Today we talk about direct sum and matrix of a linear map.

Direct Sum :-

Given two F-vector spaces V and W, we can define vector space structure on $V \times W$;

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$
$$\alpha(v, w) = (\alpha v, \alpha w), \quad \forall \alpha \in F$$

Identity is $(0, 0)$

We denote this vector space by the notation $V \oplus W$.

Remarks :- 1) We can define direct sum for any finite collection of F-vector spaces,

$$\{V_i\}_{i=1}^{n}$$

$$\bigoplus_{i=1}^{n} V_i$$

2) For a finite collecⁿ of F-vector spaces the "direct sum" of them can be called as "direct product"

Collecⁿ of V.S

Direct Sum

Direct Product

Examples :-

1) Given any field F, we know that F^n has F- vector space structure

$$F^n = F \oplus F \oplus F \oplus \cdots \oplus F$$

2) V is a F-vector space and $W \subset V$ is a subspace.

$W_F^c + W = V$ and $W_F^c \cap W = \{0\}$

$$W_F^c \oplus W \cong V = W_F^c + W$$

$$(\omega', \omega) \xmapsto{\varphi} \omega' + \omega$$

This map φ is surjective

injective $\left(\omega_1' + \omega_1 = \omega_2' + \omega_2 \Rightarrow \underbrace{\omega_1' - \omega_2'}_{W_F^c} = \underbrace{\omega_2 - \omega_1}_{W} \right.$

$\Rightarrow \omega_1' - \omega_2' = \omega_2 - \omega_1 = 0$

$\left. \Rightarrow \omega_1 = \omega_2 , \omega_1' = \omega_2' \right)$

Linear map

$\varphi : W \oplus U \longrightarrow W + U$ where W and U are subspaces of V

$$(\omega, u) \longmapsto \omega + u$$

φ is linear map

surjective

In order to have φ injective we need $W \cap U = \{0\}$

Only nontrivial fact is that φ is injective.

Suppose φ is injective $(\omega_1, u_1), (\omega_2, u_2)$

$\varphi(\omega_1, u_1) = \varphi(\omega_2, u_2) \Rightarrow \omega_1 = \omega_2 , u_1 = u_2$

$\Rightarrow \{\omega_1 + u_1 = \omega_2 + u_2 \Rightarrow \omega_1 = \omega_2 , u_1 = u_2\}$

$\Rightarrow \{\omega_1 - \omega_2 = u_2 - u_1 \Rightarrow \omega_1 = \omega_2 , u_1 = u_2\}$

1) V and W are F-vector space and let both V and W be finite dim. Then $V \oplus W$ finite dimensional with

$$\dim(V \oplus W) = \dim V + \dim W$$

Proof:- $\{v_1, \ldots, v_n\}$ be a basis for V, $\dim V = n$

$\{w_1, \ldots, w_m\}$ " " " " W, $\dim W = m$

$$\{\boxed{(v_1, 0), (v_2, 0), \ldots, (v_n, 0)}, \boxed{(0, w_1), \ldots, (0, w_m)}\}.$$

This is a basis for $V \oplus W$.

Note that $\exists$ injective map

$$\sigma_1 : V \longrightarrow V \oplus W$$
$$v \longmapsto (v, 0)$$

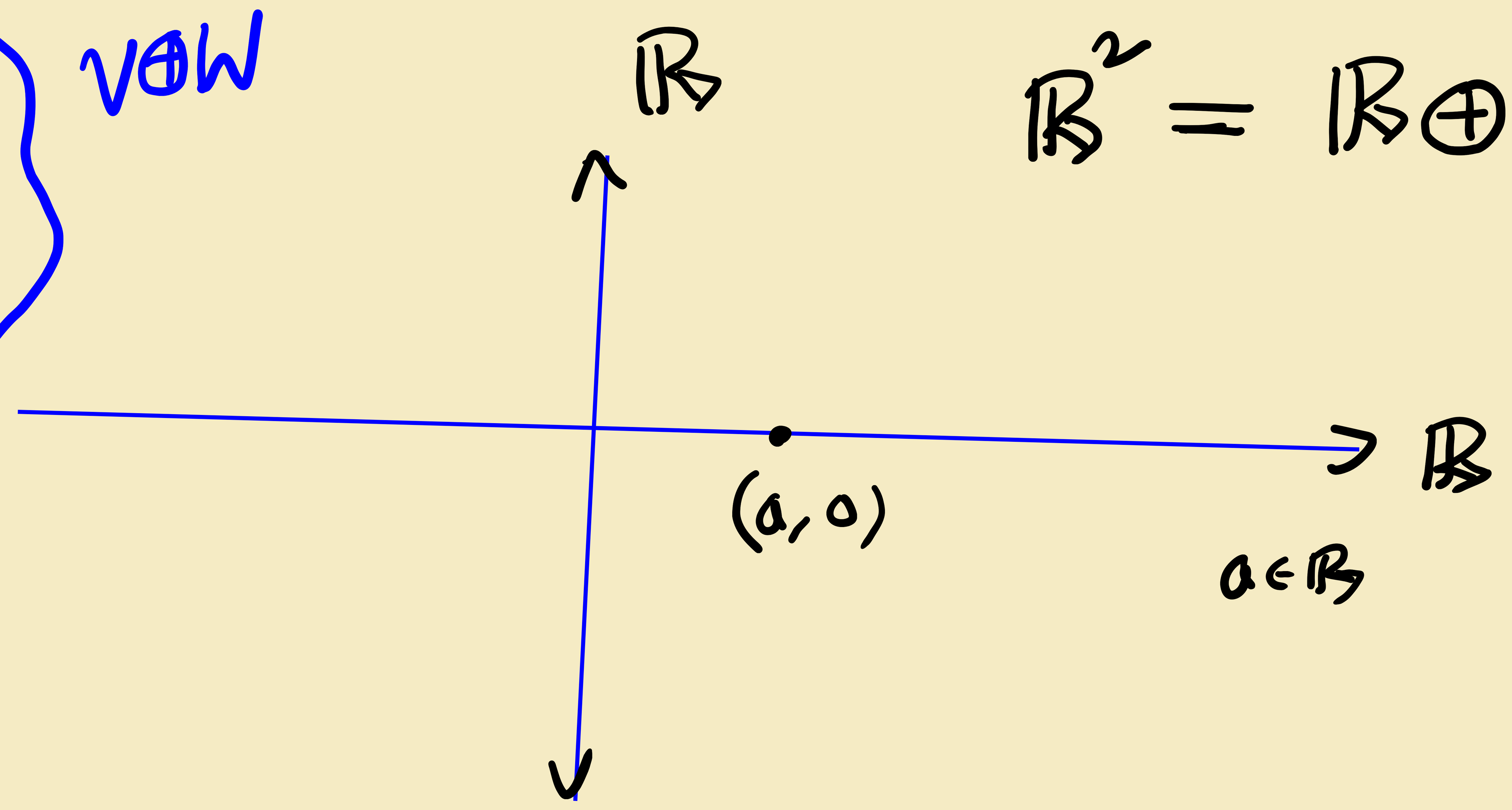
$$v_1 + v_2 \longmapsto (v_1 + v_2, 0) = (v_1, 0) + (v_2, 0)$$

$$\sigma_2 : W \longrightarrow V \oplus W$$
$$w \longmapsto (0, w)$$

$\sigma_1(V) \subset V \oplus W$

$$V \underset{\sigma_1}{\cong} \sigma_1(V)$$

$$\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$$



Summary:- An isomorphic copy of V (and W as well) sits inside $V \oplus W$ as one of its subspace

$V = W \oplus U \Rightarrow$ Some isomorphic copy of W sits inside V as subspace

$$\sigma_1 : W \longrightarrow W \oplus U \qquad w \longmapsto (w, 0)$$
$$\sigma_2 : U \longrightarrow W \oplus U \qquad u \longmapsto (0, u)$$

$\sigma_1(W)$ and $\sigma_2(U)$ are subspaces of $V = W \oplus U$

$$\sigma_1(W) \cap \sigma_2(U) = \{(0,0)\}$$

$$\sigma_1(W) + \sigma_2(U) = V = W \oplus U$$

$$(w, u) = (w, 0) + (0, u)$$
$$= \sigma_1(w) + \sigma_2(u)$$

$$\boxed{\sigma_1(W)_F^c = \sigma_2(U)}$$

Summary :- Let $V = W \oplus U$, an isomorphic copy of W sits inside V as a subspace and an isomorphic copy of U sits inside V as subspace and these two isomorphic copies are algebraically complement to each other in V.

Going in Reverse :-

Suppose $V = W + W_F^c$

$$V \cong W \oplus W_F^c$$

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Theorem :-

Suppose V is F-vector space and $\{W_i\}_{i=1}^{k}$ are subspaces

Then TFAE

i) The linear map

$$\pi : W_1 \times W_2 \times \cdots \times W_k \longrightarrow W_1 + W_2 + \cdots + W_k$$

$$(\omega_1, \cdots, \omega_k) \longmapsto \omega_1 + \omega_2 + \cdots + \omega_k$$

is an isomorphism

ii) $W_i \cap (W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_k) = \{0\}$

for every $1 \leq i \leq k$

iii) For every element ω in $W_1 + \cdots + W_k$ $\exists$ unique $\omega_i \in W_i$ for $i = 1, \cdots, k$ such that

$$\omega = \omega_1 + \cdots + \omega_k$$

$$W_1 + \cdots + W_k \cong W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

$$W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

Corollary :- $W \subset V$, $\quad W_F^c \cong V/W$

$$V = W \oplus W_F^c = W \oplus V/W$$

$$\varphi : V \longrightarrow W \Rightarrow V = \ker \varphi \oplus \operatorname{Im} \varphi$$