E9 246 Advanced Image Processing

Assignment 03

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Course: **MTech AI**

SR No.: 23112

1: MMSE estimation for Laplacian source.

In this question we will find the MMSE estimate of clean image with Gaussian noise and Laplacian prior. Mathematically, Y=X+Z where Z is zero-mean Gaussian with variance=0.1 and X is zero-mean Laplacian with variance=2. Following is the derivation of MMSE estimate expression in terms of the CDF of normal distribution.

$$Y = X + Z$$

$$f_X(x) = \frac{1}{2\sigma_x} \exp\left(-\frac{|x|}{\sigma_x}\right) \Rightarrow Var(X) = 2\sigma_x^2 = 2$$

$$Z \sim N(0, \sigma_z^2) \Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left(-\frac{(z)^2}{\sigma_z^2}\right)$$

Now, $\hat{x}_{mmse}(Y=y) = E[X \mid Y=y]$

$$= \int x f_{X|Y}(x|y)\,dx = \int x \frac{f_{XY}(x,y)}{f_Y(y)}\,dx$$

$$\therefore \hat{x}_{mmse}(Y=y) = \frac{\int x f_{XY}(x,y)\,dx}{\int f_{XY}(x,y)\,dx}, \text{ where } f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x)$$

$$\Rightarrow f_Y(y) = \int f_{Y|X}(y|x) f_X(x)\,dx$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left(-\frac{(y-x)^2}{2\sigma_z^2}\right)$$

$$\therefore f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_z^2}} \cdot e^{-\frac{(y-x)^2}{2\sigma_z^2}} \cdot \frac{1}{2\sigma_x} e^{-\frac{|x|}{\sigma_x}}\,dx$$

$$= \frac{1}{2\sigma_x\sqrt{2\pi\sigma_z^2}} \Bigg[\underbrace{\int_{-\infty}^{0} e^{-\frac{(y-x)^2}{2\sigma_z^2} + \frac{x}{\sigma_x}}\,dx}_{I_1} + \underbrace{\int_{0}^{\infty} e^{-\frac{(y-x)^2}{2\sigma_z^2} - \frac{x}{\sigma_x}}\,dx}_{I_2} \Bigg]$$

$$I_1 = \int_{-\infty}^{0} e^{-\frac{1}{2\sigma_z^2}\left[y^2 + x^2 - 2yx - \left(\frac{2\sigma_z^2}{\sigma_x}\right)x\right]}\,dx$$

$$= \int_{-\infty}^{0} e^{-\frac{1}{2\sigma_z^2}\left[x - \left(y + \frac{\sigma_z^2}{\sigma_x}\right)\right]^2} e^{-\frac{1}{2\sigma_z^2}\left[y^2 - \left(y + \frac{\sigma_z^2}{\sigma_x}\right)^2\right]}\,dx$$

$$= e^{\left[\left(y + \frac{\sigma_z^2}{\sigma_x}\right)^2 - y^2\right] \cdot \frac{1}{2\sigma_z^2}} \int_{-\infty}^{0} \frac{e^{-\frac{\left[x - \left(y + \frac{\sigma_z^2}{\sigma_x}\right)\right]^2}{2\sigma_z^2}}}{\sqrt{2\pi\sigma_z^2}}\,dx \times \sqrt{2\pi\sigma_z^2}$$

$$= \sqrt{2\pi\sigma_z^2}\, e^{\frac{1}{2\sigma_z^2}\left[\left(y + \frac{\sigma_z^2}{\sigma_x}\right)^2 - y^2\right]} \times \Phi\left(\frac{-\left(y + \frac{\sigma_z^2}{\sigma_x}\right)}{\sigma_z}\right) \quad ; \quad \Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$$

(cdf of std. normal)

Similarly,

$$I_2 = \sqrt{2\pi\sigma_z^2}\, e^{\frac{1}{2\sigma_z^2}\left[\left(y - \frac{\sigma_z^2}{\sigma_x}\right)^2 - y^2\right]} \Phi\left(\frac{y - \frac{\sigma_z^2}{\sigma_x}}{\sigma_z}\right)$$

$$\therefore f_Y(y) = \frac{e^{\sigma_z^2/2\sigma_x^2}}{2\sigma_x} \left[e^{y/\sigma_x} \Phi\left(-\frac{y}{\sigma_z} - \frac{\sigma_z}{\sigma_x}\right) + e^{-y/\sigma_x} \Phi\left(\frac{y}{\sigma_z} - \frac{\sigma_z}{\sigma_x}\right) \right]$$

Now, to compute

$$\int x f_{Y|X}(y|x) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{e^{-\frac{(y-x)^2}{2\sigma_z^2}}}{\sqrt{2\pi\sigma_z^2}} \cdot \frac{1}{2\sigma_x} e^{-\frac{|x|}{\sigma_x}}\,dx$$

$$= \frac{1}{2\sigma_x\sqrt{2\pi\sigma_z^2}} \Big[\underbrace{\int_{-\infty}^{0} x e^{-\frac{1}{2\sigma_z^2}\left[y^2+x^2-2xy+\frac{2x\sigma_z^2}{\sigma_x}\right]}}_{I_1}\,dx + \underbrace{\int_{0}^{\infty} x e^{-\frac{1}{2\sigma_z^2}\left[(y-x)^2-\frac{2x\sigma_z^2}{\sigma_x}\right]}\,dx}_{I_2} \Big]$$

$$I_1 = \int_{-\infty}^{0} x e^{-\frac{1}{2\sigma_z^2}\left[x-\left(y+\frac{\sigma_z^2}{\sigma_x}\right)\right]^2} e^{-\frac{1}{2\sigma_z^2}\left[y^2-\left(y+\frac{\sigma_z^2}{\sigma_x}\right)^2\right]}\,dx$$

Now $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{0} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{0}(x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx + \int_{-\infty}^{0}\mu \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dx$

$$= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{-\mu} \frac{-z}{\sigma^2} e^{-z^2/2\sigma^2}\,dz + \mu\,\Phi\left(\frac{-\mu}{\sigma}\right)$$

$$= \frac{-\sigma}{\sqrt{2\pi}} e^{-\mu^2/2\sigma^2} + \mu\,\Phi\left(\frac{-\mu}{\sigma}\right)$$

$$\therefore\ I_1 = e^{-\frac{1}{2\sigma_z^2}\left[y^2-\left(y+\frac{\sigma_z^2}{\sigma_x}\right)^2\right]} \times \sqrt{2\pi\sigma_z^2}\left[\frac{-\sigma_z}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma_z^2}\left(y+\frac{\sigma_z^2}{\sigma_x}\right)^2} + \left(y+\frac{\sigma_z^2}{\sigma_x}\right)\Phi\left(\frac{-y}{\sigma_z}-\frac{\sigma_z}{\sigma_x}\right)\right]$$

Similarly

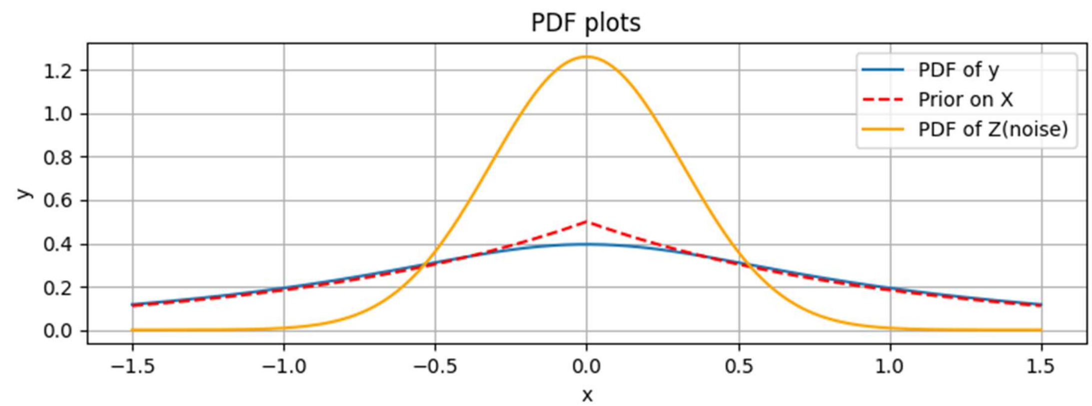
$$I_2 = \sqrt{2\pi\sigma_z^2}\, e^{-\frac{1}{2\sigma_z^2}\left[y-\left(y-\frac{\sigma_z^2}{\sigma_x}\right)^2\right]}\left[\frac{-\sigma_z}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma_z^2}\left(y-\frac{\sigma_z^2}{\sigma_x}\right)^2} + \left(y-\frac{\sigma_z^2}{\sigma_x}\right)\Phi\left(\frac{y}{\sigma_z}-\frac{\sigma_z}{\sigma_x}\right)\right]$$

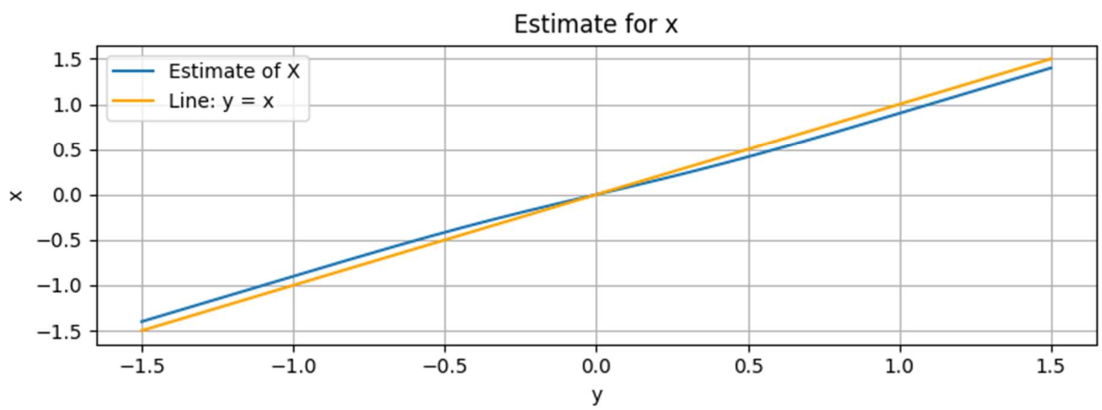
$$\therefore\ \hat{x}(y) = \frac{\int x f_{Y|X}(y|x) f_X(x)\,dx}{\int f_{Y|X}(y|x) f_X(x)\,dx} = \frac{e^{\frac{y}{\sigma_x}}\left[\frac{-\sigma_z}{\sqrt{2\pi}} e^{-\frac{\left(y+\frac{\sigma_z^2}{\sigma_x}\right)^2}{2\sigma_z^2}} + \left(y+\frac{\sigma_z^2}{\sigma_x}\right)\Phi\left(\frac{-y}{\sigma_z}-\frac{\sigma_z}{\sigma_x}\right)\right] + e^{-\frac{y}{\sigma_x}}\left[\frac{\sigma_z}{\sqrt{2\pi}} e^{-\frac{\left(y-\frac{\sigma_z^2}{\sigma_x}\right)^2}{2\sigma_z^2}} + \left(y-\frac{\sigma_z^2}{\sigma_x}\right)\Phi\left(\frac{y}{\sigma_z}-\frac{\sigma_z}{\sigma_x}\right)\right]}{e^{y/\sigma_x}\,\Phi\left(-\frac{y}{\sigma_z}-\frac{\sigma_z}{\sigma_x}\right) + e^{-y/\sigma_x}\,\Phi\left(\frac{y}{\sigma_z}-\frac{\sigma_z}{\sigma_x}\right)}$$

— x -

This formula is used to plot the estimate of X as a function of Y and ϕ(x) is implemented from SciPy library function norm.cdf().

Following are the relevant plots including the estimate of X as a function of Y.





From the figure above, it can be observed that the estimate is similar to a smooth version of the shrinkage estimate plot and hence is equivalent to a soft threshold estimate unlike shrinkage estimator where hard thresholding is done.

2. Image Denoising

1. Low pass Gaussian filter: In this part, we denoise the noisy image by Gaussian smoothing filter and hence extracting only the low pass component of the image. From the given set of filter sizes: {3,7,11} and standard deviations: {0.1,1,2,4,8}, we find the combination that gives the minimum MSE in the denoised image obtained. Following image shows the zoomed in denoised image samples with different smoothing kernels.

Filter size/Standard deviation/Obtained MSE: Left:3/0.1/99.96, Middle: 3/1/ 89.99, Right: 5/4/266.80

As obtained from the code, the best filter with least MSE is 89.99 having filter size=3 and standard deviation=1.

2. Adaptive MMSE: In this method, we process the pass component of the image too. From the noisy image we subtract the low pass smoothened image to obtain the high pass component. On this we take patches of size 32x32 and overlap of 16 (stride=16) and process them by multiplying a factor of: (Variance (Y1) - Variance(Z1)) / Variance (Y1), where Y1 is the patch, Z1 is the high pass component of noise.

$$E[X_1 \mid Y_1 = y] = \frac{\sigma_{X_1}^2}{\sigma_{X_1}^2 + \sigma_{Z_1}^2} y_1 \quad ; \quad Y_1 = X_1 + Z_1$$

$$\Rightarrow \text{Var } Y_1 = \text{Var } X_1 + \text{Var } Z_1$$

$$\text{Var } Y_1 = \frac{1}{32 \times 32} \sum_{m=1}^{32} \sum_{n=1}^{32} Y_1^2(m,n) - \left(\frac{1}{32 \times 32} \sum_{m=1}^{32} \sum_{n=1}^{32} Y_1(m,n) \right)^2$$

As, $Z_1 = Z - Z_{smooth}$.

$$\Rightarrow Z_1(m,n) = Z(m,n) - \sum_{k=-p}^{p} \sum_{l=-p}^{p} Z(m+k, n+l)\, \omega(k,l) \quad ; \quad \omega(\cdot) = \text{smoothing kernel used}$$

$$= (1 - \omega(0,0))\, Z(0,0) - \sum_{\substack{k=-p \\ k \neq 0}}^{p} \sum_{\substack{l=-p \\ l \neq 0}}^{p} Z(m+k, n+l)\, \omega(k,l)$$

; Z_{smooth} = smooth / low pass comp. of noisy image

; p: kenel size

Since, $Z \sim N(0, \sigma_Z^2)$, $\sigma_Z^2 = 100$ (given)

$$\therefore \sigma_{Z_1}^2 = \left[(1 - \omega(0,0))^2 + \sum_{\substack{k=-p \\ k \neq 0}}^{p} \sum_{\substack{l=-p \\ l \neq 0}}^{p} (\omega(k,l))^2 \right] \sigma_Z^2 = \text{Var } Z_1$$

$$\therefore E[X \mid Y_1] = \left(\frac{\text{Var } Y_1 - \text{Var } Z_1}{\text{Var } Y_1} \right) Y_1 = \hat{X}_1(Y_1)$$

This gives a modified patch of high pass components which when combined with all other patches and averaging out (for which another count matrix is made and divided elementwise) gives the final processed high frequency noisy image component. This is added to the low pass component to obtain the final denoised image.

Left: Smoothened estimate (MSE=266.68), Right: Corresponding MMSE estimate (MSE=46.13).

3. Adaptive Shrinkage: In this method we again process the high frequency component separately after separating from the smoothened image. After taking patches of 32x32 and overlap of 16 we instead apply the shrinkage estimator formula where the threshold for each patch is obtained by optimizing the SURE(t) function.

$$\hat{x}_1(m,n) = \text{sign}(Y_1(m,n)) \cdot \max\left[0,\ |Y_1(m,n)| - t^*\right]$$

$(M=N=32)$

$$t^* = \underset{t \geq 0}{\arg\min}\ \text{SURE}(t; Y_1)$$

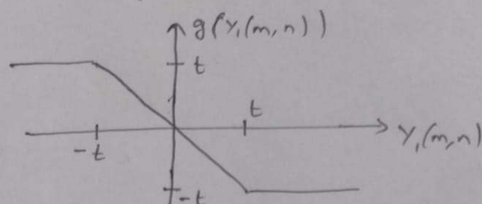
$\{ Y_1$: Patch of size 32×32

$$\text{SURE}(t; Y_1) = MN\sigma_{z_1}^2 + \sum_{m=1}^{M}\sum_{n=1}^{N}\left(g[Y_1(m,n)]\right)^2 + 2\sigma_{z_1}^2 \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{\partial g[Y_1(m,n)]}{\partial Y_1(m,n)}$$

(For each patch)

where, $g[Y_1(m,n)] = \text{sign}[Y_1(m,n)] \cdot \max[0, |Y_1(m,n)| - t] - Y_1(m,n)$

Now, $g(Y_1(m,n))$ has the following plot



∴ Clearly,

$$\frac{\partial g[Y_1(m,n)]}{\partial Y_1(m,n)} = \begin{cases} -1, & |Y_1(m,n)| \leq t \\ 0, & \text{o/w.} \end{cases}$$

Also,

$$[g(Y_1(m,n))]^2 = [\min(|Y_1(m,n)|, t)]^2$$

$$\therefore \text{SURE}(t; Y_1) = MN\sigma_{z_1}^2 + \sum_{m=1}^{M}\sum_{n=1}^{N}\left(\min(|Y_1(m,n)|, t)\right)^2 - 2\sigma_{z_1}^2 \sum_{m=1}^{M}\sum_{n=1}^{N} \mathbb{I}_{\{|Y_1(m,n)| \leq t\}}$$

Hence,

$$t^* = \underset{t \geq 0}{\arg\min} \sum_{m=1}^{M}\sum_{n=1}^{N}[\min(|Y_1(m,n)|, t)]^2 - 2\sigma_{z_1}^2 \sum_{m=1}^{M}\sum_{n=1}^{N} \mathbb{I}_{\{|Y_1(m,n)| \leq t\}}$$

This optimization is done in a brute force manner for every patch to find the optimum threshold as the function is not smooth and then finally the processed high pass component is added to the smooth version after averaging the pixels in multiple patches as before.

Left: MMSE estimate (MSE=46.13), Right: Corresponding Shrinkage estimate (MSE=46.32), (Zoomed in).

Apparently, the MSE of these estimates don't differ very much, but we can observe some differences in the high pass components of the same.

High pass components: Left: MMSE, right: Shrinkage.

Although not very clear here (refer code), the high pass component of MMSE is noisier compared to the SURE which can be seen from the graininess around the lighthouse as shown above. Hence Shrinkage estimator does a better job overall in denoising the image.