

# Real Analysis

## Problem Set 6

1. Using the condensation test, determine whether  $\sum x_n \in \mathbb{R}$ , where  $x_n$  are as follows:

(a)  $x_n = \frac{1}{n}$   
 $\sum_n \frac{1}{n} < \infty \iff \sum_n \frac{2^n}{2^n} = \sum_n 1 < \infty$ . Therefore diverges.

(b)  $x_n = \frac{1}{(n+1) \log(n+1)}$   
 $\sum_{n=2}^{\infty} \frac{1}{n \log n} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{2^n \log 2^n} = \sum_{n=2}^{\infty} \frac{1}{n} < \infty$ . Therefore diverges.

(c)  $x_n = \frac{1}{n^2}$   
 $\sum_n \frac{1}{n^2} < \infty \iff \sum_n \frac{2^n}{2^n \times 2^{2n}} = \sum_n \frac{1}{2^n} = 1 < \infty$ . Therefore converges.

(d)  $x_n = \frac{1}{(\log(n+1))^2}$   
 $\sum_{n=2}^{\infty} \frac{1}{(\log n)^2} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{n^2 (\log 2)^2} < \infty$ . Therefore diverges.

(e)  $x_n = \frac{1}{(n+1) (\log(n+1))^2}$   
 $\sum_{n=2}^{\infty} \frac{1}{n (\log n)^2} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{2^n \cdot n^2 (\log 2)^2} = \sum_{n=2}^{\infty} \frac{1}{n^2 (\log 2)^2} < \infty$ . Therefore converges.

(f)  $x_n = \frac{\log n}{n^2}$   
 $\sum_n \frac{\log n}{n^2} < \infty \iff \sum_n \frac{2^n \cdot n}{4^n} = \sum_n \frac{n}{2^n} < \infty$ . Therefore converges.

(g)  $x_{n-15} = \frac{1}{n (\log n) (\log \log n) (\log \log \log n)^p}$  if  $p > 1$

(h)  $x_{n-15} = \frac{1}{n (\log n) (\log \log n) (\log \log \log n)^p}$  if  $p \leq 1$

(i)  $x_n = \frac{1}{n^p}$  if  $p > 1$

(j)  $x_n = \frac{1}{n^p}$  if  $0 < p \leq 1$

2. Determine whether the following sequences converge in  $\mathbb{R}$ :

(a)  $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n^2 + 1}$

Check that  $\frac{n}{3n^2 + 1} \geq \frac{n+1}{3(n+1)^2 + 1} \forall n \in \mathbb{N}$  and that  $\lim_n \frac{n}{3n^2 + 1} = 0$ . Conclude that the series converges, by alternating series test. The sequence does not absolutely converge because

$$\sum_{n=1}^{\infty} \frac{n}{3n^2 + 1} \geq \sum_{n=1}^{\infty} \frac{n+1}{3(n+1)^2} = \sum_{n=2}^{\infty} \frac{1}{3n} = \infty.$$

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{3n^2 + 1}$

Note that  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2 + 1} < \infty \iff \sum_{n=1}^{\infty} \frac{2^n \cdot 2^{n/2}}{3 \cdot 2^{2n} + 1} < \infty$ . But  $\sum_{n=1}^{\infty} \frac{2^n \cdot 2^{n/2}}{3 \cdot 2^{2n} + 1} \leq \sum_{n=1}^{\infty} \frac{2^{1.5n}}{2 \cdot 2^{2n}} =$

$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}^n} < \infty$ . The series is thus (absolutely) convergent.

(c)  $\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{2^n}{1 + 4^n}}$

Note that  $\sum_{n=1}^{\infty} \sqrt{\frac{2^n}{1 + 4^n}} \leq \sum_{n=1}^{\infty} \sqrt{\frac{2^n}{4^n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}^n} < \infty$ . The series is thus absolutely convergent (so convergent).

(d)  $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$  where  $p > 0$

$\sum_{n=2}^{\infty} \frac{1}{(\log n)^p} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{n^p (\log 2)^p} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{n^p} < \infty$ . The series diverges.

(e)  $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Note that  $\limsup \frac{(n+1)! \cdot n^n}{(n+1)^{n+1} \cdot n!} = \limsup \left( \frac{n}{n+1} \right)^n = \lim_n \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$  whence by ratio test, the series (absolutely) converges.

(f)  $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\log n)^n}$

$\limsup \left| \frac{1}{(\log n)^n} \right|^{\frac{1}{n}} = \limsup \frac{1}{(\log n)} = \lim \frac{1}{\log n} = 0 < 1$  whence by root test, the series (absolutely) converges.

(g)  $\sum_{n=3}^{\infty} \frac{1}{n (\log n) (\log \log n)^p}$  where  $p > 0$

(h)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}!}$

(i)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(j)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\sqrt{n} - (-1)^n)}{n}$

3. Let  $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$ ,  $A_n := \sum_{i=1}^n a_i$  and let  $\alpha : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$  be a function. Prove that

$$(a) \sum_{j=1}^n \sum_{i=1}^j (\alpha(i, j)) = \sum_{i=1}^n \sum_{j=i}^n (\alpha(i, j))$$

Define  $\mathbf{1}_{i \leq j} := \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i \leq j \end{cases}$ . Now

$$\sum_{j=1}^n \sum_{i=1}^j \alpha(i, j) = \sum_{j=1}^n \sum_{i=1}^n \mathbf{1}_{i \leq j} \cdot \alpha(i, j) = \sum_{i=1}^n \sum_{j=1}^n \mathbf{1}_{i \leq j} \cdot \alpha(i, j) = \sum_{i=1}^n \sum_{j=i}^n \alpha(i, j) = \sum_{i=1}^n \sum_{j=i}^n \alpha(i, j).$$

$$(b) \sum_{i=1}^n a_i b_i = b_{n+1} A_n - \sum_{i=1}^n A_i (b_{i+1} - b_i)$$

$$\begin{aligned} b_{n+1} A_n - \sum_{i=1}^n A_i (b_{i+1} - b_i) &= b_{n+1} A_n - \sum_{i=1}^n \sum_{j=1}^i a_j (b_{i+1} - b_i) = b_{n+1} A_n - \sum_{j=1}^n \sum_{i=j}^n a_j (b_{i+1} - b_i) \\ &= b_{n+1} A_n - \sum_{j=1}^n a_j (b_{n+1} - b_j) = \sum_{j=1}^n b_{n+1} a_j - \sum_{j=1}^n a_j b_{n+1} + \sum_{j=1}^n a_j b_j \\ &= \sum_{j=1}^n a_j b_j. \end{aligned}$$

4. Let  $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$ ,  $A_n := \sum_{i=1}^n a_i$ . Suppose  $(A_n)$  is bounded. It is given that  $\sum_{i=1}^n (b_{i+1} - b_i)$  converges absolutely and  $\lim_{n \rightarrow \infty} b_n = 0$ .

$$(a) \text{ Show that } \lim_{n \rightarrow \infty} A_n b_{n+1} = 0.$$

$$(b) \text{ Show that } \sum_n A_n (b_{n+1} - b_n) \text{ is convergent.}$$

$$(c) \text{ Conclude that } \sum_n a_n b_n \text{ converges.}$$

Let  $B > 0$  be such that  $|A_n| \leq B \forall n$ . For  $\varepsilon > 0 \exists N > 0$  such that  $|b_n| < \frac{\varepsilon}{B} \forall n > N$  whence  $|A_n b_{n+1}| < \varepsilon \forall n \geq N$ . By definition,  $\lim_{n \rightarrow \infty} A_n b_{n+1} = 0$ .

$\sum (b_{n+1} - b_n)$  converges absolutely, say the limit is  $L$ . Note that  $\sum_{i=1}^n |A_i (b_{i+1} - b_i)| \leq B \sum_{i=1}^n |(b_{i+1} - b_i)| \leq BL$ . It follows that the sequence  $\left\{ \sum_{i=1}^n |(b_{i+1} - b_i)| \right\}_n$  is monotonous and bounded, whence it converges. It follows that  $\sum A_n (b_{n+1} - b_n)$  converges.

Combining these along with the Abel summation formula, conclude that  $\sum a_n b_n$  converges.

5. Prove using the above

$$(a) \text{ If } (x_n) \in (\mathbb{R}_{\geq 0})^{\mathbb{N}} \text{ is decreasing and } \lim x_n = 0 \text{ then } \sum (-1)^n x_n < \infty.$$

Use the above with  $a_n = (-1)^n$ ,  $b_n = x_n$ .

$$(b) \text{ If } (x_n) \in (\mathbb{R}_{\geq 0})^{\mathbb{N}} \text{ is such that } \exists B > 0 \text{ satisfying } \sum_{i=1}^n x_i \leq B \forall n, \text{ then } \sum \frac{a_n}{n} < \infty.$$

Use the above with  $a_n$  as it is, and  $b_n = 1/n$ .

6. Let  $a > 0$ . Prove  $\sum_{n=1}^{\infty} \frac{1}{(a+n+1)(a+n)} < \infty$ . Find the limit.

7. Let  $a > 0$  and  $m \in \mathbb{N}$ .

(a) Show that  $\sum_{k=1}^n \frac{m}{\prod_{j=0}^m (a+k+j)} = \frac{1}{\prod_{j=1}^m (a+j)} - \frac{1}{\prod_{j=1}^m (a+n+j)}$ .

**Hint:** Induct on  $n$ .

(b) Show that  $\sum_{n=1}^{\infty} \frac{1}{\prod_{j=0}^m (a+n+j)} = \frac{1}{m \prod_{j=1}^m (a+j)}$

8. Let  $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$ ,  $A_n := \sum_{i=1}^n a_i$ ,  $B_n := \sum_{i=1}^n b_i$ . Prove that <sup>1</sup>

$$\sum_{k=n+1}^m a_k b_k = A_m B_m - A_n B_{n+1} - \sum_{k=n+1}^{m-1} A_k b_{k+1}$$

9. (Use your knowledge of high-school integration) Let  $(a_n) \in \mathbb{R}^{\mathbb{N} \cup \{0\}}$  be a sequence and let its partial sums be  $A(n) := \sum_{k=0}^n a_k$ . Fix real numbers  $x < y$ .  $\varphi : [x, y] \rightarrow \mathbb{R}$  is a continuously differentiable function. Show that

$$\sum_{n=\lfloor x \rfloor + 1}^{\lfloor y \rfloor} a_n \varphi(n) = A(\lfloor y \rfloor) \varphi(y) - A(\lfloor x \rfloor) \varphi(x) - \int_x^y A(\lfloor t \rfloor) \varphi'(t) dt$$

Fix some  $y \in \mathbb{R}_{\geq 0}$ . Let  $k = \lfloor y \rfloor$ . Clearly  $\sum_{n=0}^k a_n \varphi(n) = A(k) \varphi(k+1) - \sum_{n=0}^k A(n) (\varphi(n+1) - \varphi(n)) =$

$A(k) \varphi(k) - \sum_{n=0}^{k-1} A(n) (\varphi(n+1) - \varphi(n))$ . But  $\sum_{n=0}^{k-1} A(n) (\varphi(n+1) - \varphi(n)) = \sum_{n=0}^{k-1} A(n) \int_n^{n+1} \varphi'(t) dt =$

$$\sum_{n=0}^{k-1} \int_n^{n+1} A(\lfloor t \rfloor) \varphi'(t) dt = \int_0^k A(\lfloor t \rfloor) \varphi'(t) dt = \int_0^y A(\lfloor t \rfloor) \varphi'(t) dt - \int_{\lfloor y \rfloor}^y A(\lfloor t \rfloor) \varphi'(t) dt$$

$$= \int_0^y A(\lfloor t \rfloor) \varphi'(t) dt - A(\lfloor y \rfloor) \int_{\lfloor y \rfloor}^y \varphi'(t) dt = \int_0^y A(\lfloor t \rfloor) \varphi'(t) dt - A(\lfloor y \rfloor) \varphi(y) + A(\lfloor y \rfloor) \varphi(\lfloor y \rfloor).$$

Putting together,  $\sum_{n=0}^{\lfloor y \rfloor} a_n \varphi(n) = A(\lfloor y \rfloor) \varphi(y) - \int_0^y A(\lfloor t \rfloor) \varphi'(t) dt.$

The required identity trivially follows.

10. Let  $(a_n)$  be a sequence of non-negative reals such that  $\sum a_n \in \mathbb{R}$ . Let  $p \geq 0$ . Show that  $\sum \sqrt{a_n} \cdot n^{-p} \in \mathbb{R}$  if  $p > \frac{1}{2}$ . Find a counterexample if  $p = \frac{1}{2}$ .

---

<sup>1</sup>(Maybe a hint) One is tempted to recall the integration by parts formula. Let  $F(x) := \int_a^x f(x) dx$ ,  $G(x) := \int_a^x g(x) dx$ . Then

$$\int_a^b f(x) G(x) dx = F(b) G(b) - F(a) G(a) - \int_a^b F(x) g(x) dx$$

$\sqrt{a_n}n^{-p} = \sqrt{a_n n^{-2p}} \leq \frac{a_n + n^{-2p}}{2}$ .  $\sum \frac{a_n + n^{-2p}}{2} \in \mathbb{R} \because 2p > 1$ . It follows by comparison that  $\sum \sqrt{a_n}n^{-p} \in \mathbb{R}$ .

Suppose  $p = \frac{1}{2}$ . Consider  $a_n = \frac{1}{n(\log n)^2}$ . Note that  $a_n \geq 0$  and  $\sum a_n \in \mathbb{R}$  by Cauchy condensation test. But  $\sum \sqrt{a_n}n^{-\frac{1}{2}}$  diverges by Cauchy condensation test.