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The K-moment problem for compact semi-algebraic sets

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Let K be a closed subset of $\mathbb{R}^d$. A function $s:\mathbb{N}_0^d\to R$ is called a *K-moment sequence* if there exists a positive Borel measure $\mu\in M(\mathbb{R}^d)$ supported by K such that $s(\alpha)$ is the α-th moment of μ, i.e. $s(\alpha)=\int x^\alpha d\mu$, for all $\alpha\in\mathbb{N}_0^d$. The main result of this note characterizes the K-moment sequences for compact semi-algebraic sets K. In particular, it proves a conjecture of Berg and Maserick (cf. [B–M, p. 495] and [B, p. 119]) and it subsumes a number of known results for special sets K (see e.g. [A, B–M, C, M]). Our results come out as an interplay between the multidimensional moment problem and semi-algebraic geometry. First we apply the positivstellensatz to solve the K-moment problem, while then the moment problem is used to obtain a representation for the positive polynomials on K.

We collect a few standard notations, cf. [B, F]. Let $M(\mathbb{R}^d)$ denote the set of positive Borel measures μ on $\mathbb{R}^d$ which have moments of all order, i.e. $x^\alpha\in L^1(\mu)$ for all $\alpha\in\mathbb{N}_0^d$. We write $x^\alpha=x_1^{\alpha_1}\dots x_d^{\alpha_d}$ for $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$, where $x_j^0:=1$. $\mathbb{R}[x_1,\dots,x_d]$ and $\mathbb{C}[x_1,\dots,x_d]$ are the real resp. complex polynomials in d indeterminates $x_1,\dots,x_d$. Let s be a function $s:\mathbb{N}_0^d\to\mathbb{R}$ and p a complex polynomial $p(x)=\sum a_\alpha x^\alpha$. We define a polynomial $\bar{p}$ by $\bar{p}(x):=\sum \overline{a_\alpha}x^\alpha$ and a function $p(E)s:\mathbb{N}_0^d\to\mathbb{C}$ by $(p(E)s)(\beta):=\sum a_\alpha s(\alpha+\beta)$, $\beta\in\mathbb{N}_0^d$. We say that s is *positive definite* if $\sum_{k,l=1}^{n} s(\alpha_k+\alpha_l)c_k\overline{c_l}\geqq 0$ for arbitrary $\alpha_1,\dots,\alpha_n\in\mathbb{N}_0^d$, $c_1,\dots,c_n\in\mathbb{C}$ and $n\in\mathbb{N}$.

Theorem 1. *Let $R=\{r_1,\dots,r_m\}$ be a finite subset of $\mathbb{R}[x_1,\dots,x_d]$. Suppose that the semi-algebraic set $K_R:=\{x\in\mathbb{R}^d: r_j(x)\geqq 0$ for $j=1,\dots,m\}$ is compact. Then a function $s:\mathbb{N}_0^d\to\mathbb{R}$ is a K_R-moment function if and only if s and $(r_{j_1}\dots r_{j_k})(E)s$ are positive definite for all possible choices $j_1,\dots,j_k$ of pairwise different numbers from $\{1,\dots,m\}$.*

Proof. That the above condition is necessary follows easily from the representation $s(\alpha)=\int x^\alpha d\mu$ combined with the fact that $\operatorname{supp}\mu\subseteqq K_R$. We prove its sufficiency.

Let Σ_R denote the set of all finite sums of elements p^2 and $p^2 r_{j_1}\dots r_{j_k}$, where $p\in\mathbb{R}[x_1,\dots,x_d]$ and $j_1,\dots,j_k\in\{1,\dots,m\}$. Then Σ_R is a cone in $\mathbb{R}[x_1,\dots,x_d]$ and closed under multiplication of polynomials. Since K_R is compact, there is a $\varrho > 0$ such that $|x|^2 := x_1^2 + \ldots + x_d^2 < \varrho^2$ for all $x = (x_1, \ldots, x_d) \in K_R$, i.e. $\varrho^2 - |x|^2$ is positive on K_R. From the positivstellensatz in semi-algebraic geometry ([B–C–R, Corollaire 4.4.3, (ii)], cf. [S]) we conclude that there exist polynomials $g, h \in \Sigma_R$ such that $(\varrho^2 - |x|^2)g = 1 + h$.

Let L_s be the (complex) linear functional on $\mathbb{C}[x_1, \ldots, x_d]$ defined by $L_s(x^\alpha) = s(\alpha)$, $\alpha \in \mathbb{N}_0^d$, and let $\mathscr{H}_s$ be the canonical Hilbert space associated with the positive definite function s, see e.g. [F, Sect. 4]. Scalar product and norm of $\mathscr{H}_s$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. For notational simplicity we shall consider the polynomials of $\mathbb{C}[x_1, \ldots, x_d]$ directly as elements of $\mathscr{H}_s$. (Strictly speaking, we have to take the corresponding elements of the quotient space $\mathbb{C}[x_1, \ldots, x_d]/N$, where

$$N := \{p \in \mathbb{C}[x_1, \ldots, x_d] : L_s(p\bar{p}) = 0\}.)$$

Then, by this convention and by the construction of $\mathscr{H}_s$, $\mathbb{C}[x_1, \ldots, x_d]$ is a dense linear subspace of $\mathscr{H}_s$ and $\langle p, q \rangle = L_s(p\bar{q})$ for $p, q \in \mathbb{C}[x_1, \ldots, x_d]$. The assumption that s and $(r_{j_1} \ldots r_{j_k})(E)s$ are positive definite implies that L_s is non-negative on the cone Σ_R. This fact will be used in the sequel without mention.

Suppose $p \in \mathbb{C}[x_1, \ldots, x_d]$ and $j \in \{1, \ldots, d\}$. The crucial step in this proof is to show that

$$\|x_j p\| \leqq \varrho \|p\| . \tag{1}$$

Since obviously $p\bar{p} = p_1^2 + p_2^2$ with $p_1, p_2 \in \mathbb{R}[x_1, \ldots, x_d]$, we have $p\bar{p} \in \Sigma_R$. Hence we obtain

$$L_s(|x|^{2n} p\bar{p}g) \leqq L_s(|x|^{2(n-1)} p\bar{p}(|x|^2 g + 1 + h)) = \varrho^2 L_s(|x|^{2(n-1)} p\bar{p}g)$$

for $n \in \mathbb{N}$, so that

$$L_s(|x|^{2n} p\bar{p}g) \leqq \varrho^{2n} L_s(p\bar{p}g), \qquad n \in \mathbb{N}. \tag{2}$$

Let $\nu_{p,j} \in M(\mathbb{R})$ be a measure with moments $L_s(x_j^n p\bar{p})$, $n \in \mathbb{N}_0$. We denote by χ_λ the characteristic function of $(-\infty, -\lambda) \cup (\lambda, +\infty)$, where $\lambda > 0$. Then we have for $n \in \mathbb{N}$

$$\begin{aligned}\lambda^{2n} \int_{\chi_\lambda} d\nu_{p,j} &\leqq \int_{\mathbb{R}} t^{2n} d\nu_{p,j}(t) = L_s(x_j^{2n} p\bar{p}) \leqq L_s(x_j^{2n} p\bar{p}(|x|^2 g + 1 + h)) \\ &= \varrho^2 L_s(x_j^{2n} p\bar{p}g) \leqq \varrho^2 L_s(|x|^{2n} p\bar{p}g) \leqq \varrho^{2n} \varrho^2 L_s(p\bar{p}g),\end{aligned}$$

where the last inequality follows from (2). If $\lambda > \varrho$, the preceding estimate implies that $\int_{\chi_\lambda} d\nu_{p,j} = 0$. Therefore, $\operatorname{supp} \nu_{p,j} \subseteqq [-\varrho, \varrho]$. Hence we get

$$\|x_j p\|^2 = L_s(x_j^2 p\bar{p}) = \int t^2 d\nu_{p,j}(t) \leqq \varrho^2 \int d\nu_{p,j} = \varrho^2 L_s(p\bar{p}) = \varrho^2 \|p\|^2$$

which proves (1).

Let X_j denote the multiplication operator by the coordinate x_j on the domain $\mathbb{C}[x_1, \ldots, x_d]$ of $\mathscr{H}_s$. By (1), X_j is bounded. The operators X_j, $j = 1, \ldots, d$, are symmetric and they pairwise commute on $\mathbb{C}[x_1, \ldots, x_d]$, hence their closures $\overline{X_j}$ are commuting bounded self-adjoint operators on $\mathscr{H}_s$. If E denotes the spectral measure of this family, $\mu(\cdot):=\langle E(\cdot)1,1\rangle$ is a measure of $M(\mathbb{R}^d)$ with moments $L_s(x^\alpha)=s(\alpha)$, $\alpha\in\mathbb{N}_0^d$. Since $\|\overline{X_j}\|\leqq\varrho$ by (1),

$$\operatorname{supp}\mu\subseteqq Q:=[-\varrho,\varrho]\times\ldots\times[-\varrho,\varrho].$$

Thus we have $L_s(r_kp^2)=\int_Q r_kp^2d\mu\geqq 0$ for each polynomial $p\in\mathbb{R}[x_1,\ldots,x_d]$ and $k=1,\ldots,m$. From this and the Weierstraß approximation theorem we conclude that $\operatorname{supp}\mu\subseteqq\{x\in\mathbb{R}^d\colon r_k(x)\geqq 0\}$. Hence $\operatorname{supp}\mu\subseteqq K_R$. □

Remarks. We sketch a second proof which is less elementary and perhaps less instructive than the previous one. We have

$$\begin{aligned}\|x_j^n\|^2&=L_s(x_j^{2n})\leqq L_s(x_j^{2n}(|x|^2g+1+h))=\varrho^2L_s(x_j^{2n}g)\\&\leqq\varrho^2L_s(|x|^{2n}g)\leqq\varrho^{2n+2}L_s(g),\end{aligned}$$

where the last inequality follows from (2) applied with $p=1$. Therefore, the sequence

$$\gamma_{2n}:=s((2n,0,\ldots,0))+s((0,2n,0,\ldots,0))+\ldots+s((0,\ldots,0,2n)),\qquad n\in\mathbb{N},$$

satisfies Carleman's condition $\sum\gamma_{2n}^{-1/2n}=+\infty$, hence s is a moment sequence by a result of Nussbaum [N, p. 189]. With a bit more work [by verifying first the density of polynomials in $L^p(\mu)$ for $p\geqq 2$] it then follows that $\operatorname{supp}\mu\subseteqq K_R$.

Let us retain the assumptions of Theorem 1 and the notation introduced above. An immediate consequence of Theorem 1 and of the Hahn-Banach separation theorem for convex sets is

Corollary 2. *Each polynomial $p\in\mathbb{R}[x_1,\ldots,x_d]$ which is non-negative on K_R belongs to the closure of Σ_R in the finest locally convex topology on $\mathbb{R}[x_1,\ldots,x_d]$.*

Corollary 3. *If $p\in\mathbb{R}[x_1,\ldots,x_d]$ is positive on the set K_R, then $p\in\Sigma_R$.*

Proof. We argue in a similar way as in [C, p. 260]. For $n\in\mathbb{N}$, let P_n be the polynomials $p\in\mathbb{R}[x_1,\ldots,x_d]$ with degree $p\leqq 2n$, and let Σ_n^i be the interior of $\Sigma_n:=\Sigma_R\cap P_n$ in the finite dimensional space P_n. For $k,l\in\{1,\ldots,n\}$ and $j_1,\ldots,j_{k+l}\in\{1,\ldots,d\}$, we have

$$2x_{j_1}\ldots x_{j_{k+l}}=(x_{j_1}\ldots x_{j_k}+x_{j_{k+1}}\ldots x_{j_{k+l}})^2-(x_{j_1}\ldots x_{j_k})^2-(x_{j_{k+1}}\ldots x_{j_{k+l}})^2.$$

Hence $P_n=\Sigma_n-\Sigma_n$. From the latter it follows that $\Sigma_n^i\neq\emptyset$.

Now assume to the contrary that p is not in Σ_R. Then $p\in P_m$ and $p\notin\Sigma_m^i$ for some $m\in\mathbb{N}$. We show by induction that there are linear functionals F_n, $n\geqq m$, on P_n such that $F_n(\cdot)>0$ on Σ_n^i, $F_n(p)\leqq 0$ and $F_n\upharpoonright P_{n-1}=F_{n-1}$ if $n-1\geqq m$. The existence of F_n follows from the separation theorem for convex sets in P_n (recall that $\Sigma_n^i\neq\emptyset$). For $n=m$ we use the assumption $p\notin\Sigma_m^i$. If F_n is chosen, then we have $(\ker F_n)\cap\Sigma_{n+1}^i=\emptyset$, so that there exists a linear functional G_{n+1} on P_{n+1} such that $G_{n+1}(\cdot)=0$ on $\ker F_n$ and $G_{n+1}(\cdot)>0$ on Σ_{n+1}^i. Then clearly $G_{n+1}\upharpoonright P_n=\gamma F_n$ for some $\gamma>0$, so $F_{n+1}:=\gamma^{-1}G_{n+1}$ has the desired properties. Now $F(q):=F_n(q)$, $q\in P_n$ and $n\in\mathbb{N}$, defines, unambiguously, a linear functional on $\mathbb{R}[x_1,\ldots,x_d]$. Since $\Sigma_n^i\neq\emptyset$, Σ_n is contained in the closure of Σ_n^i in P_n, so $F_n(\cdot)\geqq 0$ on Σ_n. Thus $F(\cdot)\geqq 0$ on Σ_R and $s(\alpha):=F(x^\alpha)$, $\alpha\in\mathbb{N}_0^d$, satisfies the condition required in Theorem 1. Hence there exists a measure $\mu\in M(\mathbb{R}^d)$ such that $\operatorname{supp}\mu\subseteqq K_R$ and $F(q)=\int q\,d\mu$ for all $q\in\mathbb{R}[x_1,\ldots,x_d]$. Since $F(p)\leqq 0$ by construction and p is positive on the compact set K_R, we have the desired contradiction. □

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