Logarithm and exponential

$a > 0$, $r \in \mathbb{R}$ then we know a^r.

$\log_a y = x \iff a^x = y$ (definition)

$\log_a(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}$.

We can show (indeed, we know)

$$\log_a(p \cdot q) = \log_a p + \log_a q \qquad — (*)$$

Say $f(x) = \log_a(x)$. Then $f(xy) = f(x) + f(y)$.

(1) $(\mathbb{R}^+, \cdot)$ is a group.

(2) $(\mathbb{R}, +)$ is a group.

$\therefore$ $(*)$ just says f is a group homomorphism.

Further, f is a bijection.

We say "$\log_a(\cdot)$ is a continuous group homomorphism".

Define $F(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$ (as a formal object).

Let $a \in \mathbb{R}$, $a_n = \frac{a^n}{n!}$ (and $a_0 = 1$)

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{a}{n+1}\right| = 0 < 1$$

Ratio Test $\Rightarrow \sum_{n=0}^{\infty} a_n \in \mathbb{R}$

$\Rightarrow F(a) \in \mathbb{R}$.

$\therefore$ F gives a function $\mathbb{R} \to \mathbb{R}$.

x is + indeterminate

Some properties that we know.

(1) $F(x) \cdot F(y) = F(x+y)$

(2) $F(0) = 1$

(3) $F(x) = F\left(\frac{x}{2} + \frac{x}{2}\right) = \left(F(x/2)\right)^2 \geq 0$

$\boxed{\mathbb{Z}_p}$ (p prime)

$$\left.\begin{aligned} P(x) &= x^p - x \\ Q(x) &= 0 \end{aligned}\right\} \text{(over } \mathbb{Z}_p)$$

$a \in \mathbb{Z}_p \Rightarrow a^p = a$
$\Rightarrow P(a) = 0$

~~Same 3~~

✓ Diff 0

When are 2 polynomials equal?

~~Same set of roots~~

coefficients same

(4) $F(x) \cdot F(-x) = F(x-x) = 1$

$\Rightarrow F(x) \neq 0 \Rightarrow F(x) > 0$

$\therefore F : \mathbb{R} \longrightarrow \mathbb{R}^+$. In fact, a continuous group isomorphism.

We define $e := F(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$

Guess (know from experience) : $F(a) = e^a \quad \forall\, a \in \mathbb{R}$.

Step 1: Show for $a \in \mathbb{N}$. (Hint : Induction)

Step 2: Show for $a \in (-\mathbb{N}) \cup \{0\}$.

Step 3 : Show for rationals

Step 4 : Show for $\mathbb{R}$ (use : $\mathbb{Q}$ is dense in $\mathbb{R}$).

TOPOLOGY ON $\mathbb{R}$

Open intervals in $\mathbb{R}$: $(a,b) = \{x \in \mathbb{R} : a < x < b\} \subseteq \mathbb{R}$

where $-\infty \leq a \leq b \leq \infty$.

Open ball : $B_r(a) = \{z \in \mathbb{R} : |z-a| < r\}$
(where $a \in \mathbb{R}$, $r \geq 0$) $= (a-r, a+r)$.

$B_r(a) = \{z \in X : d(a,z) < r\}$
$(a \in X, r \geq 0)$
$d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$

Def (open set): A set $U \subseteq \mathbb{R}$ is said to be an <u>open set</u> if U can be written as a union of open balls.

Example:

(1) $(0,2) = \bigcup_{S \in \mathcal{U}} S$ where $\mathcal{U} = \{B_1(1)\}$

$\overset{\text{Check?}}{=} \bigcup_{r<1} B_r(1) = \bigcup_{S \in \mathcal{V}} S$ where $\mathcal{V} = \{B_r(1) : r < 1\}$

$(0,2) \overset{?}{=} \bigcup_{r<1} B_r(1)$

$B_r(1) \subseteq (0,2) \ \forall \ r < 1 \Rightarrow \bigcup_{r<1} B_r(1) \subseteq (0,2)$

Let $p \in (0,2)$. $d := |1-p| < 1$. $d' = \frac{1+d}{2}$.

$d < d' < 1$. $|p-1| = d < d' (<1) \Rightarrow p \in B_{d'}(1)$

$\Rightarrow p \in \bigcup_{r<1} B_r(1)$.

Since p was arbitrary, conclude $(0,2) \subseteq \bigcup_{r<1} B_r(1)$.

(2) $\mathbb{R} = \bigcup_{S \in \mathcal{U}} S$ $\qquad \mathcal{U} = \{ B_r(p) : p \in \mathbb{R}, r > 0 \}$

pf: $S \in \mathcal{U} \Rightarrow S \subseteq \mathbb{R} \Rightarrow \bigcup_{S \in \mathcal{U}} S \subseteq \mathbb{R}$

$p \in \mathbb{R} \Rightarrow p \in B_1(p) \in \mathcal{U} \Rightarrow \mathbb{R} \subseteq \bigcup_{S \in \mathcal{U}} S$.

(3) $\mathcal{U} = \{ \}$. $\bigcup_{S \in \mathcal{U}} S = \phi$.