Exercises for equation of state and neutron star structure

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EXERCISES

1. Consider the TOV equations and a power solution of them valid near $r = 0$, of the form

$$M(r) = \sum_i M_j r^i$$
$$P(r) = \sum_i P_j r^i$$
$$\mathcal{E}(r) = \sum_i \mathcal{E}_j r^i \tag{1}$$

Assume that the equation of state $P = P(\mathcal{E})$ has the expansion near the central energy density $\mathcal{E}_c$

$$P = P(\mathcal{E}_c) + \frac{P_c \gamma_c}{\mathcal{E}_c}(\mathcal{E} - \mathcal{E}_c) + \cdots \tag{2}$$

where γ_c is the adiabatic index evaluated at $\mathcal{E}_c$. Find the first two non-vanishing terms in each power series in Eq. (1).

2. Concerning the Buchdahl solution derive the restrictions

$$P < P^*, \qquad \mathcal{E} < 7P^*$$

due to causality requirements.

3. In the case of the Buchdahl solution prove that the surface red-shift of the star is given by

$$z_s = (1 - 2\beta)^{-1/2} - 1$$

and the mass of the star by

$$M = \frac{\pi\beta(1-\beta)}{(1-2\beta)A}$$

4. Prove that in the case of the uniform solution the constraints that the pressure must be finite every where leads to the upper bound on the compactness parameter that is

$$\beta \equiv \frac{GM}{Rc^2} < \frac{4}{9}$$

5. Is there an absolute upper limit to $\beta \equiv \frac{GM}{Rc^2}$ imposed by the structure of the TOV equations, irrespective of the equation of state ?

6. Prove, employing the TOV equations, that the mass of the crust is given by

$$M_{\text{crust}} = \frac{4\pi P_{\text{t}} R_{\text{core}}^4}{G M_{\text{core}}} \left(1 - \frac{2 G M_{\text{core}}}{R_{\text{core}} c^2}\right).$$

where P_{t} is the transition pressure at the crust-core interface, M_{core}, R_{core} the mass and the radius of the core.

7. Employ the TOV equations, the equation of state $P(\mathcal{E}) = K\mathcal{E}^{4/3}$ and also the approximation that $M_{\text{core}} \simeq M$, $P(r) \ll \mathcal{E}(r)$ and $4\pi r^3 P(r)/M(r) \ll 1$ to prove that the moment of inertia and the mass of the crust which defined respectively as

$$I_{\text{crust}} = \frac{16\pi}{3R_s}\left[1 - \left(\frac{R_s}{R}\right)\left(\frac{I}{MR^2}\right)\right]\int_0^{P_t} r^6 dP$$

$$M_{\text{crust}} = 4\pi \int_{R_{\text{core}}}^{R} r^2 \mathcal{E}(r) dr$$

are given by the following expansion

$$I_{\text{crust}} = \frac{16\pi}{3}\frac{R_{\text{core}}^6 P_{\text{t}}}{R_s}\left[1 - \frac{R_s}{R}\frac{I}{MR^2}\right]\left[1 + \frac{48}{5}\left(\frac{R_{\text{core}}}{R_s} - 1\right)\frac{P_{\text{t}}}{\mathcal{E}_{\text{t}}} + \dots\right]$$

$$M_{\text{crust}} \approx 8\pi R_{\text{core}}^3 P_{\text{t}} \left(\frac{R_{\text{core}}}{R_s} - 1\right)\left[1 + \frac{32}{5}\left(\frac{R_{\text{core}}}{R_s} - \frac{3}{4}\right)\frac{P_{\text{t}}}{\mathcal{E}_{\text{t}}} + \dots\right]$$

where $R_s = 2GM/c^2$, I the total moment of inertia and $P_{\text{t}}, \mathcal{E}_{\text{t}}$ the pressure and the energy density at the interface between crust and core.

NOTES

1 Neutron star structure

The starting point for determining the mechanical equilibrium of neutron star matter is the well known Tolman-Oppenheimer-Volkoff (TOV) equations. This set of differential equations describes the structure of a neutron star. Now, considering that for a static spherical symmetric system, the metric can be written as follow

$$ds^2 = -e^{\nu(r)} c^2 dt^2 + e^{\lambda(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3}$$

the TOV equations have the form

$$\frac{dP(r)}{dr} = -\frac{G\mathcal{E}(r)M(r)}{c^2 r^2}\left(1 + \frac{P(r)}{\mathcal{E}(r)}\right)\left(1 + \frac{4\pi P(r) r^3}{M(r) c^2}\right)\left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}, \tag{4}$$

$$\frac{dM(r)}{dr} = \frac{4\pi r^2}{c^2}\mathcal{E}(r). \tag{5}$$

By introducing an EoS (e.g. a dependence on the form $P = P(\mathcal{E})$) we solve numerically the TOV equations. This EoS provides the relation between pressure and density of the neutron star matter. The other possibility, is to try find out analytical solutions of the TOV equations with the risk of obtaining solutions without physical interest. It has been remarked by Tolman that the static character of an analytical solution is self sufficient to assure that the solution describes a possible state of equilibrium for a fluid, but is not sufficient to tell us if the state of equilibrium would be stable towards perturbations; the same holds for any numerical solution. In addition, the question of stability is important since an unstable solution cannot be able to describe a physically steady state and consequently, is not of physical interest.

2 Analytical solutions of Einstein's field equations

2.1 Schwarzschild constant-density interior solution

In the case of the Schwarzschild interior solution, the density is constant through the star. There is no physical justification of this solution, actually it is not a real solution. However this solution is very interesting since it combines the following: a) in the interior of neutron stars the density is almost constant, b) it is very simple to allow an exact solution of Einstein's equations, c) it is useful for pedagogic reasons. In the present work we will use this solution in comparison with the three others. Actually, the definition of the critical adiabatic index has been introduced by Chandrasekhar by employing the constant interior solution. Briefly we present below the basic ingredients of the Schwarzschild interior solution. The metric functions are defined as

$$e^{-\lambda} = 1 - 2\beta x^2, \qquad e^{\nu} = \left(\frac{3}{2}\sqrt{1-2\beta} - \frac{1}{2}\sqrt{1-2\beta x^2}\right)^2, \quad x = r/R. \tag{6}$$

The energy density and the pressure read

$$\mathcal{E} = \mathcal{E}_c = \frac{3Mc^2}{4\pi R^3}, \tag{7}$$

$$\frac{P(x)}{\mathcal{E}_c} = \frac{\sqrt{1-2\beta} - \sqrt{1-2\beta x^2}}{\sqrt{1-2\beta x^2} - 3\sqrt{1-2\beta}}. \tag{8}$$

The ratio of the central pressure to central energy density $P_c/\mathcal{E}_c$, which plays important role on the stability condition is given by the expression

$$\frac{P_c}{\mathcal{E}_c} \equiv \frac{P(0)}{\mathcal{E}(0)} = \frac{\sqrt{1-2\beta} - 1}{1 - 3\sqrt{1-2\beta}}. \tag{9}$$

According to this solution the central pressure becomes infinite when $\beta = 4/9$ (actually, this upper limit holds for any star. The main drawback of this solution is the infinite value of the speed of sound.

2.2 Tolman VII solution

The Tolman VII solution is the most famous of the analytical solutions. It is of great interest since it has the specific property the pressure and the density to vanish at the surface of the star. This solution has been extensively employed to neutron star studies while its physical realization has been examined in detail very recently. Actually, the stability of this solution has been examined by Negi. In the present work the stability is reexamined and in addition the results are compared with the other similar solutions. The basic ingredients of the Tolman VII solutions are presented below. The metric functions are defined as follows

$$e^{-\lambda} = 1 - \beta x^2(5 - 3x^2), \qquad e^{\nu} = \left(1 - \frac{5\beta}{3}\right)\cos^2\phi, \quad x = \frac{r}{R} \tag{10}$$

where

$$\phi = \frac{w_1 - w}{2} + \phi_1, \qquad \phi_1 = \tan^{-1}\sqrt{\frac{\beta}{3(1-2\beta)}}$$

and

$$w = \ln\left(x^2 - \frac{5}{6} + \sqrt{\frac{e^{-\lambda}}{3\beta}}\right), \quad w_1 = \ln\left(\frac{1}{6} + \sqrt{\frac{1-2\beta}{3\beta}}\right).$$

The energy density and the pressure read

$$\frac{\mathcal{E}(x)}{\mathcal{E}_c} = (1 - x^2), \quad \mathcal{E}_c = \frac{15Mc^2}{8\pi R^3}, \tag{11}$$

$$\frac{P(x)}{\mathcal{E}_c} = \frac{2}{15}\sqrt{\frac{3e^{-\lambda}}{\beta}}\tan\phi - \frac{1}{3} + \frac{x^2}{5}. \tag{12}$$

There are some constraints related with the validity of the Tolman VII solution. In particular, the central value of pressure becomes infinite for $\beta = 0.3862$ while the speed of sound remains less than the speed of light only for $\beta < 0.2698$.

2.3 Buchdahl solution

The Buchdahl's solution has no particular physical basis. However, it does have two specific properties: (i) it can be made casual everywhere in the star by demanding that the local speed of sound is less than one and (ii) for small values of the pressure P it reduces to $\mathcal{E} = 12\sqrt{P^*P}$, which, in the newtonian theory of stellar structure is the well known $n = 1$ polytrope. So, Buchdahl's solution may be regarded as its relativistic generalization. The equation of state, in Buchdahl's solution, has the following simple form

$$\mathcal{E}(P) = 12\sqrt{P^*P} - 5P, \tag{13}$$

where P is the local pressure and P^* is a parameter. In particular the metric functions are defined as

$$e^{\lambda(r')} = \frac{(1-2\beta)(1-\beta+u(r'))}{(1-\beta-u(r'))(1-\beta+\beta\cos(Ar'))^2}, \qquad e^{\nu(r')} = \frac{(1-2\beta)(1-\beta-u(r'))}{1-\beta+u(r')}, \tag{14}$$

where

$$r' = \frac{r(1-2\beta)}{1-\beta+u(r')}, \qquad u(r') = \beta\frac{\sin(Ar')}{Ar'}, \qquad A^2 = \frac{288\pi P^* G}{c^4(1-2\beta)}.$$

The energy density and the pressure read

$$\frac{\mathcal{E}(r')}{\mathcal{E}_c} = \frac{(2-2\beta-3u(r'))}{(2-5\beta)(1-\beta+u(r'))^2}\frac{u(r')}{\beta}, \tag{15}$$

$$\frac{P(r')}{\mathcal{E}_c} = \frac{\beta}{(1-\beta+u(r'))^2(2-5\beta)}\left(\frac{u(r')}{\beta}\right)^2, \tag{16}$$

where

$$P_c = 36P^*\beta^2, \qquad \mathcal{E}_c = 72P^*\beta(1-5\beta/2), \qquad \frac{P_c}{\mathcal{E}_c} = \frac{\beta}{2-5\beta}. \tag{17}$$

It is more convenient to use the variable $x' = r'/R$ instead of $x = r/R$ and in this case we have

$$x = \frac{1-\beta+u(x')}{1-2\beta}x', \tag{18}$$

where

$$u(x') = \beta\frac{\sin(ARx')}{ARx'}, \qquad AR = \frac{1-\beta}{1-2\beta}\pi. \tag{19}$$

The Buchdahl solution is physically meaningful in restricted domains. These are: a) in order to ensure that $\mathcal{E}(r) > 0$ then must $\beta < 0.4$, b) the causality condition $v_s < c$ demands that $\beta < 1/6$ and c) the condition $v_s^2 > 0$ satisfied only when $\beta < 1/5$.

3 Nuclear Equation of State

3.1 Symmetry energy

The symmetry energy plays an important role on the determination of the transition density and the corresponding pressure and is a key quantity to explain in general many neutron star properties and dynamical processes. We consider that the energy per particle of nuclear matter $E_b(n, I)$ can be expanded around the asymmetry parameter I as

$$E_b(n, I) = E_b(n, I = 0) + E_{\rm sym,2}(n)I^2 + E_{\rm sym,4}(n)I^4 + \cdots + E_{\rm sym,2k}(n)I^{2k} + \cdots \tag{20}$$

where $I = (n_n - n_p)/n = 1 - 2x$ (x is the proton fraction n_p/n). The coefficients of the expansion (20) are given by the expression

$$E_{\rm sym,2k}(n) = \frac{1}{(2k)!}\frac{\partial^{2k} E_b(n, I)}{\partial I^{2k}}|_{I=0}. \tag{21}$$

The nuclear symmetry energy $E_{\rm sym}(n)$ is defined as the coefficient of the quadratic term, that is

$$E_{\rm sym}(n) \equiv E_{\rm sym,2}(n) = \frac{1}{2!}\frac{\partial^2 E_b(n, I)}{\partial I^2}|_{I=0} \tag{22}$$

and the slope of the symmetry energy L at the nuclear saturation density n_s, which is an indicator of the stiffness of the EoS, is defined as

$$L = 3n_s \frac{dE_{\rm sym}(n)}{dn}|_{n=n_s}. \tag{23}$$

In the framework of the parabolic approximation (PA) the energy per particle is given by the expression

$$E_b(n, x) \simeq E_b\,(n, I = 0) + I^2 E_{sym}^{PA}(n), \tag{24}$$

where $E_{\rm sym}^{PA}(n)$ is simply defined as

$$E_{\rm sym}^{PA}(n) = E_b(n, I = 1) - E_b\,(n, I = 0)\,. \tag{25}$$

In β-stable nuclear matter the following processes take place simultaneously

$$n \to p + e^- + \bar{\nu}_e, \quad p + e^- \to n + \nu_e \tag{26}$$

and considering that neutrinos generated in these reactions have left the system, the chemical equilibrium condition takes the form

$$\mu_n = \mu_p + \mu_e. \tag{27}$$

It is easy to show that after some algebra we get

$$\mu_n - \mu_p = \left(-\frac{\partial E_b}{\partial x}\right)_n. \tag{28}$$

Finally, we found

$$\left(\frac{\partial E_b}{\partial x}\right)_n = -\hbar c(3\pi^2 x n)^{1/3}. \tag{29}$$

Equation (29) is the most general relation that determines the proton fraction of β-stable matter and we will mention it hereafter as a full expansion (FE). Now the total energy per particle of neutron star matter $E(n, x)$ will be given by the sum of the energy per baryon and electron energy, that is

$$E(n, x) = E_b(n, x) + E_e(n, x), \tag{30}$$

where the fraction x is determined, in general, by Eq.(29). The electrons are considered as a non-interacting Fermi gas and consequently

$$E_e(n, x) = \frac{3}{4}\hbar c\left(3\pi^2 x^4 n^4\right)^{1/3}. \tag{31}$$

Accordingly, the total pressure is decomposed also into baryon and lepton contributions

$$P(n, x) = P_b(n, x) + P_e(n, x), \tag{32}$$

where by definition

$$P_b(n,x) = n^2 \left(\frac{\partial E_b}{\partial n}\right)_x . \tag{33}$$

The contribution of the electrons to the total pressure is equal to

$$P_e(n,x) = \frac{1}{12\pi^2}\frac{\mu_e^4}{(\hbar c)^3} = \frac{\hbar c}{12\pi^2}\left(3\pi^2 x n\right)^{4/3} . \tag{34}$$

Now, the transition pressure $P_{\rm t}$ in the case of the FE, is given by the equation

$$P_{\rm t}^{FE}(n_{\rm t},x_{\rm t}) = n_{\rm t}^2 \left(\frac{\partial E_b}{\partial n}\right)_{n=n_{\rm t}} + \frac{\hbar c}{12\pi^2}\left(3\pi^2 x_{\rm t} n_{\rm t}\right)^{4/3} . \tag{35}$$

In the case of the parabolic approximation, the use of Eq. (29) with the definition (24) leads to the determination of the proton fraction by the equation

$$4(1-2x)E_{\rm sym}^{PA}(n) = \hbar c(3\pi^2 n x)^{1/3}. \tag{36}$$

In this case the transition pressure $P_{\rm t}^{PA}$ is given by the relation

$$P_{\rm t}^{PA}(n_{\rm t},x_{\rm t}) = n_{\rm t}^2 \left[\left(\frac{{\rm d}E_b(n,x=0.5)}{{\rm d}n}\right)_{n=n_{\rm t}} + \left(\frac{{\rm d}E_{\rm sym}^{PA}(n)}{{\rm d}n}\right)_{n=n_{\rm t}} (1-2x_{\rm t})^2\right] + \frac{\hbar c}{12\pi^2}\left(3\pi^2 x_{\rm t} n_{\rm t}\right)^{4/3} . \tag{37}$$

4 The Models

In the present work we employed various nuclear models, which are suitable for reproducing the bulk properties of nuclear matter at low densities, close to saturation density as well as the maximum observational neutron star mass. In particular, in each case, the energy per particle of nuclear matter $E_b(n,I)$ is given as a function of the baryonic number density n and the asymmetry parameter I (or the proton fraction x).

4.1 MDI model

The momentum-dependent interaction (MDI) model used here, was already presented and analyzed in a previous paper. The MDI model is designed to reproduce the results of the microscopic calculations of both nuclear and neutron rich matter at zero temperature and it can be extended to finite temperature. The energy per baryon at $T=0$, is given by

$$\begin{aligned} E_b(n,I) &= \frac{3}{10}E_F^0 u^{2/3}\left[(1+I)^{5/3}+(1-I)^{5/3}\right] + \frac{1}{3}A\left[\frac{3}{2}-(\frac{1}{2}+x_0)I^2\right]u + \frac{\frac{2}{3}B\left[\frac{3}{2}-(\frac{1}{2}+x_3)I^2\right]u^{\sigma}}{1+\frac{2}{3}B'\left[\frac{3}{2}-(\frac{1}{2}+x_3)I^2\right]u^{\sigma-1}} \\ &+ \frac{3}{2}\sum_{i=1,2}\left[C_i + \frac{C_i-8Z_i}{5}I\right]\left(\frac{\Lambda_i}{k_F^0}\right)^3 \left(\frac{((1+I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}} - \tan^{-1}\frac{((1+I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}}\right) \\ &+ \frac{3}{2}\sum_{i=1,2}\left[C_i - \frac{C_i-8Z_i}{5}I\right]\left(\frac{\Lambda_i}{k_F^0}\right)^3 \left(\frac{((1-I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}} - \tan^{-1}\frac{((1-I)u)^{1/3}}{\frac{\Lambda_i}{k_F^0}}\right) . \end{aligned} \tag{38}$$

In Eq. (38) the ratio u is defined as $u=n/n_s$, with n_s denoting the equilibrium symmetric nuclear matter density (or saturation density), $n_s = 0.16$ fm^{-3}. The parameters A, B, σ, C_1, C_2 and B' which appear in the description of symmetric nuclear matter are determined in order that $E(n=n_0)-mc^2=-16$ MeV, $n_0 = 0.16$ fm^{-3}, and the incompressibility is $K=240$ MeV and have the values $A=-46.65$, $B=39.45$, $\sigma=1.663$, $C_1=-83.84$, $C_2=23$ and $B'=0.3$. The finite range parameters are $\Lambda_1=1.5k_F^0$ and $\Lambda_2=3k_F^0$ with k_F^0 being the Fermi momentum at the saturation density n_s. By suitably choosing the parameters x_0, x_3, Z_1, and Z_2, it is possible to obtain different forms for the density dependence of the symmetry energy as well as on the value of the slope parameter L and the value of the symmetry energy at the saturation density. Actually, for each value of L the density dependence of the symmetry energy is adjusted so that the energy of pure neutron matter is comparable with those of existing state-of-the-art calculations.

4.2 Skyrme model

The Skyrme functional providing the energy per baryon of asymmetric nuclear matter is given by the formula

$$
\begin{aligned}
E_b(n,I) &= \frac{3}{10}\frac{\hbar^2c^2}{m}\left(\frac{3\pi^2}{2}\right)^{2/3} n^{2/3}F_{5/3}(I) + \frac{1}{8}t_0 n\left[2(x_0+2)-(2x_0+1)F_2(I)\right] \\
&+ \frac{1}{48}t_3 n^{\sigma+1}\left[2(x_3+2)-(2x_3+1)F_2(I)\right] \\
&+ \frac{3}{40}\left(\frac{3\pi^2}{2}\right)^{2/3} n^{5/3}\left[\left(t_1(x_1+2)+t_2(x_2+2)\right)F_{5/3}(I)+\frac{1}{2}\left(t_2(2x_2+1)-t_1(2x_1+1)\right)F_{8/3}(I)\right],
\end{aligned}
\tag{39}
$$

where $F_m(I)=\frac{1}{2}\left[(1+I)^m+(1-I)^m\right]$.

4.3 The HLPS model

Recently, Hebeler *et al.* performed microscopic calculations based on chiral effective field theory interactions to constrain the properties of neutron-rich matter below nuclear densities. It explains the massive neutron stars of $M = 2M_{\odot}$. In this model the energy per particle is given by

$$
\begin{aligned}
E_b(u,x) &= \frac{3T_0}{5}\left(x^{5/3}+(1-x)^{5/3}\right)(2u)^{2/3} - T_0\left[(2\alpha-4\alpha_L)x(1-x)+\alpha_L\right]u \\
&+ T_0\left[(2\eta-4\eta_L)x(1-x)+\eta_L\right]u^{\gamma},
\end{aligned}
\tag{40}
$$

where $T_0 = (3\pi^2 n_0/2)^{2/3}\hbar^2/(2m) = 36.84$ MeV. The parameters α, η, α_L and η_L are determined by combining the saturation properties of symmetric nuclear matter and the microscopic calculations for neutron matter. The parameter γ is used to adjust the values of the incompressibility K and influences the range of the values of the symmetry energy and its density derivative. In the present work we employ the values $\gamma = 4/3$, $\alpha = 5.87$, $\eta = 3.81$ and also $\alpha_L = 1.3631$ with $\eta_L = 0.7596$ (soft and intermediate equation of state) and $\alpha_L = 1.53148$ with $\eta_L = 1.02084$ (stiff equation of state).

5 Hot equation of state

5.1 The nuclear equation of state

The model we use here, is designed to reproduce the results of the microscopic calculations of both nuclear and neutron-rich matter at zero temperature and can be extended to finite temperature [6]. We provide the main characteristics of the model as follows:

The energy density of the asymmetric nuclear matter (ANM) is given by the relation

$$
\epsilon(n_n,n_p,T) = \epsilon^n_{kin}(n_n,T)+\epsilon^p_{kin}(n_p,T)+V_{int}(n_n,n_p,T), \tag{41}
$$

where n_n (n_p) is the neutron (proton) density and the total baryon density is $n = n_n + n_p$. The contribution of each kinetic term is

$$
\epsilon^\tau_{kin}(n_\tau,T) = 2\int \frac{d^3k}{(2\pi)^3}\frac{\hbar^2k^2}{2m}\, f_\tau(n_\tau,k,T), \tag{42}
$$

where f_τ, (for $\tau = n, p$) is the Fermi-Dirac distribution function. where f_τ, (for $\tau = n, p$) is the Fermi-Dirac distribution function with the form

$$f_\tau(n_\tau, k, T) = \left[1 + \exp\left(\frac{e_\tau(n_\tau, k, T) - \mu_\tau(n_\tau, T)}{T}\right)\right]^{-1}. \tag{43}$$

The nucleon density n_τ is evaluated from the following integral

$$n_\tau = 2\int \frac{d^3k}{(2\pi)^3} f_\tau(n_\tau, k, T) \tag{44}$$

In Eq. (43), $e_\tau(n_\tau, k, T)$ is the single particle energy (SPE) and $\mu_\tau(n_\tau, T)$ stands for the chemical potential of each species. The SPE has the form

$$e_\tau(n_\tau, k, T) = \frac{\hbar^2 k^2}{2m} + U_\tau(n_\tau, k, T), \tag{45}$$

where the single particle potential $U_\tau(n_\tau, k, T)$ is obtained by the functional derivative of the interaction part of the energy density with respect to the distribution function f_τ.

Including the effect of finite-range forces between nucleons, the potential contribution is parameterized as follows

$$\begin{aligned} V_{int}(n_n, n_p, T) &= \frac{1}{3}An_0\left[\frac{3}{2} - (\frac{1}{2} + x_0)I^2\right]u^2 + \frac{\frac{2}{3}Bn_0\left[\frac{3}{2} - (\frac{1}{2} + x_3)I^2\right]u^{\sigma+1}}{1 + \frac{2}{3}B'\left[\frac{3}{2} - (\frac{1}{2} + x_3)I^2\right]u^{\sigma-1}} \\ &+ u\sum_{i=1,2}\left[C_i\left(\mathcal{J}_n^i + \mathcal{J}_p^i\right) + \frac{(C_i - 8Z_i)}{5}I\left(\mathcal{J}_n^i - \mathcal{J}_p^i\right)\right], \end{aligned} \tag{46}$$

where

$$\mathcal{J}_\tau^i = 2\int \frac{d^3k}{(2\pi)^3} g(k, \Lambda_i) f_\tau(n_\tau, k, T). \tag{47}$$

In Eq. (46), I is the asymmetry parameter ($I = (n_n - n_p)/n$) and $u = n/n_0$, with n_0 denoting the equilibrium symmetric nuclear matter density, $n_0 = 0.16$ fm^{-3}. The asymmetry parameter I is related to the proton fraction Y_p by the equation $I = (1 - 2Y_p)$. The parameters A, B, σ, C_1, C_2 and B' which appear in the description of symmetric nuclear matter are determined in order that $E(n = n_0) - mc^2 = -16$ MeV, $n_0 = 0.16$ fm^{-3}, and the incompressibility are $K = 240$ MeV. The additional parameters x_0, x_3, Z_1, and Z_2 used to determine the properties of asymmetric nuclear matter are treated as parameters constrained by empirical knowledge.

The first two terms of the right-hand side of Eq. (46) arise from local contact nuclear interaction which lead to power density contributions as in the standard Skyrme equation of state. The first one corresponds to attractive interaction and the second one to repulsive. They are assumed to be temperature independent. The third term describes the effects of finite range interactions according to the choice of the function $g(k, \Lambda_i)$, and is the temperature dependent part of the interaction. This interaction is attractive and important at low momentum, but it weakens and disappears at very high momentum. The function, $g(k, \Lambda_i)$, suitably chosen to simulate finite range effects, has the following form

$$g(k, \Lambda_i) = \left[1 + \left(\frac{k}{\Lambda_i}\right)^2\right]^{-1}, \tag{48}$$

where the finite range parameters are $\Lambda_1 = 1.5k_F^0$ and $\Lambda_2 = 3k_F^0$ and k_F^0 is the Fermi momentum at the saturation point n_0.

5.2 Thermodynamic description of hot nuclear matter

In order to study the properties of nuclear matter at finite temperature, we need to introduce the Helmholtz free energy F which is written as

$$F(n,T,I) = E(n,T,I) - TS(n,T,I). \tag{49}$$

In Eq. (49), E is the internal energy per particle, $E = \epsilon/n$, and S is the entropy per particle, $S = s/n$. From Eq. (49) it is also concluded that for $T = 0$, the free energy F and the internal energy E coincide.

The entropy density s has the same functional form as that of a non-interacting gas system, given by the equation

$$s_\tau(n,T,I) = -2\int \frac{d^3k}{(2\pi)^3}\left[f_\tau \ln f_\tau + (1-f_\tau)\ln(1-f_\tau)\right], \tag{50}$$

while the pressure and the chemical potentials defined as follows

$$P = n^2\left(\frac{\partial \epsilon/n}{\partial n}\right)_{S,N_i}, \qquad \mu_i = \left(\frac{\partial \epsilon}{\partial n_i}\right)_{S,V,n_{j\neq i}}. \tag{51}$$

At this point we shall examine the properties and the EOS of nuclear matter by considering an isothermal process. In this case, the pressure and the chemical potentials are related to the derivative of the total free energy density $f = F/n$. More specifically, they are defined as follows

$$P = n^2\left(\frac{\partial f/n}{\partial n}\right)_{T,N_i}, \qquad \mu_i = \left(\frac{\partial f}{\partial n_i}\right)_{T,V,n_{j\neq i}}. \tag{52}$$

The pressure P can also be calculated from the equation

$$P = Ts - \epsilon + \sum_i \mu_i n_i. \tag{53}$$

It is also possible to calculate the entropy per particle $S(n,T)$ by differentiating the free energy density f with respect to the temperature, that is

$$S(n,T) = -\left(\frac{\partial f/n}{\partial T}\right)_{V,N_i}. \tag{54}$$

The comparison of the two entropies, that is from Eqs. (50) and (54), provides a test of the approximation used in the present work. It is easy to demonstrate by applying Eq. (52) that

$$\begin{aligned}
\mu_n &= F + u\left(\frac{\partial F}{\partial u}\right)_{Y_p,T} - Y_p\left(\frac{\partial F}{\partial Y_p}\right)_{n,T},\\
\mu_p &= \mu_n + \left(\frac{\partial F}{\partial Y_p}\right)_{n,T},\\
\hat{\mu} &= \mu_n - \mu_p = -\left(\frac{\partial F}{\partial Y_p}\right)_{n,T}.
\end{aligned} \tag{55}$$

We can define the symmetry free energy per particle $F_{sym}(n,T)$ by the following parabolic approximation

$$F(n,T,I) = F(n,T,I=0) + I^2 F_{sym}(n,T) = F(n,T,I=0) + (1-2Y_p)^2 F_{sym}(n,T), \tag{56}$$

where

$$F_{sym}(n,T) = F(n,T,I=1) - F(n,T,I=0). \tag{57}$$

It is worth noting that the above approximation is not valid from the beginning, but one needs to check the validity of the parabolic law in the present model before using it.

Now, by applying Eq. (56) in Eq. (55), we obtain the key relation

$$\hat{\mu} = \mu_n - \mu_p = 4(1-2Y_p)F_{sym}(n,T). \tag{58}$$

The above equation is similar to that obtained for cold nuclear matter by replacing $E_{sym}(n)$ with $F_{sym}(n,T)$. The finding that, for both quantities ($E(n,T,I)$ and $F(n,T,I)$), the dependence of the asymmetry parameter I can be approximated very well by a quadratic dependence leads to the conclusion that the entropy $S(n,T,I)$ must also exhibit quadratic dependence of I that is

$$S(n,T,I) = S(n,T,I=0) + I^2 S_{sym}(n,T) \tag{59}$$

where

$$S_{sym}(n,T) = S(n,T,I=1) - S(n,T,I=0) = \frac{1}{T}(E_{sym}(n,T) - F_{sym}(n,T)). \tag{60}$$

In order to check the validity of the parabolic approximation (60), we plot in Fig. 1 the difference $S(n,T,I=1) - S(n,T,I=0)$ as a function of I^2 at temperature $T = 10$ and $T = 30$ MeV for two baryon densities, i.e., $n = 0.2$ fm^{-3} and $n = 0.4$ fm^{-3}. It is thus evident that in a good approximation, an almost linear relation holds between $S(n,T,I=1) - S(n,T,I=0)$ and I^2, especially for low values of I^2.

Also noteworthy in the present model is that due to temperature dependence of the interaction part of the energy density, the temperature affects both the kinetic part contribution on the entropy S and the interaction part. Hence, the total entropy per baryon must be written as follow $S_{tot} = S_{kin} + S_{int}$.

5.3 β-equilibrium in hot proto-neutron star and supernova

Stable high density nuclear matter must be in chemical equilibrium with all type of reactions, including the weak interactions in which β decay and electron capture take place simultaneously

$$n \longrightarrow p + e^- + \bar{\nu}_e, \qquad p + e^- \longrightarrow n + \nu_e. \tag{61}$$

Both types of reactions change the electron per nucleon fraction, Y_e and thus affect the equation of state. In a previous study, we assumed that neutrinos generated in these reactions left the system. The absence of neutrino-trapping has a dramatic effect on the equation of state and is the main cause of a significant reduction in the values of the proton fraction Y_p.

The chemical equilibrium of reactions (61) can be expressed in terms of the chemical potentials for the four species

$$\mu_n + \mu_{\nu_e} = \mu_p + \mu_e. \tag{62}$$

The charge neutrality condition provides us with the equation

$$Y_p = Y_e, \tag{63}$$

while the total fraction of leptons is given by

$$Y_l = Y_e + Y_{\nu_e}. \tag{64}$$

Now, according to Eqs. (58) and (62) we have

$$\mu_e - \mu_{\nu_e} = \mu_n - \mu_p = 4\,(1-2Y_p)\,F_{sym}(n,T). \tag{65}$$

Moreover, the leptons (electrons, muons and neutrinos) density is given by the expression

$$n_l = \frac{g}{(2\pi)^3}\int \frac{\mathrm{d}\mathbf{k}}{1+\exp\left[\frac{\sqrt{\hbar^2k^2c^2+m_l^2c^4}-\mu_l}{T}\right]}, \tag{66}$$

where g stands for the spin degeneracy ($g = 2$ for electrons and muons and $g = 1$ for neutrinos). One can self-consistently solve Eqs. (63), (64), (65) and (66) in order to calculate the proton fraction $Y_p(=Y_e)$, the neutrino fractions Y_{ν_e}, as well as the electron chemical potential μ_e as a function of the baryon density n for various values of temperature T.

Depending on the form of the symmetry energy, muons can appear at high density. Prakash has shown that the more rapidly $F(u)$ increases with density, the lower the density at which muons appear. For example, with $F(u)=u$, muons appear at $u\sim 4$, while with the choice $F(u)=\sqrt{u}$, muons cannot appear till $u\sim 8$. However, the presence of muons has very little effect on the electron lepton fractions (compared to the case without the inclusion of muons) since Y_μ remains extremely small ($\sim 10^{-4}$) over a wide range of densities. Thus, we do not include the contribution of muons in our study.

The next step is to calculate the energy density and pressure of leptons given by the following formulae

$$\epsilon_l(n_l,T)=\frac{g}{(2\pi)^3}\int\frac{\sqrt{\hbar^2k^2c^2+m_l^2c^4}\ \mathrm{d}\mathbf{k}}{1+\exp\left[\frac{\sqrt{\hbar^2k^2c^2+m_l^2c^4}-\mu_l}{T}\right]}, \tag{67}$$

$$P_l(n_l,T)=\frac{1}{3}\frac{g(\hbar c)^2}{(2\pi)^3}\int\frac{1}{\sqrt{\hbar^2k^2c^2+m_l^2c^4}}\frac{k^2\ \mathrm{d}\mathbf{k}}{1+\exp\left[\frac{\sqrt{\hbar^2k^2c^2+m_l^2c^4}-\mu_l}{T}\right]}. \tag{68}$$

The entropy density s has the same functional form as that of a non-interacting gas system, given by the equation

$$s_l(n,T,I)=-g\int\frac{d^3k}{(2\pi)^3}\left[f_l\ln f_l+(1-f_l)\ln(1-f_l)\right]. \tag{69}$$

The equation of state of hot nuclear matter in β-equilibrium (considering that it consists of neutrons, protons, electrons and neutrinos) can be obtained by calculating the total energy density ϵ_{tot} as well as the total pressure P_{tot}. The total energy density is given by

$$\epsilon_{tot}(n,T,I)=\epsilon_b(n,T,I)+\sum_{l=e,\nu_e}\epsilon_l(n,T,I), \tag{70}$$

where $\epsilon_b(n,T,I)$ and $\epsilon_l(n,T,I)$ are the contributions of baryons and leptons respectively. The total pressure is

$$P_{tot}(n,T,I)=P_b(n,T,I)+\sum_{l=e,\nu_e}P_l(n,T,I), \tag{71}$$

where $P_b(n,T,I)$ is the contribution of the baryons (see Eq. (53)) i.e.

$$P_b(n,T,I)=T\sum_{\tau=p,n}s_\tau(n,T,I)+\sum_{\tau=n,p}n_\tau\mu_\tau(n,T,I)-\epsilon_b(n,T,I), \tag{72}$$

while $P_l(n,T,I)$ is the contribution of the leptons (see Eq. (68)). From Eqs. (70) and (71) we can construct the isothermal curves for energy and pressure and finally derive the isothermal behavior of the equation of state of hot nuclear matter under β-equilibrium.

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