Real Analysis

Problem Set 3

May 21, 2021

1. Prove that $(a_n)_{n\in\mathbb{N}}$ converges if each a_n is:

   (a) $\dfrac{1}{n}$

   (b) $\dfrac{1}{n^2+1}$

   (c) $\dfrac{6n+5}{n+1}$

   (d) $\dfrac{pn+q}{rn+s}$ where $r \neq 0$

   (e) $\dfrac{pn+q}{n^2-101}$

   (f) x^n where $x \in (0,1)$

   (g) $x^{\frac{1}{n}}$ where $x > 0$

   (h) $n^{\frac{1}{n}}$

   (i) $\dfrac{(-1)^n}{n}$

2. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge, if each a_n is:

   (a) $100 + (-1)^n$

   (b) $(1)^n \cdot n$

3. Determine whether $(a_n)_{n\in\mathbb{N}}$ converges, if each a_n is:

   (a) $\sin n$

   (b) $\dfrac{\lfloor 10^n \cdot x \rfloor}{10^n}$ where $x \in \mathbb{R}$

   (c) $\dfrac{x}{n}$ where $x \in \mathbb{R}$

   (d) $\dfrac{n}{n+1}$

   (e) $\frac{n^2}{n!}$

   (f) $\dfrac{k^n}{n!}$ where $k \in \mathbb{N}$

4. Let (a_n) , (b_n) be sequences which converge to $a, b \in \mathbb{R}$ respectively. Prove the following. (Use only the $\varepsilon - \delta$ definition please, for your own practice and internalization).

   (a) If $a = 0$ then $(|a_n|)_{n\in\mathbb{N}}$ converges.

   (b) $xa_n + yb_n \longrightarrow xa + yb \; \forall x, y \in \mathbb{R}$.

   (c) $a_n b_n \longrightarrow ab$.

   (d) If $a_n \neq 0 \forall n$ and $a \neq 0$ then $\frac{b_n}{a_n} \longrightarrow \frac{b}{a}$.

   (e) If $a_n \geq 0 \forall n$ then $a \geq 0$.

   (f) If $a_n \geq b_n \forall n$ then $a \geq b$

5. For a sequence (a_n) show that $\lim\limits_{n\to\infty} a_n = a \iff \lim\limits_{n\to\infty}(a_n - a) = 0$.

6. Find a sequence (a_n) such that $(|a_n|)$ converges but a_n does not.

7. State and prove the Sandwich theorem.

8. Suppose (a_n) is a sequence such that $\lim_{n\to\infty} a_n \in \{\pm\infty\}$ and $a_n \neq 0 \forall n$. Show that $\frac{1}{a_n} \longrightarrow 0$.

9. Let (a_n) be a convergent sequence. Show that $\exists M > 0$ such that $|a_n| < M \forall n$.

10. Let $a_n = \dfrac{1}{n}$ for $n \in \mathbb{N}$. Define $s(n) = \displaystyle\sum_{j=1}^{n} a_j$ for $n \in \mathbb{N}$.

    (a) Show that $s(2^n) \geq 1 + \dfrac{n}{2} \; \forall \; n \geq 0$.

    (b) Show that the sequence $S = (s(n))$ does not converge.