Homework 4 for SI211: Numerical Analysis

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Abstract

This is the solution for Homework 4 of SI211: Numerical Analysis, which is taught by Boris Houska.

1 Problem 1

Prove that for all $x \in \mathbb{R}^n$ the inequality

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty \tag{1}$$

holds.

Solution 1. Suppose $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, then

$$\|x\|_\infty = \max_{i=1,\dots,n} |x_i| = \sqrt{\max_{i=1,\dots,n} x_i^2} \leq \sqrt{\sum_{i=1}^n x_i^2} = \|x\|_2$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n x_i^2 + 2\sum_{i\neq j} |x_i x_j|} = \sum_{i=1}^n |x_i| = \|x\|_1$$

$$\|x\|_1 = \sum_{i=1}^n |x_i| \leq n \max_{i=1,\dots,n} |x_i| = n\|x\|_\infty$$

2 Problem 2

Let $H, \langle \cdot, \cdot \rangle$ be a Hilbert space with norm $\|x\| = \sqrt{\langle x, x \rangle}$. Prove that $\langle x, y \rangle = 0$ if and only if we have $\|x + \alpha y\| = \|x - \alpha y\|$ for all scalars α.

Solution 2. If $\langle x, y \rangle = 0$, then for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\langle x, x \rangle + 2\alpha\langle x, y \rangle + \langle y, y \rangle = \langle x, x \rangle - 2\alpha\langle x, y \rangle + \langle y, y \rangle \\
\Rightarrow &\|x + \alpha y\|^2 = \|x - \alpha y\|^2 \\
\Rightarrow &\|x + \alpha y\| = \|x - \alpha y\|
\end{aligned}$$

where the last equality is because $\|\cdot\|$ is nonnegative.

Now if we have $\|x + \alpha y\| = \|x - \alpha y\|$ for any $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}
&\|x + \alpha y\| = \|x - \alpha y\| \\
\Rightarrow &\|x + \alpha y\|^2 = \|x - \alpha y\|^2 \\
\Rightarrow &\langle x, x \rangle + 2\alpha\langle x, y \rangle + \langle y, y \rangle = \langle x, x \rangle - 2\alpha\langle x, y \rangle + \langle y, y \rangle \\
\Rightarrow &\alpha\langle x, y \rangle = 0
\end{aligned}$$

Since $\alpha \in \mathbb{R}$ is arbitrary, we have $\langle x, y \rangle = 0$.

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3 Problem 3

Prove that the Legendre polynomials

$$P_n = \frac{1}{2^n n!} \frac{\partial^n}{\partial x^n} (x^2 - 1)^n \tag{2}$$

are orthogonal with respect to L_2-scalar product on the interval $[-1, 1]$.

Solution 3. The Legendre polynomials are defined by Legendre's differential equation

$$\left[(1 - x^2) P_k'(x)\right]' + k(k+1) P_k(x) = 0, k \in \mathbb{N} \tag{3}$$

For any $m, n \in \mathbb{N}, m \neq n$, first we let $k = n$ and multiply the equation by $P_m(x)$:

$$P_m(x) \left[(1 - x^2) P_n'(x)\right]' + n(n+1) P_m(x) P_n(x) = 0 \tag{4}$$

then we let $k = m$ and multiply the equation by $P_n(x)$:

$$P_n(x) \left[(1 - x^2) P_m'(x)\right]' + m(m+1) P_n(x) P_m(x) = 0 \tag{5}$$

subtracting the above equations yields:

$$P_m(x) \left[(1 - x^2) P_n'(x)\right]' - P_n(x) \left[(1 - x^2) P_m'(x)\right]' + \left[n(n+1) - m(m+1)\right] P_m(x) P_n(x) = 0$$
$$\Rightarrow \left\{(1 - x^2) \left[P_m(x) P_n'(x) - P_n(x) P_m'(x)\right]\right\}' + \left[n(n+1) - m(m+1)\right] P_m(x) P_n(x) = 0$$

Since

$$\begin{aligned} &\int_{-1}^{1} \left\{(1 - x^2) \left[P_m(x) P_n'(x) - P_n(x) P_m'(x)\right]\right\}' \mathrm{d}x \\ =& \left\{(1 - x^2) \left[P_m(x) P_n'(x) - P_n(x) P_m'(x)\right]\right\} \Big|_{-1}^{1} \\ =& 0 \end{aligned}$$

we have

$$0 = \int_{-1}^{1} \left[n(n+1) - m(m+1)\right] P_m(x) P_n(x) \, \mathrm{d}x = \int_{-1}^{1} P_m(x) P_n(x) \, \mathrm{d}x \tag{6}$$

which means $P_m(x)$ and $P_n(x)$ $(n \neq m)$ are orthogonal with respect to L_2-scalar product on the interval $[-1, 1]$.

4 Problem 4

Solve the least-squares optimization problem

$$\min_{p \in P_2} \int_{1}^{2} |f(x) - p(x)|^2 \, \mathrm{d}x \tag{7}$$

for $f(x) = e^x$ by using Legendre polynomials. Here, $P_2 2$ denotes the set of polynomials of order 2.

Solution 4. Since the Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{8}$$

form an orthogonal basis of the space of polynomials of order less than 2 in $[-1, 1]$, applying the shifting function $x \to 2x - 3$, we obtain an orthogonal basis of the space of polynomials of order less than 2 in $[1, 2]$:

$$\tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = 2x - 3, \quad \tilde{P}_2(x) = 6x^2 - 18x + 13 \tag{9}$$

after nomarlization, we have

$$L_0(x) = 1, \quad L_1(x) = \sqrt{3}(2x - 3), \quad L_2(x) = \sqrt{5}(6x^2 - 18x + 13) \tag{10}$$

Thus $L_1(x), L_2(x), L_3(x)$ form an orthonormal basis for the space of of polynomials of order less than 2 in $[1, 2]$. The solution is then given by

$$p(x) = \sum_{i=0}^{2} c_k L_k(x) \tag{11}$$

where

$$\begin{aligned}
c_0(x) &= \int_1^2 L_0(x) f(x)\,\mathrm{d}x = e^2 - e \\
c_1(x) &= \int_1^2 L_1(x) f(x)\,\mathrm{d}x = \sqrt{3}(3e - e^2) \\
c_2(x) &= \int_1^2 L_2(x) f(x)\,\mathrm{d}x = \sqrt{5}(7e^2 - 19e)
\end{aligned}$$

as a result the solution is:

$$\begin{aligned}
p(x) &= \sum_{i=0}^{2} c_k L_k(x) = e^2 - e + 3(3e - e^2)(2x - 3) + 5(7e^2 - 19e)(6x^2 - 18x + 13) \\
&= 3e(70ex^2 - 190x^2 - 212ex + 576x + 155e - 421)
\end{aligned}$$

5 Problem 5

Solve the least-squares optimization problem

$$\min_{p \in P_1} \int_0^\infty |f(x) - p(x)|^2 e^{-x}\,\mathrm{d}x \tag{12}$$

for $f(x) = x^2$. Here, P_1 denotes the set of polynomials of order 1.

Solution 5. There are two kind of solutions.

Mehtod 1 Since the space is simple, we assume $p(x) = ax + b$, then

$$\begin{aligned}
I &= \int_0^\infty |f(x) - p(x)|^2 e^{-x}\,\mathrm{d}x \\
&= \int_0^\infty |x^2 - (ax + b)|^2 e^{-x}\,\mathrm{d}x \\
&= b^2 + (2a - 4)b + 2a^2 - 12a + 24 \\
&= (b + (a - 2))^2 + (a - 4)^2 + 4
\end{aligned}$$

thus I is minimized when $a = 4, b = -2$. which means the best approximation polynomial is given by

$$p(x) = 4x - 2 \tag{13}$$

Method 2 Consider the Laguerre polynomials, the Laguerre polynomials are an orthonormal basis for polynomial space in $[0, \infty)$ with respect to the following inner product

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) e^{-x}\,\mathrm{d}x \tag{14}$$

Thus the best approximation is given by

$$p(x) = c_0 L_0(x) + c_1 L_1(x) \tag{15}$$

where $L_0(x) = 1$, $L_1(x) = -x + 1$, and

$$\begin{aligned}
c_0 &= \langle f(x), L_0(x) \rangle = \int_0^\infty x^2 e^{-x}\,\mathrm{d}x = 2 \\
c_1 &= \langle f(x), L_1(x) \rangle = \int_0^\infty x^2(-x + 1) e^{-x}\,\mathrm{d}x = -4
\end{aligned}$$

Thus the solution is given by

$$p(x) = 2 - 4(-x + 1) = 4x - 2 \tag{16}$$

which is consistent with the Method 1.