Real Analysis

Problem Set 1: Hints to selected problems

1. **Problem.** Let $r \in \mathbb{Q} \setminus \{0\}$, $k \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\frac{1}{k}$, $r+k$, $rk \in \mathbb{R} \setminus \mathbb{Q}$.

   Solution. Assume the contrary. It is clear that $k \neq 0$.

   - $\dfrac{1}{k} \in \mathbb{Q} \implies k \in \mathbb{Q}$.
   - $r + k \in \mathbb{Q} \implies k = (r+k) - r \in \mathbb{Q}$.
   - $r \cdot k \in \mathbb{Q} \implies k = r^{-1} \cdot r \cdot k \in \mathbb{Q}$

2. Define $f : \mathbb{Q} \to \mathbb{Q}$ by $f(x) = x^2$. Show that $f^{-1}(2) = \emptyset$. You may assume properties of integers and natural numbers.

3. Let K be an ordered field. Show that $1 > 0$. It can be shown that $x^2 \geq 0$ with equality iff $x = 0$.

4. Let K be an ordered field and $\emptyset \neq S \subseteq K$ which is bounded above. Show that if l and l' are both least upper bounds of S, then $l = l'$.

5. Let K be an ordered field. We can define the *greatest lower bound* (*glb*) of a nonempty subset of K, bounded below, similar to the least upper bound. Come up with such a definition. The *glb* will be referred to as the *infimum*.
   When do we say K has the *glb* property? Come up with a definition. Build a similar problem like Problem 4 and convince yourself that it's true.

6. Let K be an ordered field with the *lub* property. Let S be a non-empty subset of K which is bounded above. Let $-S := \{-x : x \in S\}$. Here $-x$ denotes the additive inverse of x in K. You may assume that such an additive inverse always exists and is unique.

   (a) Does $-S$ have a *glb*?

   (b) Every nonempty subset of K bounded above has an *lub* $\iff$ every nonempty subset of K bounded below has a *glb*. Prove or disprove. If false, suggest a reasonable salvage and prove it.

7. Let $a, b, c, d \in \mathbb{R}$. Prove the following.

   (a) If $a < b$ and $c \leq d$ then $a + c < b + d$.

   (b) If $0 < a < b$ and $0 < c < d$ then $ac < bd$.

   (c) If $a, b, c, d \in \mathbb{R}^+$ and $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

8. Consider the function $f : \mathbb{R}^\times \to \mathbb{R}^\times$ given by $f(x) = \frac{1}{x}$. Assume algebraic properties. Prove the following.

   (a) If $a > 0$ then $f(a) > 0$.

   (b) f is a bijection.

9. Prove the following using the principle of mathematical induction:

   (a) $\displaystyle\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$. Notice that $n(n+2) = (n+1)^2$.

   (b) $n < 2^n \ \forall\, n \in \mathbb{Z}, n \geq 0$

   (c) Any nonempty subset of $\mathbb{N}_0$ has a least element. Known as the Well-Ordering principle. The equivalence of this statement with the principle of induction can be found in any standard textbook.

   (d) If $x > -1$ then $(1+x)^n \geq 1 + nx \ \forall\, n \in \mathbb{Z}_{\geq 1}$. Known as Bernouli's inequality. Look up Bartle Sherbert's book.

Definition. 1. The empty set $\emptyset$ is said to have cardinality 0.

2. A set S is said to have cardinality $n \in \mathbb{Z}_{\geq 1}$ if $\exists$ a bijection $f : S \longrightarrow \{1, 2, \cdots, n\}$.

3. A set S is said to be finite if $S = \emptyset$ or there is some $n \in \mathbb{Z}_{\geq 1}$ and a bijection $f : S \longrightarrow \{1, 2, \cdots, n\}$.

4. A set S is said to be infinite if it is not finite.

Lemma 1
Let $S \neq \emptyset$ be a finite set. Say $m, n \in \mathbb{Z}_{\geq 1}$ are such that there are bijections $f : S \longrightarrow \{1, 2, \cdots, n\}$ and $g : S \longrightarrow \{1, 2, \cdots, m\}$. Then $m = n$.

Corollary 2
The cardinality of a finite set is well-defined. Denote the cardinality of S by $|S|$.

10. **Problem.** Assume the above. $h : A \longrightarrow B$ is a bijection where A, B are finite sets. Show that $|A| = |B|$.

    Solution. Let $n = |A|, m = |B|$. We have $f : A \xrightarrow{\sim} \{1, \cdots, n\}$ and $g : B \xrightarrow{\sim} \{1, \cdots, m\}$. We know that bijections between sets have inverses which are themselves bijections, that is, $\exists\, u : B \xrightarrow{\sim} A$ such that $u(h(a)) = a \ \forall\, a \in A$, and that composition of bijections is a bijection, that is, $v := (f \circ u) : B \xrightarrow{\sim} \{1, \cdots, n\}$. So $v : B \xrightarrow{\sim} \{1, \cdots, n\}$ and $g : B \xrightarrow{\sim} \{1, \cdots, m\}$ are bijections. By lemma 1, $m = n$.

11. A, B are finite disjoint sets. Show that $|A \cup B| = |A| + |B|$.

12. Determine the set of all real numbers x that satisfy $3x + 4 \leq 5$.

13. The real numbers have the trichotomy property, which is stated as follows. For any $a \in \mathbb{R}$ exactly one of the following is true: $a < 0, a = 0, a > 0$.
    If $a, b \in \mathbb{R}$ are such that $ab > 0$ show that either $a, b \in \mathbb{R}^+$ or $a, b \in \mathbb{R}^-$.

14. Find all real numbers x satisfying $x^2 - x > 6$. Use the trichotomoy property.

15. For a positive real number a, we mean by $a^{1/n}$ (for some $n \in \mathbb{Z}_{\geq 1}$) another positive real number which when raised to the n^{th} power gives a. Assume that $a^{1/n}$ exists and is unique for all $a \in \mathbb{R}^+$. Show that $a > b \iff a^{1/n} > b^{1/n}$.

16. Assume existence of roots as before. Let $a \in \mathbb{R}^+$ and $m, n \in \mathbb{Z}_{\geq 1}$. Show that $a^{1/m} > a^{1/n} \iff n > m$.

17. **Problem.** Let $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{Z}_{\geq 1}$. Show that $\left(a^{-1}\right)^n = (a^n)^{-1}$.

Solution. We proceed by induction on n. The base case is trivial because $\left(a^{-1}\right)^{1} = a^{-1} = \left(a^{1}\right)^{-1}$. Suppose $\left(a^{-1}\right)^{k} = \left(a^{k}\right)^{-1}$ for some $k \geq 1, k \in \mathbb{Z}$. Then notice that $\left(a^{-1}\right)^{k+1}\left(a^{k+1}\right) = \left(a^{k}\right)^{-1} \cdot a^{-1} \cdot a \cdot a^{k} = \left(a^{k}\right)^{-1} \cdot a^{k} = 1$. By uniqueness of inverses, we conclude that $\left(a^{k+1}\right)^{-1} = \left(a^{-1}\right)^{k+1}$.
Notice that we tried to avoid commutativity of multiplication.

18. Let $a \in \mathbb{R} \setminus \{0\}$ and $m, n \in \mathbb{Z}$. Show that $a^m a^n = a^{m+n}$.

19. Let $a \in \mathbb{R} \setminus \{0\}$ and $m, n \in \mathbb{Z}$. Show that $(a^m)^n = a^{mn}$.

20. **Problem.** Using induction, prove the AM-GM inequality. You may assume properties of exponentiation. Here is the satement of the inequality:

    Let $a_n, \ldots, a_n \in \mathbb{R}^+ \cup \{0\}$, then $\dfrac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdot a_2 \cdot \ldots a_n)^{\frac{1}{n}}$

    Solution. Base case (one variable) is trivial. We just show the inductive step. Suppose the above statement is true for any n non-negative real numbers a_i (induction hypothesis).

    Let $x_1, \cdots, x_n, x_{n+1} \in \mathbb{R}_{\geq 0}$. WLOG, assume these are in descending order. Then their mean is $\overline{x} = \frac{1}{n+1} \sum_{i=1}^{n+1} x_i$. If $x_i = \overline{x} \forall i$, we are done. Suppose not. Then $x_1 > \overline{x}$. It follows that $x_{n+1} < \overline{x}$. Consider a new quantity $y = x_1 + x_{n+1} - \overline{x}$. Clearly $y \geq 0$. It follows that $y \cdot \overline{x} = (\overline{x} - x_{n+1})(x_1 - \overline{x}) + x_n x_{n+1} > x_n x_{n+1}$ Note that the arithmetic mean of the numbers $x_2, x_3, \cdots, x_n, y$ is $\overline{x}$. We thus have

$$\prod_{i=1}^{n+1} x_i = (x_2 \cdots x_n)(x_n x_{n+1}) < (x_2 \cdots x_n)(y \cdot \overline{x}) = (x_2 \cdots x_n \cdot y)\,\overline{x} \overset{\text{IH}}{\leq} (\overline{x})^n \cdot \overline{x} = (\overline{x})^{n+1}.$$