Homework 5 for SI211: Numerical Analysis

Mao Song
SIST
ShanghaiTech University
Student ID: 2019233134
maosong@shanghaitech.edu.cn

Abstract

This is the solution for Homework 5 of SI211: Numerical Analysis, which is taught by Boris Houska.

1 Probelm 1

The goal of this exercise is to compare different methods for approxi- mating the integral

$$\int_0^4 e^x \,\mathrm{d}x \tag{1}$$

For this aim, we first write the integral in the form

$$\int_0^4 e^x \,\mathrm{d}x = \sum_{i=0}^{N-1} \left\{ \int_{4i/N}^{4(i+1)/N} e^x \,\mathrm{d}x \right\} \tag{2}$$

then apply Simpson's rule on each of the integrals separately, and sum up the result.

1. Plot the actual error of the integral approximation versus N for $N \in \{0, 1, 2, \ldots, 100\}$.
2. Derive a theoretical bound on the integral approximation in dependence on N and plot this upper bound, too.

Solution 1. 1. The solution is shown as Fig 1.

2. By Simpson Rule:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \xi \in (a,b)\\ &:= I_{[a,b]} - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \xi \in (a,b)\end{aligned}$$

Denote $x_i = 4i/N$ and $x_{i+1} = 4(i+1)/N$, we have

$$\begin{aligned}\int_0^4 e^x \,\mathrm{d}x &= \sum_{i=0}^{N-1} \left\{ \int_{x_i}^{x_{i+1}} e^x \,\mathrm{d}x \right\}\\ &= \sum_{i=0}^{N-1} \left\{ I_{[x_i, x_{i+1}]} - \frac{16}{45N^5} \exp(\xi_i) \right\} . \xi_i \in (x_i, x_{i+1})\\ &= \sum_{i=0}^{N-1} I_{[x_i, x_{i+1}]} - \frac{16}{45N^5} \sum_{i=0}^{N-1} \exp(\xi_i)\end{aligned}$$

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which implies

$$\Big|\int_0^4 e^x \,\mathrm{d}x - \sum_{i=0}^{N-1} I_{[x_i,x_{i+1}]}\Big| = \frac{16}{45N^5}\sum_{i=0}^{N-1}\exp(\xi_i)$$
$$\le \frac{16}{45N^5}\sum_{i=0}^{N-1}\exp(x_{i+1})$$
$$= \frac{16p(1-p^N)}{45N^4(1-p)}$$

where $p = \exp(\frac{4}{N})$. The plot with the bound are shown in Fig 1.

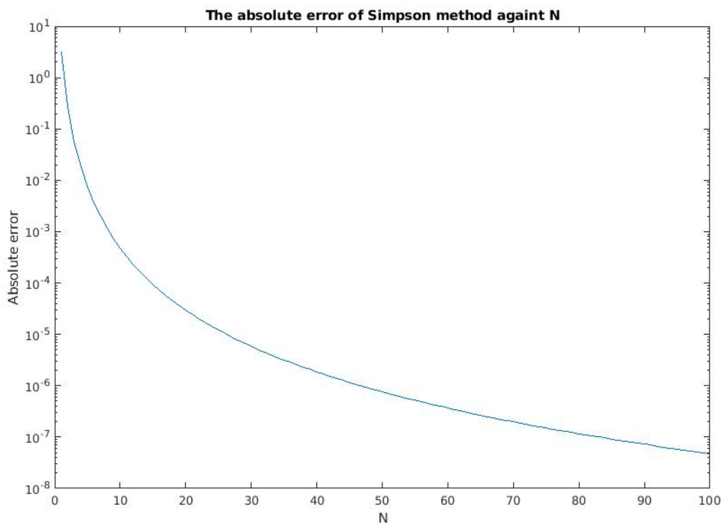

Figure 1: The absolute error of Simpson method against N

2 Probelm 2

Implement and compare the results of the closed Newton-Cotes formulas for $n = 3$ and $n = 5$ when approximating the integral

$$\int_0^{\pi/4} \sin(x)\,\mathrm{d}x = 1 - \frac{\sqrt{2}}{2} \tag{3}$$

Solution 2. Let $f(x) = \sin(x)$. For $n = 3$, we have $h = (\frac{\pi}{4} - 0)/3 = \frac{\pi}{12}$, $x_0 = 0$, $x_1 = \frac{\pi}{12}$, $x_2 = \frac{\pi}{6}$, $x_3 = \frac{\pi}{4}$. then

$$\int_0^{\frac{\pi}{4}} \sin(x)\,\mathrm{d}x \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$
$$= 0.29291070$$

The error is bounded by

$$\varepsilon_{n=3} \le \Big|\frac{3h^5}{80}f^{(4)}(\xi)\Big| = \Big|\frac{3h^5\pi^5}{80*12^5}\cos(\xi)\Big| \le \Big|\frac{3\pi^5}{80*12^5}\Big| \approx 0.00004611 \tag{4}$$

For $n = 5$, we have $h = (\frac{\pi}{4} - 0)/5 = \frac{\pi}{20}$, $x_0 = 0$, $x_1 = \frac{\pi}{20}$, $x_2 = \frac{\pi}{10}$, $x_3 = \frac{3\pi}{20}$, $x_4 = \frac{\pi}{5}$, $x_5 = \frac{\pi}{4}$. then

$$\int_0^{\frac{\pi}{4}} \sin(x)\,\mathrm{d}x \approx \frac{5h}{288}\left[19f(x_0) + 75f(x_1) + 50f(x_2) + 50f(x_3) + 75f(x_4) + 19f(x_5)\right]$$
$$= 0.29289320$$

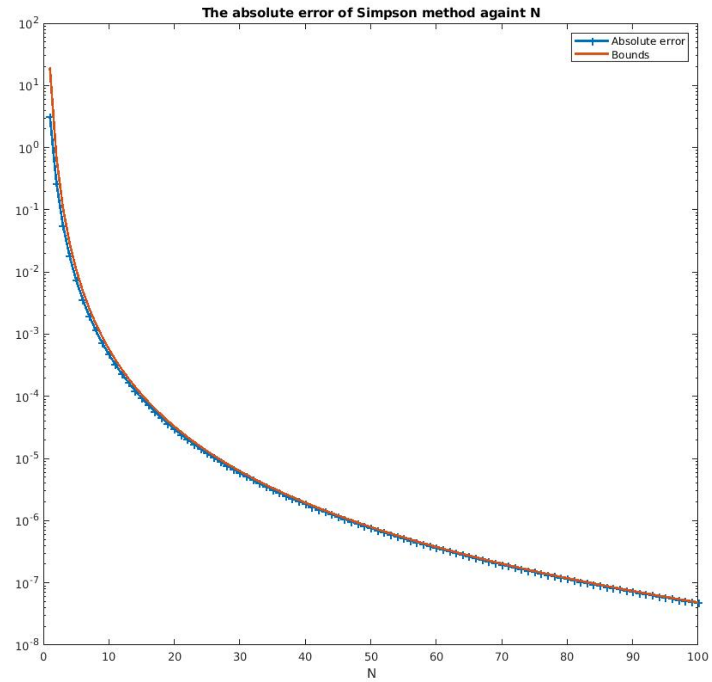

Figure 2: The absolute error of Simpson method against N

The error is bounded by

$$\varepsilon_{n=5} \leq \left|\frac{275h^7}{12096} f^{(6)}(\xi)\right| = \left|\frac{275\pi^7}{12096 * 12^7}\cos(\xi)\right| \leq \left|\frac{275\pi^7}{12096 * 20^7}\right| \approx 0.00000005 \tag{5}$$

The absolute error are

$$\varepsilon_{n=3} = |1 - \frac{\sqrt{2}}{2} - 0.29291070| = 0.00001748$$
$$\varepsilon_{n=5} = |1 - \frac{\sqrt{2}}{2} - 0.29289320| = 0.00000002$$

We can see that as n becomes large, the error and the error bound become smaller and smaller.

3 Probelm 3

The exact value of the integral

$$I(\omega) = \int_0^{\pi/4} \cos(\omega x)\,\mathrm{d}x \tag{6}$$

is given by $I(w) = \frac{1}{w}\sin(wx)$ for any $w > 0$. In the following, we test how accurate a Gauss-Quadrature of the form

$$I_1(\omega) = \sum_{i=0}^{1} \alpha_i \cos(\omega x_i) \tag{7}$$

can approximate this integral. Explain how to compute the approximation $I_1(\omega) \approx I(\omega)$. You may use that the second order q Legendre polynomial of order 2 on the interval $[-1, 1]$ has root s at $\pm\sqrt{\frac{1}{3}}$. How large is the approximation error $|I(1) - I_1(1)|$? What happens for large ω? Plot your result.

Solution 3. The first Gauss quadrature formula is given by

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right) \tag{8}$$

which is exact for polynomials of order less or equal than 3. By using the changing of variables:

$$t = \frac{8x-\pi}{\pi} \Leftrightarrow x = \frac{1}{2}\left[\frac{\pi}{4}t + \frac{\pi}{4}\right] \tag{9}$$

we have

$$\begin{aligned}
\int_0^{\pi/4} \cos(\omega x)\,\mathrm{d}x &= \int_0^{\pi/4} f(x)\,\mathrm{d}x \\
&= \int_{-1}^{1} f\left(\frac{\pi(t+1)}{8}\right)\frac{\pi}{8}\,\mathrm{d}t \\
&= \frac{\pi}{8}\int_{-1}^{1} g(t)\,\mathrm{d}t \\
&= \frac{\pi}{8}\left[g\left(-\sqrt{\frac{1}{3}}\right) + g\left(\sqrt{\frac{1}{3}}\right)\right] \\
&= \frac{\pi}{8}\left[f\left(\frac{\pi(\sqrt{3}-1)}{8\sqrt{3}}\right) + f\left(\frac{\pi(\sqrt{3}+1)}{8\sqrt{3}}\right)\right] \\
&= \frac{\pi}{8}\left[\cos\left(\omega\frac{\pi(\sqrt{3}-1)}{8\sqrt{3}}\right) + \cos\left(\omega\frac{\pi(\sqrt{3}+1)}{8\sqrt{3}}\right)\right]
\end{aligned}$$

where $f(x) = \cos(\omega x)$, $g(t) = f\left(\frac{\pi(t+1)}{8}\right)$. Compare with Eq 7, we have

$$\alpha_0 = \alpha_1 = \frac{\pi}{8}, x_0 = \frac{\pi(\sqrt{3}-1)}{8\sqrt{3}}, x_1 = \frac{\pi(\sqrt{3}+1)}{8\sqrt{3}} \tag{10}$$

The approximation error is then given by (notice that $n = 1$)

$$|I(1) - I_1(1)| \leq \left(\frac{\pi}{4}\right)^4 \approx 0.3805 \tag{11}$$

As Fig 3 shows, the error becomes large when ω becomes large, when $\omega > 10$, the error bound is meaningless, let's see the figure of exact solution and the approximate solution: As is can be seen from Fig 3, the exact solution is approximating the function $h(\omega) = \frac{1}{\omega}$, but the approximate solution changes as ω changes since it's a cos function of ω. In a word, when ω is relative small ($\omega < 10$), the approximation is accurate (the bound is meaningful). When ω is large, the approximation will fail.

4 Probelm 4

We would like to develop a new numerical integration formula by passing through the following steps:

1. compute the coefficients $c_1, \ldots, c_2, c_3$ such that

$$\forall i \in \{1, 2, 3\}, \quad f(x_i) = c_1 + c_2\sin(x_i) + c_3\cos(x_i) \tag{12}$$

   for $x_1 = a, x_2 = \frac{a+b}{2}$, and $x_3 = b$. You may assume that $b > a$ as well as $b - a < \frac{\pi}{2}$.

2. Derive an integral approximation of the form

$$\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b \left[c_1 + c_2\sin(x) + c_3\cos(x)\right]\mathrm{d}x \tag{13}$$

   by working out an explicit expression for the integral on the right side.

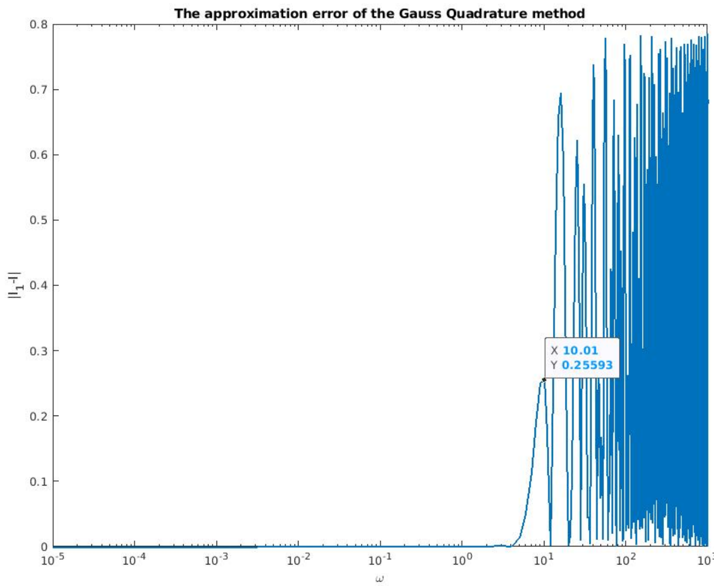

Figure 3: The error of approximation against ω

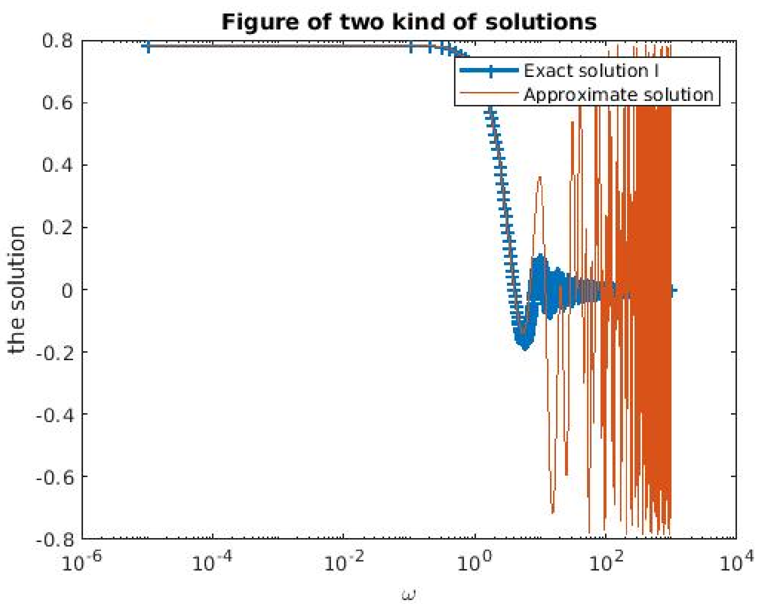

Figure 4: The figure of two kinds of solution against ω

3. Combine the above two results to show that the final numerical integration formula can be written in the form

$$\int_a^b f(x)\,\mathrm{d}x \approx \alpha_1 f(a) + \alpha_2 f\left(\frac{a+b}{2}\right) + \alpha_3 f(b) \tag{14}$$

What are the coefficients $\alpha_1, \alpha_2, \alpha_3$?

Compare the above integration formula with Simpson's formula for the integrals

$$\int_0^{0.5} \sin\left(\frac{9}{10}x\right)\mathrm{d}x, \int_0^1 x^3\,\mathrm{d}x \text{ and } \int_0^1 \cos(x)\,\mathrm{d}x \tag{15}$$

Which integration formula is better? Discuss advantages and disadvantages.

Solution 4. From step 1, c_1, c_2, c_3 are given by

$$\begin{bmatrix} 1 & \sin(a) & \cos(a) \\ 1 & \sin(\frac{a+b}{2}) & \cos(\frac{a+b}{2}) \\ 1 & \sin(b) & \cos(b) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} f(a) \\ f(\frac{a+b}{2}) \\ f(b) \end{bmatrix} \tag{16}$$

By the integral approximation form, we have

$$\int_a^b f(x)\,\mathrm{d}x \approx c_1(b-a) + c_2(\cos(a) - \cos(b)) + c_3(\sin(b) - \sin(a)) \tag{17}$$

By step 3, we have

$$\int_a^b f(x)\,\mathrm{d}x \approx \alpha_1 f(a) + \alpha_3 f(\frac{a+b}{2}) + \alpha_3 f(b) \tag{18}$$

$$= \alpha_1 \left[c_1 + c_2 \sin(a) + c_3 \cos(a)\right] + \alpha_2 \left[c_1 + c_2 \sin(\frac{a+b}{2}) c_3 \cos(\frac{a+b}{2})\right] \tag{19}$$

$$+ \alpha_3 \left[c_1 + c_2 \sin(b) + c_3 \cos(b)\right] \tag{20}$$

$$= c_1(b-a) + c_2(\cos(a) - \cos(b)) + c_3(\sin(b) - \sin(a)) \tag{21}$$

By collecting terms according to $f(a)$, $f\left(\frac{a+b}{2}\right)$, $f(b)$ in equation (19-21), we can obtain $\alpha_1, \alpha_2, \alpha_3$.

$$\alpha_1 = \frac{\cos(x_1 - x_2) - \cos(x_1 - x_3) - \cos(x_2 - x_3) - x_1 \sin(x_2 - x_3) + x_3 \sin(x_2 - x_3) + 1}{\sin(x_1 - x_2) - \sin(x_1 - x_3) + \sin(x_2 - x_3)}$$

$$\alpha_2 = \frac{2\cos(x_1 - x_3) + x_1 \sin(x_1 - x_3) - x_3 \sin(x_1 - x_3) - 2}{\sin(x_1 - x_2) - \sin(x_1 - x_3) + \sin(x_2 - x_3)}$$

$$\alpha_3 = -\frac{\cos(x_1 - x_2) + \cos(x_1 - x_3) - \cos(x_2 - x_3) + x_1 \sin(x_1 - x_2) - x_3 \sin(x_1 - x_2) - 1}{\sin(x_1 - x_2) - \sin(x_1 - x_3) + \sin(x_2 - x_3)}$$

After substituting $x_1 = a$, $x_2 = (a+b)/2$, $x_3 = b$, we have

$$\alpha_1 = -\frac{\cos(a-b) + a\sin(a/2 - b/2) - b\sin(a/2 - b/2) - 1}{2\sin(a/2 - b/2) - \sin(a-b)}$$

$$\alpha_2 = \frac{2\cos(a-b) + a\sin(a-b) - b\sin(a-b) - 2}{2\sin(a/2 - b/2) - \sin(a-b)}$$

$$\alpha_3 = -\frac{\cos(a-b) + a\sin(a/2 - b/2) - b\sin(a/2 - b/2) - 1}{2\sin(a/2 - b/2) - \sin(a-b)}$$

Denote I_1 as the above approximation integration, I_2 the Simpson's method, I the exact value of integration. For $f(x) = \sin(\frac{9x}{10})$, we have

$$I = 0.11061433, \quad I_1 = 0.11061396, \quad I_2 = 0.11061592 \tag{22}$$

The error is given by

$$\varepsilon_1 = |I_1 - I| = 3.7 * 10^{-7}, \quad \varepsilon_2 = |I_2 - I| = 1.59 * 10^{-6} \tag{23}$$

For $g(x) = x^3$, we have

$$I = 0.25000000, \quad I_1 = 0.25105105, \quad I_2 = 0.25000000 \tag{24}$$

The error is given by

$$\varepsilon_1 = |I_1 - I| = 1 * 10^{-3}, \quad \varepsilon_2 = |I_2 - I| = 0 \tag{25}$$

For $h(x) = \cos(x)$, we have

$$I = 0.84147098, \quad I_1 = 0.84147098, \quad I_2 = 0.84177209 \tag{26}$$

The error is given by

$$\varepsilon_1 = |I_1 - I| = 0, \quad \varepsilon_2 = |I_2 - I| = 3 * 10^{-4} \tag{27}$$

From the above results, we can see that when the function is trigonometric function $(\sin(x), \cos(x))$, the proposed approximation formula performs better than the Simposon method; when the function is a polynomial function, the Simpson method performs better than the proposed approximation formula. The reason is that we use $1, \cos(x), \sin(x)$ to form a basis while the Simposon method use the $1, x, x^2, \ldots$ to form a basis, when the function lies in different space, the corresponding method will be well-behaviored.