Lecture#6
Data Structures

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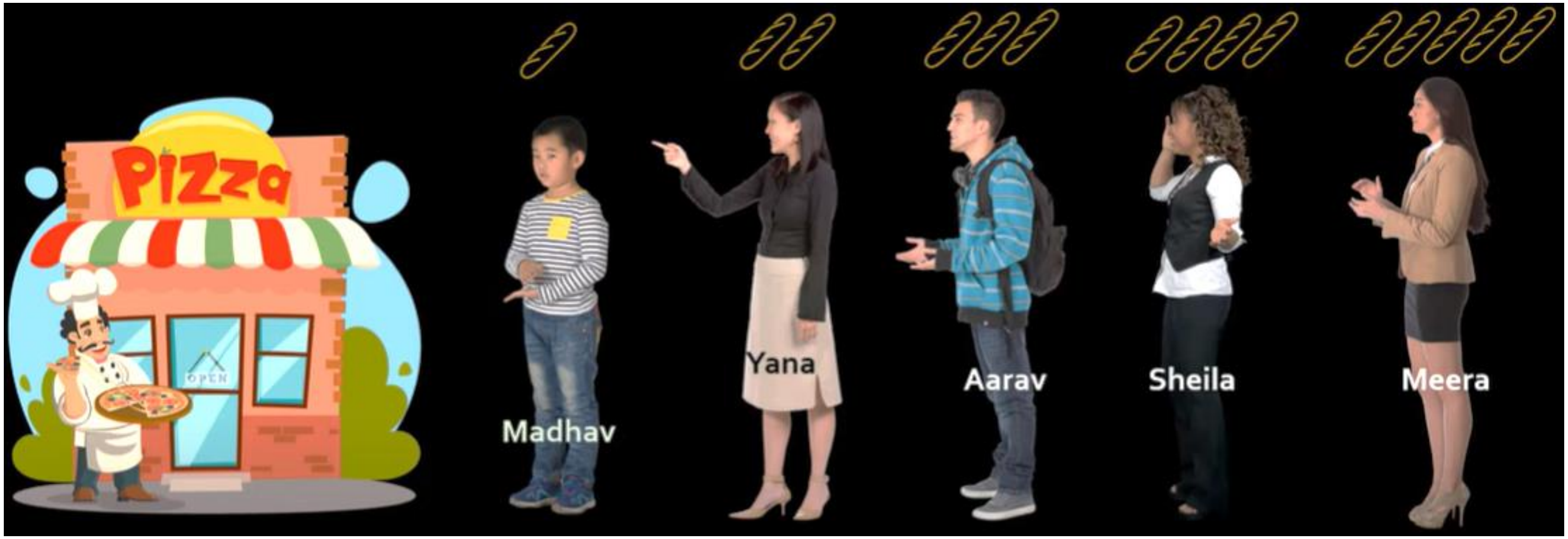
[Faculty Profile]

Recursion



Recursion

5 + find sum of numbers 1 to 4
4 + find sum of numbers 1 to 3
3 + find sum of numbers 1 to 2
2 + find sum of numbers 1 to 1

find sum of numbers 1 to 1 → 1

Recursion

Sometimes, the best way to solve a problem is by solving a **smaller version** of the exact same problem first

Recursion is a technique that solves a problem by solving a **smaller problem** of the same type

Recursion

```
void recurse() {
    ... .. ...
    recurse();
    ... .. ...
}

int main() {
    ... .. ...
    recurse();
    ... .. ...
}
```

recursive call

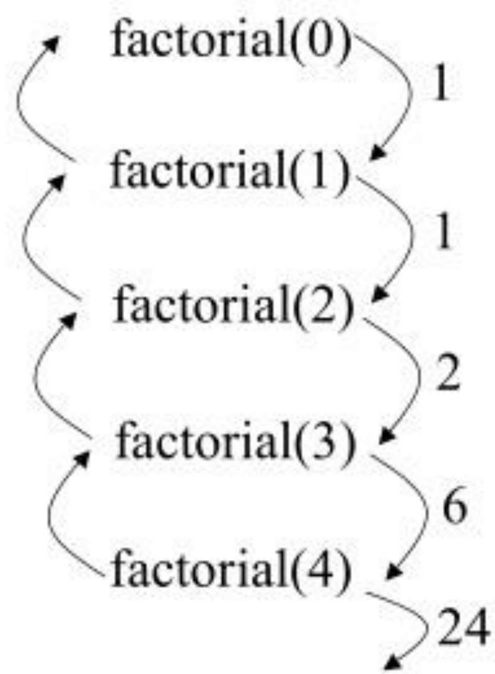
function call

Recursion

$fact(0) = 1$

$fact(n) = n * fact(n-1) \mid (n > 0)$

```
int factorial(int n) {
  if (n == 0) return 1;
  else return n*factorial(n-1);
}
```



Execution of factorial(4)

Recursion Vs. Iteration

- Recursion and iteration are *similar*
- **Iteration:**
  - Loop repetition test determines whether to exit
- **Recursion:**
  - Condition tests for a base case
- Can always write iterative solution to a problem solved recursively, *but:*
- Recursive code often simpler than iterative
  - Thus easier to write, read, and debug

Recurrence Relation

Example:

Someone deposits $10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years.

How can we determine P_n on the basis of P_{n-1}?

Recurrence Relation

We can derive the following **recurrence relation**:

$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.

The initial condition is $P_0 = 10{,}000$.

Then we have:

$P_1 = 1.05P_0$

$P_2 = 1.05P_1 = (1.05)^2P_0$

$P_3 = 1.05P_2 = (1.05)^3P_0$

…

$P_n = 1.05P_{n-1} = (1.05)^nP_0$

We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

Recurrence Relation

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$P_{30} = (1.05)^{30} \cdot 10,000 = 43,219.42$

After 30 years, the account contains $43,219.42.

Recurrence Relation

Derive a recurrence relation and then solve the below problem:

“*Currently the fuel oil price in Bangladesh is 135BDT per liter. Following the last few years, it seems that the price is yielding 51.68% per year. Estimate the plausible price of oil after 7 years.*”

```
void count(int n)
{
    static int d = 1;
    printf("%d ", n);
    printf("%d ", d);
    d++;
    if(n > 1) count(n-1);
    printf("%d ", d);
}
int main()
{
    count(3);
}
```

Popular Recurrent Problems

Tower of Hanoi Problem

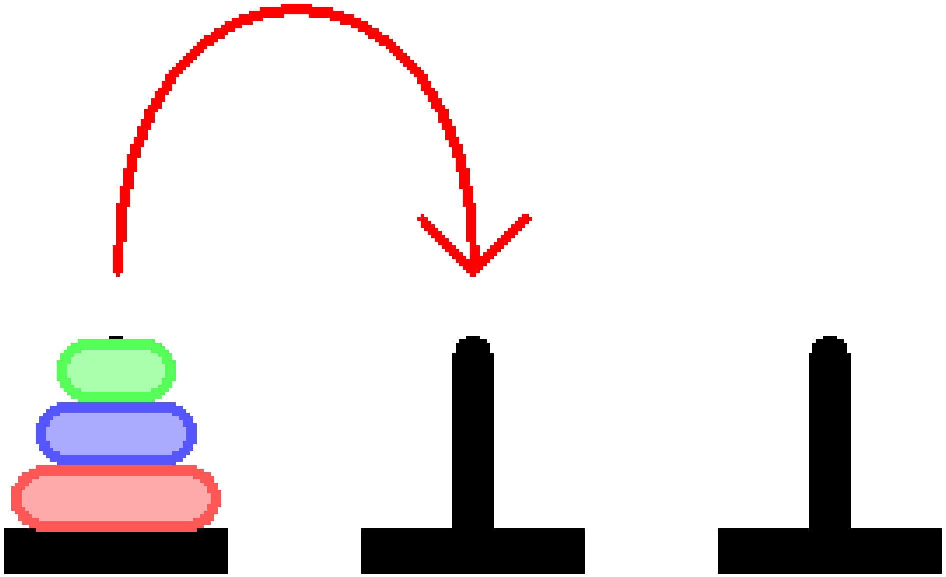
The Tower of Hanoi puzzle was invented by the French mathematician Edouard Lucas in 1883.

The objective of the puzzle is to move the entire stack to the last rod, obeying the following rules:

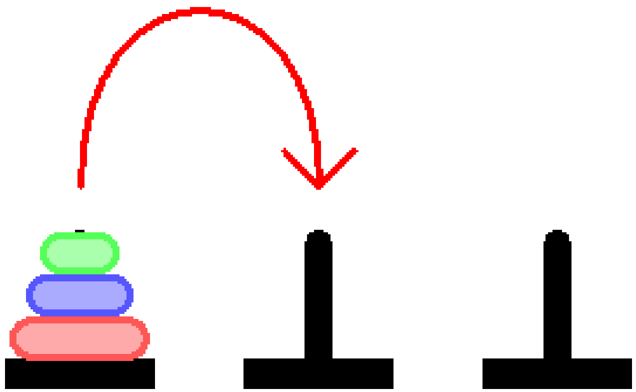
- Only one disk may be moved at a time.
- Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.
- No disk may be placed on top of a disk that is smaller than it.

Tower of Hanoi Problem

With **3** disks, the puzzle can be solved in **7** moves. The minimal number of moves required to solve a Tower of Hanoi puzzle is $2^n - 1$, where n is the number of disks.



Tower of Hanoi Problem



$T_3 = 7$, [Transferring small 2 disks to middle peg requires 3 moves. Then moving the largest disk from left peg to right peg requires only 1 move. Finally, moving 2 smaller disks from middle peg to right peg requires another 3 moves. Thus, total 3+1+3=7 moves require for moving 3 disks from one peg to another]

Similarly, for n disks to transfer from one peg to another we require:

$$T_n = T_{n-1} + 1 + T_{n-1} = 2T_{n-1} + 1$$

Tower of Hanoi Problem

Thus, the recurrence for Tower of Hanoi stands

$T_0 = 0$

$T_n = 2T_{n-1} + 1, \qquad \text{for } n > 0$

We can find out the *closed form* of any recurrence to get quick result from the problem.

$$
\begin{aligned}
T_n &= 2T_{n-1} + 1 \\
&= 2(2T_{n-2} + 1) + 1 \\
&= 2^2 T_{n-2} + 2 + 1 \\
&= 2^2(2T_{n-3} + 1) + 2 + 1 \\
&= 2^3 T_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^n T_{n-n} + 2^{(n-1)} + 2^{(n-2)} + \cdots + 2^2 + 2 + 1 \\
&= 2^n T_0 + 2^{(n-1)} + 2^{(n-2)} + \cdots + 2^2 + 2^1 + 2^0
\end{aligned}
$$

Tower of Hanoi Problem

$$T_n = 2^0 + 2^1 + 2^2 + \cdots + 2^{(n-2)} + 2^{(n-1)}$$

$$= \frac{2^n - 1}{2 - 1} \qquad [\because a^0 + a^1 + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}]$$

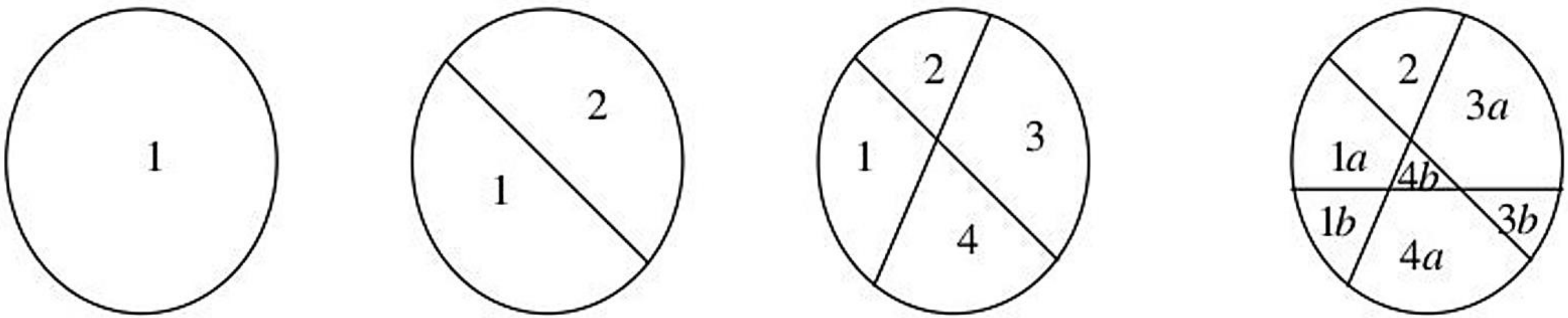
$$= 2^n - 1$$

$T_n = 2^n - 1$ is called the *closed form* solution of the "Tower of Hanoi" problem.

How many moves does it require for moving 6 disks from one peg to another considering the Tower of Hanoi problem?

Lines in the Plane

<u>Lines in the Plane:</u> We have to find out how many slices of pizza can a person obtain by making n straight cuts with a knife. Academically, what is the maximum number of regions, L_n defined by n lines in the plane? We can start by looking at small cases.



$L_0 = 1$

$L_1 = 2$
$L_1 = L_0 + 1$

$L_2 = 4$
$L_2 = L_1 + 2$

$L_3 = 4 + 3 = 7$
$L_3 = L_2 + 3$

$$L_n = L_{n-1} + n$$

Lines in the Plane

$$
\begin{aligned}
\boxed{L_n = L_{n-1} + n} \\
&= L_{n-2} + (n-1) + n \\
&= L_{n-3} + (n-2) + (n-1) + n \\
&\;\;\vdots \\
&= L_{n-n} + (n-n+1) + \cdots + (n-2) + (n-1) + n \\
&= L_0 + 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
&= S_n + 1\ , \quad \text{where } S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n
\end{aligned}
$$

$$L_n = \frac{n(n+1)}{2} + 1$$

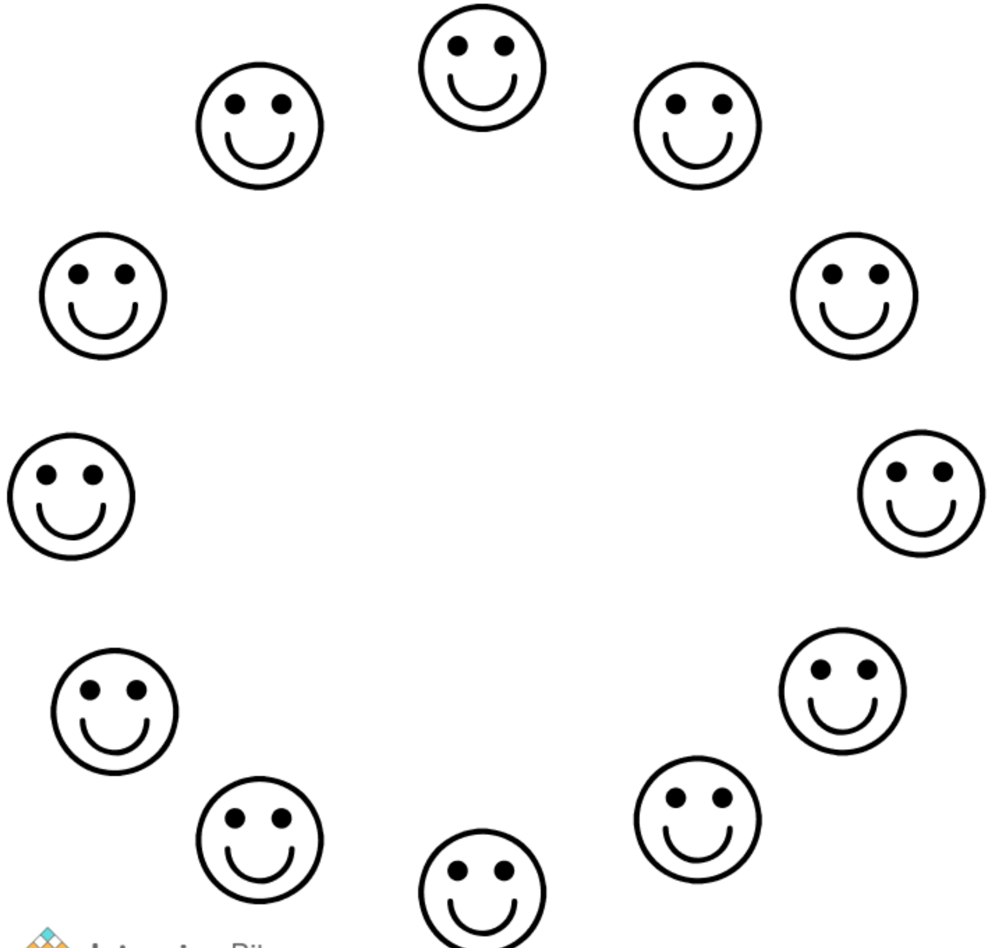
Lines in the Plane

Estimate how many slices of pizza can a person obtain by making 5 straight cuts with a knife?

Josephus Problem

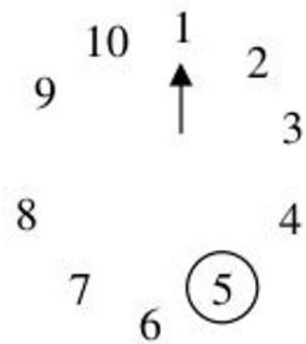
There are n people standing in a circle waiting to be executed. The counting out begins at some point in the circle and proceeds around the circle in a fixed direction. In each step, a certain number of people are skipped and the next person is executed. The elimination proceeds around the circle (which is becoming smaller and smaller as the executed people are removed), until only the last person remains, who is given freedom.

Josephus Problem



Josephus Problem

<u>Josephus Problem:</u> We start with n people numbered 1 to n around a circle and we eliminate every second remaining person until only one survives. For example, here is the starting configuration for $n = 10$.



The elimination order is $2 \to 4 \to 6 \to 8 \to 10 \to 3 \to 7 \to 1 \to 9$, so 5^{th} person survives. The problem is to determine the survivor's number, $J(n)$. In this example $J(10) = 5$.

$$J(1) = 1$$

Even $J(2n) = 2J(n) - 1$,

Odd $J(2n+1) = 2J(n) + 1$,

Josephus Problem

❖ Compute J(9) using the recurrences of the Josephus problem.

$$
\begin{aligned}
J(9) &= 2J(4) + 1 \\
&= 2\,\{2\,J(2) - 1\} + 1 \\
&= 4\,J(2) - 2 + 1 \\
&= 4\,\{2\,J(1) - 1\} - 2 + 1 \\
&= 8\,J(1) - 4 - 2 + 1 \\
&= 8 \times 1 - 4 - 2 + 1 \\
&= 3
\end{aligned}
$$

Josephus Problem

❖ Compute J(100) using the recurrences of the Josephus problem.

Find J(100) using thess recurrences:

$J(100) = 2\ J(50) - 1$

$= 4\ J(25) - 2 - 1 = 8J(12) + 4 - 2 - 1$

$= 16\ J(6) - 8 + 4 - 2 - 1$

$= 32\ J(3) - 16 - 8 + 4 - 2 - 1$

$= 64\ J(1) + 32 - 16 - 8 + 4 - 2 - 1 = 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73$ (ANS)

Josephus Problem

- Compute J(9) using the binary property of the Josephus problem.

$$J(9) = (1001)_2$$

$$= (0011)_2$$

$$= 3$$

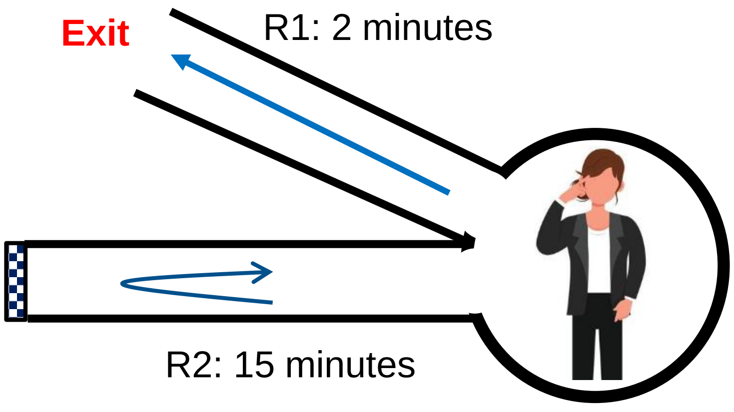
Josephus Problem

❖ Compute J(100) using the binary property of the Josephus problem.

$$J(100) = J((1100100)_2) = (1001001)_2 = 64 + 8 + 1 = 73$$

Expectation Estimation

Expectation Estimation



$P(R1) = P(R2) = \frac{1}{2}$

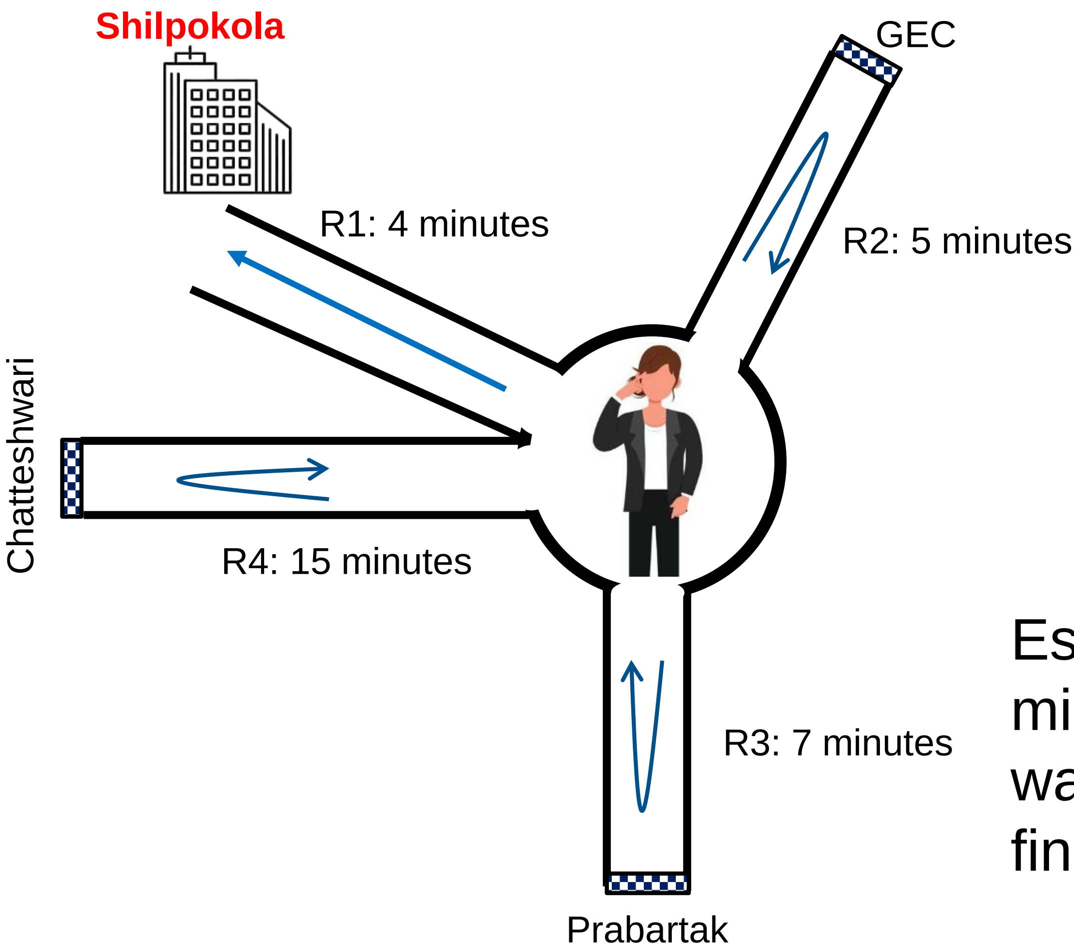
$E(T|R1) = 2$

$E(T|R2) = 15 + E(T)$

$E(T) = P(R1).E(T|R1) + P(R2).E(T|R2)$

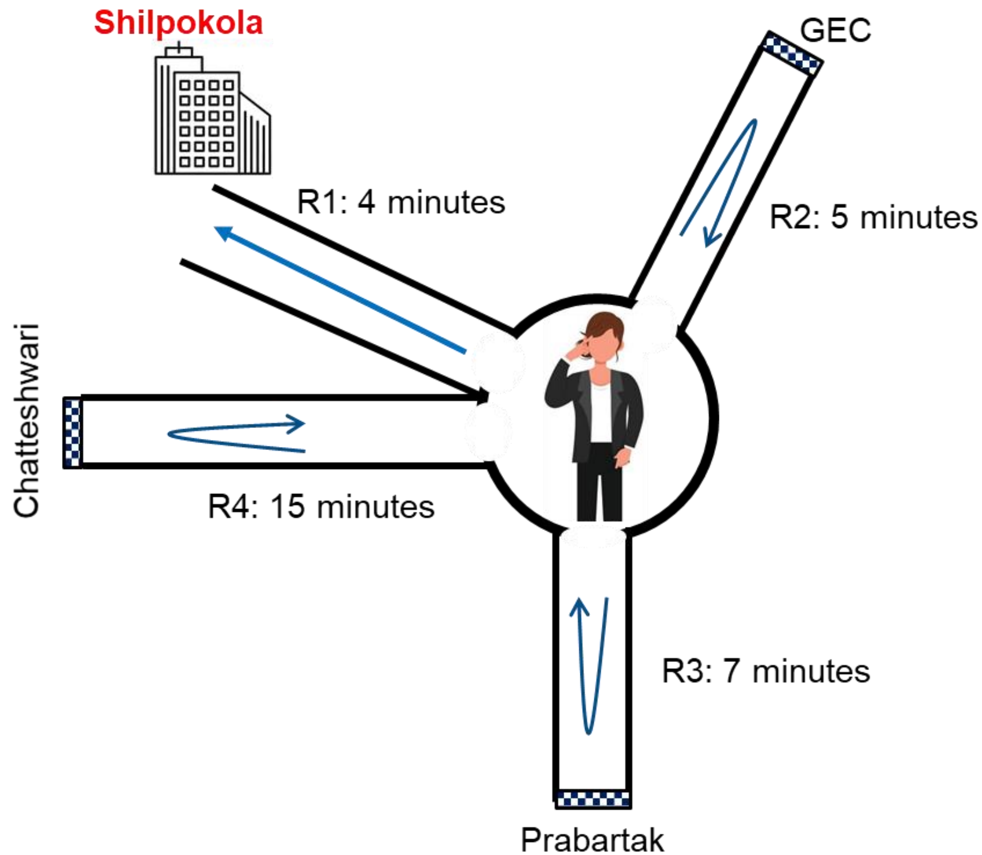
$E(T) = P(R1).E(T|R1) + P(R2).E(T|R2)$

Expectation Estimation



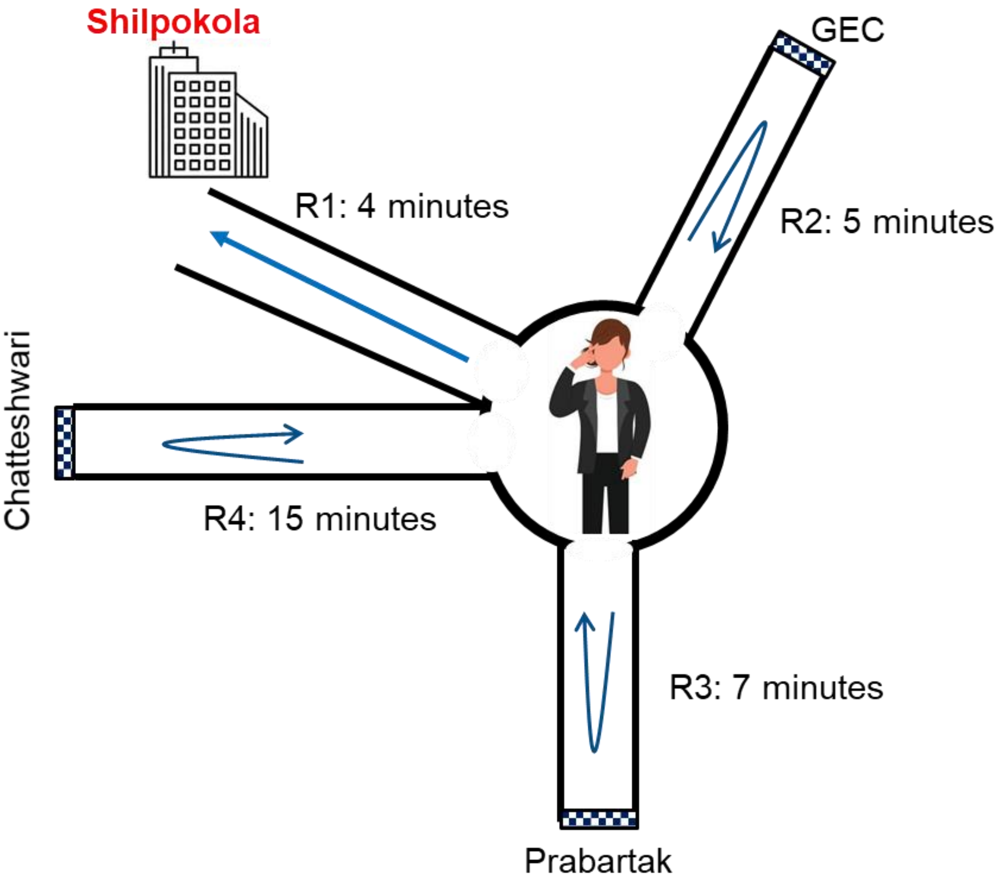
Estimate the expected number of minutes that your friends will be wandering in the Golpahar Mor to find the Shilpokola academy?

Expectation Estimation



$$E(T) = P(R1).E(T|R1) + P(R2).E(T|R2) + P(R3).E(T|R3) + P(R4).E(T|R4)$$

Expectation Estimation



$P(R1) = P(R2) = P(R3) = P(R3) = \frac{1}{4}$

$E(T|R1) = 4$

$E(T|R2) = 5 + E(T)$

$E(T|R3) = 7 + E(T)$

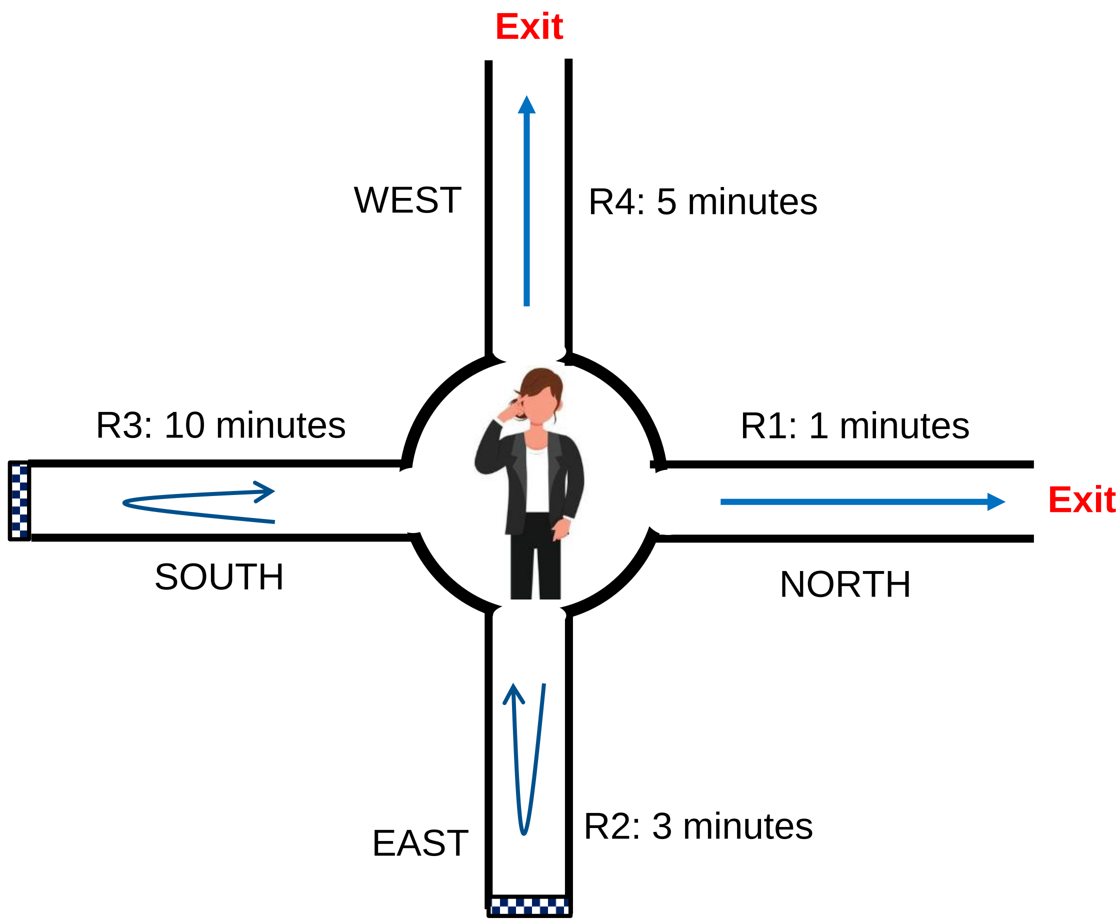
$E(T|R4) = 15 + E(T)$

$$E(T) = P(R1).E(T|R1) + P(R2).E(T|R2) + P(R3).E(T|R3) + P(R4).E(T|R4)$$

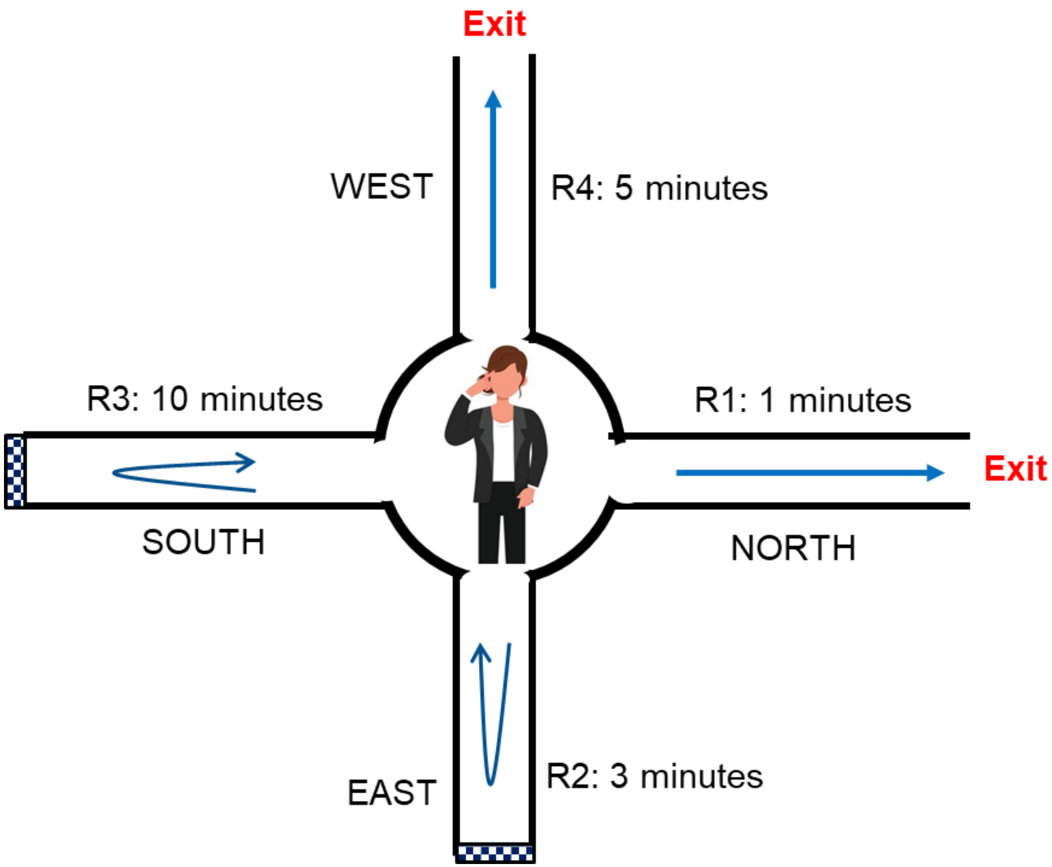
Expectation Estimation

At an unknown railway station, a person arrived at a four-way intersection point. Now, he wants to find out the exit gate, but there is no proper signboard available in this station. Therefore, initially, he has to choose one of four directions. If he goes North, then he will find the exit gate after one minute of travelling. If he goes East, he will wander around the station for three minutes and will then return to his initial position. If he goes South, he will wander around the station for ten minutes and will then return to his initial position. If he goes West, then he will find the exit gate after five minutes of travelling. Assuming that the person is at all times equally likely to choose any of the four directions. Now, estimate the expected number of minutes that the person will be trapped in the station.

Expectation Estimation



Expectation Estimation



$P(R1) = P(R2) = P(R3) = P(R3) = \frac{1}{4}$

$E(T|R1) = 1$

$E(T|R2) = 3 + E(T)$

$E(T|R3) = 10 + E(T)$

$E(T|R4) = 5$

$$E(T) = P(R1).E(T|R1) + P(R2).E(T|R2) + P(R3).E(T|R3) + P(R4).E(T|R4)$$

Expectation Estimation

A person is kept in a dark, underground prison that has three tunnels out of it. If he picks the first tunnel, he will escape to freedom after 30 minutes of walk. Taking the second tunnel would bring him back to where he is now after 1 hour of walk. He would escape to freedom after 1 and a half hours of walk by taking the third tunnel. If the person is tireless in his attempt to escape, what is the expected length of the time until he escapes the prison?

Expectation Estimation

A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third leads immediately to freedom. Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?


Thank You!