Fixed_Point_Iteration_Script

January 28, 2022

[20]:
```
# We first import the libraries we will be using in this notebook
import numpy as np                                     #numpy
import time                                            #time (to measure the time it␣
 ↪takes to execute code)
import math                                            #math functions
from scipy import io, integrate, linalg, signal #scipy has some linear algebra␣
 ↪and specialized libraries
import matplotlib.pyplot as plt                        #pyplot
from matplotlib.animation import FuncAnimation  #for animation purposes
from IPython.display import HTML, Video         #to produce video

plt.rcParams['figure.figsize'] = [15, 5]; #changes the size of the display␣
 ↪window
```

1 Fixed point iteration method

We implement the fixed point iteration method, and apply it to finding a root of the following function:

[21]:
```
def fun1(x):
    return 1 + 0.9*np.sin(x)-x;
```

[24]:
```
def g1(x):
    return 1 + 0.9*np.sin(x);
```

The following is an implementation of the fixed point iteration applied to the function g(x). If it converges, it solves the problem: find p such that g(p)=p.

[22]:
```
def fixed_point_method(g,x0,a,b,tol,nmax,vrb=False):
    # Fixed point iteration method applied to find the fixed point of g from␣
 ↪starting point x0


    # Initial values for guess xn = f(xn)
    n=0;
    xn = x0;
    # Current guess is stored at rn[n]
    rn=np.array([xn]);
```

```
    r=xn;

    if vrb:
        print("\n Fixed point method with nmax=%d and tol=%1.1e\n" % (nmax, tol));
        print("\n|--n--|----xn----|---|g(xn)|---|");

        fig, (ax1, ax2) = plt.subplots(1, 2)
        fig.suptitle('Fixed method results')
        ax1.set(xlabel='x',ylabel='y=g(x)')
        xl=np.linspace(a,b,100,endpoint=True);
        yl=g(xl);
        ax1.plot(xl,yl); #plot of y = g(x)
        ax1.plot(xl,xl); #plot of line y = x. The intersection with g is the fixed point.
        #ax1.plot(np.array([xn,xn]),np.array([0,g(xn)]),'-rs');

    while n<=nmax:
        #print and pause. Remove or comment out pause if you want the code to go faster.
        if vrb:
            print("|--%d--|%1.8f|%1.8f|" % (n,xn,np.abs(g(xn))));

            ##################################################################################
            # Plot results of fixed pt iteration on subplot 1 of 2 (horizontal). If vrb is true, pause.
            ax1.plot(np.array([xn,g(xn)]),np.array([g(xn),g(xn)]),'-rs'); #horizontal line to y=x
            ##################################################################################

        # If the estimate is approximately a root, get out of while loop
        if np.abs(g(xn)-xn)<tol:
            #(break is an instruction that gets out of the while loop)
            break;

        # update iterate xn, increase n.
        n += 1;
        xn = g(xn); #apply g (fixed point step)

        if vrb:
            ax1.plot(np.array([xn,xn]),np.array([xn,g(xn)]),'-rs'); #vertical line back to y=g(x)
```

```
        rn = np.append(rn,xn); #add new guess to list of iterates

    # Set root estimate to xn.
    r=xn;

    if vrb:
        ␣
↪##############################################################################
        # subplot 2: approximate error log-log plot
        e = np.abs(g(rn) - rn);
        #np.abs(r-rn[0:n]);
        # steps array
        ln = np.arange(0,n+1);
        #log-log plot error vs interval length
        ax2.plot(ln,np.log10(e),'r--');
        ax2.set(xlabel='n',ylabel='log10(error)');
        ␣
↪##############################################################################

    return r, rn;
```

We also write a related method, which uses the function g(x) = x + f(x) to find a root of f(x) using fixed point. Recall that this is not the only choice for g(x), so this method is less general.

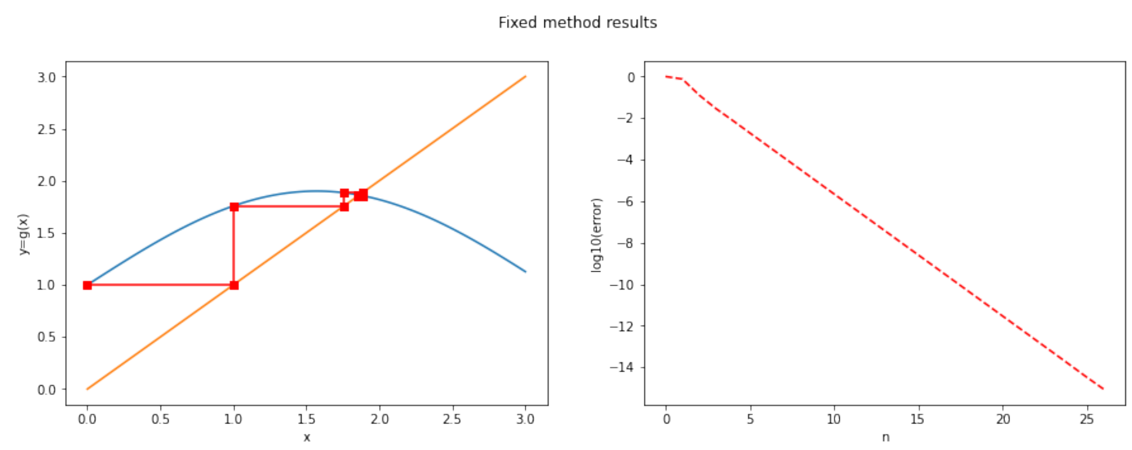
[23]:
```
def fixed_point_method_rf(f,x0,a,b,tol,nmax,vrb):
    # Fixed point iteration method applied to find the root of f from starting␣
↪point x0

    def g(x):
        return f(x)+x;
    # If g(x) = x, then f(x) = g(x) - x = 0.

    # Initial values for guess xn = f(xn)
    n=0;
    xn = x0;
    # Current guess is stored at rn[n]
    rn=np.array([xn]);
    r=xn;

    print("\n Fixed point method with nmax=%d and tol=%1.1e\n" % (nmax, tol));

    print("\n|--n--|----xn----|---|g(xn)|---|---|f(xn)|---|");

    fig, (ax1, ax2) = plt.subplots(1, 2)
    fig.suptitle('Fixed method results')
    ax1.set(xlabel='x',ylabel='y=f(x)')
    xl=np.linspace(a,b,100,endpoint=True);
```

```
    yl=f(xl);
    ax1.plot(xl,yl); #plot of y = f(x)
    ax1.plot(xl,np.zeros(np.size(xl))); #y = 0

    while n<=nmax:
        #print and pause. Remove or comment out pause if you want the code to␣
↪go faster.
        print("|--%d--|%1.8f|%1.8f|%1.8f|" % (n,xn,np.abs(g(xn)),np.
↪abs(f(xn))));

        ␣
↪#################################################################################
        # Plot results of fixed pt iteration on subplot 1 of 2 (horizontal). If␣
↪vrb is true, pause.
        ax1.plot(xn,f(xn),'rs');
        ␣
↪#################################################################################␣
↪

        # If the estimate is approximately a root, get out of while loop
        if np.abs(f(xn))<tol:
            #(break is an instruction that gets out of the while loop)
            break;

        # update iterate xn, increase n.
        n += 1;
        xn = g(xn); #apply g (fixed point step)
        rn = np.append(rn,xn); #add new guess to list of iterates

    # Set root estimate to xn.
    r=xn;

    ##########################################################################
    # subplot 2: approximate error log-log plot
    e = np.abs(f(rn));
    # steps array
    ln = np.arange(0,n+1);
    #log-log plot error vs interval length
    ax2.plot(ln,np.log10(e),'r--');
    ax2.set(xlabel='n',ylabel='log10(error)');
    ##########################################################################

    return r, rn;
```

We first apply the fixed point method to g1(x):

[25]:
```
(r,rn)=fixed_point_method(g1,0,0,3,1e-15,1000,True);
```

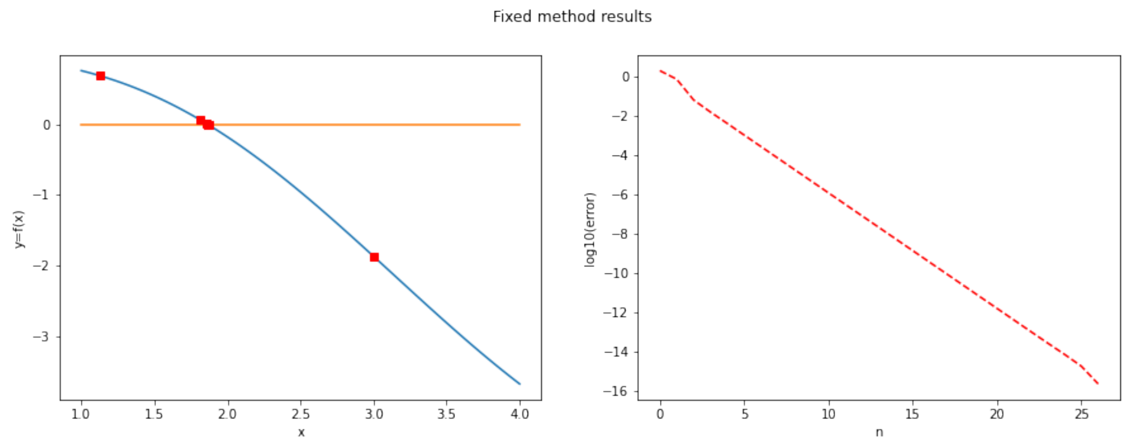
```
Fixed point method with nmax=1000 and tol=1.0e-15


|--n--|----xn----|---|g(xn)|---|
|--0--|0.00000000|1.00000000|
|--1--|1.00000000|1.75732389|
|--2--|1.75732389|1.88438870|
|--3--|1.88438870|1.85610839|
|--4--|1.85610839|1.86361648|
|--5--|1.86361648|1.86169027|
|--6--|1.86169027|1.86218908|
|--7--|1.86218908|1.86206022|
|--8--|1.86206022|1.86209353|
|--9--|1.86209353|1.86208492|
|--10--|1.86208492|1.86208714|
|--11--|1.86208714|1.86208657|
|--12--|1.86208657|1.86208672|
|--13--|1.86208672|1.86208668|
|--14--|1.86208668|1.86208669|
|--15--|1.86208669|1.86208669|
|--16--|1.86208669|1.86208669|
|--17--|1.86208669|1.86208669|
|--18--|1.86208669|1.86208669|
|--19--|1.86208669|1.86208669|
|--20--|1.86208669|1.86208669|
|--21--|1.86208669|1.86208669|
|--22--|1.86208669|1.86208669|
|--23--|1.86208669|1.86208669|
|--24--|1.86208669|1.86208669|
|--25--|1.86208669|1.86208669|
|--26--|1.86208669|1.86208669|
```



We get the exact same results if we apply our “rootfinding” method (as they are equivalent, so far)

[26]:
```
(r,rn)=fixed_point_method_rf(fun1,3,1,4,5e-16,100,True);
```

```
Fixed point method with nmax=100 and tol=5.0e-16


|--n--|----xn----|---|g(xn)|---|---|f(xn)|---|
|--0--|3.00000000|1.12700801|1.87299199|
|--1--|1.12700801|1.81281842|0.68581041|
|--2--|1.81281842|1.87376980|0.06095138|
|--3--|1.87376980|1.85900819|0.01476161|
|--4--|1.85900819|1.86287830|0.00387011|
|--5--|1.86287830|1.86188181|0.00099649|
|--6--|1.86188181|1.86213962|0.00025781|
|--7--|1.86213962|1.86207300|0.00006662|
|--8--|1.86207300|1.86209022|0.00001722|
|--9--|1.86209022|1.86208577|0.00000445|
|--10--|1.86208577|1.86208692|0.00000115|
|--11--|1.86208692|1.86208663|0.00000030|
|--12--|1.86208663|1.86208670|0.00000008|
|--13--|1.86208670|1.86208668|0.00000002|
|--14--|1.86208668|1.86208669|0.00000001|
|--15--|1.86208669|1.86208669|0.00000000|
|--16--|1.86208669|1.86208669|0.00000000|
|--17--|1.86208669|1.86208669|0.00000000|
|--18--|1.86208669|1.86208669|0.00000000|
|--19--|1.86208669|1.86208669|0.00000000|
|--20--|1.86208669|1.86208669|0.00000000|
|--21--|1.86208669|1.86208669|0.00000000|
|--22--|1.86208669|1.86208669|0.00000000|
|--23--|1.86208669|1.86208669|0.00000000|
|--24--|1.86208669|1.86208669|0.00000000|
|--25--|1.86208669|1.86208669|0.00000000|
|--26--|1.86208669|1.86208669|0.00000000|
```

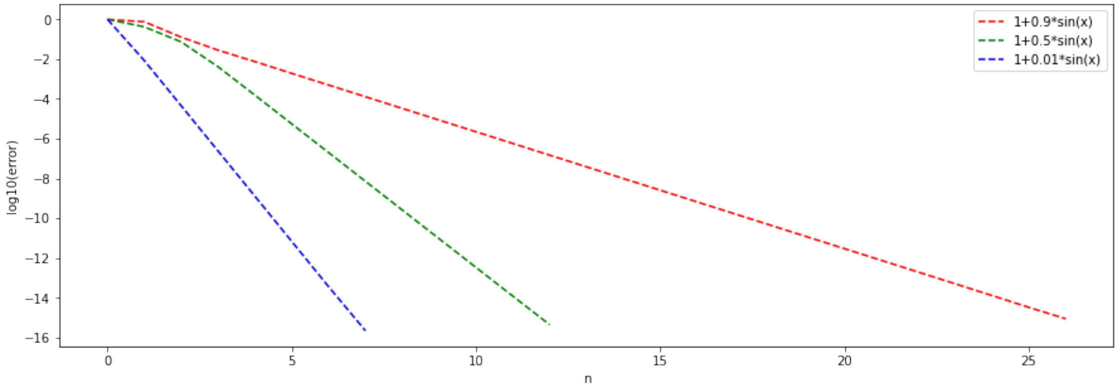


We now demonstrate the role the derivative of g(x) around the fixed point plays in the rate of convergence of the fixed point iteration method. For this purpose, we have 3 functions of the form 1 + c sin(x).

[27]:
```
def g1(x):
    return 1 + 0.9*np.sin(x);
def ga5(x):
    return 1 + 0.5*np.sin(x);
def ga1(x):
    return 1 + 0.01*np.sin(x);
```

We run the method for each of these (with vrb=False), and then compare log10(en) against number of iterations n. We clearly see a direct relationship between the observed rate of convergence and the value of g'(x) at and around the fixed point.

[28]:
```
(r1,r1n)=fixed_point_method(g1,0,0,3,1e-15,1000,False);
(r2,r2n)=fixed_point_method(ga5,0,0,3,1e-15,1000,False);
(r3,r3n)=fixed_point_method(ga1,0,0,3,1e-15,1000,False);
e1 = np.abs(r1n-g1(r1n));
e2 = np.abs(r2n-ga5(r2n));
e3 = np.abs(r3n-ga1(r3n));
ln1 = np.arange(0,len(e1));
ln2 = np.arange(0,len(e2));
ln3 = np.arange(0,len(e3));
plt.plot(ln1,np.log10(e1),'r--',label="1+0.9*sin(x)");
plt.plot(ln2,np.log10(e2),'g--',label="1+0.5*sin(x)");
plt.plot(ln3,np.log10(e3),'b--',label="1+0.01*sin(x)");
plt.xlabel('n'); plt.ylabel('log10(error)');
plt.legend();
```



[29]:
```
c1 = np.polyfit(ln1[1:len(e1)-1],np.log10(e1[1:len(e1)-1]),1);
print(10**c1);
c2 = np.polyfit(ln2[1:len(e2)-1],np.log10(e2[1:len(e2)-1]),1);
print(10**c2);
c3 = np.polyfit(ln3[1:len(e3)-1],np.log10(e3[1:len(e3)-1]),1);
print(10**c3);
```

```
[0.25680568 1.86574348]
[ 0.04030041 37.06288293]
[0.00533506 1.58299093]
```

Finally, we demonstrate what can happen if the fixed point method is applied to a function with g'(x)>1 around the fixed point. The fixed point of g(x) (or root of f) "repels" all initial points chosen near it. This usually causes divergent behavior (cycling or going to infinity)

[30]:
```
def fun2(x):
    return 3 + 2*np.sin(x)-x;
```

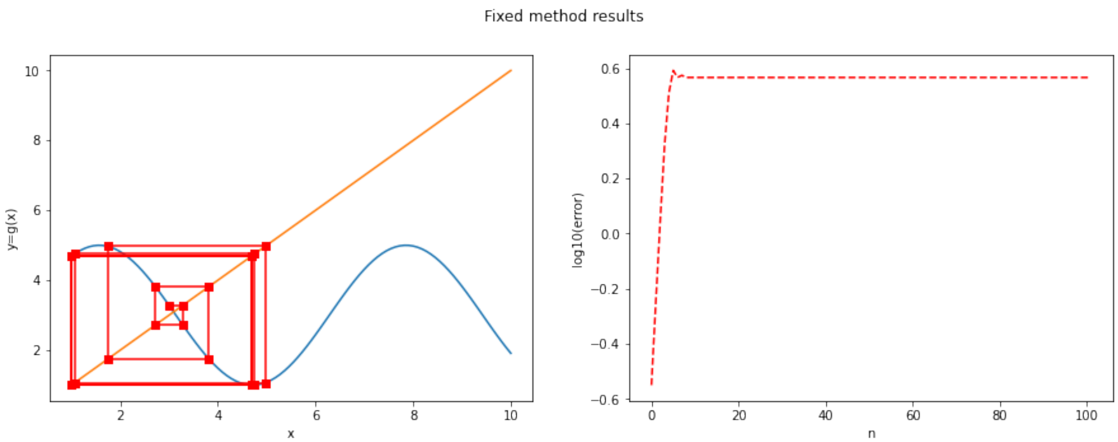
[31]:
```
def g2(x):
    return 3 + 2*np.sin(x);
```

[32]:
```
(r2,rn2)=fixed_point_method(g2,3,1,10,1e-15,100,True);
```

```
 Fixed point method with nmax=100 and tol=1.0e-15


|--n--|----xn----|---|g(xn)|---|
|--0--|3.00000000|3.28224002|
|--1--|3.28224002|2.71963177|
|--2--|2.71963177|3.81910025|
|--3--|3.81910025|1.74629390|
|--4--|1.74629390|4.96927957|
|--5--|4.96927957|1.06563065|
|--6--|1.06563065|4.75018862|
```

```
|--7--|4.75018862|1.00142864|
|--8--|1.00142864|4.68448405|
|--9--|4.68448405|1.00077863|
|--10--|1.00077863|4.68378286|
|--11--|4.68378286|1.00081825|
|--12--|1.00081825|4.68382562|
|--13--|4.68382562|1.00081581|
|--14--|1.00081581|4.68382298|
|--15--|4.68382298|1.00081596|
|--16--|1.00081596|4.68382314|
|--17--|4.68382314|1.00081595|
|--18--|1.00081595|4.68382313|
|--19--|4.68382313|1.00081595|
|--20--|1.00081595|4.68382313|
|--21--|4.68382313|1.00081595|
|--22--|1.00081595|4.68382313|
|--23--|4.68382313|1.00081595|
|--24--|1.00081595|4.68382313|
|--25--|4.68382313|1.00081595|
|--26--|1.00081595|4.68382313|
|--27--|4.68382313|1.00081595|
|--28--|1.00081595|4.68382313|
|--29--|4.68382313|1.00081595|
|--30--|1.00081595|4.68382313|
|--31--|4.68382313|1.00081595|
|--32--|1.00081595|4.68382313|
|--33--|4.68382313|1.00081595|
|--34--|1.00081595|4.68382313|
|--35--|4.68382313|1.00081595|
|--36--|1.00081595|4.68382313|
|--37--|4.68382313|1.00081595|
|--38--|1.00081595|4.68382313|
|--39--|4.68382313|1.00081595|
|--40--|1.00081595|4.68382313|
|--41--|4.68382313|1.00081595|
|--42--|1.00081595|4.68382313|
|--43--|4.68382313|1.00081595|
|--44--|1.00081595|4.68382313|
|--45--|4.68382313|1.00081595|
|--46--|1.00081595|4.68382313|
|--47--|4.68382313|1.00081595|
|--48--|1.00081595|4.68382313|
|--49--|4.68382313|1.00081595|
|--50--|1.00081595|4.68382313|
|--51--|4.68382313|1.00081595|
|--52--|1.00081595|4.68382313|
|--53--|4.68382313|1.00081595|
|--54--|1.00081595|4.68382313|
```

```
|--55--|4.68382313|1.00081595|
|--56--|1.00081595|4.68382313|
|--57--|4.68382313|1.00081595|
|--58--|1.00081595|4.68382313|
|--59--|4.68382313|1.00081595|
|--60--|1.00081595|4.68382313|
|--61--|4.68382313|1.00081595|
|--62--|1.00081595|4.68382313|
|--63--|4.68382313|1.00081595|
|--64--|1.00081595|4.68382313|
|--65--|4.68382313|1.00081595|
|--66--|1.00081595|4.68382313|
|--67--|4.68382313|1.00081595|
|--68--|1.00081595|4.68382313|
|--69--|4.68382313|1.00081595|
|--70--|1.00081595|4.68382313|
|--71--|4.68382313|1.00081595|
|--72--|1.00081595|4.68382313|
|--73--|4.68382313|1.00081595|
|--74--|1.00081595|4.68382313|
|--75--|4.68382313|1.00081595|
|--76--|1.00081595|4.68382313|
|--77--|4.68382313|1.00081595|
|--78--|1.00081595|4.68382313|
|--79--|4.68382313|1.00081595|
|--80--|1.00081595|4.68382313|
|--81--|4.68382313|1.00081595|
|--82--|1.00081595|4.68382313|
|--83--|4.68382313|1.00081595|
|--84--|1.00081595|4.68382313|
|--85--|4.68382313|1.00081595|
|--86--|1.00081595|4.68382313|
|--87--|4.68382313|1.00081595|
|--88--|1.00081595|4.68382313|
|--89--|4.68382313|1.00081595|
|--90--|1.00081595|4.68382313|
|--91--|4.68382313|1.00081595|
|--92--|1.00081595|4.68382313|
|--93--|4.68382313|1.00081595|
|--94--|1.00081595|4.68382313|
|--95--|4.68382313|1.00081595|
|--96--|1.00081595|4.68382313|
|--97--|4.68382313|1.00081595|
|--98--|1.00081595|4.68382313|
|--99--|4.68382313|1.00081595|
|--100--|1.00081595|4.68382313|
```



[39]:
```
xl = np.linspace(0,8,300);
def g2p(x):
    return 2*np.cos(x);
yl=g2(xl);
plt.plot(xl,yl,label="g(x)"); #plot of y = g(x)
plt.plot(xl,xl,label="y=x"); #plot of line y = x. The intersection with g is␣
 ↪the fixed point.
idconv = np.abs(g2p(xl))<1.0;
iddiv = np.abs(g2p(xl))>=1.0;
plt.plot(xl[idconv],g2p(xl[idconv]),'.g',label="g'(x)<1");
plt.plot(xl[iddiv],g2p(xl[iddiv]),'.r',label="g'(x)>=1");
plt.legend();
```