# Real Analysis

Baire's theorem

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**Definition 1 (Nowhere dense set)** *A subset $A$ of $\mathbb{R}$ is <u>nowhere dense</u> or <u>rare</u> if $\left(\overline{A}\right)^o = \varnothing$.*

In other words, $A$ is rare iff it is contained in a closed set with empty interior. In fact, if $A$ is rare then $A$ is contained in $F = \overline{A}$ which has empty interior. Conversely if $A$ is contained in closed $F$ with $F^o = \varnothing$ then $\left(\overline{A}\right)^o \subseteq \left(\overline{F}\right)^o = F^o = \varnothing$.

We recall what *dense* means.

**Definition 2 (Dense set)** *A subset $A$ of $\mathbb{R}$ is said to be <u>dense</u> if $\overline{A} = \mathbb{R}$.*

In case of subsets of $\mathbb{R}$, we can equivalently say that $A$ is dense iff $\forall x \in \mathbb{R}, r > 0, \exists a \in A$ such that $a \in \mathscr{B}_r(x)$.

We might guess, from the terminology, that the complement of a nowhere dense set might be dense. This is true, as we shall see in the next paragraph. One might get more bold and claim that $A$ is rare iff $A^c$ is dense. Well, not quite. Think about $A = \mathbb{R} \setminus \mathbb{Q}$ which is dense in $\mathbb{R}$. But the closure of $A^c = \mathbb{Q}$ has nonempty interior, hence not rare.

It turns out that $A$ is rare iff $\left(\overline{A}\right)^c$ is dense. Indeed recall that $\overline{S} = ((S^c)^o)^c$ for any set $S$. Take $S = \left(\overline{A}\right)^c$. This gives $(S^c)^o = \left(\overline{A}\right)^o = \varnothing \iff \overline{S} = ((S^c)^o)^c = \mathbb{R} \iff S$ is dense $\iff \left(\overline{A}\right)^c$ is dense.
Clearly $A \subseteq \overline{A} \iff \left(\overline{A}\right)^c \subseteq A^c$. It thus stands that $A$ is rare $\iff \mathbb{R} = \overline{\left(\overline{A}\right)^c} \subseteq \overline{A^c} \implies \overline{A^c} = \mathbb{R} \iff A^c$ is dense.

**Proposition 3** *(a) Any subset of a rare set is rare.*

*(b) A finite union rare sets is rare.*

*(c) The closure of a nowhere dense set is nowhere dense.*

Proof (a) Let $A \subseteq B$ where $B$ is rare. Then $\overline{A} \subseteq \overline{B}$ whence $\left(\overline{A}\right)^o \subseteq \left(\overline{B}\right)^o = \varnothing$.

(b) Let $A, B$ be rare sets. Equivalently, $\left(\overline{A}\right)^c, \left(\overline{B}\right)^c$ are dense. Let $S := A \cup B$. Let $T \neq \varnothing$ be open. $\left(\overline{A}\right)^c$ dense $\implies T \cap \left(\overline{A}\right)^c \neq \varnothing$. Further, $T \cap \left(\overline{A}\right)^c$ is a nonempty open set whence $\varnothing \neq T \cap \left(\overline{A}\right)^c \cap \left(\overline{B}\right)^c = T \cap \left(\overline{A} \cup \overline{B}\right)^c = T \cap \left(\overline{A \cup B}\right)^c = T \cap \left(\overline{S}\right)^c$ whence $S$ is rare.

(c) $A$ rare $\iff \left(\overline{A}\right)^o = \varnothing \implies \left(\overline{\left(\overline{A}\right)}\right)^o = \left(\overline{A}\right)^o = \varnothing$.

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**Exercise** *Let $A, B$ be closed sets such that $(A \cup B)^o \neq \varnothing$. Show that either $A^o \neq \varnothing$ or $B^o \neq \varnothing$.*

**Exercise** *Give examples of two sets $A, B \subseteq \mathbb{R}$ such that $(A \cup B)^o \neq \varnothing$ but $A^o = B^o = \varnothing$.*

**Exercise** *Show that either $\mathbb{Q}$ can be written as a countable union of rare sets in $\mathbb{R}$.*

The above proposition must ring a bell in your mind and raise a question like "What about the *countable* union of rare sets?" One recalls the example that $\mathbb{Q}$ is a countable union of rare sets in $\mathbb{R}$ but $\mathbb{Q}$ is not itself rare - $\overline{\mathbb{Q}} = \mathbb{R}$ whence $\left(\overline{\mathbb{Q}}\right)^o = \mathbb{R}$. Such countable unions are not dense and mathematicians gave a name for it.

**Definition 4** *Let $A$ be a subset of $\mathbb{R}$ .*
*$A$ is said to be meagre or of the first category if $A$ can be written as a countable union of rare sets in $\mathbb{R}$.*
*If $A$ is not meagre, it is said to be nonmeagre or of the second category.*
*$A$ is said to be residual if its complement is meagre.*

We further see another small, but useful result.

**Proposition 5** *The following are equivalent for $\mathbb{R}$ . Note that we are not yet claiming about their truth or falsity.*

(a) *A meagre set has empty interior.*

(b) *A countable intersection of open dense sets is dense.*

(c) *A residual set is dense.*

Proof We prove them in a circular way as follows.

$(a) \implies (b)$: Note that the complement of an open dense set is a closed rare set. Let $\mathscr{U}$ be a countable collection of open dense sets in $\mathbb{R}$ and consider $S := \bigcap_{U \in \mathscr{U}} U$. Then $S^c := \bigcup_{U \in \mathscr{U}} U^c$ is a countable union of closed rare sets. By definition, $S^c$ is meagre, whence by hypothesis, $(S^c)^o = \varnothing$. But $(S^c)^o = \left(\overline{S}\right)^c$ so that $\overline{S} = \mathbb{R}$.

$(b) \implies (c)$: By definition, a residual set is the complement of a meagre set whence it is a countable intersection of some sets with dense interiors. In other words, a rare set contains a countable intersection of open dense sets, which is dense by hypothesis. Since any superset of a dense set must be dense, conclude that a residual set is dense.

$(c) \implies (a)$: Let $S$ be meagre. Then $S^c$ is residual. By hypothesis, $\overline{S^c} = \mathbb{R} \implies S^o = \left(\overline{S^c}\right)^c = \varnothing$.

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We have built up to an important result known as the Baire category theorem.

**Theorem 6 (Baire category theorem)** *A countable intersection of open dense sets in $\mathbb{R}$ is dense.*

Proof Let $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable collection of open dense sets in $\mathbb{R}$. Let $V \neq \varnothing$ be any open set. Clearly $V \cap U_1 \neq \varnothing$. Pick a closed disc $\overline{\mathscr{B}_{r_1}(x_1)} \subset V \cap U_1$ with $r_1 < 1$. Since $U_2$ is dense, $\mathscr{B}_{r_1}(x_1) \cap U_2 \neq \varnothing$ (also open). So pick a closed disc $\overline{\mathscr{B}_{r_2}(x_2)} \subset \mathscr{B}_{r_1}(x_1) \cap U_2$ such that $r_2 < \frac{1}{2}$. Continuing this process will give us a decreasing sequence of closed balls $\mathscr{B}_{r_n}(x_n)$ with $0 < r_n < \frac{1}{n}$. Further notice that the sequence $(x_n)$ is Cauchy in $\mathbb{R}$: for any $n \in \mathbb{N}$, we can pick $N = n$ so that $p \geq q \geq N \implies d(x_p, x_q) \leq \frac{1}{q} \leq \frac{1}{n}$. By completeness, $X$ converges to a point, say $x$, in $\mathbb{R}$. By definition, for any $n \in \mathbb{N}, \exists N \geq n \in \mathbb{N}$ such that $x \in \mathscr{B}_{\frac{1}{n}}(x_k) \forall k \geq N$; but $k \geq N \geq n \implies x \in \mathscr{B}_{\frac{1}{n}}(x_k) \subseteq \overline{\mathscr{B}_{\frac{1}{n}}(x_n)} \subseteq V \cap \left(\bigcap_{i=1}^{n} U_i\right)$. This means $x \in U_i \forall i$ and $x \in V$ whence $V \cap \left(\bigcap_{i \in \mathbb{N}} U_i\right) \neq \varnothing$. Since $V$ was an arbitrary open set to start with, we conclude that $\bigcap_{i \in \mathbb{N}} U_i$ is dense in $\mathbb{R}$. ■

**Corollary 7** *One cannot write $\mathbb{R}$ as a countable union of rare sets. In other words, $\mathbb{R}$ is not meagre.*

**Corollary 8** *A residual set in $\mathbb{R}$ is not meagre.*

Proof We can note that a countable union of meagre sets is meagre ($\because$ a countable union of countable sets is countable). Let $A \subseteq \mathbb{R}$ be residual, whence $A^c$ is meagre. If $A$ were meagre, so would be $\mathbb{R} = A \cup A^c$. But $A^c$ is meagre, so that $\mathbb{R}$ is meagre which is clearly false. $\blacksquare$

**Corollary 9** $\mathbb{R} \setminus \mathbb{Q}$ *is not meagre in* $\mathbb{R}$.

Proof $\mathbb{Q}$ is meagre $\implies$ $\mathbb{R} \setminus \mathbb{Q}$ is residual and thus, by the previous corollary, not meagre. $\blacksquare$

**Corollary 10** $\mathbb{Q}$ *cannot be written as the intersection of countably many open sets in* $\mathbb{R}$.

Proof Say $\mathbb{Q} = \bigcap_{n \in \mathbb{N}} U_n$ for some collection of open sets $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ in $\mathbb{R}$. Note that $U_n \supseteq \mathbb{Q} \implies \overline{U_n} = \overline{\mathbb{Q}} = \mathbb{R} \forall n$ whence each $U_n$ is an open dense set in $\mathbb{R}$. Also note that $\mathscr{V} = \{\mathbb{R} \setminus \{q\} : q \in \mathbb{Q}\}$ is a countable collection of open dense sets in $\mathbb{R}$. Further $\bigcap_{V \in \mathscr{V}} V = \varnothing$ whence $\bigcap_{S \in \mathscr{U} \cup \mathscr{V}} S = \varnothing$ which contradicts theorem 6. $\blacksquare$