Homework 6 for SI211: Numerical Analysis

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Abstract

This is the solution for Homework 6 of SI211: Numerical Analysis, which is taught by Boris Houska.

1 Problem 1

Provide a formal proof of the fact that if $\|\cdot\|$ is a matrix norm (assume $\|I\| = 1$) and $A \in \mathbb{R}^{m\times n}$ a given matrix whose norm satisfies $\|A\| < 1$, then the matrix $I - A$ is invertible and

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|} \tag{1}$$

Solution 1. Suppose that $I - A$ is not invertible, then there exists an eigen pair $(0, v)$ such that $(I - A)v = 0v = 0$, where $v \neq 0 \in \mathbb{R}^n$, which implies that $Av = v$. So, by the definition of matrix norm, we have

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Av\|}{\|v\|} = \frac{\|v\|}{\|v\|} = 1 \tag{2}$$

which contradicts with $\|A\| < 1$. Thus $I - A$ is invertible.

From the properties of matrix norm, we have

$$(I-A)(I-A)^{-1} = I \tag{3}$$

$$\Rightarrow (I-A)^{-1} = A(I-A)^{-1} + I \tag{4}$$

$$\Rightarrow \|(I-A)^{-1}\| \leq \|A(I-A)^{-1}\| + 1 \text{ (triangular inequality)} \tag{5}$$

$$\Rightarrow \|(I-A)^{-1}\| \leq \|A\|\|(I-A)^{-1}\| + 1 \text{ } (\|AB\| \leq \|A\|\|B\|) \tag{6}$$

$$\Rightarrow \|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|} \tag{7}$$

2 Problem 2

Solve the linear equation

$$\begin{bmatrix} 2.0002 & 1.9998 \\ 1.9998 & 2.0002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \tag{8}$$

Next, add a distribution to the right, $\delta b = \begin{bmatrix} 2 * 10^{-4}, -2 * 10^{-4} \end{bmatrix}^T$, and solve the new linear equation:

$$\begin{bmatrix} 2.0002 & 1.9998 \\ 1.9998 & 2.0002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4.0002 \\ 3.9998 \end{bmatrix} \tag{9}$$

Compare both solutions and compute $\mathrm{cond}(A)$, $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$, $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$. Interpret your results.

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Solution 2. The solution to first linear equation is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{10}$$

The solution to the second equation is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{11}$$

We will use the infinity matrix norm to deduce our condition number, since

$$\|A\|_\infty = 4, \|A^{-1}\|_\infty = 2500 \tag{12}$$

we have, $\mathrm{cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty = 10000$. Moreover, we have

$$\delta x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \|\delta x\|_\infty = 1, \|x\|_\infty = 2, \frac{\|\delta x\|_\infty}{\|x\|_\infty} = 0.5 \tag{13}$$

and

$$\delta b = \begin{bmatrix} 2 * 10^{-4} \\ -2 * 10^{-4} \end{bmatrix}, \|\delta b\|_\infty = 2 * 10^{-4}, \|b\|_\infty = 4, \frac{\|\delta b\|_\infty}{\|b\|_\infty} = 0.5 * 10^{-4} \tag{14}$$

the solution shows that when the right hand side vector is purburbated, the perturbation of solution is bounded by the condition number of coefficient matrix, that is,

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq \mathrm{cond}(A) \frac{\|\delta b\|_\infty}{\|b\|_\infty} \tag{15}$$

from this example, we can see the equality can be attained.

3 Problem 3

Implement an LR-decomposition code (slides7-36) by a programming language of your choice and solve the equation system

$$x - 2y + z = 0 \tag{16}$$
$$2x + y - 3z = 5 \tag{17}$$
$$4x - 7y + z = -1 \tag{18}$$

Solution 3. Rewrite the linear equation as

$$Av = b \tag{19}$$

where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix}, v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} \tag{20}$$

The LU decomposition of A is given by (see file *mylu.m*):

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0.2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & -2 \end{bmatrix} \tag{21}$$

then we solve two triangular linear equations (see files *forward.m* and *backward.m*), $Ly = b$ and $Uv = y$, we have

$$v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{22}$$

4 Problem 4

If A is a symmetric positive band-structured matrix with bandwidth $2n+1$, then its Cholesky decomposition factor L is also a band-structured matrix. What is the bandwidth of L?

Solution 4. The bandwidth of L is n. We prove the conclusion by induction on m. For $m = n+1$, we have nothing to prove since it's just a naive Cholesky decomposition. Suppose for all $A \in \mathbb{R}^{m\times m}$, where $m \geq n+1$, the conclusion is correct, let $A \in \mathbb{R}^{(m+1)\times(m+1)}$ be symmetric positive definite. Then we can write A as

$$A = \begin{bmatrix} \alpha & v^T \\ v & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v/\alpha & I_m \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & B - vv^T/\alpha \end{bmatrix} \begin{bmatrix} 1 & v^T/\alpha \\ 0 & I_m \end{bmatrix} = R \begin{bmatrix} \alpha & 0 \\ 0 & B - vv^T/\alpha \end{bmatrix} R^T \quad (23)$$

Since A has bandwidth $2n+1$, the matrix $B - vv^T/\alpha \in \mathbb{R}^{m\times m}$ has bandwidth $2n$(only first n components of v are non-zero). Since B is a principal submatrix of $R^{-T}AR^{-1}$ and A is symmetric positive definite, we can deduce that $B - vv^T/\alpha$ is also symmetric positive definite. By induction hypothesis, suppose $B - vv^T/\alpha = G_1G_1^T$ is the Cholesky decomposition of B, where G_1 has bandwidth $n-1$, noting the sparsity of v, it follows that the matrix

$$G = \begin{bmatrix} \sqrt{\alpha} & 0 \\ v/\sqrt{\alpha} & G_1 \end{bmatrix} \quad (24)$$

has bandwidth n and satisfies $A = GG^T$.