$\exists$ a bij $f : \mathbb{N} \to \mathbb{Z}$. $f(n) = (-1)^n \lfloor \frac{n}{2} \rfloor$

$f(1) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$, ...

<u>Hilbert's hotel</u>: A hotel has rooms marked $1, 2, \ldots$. $\forall n \in \mathbb{N}$, room n is occupied. You have an announcement system which allows you to send messages to all residents of the hotel. You can only ask them to shift rooms.

- One new customer comes. Ask all residents to shift by one room, i.e., ask them to follow the rule $n \longmapsto n+1$.
- k new customers come $(k \geq 1)$. $n \longmapsto n+k$.
- A bus comes with infinitely many customers named as $1, 2, 3, \ldots$. $n \longmapsto 2n$
- Infinitely many buses $B_1, B_2, \ldots$ come where each bus carries infinitely many customers (as the previous case)

  $\exists$ a bij $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. $\exists$ a bij $g : (\mathbb{N} \cup \{0\}) \times \mathbb{N} \to \mathbb{N}$,

  $g(i, j) := f(i+1, j)$.

  (Check: g is bij)

  Think of the original customers as people of Bus 0. Identify the j^{th} person of Bus i by (i, j). Allot them the room $g(i, j)$.

- A bus comes with infinitely many people where the names of people are infinite binary strings (& for each such string, there is exactly one person with that name). Hilbert's hotel can no longer accommodate these people. (HW: Why can I not accommodate them).

Theorem: 1. Any subset of a countable set is countable.

2. Countable union of countable sets is Countable.

3. If A is countable, so is $A \times A$.

<u>Problem</u>: Suppose $f : [0,1] \to [0,\infty)$ is a func. There is a constant $M \geqslant 0$ s.t. for any $k \in \mathbb{N}$ & for any $a_1, \dots, a_k \in [0,1]$ we have $f(a_1) + \dots + f(a_k) \leq M$.

$S = \{x \in [0,1] : f(x) \neq 0\} = \{x \in [0,1] : f(x) > 0\}$.

(i) Let $A_n = \{x \in [0,1] : f(x) > \frac{1}{n}\}$ (defined for $n \in \mathbb{N}$). Show A_n is countable.

(ii) Show S is countable.

<u>Soln</u>: (i) Suppose A_n has atleast $m \geqslant Mn+1$ distinct elts. Let $x_1, \dots, x_m \in A_n$ be distinct.

$\therefore \quad M \geqslant f(x_1) + \dots + f(x_m) \geqslant \frac{m}{n}$

$\Rightarrow m \leq Mn \Rightarrow 1 \leq 0$ (Contradiction)

(ii) $S \underset{*}{=} \bigcup_{n \in \mathbb{N}} A_n$. But this a ctble union of ctble sets.

$\therefore S$ countable.

$* :\quad S = \{ x \in [0,1] : f(x) > 0 \}$

$x \in S \Rightarrow f(x) > 0 \Rightarrow \exists\, n \in \mathbb{N}$ s.t. $nf(x) \geq 1$

$\Rightarrow f(x) \geq \frac{1}{n} > \frac{1}{n+1} \Rightarrow x \in A_{n+1} \Rightarrow x \in \bigcup_{n \in \mathbb{N}} A_n .$

$\therefore S \subseteq \bigcup_{n \in \mathbb{N}} A_n$

$x \in \bigcup_{n \in \mathbb{N}} A_n \Rightarrow \exists\, k \in \mathbb{N}$ s.t. $x \in A_k$

$\Rightarrow f(x) > \frac{1}{k} > 0 \Rightarrow f(x) > 0 \Rightarrow x \in S$

$\therefore \bigcup_{n \in \mathbb{N}} A_n \subseteq S.$