MIT 6.7230
Algebraic techniques and semidefinite programming
Homework assignment # 5

Date Given: April 3, 2024
Date Due: April 12, 1:00PM

P1. [***10 pts***] Consider a univariate polynomial of degree d, that is bounded by one in absolute value on the interval $[-1, 1]$. How large can its leading coefficient be? Give an SOS formulation for this problem, and solve it numerically for $d = 2, 3, 4, 5$. Can you guess what the general solution is as a function of d? Can you characterize the optimal polynomial?

P2. [***15 pts***] Consider a given univariate rational function $r(x)$, for which we want to find a good polynomial approximation $p(x)$ of fixed degree d on the interval $[-2, 2]$.

(a) Write an SOS formulation to compute the best polynomial approximation of $r(x)$ in the supremum (or $\|\cdot\|_\infty$) norm.

(b) Same as before, but now $p(x)$ is also required to be convex.

(c) Same as before, but $p(x)$ is required to be a convex lower bound of $r(x)$ (i.e., $p(x) \le r(x)$ for all $x \in [-2, 2]$).

(d) Let $r(x) = \frac{1-2x+x^2}{1+x+x^2}$. Find the solution of the previous subproblems (for $d = 4$), and plot them.

P3. [***15 pts***] Given a monic real univariate polynomial of degree $2d$, consider the following linear algebra based algorithm for computing an SOS decomposition:

1. Form the companion matrix $\mathcal{C}_p$
2. Find a complex Schur decomposition of the companion matrix, i.e.,

$$C_p = U\Lambda U^* = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}^*,$$

   where U is unitary, Λ is upper triangular, and the spectra of Λ_{11}, Λ_{22} are complex conjugates of each other.

3. Let $q := vU_{12}^{-1}$, where v is the first row of U_{22}. Let q_r and q_i be the real and imaginary parts of q, respectively.
4. Define

$$\begin{bmatrix} q_1(x) \\ q_2(x) \end{bmatrix} = \begin{bmatrix} -q_r & 1 \\ -q_i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix}.$$

   Then, we have the SOS decomposition $p(x) = q_1^2(x) + q_2^2(x)$.

(a) Implement the algorithm in Julia (or MATLAB, etc.), and test it in a few examples.

(b) If $p(x)$ is not nonnegative, where does the algorithm fail?

(c) Prove that the algorithm is correct, i.e., it always produces a valid SOS decomposition.

Hint: what properties does the complex polynomial $q(x) = q_1(x) + iq_2(x)$ have?

P4. [***15 pts***] In general, the SOS decomposition of a univariate polynomial is not unique. Given a specific basis of $\mathbb{R}[x]$ (for instance, the standard monomial basis we have been considering), a "natural" choice can be obtained by finding a matrix Q satisfying

$$p(x) = [x]_d^T \, Q \, [x]_d, \qquad Q \succeq 0$$

and such that the determinant of Q is as large as possible. This "central solution" $Q_\star$ can be computed by solving a convex optimization problem, since $\log\det Q$ is a concave function on the region where Q is positive semidefinite. [In fact, this convex optimization problem can be reformulated a semidefinite programming problem.]

(a) Compute numerically the central solution $Q_\star$ for the polynomial $p(x) = x^6 - 6x^5 + 16x^4 - 24x^3 + 22x^2 - 12x + 4$.

(b) Show, using the KKT optimality conditions, that in general the inverse of the optimal matrix Q is a Hankel matrix. Verify this property in your example.

P5. [***20 pts***] Consider linear maps between symmetric matrices, i.e., of the form $\Lambda : \mathcal{S}^n \to \mathcal{S}^m$. A map is said to be a *positive map* if it maps the PSD cone S^n_+ into the PSD cone S^m_+ (i.e., it preserves positive semidefinite matrices).

(a) Show that any linear map of the form $A \mapsto \sum_i P_i^T A P_i$, where $P_i \in \mathbb{R}^{n\times m}$, is positive. These maps are known as *decomposable* maps.

(b) Show that the linear map $C : \mathcal{S}^3 \to \mathcal{S}^3$ (due to M.-D. Choi) given by:

$$C : A \mapsto \begin{bmatrix} 2a_{11} + a_{22} & 0 & 0 \\ 0 & 2a_{22} + a_{33} & 0 \\ 0 & 0 & 2a_{33} + a_{11} \end{bmatrix} - A.$$

is a positive map, but is not decomposable.

Hint: Consider the polynomial defined by $p(x, y) := y^T \Lambda(xx^T) y$. How can you express positivity and decomposability of the linear map Λ in terms of the polynomial p?

P6. [***20 pts***] Recall the procedure described in the lecture to recover a nonnegative measure from its moments.

(a) Prove that the procedure as described always produces a valid measure, provided the initial matrix of moments is positive definite.

(b) Find a discrete measure having the same first eight moments as a standard Gaussian distribution of zero mean and unit variance.

(c) What does the previous result imply, if we are interested in computing integrals of the type

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(x) e^{-\frac{x^2}{2}} \, dx,$$

where $p(x)$ is a polynomial of degree less than eight? What would you do if $p(x)$ is an arbitrary (smooth) function?

(d) Use these ideas to give an approximate numerical value of the definite integral

$$\int_{-\infty}^{\infty} \cos(1 + x) \, e^{-3x^2} \, dx$$

How does your approximation compare with the true value?

Note: In the general case where we are matching $2d$ moments of a standard Gaussian, it can be shown that the support of these discrete measures will be given by the d zeros of $H_d(x/\sqrt{2})$, where H_d is the standard Hermite polynomial of degree d. (Optional) Can you prove this?

P7. [***15 pts***] In this exercise we describe a procedure to generalize Chebyshev-type inequalities. For simplicity, we consider the univariate case; see the paper of Bertsimas and Popescu for extensions and more details.

Consider a univariate random variable X, with an unknown probability distribution supported on the set Ω, and for which we know its first $d+1$ moments $(\mu_0, \ldots, \mu_d)$. We want to find bounds on the probability of an event $S \subseteq \Omega$, i.e., want to bound $\mathbf{P}(X \in S)$. We assume S and Ω are given closed intervals. Consider the following optimization problem in the decision variables c_k:

$$\min \sum_{k=0}^{d} c_k \mu_k \qquad \text{s.t.} \quad \begin{cases} \sum_{k=0}^{d} c_k x^k \geq 1 & \forall x \in S \\ \sum_{k=0}^{d} c_k x^k \geq 0 & \forall x \in \Omega. \end{cases} \tag{1}$$

(a) Show that any feasible solution of (1) gives a valid upper bound on $\mathbf{P}(X \in S)$. How would you solve this problem?

(b) Assume that $\Omega = [0, 5]$, $S = [4, 5]$, and we know that the mean and variance of X are equal to 1 and $1/2$, respectively. Give upper and lower bounds on $\mathbf{P}(X \in S)$. Are these bounds tight? Can you find the worst-case distributions?

P8. [***15 pts, Optional***] A minor variation of this method can be used to obtain bounds on moments of non-polynomial functions. For instance, try to prove the following bounds on the expectation of the absolute value, valid for all random variables X for which the fourth moment exists:

$$\sqrt{\mu_2^3/\mu_4} \leq \mathbf{E}[|X|] \leq \sqrt{\mu_2}.$$

Hint: You may want to bound the absolute value with polynomials, and use homogeneity (i.e., for $t > 0$, we have $|tX| = t|X|$).