Chaos theory and Atmospheric predictability

ME5127 Term paper

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IITM

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Aperiodic Map

Let us consider a simple function:

$$\mathrm{f}(x) = 4x(1-x), \qquad \mathrm{f} : [0,1] \to [0,1]$$

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Aperiodic Map

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- $x = 0.4$
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- $\{0.4, 0.96, 0.1536, 0.520028, 0.998395, 0.00640774\}$

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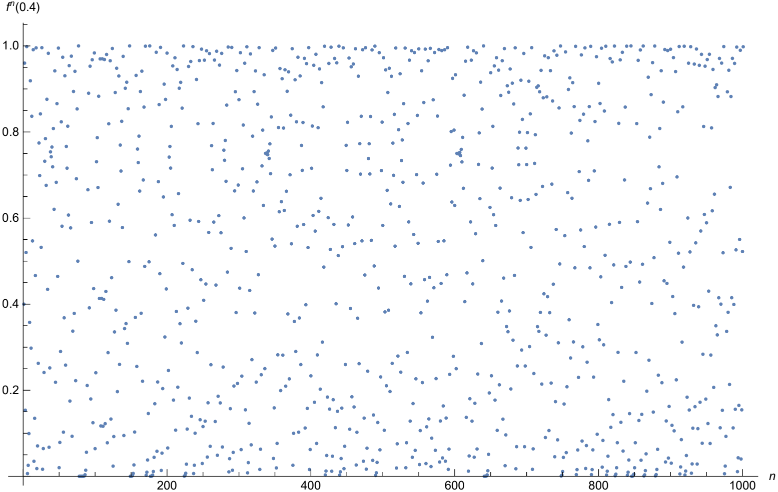

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- $f(x) = 0.96$, $f(f(x)) = 0.1536$
- $\{0.4, 0.96, 0.1536, 0.520028, 0.998395, 0.00640774\}$
- Now we also take $x_0 = 0.40001$ alongside $x_0 = 0.4$

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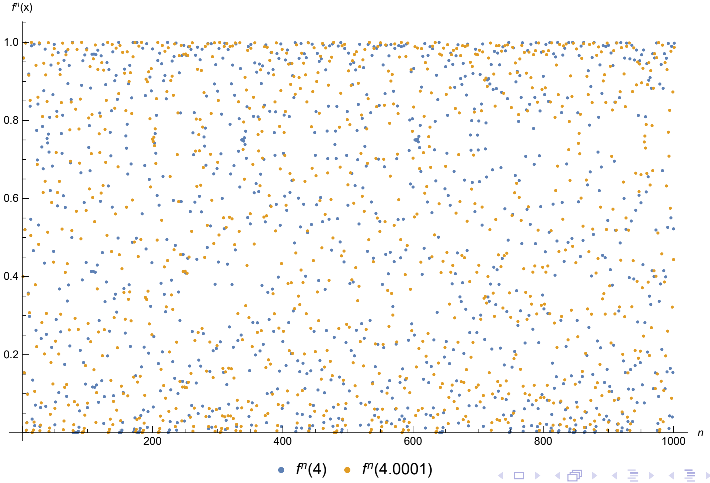
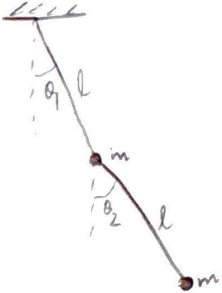


Chaos in a Double pendulum

We considered a discretised system before, now we shall look at a continuous (*in time*) system.

Chaos in a Double pendulum



We use Lagrangian mechanics for the formulation:

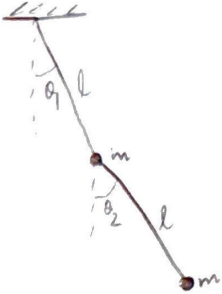
$$\mathcal{T} = \frac{1}{2} l^2 m \left(2\left(\theta_1'\right)^2 + \left(\theta_2'\right)^2 + 2\theta_2'\theta_1' \cos\left(\theta_1 - \theta_2\right)\right)$$
$$\mathcal{V} = -glm\left(2\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right)$$
$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

Now the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial q'}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

with $q = \theta_1,\, \theta_2$.
And time is also rescaled as $t \to \sqrt{\frac{l}{g}}\tau$ in the equations of motion

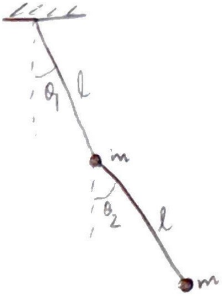
Chaos in a Double pendulum



The equations of motion are:

$$\left(\theta_2'\right)^2 \sin\left(\theta_1-\theta_2\right)+2\theta_1''+\theta_2''\cos\left(\theta_1-\theta_2\right)+2\sin\left(\theta_1\right)=0$$
$$\left(\theta_1'\right)^2 \left(-\sin\left(\theta_1-\theta_2\right)\right)+\theta_2''+\theta_1''\cos\left(\theta_1-\theta_2\right)+\sin\left(\theta_2\right)=0$$

Chaos in a Double pendulum



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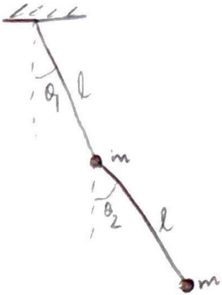
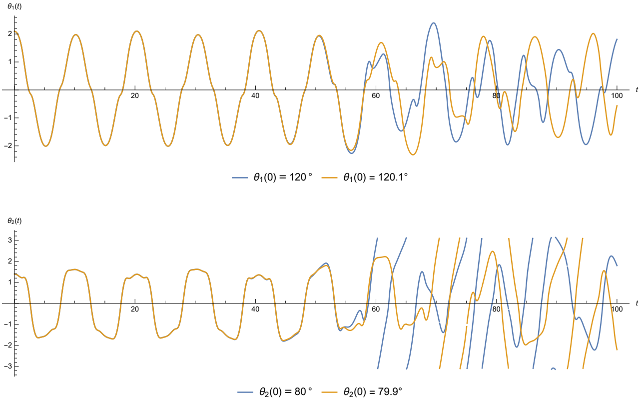
We consider the following initial conditions:

Initial conditions	$\theta_1(0)$	$\theta_2(0)$	$\theta_1'(0)$	$\theta_2'(0)$
System 1	120°	80°	0	0
System 2	120.1°	79.9°	0	0

Chaos in a Double pendulum





Features of Chaos

Definition

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

1. Aperiodic long-term behaviour
2. "Deterministic"
3. Sensitive dependence on initial conditions

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1. Aperiodic long-term behaviour
   There are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$
2. "Deterministic"
3. Sensitive dependence on initial conditions

Features of Chaos

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Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

1. Aperiodic long-term behaviour
2. "Deterministic"
   The system has no random or noisy inputs or parameters. The irregular behavior arises from the system's nonlinearity, rather than from noisy driving forces
3. Sensitive dependence on initial conditions

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Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

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   Nearby trajectories separate exponentially fast

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- He studied a skeletal form of Saltzman's convection equations:

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(x,z)} + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x}$$

$$\frac{\partial}{\partial t} = -\frac{\partial(\psi,\theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} + \kappa\nabla^2\theta$$

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- Which, after much simplification, transformed to

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

with $\sigma, r, b > 0$, all of which are parameters.

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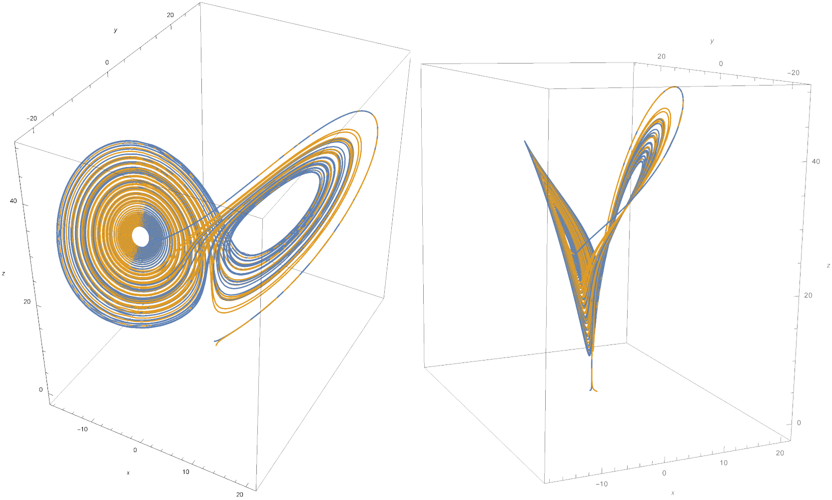

Figure: Lorenz attractors for $\sigma = 10, b = 8/3, r = 28$ and initial conditions (0,1,0) and (1,0,0)

Atmospheric predictability

Figure: Athirapally Falls on 7-3-2021, *Photo: Nikhil S*

Atmospheric predictability

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- As time goes on, the errors multiply and increase exponentially

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When our results concerning the instability of non-periodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the *sufficiently distant future is impossible by any method, unless the present conditions are known exactly*. In view of the inevitable inaccuracy and incompleteness of weather observations, **precise very-long-range forecasting would seem to be non-existent**.

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The European Centre's assessments suggested that the world saved billions of dollars each year from predictions that were statistically better than nothing. But beyond <u>two or three days</u> the world's best forecasts were speculative. ...[James Gleick, 1987]

The End

thank you!

Nikhil S
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