Homework 2 for SI211: Numerical Analysis

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Abstract

This is the solution for Homework 2 of SI211: Numerical Analysis, which is taught by Boris Houska.

1 Jacobian Matrix

Solution 1. The Jacobian matrix of $f(x)$ is

$$J = \begin{bmatrix} 2x_1x_2 + x_2^2 & x_1^2 + 2x_1x_2 \\ 2x_1 + x_2 & x_1 \end{bmatrix} \tag{1}$$

subsitute $x = (1, 2)^T$ into the above equation, we have

$$J|_{(1,2)^T} = \begin{bmatrix} 8 & 5 \\ 4 & 1 \end{bmatrix} \tag{2}$$

2 Polynomial interpolation

Solution 2. Since $p(-1) = f(-1) = 6$, $p(2) = f(2) = 12$, $p(4) = f(4) = 66$, we have

$$a_0 - a_1 - a_2 = 6 \tag{3}$$
$$a_0 + 2a_1 + 8a_2 = 12 \tag{4}$$
$$a_0 + 4a_1 + 64a_2 = 66 \tag{5}$$

rewrite it as the matrix form

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 8 \\ 1 & 4 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 66 \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix} \tag{6}$$

3 Interpolation with rational functions

Solution 3. We have that $q(-2) = f(-2) = -6$, $p(-1) = f(-1) = -3$, $p(1) = f(1) = 5$, $p(2) = f(2) = 10$, which is the same as

$$\begin{bmatrix} -1/2 & 1 & -2 & 4 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1/2 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 5 \\ 10 \end{bmatrix} \tag{7}$$

of which the solution is

$$\begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ 4 \\ 1/3 \end{bmatrix} \tag{8}$$

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4 Hermite interpolation

Solution 4. From the initial condition, $n = 1$. We first compute the Lagrange polynomials and their derivatives. This gives (We use the notation $x_0 = 1$ and $x_1 = 2$ for simplicity.)

$$L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1} = -(x-2), \quad L'_{1,0}(x) = -1$$
$$L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0} = x - 1, \quad L'_{1,1}(x) = 1$$

The polynomials $H_{1,0}(x), H_{1,1}(x)$ and $\hat{H}_{1,0}(x), \hat{H}_{1,1}(x)$ are then

$$\begin{aligned}
H_{1,0}(x) &= [1 - 2(x - x_0)L'_{1,0}(x)]L^2_{1,0}(x) = (2x-1)(x-2)^2 \\
H_{1,1}(x) &= [1 - 2(x - x_1)L'_{1,1}(x)]L^2_{1,1}(x) = (-2x+5)(x-1)^2 \\
\hat{H}_{1,0}(x) &= (x - x_0)L^2_{1,0}(x) = (x-1)(x-2)^2 \\
\hat{H}_{1,1}(x) &= (x - x_1)L^2_{1,1}(x) = (x-2)(x-1)^2
\end{aligned}$$

Finally,

$$\begin{aligned}
H_3(x) &= [p(x_0)H_{1,0}(x) + p(x_1)H_{1,1}(x)] + \left[p'(x_0)\hat{H}_{1,0}(x) + p'(x_1)\hat{H}_{1,1}(x)\right] \\
&= H_{1,0}(x) + 2H_{1,1}(x) + 2\hat{H}_{1,0}(x) + 4\hat{H}_{1,1}(x) \\
&= (2x-1)(x-2)^2 + 2(-2x+5)(x-1)^2 + 2(x-1)(x-2)^2 + 4(x-2)(x-1)^2 \\
&= (4x-3)(x-2)^2 + 2(x-1)^2 \\
&= 4x^3 - 17x^2 + 24x - 10
\end{aligned}$$

5 Polynomial Approximation error

Solution 5. The results are shown as follows:

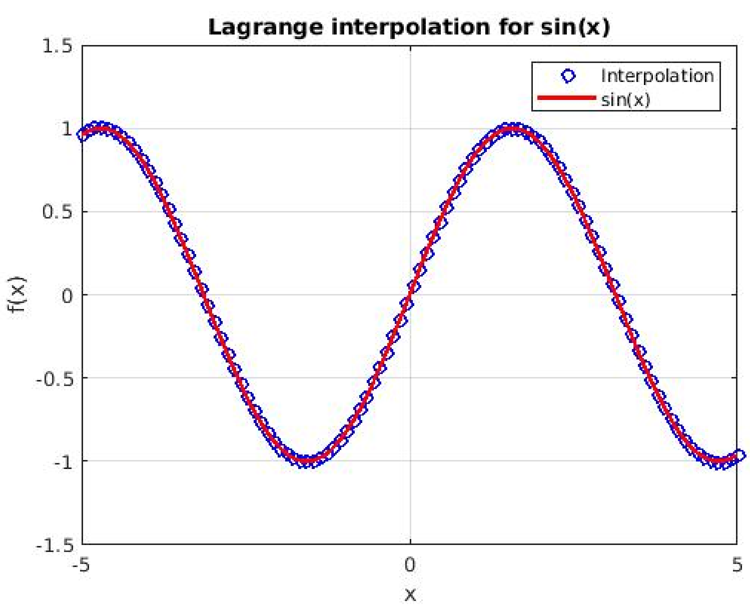

Figure 1: The interpolation for $\sin(x)$

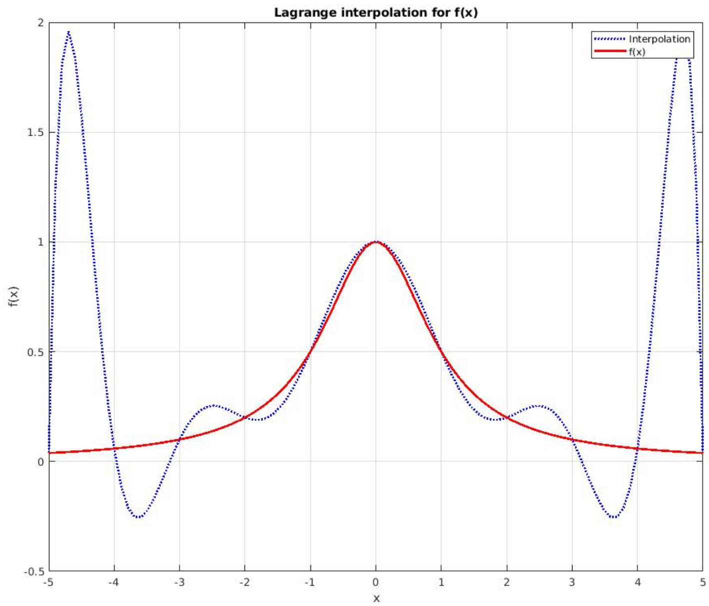

Figure 2: The interpolation for $f(x) = 1/(1 + x^2)$

Now we analysis the approximation error. By Theorem 3.3 of *Numerical Analysis, 9th ed*. We have

$$f(x) = P(x) + \frac{f^{(11)}(\xi(x))}{11!}(x+5)\cdots(x-5) \tag{9}$$

where $P(x)$ is the Lagrange interpolation polynomial, $\xi(x) \in [-5, 5]$.

1. $f(x) = \sin(x)$. We have

   $$f^{(11)}(x) = -\cos(x) \tag{10}$$

   and

   $$\frac{f^{(11)}(\xi(x))}{11!}(x+5)\cdots(x-5) = \frac{-\cos(\xi(x))}{11!}(x+5)\cdots(x-5)$$

   The maximal value of $\cos(\xi(x))$ on the interval $[-5, 5]$ is 1. Define $g(x) = (x+5)\cdots(x-5)$, $x \in [-5, 5]$. Since the mximum value of $g(x)$ is attained at $x^* \approx -4.71458$, and $g(x^*) \approx 416614$ and $g(x)$ is an odd function. Thus

   $$|g(x)| \leq 416620, x \in [-5, 5] \tag{11}$$

   Now we have

   $$\left|\frac{-\cos(\xi(x))}{11!}(x+5)\cdots(x-5)\right| \leq \frac{416620}{11!} \approx 0.01 \tag{12}$$

   which is consistent with the plot.

2. $f(x) = 1/(1 + x^2)$. We use Mathematica to obtain an upper bound for $|f^{(11)}(\xi(x))|$:

   $$|f^{(11)}(\xi(x))| \leq |f^{(11)}(0.12)| \leq 3.6 \times 10^7 \tag{13}$$

   $$\left|\frac{f^{(11)}(\xi(x))}{11!}(x-x_0)\cdots(x-x_{10})\right| \leq \frac{416620 \times 3.6 \times 10^7}{11!} \approx 3.6 \times 10^5 \tag{14}$$

   The bound indicates that the Lagrange interpolation method is unstable for $f(x) = 1/(1 + x^2)$. Actually, as n increases, the oscillations get even larger. The interpolation polynomial $P(x)$ doesn't converge to $f(x)$ uniformly.