Final Exam for SI211: Numerical Analysis

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Abstract

This is the solution for Final Exam of SI211: Numerical Analysis, which is taught by Boris Houska.

1 Problem 1

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 10 & 16 \\ 6 & 16 & 28 \end{bmatrix} \tag{1}$$

be a given matrix in $\mathbb{R}^{3\times 3}$.

1. Find a lower triangular matrix $\mathbf{L}$ (with 1s on its diagonal) and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$.
2. Find a lower triangular matrix $\mathbf{L}$ (with 1s on its diagonal) and an upper triangular matrix $\mathbf{R}$ such that $\mathbf{A} = \mathbf{L}\mathbf{R}$.

Solution 1. 1. Suppose that

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{L}\mathbf{D}\mathbf{L}^T &= \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & \ell_{21} & \ell_{31} \\ 0 & 1 & \ell_{32} \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} d_{11} & \ell_{21}d_{11} & \ell_{31}d_{11} \\ \ell_{21}d_{11} & \ell_{21}^2 d_{11} + d_{22} & \ell_{21}\ell_{31}d_{11} + \ell_{32}d_{22} \\ \ell_{31}d_{11} & \ell_{21}\ell_{31}d_{11} + \ell_{32}d_{22} & \ell_{31}^2 d_{11} + \ell_{32}^2 d_{22} + d_{33} \end{bmatrix} \end{aligned}$$

compare with the matrix $\mathbf{A}$, we obtain that

$$\begin{aligned} d_{11} = 2, \quad \ell_{21} = 2, \quad \ell_{31} &= 3 \\ d_{22} = 2, \quad \ell_{32} &= 2 \\ d_{33} &= 2 \end{aligned}$$

thus,

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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2. Let

$$\mathbf{R} = \mathbf{D}\mathbf{L}^T = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Then $\mathbf{A} = \mathbf{L}\mathbf{R}$ is the LU decomposition of $\mathbf{A}$, by the uniqueness of the LU decomposition, we obtain such $\mathbf{L}$ and $\mathbf{R}$.

2 Problem 2: Convergence of Newton's Method

Let $f : \mathbb{R} \to \mathbb{R}$ be a three times continuously differentiable function with (locally) bounded third detivatives. The first and second detivatives of f are denoted by f' and f'', respectively. We additionally assume that

1. We have $f(x^*) = 0$ and $f''(x^*) = 0$ at a point $x^* \in \mathbb{R}$, and
2. we have $f'(x^*) \neq 0$.

Prove that the iterates of the exact Newton method, which takes the form

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$$

converges locally with cubic convergence rate, that is

$$|x^{k+1} - x^k| \leq \gamma |x^k - x^*|^3$$

for a constant $\gamma < \infty$.

Solution 2. We expand $f(x^k)$ and $f(x^k)$ at the point x^*:

$$\begin{aligned} f(x^k) &= f(x^*) + f'(x^*)(x^k - x^*) + \frac{f''(x^*)}{2}(x^k - x^*)^2 + \frac{f^{(3)}(\xi)}{6}(x^k - x^*)^3 \\ &= f'(x^*)(x^k - x^*) + \frac{f^{(3)}(\xi)}{6}(x^k - x^*)^3 \\ f'(x^k) &= f'(x^*) + f''(x^*)(x^k - x^*) + \frac{f^{(3)}(\eta)}{2}(x^k - x^*)^2 \\ &= f'(x^*) + \frac{f^{(3)}(\eta)}{2}(x^k - x^*)^2 \end{aligned}$$

where ξ, η are between x^k and x^*. Thus,

$$\begin{aligned} x^{k+1} - x^* &= x^k - x^* - \frac{f'(x^*)(x^k - x^*) + \frac{f^{(3)}(\xi)}{6}(x^k - x^*)^3}{f'(x^*) + \frac{f^{(3)}(\eta)}{2}(x^k - x^*)^2} \\ &= (x^k - x^*)\left(1 - \frac{f'(x^*) + \frac{f^{(3)}(\xi)}{6}(x^k - x^*)^2}{f'(x^*) + \frac{f^{(3)}(\eta)}{2}(x^k - x^*)^2}\right) \\ &= (x^k - x^*)^3 \frac{3f^{(3)}(\eta) - f^{(3)}(\xi)}{6f'(x^*) + 3f^{(3)}(\eta)(x^k - x^*)^2} \end{aligned}$$

since $6f'(x^*) + 3f^{(3)}(\eta)(x^k - x^*)^2 \neq 0$ for x^k sufficiently close to x^*. If we let

$$\gamma = \frac{3f^{(3)}(\eta) - f^{(3)}(\xi)}{6f'(x^*) + 3f^{(3)}(\eta)(x^k - x^*)^2}$$

then

$$|x^{k+1} - x^*| \leq \gamma |x^k - x^*|^3$$

3 Problem 3: Gauss Newton Method

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with a minimizer $x^* \in \mathbb{R}^n$ at which we have $\nabla f(x^*) = 0$ and $f(x^*) > 0$. We assume that the Hessian matrix $\nabla^2 f(x^*)$ at x^* is positive definite. Moreover, we introduce the vector-valued function: $g : \mathbb{R}^n \to \mathbb{R}^{n+1}$,

$$g(x) = \begin{bmatrix} f(x) \\ \nabla f(x) \end{bmatrix} \tag{2}$$

1. Prove that the minimizer of $f(x)$ is also a minimizer of the problem

$$\min_x \quad \frac{1}{2}\|g(x)\|^2 \tag{3}$$

2. Explain how to apply a Gauss-Newton method for solving the optimization problem:

$$\min_x \quad \frac{1}{2}\|g(x)\|^2 \tag{4}$$

Discuss the advantages or the disadvantages of Gauss-Newton method for finding the minimizer of f compared to standard Newton method. for solving the equation $\nabla f(x) = 0$ directly.

Solution 3.

1. Suppose that x^* is the minimizer of $f(x)$, that is,

$$f(x^*) = \min_{x \in \mathbb{R}^n} f(x)$$

Now, we need to prove that x^* is also a minimizer of

$$h(x) = \frac{1}{2}\|g(x)\|^2$$

that is, $\forall x \in \mathbb{R}^n, h(x^*) \leq h(x)$. Since f is smooth, $\nabla f(x^*) = 0$, and

$$\begin{aligned} h(x) &= \frac{1}{2}\|g(x)\|^2 \\ &= \frac{1}{2}(f^2(x) + \|\nabla f(x)\|_2^2) \\ &\geq \frac{1}{2}(f^2(x) + 0) \\ &\geq \frac{1}{2}(f^2(x^*) + 0) \\ &= \frac{1}{2}(f^2(x^*) + \|\nabla f(x^*)\|_2^2) \\ &= h(x^*), \forall x \in \mathbb{R}^n \end{aligned}$$

where the first inequality is because the non-negativity of norm, and the second inequality is because $f(x^*) = \min_{x\in\mathbb{R}^n} f(x) > 0$

2. The Gauss-Newton method takes the form

$$\begin{aligned} x^{k+1} &= x^k - \left[\nabla g(x^k)^T \nabla g(x^k)\right]^{-1} \nabla g(x^k)^T g(x^k) \\ &= x^k - \left[\begin{bmatrix} \nabla f(x^k)^T \\ \nabla^2 f(x^k) \end{bmatrix}^T \begin{bmatrix} \nabla f(x^k)^T \\ \nabla^2 f(x^k) \end{bmatrix}\right]^{-1} \begin{bmatrix} \nabla f(x^k), & \nabla^2 f(x^k) \end{bmatrix} \begin{bmatrix} f(x^k) \\ \nabla f(x^k) \end{bmatrix} \\ &= x^k - \left[\nabla f(x^k) \nabla f(x^k)^T + \left(\nabla^2 f(x^k)\right)^2\right]^{-1} \left[f(x^k)\mathbf{I} + \nabla^2 f(x^k)\right] \nabla f(x^k) \end{aligned}$$

this is a special type of Newton-type methods, of which $M(x^k)$ takes the form:

$$M(x^k) = \left[f(x^k)\mathbf{I} + \nabla^2 f(x^k)\right]^{-1} \left[\nabla f(x^k) \nabla f(x^k)^T + \left(\nabla^2 f(x^k)\right)^2\right]$$

and the update step can be written as

$$x^{k+1} = x^k - \left[M(x^k)\right]^{-1} \nabla f(x^k)$$

The disadvantage of this Gauss-Newton method:

(a) First, it needs to compute the the inverse of a complecated matrix $M(x^k)$ at each iteration, and the matrix $M(x^k)$ becomes ill-conditioned quickly compared to the Hessian matrix.

(b) Second, since the limit of $M(x^k)$:

$$\lim_{k\to\infty} M(x^k) = \left[f(x^*)\mathbf{I} + \nabla^2 f(x^*)\right]^{-1} \left(\nabla^2 f(x^*)\right)^2$$

equals to $\nabla^2 f(x^*)$ only if $f(x^*) = 0$, that is, the Gauss-Newton methods with locally quadratic convergence rate only if $f(x^*) = 0$, but $f(x^*) \neq 0$, which means the method is only with locally linearly convergence rate.

4 Problem 4: Equality Constrained Optimization

Let us consider an equality-constrained optimization problem of the form

$$\min_{x,y} \quad y^4 - \frac{x}{2} \tag{5}$$

$$s.t. \quad x + y = 1 \tag{6}$$

with scalar optimization variables $x, y \in \mathbb{R}$.

1. What is the minimizer (x^*, y^*) of the optimization problem (5)?
2. Let λ^* be the multiplier of the equality constraint of (5). What is the value of λ^*?
3. Let us consider an exact Newton method of the form $z^{k+1} = z^k - [J(z^k)]^{-1}R(z^k)$ for solving (5). Here,

$$R(z) = \begin{bmatrix} \nabla_x(y^4 - \frac{x}{2} + \lambda(x+y-1)) \\ \nabla_y(y^4 - \frac{x}{2} + \lambda(x+y-1)) \\ x + y - 1 \end{bmatrix}, z = \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} \tag{7}$$

and $J(z^k)$ is the Jacobian matrix of $R(z)$ with respect to z at z^k. $R(z)$ denotes the KKT residuum and the stacked primal-dual iterates, respectively. Because we consider an exact Newton method, we expect locally quadratic convergence of the method. What is the convergencee rate

$$\lim_{k\to\infty} \frac{\|z^{k+1} - z^*\|}{\|z^k - z^*\|^2} \tag{8}$$

of the method? You may assume that the method started at a point $x^0 \approx x^*$ in a very small neighborhood of the point $z^* = (x^*, y^*, z^*)$ and $\|\cdot\|$ denotes the Euclidean norm.

Solution 4. 1. The Lagrangian function is given by $L(x, y, \lambda) = y^4 - \frac{x}{2} + \lambda(x + y - 1)$. Write the KKT condition for the problem (5):

$$\begin{cases} \nabla_x L(x,y,\lambda) = \lambda - \frac{1}{2} = 0 \\ \nabla_y L(x,y,\lambda) = 4y^3 + \lambda = 0 \\ \nabla_\lambda L(x,y,\lambda) = x + y - 1 = 0 \end{cases} \tag{9}$$

from which we obtain that $(x^*, y^*, \lambda^*) = (\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$.

2. From (1), we know that $\lambda^* = \frac{1}{2}$.

3. From the definition of $R(z)$, we have

$$R(z) = \begin{bmatrix} \lambda - \frac{1}{2} \\ 4y^3 + \lambda \\ x + y - 1 \end{bmatrix} \text{ and } J(z^k) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 12(y^k)^2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{10}$$

The inverse of $J(z^k)$ is given by

$$[J(z^k)]^{-1} = \frac{1}{12(y^k)^2} \begin{bmatrix} 1 & -1 & 12(y^k)^2 \\ -1 & 1 & 0 \\ 12(y^k)^2 & 0 & 0 \end{bmatrix} \tag{11}$$

then,

$$
\begin{aligned}
12(y^k)^2 z^{k+1} &= 12(y^k)^2 z^k - 12(y^k)^2 [J(z^k)]^{-1} R(z^k) \\
&= 12(y^k)^2 \begin{bmatrix} x^k \\ y^k \\ \lambda^k \end{bmatrix} - \begin{bmatrix} 1 & -1 & 12(y^k)^2 \\ -1 & 1 & 0 \\ 12(y^k)^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda^k - \frac{1}{2} \\ 4(y^k)^3 + \lambda^k \\ x^k + y^k - 1 \end{bmatrix} \\
&= \begin{bmatrix} 12(y^k)^2 - 8(y^k)^3 + \frac{1}{2} \\ 8(y^k)^3 - \frac{1}{2} \\ \frac{1}{2} 12(y^k)^2 \end{bmatrix}
\end{aligned}
$$

that is,

$$
x^{k+1} = \frac{12(y^k)^2 - 8(y^k)^3 + \frac{1}{2}}{12(y^k)^2} \tag{12}
$$

$$
y^{k+1} = \frac{8(y^k)^3 - \frac{1}{2}}{12(y^k)^2} \tag{13}
$$

$$
\lambda^{k+1} = \frac{1}{2} \tag{14}
$$

from the above equations, we also have $x^{k+1} = 1 - y^{k+1}$. Then,

$$
z^{k+1} - z^* = \begin{bmatrix} -y^{k+1} - \frac{1}{2} \\ y^{k+1} + \frac{1}{2} \\ 0 \end{bmatrix} \tag{15}
$$

which implies that

$$
\|z^{k+1} - z^*\| = 2\|y^{k+1} + \frac{1}{2}\|, \forall k \geq 0 \tag{16}
$$

on the other hands, we expand $y^{k+1} - y^*$ at the point $y^k - y^* = y^k + \frac{1}{2}$, which yields

$$
\begin{aligned}
y^{k+1} + \frac{1}{2} &= \frac{8(y^k)^3 - \frac{1}{2} + 6(y^k)^2}{12(y^k)^2} \\
&= \frac{8(y^k + \frac{1}{2})^3 - 6(y^k + \frac{1}{2})^2}{12(y^k + \frac{1}{2})^2 - 12(y^k + \frac{1}{2}) + 3} \\
&= \left(y^k + \frac{1}{2}\right)^2 \left(-2 + o(y^k + \frac{1}{2})\right) \\
&= -2\left(y^k + \frac{1}{2}\right)^2 + o\left(\left(y^k + \frac{1}{2}\right)^3\right)
\end{aligned}
$$

thus we have

$$
\begin{aligned}
\|z^{k+1} - z^*\| &= 2\|y^{k+1} + \frac{1}{2}\| \\
&= 2\left\|2\left(y^k + \frac{1}{2}\right)^2\right\|^2 + o\left(\left\|y^k + \frac{1}{2}\right\|^3\right) \\
&= 2\left\|z^k - z^*\right\|^2 + o\left(\left\|z^k - z^*\right\|^3\right)
\end{aligned}
$$

which implies that

$$
\lim_{k \to \infty} \frac{\|z^{k+1} - z^*\|}{\|z^k - z^*\|^2} = 2
$$