Homework 3 for SI211: Numerical Analysis

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Abstract

This is the solution for Homework 3 of SI211: Numerical Analysis, which is taught by Boris Houska.

1 Natural Spline

Write a computer code in Julia, Matlab, Python, C++, (or a similar programming language), which returns a natural spline that interpolates the function $f : [x_0, x_N] \to \mathbb{R}$ the equidistantpoints

$$\forall i \in \{0, 1, \ldots, N\}, x_i = x_0 + hi \tag{1}$$

where

$$h = \frac{x_N - x_0}{N} \tag{2}$$

Solution 1. The Matlab code is listed as follows:

```
function [a, b, c, d] = natual_spline(x0, xN, N, init_func)
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  % This function aims to implement the algorithm 3.4:
  % Natural Cubic Spline in <<Numerical Anslysis>> 9th.
  % But with a interval [x0, xN] and a fix length h of subinterval
  % that is, x_i = x0 + i*h

  % Parameters:
  %   x0: left endpoint of the interval
  %   xN: right endpoint of the interval
  %   N: The number of intervals in [x0, xN]
  %   init_func: function handle
  %              the function we need to interpolate

  % Return
  %   a, b, c, d: an array-like, shape=[N, 1]
  %       the cubic spline function coefficient in each subinterval
  %           [x_i,x_{i+1}]
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

  % step 1
  h = (xN - x0) / N;
  x = x0: h: xN;
  a = init_func(x);
  % step 2
  alpha = 3/h * (a(3:end) - 2 * a(2:end-1) + a(1:end-2));

  % Use Theorem 3.11 to solve c
```

34th Conference on Neural Information Processing Systems (NeurIPS 2020), Vancouver, Canada.

```
    % the coefficient matrix
    A = diag(4*h*ones(N-1, 1)) + diag(h*ones(N-2, 1), 1) + diag(h*ones(N-2, 1), -1);
    c = (A\alpha')';
    c = [0, c, 0];

    % step 6
    b = (a(2:end) - a(1:end-1)) / h - h/3 * (c(2:end) + 2*c(1:end-1));
    d = (c(2:end) - c(1:end-1)) / (3 * h);
end
```

2 Comparision of Interpolation and Natural Spline

We have constructed a computer program that interpolates a function $f(x) = 1/(1+x^2)$ with a polynomialof order 10. Use your code from the first exercise to interpolates this function

$$f(x) = \frac{1}{1+x^2} \tag{3}$$

at the points $x_1 = -5, x_2 = -4, x_3 = -3, \ldots, x_{10} = 4, x_{11} = 5$. Plot the functionf as well as the natural spline that interpolates f and the interpolating polynomials (HW2).

Solution 2. As figure 2 shows, the natural spline approximates the function f extremely well, while the Lagrange interpolation polynomial performance badlly when the x is far away from 0.

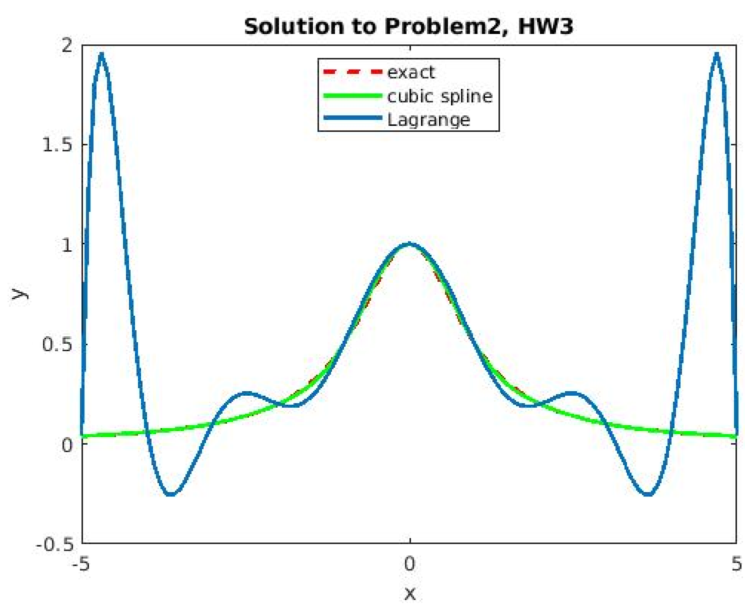

Figure 1: The interpolation for $f(x) = 1/(1+x^2)$

3 Comparision of Interpolation and Natural Spline (continued)

For the function f, the interpolating polynomial p, and the interpolating spline s from the previous exercise, compute

1. the value of the integral

$$\int_{-5}^{5} \left[f''(x)\right]^2 \mathrm{d}x \tag{4}$$

2. the value of the integral

$$\int_{-5}^{5} [p''(x)]^2 \,\mathrm{d}x \tag{5}$$

3. the value of the integral

$$\int_{-5}^{5} [s''(x)]^2 \,\mathrm{d}x \tag{6}$$

Interpret your results.

Solution 3. 1. Since

$$f'(x) = -\frac{2x}{(1+x^2)^2}, \quad f''(x) = \frac{8x^2 - 2(1+x^2)}{(1+x^2)^3} \tag{7}$$

Thus

$$\begin{aligned}
\int_{-5}^{5} [f''(x)]^2 \,\mathrm{d}x &= \int_{-5}^{5} \left[\frac{8x^2 - 2(1+x^2)}{(1+x^2)^3}\right]^2 \mathrm{d}x \\
&= \frac{1}{20}\left[\frac{x(15x^8 + 70x^6 + 128x^4 + 10x^2 + 65)}{(x^2+1)^5} + 15\arctan(x)\right]_{-5}^{5} \\
&= \frac{219795}{742586} + \frac{3}{2}\arctan(5) \\
&\approx 2.3561
\end{aligned}$$

2. Since

$$p(x) = \sum_{k=0}^{N} f(x_k) \prod_{i=0, i\neq k}^{N} \frac{(x - x_i)}{(x_k - x_i)} \tag{8}$$

We use the Matlab *diff* and *int* function to calculate the result, the code is in *solution_to_problem_3.m*. The result is

$$\int_{-5}^{5} [p''(x)]^2 \,\mathrm{d}x \approx 2007.70 \tag{9}$$

3. Since

$$s(x) = s_i(x), x \in [-6+i, -5+i], \quad i = 1, \dots, 10 \tag{10}$$

where $s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$. we have

$$\begin{aligned}
\int_{-5}^{5} [s''(x)]^2 \,\mathrm{d}x &= \sum_{i=1}^{10} \int_{-6+i}^{-5+i} [s_i''(x)]^2 \,\mathrm{d}x \\
&= \sum_{i=1}^{10} \int_{-6+i}^{-5+i} [2c_i + 6d_i(x - x_i)]^2 \,\mathrm{d}x \\
&= 4\sum_{i=1}^{10} c_i^2 + 12\sum_{i=1}^{10} c_i d_i + 12\sum_{i=1}^{10} d_i^2 \\
&\approx 2.2161
\end{aligned}$$

where in the last step we use the numerical result computed by Problem 2.

Interpretation:

1. The natural cubic spline s never oscillates more than the function f. That is,

$$\int_{x_0}^{x_N} [s''(x)]^2 \,\mathrm{d}x \le \int_{x_0}^{x_N} [f''(x)]^2 \,\mathrm{d}x \tag{11}$$

2. High order interpolating polynomials tends to oscillate “unreasonably”.