SI214C Project: Multigrid Method

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Abstract

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Index Terms

multigrid method, acceleration, partial differential equation

I. Introduction

This document is a model and instructions for LaTeX. Please observe the conference page limits.

II. Problem formulation

A. Finite difference method

To solve a one dimensional boundary value problem:

$$-u''(x) + u(x) = f(x), \quad 0 < x < 1 \tag{1}$$
$$u(0) = u(1) = 0 \tag{2}$$

one could use the central difference instead of the exact dirivative:

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + u_i = f_i, \quad i = 1, \dots, N-1 \tag{3}$$
$$u_0 = u_N = 0 \tag{4}$$

where N is a given positive integer, $h = 1/N$, $x_i = ih$ for $i = 0, \dots, N$, $f_i = f(x_i)$. $u_i \approx u(x_i)$ is the approximate solution. The above equations can be rewritten as

$$\mathbf{A}\mathbf{u} = \mathbf{f} \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} 2+h^2 & -1 & & \\ -1 & 2+h^2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2+h^2 \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}, \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N-1}, \mathbf{f} = \begin{bmatrix} h^2 f_1 \\ \vdots \\ h^2 f_{N-1} \end{bmatrix} \in \mathbb{R}^{N-1} \tag{6}$$

Similarly, we can transform a 2-dimensional boundary value problem:

$$-u_{xx} + -u_{yy} = f(x,y), \quad 0 < x < 1, \quad 0 < y < 1 \tag{7}$$
$$u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0, \quad 0 \le x \le 1, \quad 0 \le y \le 1 \tag{8}$$

into a similar block-tridiagonal system as (5) TBD, make the system more clear. The problem now focuses on how to solve the problem (5)

B. Iterative method

In this section, we are going to solve the linear system:

$$\mathbf{A}\mathbf{u} = \mathbf{f} \tag{9}$$

There are two ways to solve such a linear system: direct methods and the iterative methods. Since the special structure of the problem ($\mathbf{A}$ is a tri-diagonal matrix) and N is usually very large, one often choses the iterative method as the solver. TBD, We will compare two kind of methods in Appendix A.

Now we introduce three basic iterative solvers:

1) The first one is the Jacobi method, which takes the form:

$$u_i^{k+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} u_j^k \right), \quad i = 1, \ldots, N-1 \tag{10}$$

2) The second one is the Gauss-Seidel method, which takes the form:

$$u_i^{k+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} u_j^{k+1} - \sum_{j>i} a_{ij} u_j^k \right), \quad i = 1, \ldots, N-1 \tag{11}$$

3) The third one is the SOR method, which involves a para,eterized splitting of $\mathbf{A}$:

$$u_i^{k+1} = \omega \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} u_j^{k+1} - \sum_{j>i} a_{ij} u_j^k \right) + (1-\omega) x_i^k, \quad i = 1, \ldots, N-1 \tag{12}$$

where $\omega > 0$ is the parameter.

The iterative methods can actually be seen as ways to decompose the matrix $\mathbf{A}$: If we write A as

$$\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$$

where $\mathbf{D}$ is the diagonal, $\mathbf{L}$ and $\mathbf{U}$ are the strictly lower and upper parts of $\mathbf{A}$. Then the above three methods can be rewritten as

$$(Jacobi) \quad \mathbf{u}^{k+1} = \mathbf{D}^{-1} (\mathbf{L} + \mathbf{U}) \mathbf{u}^k + \mathbf{D}^{-1} \mathbf{f} \tag{13}$$

$$(Gauss - Seidel) \quad \mathbf{u}^{k+1} = (\mathbf{D} - \mathbf{L})^{-1} \mathbf{U} \mathbf{u}^k + (\mathbf{D} - \mathbf{L})^{-1} \mathbf{f} \tag{14}$$

$$(SOR) \quad \mathbf{u}^{k+1} = \left(\frac{1}{\omega} \mathbf{D} + \mathbf{L} \right)^{-1} \left(\left(\frac{1}{\omega} - 1 \right) \mathbf{D} + \mathbf{U} \right) \mathbf{u}^k + \left(\frac{1}{\omega} \mathbf{D} + \mathbf{L} \right)^{-1} \mathbf{f} \tag{15}$$

III. Multigrid Method

A. Error Analysis

With the iterative solvers for the linear system, we now analyze the convergence of three methods: Define the error term

$$\|\mathbf{e}^k\| := \|\mathbf{u} - \mathbf{u}^k\| \tag{16}$$

where the norm usually set as 1, 2 or $\inf$ norm . The iterative method can be seen as minimize the residual:

$$\|\mathbf{r}\| := \|\mathbf{f} - \mathbf{A}\mathbf{u}\| \tag{17}$$

By the definition of the residual, we obtain the relationship defined by the three iterative methods:

$$(Jacobi) \quad e^{k+1} = \mathbf{R}_J \mathbf{e}^k \tag{18}$$

$$(Gauss - Seidel) \quad e^{k+1} = \mathbf{R}_G \mathbf{e}^k \tag{19}$$

$$(SOR) \quad e^{k+1} = \mathbf{R}_\omega \mathbf{e}^k \tag{20}$$

where $\mathbf{R}_J = \mathbf{D}^{-1} (\mathbf{L} + \mathbf{U})$, $\mathbf{R}_G = (\mathbf{D} - \mathbf{L})^{-1} \mathbf{U}$ and $\mathbf{R}_\omega = \left(\frac{1}{\omega} \mathbf{D} + \mathbf{L} \right)^{-1} \left(\left(\frac{1}{\omega} - 1 \right) \mathbf{D} + \mathbf{U} \right)$.

To ensure the convergence of the iterative methods, we have the following lemma

Lemma 1. *The iterative methods are convergent if and only if*

$$\rho(\mathbf{R}) < 1 \tag{21}$$

where $\mathbf{R} = \mathbf{R}_J$ for Jacobi mathod, $\mathbf{R} = \mathbf{R}_G$ for Gauss-Seidel mathod, $\mathbf{R} = \mathbf{R}_\omega$ for SOR mathod. $\rho(\mathbf{R})$ is the spectral radius of $\mathbf{R}$, that is, if we denote $\lambda_1, \ldots, \lambda_{N-1}$ as the eigenvalues of $\mathbf{R}$, then

$$\rho(\mathbf{R}) = \max\{|\lambda_1|, \ldots, |\lambda_{N-1}|\} \tag{22}$$

Actually, the convergence rate is dominant by $\rho(\mathbf{R})$, which means

$$\frac{\|\mathbf{e}^M\|}{\|\mathbf{e}^0\|} \leq \|\mathbf{R}\|^M \approx (\rho(\mathbf{R}))^M \tag{23}$$

Now we analyze the convergence of the Gauss-Seidel method, the analysis to other two methods are similar. The eigenvalues and the corresponding eigenvectors of $\mathbf{R}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U}$ are given by

$$\lambda_k(\mathbf{R}_G) = \cos^2\left(\frac{k\pi}{N}\right), \mathbf{v}_k^j = (\lambda_k)^{j/2}\sin\left(\frac{jk\pi}{N}\right), \quad j,k = 1,\ldots,N-1 \tag{24}$$

thus $\rho(\mathbf{R}_G) = \cos^2\left(\frac{1\pi}{N}\right)$. Now, we use the eigenvectors of $\mathbf{R}_G$ as the initial guess for the equation $\mathbf{Re} = 0$, the number of

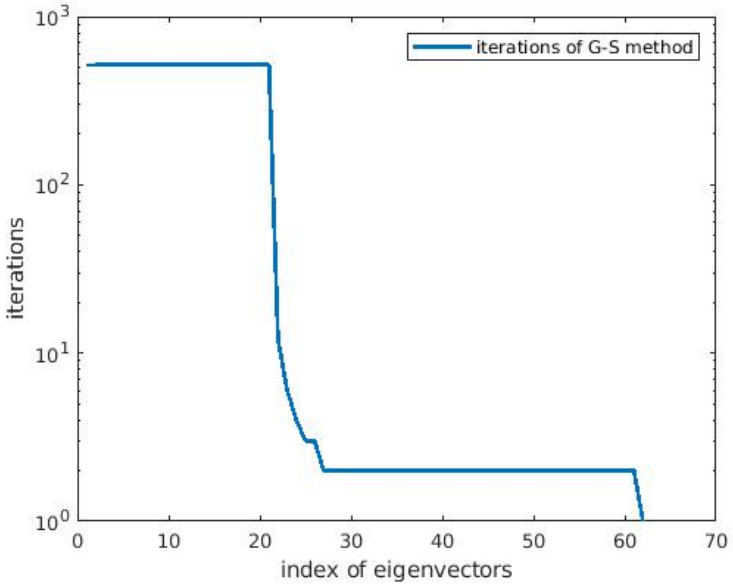

Fig. 1. The relationship between initial guess and the convergent iterations

iterations required to reduce to $\|\mathbf{e}\|_\infty \leq 0.1$ is shown in 1.

Lemma 2. *The eigenvectors corresponding to different eigenvalues are linearly independent.*

By the above lemma, for any initial guess $\mathbf{w}$ of Gauss-Seidel method, it can be written as

$$\mathbf{e}^0 = \sum_{j=1}^{N-1} \alpha_j \mathbf{v}_j \tag{25}$$

As shown in 1, now the convergence rate depends on the low frequency term (j is small). So, the problem is: Can we accelerate the convergence of Gauss-Seidel method? That is how multigrid method comes out.

B. Multigrid method

The basic idea is that we consider the linear system (5) in the *coarse grid*, where low frequency term acts like high frequency term, that way, the iterative method is accelerated. The main steps are show in Algorithm III-B, we explain in detail for 1 dimensional and 2 dimensional cases.

Algorithm 1 Multigrid Method

Require: Generating meshes and their relationships and set the initial guess $\mathbf{u}^0$ for $\mathbf{Au} = \mathbf{f}$

1: **while** NOT convergent **do**
2: Smoothing on the fine grid via iterative method
3: Compute the residual on fine grid
4: Transfer of fine-grid residual to coarse grid (restriction)
5: Smoothing on coarse grid
6: Transfer of coarse-grid correction to fine grid (prolongation)
7: Update of fine-grid solution
8: **end while**

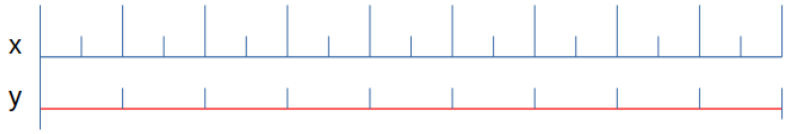

Fig. 2. Coarse and fine grid in ta Two-Grid algorithm with Uniform Mesh Spacing

1) 1-d case: Mathematically, we first use Gauss-Seidel method to reduce the errror on the fine grid $x_0, \ldots, x_N$, then we consider the coarse grid, of which intervals have length $2h$, we denote the coarse grid as $y_1, \ldots, y_{N/2}$ (we assume that $N/2$ is the smallest integer that is larger than $N/2$, we still write it in the form $N/2$ for simplicity), it can be seen that $x_{2j} = y_j$ for $j = 1, \ldots, N/2$. We then transfer the residual to the coarse grid, which can be done by just coping the residuals on the fine grid to the coarse grid. Now, we use the Gauss-Seidel method to reduce the errror again, note that the convergence rate is dominant by $\rho(\mathbf{R}_G^{2h}) = \cos^2\left(\frac{2\pi}{N}\right)$, whici is less than $\rho(\mathbf{R}_G^h) = \rho(\mathbf{R}_G)$. Now the iterative method converges faster than that on the fine grid. Next step, we transfer the results of coarse grid back to fine grid, where we need to interpolate the residuals obtained on the coarse grid. At last, we check for the convergence to decide if we need to run the before processes again.

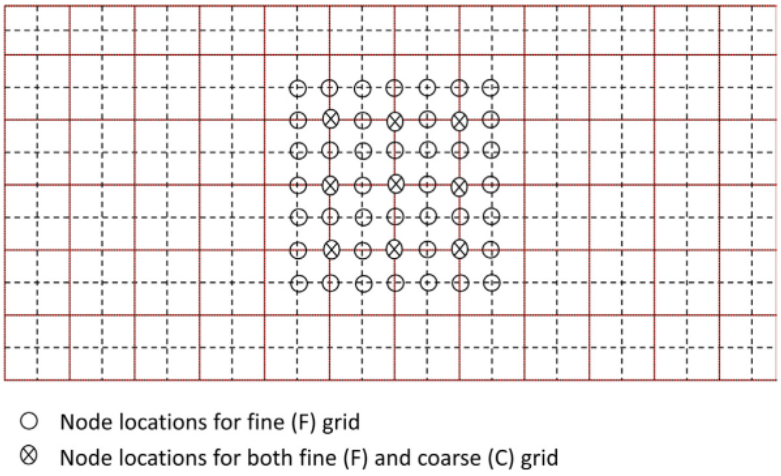

Fig. 3. Coarse and fine grid in ta Two-Grid algorithm with Uniform Mesh Spacing

2) 2-d case: The analysis in 2-d case is similar to 1-d case. We first denote the the fine grid and the coarse grid by Ω^h and Ω^{2h}, and we also de define the pair of find grid as (i, j) and the coarse grid (I, J). Using the same configuration as 1-d case, we can express the find grid as the coarse grid as follows:

$$(i, j) = (2I - 1, 2J - 1) \tag{26}$$

The Fig 3 shows the relationship between the fine grid and the coarse grid in 2-d case.

C. Recursive multigrid method

Note that we can actually use multi-level multigrid method to make the iterative methods converges faster. The idea is that we consider the coarser grid on the coarse grid,

D. Algebraic multigrid method

Though (Geometric) multigrid methods is easy to understand and inplement, it has some disadvantages:

- aaa

To solve these problems, algebraic multigrid methods are proposed. The basic idea is that

IV. Experiment

V. Conclusion

VI. Ease of Use

A. Maintaining the Integrity of the Specifications

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Define abbreviations and acronyms the first time they are used in the text, even after they have been defined in the abstract. Abbreviations such as IEEE, SI, MKS, CGS, ac, dc, and rms do not have to be defined. Do not use abbreviations in the title or heads unless they are unavoidable.

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Number equations consecutively. To make your equations more compact, you may use the solidus (/), the exp function, or appropriate exponents. Italicize Roman symbols for quantities and variables, but not Greek symbols. Use a long dash rather than a hyphen for a minus sign. Punctuate equations with commas or periods when they are part of a sentence, as in:

$$a + b = \gamma \tag{27}$$

Be sure that the symbols in your equation have been defined before or immediately following the equation. Use "(27)", not "Eq. (27)" or "equation (27)", except at the beginning of a sentence: "Equation (27) is . . ."

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Please use "soft" (e.g., `\eqref{Eq}`) cross references instead of "hard" references (e.g., `(1)`). That will make it possible to combine sections, add equations, or change the order of figures or citations without having to go through the file line by line.

Please don't use the `{eqnarray}` equation environment. Use `{align}` or `{IEEEeqnarray}` instead. The `{eqnarray}` environment leaves unsightly spaces around relation symbols.

Please note that the `{subequations}` environment in LaTeX will increment the main equation counter even when there are no equation numbers displayed. If you forget that, you might write an article in which the equation numbers skip from (17) to (20), causing the copy editors to wonder if you've discovered a new method of counting.

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- In your paper title, if the words "that uses" can accurately replace the word "using", capitalize the "u"; if not, keep using lower-cased.
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TABLE I
TABLE TYPE STYLES

Table Head	**Table Column Head**		
	Table column subhead	***Subhead***	***Subhead***
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[a]Sample of a Table footnote.

Fig. 4. Example of a figure caption.

Figure Labels: Use 8 point Times New Roman for Figure labels. Use words rather than symbols or abbreviations when writing Figure axis labels to avoid confusing the reader. As an example, write the quantity "Magnetization", or "Magnetization, M", not just "M". If including units in the label, present them within parentheses. Do not label axes only with units. In the example, write "Magnetization (A/m)" or "Magnetization {A[m(1)]}", not just "A/m". Do not label axes with a ratio of quantities and units. For example, write "Temperature (K)", not "Temperature/K".

ACKNOWLEDGMENT

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REFERENCES

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Number footnotes separately in superscripts. Place the actual footnote at the bottom of the column in which it was cited. Do not put footnotes in the abstract or reference list. Use letters for table footnotes.

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