Quiver representations: a geometric view

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May 5, 2023

1 Representation spaces

Fix a quiver $Q = (Q_0, Q_1, s, t)$ and a dimension vector $\boldsymbol{n} = (n_i)_{i \in Q_0}$.

Definition 1 (Representation space)**.** The representation space of the quiver Q for the dimension vector $\boldsymbol{n}$ is

$$\operatorname{Rep}(Q, \boldsymbol{n}) \coloneqq \bigoplus_{\{i \to j\} \in Q_1} \operatorname{Mat}_{n_i \times n_j}(k).$$

This is called the *representation space* because every point $x \in \operatorname{Rep}(Q, \boldsymbol{n})$ corresponds to a representation V_x of Q with dimension vector $\boldsymbol{n}$. Clearly $\dim \operatorname{Rep}(Q, \boldsymbol{n}) = \displaystyle\sum_{\{i \to j\} \in Q_1} n_i n_j$. An object $\boldsymbol{x} \in \operatorname{Rep}(Q, \boldsymbol{n})$ with be denoted by $(x_\alpha)_{\left(i \xrightarrow{\alpha} j\right) \in Q_1}$ where $x_\alpha \in \operatorname{Hom}(k^{n_{s(\alpha)}}, k^{n_{t(\alpha)}})$. The group

$$\operatorname{GL}(\boldsymbol{n}) \coloneqq \prod_{i \in Q_0} \operatorname{GL}(n_i)$$

acts on each $\operatorname{Mat}_{n_i \times n_j}(k)$ by $(g_i)_{i \in Q_0} \cdot x_\alpha = g_j x_\alpha g_i^{-1}$, and thus extends to an action on $\operatorname{Rep}(Q, \boldsymbol{n})$. It is not hard to see that $k^* \cong k^*(\mathbf{1}_{n_i})_{i \in Q_0}$ is a normal subgroup of $\operatorname{GL}(\boldsymbol{n})$ and acts trivially on $\operatorname{Rep}(Q, \boldsymbol{n})$. This gives an action of $\operatorname{PGL}(\boldsymbol{n}) = \operatorname{GL}(\boldsymbol{n})/k^*$ on $\operatorname{Rep}(Q, \boldsymbol{n})$. We note that the representations V_x, V_y for two points $x, y \in \operatorname{Rep}(Q, \boldsymbol{n})$ are isomorphic iff x, y are in the same orbit of $\operatorname{GL}(\boldsymbol{n})$ (equivalently, $\operatorname{PGL}(\boldsymbol{n})$). This is made more formal and informative in the following lemma:

Lemma 1.1. *The assignment $x \mapsto V_x$ gives a one-one correspondence between the orbits* $\mathrm{GL}(\boldsymbol{n})$ *acting on* $\mathrm{Rep}(Q, \boldsymbol{n})$ *and the set of isomorphism classes of representations of Q with dimension vector $\boldsymbol{n}$. The stabilizer or the isotropy group* $\mathrm{GL}(\boldsymbol{n})_x = \{g \in \mathrm{GL}(\boldsymbol{n}) : g \cdot x = x\}$ *is isomorphic to the automorphism group* $\mathrm{Aut}_Q(V_x)$.

Example 1.2. Consider the following quiver

$$\begin{tikzcd} k \arrow[rr] & & k^n \arrow[loop above, "\alpha_1"] \arrow[loop, out=60, in=20, distance=2em, "\alpha_2"] \arrow[loop, out=-20, in=-60, distance=2em, "\cdots"] \arrow[loop below, "\alpha_r"] \end{tikzcd}$$

where $1, n$ denote the dimensions at the respective vertices, so our dimension vector is $\boldsymbol{n} = (1, n)$. Call it H_r. Then a typical point in $\mathrm{Rep}(H_r, \boldsymbol{n})$ looks like $(M, M_1, \cdots, M_r)$ where $X \in \mathrm{Mat}_{n\times 1}(k) = k^n, M_i \in \mathrm{Mat}_{n\times n}(k)$. Here $\mathrm{GL}(\boldsymbol{n}) = \mathrm{GL}(1) \times \mathrm{GL}(n) = k^* \times \mathrm{GL}(n)$ whose action on $\mathrm{Rep}(H_r, \boldsymbol{n})$ is given by $(c, g) \cdot (M, M_1, \cdots, M_r) = (gMt^{-1}, gM_1g^{-1}, \cdots, gM_rg^{-1})$. Such a point corresponds to a representation

$$\begin{tikzcd} k \arrow[r, "M"] & k^n \arrow[loop above, "M_1"] \arrow[loop, out=60, in=20, distance=2em, "M_2"] \arrow[loop, out=-20, in=-60, distance=2em, "\cdots"] \arrow[loop below, "M_r"] \end{tikzcd}$$

The isomorphism classes of representations of the above quiver with the aforementioned dimension vector is parameterized by the orbits of the action of $\mathrm{GL}(n) = \{1\} \times \mathrm{GL}(n)$ (not just $\mathrm{GL}(\boldsymbol{n})$) because (t, g) and $(1, t^{-1}g)$ have the same action. Basically the action of the k^* component in $\mathrm{GL}(\boldsymbol{n})$ is insignificant in the sense that $(Mt^{-1}, M_1, \cdots, M_n)$ and $(M, M_1, \cdots, M_n)$ belong to the same orbit $-$ we can go from the former to the latter by the action of (t^{-1}, Id). Alternately, such a representation is described by a $k-$algebra homomorphism $f : k\langle X_1, \cdots, X_n\rangle \to \mathrm{Mat}_{n\times n}(k), X_i \mapsto M_i$, together with an element $M \in k^n$.

We will call an element $(M, M_1, \cdots, M_r)$ *cyclic* if M generates k^n as a $k\langle X_1, \cdots, X_r\rangle-$module. Collect all such cyclic elements to form the set $\mathrm{Rep}(H_r, \boldsymbol{n})^{\mathrm{cyc}}$. It is clear that $\mathrm{Rep}(H_r, \boldsymbol{n})^{\mathrm{cyc}}$ is $\mathrm{GL}(n)-$stable. Further if $\boldsymbol{M} = (M, M_1, \cdots, M_r)$ is a cyclic tuple, then $\mathrm{GL}(n)_{\boldsymbol{M}}$ is trivial. This is seen as follows: If $(\boldsymbol{M})$ is cyclic, then there are constants $\lambda_i^{(j)} \in k$ such that $\sum_i \lambda_i^{(j)} M_i M = \boldsymbol{e}_j \in k^n$. If $g \in \mathrm{GL}(n)_{\boldsymbol{M}}$ then $g\boldsymbol{e}_j = \sum_i \lambda_i^{(j)} g M_i M = \sum_i \lambda_i^{(j)} M_i g M = \sum_i \lambda_i^{(j)} M_i M = \boldsymbol{e}_j$. Since this is true for every coordinate vector, we must have $g = \mathrm{Id}$.

Let's talk about the orbit space $\mathrm{Rep}(H_r, \boldsymbol{n})^{\mathrm{cyc}} / \mathrm{GL}(n)$. Here we will view the points of representation space as an algebra homomorphism $k[X_1, \cdots, X_r] \to \mathrm{Mat}_{n\times n}(k)$ together with an element of k^n. This means $\mathrm{Rep}(H_r, \boldsymbol{n})^{\mathrm{cyc}} = \{(f, v) \in \mathrm{Hom}\left(k\left\langle X_1, \cdots, X_r\right\rangle, \mathrm{Mat}_{n\times n}(k)\right) \times k^n : f\left(k\left\langle X_i\right\rangle\right) v = k^n\}$. Note that Two points $(M, \mu), (N, \nu) \in \mathrm{Rep}(H_r, \boldsymbol{n})^{\mathrm{cyc}}$ are in the same orbit iff $M = gN$ and $\mu(X_i) = g\nu(X_i)g^{-1}$ for some $g \in GL(n)$. Just to repeat, μ, ν are algebra homomorphisms of the above type. Given any $(f, v) \in \mathrm{Rep}(H_r, \boldsymbol{n})^{\mathrm{cyc}}$, we can put a ring structure on k^n given as follows: for $u_1, u_2 \in k^n$ there are polynomials $P_1, P_2 \in k\left\langle X_1, \cdots, X_r\right\rangle$ such that $f(P_i)v = u_i$, and so define $u_1u_2 = f(P_1P_2)v$.[1] The kernel is $I(f, v) = \{P \in k\left\langle X_1 \cdots, X_n\right\rangle : f(P)v = 0\}$. One should check that the following is a bijective correspondence between $\mathrm{Rep}(H_r, \boldsymbol{n})^{\mathrm{cyc}} / \mathrm{GL}(n)$ and $\{$left ideals $\subseteq k[X_1, \cdots, X_n]$ of codimension $n\}$:

$$\begin{aligned} (f, v) &\mapsto I(f, v) \\ (P \mapsto (\pi(Q) \mapsto \pi(PQ)), \pi(1)) &\hookleftarrow I \end{aligned}$$

$PQ - P'Q = PQ - PQ' + PQ' - P'Q = PQ' - P'Q$

Definition 2. An *(affine) algebraic group* is an (affine) algebraic variety G equipped with a group structure such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are morphisms of varieties.
An *algebraic action* of an algebraic group G on a variety X is a group action $G \times X \to X$ which is also a morphism of varieties.

Proposition 1.3. *Let G have an algebraic action on a variety X. Fix $x \in X$.*

(a) $G_x = \{g \in G : g \cdot x = x\}$ is closed in G.

(b) $G \cdot x$ is a locally closed, non-singular subvariety of X. All connected components of $G \cdot x$ have dimension $\dim G - \dim G_x$.

(c) The orbit closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and of orbits of smaller dimension; it contains at least one closed orbit.

[1] I couldn't verify that this is well defined because of the non-commutativity of the X_i's.

(d) *The variety G is connected if and only if it is irreducible; then the orbit $G \cdot x$ and its closure are irreducible as well.*

Now consider a group homomorphism $\varphi : G \to H$ of algebraic groups. This gives an action of G on H given by $g \cdot h := \varphi(g)h$. This is an algebraic action and its orbits are $G \cdot h = (\operatorname{Im}\varphi) \cdot h$. There is at least one closed orbit (contained in $\overline{(\operatorname{Im}\varphi) \cdot h}$ for some $h \in H$). But the orbits are permuted transitively by the action of H on tiself by right multiplication, thus implying that all orbits (that is, cosets) are closed. This means $\operatorname{Im}\varphi$ is closed. Now note that $G_{1_H} = \{g \in G : \varphi(g) = 1\} = \ker\varphi$, which is also closed. Thus $\ker\varphi, \operatorname{Im}\varphi$ are closed in G, H respectively. Finally we get that $\dim \operatorname{Im}\varphi = \dim(G \cdot 1_H) = \dim G - \dim G_{1_H} = \dim G - \dim\ker\varphi$.

2 Isotropy groups

Proposition 2.1. *Let M be a finite-dimensional representation of Q.*

(a) *The automorphism group $\operatorname{Aut}_Q(M)$ is an open affine subset of $\operatorname{End}_Q(M)$. As a consequence, $\operatorname{Aut}_Q(M)$ is a connected linear algebraic group.*

(b) *There exists a decomposition $\operatorname{Aut}_Q(M) \cong U \rtimes \prod_{i=1}^{r} \mathrm{GL}(m_i)$ where U is a s a closed normal unipotent subgroup and $m_1, \cdots, m_r$ denote the multiplicities of the indecomposable summands of M.*

We will allude to a theorem for finite-dimensional representations of associative algebras, and leave it as an exercise to the reader to prove proposition 2.1.

Theorem 2.2. *Let M be a finite-dimensional module over an algebra A. Then there is a decomposition of A-modules*

$$M \cong \bigoplus_{i=1}^{r} M_i^{m_i}$$

where $M_1, \cdots, M_r$ are indecomposable and pairwise non-isomorphic, and $m_1, \cdots, m_r$ are positive integers. Moreover, the indecomposable summands M_i and their multiplicities m_i are uniquely determined up to reordering. We also have a decomposition of vector spaces

$$\operatorname{End}_A(M) \cong I \oplus \prod_{i=1}^{r} \operatorname{Mat}_{m_i \times m_i}(k)$$

where I is a nilpotent ideal.

Proof sketch of proposition 2.1. The first part is immediate by the observation that $\mathrm{Aut}_Q(M) = \mathrm{End}_Q(M) \smallsetminus V(\det) = D(\det)$.

For the next part, we start with the split surjective algebra-homomorphism $\mathrm{End}_Q(M) \to \mathrm{End}_Q(M)/I \cong \prod_{i=1}^{r} \mathrm{Mat}_{m_i \times m_i}(k)$ which, in turn, gives a split surjective algebra-homomorphism $\mathrm{Aut}_Q(M) \to \prod_{i=1}^{r} \mathrm{GL}(m_i)$. The kernel of this map is $\mathrm{Id}_M + I$. Thus, $\mathrm{Id}_M + I$ is a closed connected normal subgroup of $\mathrm{Aut}_Q(M)$.
Next consider the linear action of $\mathrm{Id}_M + I$ on $k\,\mathrm{Id}_M \oplus I$ by left multiplication. Since the orbit of Id_M is isomorphic to the affine space $\mathrm{Id}_M + I$, this action yields a closed embedding $\mathrm{Id}_M + I \hookrightarrow \mathrm{GL}(k\,\mathrm{Id}_M \oplus I)$. The powers I^n form a decreasing filtration of the vector space $k\,\mathrm{Id}_M \oplus I$, and they stabilize to 0. Any I^n is stable under the action of $I + \mathrm{Id}_M$ and this action fixes the associated grades I^n/I^{n+1} and the quotient $(k\,\mathrm{Id}_M \oplus I)/I$. This establishes $\mathrm{Id}_M + I$ as a unipotent subgroup of $\mathrm{GL}(k\,\mathrm{Id}_M \oplus I)$, by choosing a basis of $k\,\mathrm{Id}_M \oplus I$ compatible with the filtration $(I^n)_{n\geq 1}$. ■

Corollary 2.3. *The representation V_x, for $x \in \mathrm{Rep}(Q, \boldsymbol{n})$ is is indecomposable if and only if the isotropy group $\mathrm{GL}(\boldsymbol{n})_x$ is the semi-direct product of a unipotent subgroup with the group $k^* \,\mathrm{Id}_{\boldsymbol{n}}$; equivalently, $\mathrm{PGL}(\boldsymbol{n})_x$ is unipotent.*

Now, when studying homological aspects, one comes across the following exact sequence

$$0 \to \mathrm{End}_Q(M) \to \prod_{i \in Q_0} \mathrm{End}(V_i) \to \prod_{\alpha \in Q_1} \mathrm{Hom}(V_{s(\alpha)}, V_{t(\alpha)}) \to \mathrm{Ext}^1_Q(M, M) \to 0.$$

The above discussion helps put this exact sequence in a nice geometric framework.

Theorem 2.4. *Let $x \in \mathrm{Rep}(Q, \boldsymbol{n})$ and denote by $M = V_x$ the corresponding representation of Q.*

(a) There is an exact sequence

$$0 \longrightarrow \mathrm{End}_Q(M) \longrightarrow \mathrm{End}(\boldsymbol{n}) \xrightarrow{c_x} \mathrm{Rep}(Q, \boldsymbol{n}) \longrightarrow \mathrm{Ext}^1_Q(M, M) \longrightarrow 0$$

with $c_x((f_i)_{i\in Q_0}) = (f_{t(\alpha)}x_\alpha - x_\alpha f_{s(\alpha)})_\alpha$.

(b) c_x *may be identified with the differential at the identity of the orbit map* $\varphi_x : \mathrm{GL}(\boldsymbol{n}) \to \mathrm{Rep}(Q, \boldsymbol{n}), g \mapsto g \cdot x$.

(c) The image of c_x *is the Zariski tangent space* $T_x\left(\mathrm{GL}(\boldsymbol{n}) \cdot x\right)$ *viewed as a subspace of* $T_x\left(\mathrm{Rep}(Q, \boldsymbol{n})\right) \cong \mathrm{Rep}(Q, \boldsymbol{n})$.

Proof.(b) $\mathrm{GL}(\boldsymbol{n}) \underset{\text{open}}{\subset} \mathrm{End}(\boldsymbol{n})$. So the Zariski tangent space[2] to this group at $\mathrm{Id}_{\boldsymbol{n}}$ may be identified with the vector space $\mathrm{End}(\boldsymbol{n})$. The tangent space to $\mathrm{Aut}_Q(M)$ at $\mathrm{Id}_{\boldsymbol{n}}$ is $\mathrm{End}_Q(M)$. The action of $\mathrm{GL}(\boldsymbol{n})$ is given by

$$\begin{aligned} \mathrm{GL}(n_i) \times \mathrm{GL}(n_j) &\longrightarrow \mathrm{Mat}_{n_i\times n_j}(k) \\ (g, h) &\longmapsto h x_{i\to j} g^{-1} \end{aligned}$$

c_x immediately comes from the differential of this map

$$\begin{aligned} \mathrm{Mat}_{n_i\times n_i} \times \mathrm{Mat}_{n_j\times n_j} &\longrightarrow \mathrm{Mat}_{n_i\times n_j}(k) \\ (f_i, f_j) &\longmapsto f_j x_{i\to j} - x_{i\to j} f_i. \end{aligned}$$

(c) From proposition 1.3 we get $\dim(\mathrm{GL}(\boldsymbol{n}) \cdot x) = \dim \mathrm{GL}(\boldsymbol{n}) - \dim \mathrm{GL}(\boldsymbol{n})_x$. But $\dim(\mathrm{GL}(\boldsymbol{n}) \cdot x) = \dim\left[T_x(\mathrm{GL}(\boldsymbol{n}) \cdot x)\right]$. But $\mathrm{GL}(\boldsymbol{n})_x$ comprise of the invertible intertwiners for the module $M = V_x$, and thus $\dim \mathrm{GL}(\boldsymbol{n})_x = \dim \mathrm{Aut}_Q(M) = \dim \mathrm{End}_Q(M)$. Also $\dim \mathrm{GL}(\boldsymbol{n}) = \dim \mathrm{End}(\boldsymbol{n})$. The last two equalities follow from the fact that $\mathrm{GL} \underset{\text{open}}{\subset} \mathrm{End}$ which is proposition 2.1. Combining these gives $\dim\left[T_x(\mathrm{GL}(\boldsymbol{n}) \cdot x)\right] = \dim \mathrm{End}(\boldsymbol{n}) - \dim \mathrm{End}_Q(M)$. By *(b)* and the above exact sequence, $\ker c_x = \mathrm{End}_Q(M)$. This means that $T_x(\mathrm{GL}(\boldsymbol{n}) \cdot x)$ is the entire image of c_x.

■

[2]This is a technical term which can be defined without using differential geometry concepts and simply by linearizing things using ‘abstract’ algebra.
A similar definition in this spirit is the tangent space for a local ring $(R, \mathfrak{m})$ which is $\mathfrak{m}/\mathfrak{m}^2-$ this essentially keeps only linear terms.

References

[1] M. Brion, "Representations of quivers," 2008. [Online]. Available: https://www-fourier.ujf-grenoble.fr/~mbrion/notes_quivers_rev.pdf