③ $F \subseteq \mathbb{R}$ closed. $U_n := \bigcup_{x \in F} B_{y_n}(x)$. Claim: $F = \bigcap_{n \in \mathbb{N}} U_n$.

# Real Analysis

Problem Set 7

August 16, 2021

1. Let $U \subseteq \mathbb{R}$ be nonempty and open. Show that $\exists r \in \mathbb{Q}, s \in \mathbb{R} \setminus \mathbb{Q}$ such that $r, s \in U$.
2. Let $U \subseteq \mathbb{R}$ be clopen (i.e., both open and closed). Show that $U$ is either $\varnothing$ or $\mathbb{R}$.
3. Prove that every closed set in $\mathbb{R}$ is the intersection of a countable collection of open sets.
4. Let $U, V \subseteq \mathbb{R}$. Show that $(U \cap V)^o = U^o \cap V^o, (U \cup V)^o \supseteq U^o \cup V^o$ and $(U \cup V)' = U' \cup V'$.
5. Show that $S'$ is closed for any $S \subseteq \mathbb{R}$.
6. Let $S \subseteq \mathbb{R}$ be a bounded set containing infinitely many points.
   (a) Show that there must be reals $a, b \in \mathbb{R}$ such that $S \subseteq [a, b]$.
   (b) Show that we can find an increasing sequence $(a_n)$ and a decreasing sequence $(b_n)$ such that
      - $a \leq a_1 \leq b_1 \leq b$
      - $b_n - a_n = \dfrac{b-a}{2^n}\ \forall n$
      - $[a_n, b_n] \cap S$ is an infinite set $\forall n$.

   (c) Show that $\sup a_n = \inf b_n$. Call this $l$.
   (d) Conclude that $S$ has a limit point. (**Hint:** $l$ will be a limit point of $S$).
7. Let $S \subseteq [a, b]$ be a set with no limit point.
   (a) Let $x \in [a, b]$. Show that $\exists$ an open set $U_x \subseteq \mathbb{R}$ such that $x \in U_x$ and $U_x \cap S \subseteq \{x\}$.
   (b) Conclude that $S$ is finite. (**Hint:** Compactness of closed intervals).
8. Let $S \subseteq [a, b]$ be an infinite set.
   (a) Prove that there is a sequence in $[a, b]$, all of whose terms are in $S$ with no repeated terms.
   (b) Show that the above sequence has a limit point $l \in [a, b]$.
   (c) Conclude that $S$ has a limit point. (**Hint:** $l$ will be a limit point of $S$).
9. $S \subseteq \mathbb{R}$ is a bounded infinite set. Let $T := \{x \in \mathbb{R} : \text{there are infinitely many points in } S \text{ more than } x\}$.
   (a) Show that $T \neq \varnothing$ and $T$ is bounded above. Let $s := \sup T$. Clearly $s \in \mathbb{R}$.
   (b) Let $a \in \mathbb{R} \setminus T$. Show that $a$ is an upper bound of $T$.
   (c) Show that $s$ is a limit point of $S$.

Let $C_1 \supseteq C_2 \supseteq \cdots$ be a decreasing seq of nonempty compact sets of $\mathbb{R}$.
Then $\bigcap_{n\in\mathbb{N}} C_n \neq \phi$.

10. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a decreasing (under containment) sequence of compact sets of $\mathbb{R}$. Suppose $\bigcap_{n\in\mathbb{N}} \mathfrak{C}_n = \varnothing$.

    (a) Show that $\mathscr{U} := \{\mathbb{R} \setminus \mathfrak{C}_n : n \in \mathbb{N}\}$ is an open cover of $\mathfrak{C}_1$.

    (b) Show that $\exists K \in \mathbb{N}$ such that $k \geq K \implies \mathfrak{C}_k = \varnothing$.

11. For a bounded set $S \subseteq \mathbb{R}$ define

$$\operatorname{diam} S := \sup_{x,y\in S} |x - y|.$$

Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a decreasing sequence of nonempty compact sets of $\mathbb{R}$ such that $\lim_{n\to 0}(\operatorname{diam} \mathfrak{C}_n) = 0$. Show that $\bigcap_{n\in\mathbb{N}} \mathfrak{C}_n$ is a singleton.

12. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a sequence of closed subsets of compact $\mathfrak{C} \subseteq \mathbb{R}$ such that $\bigcap_{i\in A} \mathfrak{C}_i \neq \varnothing$ for any finite $A \subseteq \mathbb{N}$. Show $\bigcap_{n\in\mathbb{N}} \mathfrak{C}_n \neq \varnothing$.

    (**Hint:** Use a similar construction as in problem 10).

13. For $S \subseteq \mathbb{R}$, show that $\mathbb{R} \setminus (\overline{S}) = (\mathbb{R} \setminus S)^o$. $\quad (\overline{S})^c = (S^c)^o$

14. (Something from sequences and series) Let $(a_n)$ be a sequence of real numbers converging to $a$. Define a sequence $(b_n)$ by $b_n := \dfrac{\sum_{i=1}^{n} i \cdot a_i}{n(n+1)}$. Prove that $\lim_{n\to\infty} b_n = \dfrac{a}{2}$.

② $\mathbb{R}$ cannot be written as the disjoint union of 2 nonempty open sets.

Pf: Suppose $\mathbb{R} = U \cup V$ s.t. $U, V$ open & $U \cap V = \phi$, $U \neq \phi \neq V$.

Let $x \in U$, $y \in V$, can assume $x < y$.

$A := \{ t \in \mathbb{R} : [x, t] \subseteq U \}$. $A \neq \phi$ (Reason: $x \in A$).

$A$ bdd above (Reason: $t \in A \Rightarrow t < y$).

$s := \sup A \in \mathbb{R} = U \cup V$.

So $s \in U$ or $s \in V$.

→ Say $s \in U$. So $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(s) \subseteq U \Rightarrow s + \frac{\varepsilon}{2} \in U$

$\Rightarrow s$ not u.b. of $U$.

(Contradiction).

So $s \notin U$.

→ Say $s \in V$. So $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(s) \subseteq V$.

$$\left(s - \frac{\varepsilon}{2}, s\right] \cap V = \left(s - \frac{\varepsilon}{2}, s\right]$$

$$\left(s - \frac{\varepsilon}{2}, s\right] \cap U \neq \phi.$$

Reason: $s - \frac{\varepsilon}{2}$ not u.b. of $U$

$\Rightarrow \exists\ t \in (s - \varepsilon/2, s]$ s.t. $t \in U$.

$$(s - \varepsilon/2, s] \cap V \cap U \neq \phi$$

$$\Rightarrow V \cap U \neq \phi \quad \text{(contradiction)}.$$

It finally stands that $s \notin \mathbb{R}$. □

Main problem: $U \subseteq \mathbb{R}$, $U$ clopen. Let $V = \mathbb{R} \setminus U = U^c$.
So $V$ is open. Also, $U \cap V = \phi$.
But $\mathbb{R} = U \cup V$.
$\therefore$ Either $U = \phi$ or $V = \phi$ ($\Leftrightarrow U = \mathbb{R}$).

(8) (a) Choose $x_1 \in S$. $F_1 := S \setminus \{x_1\}$. Choose $x_2 \in F_1$.

Inductively after having chosen $x_n$, define $F_n := S \setminus \{x_1, \ldots, x_n\}$ & choose $x_{n+1} \in F_n$.

By construction & induction:

- $x_n \in S \quad \forall n$
- $x_i \neq x_j \quad \forall i \neq j$.

$\therefore (x_n) = X$ is a seq in $S$ s.t. all terms distinct.

(b) By Bolzano Weierstraß, there is a convergent subseq $(x_{n_k})_{k \in \mathbb{N}}$, converging to $l \in [a,b]$.

$\because [a,b]$ closed.

(c) For $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $k \geqslant K \Rightarrow |x_{n_k} - l| < \varepsilon$.

So $\{x_{n_k}\}_{k \geqslant K}$ is an infinite subset ($\because x_n$ distinct) of $S \cap B_\varepsilon(l)$. $\therefore l \in S'$.

# Real Analysis

Baire's theorem

August 16, 2021

$\overline{S}^c = (S^c)^o$

$\overline{S^c}^c = S^o \Rightarrow \overline{S^c} = (S^o)^c$

**Definition 1 (Nowhere dense set)** *A subset $A$ of $\mathbb{R}$ is* <u>*nowhere dense*</u> *or* <u>*rare*</u> *if $(\overline{A})^o = \varnothing$.*

In other words, $A$ is rare iff it is contained in a closed set with empty interior. In fact, if $A$ is rare then $A$ is contained in $F = \overline{A}$ which has empty interior. Conversely if $A$ is contained in closed $F$ with $F^o = \varnothing$ then $(\overline{A})^o \subseteq (\overline{F})^o = F^o = \varnothing$.

We recall what *dense* means.

**Definition 2 (Dense set)** *A subset $A$ of $\mathbb{R}$ is said to be* <u>*dense*</u> *if $\overline{A} = \mathbb{R}$.*

In case of subsets of $\mathbb{R}$, we can equivalently say that $A$ is dense iff $\forall x \in \mathbb{R}, r > 0, \exists a \in A$ such that $a \in \mathscr{B}_r(x)$.

We might guess, from the terminology, that the complement of a nowhere dense set might be dense. This is true, as we shall see in the next paragraph. One might get more bold and claim that $A$ is rare iff $A^c$ is dense. Well, not quite. Think about $A = \mathbb{R} \setminus \mathbb{Q}$ which is dense in $\mathbb{R}$. But the closure of $A^c = \mathbb{Q}$ has nonempty interior, hence not rare.

$\overline{\overline{A}^c} = ((\overline{A})^o)^c \Rightarrow (\overline{A})^o = (\overline{\overline{A}^c})^c$

It turns out that $A$ is rare iff $(\overline{A})^c$ is dense. Indeed recall that $\overline{S} = ((S^c)^o)^c$ for any set $S$. Take $S = (\overline{A})^c$. This gives $(S^c)^o = (\overline{A})^o = \varnothing \iff \overline{S} = ((S^c)^o)^c = \mathbb{R} \iff S$ is dense $\iff (\overline{A})^c$ is dense.

Clearly $A \subseteq \overline{A} \iff (\overline{A})^c \subseteq A^c$. It thus stands that $A$ is rare $\iff \mathbb{R} = \overline{(\overline{A})^c} \subseteq \overline{A^c} \implies \overline{A^c} = \mathbb{R} \iff A^c$ is dense.

A rare $\Leftrightarrow \overline{A}^o = \phi \Leftrightarrow \overline{\overline{A}^c} = \mathbb{R}$

$\Leftrightarrow \overline{A}^c$ dense

A rare $\Rightarrow \overline{A^c} = \mathbb{R}$

$\overline{A} \supseteq A$

$\Rightarrow \overline{A}^c \subseteq A^c$

$\Rightarrow \mathbb{R} \subseteq \overline{A^c} \Rightarrow \overline{A^c} = \mathbb{R}$

**Proposition 3** *(a) Any subset of a rare set is rare.*

*(b) A finite union rare sets is rare.*

*(c) The closure of a nowhere dense set is nowhere dense.*

Proof (a) Let $A \subseteq B$ where $B$ is rare. Then $\overline{A} \subseteq \overline{B}$ whence $(\overline{A})^o \subseteq (\overline{B})^o = \varnothing$.

(b) Let $A, B$ be rare sets. Equivalently, $(\overline{A})^c, (\overline{B})^c$ are dense. Let $S := A \cup B$. Let $T \neq \varnothing$ be open. $(\overline{A})^c$ dense $\implies T \cap (\overline{A})^c \neq \varnothing$. Further, $T \cap (\overline{A})^c$ is a nonempty open set whence $\varnothing \neq T \cap (\overline{A})^c \cap (\overline{B})^c = T \cap (\overline{A} \cup \overline{B})^c = T \cap (\overline{A \cup B})^c = T \cap (\overline{S})^c$ whence $S$ is rare.

(c) $A$ rare $\iff (\overline{A})^o = \varnothing \implies \left(\overline{(\overline{A})}\right)^o = (\overline{A})^o = \varnothing$.

■

**Exercise** *Let $A, B$ be closed sets such that $(A \cup B)^o \neq \varnothing$. Show that either $A^o \neq \varnothing$ or $B^o \neq \varnothing$.*

**Exercise** *Give examples of two sets $A, B \subseteq \mathbb{R}$ such that $(A \cup B)^o \neq \varnothing$ but $A^o = B^o = \varnothing$.*

**Exercise** *Show that $\mathbb{Q}$ can be written as a countable union of rare sets in $\mathbb{R}$.*

$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$

The above proposition must ring a bell in your mind and raise a question like "What about the *countable* union of rare sets?" One recalls the example that $\mathbb{Q}$ is a countable union of rare sets in $\mathbb{R}$ but $\mathbb{Q}$ is not itself rare - $\overline{\mathbb{Q}} = \mathbb{R}$ whence $\left(\overline{\mathbb{Q}}\right)^o = \mathbb{R}$. Such countable unions are not dense and mathematicians gave a name for it.

**Definition 4** *Let $A$ be a subset of $\mathbb{R}$ .*
*$A$ is said to be meagre or of the first category if $A$ can be written as a countable union of rare sets in $\mathbb{R}$.*
*If $A$ is not meagre, it is said to be nonmeagre or of the second category.*
*$A$ is said to be residual if its complement is meagre.*

We further see another small, but useful result.

**Proposition 5** *The following are equivalent for $\mathbb{R}$ . Note that we are not yet claiming about their truth or falsity.*

*(a) A meagre set has empty interior.*

*(b) A countable intersection of open dense sets is dense.*

*(c) A residual set is dense.*

Proof We prove them in a circular way as follows.

$(a) \implies (b)$: Note that the complement of an open dense set is a closed rare set. Let $\mathscr{U}$ be a countable collection of open dense sets in $\mathbb{R}$ and consider $S := \bigcap_{U\in\mathscr{U}} U$. Then $S^c := \bigcup_{U\in\mathscr{U}} U^c$ is a countable union of closed rare sets. By definition, $S^c$ is meagre, whence by hypothesis, $(S^c)^o = \varnothing$. But $(S^c)^o = \left(\overline{S}\right)^c$ so that $\overline{S} = \mathbb{R}$.

$(b) \implies (c)$: By definition, a residual set is the complement of a meagre set whence it is a countable intersection of some sets with dense interiors. In other words, a rare set contains a countable intersection of open dense sets, which is dense by hypothesis. Since any superset of a dense set must be dense, conclude that a residual set is dense.

$(c) \implies (a)$: Let $S$ be meagre. Then $S^c$ is residual. By hypothesis, $\overline{S^c} = \mathbb{R} \implies S^o = \left(\overline{S^c}\right)^c = \varnothing$.

■

We have built up to an important result known as the Baire category theorem.

**Theorem 6 (Baire category theorem)** *A countable intersection of open dense sets in $\mathbb{R}$ is dense.*

Proof Let $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable collection of open dense sets in $\mathbb{R}$. Let $V \neq \varnothing$ be any open set. Clearly $V \cap U_1 \neq \varnothing$. Pick a closed disc $\overline{\mathscr{B}_{r_1}(x_1)} \subset V \cap U_1$ with $r_1 < 1$. Since $U_2$ is dense, $\mathscr{B}_{r_1}(x_1) \cap U_2 \neq \varnothing$ (also open). So pick a closed disc $\overline{\mathscr{B}_{r_2}(x_2)} \subset \mathscr{B}_{r_1}(x_1) \cap U_2$ such that $r_2 < \frac{1}{2}$. Continuing this process will give us a decreasing sequence of closed balls $\mathscr{B}_{r_n}(x_n)$ with $0 < r_n < \frac{1}{n}$. Further notice that the sequence $(x_n)$ is Cauchy in $\mathbb{R}$: for any $n \in \mathbb{N}$, we can pick $N = n$ so that $p \geq q \geq N \implies d(x_p, x_q) \leq \frac{1}{q} \leq \frac{1}{n}$. By completeness, $X$ converges to a point, say $x$, in $\mathbb{R}$. By definition, for any $n \in \mathbb{N}, \exists N \geq n \in \mathbb{N}$ such that $x \in \mathscr{B}_{\frac{1}{n}}(x_k) \forall k \geq N$; but $k \geq N \geq n \implies x \in \mathscr{B}_{\frac{1}{n}}(x_k) \subseteq \overline{\mathscr{B}_{\frac{1}{n}}(x_n)} \subseteq V \cap \left(\bigcap_{i=1}^{n} U_i\right)$. This means $x \in U_i \forall i$ and $x \in V$ whence $V \cap \left(\bigcap_{i\in\mathbb{N}} U_i\right) \neq \varnothing$. Since $V$ was an arbitrary open set to start with, we conclude that $\bigcap_{i\in\mathbb{N}} U_i$ is dense in $\mathbb{R}$. ■

**Corollary 7** *One cannot write $\mathbb{R}$ as a countable union of rare sets. In other words, $\mathbb{R}$ is not meagre.* $\because \mathbb{R}^o \neq \phi$.

**Corollary 8** *A residual set in $\mathbb{R}$ is not meagre.*

Proof We can note that a countable union of meagre sets is meagre ($\because$ a countable union of countable sets is countable). Let $A \subseteq \mathbb{R}$ be residual, whence $A^c$ is meagre. If $A$ were meagre, so would be $\mathbb{R} = A \cup A^c$. But $A^c$ is meagre, so that $\mathbb{R}$ is meagre which is clearly false. $\blacksquare$

**Corollary 9** $\mathbb{R} \setminus \mathbb{Q}$ *is not meagre in* $\mathbb{R}$.

Proof $\mathbb{Q}$ is meagre $\implies$ $\mathbb{R} \setminus \mathbb{Q}$ is residual and thus, by the previous corollary, not meagre. $\blacksquare$

**Corollary 10** $\mathbb{Q}$ *cannot be written as the intersection of countably many open sets in* $\mathbb{R}$.

Proof Say $\mathbb{Q} = \bigcap_{n \in \mathbb{N}} U_n$ for some collection of open sets $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ in $\mathbb{R}$. Note that $U_n \supseteq \mathbb{Q} \implies \overline{U_n} = \overline{\mathbb{Q}} = \mathbb{R} \forall n$ whence each $U_n$ is an open dense set in $\mathbb{R}$. Also note that $\mathscr{V} = \{\mathbb{R} \setminus \{q\} : q \in \mathbb{Q}\}$ is a countable collection of open dense sets in $\mathbb{R}$. Further $\bigcap_{V \in \mathscr{V}} V = \varnothing$ whence $\bigcap_{S \in \mathscr{U} \cup \mathscr{V}} S = \varnothing$ which contradicts theorem 6. $\blacksquare$