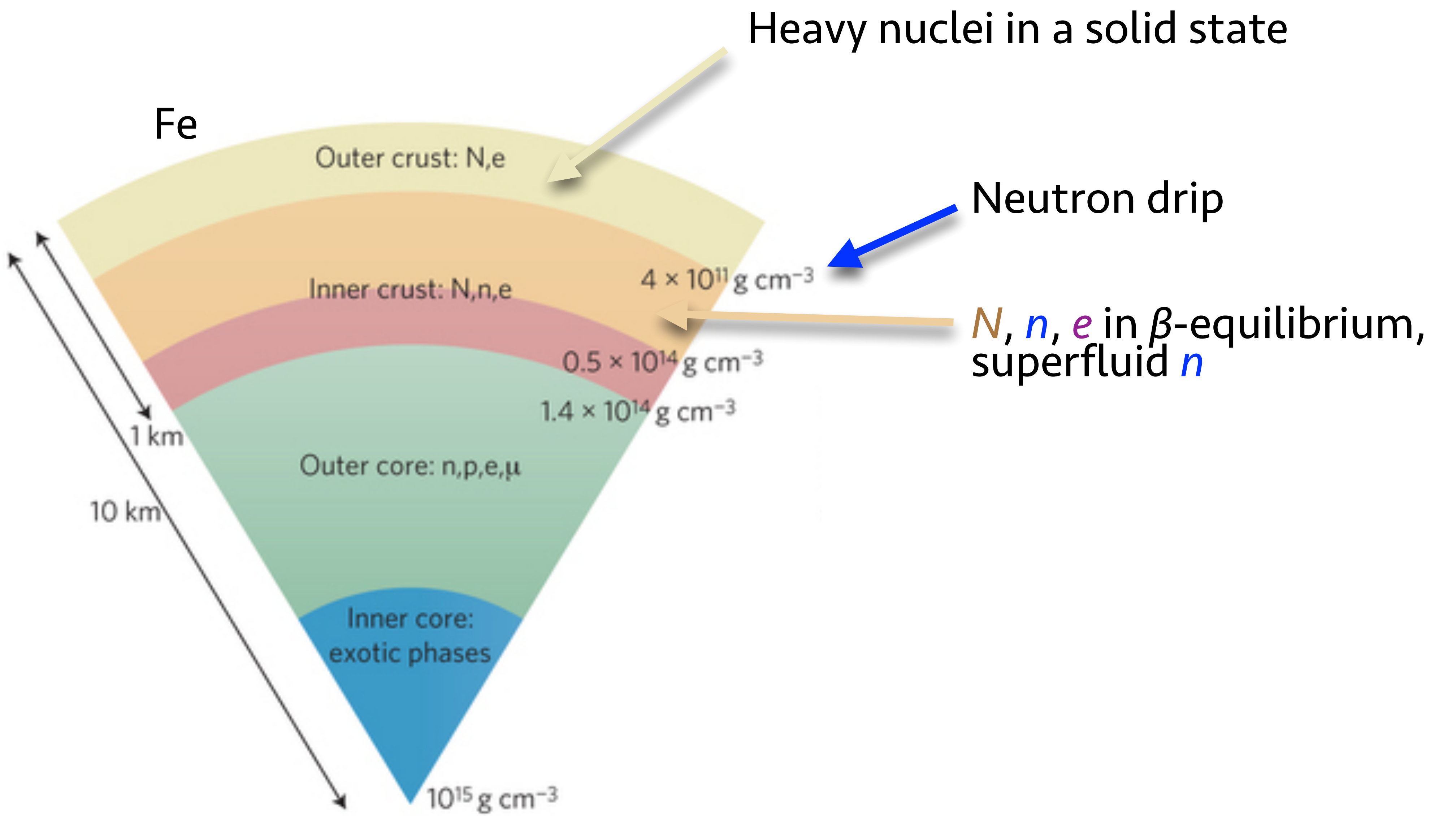

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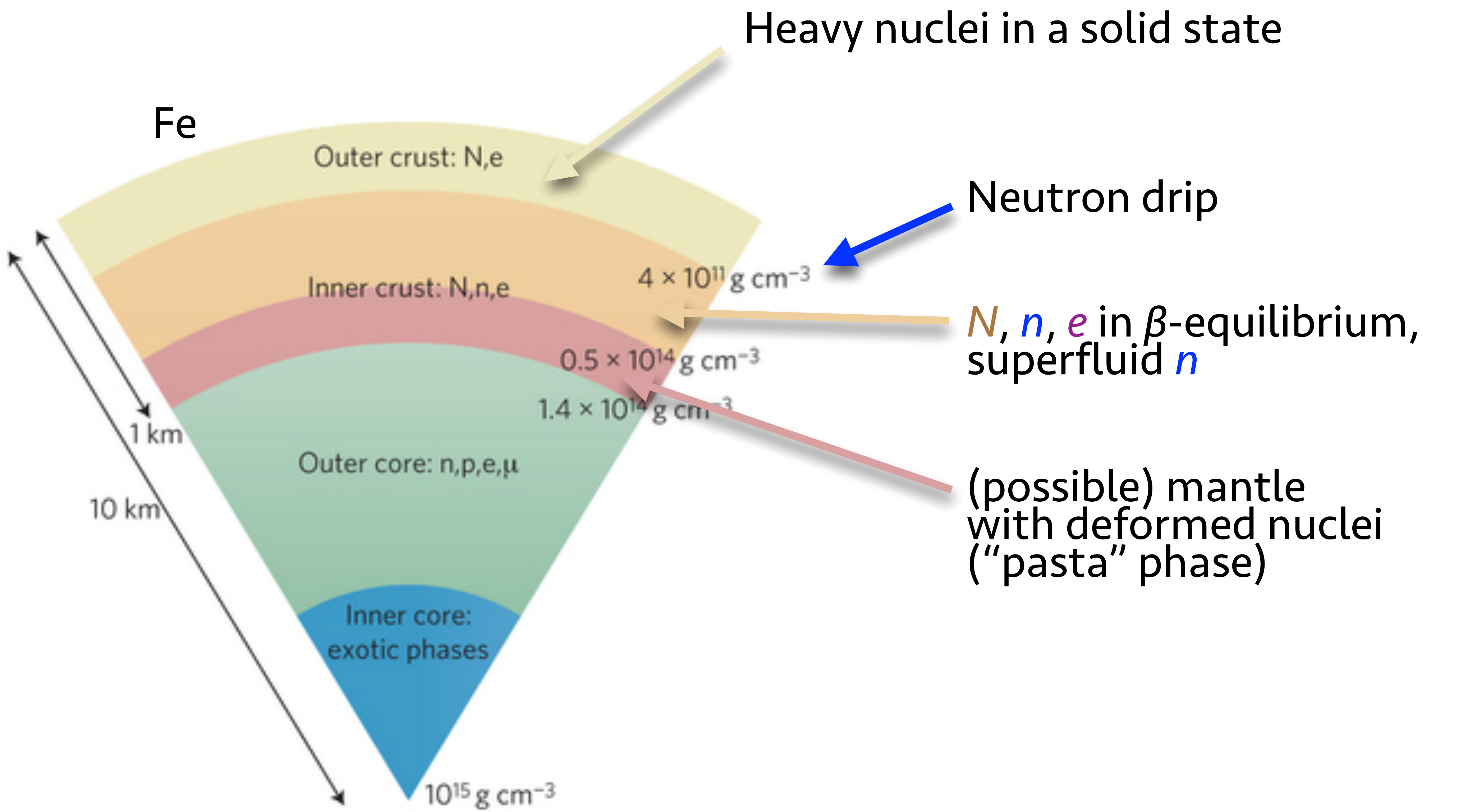

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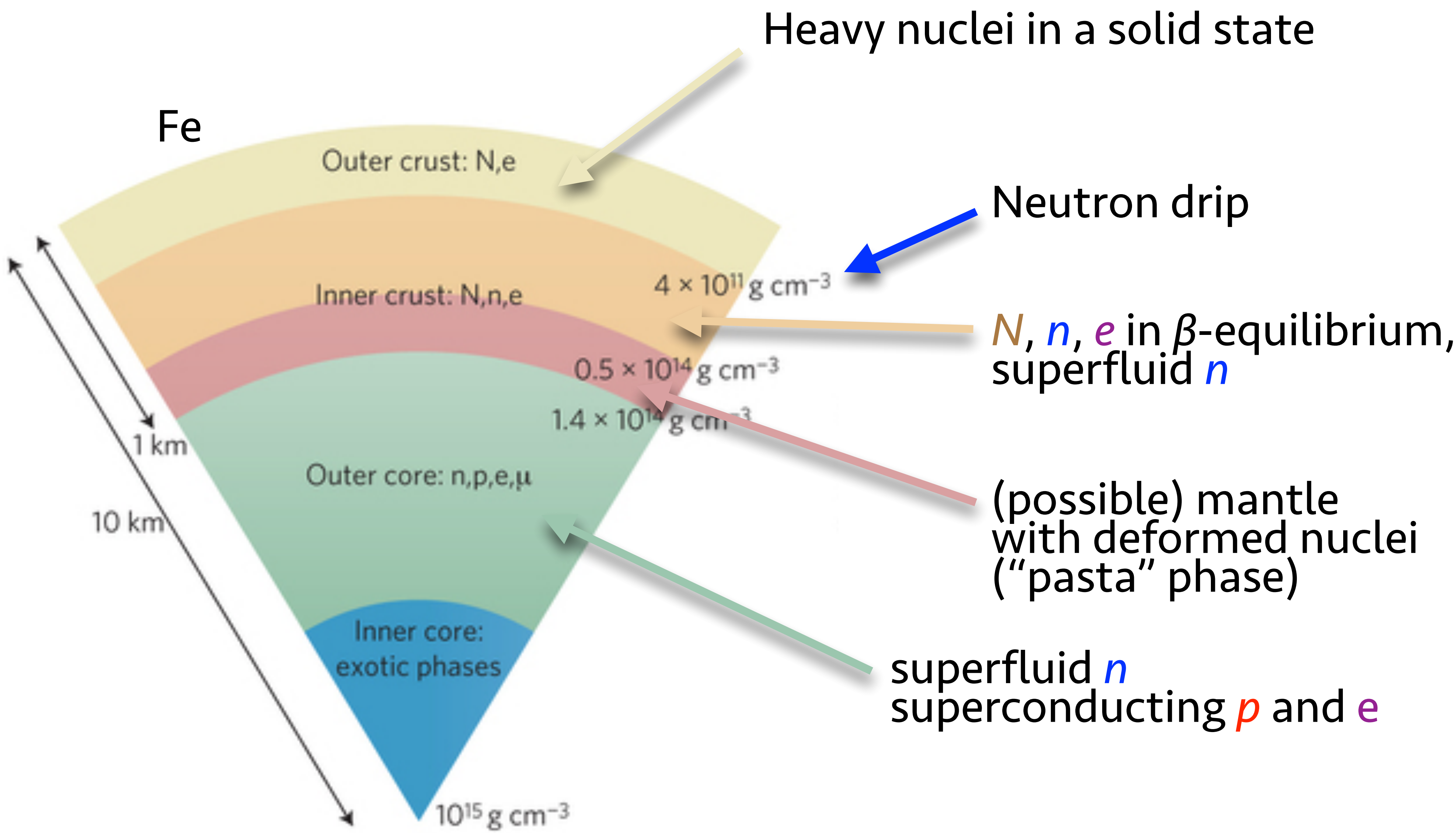

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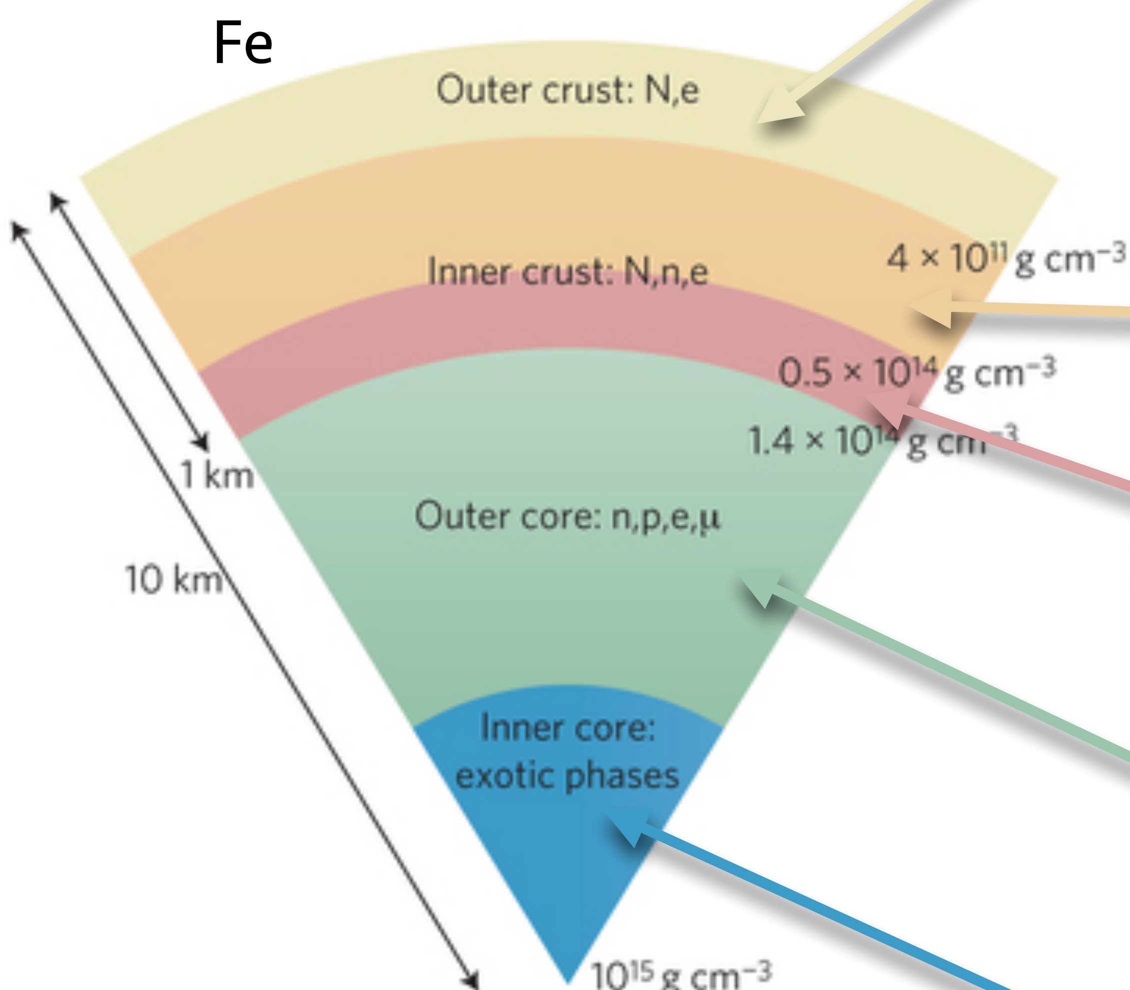


Heavy nuclei in a solid state

Neutron drip

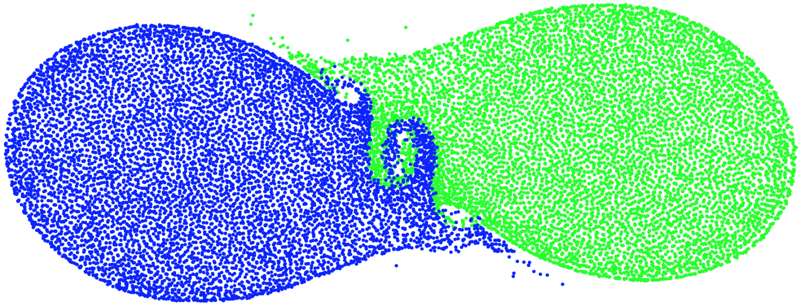
N, *n*, *e* in β-equilibrium,
superfluid *n*

(possible) mantle
with deformed nuclei
("pasta" phase)

superfluid n
superconducting p and e

ρ >> nuclear saturation density
(~2×10^{14} g/cm^3)

Outcome of Binary NS Mergers



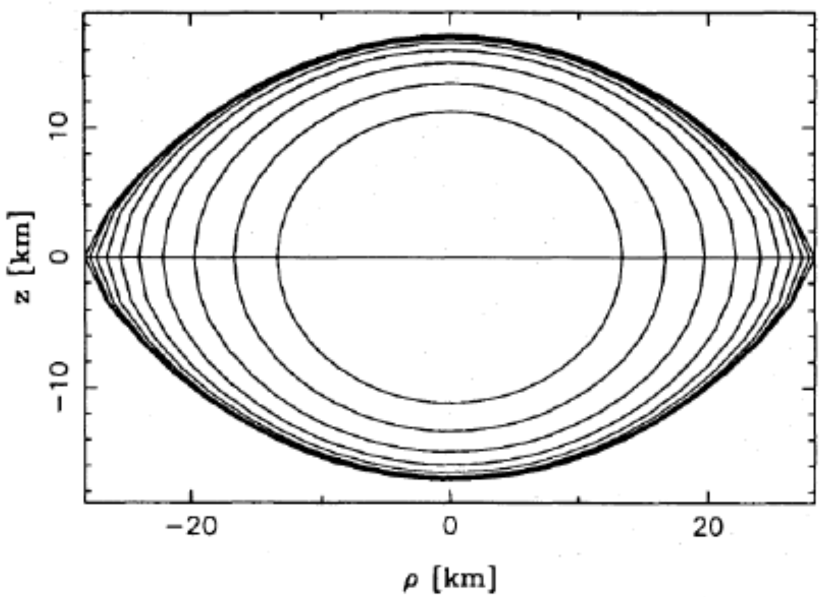
Most likely range of total mass for binary system:

$$2.4M_\odot < M_{\rm tot} < 3M_\odot$$

Because nonrotating $M_{\rm max} > 2M_\odot$ (as required by observations), a long-lived (τ >10ms) remnant is likely to be formed.

The remnant is a *hypermassive neutron star* (HMNS), supported by *differential rotation*, with a mass larger than the maximum mass allowed for uniform rotation.

Examples of Equilibrium Models



Bonazzola et al. 1993

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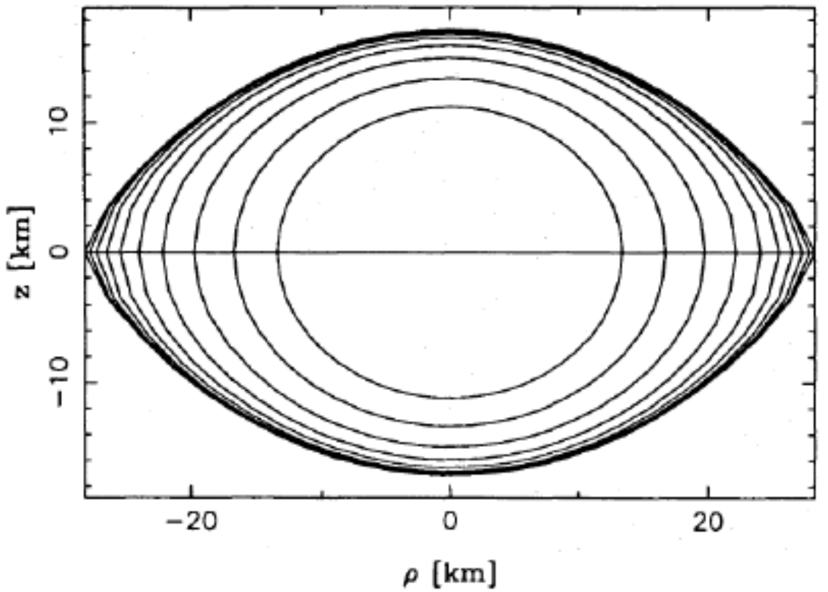

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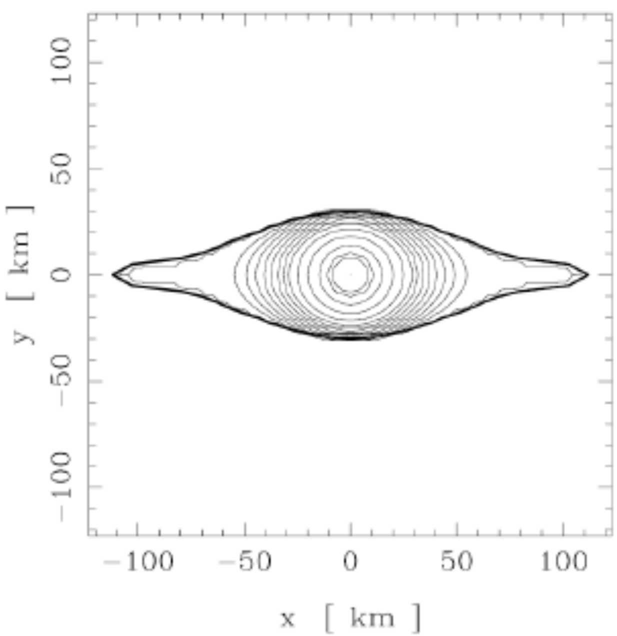

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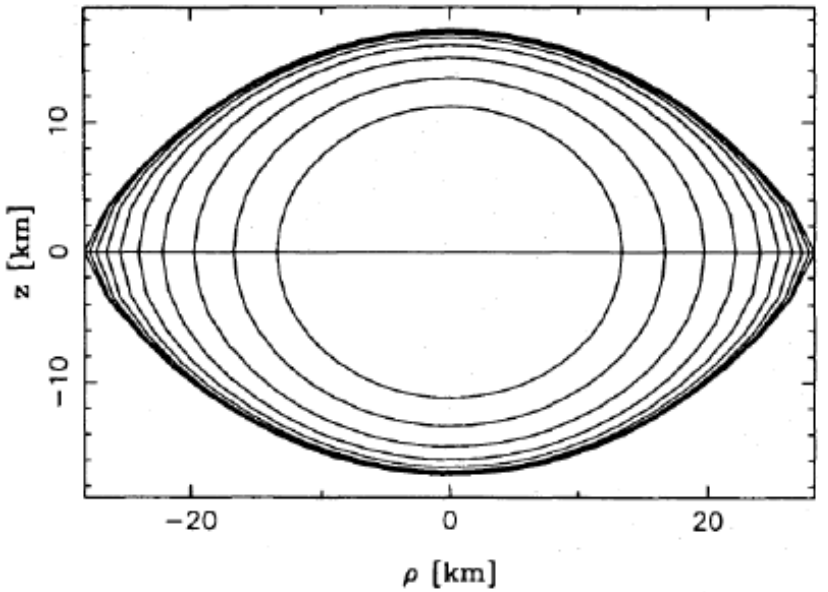

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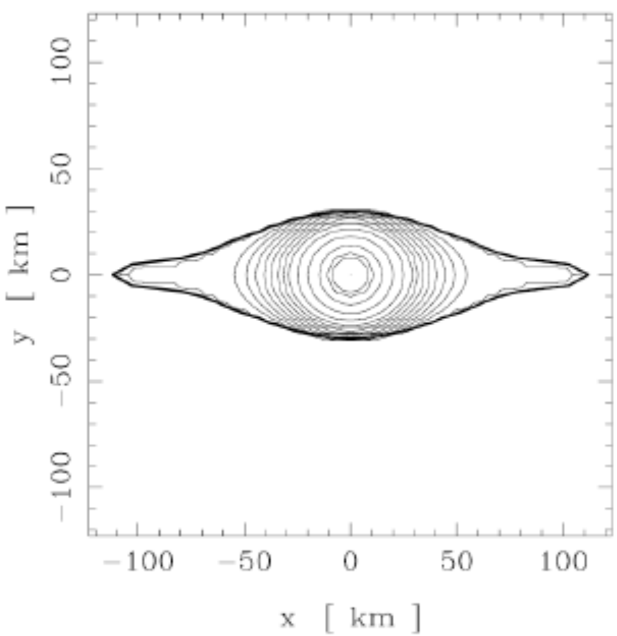

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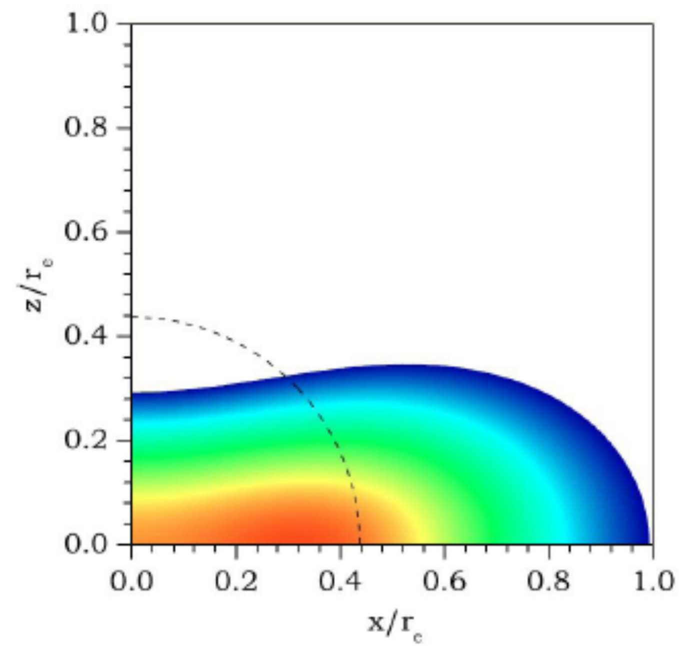

N. S., Apostolatos & Font, 2004

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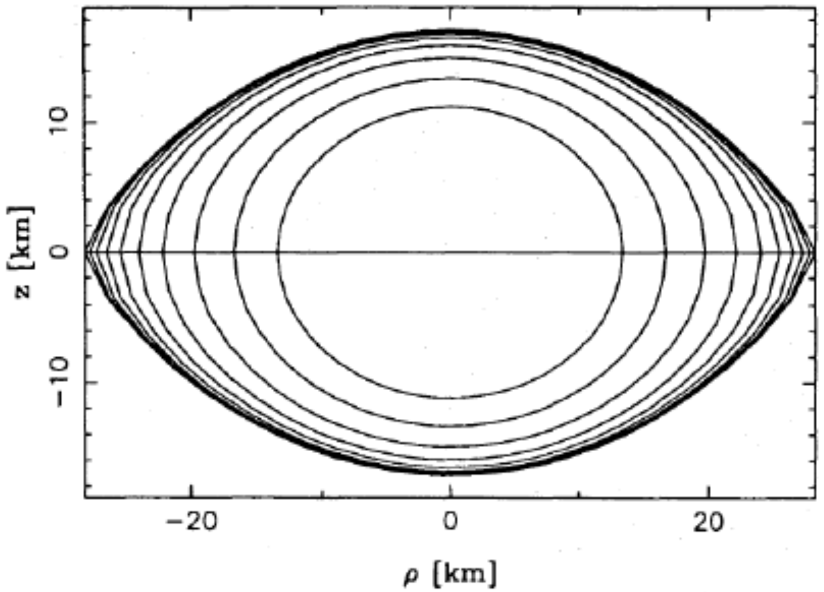

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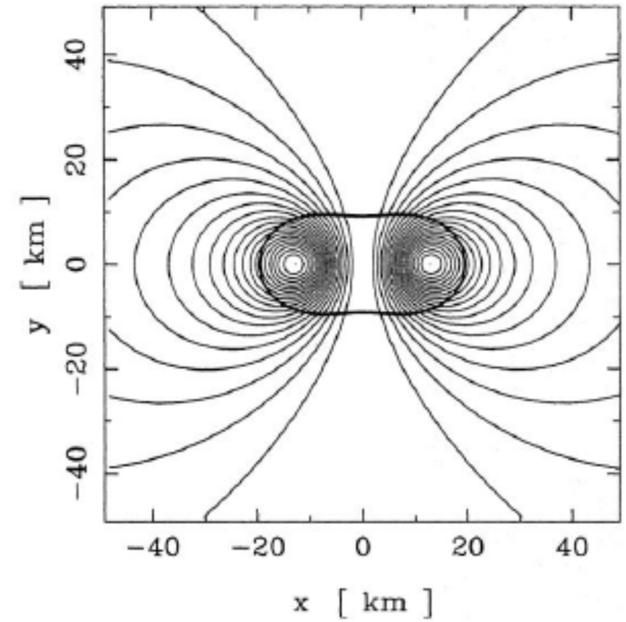
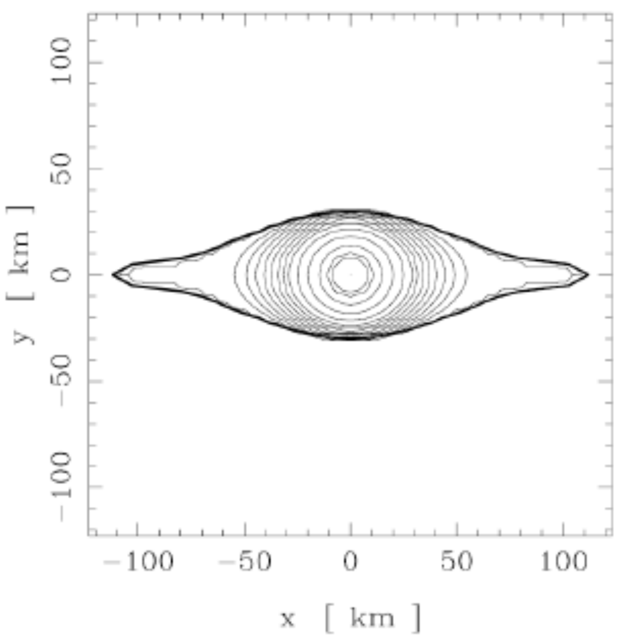


Bocquet et al. 1995



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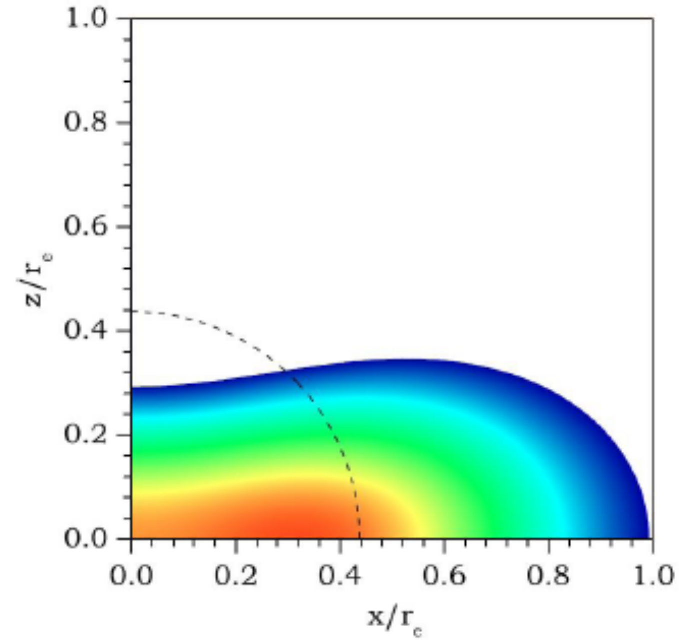

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Rotating Equilibria in General Relativity

Assumptions on spacetime:

1. Stationary: Killing vector t^α which is timelike at spatial infinity.

2. Axisymmetric: Killing vector ϕ^α which is *spacelike* everywhere, vanishes on a *symmetry axis* and whose orbits are *closed curves*.

3. Asymptotically flat: $t^\alpha t_\alpha = -1 \quad \phi^\alpha \phi_\alpha = 1 \qquad t^\alpha \phi_\alpha = 0$

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$\longrightarrow$ We can *choose* coordinates t and ϕ such that t^α and ϕ^α are *coordinate vectors*:

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4. Circularity (no meridional currents): t^α and ϕ^α are everywhere *orthogonal* to the 2-surfaces formed by the integral curves of the remaining two coordinates x^1 and x^2.

Quasi-Isotropic Coordinates

The nonzero components of the metric involving *t* and *φ* are written as *invariant combinations* of the Killing vectors:

$$
\begin{aligned}
g_{tt} &= t^\alpha t_\alpha = -e^{2\nu} + \omega^2 e^{2\psi} \\
g_{t\phi} &= t^\alpha \phi_\alpha = -\omega e^{2\psi} \\
g_{\phi\phi} &= \phi^\alpha \phi_\alpha = e^{2\psi}
\end{aligned}
$$

Then

$$
g_{\alpha\beta} = \begin{bmatrix} -e^{2\nu} + \omega^2 e^{2\psi} & 0 & 0 & -\omega e^{2\psi} \\ & g_{11} & g_{12} & 0 \\ \text{sym.} & & g_{22} & 0 \\ & & & e^{2\psi} \end{bmatrix}
$$

Quasi-Isotropic Coordinates

If one chooses *orthogonal coordinates* x^1 and x^2, then $g_{12}=0$.

For the remaining components, one can choose an *isotropic gauge,* in which the (x^1, x^2) sub-space is *conformally flat*:

e.g. *cylindrical-like* coordinates:

$$e^{2\mu}(d\varpi^2 + dz^2)$$

so that, finally:

$$\boxed{ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2)}$$

Rotation of the Fluid

The ratio of the two components of the 4-velocity

$$\Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{d\phi}{dt}$$

is the *angular velocity* as seen by a nonrotating observer at infinity.

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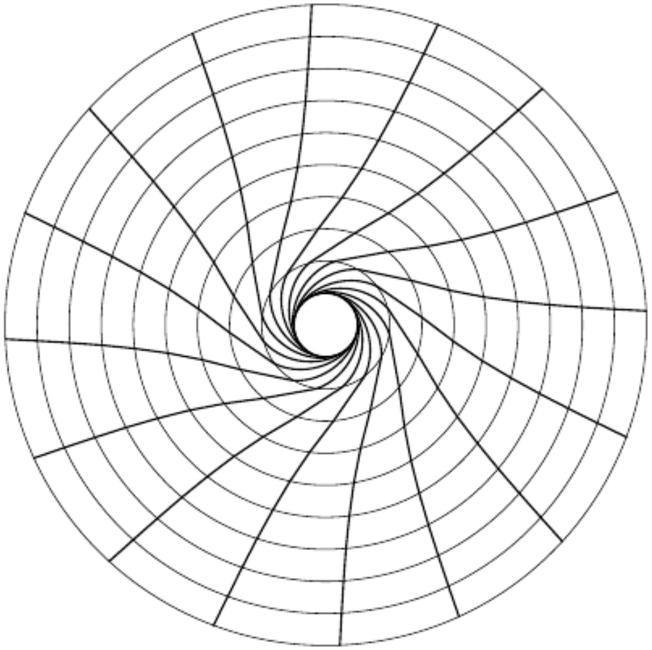
Thus, an observer with *zero angular momentum* $u_\varphi=0$ *rotates with nonzero angular velocity* $\Omega=\omega$ w.r.t. infinity.

This relativistic effect is called *dragging of inertial frames*.

Frame Dragging

Freely falling observers with conserved zero angular momentum u_φ=0 follow inwards-falling spiral paths.



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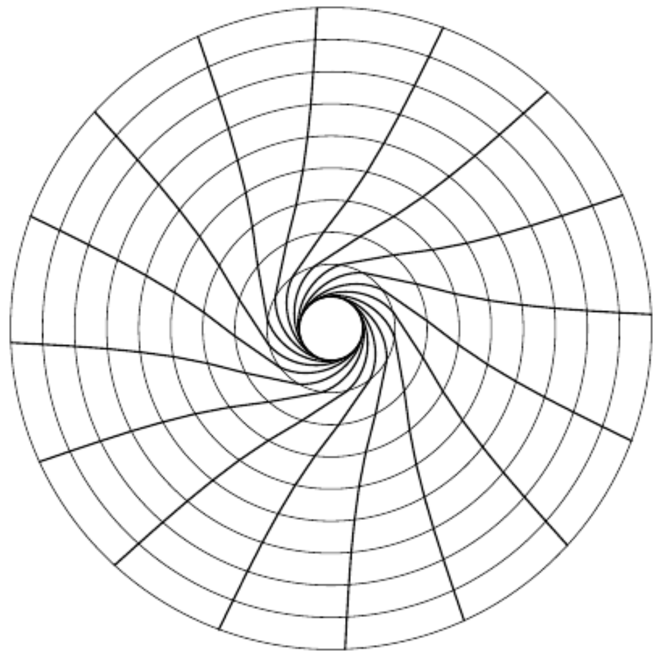


ZAMOs are *defined* as *local* observers with u_φ=0 with

4-velocity $$u^\alpha_{\rm ZAMO} = u^t(t^\alpha + \omega\phi^\alpha) = e^{-\nu}(t^\alpha + \omega\phi^\alpha)$$

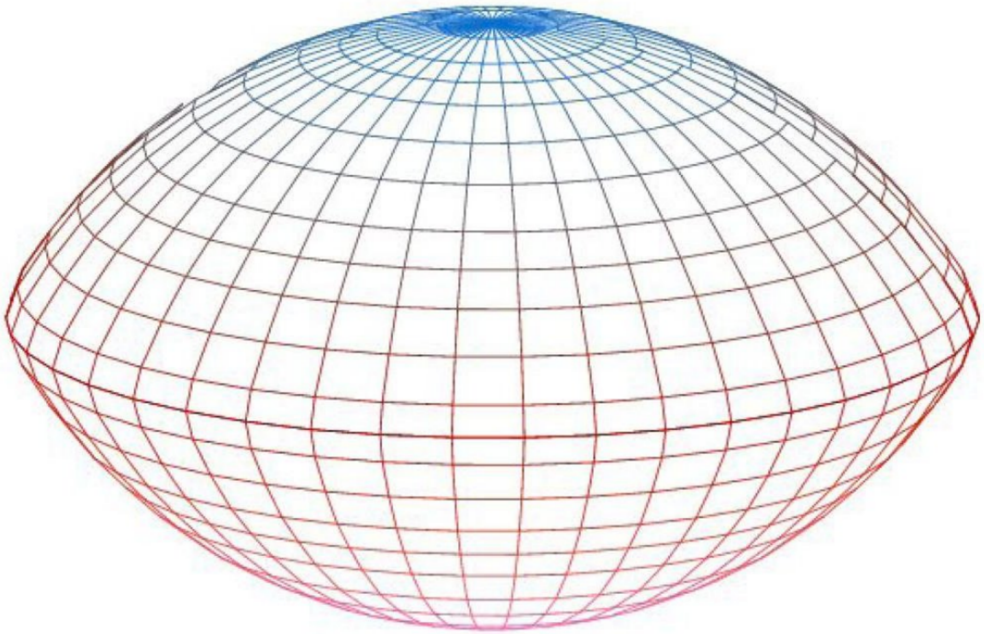
3-velocity $$v = 0$$

angular velocity $$\Omega = \omega$$

Radius of a Rotating Star

Consider the metric in quasi-isotropic coordinates

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2)$$



Q: What is the **radius** of the star?

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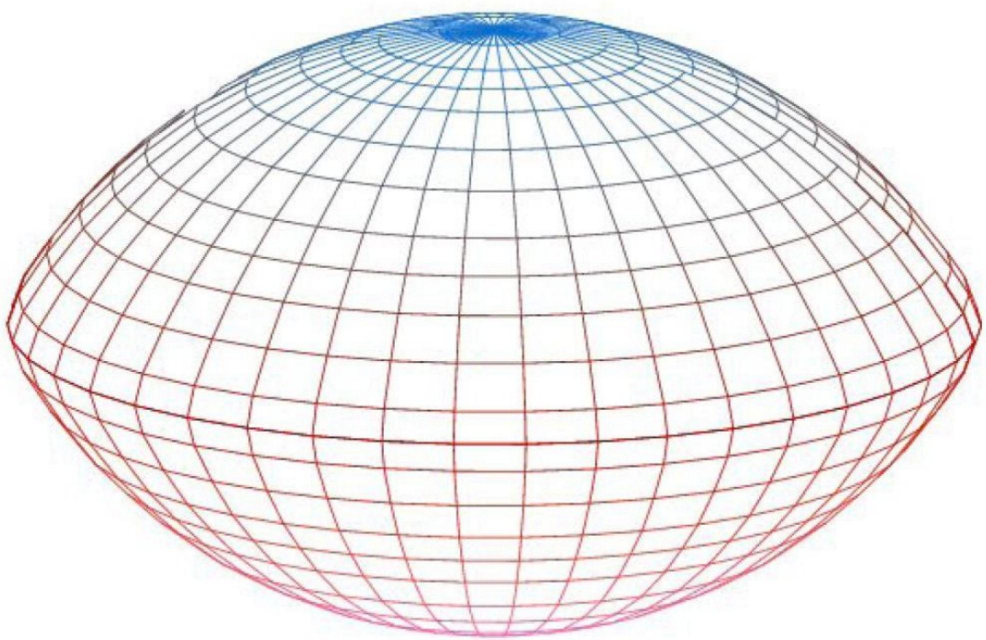
Consider the metric in quasi-isotropic coordinates

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2)$$



Q: What is the **radius** of the star?

A: At fixed t, ϖ, z integrate the proper length along the equator and divide by 2π.

Circumferential Radius

The proper circumference of a spatial circle at fixed t, ϖ, z is

$$\mathcal{C} = \oint ds = \int_0^{2\pi} \sqrt{g_{\phi\phi}} d\phi = 2\pi e^{\psi}$$

Thus, the *circumferential radius* is defined as

$$R := \mathcal{C}/(2\pi) = e^{\psi}$$

Summary, so far

The metric of a stationary, axisymmetric star with purely circular flow is:

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2)$$

where three metric functions have an invariant meaning:

$e^{-\nu}$ *(time dilation factor)*

e^{ψ} *(circumferential radius)*

ω *(dragging of inertial frames)*

while $e^{2\mu}$ is a conformal factor for the geometry of the (ϖ,z) 2-planes.

The angular velocity Ω is measured by an observer at infinity

$$\Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi}{dt}$$

while the 3-velocity v is measured by the local zero-angular-momentum observer (ZAMO).

Field Equations

The field equations are most easily derived in a ZAMO orthonormal tetrad, where the components of the stress-energy tensor are

$$T^{\hat{0}\hat{0}} = \frac{\epsilon + pv^2}{1 - v^2}, \qquad T^{\hat{0}\hat{1}} = \epsilon + p\frac{v}{1 - v^2},$$

$$T^{\hat{1}\hat{1}} = \frac{\epsilon v^2 + p}{1 - v^2}, \qquad T^{\hat{2}\hat{2}} = T^{\hat{3}\hat{3}} = p.$$

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Then the $R_{\hat{0}\hat{0}}, R_{\hat{0}\hat{3}},$ and $R_{\hat{0}\hat{0}} - R_{\hat{3}\hat{3}}$ components of the field equations become

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(B\boldsymbol{\nabla}\nu) &= \frac{1}{2}r^2\sin^2\theta B^3 e^{-4\nu}\boldsymbol{\nabla}\omega\cdot\boldsymbol{\nabla}\omega \\
&\quad +4\pi B e^{2\zeta-2\nu}\left[\frac{(\epsilon+p)(1+v^2)}{1-v^2}+2p\right], \\
\boldsymbol{\nabla}\cdot(r^2\sin^2\theta B^3 e^{-4\nu}\boldsymbol{\nabla}\omega) &= -16\pi r\sin\theta B^2 e^{2\zeta-4\nu}\frac{(\epsilon+p)v}{1-v^2}, \\
\boldsymbol{\nabla}\cdot(r\sin\theta\boldsymbol{\nabla}B) &= 16\pi r\sin\theta B e^{2\zeta-2\nu}p,
\end{aligned}$$

with $\zeta = \mu + \nu$ and ∇ is the *flat-space Laplacian.*

Relativistic Euler Equation

The projection the conservation of the stress energy tensor *normal* to the flow is:

$$\frac{\nabla_\alpha p}{(\epsilon + p)} = -u^\beta \nabla_\beta u_\alpha$$

which becomes:

$$\frac{\nabla p}{(\epsilon + p)} = \nabla \ln u^t - u^t u_\phi \nabla \Omega$$

First Integral

Define the log-enthalpy

$$H(p) := \int_0^p \frac{dp'}{\epsilon(p') + p'},$$

Then

$$\nabla(H - \ln u^t) = -F\nabla\Omega$$

where $F := u^t u_\phi$

A *first integral* exists when **either** Ω = const. or $F = F(\Omega)$

$$H - \ln u^t + \int_{\Omega_0}^{\Omega} F(\Omega')d\Omega' = \text{constant}$$

Numerical Method

Metric:

$$ds^2 = -e^{\gamma+\rho}dt^2 + e^{\gamma-\rho}r^2\sin^2\theta(d\phi - \omega dt)^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2)$$

Field Equations:
integral form

$$\Delta[\rho e^{\gamma/2}] = S_\rho(r,\mu),$$

$$\left(\Delta + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\mu\frac{\partial}{\partial \mu}\right)\gamma e^{\gamma/2} = S_\gamma(r,\mu),$$

$$\left(\Delta + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\mu\frac{\partial}{\partial \mu}\right)\omega e^{(\gamma-2\rho)/2} = S_\omega(r,\mu),$$

$$\rho = -\frac{1}{4\pi}e^{-\gamma/2}\int_0^\infty dr'\int_{-1}^{1} d\mu'\int_0^{2\pi} d\phi' r'^2 S_\rho(r',\mu')\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

$$r\sin\theta\,\gamma = \frac{1}{2\pi}e^{-\gamma/2}\int_0^\infty dr'\int_0^{2\pi} d\theta' r'^2\sin\theta' S_\gamma(r',\theta')\log|\mathbf{r}-\mathbf{r}'|,$$

$$r\sin\theta\cos\phi\,\omega = -\frac{1}{4\pi}e^{(2\rho-\gamma)/2}\int_0^\infty dr'\int_0^{\pi} d\theta'\int_0^{2\pi} d\phi' r'^3\sin^2\theta'\cos\phi' S_\omega(r',\theta')\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

RNS code, NS & Friedman

Komatsu, Eriguchi & Hachisu (1989) method

Cook, Shapiro & Teukolsky (1994)
compactified radial coordinate

$$\begin{aligned}
\alpha_{,\mu} = &-\nu_{,\mu} - \{(1-\mu^2)(1+rB^{-1}B_{,r})^2 + [\mu-(1-\mu^2)B^{-1}B_{,\mu}]^2\}^{-1} \\
&[\![\tfrac{1}{2}B^{-1}\{r^2B_{,rr} - [(1-\mu^2)B_{,\mu}]_{,\mu} - 2\mu B_{,\mu}\}[-\mu+(1-\mu^2)B^{-1}B_{,\mu}] \\
&+ rB^{-1}B_{,r}[\tfrac{1}{2}\mu + \mu rB^{-1}B_{,r} + \tfrac{1}{2}(1-\mu^2)B^{-1}B_{,\mu}] \\
&+ \tfrac{3}{2}B^{-1}B_{,\mu}[-\mu^2 + \mu(1-\mu^2)B^{-1}B_{,\mu}] - (1-\mu^2)rB^{-1}B_{,\mu r}(1+rB^{-1}B_{,r}) \\
&- \mu r^2\nu_{,r}^2 - 2(1-\mu^2)r\nu_{,\mu}\nu_{,r} + \mu(1-\mu^2)\nu_{,\mu}^2 - 2(1-\mu^2)r^2B^{-1}B_{,r} \\
&\times \nu_{,\mu}\nu_{,r} + (1-\mu^2)B^{-1}B_{,\mu}[r^2\nu_{,r}^2 - (1-\mu^2)\nu_{,\mu}^2] + (1-\mu^2)B^2e^{-4\nu} \\
&\times [\tfrac{1}{4}\mu r^4\omega_{,r}^2 + \tfrac{1}{2}(1-\mu^2)r^3\omega_{,\mu}\omega_{,r} - \tfrac{1}{4}\mu(1-\mu^2)r^2\omega_{,\mu}^2 + \tfrac{1}{2}(1-\mu^2) \\
&\times r^4B^{-1}B_{,r}\omega_{,\mu}\omega_{,r} - \tfrac{1}{4}(1-\mu^2)r^2B^{-1}B_{,\mu}[r^2\omega_{,r}^2 - (1-\mu^2)\omega_{,\mu}^2]]\}]\!],
\end{aligned}$$

Differential Rotation

For polytropic EOS the specific angular momentum measured by proper time of matter is

$$\boxed{j \equiv u^t u_\phi = j(\Omega)}$$

Rotation Law:

$$\boxed{\Omega = \Omega_c - \frac{(\Omega-\omega) r^2 \sin^2\theta \, e^{-2\rho}}{\mathrm{A}^2\left[1-(\Omega-\omega)^2 r^2 \sin^2\theta \, e^{-2\rho}\right]}}$$

Dimensionless constant:

$$\boxed{\hat{\mathrm{A}} = \mathrm{A}/r_e}$$

Limits:

$$\boxed{\hat{\mathrm{A}}^{-1} \rightarrow \begin{cases} 0 & \text{uniform rotation} \\ \infty & j-\text{constant rotation law} \end{cases}}$$

Specific angular momentum conserved during homologous collapse:

$$\boxed{\tilde{j} \equiv u_\phi \left(\frac{\varepsilon + p}{\rho_0}\right)}$$

Satisfies Rayleigh criterion for local dynamical stability to axisymmetric perturbations:

$$\boxed{\frac{d\tilde{j}}{d\Omega} < 0}$$

Equilibrium Quantities

Using the unit normal $\hat{n}^\alpha$ to the *t*=const. spacelike surfaces and the proper volume

$$dV = \sqrt{|^3g|}d^3x$$

one can define various *extensive* equilibrium quantities

$$
\begin{aligned}
M &= \int \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) t^\alpha \hat{n}^\beta dV && \textit{gravitational mass} \\
M_0 &= \int \rho u_\alpha \hat{n}^\alpha dV && \textit{baryon mass} \\
U &= \int \rho\epsilon u_\alpha \hat{n}^\alpha dV && \textit{internal energy} \\
J &= \int T_{\alpha\beta}\phi^\alpha n^\beta dV && \textit{angular momentum} \\
T &= \frac{1}{2}\int \Omega dJ && \textit{rotational energy} \\
W &= M - (M_0 + U + T) && \textit{gravitational binding energy} \\
I &= J/\Omega && \textit{moment of inertia}
\end{aligned}
$$

The Virial Theorems

The well-known Newtonian Virial theorem for equilibrium configurations

$$2T + (3\gamma - 1)U + W = 0$$

has been generalized in GR by Bonazzola & Gourgoulhon

$$\begin{aligned}
\lambda_3 \equiv\ & 4\pi \int_0^{+\infty}\int_0^{\pi}\left[3p + (\varepsilon + p)\frac{v^2}{1-v^2}\right] \mathrm{e}^{2\mu+\psi}\, r\,\mathrm{d}r\,\mathrm{d}\theta \\
& \times\left\{\int_0^{+\infty}\int_0^{\pi}\left[\partial\nu\partial\nu - \frac{1}{2}\partial\mu\partial\psi\right.\right. \\
& + \frac{\mathrm{e}^{2\mu-2\psi}}{2} r\sin^2\theta\left(\frac{\partial\mu}{\partial r} + \frac{1}{r\tan\theta}\frac{\partial\mu}{\partial\theta}\right) \\
& + \frac{1}{4r}\left(1 - \mathrm{e}^{2\mu-2\psi} r^2\sin^2\theta\right)\left(\frac{\partial\psi}{\partial r} + \frac{1}{r\tan\theta}\frac{\partial\psi}{\partial\theta}\right. \\
& \left.\left.\left. - \frac{1}{r\sin^2\theta}\right) - \frac{3}{8}\mathrm{e}^{2\psi-2\nu}\partial\omega\partial\omega\right]\mathrm{e}^{\psi}\, r\,\mathrm{d}r\,\mathrm{d}\theta\right\}^{-1},
\end{aligned}$$

Comparison of Different Codes

N.S., Living Reviews in Relativity (2003)

	AKM	Lorene/ rotstar	SF (260 × 400)	SF (70 × 200)	BGSM	KEH
$\bar{p}_{\rm c}$	1.0					
$r_{\rm p}/r_{\rm e}$	0.7				1×10^{-3}	
$\bar{\Omega}$	1.41170848318	9×10^{-6}	3×10^{-4}	3×10^{-3}	1×10^{-2}	1×10^{-2}
$\bar{M}$	0.135798178809	2×10^{-4}	2×10^{-5}	2×10^{-3}	9×10^{-3}	2×10^{-2}
$\bar{M}_0$	0.186338658186	2×10^{-4}	2×10^{-4}	3×10^{-3}	1×10^{-2}	2×10^{-3}
$\bar{R}_{\rm circ}$	0.345476187602	5×10^{-5}	3×10^{-5}	5×10^{-4}	3×10^{-3}	1×10^{-3}
$\bar{J}$	0.0140585992949	2×10^{-5}	4×10^{-4}	5×10^{-4}	2×10^{-2}	2×10^{-2}
$Z_{\rm p}$	1.70735395213	1×10^{-5}	4×10^{-5}	1×10^{-4}	2×10^{-2}	6×10^{-2}
$Z_{\rm eq}^{\rm f}$	−0.162534082217	2×10^{-4}	2×10^{-3}	2×10^{-2}	4×10^{-2}	2×10^{-2}
$Z_{\rm eq}^{\rm b}$	11.3539142587	7×10^{-6}	7×10^{-5}	1×10^{-3}	8×10^{-2}	2×10^{-1}
$\lvert$GRV3$\rvert$	4×10^{-13}	3×10^{-6}	3×10^{-5}	1×10^{-3}	4×10^{-3}	1×10^{-1}

AKM: Ansorg et al.

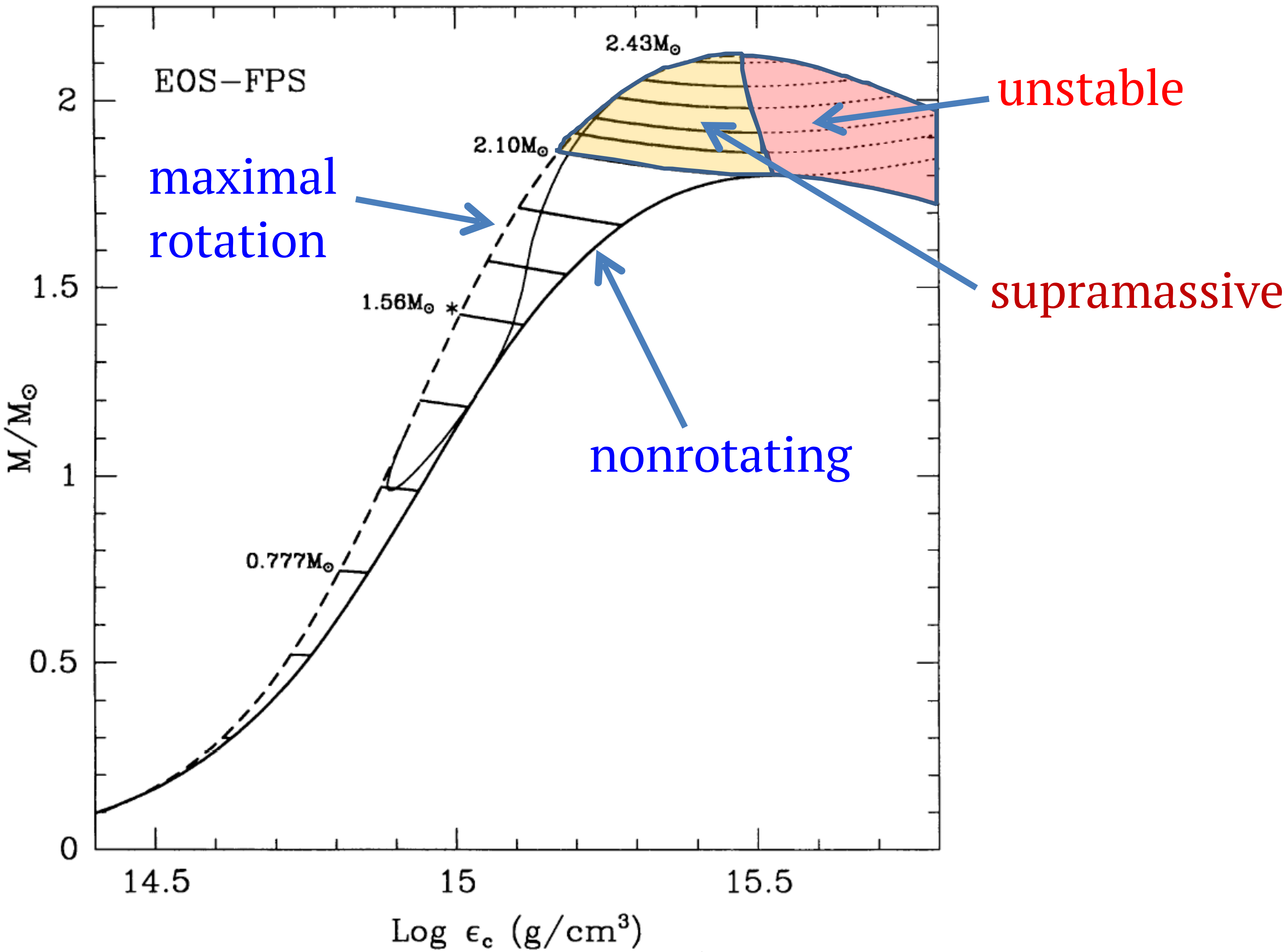
Lorene/rotstar + BGSM: Meudon group

SF: RNS code

KEH: original KEH code (not compactified)

Equilibria of Rotating Stars

Uniformly rotating equilibrium models for a realistic neutron star equation of state.

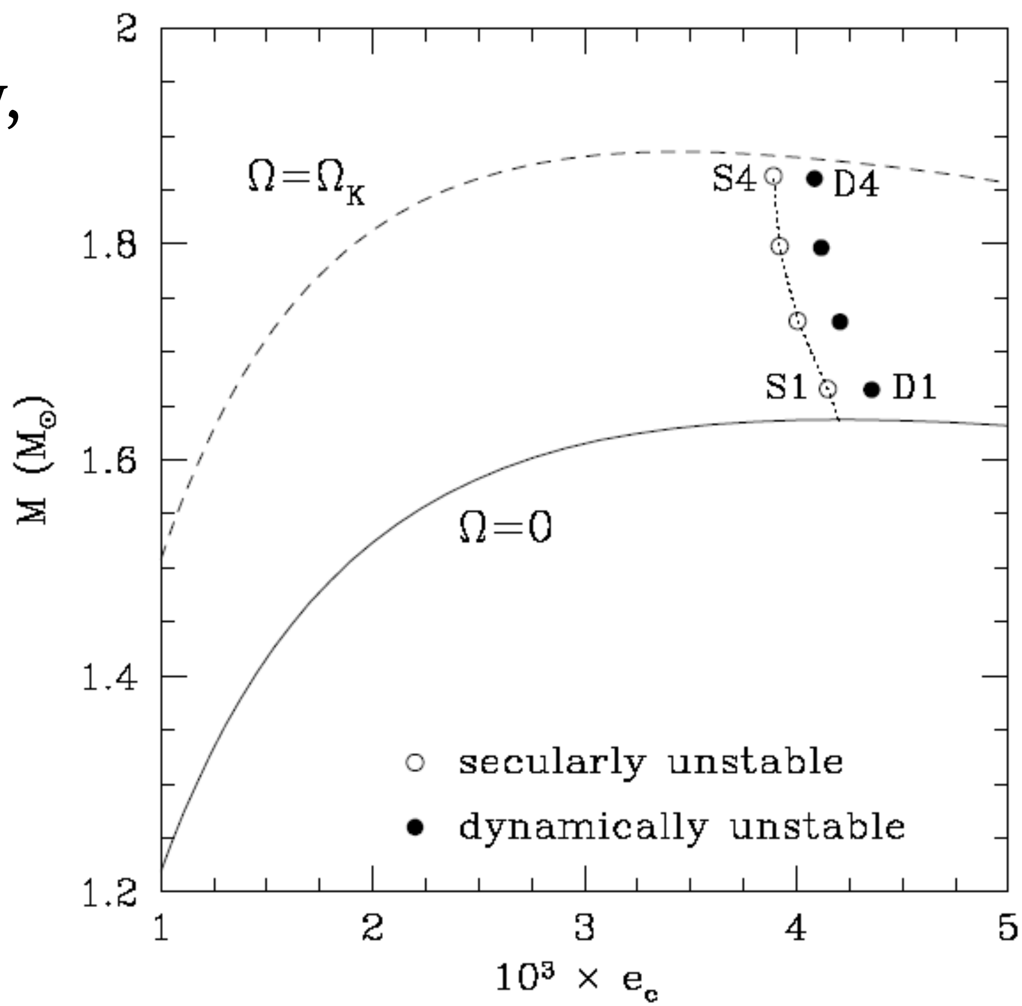


Axisymmetric Instability to Collapse

Rotating stars are subject to a ***secular*** axisymmetric instability, if:

$$\left(\frac{\partial M}{\partial \epsilon_c}\right)_J < 0$$

(Friedman, Ipser & Sorkin, 1988).



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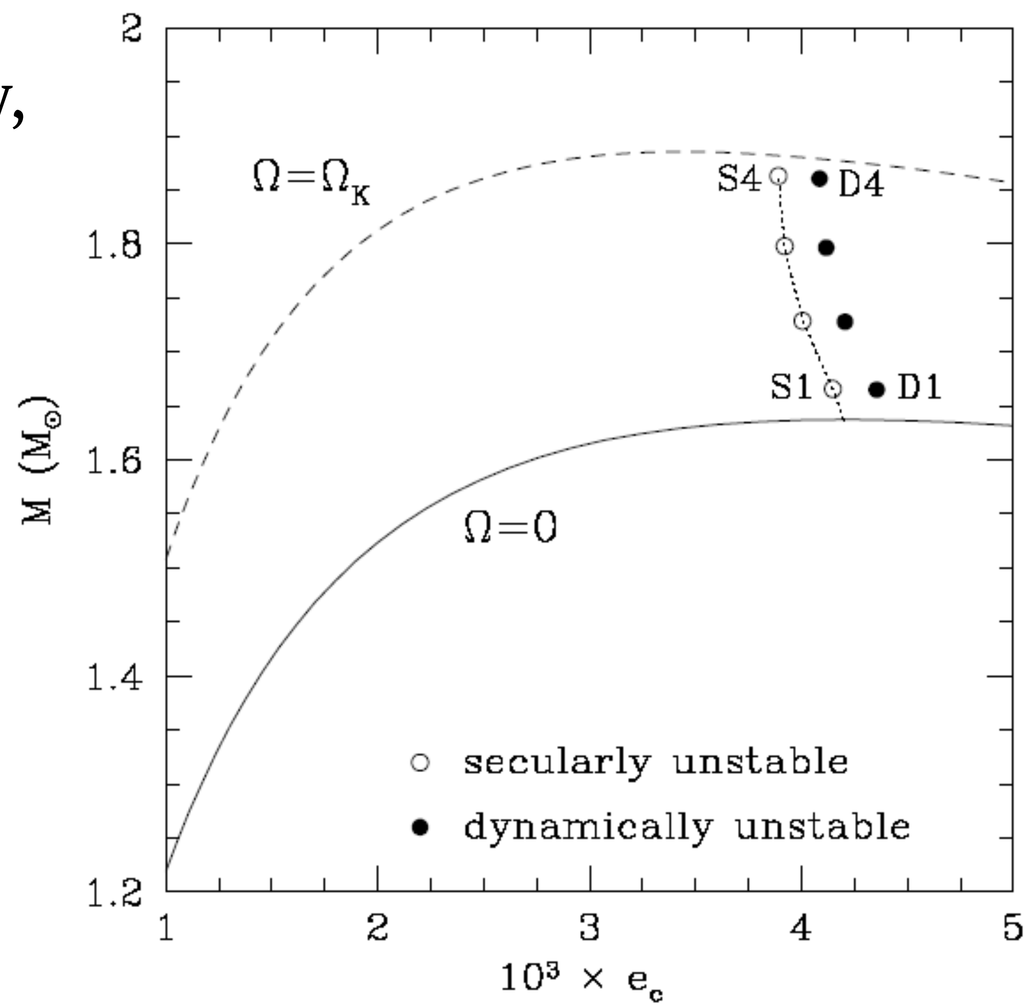

Dynamical instability soon after onset of secular instability.

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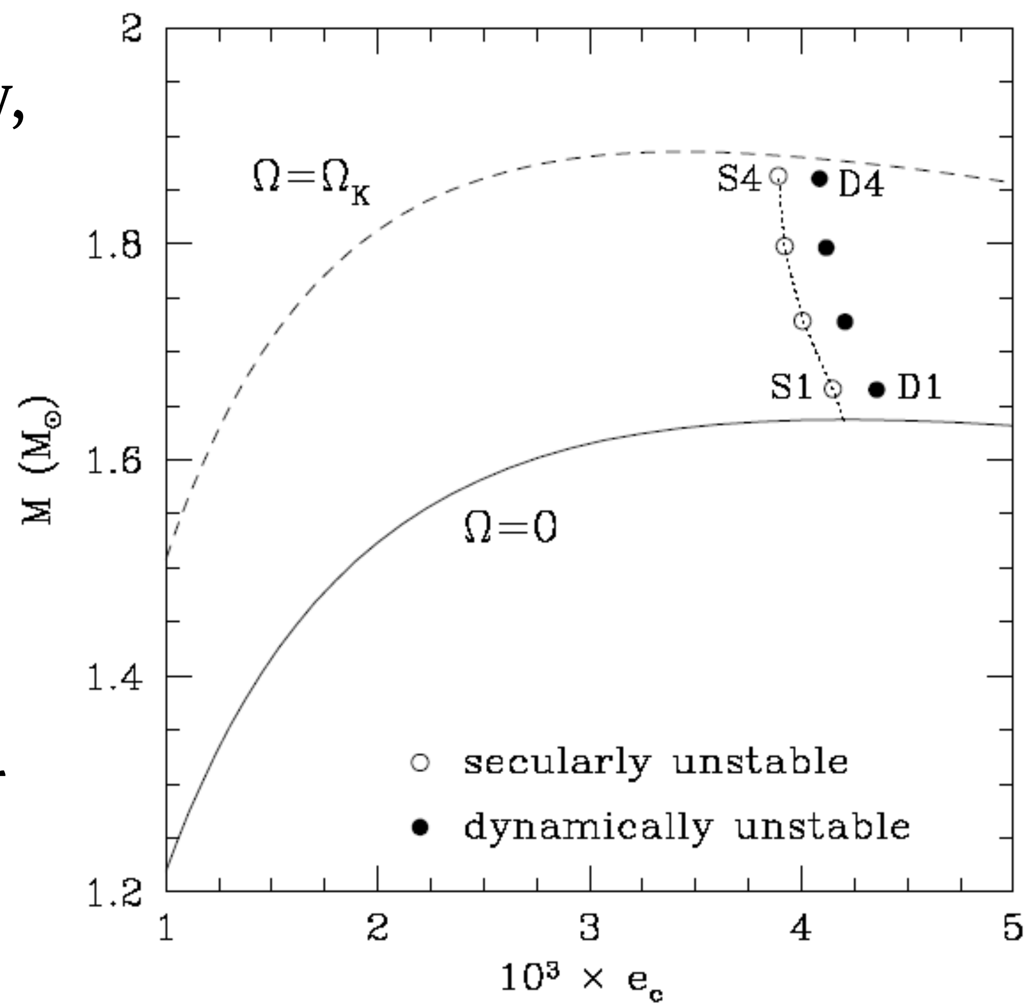
A neutron star can collapse to a Kerr BH during:

a) *Core collapse* of massive star

b) *Accretion-induced collapse*

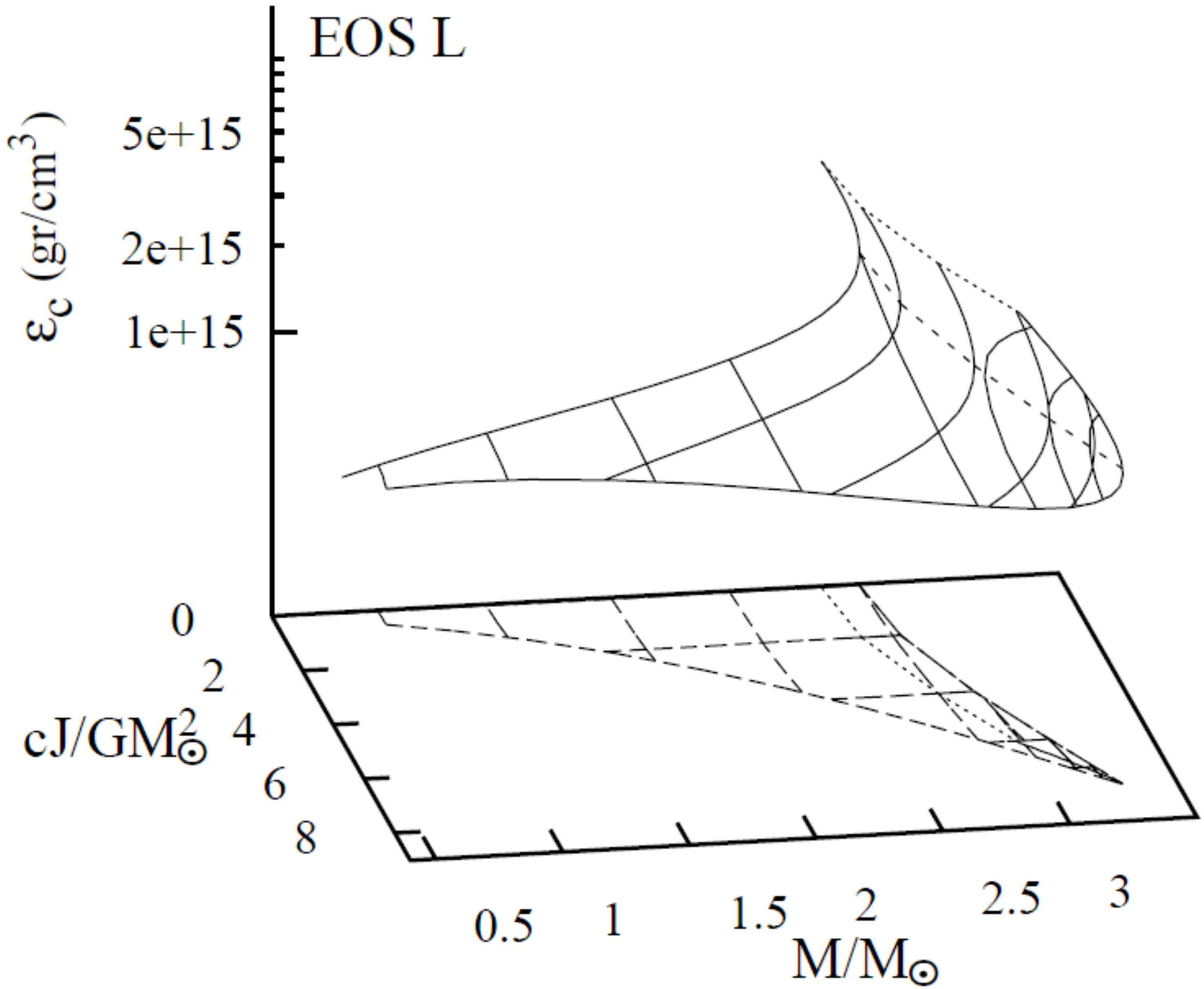
c) *Binary neutron star merger*

d) *Spin-down* of a supramassive pulsar



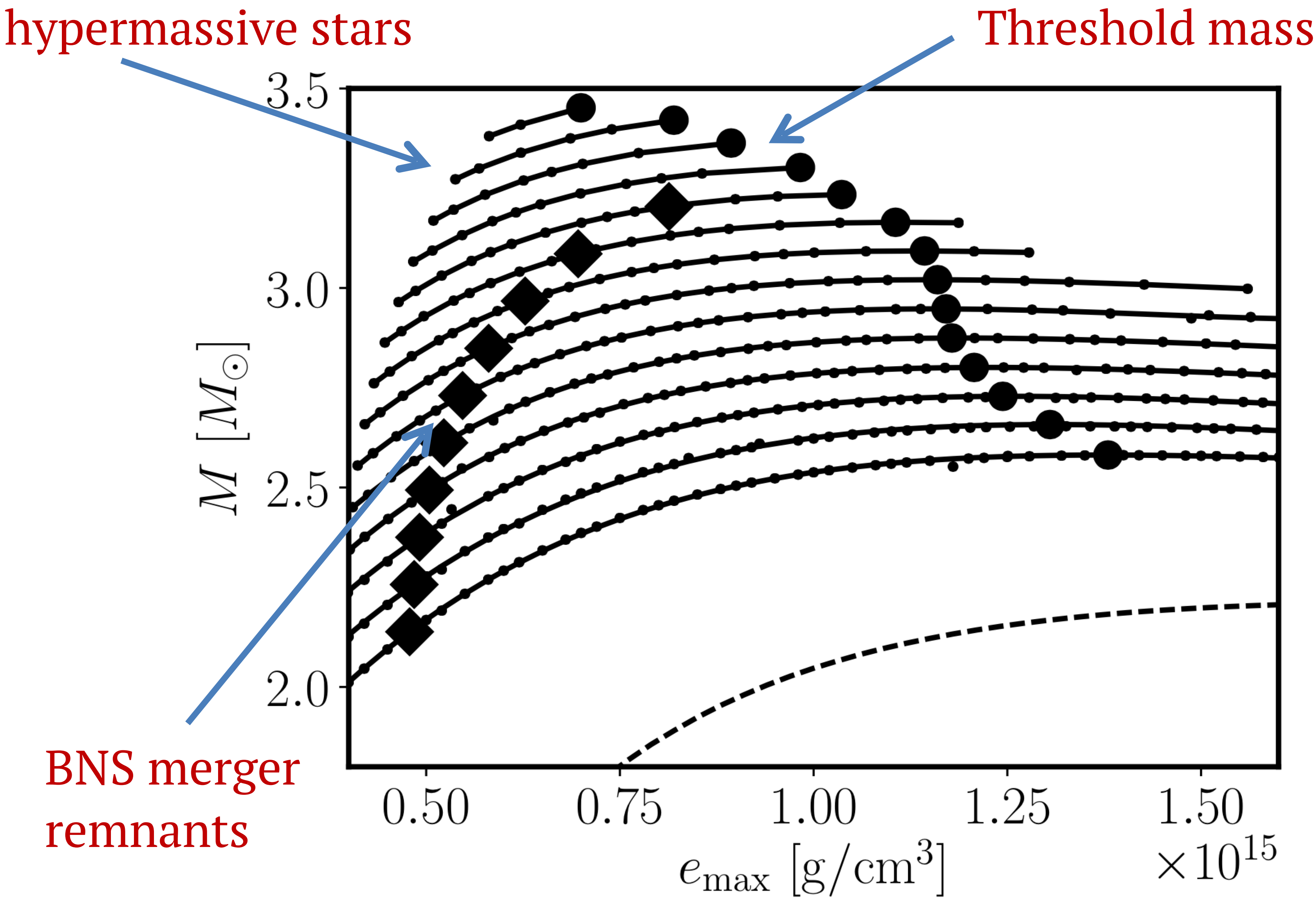
Dynamical instability soon after onset of secular instability.

Equilibrium Sequences (II)



(NS & Friedman 1995)

Differentially Rotating Models



(Bauswein & NS 2017)

Exercises

1) Consider nonrotating stars constructed with the maximally-compact EOS

$$p(\epsilon) = \begin{cases} 0 & \epsilon \leq \epsilon_C \\ \epsilon - \epsilon_C & \epsilon \geq \epsilon_C \end{cases}$$

where ε_C is the surface energy density. Assume units of c=G=1 and define dimensionless radius, mass and pressure as

$$x = r\epsilon_C^{1/2} \qquad y = m\epsilon_C^{1/2} \qquad q = p\epsilon_C^{-1}$$

a) Express the TOV equations w.r.t. these variables.

b) Find the properties of the model with maximum mass.

c) Show that it has a minimum radius independent of ε_C :

$$R_{\mathrm{min}} = 2.825 M_{\mathrm{max}}$$

Exercises

2) Construct equilibrium sequences of nonrotating neutron stars with *polytropic equations of state* and find the maximum mass (in units of c=G=K=1) for N=0.5 and N=1.0 sequences. For the numerical solutions, use the **pyTOV** code (https://github.com/niksterg/pyTOV).

3) Repeat for several *piecewise polytropic* equations of state using the **pyTOVpp** code (https://github.com/niksterg/pyTOVpp).

4) Construct sequences of rotating models (with constant baryon mass or constant angular momentum) using the **RNS** code (http://www.gravity.phys.uwm.edu/rns).