Isomorphism Theorems :-

1) First Isomorphism Thm :-

Let $T : V \to W$, and V_1 be a subspace of $\ker T$.
Then there exist surjective linear map $\tilde{T} : V/V_1 \to W$ s.t the following diagram commute

$$\begin{tikzcd}
V \arrow[rr, "T"] \arrow[dr, "\pi"'] & & \mathrm{Im}(T) \\
& V/V_1 \arrow[ur, "\tilde{T}"'] &
\end{tikzcd}$$

$$\boxed{\tilde{T} \circ \pi = T}$$

The map $\tilde{T}$ is injective iff $V_1 = \ker T$

$$\boxed{\begin{aligned} \pi : V &\to V/V_1 \\ v &\to v + V_1 \end{aligned}}$$

Proof:- Define

$$\tilde{T} : V/V_1 \longrightarrow \mathrm{Im}\, T$$

$$\tilde{T}(\underset{\pi(u)}{\underset{\parallel}{u + V_1}}) = T(u)$$

Check that $\tilde{T}$ is well defined.
Check " " " Surjective

$T(u) \in \mathrm{Im}\, T$, $\quad \tilde{T}(u + V_1) = T(u)$

Let $\tilde{T}$ be injective. To prove that $V_1 = \ker T$

On contrary assume that $V_1 \subset \ker T$

$\Rightarrow \quad \exists\, v \in \ker T$ s.t $v \notin V_1$

$\Rightarrow \tilde{T}(v + V_1) = T(v) = 0 = T(0) = \tilde{T}(0 + V_1)$

but $\tilde{T}$ is injective $\Rightarrow v + V_1 = 0 + V_1$

$\Rightarrow v - 0 = v \in V_1$ $(\rightarrow \leftarrow)$

$V_1 = \ker T \Rightarrow \tilde{T}$ injective (Done in last class)

2) 2nd Isomorphism Theorem :-

Let V_1 and V_2 be subspaces of V. Then

$$\frac{V_1+V_2}{V_2} \cong \frac{V_1}{V_1 \cap V_2}$$

Proof → Here $V_1+V_2 := \{ v_1+v_2 , \forall v_1 \in V_1, v_2 \in V_2 \}$

** Note that V_1+V_2 is a subspace of V.

$$T : V_1 \longrightarrow \frac{V_1+V_2}{V_2}$$

$$v \longrightarrow v+V_2$$

$$\ker T := \{ v \in V_1 : v+V_2 = 0+V_2 \Leftrightarrow v-0 \in V_2 \Leftrightarrow v \in V_2 \}$$

$$= V_1 \cap V_2$$

$$T/\ker T \cong \operatorname{Im} T \Rightarrow \frac{V_1}{V_1 \cap V_2} \cong \operatorname{Im} T$$

T is surjective. $(v_1+v_2)+V_2 := \{ v_1+v_2+v \mid v \in V_2 \}$

$$= \{ v_1+v \mid v \in V_2 \}$$

$$= v_1+V_2$$

$$T(v_1) = v_1+V_2 = (v_1+v_2)+V_2$$

3) 3rd Isomorphism Thm :-

$U \subseteq W$ be subspaces of V. Then

$$\frac{V/U}{W/U} \cong V/W$$

$\longrightarrow \quad T: V/U \longrightarrow V/W$ $\longrightarrow$ Linear map T

$v + U \longrightarrow v + W$ $\longrightarrow$ Surjective

$\searrow \left\{ v+U : T(v+U) = 0 + W \iff v + W = 0 + W \iff v - 0 = v \in W \right\}$

$$\ker T := \{ v + U : v \in W \} = W/U$$

$$\frac{V/U}{W/U} \cong V/W$$

4) Lattice Isomorphism :-

$W \subset V$, subspace

$$\left\{ \begin{array}{l} \text{Subspaces of } V, \\ \text{containing } W \end{array} \right\} \xleftrightarrow{T} \left\{ \text{Subspaces of } V/W \right\}$$

$$A \xleftrightarrow{T} A/W$$

$T(A) = A/W$ is a set bijection

$A \supset W$, $B \supset W$

$$T(A+B) = \frac{A}{W} + \frac{B}{W} = \frac{A+B}{W}$$

$A \supset W$, $B \supset W$

$$T(A \cap B) = T(A) \cap T(B)$$