NUCLEAR EQUATION OF STATE FOR NEUTRON STAR AND SUPERNOVA

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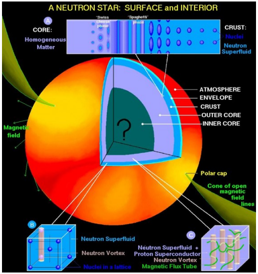
3rd HEL.A.S. SUMMER SCHOOL AND DAAD SCHOOL
NEUTRON STARS AND GRAVITATIONAL WAVES

Outline of the lecture

- Structure and the bulk neutron stars properties
- The Tolman-Oppenheimer-Volkov (TOV) equations
- Analytical solutions of TOV equations
- Equation of state of nuclear matter and neutron star matter
- Nuclear models for finite nuclei and infinite nuclear matter
- Hot equation of state
- Applications
- Conclusions and perspectives

Bulk neutron star properties

Radius: $R \sim 10 - 15$ km, Mass: $M \sim 1 - 2\ M_{\odot}$,
Mean density: $\rho \sim 4x10^{14} g/cm^3$, Frequency: f $\sim$ few $Hz - 700Hz$,
Magnetic field: $B \sim 10^{12} - 10^{18} Gauss$, $T < 1MeV(10^{10}K)$



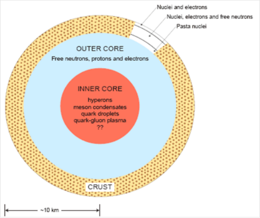
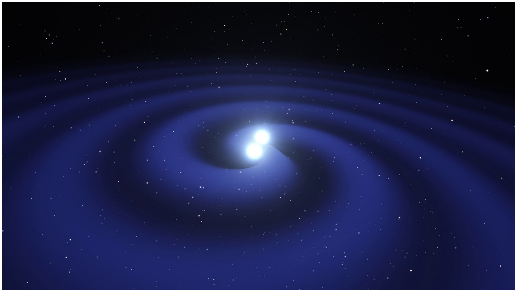


Figure: Neutron star structure and size

Supernova ∗ Proto-neutron star ∗ Neutron stars merger



Two recent important discoveries

The most famous examples are the recent discoveries of massive neutron stars with gravitational masses

- $M = 1.97 \pm 0.04$ (PRS J1614-2230): P. Demorest, T. Pennucci, S. Ransom, M. Roberts, and J. Hessels, Nature 467, 1081 (2010).
- $M = 2.01 \pm 0.04$ (PSR J0348+0432): J. Antoniadis, P.C. Freire, N. Wex, T.M. Tauris, R.S. Lynch, et al., Science 340, 1233232 (2013).

Maximum mass of neutron stars

Maximum mass of white dwarfs and neutron stars

- The maximum possible mass of a white dwarf is well defined ($M_{max} = 1.4 M_\odot$ (Chandrasekhar limit)) since the equation of state of the white dwarf is well defined.
- However, the maximum mass of a neutron star still remain unknown due to the lack of knowledge of equation of state at high densities. Theoretical predictions gives $M_{max} = 2 - 3 M_\odot$

The knowledge of maximum neutron star mass is important since:

- Helps to identify a compact object as a black hole.
- The accurate calculation of M_{max} strongly depends on the knowledge of the nuclear equation of state up to very high densities.
- Helps to understand some of the more extreme NS related processes like core-collapse supernovae, magnetar flares, and NS mergers.

The metric for a spherically symmetric configurations and for the static case

$$ds^2 = e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - e^{\nu(r)}dt^2$$

The energy density distribution $\mathcal{E}(r)$ and the local pressure $P(r)$ are related to the metric functions $\lambda(r)$ and $\nu(r)$ as

$$\frac{8\pi G}{c^4}\mathcal{E}(r) = \frac{1}{r^2}\left(1 - e^{-\lambda(r)}\right) + e^{-\lambda(r)}\frac{\lambda'(r)}{r},$$

$$\frac{8\pi G}{c^4}P(r) = -\frac{1}{r^2}\left(1 - e^{-\lambda(r)}\right) + e^{-\lambda(r)}\frac{\nu'(r)}{r},$$

$$P'(r) = -\frac{P(r) + \mathcal{E}(r)}{2}\nu'(r)$$

where derivatives with respect to the radius are denoted by $'$. The mechanical equilibrium of the star matter is determined by the three equations together with the equation of state $P = P(\mathcal{E})$ of the fluid.

The Tolman-Oppenheimer-Volkov (TOV) equations of the equilibrium

$$\frac{dP(r)}{dr} = -\frac{G}{r^2}\left[\rho(r) + \frac{P(r)}{c^2}\right]\left[m(r) + \frac{4\pi r^3 P(r)}{c^2}\left[\left[1 - \frac{2Gm(r)}{rc^2}\right]^{-1}\right.\right.$$

$$\frac{dm(r)}{dr} = 4\pi^2 r^2 \rho(r), \qquad \rho(r) = \mathcal{E}(r)/c^2$$

There are two ways to solve the TOV equations:
a) By employing possible analytical solutions which mainly have their origin on the definition of the metric functions $e^{\lambda(r)}$ and $e^{\nu(r)}$ and
b) By employing various equations of state $P = P(\mathcal{E})$ for the neutron star matter originated from the nuclear theory.

I) Analytical solutions-Numerical solutions

Analytical solutions:

1. Advantage: It is relatively easy to examine the implied physics that could be expected for actual fluids
2. Disadvantage: There is a risk obtaining solutions which may not be physically interesting or even physically possible

Numerical solutions (employing physically motivated equation of state of neutron star matter):

1. Advantage: The solutions are realistic
2. Disadvantage: The equation of state of nuclear matter is almost unknown at high densities (neutron star core)

II) Analytical solutions-Numerical solutions

1. The static character of any solution (analytical or numerical) is in itself only sufficient to assure us that the solution describes a possible state of equilibrium for fluid, but is not sufficient to tell us whether or not that state of equilibrium would be stable towards disturbances
2. Further investigation is necessary to settle the equation of stability (under any given circumstances)
3. Solutions which leads to unstable configurations cannot regard as representing a physically permanent state of the fluid

Schwarzschild constant-density interior solution

The energy density and the pressure read

$$\mathcal{E} = \mathcal{E}_c = \frac{3Mc^2}{4\pi R^3}, \qquad \frac{P(x)}{\mathcal{E}_c} = \frac{\sqrt{1-2\beta} - \sqrt{1-2\beta x^2}}{\sqrt{1-2\beta x^2} - 3\sqrt{1-2\beta}}, \quad x = r/R$$

The ratio of the central pressure to central energy density $P_c/\mathcal{E}_c$, which plays important role on the stability condition is given by the expression

$$\frac{P_c}{\mathcal{E}_c} \equiv \frac{P(0)}{\mathcal{E}(0)} = \frac{\sqrt{1-2\beta} - 1}{1 - 3\sqrt{1-2\beta}}$$

According to this solution the central pressure becomes infinite when $\beta = 4/9$ (where $\beta = GM/Rc^2$ the compactness parameter) (actually, this upper limit holds for any star). The main drawback of this solution is the infinite value of the speed of sound.

Tolman VII solution

The Tolman VII solution is the most famous of the analytical solutions. The energy density and the pressure read

$$\frac{\mathcal{E}(x)}{\mathcal{E}_c} = (1 - x^2), \qquad \frac{P(x)}{\mathcal{E}_c} = \frac{2}{15}\sqrt{\frac{3e^{-\lambda}}{\beta}}\tan\phi - \frac{1}{3} + \frac{x^2}{5}, \qquad x = r/R$$

The Tolman VII solution has the correct behavior not only on extreme limit $r = 0, R$ but also in the intermediate regions. There are some constraints related with the validity of the Tolman VII solution. In particular, the central value of pressure becomes infinite for $\beta = 0.3862$ while the speed of sound remains less than the speed of light only for $\beta < 0.2698$.

Buchdahl solution

The equation of state, in Buchdahl's solution, has the following simple form

$$\mathcal{E}(P) = 12\sqrt{P^* P} - 5P$$

While Buchdahl's solution has no particular physical basis, it does have two specific properties: (i) it can be made casual everywhere in the star by demanding that the local speed of sound v_c be less than the spedd of light c and (ii) for small values of the pressure P it reduces to the polytropic equation of state one with $n = 1$ (suitable for white dwarfs).

Equation of state of cold neutron star matter

We consider that the neutron star matter consists by neutrons, protons and electron in chemical equilibrium at zero temperature (cold matter)

$$n \leftrightarrows p + e^-, \qquad \mu_n = \mu_p + \mu_e$$

The total energy is

$$E = E_b(n,x) + E_e(n,x), \quad x = n_p/n \text{ is the proton fraction}$$

The same for the total pressure

$$P(n,x) = P_b(n,x) + P_e(n,x)$$

where

$$P_b = n^2 \frac{\partial E_b(n,x)}{\partial n}, \quad P_e(n,x) = \frac{\hbar c}{12\pi^2}\left(3\pi^2 x n\right)^{4/3}.$$

Finally the proton fraction is found from the β-equilibrium and is given by

$$\mu_e = \mu_n - \mu_p = \left(-\frac{\partial E}{\partial x}\right)_n \Rightarrow \left(\frac{\partial E}{\partial x}\right)_n = -\hbar c (3\pi^2 x n)^{1/3}$$

Nuclear symmetry energy

The baryons contribution on the total energy per particle takes the form

$$E_b(n,x) = E_b(n,0.5) + (1-2x)^2 S_u(n)$$

$S_u(n)$ is the nuclear symmetry energy

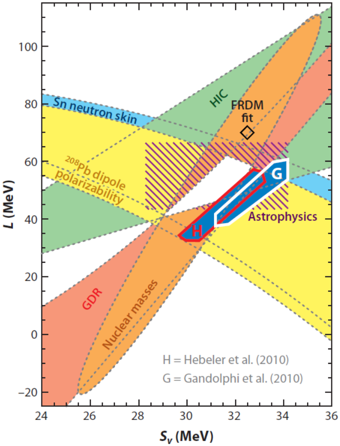
$$S_u(n) = E_b(n,0) - E_b(n,0.5)$$

The stiffness of the EoS characterized by the the slope parameter L and the incompressibility K

$$L = 3n_0 \left(\frac{dS_u(n)}{dn}\right)_{n=n_0}, \qquad K = 9n^2 \left(\frac{d^2 E_b(n,0.5)}{dn^2}\right)_{n=n_0}$$

K constrained by experiments $K = 240 \pm 20$ MeV. However, both $S_u(n)$ and L are constrained only for low values of density up to $1.5n_0$
Conclusion: The EoS of neutron star matter is unknown at high densities $n > 1.5n_0$

Constraints on $S_u(n_0)$ and L from nuclear experiments and astrophysical observations



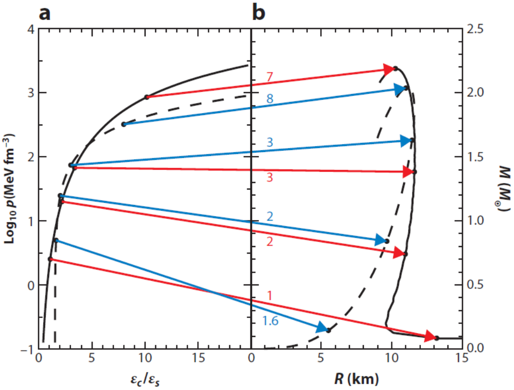
Theoretical nuclear models for the equations of state

There are various theoretical models which may categorized in two main classes:

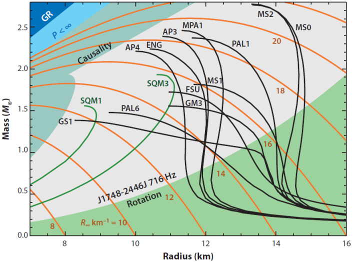
- **Potential (non relativistic) models and Field theoretical models (relativistic)**: in both the Hohenberg-Kohn-Sham theorem allows the total Hamiltonian density to be expressed in terms of the local particle number and kinetic energy density of the various species.This simplification leads to the study both of heavy nuclei and infinite nuclear matter properties.
- **Variational microscopic calculations/Monde Carlo models**: starting from a Hamiltonian constructed on the basis of nucleon-nucleon and many-nucleon interactions. The disadvantage: the calculations are computer intensive and are only beginning to be undertaken for heavy nuclei.

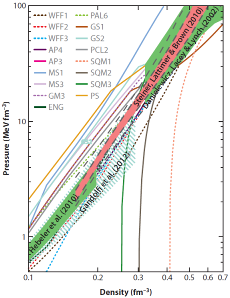
The basic demand: Each model must be able to reproduces experimental data of both finite nuclei and infinite nuclear matter

Schematic equation of state and $M - R$ diagram

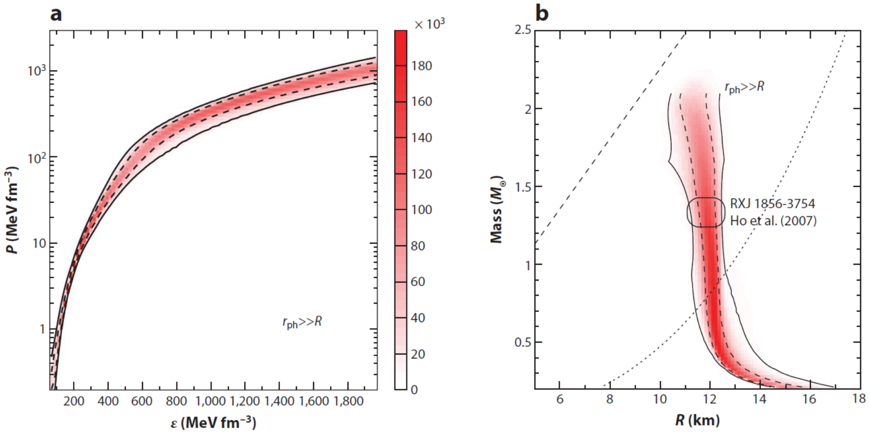


Realistic equations of state and $M - R$ diagram





Constraints on the equation of state and $M - R$ diagram



Nuclear models

The momentum dependent interaction model MDI[a]

[a] M. Prakash, I. Bombaci, M. Prakash, P. J. Ellis, J. M. Lattimer, and R. Knorren, Phys. Rep. **280**, 1 (1997).
Ch.C. Moustakidis and C.P. Panos, Phys. Rev. C **79**, 045806 (2009).

$$
\begin{aligned}
E_b(n,I) &= \frac{3}{10}E_F^0 u^{2/3}\left[(1+I)^{5/3}+(1-I)^{5/3}\right]+\frac{1}{3}A\left[\frac{3}{2}-(\frac{1}{2}+x_0)I^2\right]u \\
&+ \frac{\frac{2}{3}B\left[\frac{3}{2}-(\frac{1}{2}+x_3)I^2\right]u^{\sigma}}{1+\frac{2}{3}B'\left[\frac{3}{2}-(\frac{1}{2}+x_3)I^2\right]u^{\sigma-1}} \\
&+ \frac{3}{2}\sum_{i=1,2}\left[C_i+\frac{C_i-8Z_i}{5}I\right]\left(\frac{\Lambda_i}{k_F^0}\right)^3\left(\frac{((1+I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}}-\tan^{-1}\frac{((1+I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}}\right) \\
&+ \frac{3}{2}\sum_{i=1,2}\left[C_i-\frac{C_i-8Z_i}{5}I\right]\left(\frac{\Lambda_i}{k_F^0}\right)^3\left(\frac{((1-I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}}-\tan^{-1}\frac{((1-I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}}\right)
\end{aligned}
$$

Nuclear models

The Skyrme model[a]

[a] E. Chabanat, P. Bonche, P. Haensel, J. Meyer, and R. Schaeffer, Nucl. Phys. A **627**, 710 (1997).
M. Farine, J. M. Pearson, and F. Tondeur, Nucl. Phys. A **615**, 135 (1997).

$$
\begin{aligned}
E_b(n,I) &= \frac{3}{10}\frac{\hbar^2 c^2}{m}\left(\frac{3\pi^2}{2}\right)^{2/3} n^{2/3} F_{5/3}(I) + \frac{1}{8} t_0 n \left[2(x_0+2) - (2x_0+1)F_2(I)\right] \\
&+ \frac{1}{48} t_3 n^{\sigma+1}\left[2(x_3+2) - (2x_3+1)F_2(I)\right] \\
&+ \frac{3}{40}\left(\frac{3\pi^2}{2}\right)^{2/3} n^{5/3}\Big[\left(t_1(x_1+2) + t_2(x_2+2)\right) F_{5/3}(I) \\
&+ \frac{1}{2}\left(t_2(2x_2+1) - t_1(2x_1+1)\right) F_{8/3}(I)\Big]
\end{aligned}
$$

Nuclear models

Recently, Hebeler et alperformed microscopic calculations based on chiral effective field theory interactions to constrain the properties of neutron-rich matter below nuclear densities. It explains the massive neutron stars of $M = 2M_{\odot}$. In this model the energy per particle is given by(hereafter HLPS model[a])

[a] K. Hebeler, J.M. Lattimer, C.J. Pethick, and A. Schwenk, Astroph. J. **773**, 11 (2013).

$$
\begin{aligned}
E_b(u,x) &= \frac{3T_0}{5}\left(x^{5/3}+(1-x)^{5/3}\right)(2u)^{2/3} - T_0\left[(2\alpha-4\alpha_L)x(1-x)+\alpha_L\right]u \\
&+ T_0\left[(2\eta-4\eta_L)x(1-x)+\eta_L\right]u^{\gamma}
\end{aligned}
$$

The role of the symmetry energy on the crust-core interface

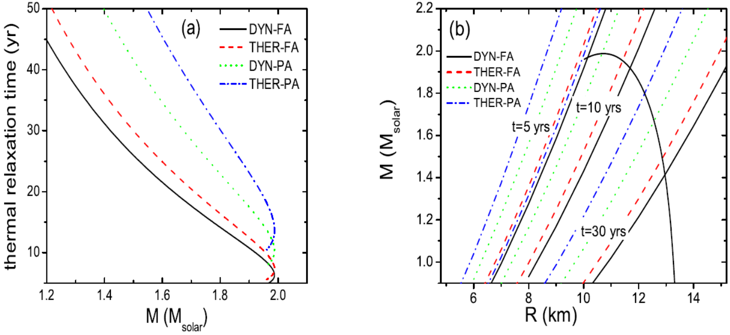


Figure: (a) The thermal relaxation time t_w as a function of the total mass. (b) Constraints on the M-R diagram form the thermal relaxation time t_w. The M-R dependence for the MDI model ($L = 80$ MeV) has been included also.

The crustal fraction of the moment of the inertia

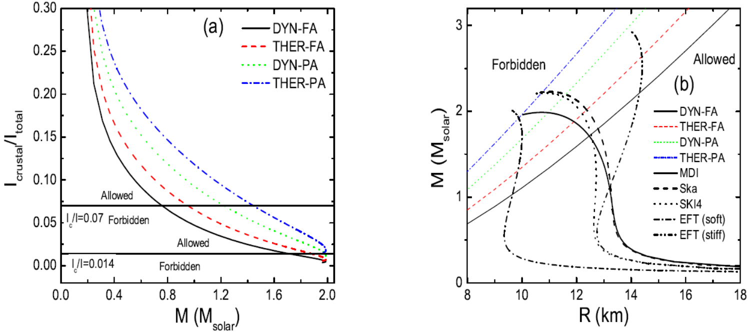


Figure: (a) The crustal fraction of the moment of the inertia as a function of mass. The horizontal lines, each one representing a possible $I_{\rm crust}/I$ constraint, deduced for the Vela pulsar (assuming a mass $M = 1.4 M_\odot$). (b) The mass-radius diagram for various nuclear EoS and the constraints $I_{\rm crust}/I = 0.014$.

The critical frequency-temperature dependence

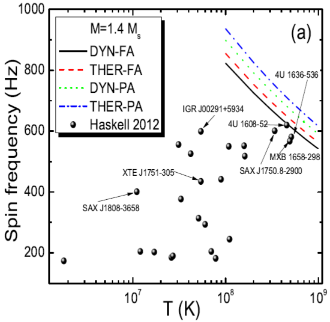


Figure: The critical frequency-temperature dependence for a neutron star with mass $M = 1.4M_\odot$ (a) and $M = 1.8M_\odot$ (b). The observed cases of LMXBs and MSRPs and the cases IGR J00291+5934, XTE J1751-305, and SAX J1808-3658 with well-known observation spin-down rate, are also indicated.

Hot equation of state: why we need it ?

Table: Ranges of baryon number density n (in units of saturation densityn_s), temperature T, electron fraction $Y_e = n_e/n$ and entropy per baryon S related with the corresponding astrophysical phenomena.

	supernovae	Proto-Neutron stars	Merger of NS
n/n_s	$10^{-8} - 10$	$10^{-8} - 10$	$10^{-8} - 10$
T (MeV)	0 - 30	0 - 50	0 - 100
Y_e (MeV)	0.35 - 0.45	0.01 - 0.30	0.01 - 0.60
S (k_B)	0.5 - 10	0 - 10	0 - 100

Hot equation of state

The energy density of the asymmetric nuclear matter (ANM) is given by the relation

$$\epsilon(n_n, n_p, T) = \epsilon^n_{kin}(n_n, T) + \epsilon^p_{kin}(n_p, T) + V_{int}(n_n, n_p, T)$$

$$\epsilon^\tau_{kin}(n_\tau, T) = 2\int \frac{d^3k}{(2\pi)^3}\frac{\hbar^2 k^2}{2m}\; f_\tau(n_\tau, k, T)$$

$$f_\tau(n_\tau, k, T) = \left[1 + \exp\left(\frac{e_\tau(n_\tau, k, T) - \mu_\tau(n_\tau, T)}{T}\right)\right]^{-1}$$

$$\begin{aligned} V_{int}(n_n, n_p, T) &= \frac{1}{3}An_0\left[\frac{3}{2} - (\frac{1}{2} + x_0)I^2\right]u^2 + \frac{\frac{2}{3}Bn_0\left[\frac{3}{2} - (\frac{1}{2} + x_3)I^2\right]u^{\sigma+1}}{1 + \frac{2}{3}B'\left[\frac{3}{2} - (\frac{1}{2} + x_3)I^2\right]u^{\sigma-1}} \\ &+ u\sum_{i=1,2}\left[C_i\left(\mathcal{J}^i_n + \mathcal{J}^i_p\right) + \frac{(C_i - 8Z_i)}{5}I\left(\mathcal{J}^i_n - \mathcal{J}^i_p\right)\right] \end{aligned}$$

where

$$\mathcal{J}^i_\tau = 2\int \frac{d^3k}{(2\pi)^3}g(k, \Lambda_i)f_\tau(n_\tau, k, T)$$

Thermodynamical description of hot equation of state

The Helmholtz free energy F which is written as

$$F(n, T, I) = E(n, T, I) - TS(n, T, I)$$

The entropy density $s = Sn$ has the same functional form as that of a non-interacting gas system, given by the equation

$$s_\tau(n, T, I) = -2 \int \frac{d^3k}{(2\pi)^3} \left[f_\tau \ln f_\tau + (1 - f_\tau) \ln(1 - f_\tau) \right]$$

The pressure and the chemical potentials defined as follows

$$P = n^2 \left(\frac{\partial \epsilon / n}{\partial n} \right)_{S, N_i}, \qquad \mu_i = \left(\frac{\partial \epsilon}{\partial n_i} \right)_{S, V, n_{j \neq i}}$$

The pressure P can also be calculated from the equation

$$P = Ts - \epsilon + \sum_i \mu_i n_i$$

Thermodynamical description of hot equation of state

$$\mu_n = F + u\left(\frac{\partial F}{\partial u}\right)_{Y_p,T} - Y_p\left(\frac{\partial F}{\partial Y_p}\right)_{n,T}, \quad \mu_p = \mu_n + \left(\frac{\partial F}{\partial Y_p}\right)_{n,T}$$

$$\hat{\mu} = \mu_n - \mu_p = -\left(\frac{\partial F}{\partial Y_p}\right)_{n,T}$$

$$F(n,T,I) = F(n,T,I=0) + I^2 F_{sym}(n,T)$$

$$\hat{\mu} = \mu_n - \mu_p = 4(1-2Y_p)F_{sym}(n,T)$$

$$S(n,T,I) = S(n,T,I=0) + I^2 S_{sym}(n,T)$$

where

$$\begin{aligned} S_{sym}(n,T) &= S(n,T,I=1) - S(n,T,I=0) \\ &= \frac{1}{T}(E_{sym}(n,T) - F_{sym}(n,T)) \end{aligned}$$

$\beta-$equilibrium in hot proto-neutron star and supernova

$$n \longrightarrow p + e^- + \bar{\nu}_e, \qquad\qquad p + e^- \longrightarrow n + \nu_e$$

$$\mu_n + \mu_{\nu_e} = \mu_p + \mu_e \qquad Y_p = Y_e, \qquad Y_l = Y_e + Y_{\nu_e}$$

$$\mu_e - \mu_{\nu_e} = \mu_n - \mu_p = 4\,(1 - 2Y_p)\,F_{sym}(n, T)$$

the energy density and pressure of leptons given by the following formulae

$$\epsilon_l(n_l, T) = \frac{g}{(2\pi)^3} \int \frac{\sqrt{\hbar^2 k^2 c^2 + m_l^2 c^4}\ \mathrm{d}\mathbf{k}}{1 + \exp\left[\frac{\sqrt{\hbar^2 k^2 c^2 + m_l^2 c^4} - \mu_l}{T}\right]}$$

$$P_l(n_l, T) = \frac{1}{3}\frac{g(\hbar c)^2}{(2\pi)^3} \int \frac{1}{\sqrt{\hbar^2 k^2 c^2 + m_l^2 c^4}} \frac{k^2\ \mathrm{d}\mathbf{k}}{1 + \exp\left[\frac{\sqrt{\hbar^2 k^2 c^2 + m_l^2 c^4} - \mu_l}{T}\right]}$$

$$s_l(n, T, l) = -g \int \frac{d^3k}{(2\pi)^3} \left[f_l \ln f_l + (1 - f_l)\ln(1 - f_l)\right]$$

$\beta-$equilibrium in hot proto-neutron star and supernova

The equation of state of hot nuclear matter in β-equilibrium (considering that it consists of neutrons, protons, electrons and neutrinos) is given by

$$\epsilon_{tot}(n, T, I) = \epsilon_b(n, T, I) + \sum_{l=e,\nu_e} \epsilon_l(n, T, I), \tag{1}$$

where $\epsilon_b(n, T, I)$ and $\epsilon_l(n, T, I)$ are the contributions of baryons and leptons respectively. The total pressure is

$$P_{tot}(n, T, I) = P_b(n, T, I) + \sum_{l=e,\nu_e} P_l(n, T, I), \tag{2}$$

where $P_b(n, T, I)$ is the contribution of the baryons (see Eq. (1)) i.e.

$$P_b(n, T, I) = T \sum_{\tau=p,n} s_\tau(n, T, I) + \sum_{\tau=n,p} n_\tau \mu_\tau(n, T, I) - \epsilon_b(n, T, I), \tag{3}$$

while $P_l(n, T, I)$ is the contribution of the leptons (see Eq. (1)). From Eqs. (1) and (2) we can construct the isothermal curves for energy and pressure and finally derive the isothermal behavior of the equation of state of hot nuclear matter under β-equilibrium.

Adiabatic equations of state

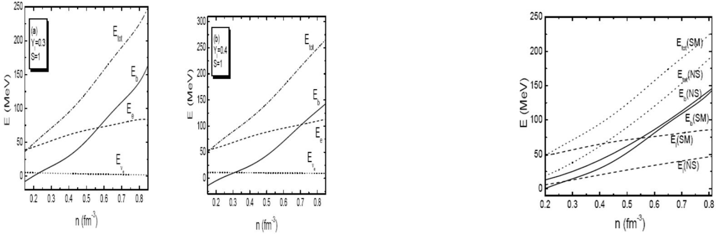


Figure: Hot equations of state

General Perspectives

- The maximum and the minimum limit of neutron star mass related with the nuclear equation of state at supranuclear and subnuclear densities respectively
- Precise measurements of masses and radii for several individual neutron stars would pin down the equation of state without recourse to models
- The hadrons-quark phase transition at high densities
- Hot equations of state (isothermal and adiabatic) are necessary to study the core-collapse supernovae, the structure and evolution of proto-neutron star and neutron star merger
- Rich information for neutron star equation of state with gravitational waves observations
- Alternative theories (from general relativity) concerning mainly the strong-field regime

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