Few words before we begin :-

1) Recall that eigenvector associated to distinct eigenvalues are L.I.

2) Given a linear map

$$T : V \longrightarrow V$$

where V is finite dimensional F vector space, characteristic polynomial of T is

$$\text{char}_T(x) = \det(A - xI)$$

where A is some matrix representating T. degree of this polynomial is $n = \dim V$

Also remember that $\lambda \in F$ is an eigenvalue of T iff λ is a root of $\text{char}_T(x)$

Find char. poly of this matrix

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\text{char}_T(x) = (1-x)(x^2+1)$

$$\det(A - xI) = \det \begin{bmatrix} -x & -1 & 0 \\ 1 & -x & 0 \\ 0 & 0 & 1-x \end{bmatrix}$$

$$= x^2(1-x) - (-1)(1-x)$$

$$= (1-x)(x^2+1)$$

So note that given a linear map $T : V \longrightarrow V$, it might not admit n eigenvalues in F

Def$^{\underline{n}}$:- Let $T: V \to V$, $\dim V = n$

T is diagonalizable iff $\exists$ a basis wrt. which the matrix of T is diagonal.

Proposition :- T is diagonalizable iff $\exists$ an eigenbasis wrt. T

$$M_{\underbrace{B}_{\text{Eigenbasis}}}^{B}(T) = \operatorname{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$$

Ex :- $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T: \mathbb{R}^2 \to \mathbb{R}^2$

$\mathrm{char}_T(x) = x^2 + 1$

Diagonalizble? NO

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: T_c : \mathbb{C}^2 \to \mathbb{C}^2$

$R = M_{st}^{st}(T_c)$ $\quad$ i: $\begin{pmatrix} i \\ 1 \end{pmatrix}$ $\quad$ $-i$: $\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = M_B^{st}(id)$

$$P^{-1} R P = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$M_{st}^{B}\, M_{st}^{st}(T_c)\, M_B^{st} = M_B^B$$

- Note that if a linear map is diagonalizable, then all its eigenvalues lies in F

Is the converse true?

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \longrightarrow (x-1)^2$$

Thm: A linear map $T: V \to V$, $char_T(x)$ is prod. of distinct linear factors in $F[x]$ then T is diagonalizable.

Proof → $\deg(char_T(x)) = n = \dim V$

$$char_T(x) = a(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

$\{\lambda_1, \dots, \lambda_n\}$ are distinct eigenvalues of T

$B = \{v_1, v_2, \dots, v_n\}$ is L.I.

but $\dim V = n$

So B is a basis.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$x(x-1) - 1$

$= x^2 - x - 1$

$= (x-\alpha)(x-\beta)$ $\alpha \neq \beta$

Suppose $T: V \to V$ linear map, V is finite dim. F-vec. space

$p(x) \in F[x]$, $p(x) = a_k x^k + \dots + a_1 x + a_0$, $a_i \in F$

$$p(x) * T = a_k T^k + a_{k-1} T^{k-1} + \dots + a_1 T + a_0 : V \to V$$

$m \in \mathbb{N}$

$$T^m = \underbrace{(T \circ T \circ \dots \circ T)}_{m\text{-times}}$$

$p(x)$, $q(x)$, $\quad T, S$ linear map $V \to V$

$$\big(p(x)+q(x)\big)T = p(x)\cdot T + q(x)\cdot T$$

$$\big(p(x)\cdot q(x)\big)*T = p(x)*\big(q(x)*T\big)$$

$$p(x)\big(T+S\big) = \big(p(x)T\big)+\big(p(x)\cdot S\big)$$

$\alpha(v+w)$
$=\alpha v+\alpha w$

Defn :- Let V be F- vector space
$Hom(V,V)$ forms a group under addition.

$$(T+S)(v) = T(v)+S(v)$$

$F[x]$ · for every $p(x) \in F[x]$, $p(x) = a_k x^k + \cdots + a_1 x + a_0$

$$p(x)\cdot T = a_k T^k + \cdots + a_1 T + a_0$$

$$T^m = \underbrace{T \circ T \circ \cdots \circ T}_{m \text{ times}}, \quad m \in \mathbb{N}$$

$Hom(V,V)$ is a $F[x]$-module