MIT 6.7230/18.456
Algebraic techniques and semidefinite programming
Homework assignment # 6

Date Given: Friday April 19, 2024
Date Due: Wednesday May 1, 1PM.

P1. [***5 pts***] Let $M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ be the Motzkin polynomial. Show that $M(x,y,z)$ is not SOS, but $(x^2+y^2+z^2) \cdot M(x,y,z)$ is. If you use numerical software to solve this, explain whether/how your answer is affected by possible numerical precision issues.

P2. [***5 pts***] Consider a single quadratic polynomial equation $ax^2 + bx + c = 0$. What conditions must (a,b,c) satisfy for this equation to have no real solutions? Assuming this condition holds, give a Positivstellensatz certificate of the nonexistence of real solutions.

P3. [***20 pts***] Consider the polynomial system $\{x + y^3 = 2,\ x^2 + y^2 = 1\}$.

- **(a)** Compute a Groebner basis for this system. Is the ideal zero-dimensional?
- **(b)** Is it feasible over $\mathbb{C}$? How many solutions are there?
- **(c)** Solve this polynomial system numerically, using the eigenvalue method (i.e., compute the multiplication matrices M_x, M_y, and simultaneously diagonalize them).
- **(d)** Is it feasible over $\mathbb{R}$? If not, find using SDP a Positivstellensatz infeasibility certificate.

P4. [***20 pts***] Consider a trigonometric polynomial of the form

$$p(t) = a_0 + \sum_{k=1}^{d} a_k \cos(kt) + \sum_{k=1}^{d} b_k \sin(kt).$$

In this exercise, we will develop a "companion matrix" method for computing its zeros, i.e., the values of $t \in [-\pi, \pi]$ for which $p(t) = 0$.

- **(a)** Write a polynomial system whose solution are the zeros of $p(t)$.
- **(b)** What is the maximum number of zeros that this trigonometric polynomial can have?
- **(c)** Find a Groebner basis for this system. What are the standard monomials?
- **(d)** Give a "companion matrix" based algorithm to produce the zeros, and test it in the following example:

$$p(t) = 1 + \sum_{k=1}^{7} \sqrt{k-1}\cos(kt) + \sum_{k=1}^{5} \frac{1}{\sqrt{k}} \sin(kt).$$

Hint: Depending on your specific approach (there are at least a couple of possibilities), you may find useful to think about Chebyshev polynomials.

P5. [***15 pts***] The *stability number* $\alpha(G)$ of a graph G is the cardinality of its largest stable set. Define the ideal $I = \langle x_i^2 - x_i \quad i \in V\rangle + \langle x_i x_j \quad (i,j) \in E\rangle$.

- **(a)** Show that $\alpha(G)$ is *exactly* given by

$$\min \gamma \qquad \gamma - \sum_{i \in V} x_i \quad \text{is SOS mod } I.$$

[Hint: recall (or prove!) that if I is zero-dimensional and radical, then $p(x) \geq 0$ on $V(I)$ if and only if $p(x)$ is SOS mod I.]

(b) A polynomial is 1-SOS if it can be written as a sum of squares of affine (degree 1) polynomials. Show that an upper bound on $\alpha(G)$ can be obtained by solving

$$\min \gamma \qquad \gamma - \sum_{i \in V} x_i \quad \text{is 1-SOS mod } I. \tag{1}$$

What is the relationship between this upper bound and Lovasz's theta number?

P6. [***15 pts***] In this exercise we compare the relative power of Nullstellensatz and Positivstellensatz based proofs, in the context of a specific example. Consider the set of equations in n variables given by $\{\sum_{i=1}^{n} x_i = 1,\ x_i^2 = 0 \text{ for } i = 1, \ldots, n\}$.

(a) Show that the given equations are infeasible (either over $\mathbb{C}$ or $\mathbb{R}$).

(b) Give a short Positivstellensatz proof of infeasibility (degree 2 should be enough).

(c) Show that every Nullstellensatz proof of infeasibility must have degree greater than or equal to n. (If this gives you trouble, just prove it for a few small values of n).

(d) Use any computational software (e.g., Maple, Mathematica, Macaulay2) to compute a Groebner basis, or numerically solve these equations for increasing values of n. Plot the running time as a function of n. What do you observe?