Subsequences: $(a_n) \in \mathbb{R}^{\mathbb{N}}$. $g: \mathbb{N} \to \mathbb{N}$ is a strictly increasing func ($m > n \iff g(m) > g(n)$)

Exercise: if $g: \mathbb{N} \to \mathbb{N}$ is strictly increasing then g is one-one.

Pf: $\exists$ $m, n \in \mathbb{N}$, $m \neq n$ s.t. $g(m) = g(n)$.

WLOG $m > n$. So $g(m) > g(n) = g(m)$.

This contradicts trichotomy.

Then $(a_{g(n)}) \in \mathbb{R}^{\mathbb{N}}$ is said to be a subseq of (a_n).

CONVERGENCE OF SEQUENCES in $\mathbb{R}^{\mathbb{N}}$:

Def: Let $A = (a_n) \in \mathbb{R}^{\mathbb{N}}$ be a sequence. We say A converges in $\mathbb{R}$ (or simply "converges") if $\exists$ $a \in \mathbb{R}$

P1 s.t. $\forall$ $\varepsilon > 0$ there are only finitely many a_i's which are not in $(a-\varepsilon, a+\varepsilon)$

or,

P2 $\forall$ $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t. $a_n \in (a-\varepsilon, a+\varepsilon)$ $\forall$ $n \geq N$

In that case we say $a \in \mathbb{R}$ is a limit of the seq A.

Let $(a_n) \in \mathbb{R}^{\mathbb{N}}$ be a sequence. $a \in \mathbb{R}$. It has P1 $\iff$ it has P2.

Pf: Suppose a has P1. Let $\varepsilon > 0$ given. Then I can find finitely many points $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ for which $a_{i_j} \notin (a-\varepsilon, a+\varepsilon)$ $(\forall j = 1, \dots, k)$.

Choose $N = 1 + \max\{i_1, \dots, i_k\}$. So $n \geq N$

$\Rightarrow a_n \in (a-\varepsilon, a+\varepsilon)$. $\therefore (a_n)$ has P2.

(a_n) has P2. Let $\varepsilon > 0$. $\exists\ N \in \mathbb{N}$ for which

$a_n \in (a-\varepsilon, a+\varepsilon)\ \forall\ n \geq N$. $\therefore$ If some $a_i \notin (a-\varepsilon, a+\varepsilon)$

then $i < N$. $\therefore \{ i \in \mathbb{N} : a_i \notin (a-\varepsilon, a+\varepsilon) \} \subseteq \{1, \ldots, N-1\}$

$\therefore$ Only finitely many terms are outside $(a-\varepsilon, a+\varepsilon)$.

$\therefore (a_n)$ has P1. $\square$

Examples: (1) $(a_n) = \left(\frac{1}{n}\right)$. This converges to $a = 0$.

Why? $\varepsilon > 0$ given. Then choose $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 100 > \frac{1}{\varepsilon}$

Then $n \geq N \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon$

(2) $(a_n) = \left(\frac{1}{2^n}\right)$. Converges to 0.

Why? Let $\varepsilon > 0$ given. Can find N s.t. $2^N \varepsilon \geq N\varepsilon > 1$

$\therefore n \geq N \Rightarrow 2^n \cdot \varepsilon > 1 \Rightarrow \varepsilon > \frac{1}{2^n} = \left| \frac{1}{2^n} - 0 \right|$.

(3) $(a_n) = ((-1)^n)$. Does not converge.

Why? Suppose $x \in \mathbb{R}$ is a limit of the given seq.

Let $d = \max\{ |x-1|, |x-(-1)| \}$.

WLOG $|x-1| \geq |x-(-1)|$. So $d = |x-1| > 0$.

Clearly $d > 0$. Let $\varepsilon = \frac{d}{2}$. Then $\exists\ N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow |(-1)^n - x| < \varepsilon$

$\therefore$ This must also be true for $n = 2N \geq N$

i.e., $|x-1| = |(-1)^{2N} - x| < \varepsilon = \frac{|x-1|}{2}$

$\Rightarrow 1 < \frac{1}{2}$ $\square$

Lemma : Let $A = (a_n)$ be a sequence. And $(a_{g(n)})$ is a subseq. Say a_n cgs to $a \in \mathbb{R}$. Then $a_{g(n)}$ cgs to a.

Pf : g is inc. Let $\varepsilon > 0$ be given. $\exists\ N$ s.t. $a_n \in (a-\varepsilon, a+\varepsilon)\ \forall\ n \geq N$.

$$\therefore n \geq N \Rightarrow \underbrace{g(n) \geq n}_{(\because g \text{ inc})} \underbrace{\geq N}_{(\because n \geq N)} \Rightarrow a_{g(n)} \in (a-\varepsilon, a+\varepsilon)$$

$\Rightarrow \left(a_{g(n)}\right)$ converges to a. $\square$

Lemma : Say $(a_n) \in \mathbb{R}^{\mathbb{N}}$ is a convergent seq s.t. it converges to $a \in \mathbb{R}$, $b \in \mathbb{R}$. Then $a = b$.

Pf: Suppose $a \neq b$. Choose $\varepsilon = \frac{|a-b|}{2} > 0$.

$\exists\ N_1$ s.t. $|a_n - a| < \varepsilon \quad \forall\ n \geq N_1$.

$\exists\ N_2$ s.t. $|a_n - b| < \varepsilon \quad \forall\ n \geq N_2$.

Take $N = \max\{N_1, N_2\}$.

$$\begin{aligned} \therefore n \geq N \Rightarrow 2\varepsilon = |a-b| &= |a - a_n + a_n - b| \\ &\leq |a_n - a| + |a_n - b| \\ &< 2\varepsilon \end{aligned}$$

$\Rightarrow \varepsilon < \varepsilon$ (contradiction)

$\therefore$ Limit (if exists) is unique.