SI214C Project: Multigrid Method

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Abstract

This document is a model and instructions for LaTeX. This and the IEEEtran.cls file define the components of your paper [title, text, heads, etc.]. *CRITICAL: Do Not Use Symbols, Special Characters, Footnotes, or Math in Paper Title or Abstract.

Index Terms

multigrid method, acceleration, partial differential equation

I. Introduction

This document is a model and instructions for LaTeX. Please observe the conference page limits.

II. Problem formulation

A. Finite difference method

To solve a one dimensional boundary value problem:

$$-u''(x) + u(x) = f(x), \quad 0 < x < 1 \tag{1}$$
$$u(0) = u(1) = 0 \tag{2}$$

one could use the central difference instead of the exact dirivative:

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + u_i = f_i, \quad i = 1, \dots, N-1 \tag{3}$$
$$u_0 = u_N = 0 \tag{4}$$

where N is a given positive integer, $h = 1/N$, $x_i = ih$ for $i = 0, \dots, N$, $f_i = f(x_i)$. $u_i \approx u(x_i)$ is the approximate solution. The above equations can be rewritten as

$$\mathbf{A}\mathbf{u} = \mathbf{f} \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} 2+h^2 & -1 & & \\ -1 & 2+h^2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2+h^2 \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}, \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N-1}, \mathbf{f} = \begin{bmatrix} h^2 f_1 \\ \vdots \\ h^2 f_{N-1} \end{bmatrix} \in \mathbb{R}^{N-1} \tag{6}$$

Similarly, we can transform a 2-dimensional boundary value problem:

$$-u_{xx} + -u_{yy} = f(x,y), \quad 0 < x < 1, \quad 0 < y < 1 \tag{7}$$
$$u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0, \quad 0 \le x \le 1, \quad 0 \le y \le 1 \tag{8}$$

into a similar block-tridiagonal system as (5) TBD, make the system more clear. The problem now focuses on how to solve the problem (5)

B. Iterative method

In this section, we are going to solve the linear system:

$$\mathbf{A}\mathbf{u} = \mathbf{f} \tag{9}$$

There are two ways to solve such a linear system: direct methods and the iterative methods. Since the special structure of the problem ($\mathbf{A}$ is a tri-diagonal matrix) and N is usually very large, one often choses the iterative method as the solver. TBD, We will compare two kind of methods in Appendix A.

Now we introduce three basic iterative solvers:

1) The first one is the Jacobi method, which takes the form:

$$u_i^{k+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} u_j^k \right), \quad i = 1, \ldots, N-1 \tag{10}$$

2) The second one is the Gauss-Seidel method, which takes the form:

$$u_i^{k+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} u_j^{k+1} - \sum_{j>i} a_{ij} u_j^k \right), \quad i = 1, \ldots, N-1 \tag{11}$$

3) The third one is the SOR method, which involves a para,eterized splitting of $\mathbf{A}$:

$$u_i^{k+1} = \omega \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} u_j^{k+1} - \sum_{j>i} a_{ij} u_j^k \right) + (1-\omega) x_i^k, \quad i = 1, \ldots, N-1 \tag{12}$$

where $\omega > 0$ is the parameter.

The iterative methods can actually be seen as ways to decompose the matrix $\mathbf{A}$: If we write A as

$$\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$$

where $\mathbf{D}$ is the diagonal, $\mathbf{L}$ and $\mathbf{U}$ are the strictly lower and upper parts of $\mathbf{A}$. Then the above three methods can be rewritten as

$$(Jacobi) \quad \mathbf{u}^{k+1} = \mathbf{D}^{-1} (\mathbf{L} + \mathbf{U}) \mathbf{u}^k + \mathbf{D}^{-1} \mathbf{f} \tag{13}$$

$$(Gauss - Seidel) \quad \mathbf{u}^{k+1} = (\mathbf{D} - \mathbf{L})^{-1} \mathbf{U} \mathbf{u}^k + (\mathbf{D} - \mathbf{L})^{-1} \mathbf{f} \tag{14}$$

$$(SOR) \quad \mathbf{u}^{k+1} = \left(\frac{1}{\omega} \mathbf{D} + \mathbf{L} \right)^{-1} \left(\left(\frac{1}{\omega} - 1 \right) \mathbf{D} + \mathbf{U} \right) \mathbf{u}^k + \left(\frac{1}{\omega} \mathbf{D} + \mathbf{L} \right)^{-1} \mathbf{f} \tag{15}$$

III. Multigrid Method

A. Error Analysis

With the iterative solvers for the linear system, we now analyze the convergence of three methods: Define the error term

$$\|\mathbf{e}^k\| := \|\mathbf{u} - \mathbf{u}^k\| \tag{16}$$

where the norm usually set as 1, 2 or $\inf$ norm . The iterative method can be seen as minimize the residual:

$$\|\mathbf{r}\| := \|\mathbf{f} - \mathbf{A}\mathbf{u}\| \tag{17}$$

By the definition of the residual, we obtain the relationship defined by the three iterative methods:

$$(Jacobi) \quad e^{k+1} = \mathbf{R}_J \mathbf{e}^k \tag{18}$$

$$(Gauss - Seidel) \quad e^{k+1} = \mathbf{R}_G \mathbf{e}^k \tag{19}$$

$$(SOR) \quad e^{k+1} = \mathbf{R}_\omega \mathbf{e}^k \tag{20}$$

where $\mathbf{R}_J = \mathbf{D}^{-1} (\mathbf{L} + \mathbf{U})$, $\mathbf{R}_G = (\mathbf{D} - \mathbf{L})^{-1} \mathbf{U}$ and $\mathbf{R}_\omega = \left(\frac{1}{\omega} \mathbf{D} + \mathbf{L} \right)^{-1} \left(\left(\frac{1}{\omega} - 1 \right) \mathbf{D} + \mathbf{U} \right)$.

To ensure the convergence of the iterative methods, we have the following lemma

Lemma 1. *The iterative methods are convergent if and only if*

$$\rho(\mathbf{R}) < 1 \tag{21}$$

where $\mathbf{R} = \mathbf{R}_J$ for Jacobi mathod, $\mathbf{R} = \mathbf{R}_G$ for Gauss-Seidel mathod, $\mathbf{R} = \mathbf{R}_\omega$ for SOR mathod. $\rho(\mathbf{R})$ is the spectral radius of $\mathbf{R}$, that is, if we denote $\lambda_1, \ldots, \lambda_{N-1}$ as the eigenvalues of $\mathbf{R}$, then

$$\rho(\mathbf{R}) = \max\{|\lambda_1|, \ldots, |\lambda_{N-1}|\} \tag{22}$$

Actually, the convergence rate is dominant by $\rho(\mathbf{R})$, which means

$$\frac{\|\mathbf{e}^M\|}{\|\mathbf{e}^0\|} \leq \|\mathbf{R}\|^M \approx (\rho(\mathbf{R}))^M \tag{23}$$

Now we analyze the convergence of the Gauss-Seidel method, the analysis to other two methods are similar. The eigenvalues and the corresponding eigenvectors of $\mathbf{R}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U}$ are given by

$$\lambda_k(\mathbf{R}_G) = \cos^2\left(\frac{k\pi}{N}\right), \mathbf{v}_k^j = (\lambda_k)^{j/2}\sin\left(\frac{jk\pi}{N}\right), \quad j,k = 1,\ldots,N-1 \tag{24}$$

Now, we use the eigenvectors of $\mathbf{R}_G$ as the initial guess for the equation $\mathbf{Re} = 0$, the number of iterations required to reduce

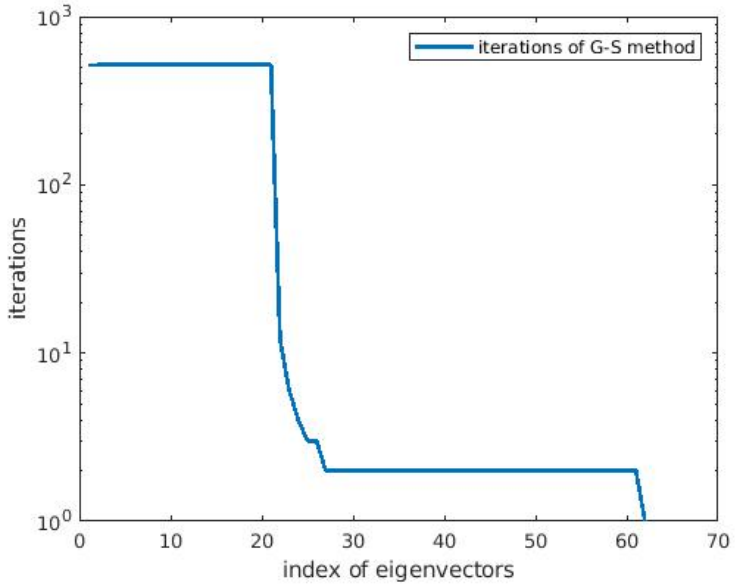

Fig. 1. The relationship between initial guess and the convergent iterations

to $\|\mathbf{e}\|_\infty \leq 0.1$ is shown in 1.

Lemma 2. *The eigenvectors corresponding to different eigenvalues are linearly independent.*

By the above lemma, for any initial guess $\mathbf{w}$ of Gauss-Seidel method, it can be written as

$$\mathbf{e}^0 = \sum_{j=1}^{N-1} \alpha_j \mathbf{v}_j \tag{25}$$

As shown in 1, now the convergence rate depends on the low frequency term (j is small). So, the problem is: Can we accelerate the convergence of Gauss-Seidel method? That is how multigrid method comes out.

B. Multigrid method

The basic idea is that we consider the linear system (5) in the *coarse grid*, where low frequency term acts like high frequency term, that way, the iterative method is accelerated.

1) 1-d case: Mathematically, we first use Gauss-Seidel method to reduce the errror on the fine grid $x_0,\ldots,x_N$, then we consider the coarse grid, of which intervals have length $2h$, we denote the coarse grid as $y_1,\ldots,y_{N/2}$ (we assume that $N/2$ is the smallest integer that is larger than $N/2$, we still write it in the form $N/2$ for simplicity), it can be seen that $x_{2j} = y_j$ for $j = 1,\ldots,N/2$.

In 2-d cases, we first

The algorithm is given as follows:

C. Recursive multigrid method

Note that we can actually use multi-level multigrid method to make the iterative methods converges faster. The idea is

D. Algebraic multigrid method

Though (Geometric) multigrid methods is easy to understand and inplement, it has some disadvantages:

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To solve these problems, algebraic multigrid methods are proposed. The basic idea is that

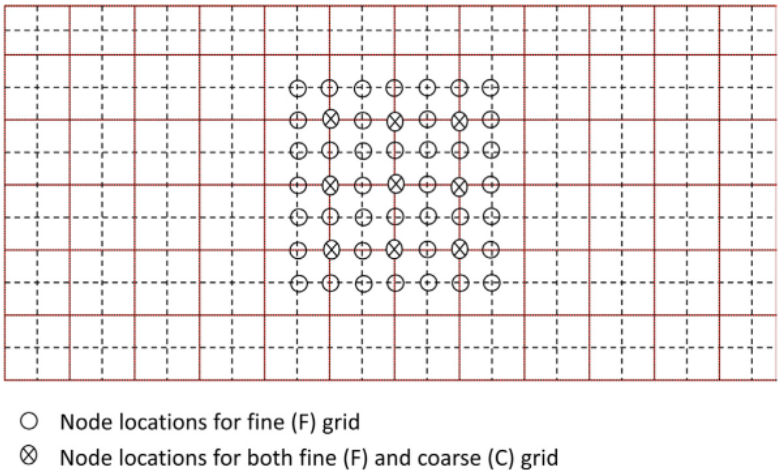

Fig. 2. Coarse and fine grid in ta Two-Grid algorithm with Uniform Mesh Spacing

Algorithm 1 Multigrid Method

Require: Generating meshes and their relationships and set the initial guess $\mathbf{u}^0$ for $\mathbf{A}\mathbf{u} = \mathbf{f}$

while NOT convergent **do**
  Smoothing on the fine grid via iterative method
  Compute the residual on fine grid
  Transfer of fine-grid residual to coarse grid (restriction)
  Smoothing on coarse grid
  Transfer of coarse-grid correction to fine grid (prolongation)
  Update of fine-grid solution
end while

IV. Experiment

V. Conclusion

VI. Ease of Use

A. Maintaining the Integrity of the Specifications

The IEEEtran class file is used to format your paper and style the text. All margins, column widths, line spaces, and text fonts are prescribed; please do not alter them. You may note peculiarities. For example, the head margin measures proportionately more than is customary. This measurement and others are deliberate, using specifications that anticipate your paper as one part of the entire proceedings, and not as an independent document. Please do not revise any of the current designations.

VII. Prepare Your Paper Before Styling

Before you begin to format your paper, first write and save the content as a separate text file. Complete all content and organizational editing before formatting. Please note sections VII-A–VII-E below for more information on proofreading, spelling and grammar.

Keep your text and graphic files separate until after the text has been formatted and styled. Do not number text heads—LaTeX will do that for you.

A. Abbreviations and Acronyms

Define abbreviations and acronyms the first time they are used in the text, even after they have been defined in the abstract. Abbreviations such as IEEE, SI, MKS, CGS, ac, dc, and rms do not have to be defined. Do not use abbreviations in the title or heads unless they are unavoidable.

B. Units

- Use either SI (MKS) or CGS as primary units. (SI units are encouraged.) English units may be used as secondary units (in parentheses). An exception would be the use of English units as identifiers in trade, such as "3.5-inch disk drive".
- Avoid combining SI and CGS units, such as current in amperes and magnetic field in oersteds. This often leads to confusion because equations do not balance dimensionally. If you must use mixed units, clearly state the units for each quantity that you use in an equation.
- Do not mix complete spellings and abbreviations of units: "Wb/m^2" or "webers per square meter", not "webers/m^2". Spell out units when they appear in text: ". . . a few henries", not ". . . a few H".
- Use a zero before decimal points: "0.25", not ".25". Use "cm^3", not "cc".)

C. Equations

Number equations consecutively. To make your equations more compact, you may use the solidus (/), the exp function, or appropriate exponents. Italicize Roman symbols for quantities and variables, but not Greek symbols. Use a long dash rather than a hyphen for a minus sign. Punctuate equations with commas or periods when they are part of a sentence, as in:

$$a + b = \gamma \tag{26}$$

Be sure that the symbols in your equation have been defined before or immediately following the equation. Use "(26)", not "Eq. (26)" or "equation (26)", except at the beginning of a sentence: "Equation (26) is . . ."

D. LaTeX-Specific Advice

Please use "soft" (e.g., `\eqref{Eq}`) cross references instead of "hard" references (e.g., `(1)`). That will make it possible to combine sections, add equations, or change the order of figures or citations without having to go through the file line by line.

Please don't use the `{eqnarray}` equation environment. Use `{align}` or `{IEEEeqnarray}` instead. The `{eqnarray}` environment leaves unsightly spaces around relation symbols.

Please note that the `{subequations}` environment in LaTeX will increment the main equation counter even when there are no equation numbers displayed. If you forget that, you might write an article in which the equation numbers skip from (17) to (20), causing the copy editors to wonder if you've discovered a new method of counting.

BibTeX does not work by magic. It doesn't get the bibliographic data from thin air but from .bib files. If you use BibTeX to produce a bibliography you must send the .bib files.

LaTeX can't read your mind. If you assign the same label to a subsubsection and a table, you might find that Table I has been cross referenced as Table IV-B3.

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Do not use `\nonumber` inside the `{array}` environment. It will not stop equation numbers inside `{array}` (there won't be any anyway) and it might stop a wanted equation number in the surrounding equation.

E. Some Common Mistakes

- The word "data" is plural, not singular.
- The subscript for the permeability of vacuum μ_0, and other common scientific constants, is zero with subscript formatting, not a lowercase letter "o".
- In American English, commas, semicolons, periods, question and exclamation marks are located within quotation marks only when a complete thought or name is cited, such as a title or full quotation. When quotation marks are used, instead of a bold or italic typeface, to highlight a word or phrase, punctuation should appear outside of the quotation marks. A parenthetical phrase or statement at the end of a sentence is punctuated outside of the closing parenthesis (like this). (A parenthetical sentence is punctuated within the parentheses.)
- A graph within a graph is an "inset", not an "insert". The word alternatively is preferred to the word "alternately" (unless you really mean something that alternates).
- Do not use the word "essentially" to mean "approximately" or "effectively".
- In your paper title, if the words "that uses" can accurately replace the word "using", capitalize the "u"; if not, keep using lower-cased.
- Be aware of the different meanings of the homophones "affect" and "effect", "complement" and "compliment", "discreet" and "discrete", "principal" and "principle".
- Do not confuse "imply" and "infer".
- The prefix "non" is not a word; it should be joined to the word it modifies, usually without a hyphen.

- There is no period after the “et” in the Latin abbreviation “et al.”.
- The abbreviation “i.e.” means “that is”, and the abbreviation “e.g.” means “for example”.

An excellent style manual for science writers is [7].

F. Authors and Affiliations

The class file is designed for, but not limited to, six authors. A minimum of one author is required for all conference articles. Author names should be listed starting from left to right and then moving down to the next line. This is the author sequence that will be used in future citations and by indexing services. Names should not be listed in columns nor group by affiliation. Please keep your affiliations as succinct as possible (for example, do not differentiate among departments of the same organization).

G. Identify the Headings

Headings, or heads, are organizational devices that guide the reader through your paper. There are two types: component heads and text heads.

Component heads identify the different components of your paper and are not topically subordinate to each other. Examples include Acknowledgments and References and, for these, the correct style to use is “Heading 5”. Use “figure caption” for your Figure captions, and “table head” for your table title. Run-in heads, such as “Abstract”, will require you to apply a style (in this case, italic) in addition to the style provided by the drop down menu to differentiate the head from the text.

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H. Figures and Tables

a) Positioning Figures and Tables: Place figures and tables at the top and bottom of columns. Avoid placing them in the middle of columns. Large figures and tables may span across both columns. Figure captions should be below the figures; table heads should appear above the tables. Insert figures and tables after they are cited in the text. Use the abbreviation “Fig. 3”, even at the beginning of a sentence.

TABLE I
TABLE TYPE STYLES

Table Head	**Table Column Head**		
	Table column subhead	***Subhead***	***Subhead***
copy	More table copy[a]		

[a]Sample of a Table footnote.

Fig. 3. Example of a figure caption.

Figure Labels: Use 8 point Times New Roman for Figure labels. Use words rather than symbols or abbreviations when writing Figure axis labels to avoid confusing the reader. As an example, write the quantity “Magnetization”, or “Magnetization, M”, not just “M”. If including units in the label, present them within parentheses. Do not label axes only with units. In the example, write “Magnetization (A/m)” or “Magnetization {A[m(1)]}”, not just “A/m”. Do not label axes with a ratio of quantities and units. For example, write “Temperature (K)”, not “Temperature/K”.

ACKNOWLEDGMENT

The preferred spelling of the word “acknowledgment” in America is without an “e” after the “g”. Avoid the stilted expression “one of us (R. B. G.) thanks . . .”. Instead, try “R. B. G. thanks. . .”. Put sponsor acknowledgments in the unnumbered footnote on the first page.

References

Please number citations consecutively within brackets [1]. The sentence punctuation follows the bracket [2]. Refer simply to the reference number, as in [3]—do not use "Ref. [3]" or "reference [3]" except at the beginning of a sentence: "Reference [3] was the first ..."

Number footnotes separately in superscripts. Place the actual footnote at the bottom of the column in which it was cited. Do not put footnotes in the abstract or reference list. Use letters for table footnotes.

Unless there are six authors or more give all authors' names; do not use "et al.". Papers that have not been published, even if they have been submitted for publication, should be cited as "unpublished" [4]. Papers that have been accepted for publication should be cited as "in press" [5]. Capitalize only the first word in a paper title, except for proper nouns and element symbols.

For papers published in translation journals, please give the English citation first, followed by the original foreign-language citation [6].

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