Real Analysis

Problem Set 8

August 30, 2021

Assume that $\varnothing \subseteq \mathbb{R}$ is connected.

1 State true or false:

1. $S := \left\{(x_n) \in \mathbb{Q}^{\mathbb{N}} : \lim x_n = 0\right\}$ is countable.
2. $S := \left\{(x_n) \in \mathbb{R}^{\mathbb{N}} : x_n(x_n - 1) = 0 \forall n \in \mathbb{N}\right\}$ is countable.
3. One can find a nonempty set S such that there is a surjection $f : S \twoheadrightarrow 2^S$.
4. Let $A_1 \subseteq A_2 \subseteq \cdots$ be a countable collection of countable sets. Then $\bigcap_{n\in\mathbb{N}} (\mathbb{R} \setminus A_n)$ is uncountable.
5. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a countable collection of uncountable sets. Then $\bigcap_{n\in\mathbb{N}} A_n$ is uncountable.
6. For any $X \subseteq \mathbb{R}$ we define $\mathscr{T}(X)$ to be the collection of all open sets in X, i.e., all $S \subseteq X$ such that S is open in X. Then $\mathscr{T}(X) \subseteq \mathscr{T}(\mathbb{R}) \forall X \subseteq \mathbb{R}$.
7. For any $X \subseteq \mathbb{R}$ we define $\mathscr{T}(X)$ to be the collection of all open sets in X, i.e., all $S \subseteq X$ such that S is open in X.
   If $X \subseteq \mathbb{R}$ is such that $\mathscr{T}(X) \subseteq \mathscr{T}(\mathbb{R})$ then $X \in \mathscr{T}(\mathbb{R})$.
8. For any $X \subseteq \mathbb{R}$ we define $\mathscr{T}(X)$ to be the collection of all open sets in X, i.e., all $S \subseteq X$ such that S is open in X.
   If $X \subseteq \mathbb{R}$ is finite, then $\mathscr{T}(X) = 2^X$.
9. Let $A \subseteq \mathbb{R}$ be such that A^o is connected. Then A is connected.
10. For any $X \subseteq \mathbb{R}, X^o = \left(\overline{X}\right)^o$.

2 Choose the correct options

1. Which of the following have non-empty interior in $\mathbb{R}$?
   (a) $\mathbb{R} \setminus \mathbb{Q}$
   (b) $\{x \in \mathbb{R} : \sin(x) = 1\}$
   (c) $\{b \in \mathbb{R} : x^2 + bx + 1 = 0 \text{ has distinct roots in } \mathbb{R}\}$
   (d) $\mathbb{Q}$

2. How many bijective maps $f:\mathbb{N}\to\mathbb{N}$ are there such that $\sum_{n=1}^{\infty}\frac{f(n)}{n^2}\in\mathbb{R}$?

(a) Zero.

(b) Exactly one.

(c) More than one but finitely many.

(d) Infinitely many.

3. Evaluate $\lim_{n\to\infty}\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)$.

(a) $\frac{1}{2}$.

(b) $\frac{1}{4}$.

(c) $\frac{3}{4}$.

(d) 1.

4. (a_n) is a sequence of positive reals such that $l := \lim\frac{a_{n+1}}{a_n}$. Which of the following is/are true?

(a) $l = 1 \implies \lim a_n = 1$.

(b) $l = 1 \implies \lim a_n = 0$.

(c) $l < 1 \implies \lim a_n = 1$.

(d) $l = 1 \implies \lim a_n = 0$.

5. Define a sequence (s_n) inductively by $s_1 := a > 0$ and $s_{n+1} := \sqrt{\frac{1+s_n^2}{1+a^2}}\forall n \geq 1$. Choose the correct options.

(a) $as_n^2 < 1\forall n \implies (s_n)\uparrow$ and $\lim s_n = \frac{1}{\sqrt{a}}$.

(b) $as_n^2 < 1\forall n \implies (s_n)\downarrow$ and $\lim s_n = \frac{1}{a}$.

(c) $as_n^2 > 1\forall n \implies (s_n)\uparrow$ and $\lim s_n = \frac{1}{\sqrt{a}}$.

(d) $as_n^2 > 1\forall n \implies (s_n)\downarrow$ and $\lim s_n = \frac{1}{a}$.

6. Let (a_n) be a real sequence such that that $S := \sum a_n \in \mathbb{R}$. Define $t_n := a_n + a_{n+1} + a_{n+2}$. Then $\left(\sum_{j=1}^{n} t_j\right)_{n\in\mathbb{N}}$

(a) converges to $3S - a_1 - a_2$.

(b) converges to $3S - a_1 - 2a_2$.

(c) converges to $3S - 2a_1 - a_2$.

(d) diverges.

7. Let $s_n := \frac{(-1)^n}{2^n+3}, t_n := \frac{(-1)^n}{4n-1}$. Then

(a) $\sum s_n$ is absolutely convergent.

(b) $\sum s_n$ is convergent.

(c) $\sum t_n$ is absolutely convergent.

(d) $\sum t_n$ is convergent.

8. Let $a = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{n^2}\right)$, $b = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+n}\right)$. Then

(a) $a > b$.

(b) $b > a$.

(c) $ab = \ln\sqrt{2}$.

(d) $\frac{a}{b} = \ln\sqrt{2}$.

9. Evaluate $\sum_{n=0}^{\infty}\frac{1}{(n+2)n!}$.

(a) e.

(b) 1.

(c) $\frac{e}{2}$.

(d) $\frac{1}{2}$.

10. Evaluate $\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}$.

(a) $\frac{1}{2}$.

(b) 1.

(c) $\frac{1}{4}$.

(d) $\frac{3}{2}$.