Graph Theory: Homework 1

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Problem 1

$$MM^T = A + D_V \tag{1}$$

where D_V is a diagonal matrix where the degree of the vertex defined by row i in M is the (i, i) entry in D_V.

Let M be a q by r size matrix and let A be a q by q size matrix. Let $a_{i,j}$ be the entry in A at row i and column j. This entry is defined by $M_i M_j^T$, where M_i is the row vector at i and M_j^T is a column vector in M^T. This product is equivalent to $M_i \cdot M_j$ since a column vector at j in M^T is equivalent to a row vector at j in M. Thus, we can define $a_{i,j}$ as follows where n is the length of the row vector.

$$\forall \underset{i \neq j}{i, j} \in [1, q] \quad a_{i,j} = M_i \cdot M_j = \sum_{t=1}^{r} m_{i,t} m_{j,t} \tag{2}$$

By definition of an incidence matrix, we know that each entry in M will be a 1 if the vertex defined by the entry's row number is incident to the edge defined by the entry's column number and a 0 otherwise. When $a_{i,j}$ is not an entry along the diagonal of A, there are two possible products. The first product is when both $m_{i,t}$ and $m_{j,t}$ are both 1. This will yield a 1 to the sum indicative that both vertex i and vertex j are connected to the same edge t. The second possible product is a 0, which is when either $m_{i,t}$ or $m_{j,t}$ is a 1 (indicative that one of the vertices is connected to edge t, but not the other) or when both $m_{i,t}$ and $m_{j,t}$ are 0, which is indicative that neither vertex is connected to edge t. The resulting sum will be total walks of length 1 between vertex i and vertex j.

$$\forall \underset{i \neq j}{i, j} \in [1, q] \quad a_{i,j} = \sum_{t=1}^{r} m_{i,t} m_{j,t} \tag{3}$$

This sum is bounded by r, or in other words $\sum_{t=1}^{r} m_{i,t} m_{j,t} \in [0, r]$.

$$\forall i \in [1, q] \quad d_{V\,i,i} = M_i \cdot M_i = \sum_{t=1}^{r} m_{i,t}^2 \tag{4}$$

Let $d_{V\,i,i}$ be the entry at (i, i) in the diagonal matrix D_V. When we take the dot product of the same row vector in the incidence matrix we get that each entry in the row vector will be squared then summed. Since each entry is 1 or 0, we are effectively taking the sum of all the entries in the row vector. Entries will only have a 1 if the edge defined by its column is incident to the vertex defined by the row, thus the sum will give us the degree since we are just counting the total number of edges that are incident to vertex i. This will yield the some integer between 0 and r.