MIT 6.7230
Algebraic techniques and semidefinite programming
Homework assignment # 4

Date Given: March 22nd, 2024
Date Due: April 3rd, 1PM

P1. [***15 pts***] Consider the polynomial system

$$p(x,y) = (x^2 + y^2 - 4)(x^2 + 2y^2 - 9) - 1$$
$$q(x,y) = x^2 - y^2 + x + 2y.$$

(a) Plot the zero sets of each of these polynomials.

(b) Use resultants to numerically solve the system $\{p(x,y) = 0,\ q(x,y) = 0\}$.

(c) How many solutions did you find? How many of them are real?

(d) Give a graphical interpretation of your solution.

P2. [***15 pts***] Consider the sparse polynomial system given by:

$$1 + x^2y - 5xy^2 = 0$$
$$2 - 3xy + y^3 = 0.$$

(a) What is the Bézout bound on the number of solutions?

(b) What is the corresponding BKK bound?

(c) Using resultants, find all the solutions of this system by eliminating the variable x. Do "extraneous" solutions appear? How do you explain this?

Note: We haven't quite learned any systematic way to compute mixed volumes, so you'll have to improvise here... ;)

P3. [***15 pts***] Consider the *Huber loss* function, which is sometimes used for robust regression in statistics and machine learning. It is defined by the piecewise expression

$$L_\gamma(x) = \begin{cases} x^2/2 & \text{if } |x| \leq \gamma \\ \gamma|x| - \gamma^2/2 & \text{if } |x| > \gamma \end{cases}$$

where $\gamma > 0$ is a fixed parameter. It can be shown that (similarly as what was discussed in the lecture for univariate polynomials) the epigraph of the Huber loss is not basic semialgebraic. Nevertheless, by adding additional variables we can obtain "nice" conic descriptions for it.

(a) Sketch the Huber loss, and its epigraph. What is the role of γ?

(b) Find a semidefinite (or SOCP) representation of the Huber loss.

(c) Use your representation to solve numerical an underconstrained regression problem of the form

$$\min_{x \in \mathbb{R}^m} \sum_{i=1}^{n} L_\gamma(a_i^T x - b_i)$$

where a_i, b_i are randomly generated (take e.g., $n = 20$, $m = 5$). What are the differences between using the least-squares loss vs. the Huber loss, particularly with respect to outliers?

P4. [***15 pts***] Prove the following:

(a) Show that the discriminant of a polynomial $p(x)$ satisfies

$$\mathrm{Dis}_x(p) = p_n^{2n-2} \det H_1(p).$$

Hint: Consider the Vandermonde matrix, and its determinant. State clearly your assumptions, if any.

(b) What is the interpretation of this result? Why can one "intuitively" expect it to be true?

(c) Generalize the previous result, by finding (and proving) a nice expression for the determinant of the Hermite matrix $H_q(p)$, in terms of resultants and discriminants. What factors do you expect? When is $H_q(p)$ singular?

P5. [***15 pts***] Consider the set $S \subset \mathbb{R}^2$ that is the union of three quadrants in $\mathbb{R}^2$, i.e.,

$$S = \{(x, y) : x \leq 0\} \cup \{(x, y) : y \leq 0\}.$$

(a) Is this a semialgebraic set?

(b) Show that S is *not* basic semialgebraic, i.e., there is no description of the form

$$S = \{(x, y) : g_i(x, y) \leq 0,\ i = 1 \ldots, m\}, \tag{1}$$

for any finite set of polynomials $g_1, \ldots, g_m$.

(c) Show that S *can* be written as in (1), for $m = 1$ and a *non-polynomial* function g_1 that is continuously differentiable on all of $\mathbb{R}^2$. (Hint: Consider a piecewise-quadratic function.)

P6. [***10 pts***] In this exercise we investigate two special cases of mixed volumes.

(a) Let $P_1, \ldots, P_n$ be line segments, i.e., $P_i = \mathrm{conv}\{v_i, w_i\}$ for some vectors $v_i, w_i \in \mathbb{R}^n$. What is $MV(P_1, \ldots, P_n)$? How does this relate to the number of solutions of binomial equations?

(b) Let $P_1, \ldots, P_n$ be axes-parallel parallelotopes, i.e., polytopes of the form $C_a = \{x \in \mathbb{R}^n : 0 \leq x \leq a\}$ where $a \in \mathbb{R}^n$ is a given vector, and the inequalities are componentwise. Given $\{v_1, \ldots, v_n\} \subset \mathbb{R}^n$, what is $MV(C_{v_1}, \ldots, C_{v_n})$? Hint: look up "matrix permanent"

P7. [***Optional, 15 pts***] There is a nice relationship between the Bézout matrix and *interlacing* of polynomials. Given two univariate polynomials p and q of degree d, we say that p *interlaces* q if they have real roots $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$, respectively, and

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_d < \alpha_d.$$

Consider now two monic univariate polynomials f and g of degree d. Assume that f has only real roots, and that f and g have no common zeros.

(a) Find a congruence transformation of $\mathrm{Bez}(f, g)$ that will diagonalize it. What is the corresponding diagonal matrix?
Hint: consider the Vandermonde matrix V associated to the roots of f (or g).

(b) Using these results, show that f interlaces g if and only if $\mathrm{Bez}(f, g) \succ 0$.