$x^2 - 2 = 0$ $\qquad$ $\mathbb{Q}$ ~ rationals

$\qquad\qquad\qquad\qquad$ $\mathbb{R}$ ~ reals

no solutions in $\mathbb{Q}$.

exactly 2 solutions in $\mathbb{R}$ — one of them +ve.

$x^2+1 = 0$ $\qquad$ no solution in $\mathbb{R}$

$\{x \in \mathbb{Q} \mid x^2 - 2 < 0\} \neq \phi$

$\{x \in \mathbb{R} \mid x^2 + 1 < 0\} = \phi$

I. Preliminaries

Partially ordered set: $(P, \leq)$, where P is a set and $\leq$ is a relation on P, is called a partially ordered set (Poset) if:

(Reflexivity)	$a \leq a$	$\forall\, a \in P$
(Antisymmetry)	$a \leq b$ and $b \leq a \Rightarrow a = b$	$\forall\, a, b \in P$
(Transitivity)	$a \leq b,\ b \leq c \Rightarrow a \leq c$	$\forall\, a, b, c \in P$

In this case, $\leq$ is said to be a partial order on P.

* $P = \{1, 2, 3, 6\}$. $a \leq b$ iff $a \mid b$. $2 \nleq 3$ and $3 \nleq 2$.
  Verify $(P, \leq)$ is a poset.

* $P = \mathbb{R}$. $\leq$ is the usual comparison. $(P, \leq)$ is a poset.

Define $<$: $\quad a < b \iff (a \leq b \text{ and } a \neq b)$

Totally ordered set: $(P, \leq)$ is said to be a totally ordered set if $\leq$ is a partial order on P and $\forall\, a, b \in P$ either $a \leq b$ or $b \leq a$.
In this case, $\leq$ is said to be a total order on P.

(Totally) ordered field: Let K be a field. $(K, <)$ is said to be a totally ordered field if $<$ is a total order on K and the following hold true:

- $a < b \Rightarrow a + c < b + c \quad \forall\, a, b, c \in K$
- $a > 0,\ b > 0 \Rightarrow ab > 0 \quad \forall\, a, b \in K$.

In what follows, $(K, <)$ will always denote an ordered field.

1) $K = K^+ \sqcup K^- \sqcup \{0\}$ $\quad$ disjoint union

2) $K^\times = K^+ \sqcup K^-$ is a group (under multiplication) Verify as HW.

3) K^+ is a group (under mult).
   In fact, K^+ is a subgp of $K^\times$.

Fact : $a^2 > 0 \quad \forall\ a \in K^\times$

Pf : $\to\ a > 0$: $a^2 = a \cdot a > 0$

$\to\ a < 0 \Rightarrow 0 = a + (-a) < 0 + (-a) = -a$

$\Rightarrow a^2 = (-a)\cdot(-a) > 0$

Fact : $1 > 0$

Pf : HW

$$\underset{\substack{\Psi \\ 0}}{\mathbb{N}} \longrightarrow \mathbb{Z} \xrightarrow{\text{take fractions}} \underset{\substack{\downarrow \\ \text{field}}}{\mathbb{Q}}$$

$f : \mathbb{Q} \to \mathbb{Q}$ given by $x \mapsto x^2$ is not surjective.
$f^{-1}(2) = \phi$.

Let $S \subseteq K$, $S \neq \phi$. We say $l \in K$ is an upper bound of S if $l \geq x \quad \forall\ x \in S$.

$l \in K$ is said to be a least upper bound (lub) (or supremum) of S if

- l is an u.b. of S
- if $l' \in K$ is s.t. $l' < l$ then l' is not an u.b. of S.

K is said to have the lub property if every subset $S \subseteq K$, $S \neq \phi$, which is bounded above, has a supremum.

$\mathbb{Q}$ does not have the lub property. $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$

Theorem (Dedekind / Cauchy / Cantor) : There is an ordered field $\mathbb{R}$ such that

i) $\mathbb{Q}$ is an ordered subfield of $\mathbb{R}$, i.e.,
$\exists$ a map $i : \mathbb{Q} \hookrightarrow \mathbb{R}$ s.t.
$$i(a+b) = i(a) + i(b)$$
$$i(ab) = i(a)\cdot i(b)$$
$$a \leq b \Rightarrow i(a) < i(b)$$
For your comfort, you may think that $\mathbb{Q} \subseteq \mathbb{R}$ and order is preserved inside $\mathbb{R}$.

ii) $\mathbb{R}$ has the lub property.

I. Properties of $\mathbb{R}$:

① (Archimedean property) Let $x \in \mathbb{R}^+$, $y \in \mathbb{R}$. Then $\exists\ n \in \mathbb{Z}^+$ s.t. $nx > y$.

Pf : $S = \{nx \mid n \in \mathbb{Z}^+\} \subseteq \mathbb{R}$

(*) — Suppose $t \leq y \quad \forall\ t \in S$. So S is bdd above. $S \neq \phi$.
By lub property of $\mathbb{R}$, S has a supremum, say u.

$x > 0 \Rightarrow -x < 0 \Rightarrow u - x < u \Rightarrow u - x$ is not an u.b. of S.

$\Rightarrow \exists\ w \in S$ s.t. $w > u - x$
($w = kx$ for some $k \in \mathbb{Z}^+$) $\Rightarrow \underset{\overset{\uparrow}{S}}{(k+1)x} > u$

This is a contradiction ($\because$ u is an u.b. of S)

So $(*)$ is false $\Rightarrow \exists\ n \in \mathbb{Z}^+$ s.t. $nx > y$. □

Cor: $\mathbb{Z}$ is unbounded above & unbounded below.

② ($\mathbb{Q}$ is dense in $\mathbb{R}$) Given $\alpha, \beta \in \mathbb{R}$ $(\alpha < \beta)$ $\exists\ a \in \mathbb{Q}$ s.t. $\alpha < a < \beta$.

Pf: Try as exercise / look up any standard book or internet. □

Open interval: $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$
$(a < b \in \mathbb{R})$

Cor: $(a, b) \cap \mathbb{Q}$ is an infinite set.
Pf: HW (important)

Exercise: (1) If $a \in \mathbb{R} \setminus \mathbb{Q}$ then $\frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$

(2) There is no total ordering on $\mathbb{C}$.

(3) $a, b, c, d \in \mathbb{R}^+$ s.t. $a > b$, $c > d$. Show $ac > bd$.

II. Roots of +ve reals:

Let $\alpha \in \mathbb{R}^+$. A real n^{th} root of α is some $x \in \mathbb{R}$ s.t. $x^n = \alpha$.

Claim: Atmost one such $x > 0$ exists.

Pf: WLOG, $\alpha > 1$. Suppose $\exists\ x > y > 0$ in $\mathbb{R}^+$ s.t. $x^n = \alpha$, $y^n = \alpha$.

$x^n > y^n$ (use induction + exercise (3))

$\Rightarrow \alpha > \alpha$ (contradiction) □

Finite set: S is said to be "finite" if $\exists\ n \in \mathbb{N}_0$ and a bijection $f : S \to \{1, 2, \dots, n\}$.