CHENNAI MATHEMATICAL INSTITUTE

Discrete Mathematics

Quiz 4:
Date: July 9, 2021.

(1) Let $n \geq 2$. Compute the number of spanning trees on $[n]$ which have the edge $[1,2]$. Compute the number of spanning trees where vertex 1 has degree $n-3$.   5 marks

(2) A vertex v in a graph is said to be a cut vertex if $G\backslash\{v\}$ has at least one more component than G. Is the following statement true or false? Every graph on $n \geq 2$ vertices has at least 2 vertices that are not cut vertices. If it is true, give a proof. If it is false provide a counterexample. Give an example of a graph which has a cut vertex. 5 marks

(3) We build a graph on $[n]$ in the following way.

```
Buildgraph[n] {
    V={1,2,..,n}; E={};  G=(V,E);
    do until G is connected
        select e={i,j} not in E uniformly at random;
        add e to E;
    return(G);
}
```

   i What is the probability that the second edge added decreases the number of components?   1 mark

  ii Suppose at some stage G has k components and edges E. Fix $i \in V$. Give a reasonable lower bound on the probability that $\{i,j\} \notin E$ reduces the number of components from k to $k-1$. (hint: Give a bound which works for all vertices i.)   3 marks

iiii If a coin has probability p of showing up heads, what is the expected number of coin tosses before you see a head? 2 marks

  iv Let X_k denote the number of edges added to G before the number of components reduces from k to $k-1$. What is $\mathbb{E}(X_k)$? 2 marks

   v With respect to the program above, let X be the number of edges in G returned by the program. Express X in terms of X_k. What is the $\mathbb{E}(X)$? Simplify the expression.   2 marks