COMPACTNESS

$K \subseteq \mathbb{R}$ is said to be <u>compact</u> if every open cover of K has a finite subcover.

<u>Note</u>

(1) $\{(n, n+1)\}_{n\in\mathbb{Z}}$ is an open cover for every compact set $K \subseteq \mathbb{R}$. By definition, and looking at this open cover, every compact set is bounded.

(2) Let $K \subseteq \mathbb{R}$ be finite. Then K compact.

Pf: Let $\mathcal{U} = \{V_\lambda\}_{\lambda\in\Lambda}$ is an open cover for K.

In other words, $K \subseteq \bigcup_{\lambda\in\Lambda} V_\lambda$.

$\forall x \in K$, $\exists T_x \in \mathcal{U}$ s.t. $x \in T_x$.

So $K \subseteq \bigcup_{x\in K} T_x$. But $\{T_x : x \in K\}$ is a finite subcover for K.

(3) Finite union of compact sets is compact.

(4) <u>Relative compactness</u> : $K \subseteq X \subseteq \mathbb{R}$. K is said to be compact in X if every open cover of K in X has a finite subcover.

This is "non sense"

Why is the above "non-sense"? Because of the following claim: Let $K \subseteq X \subseteq Y \subseteq \mathbb{R}$. K is compact in X iff K is compact in Y. → look up the proof in Rudin

In other words, compactness is an <u>intrinsic</u> property of a set.

(Thm 2.33 of Rudin page 37).

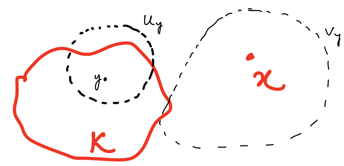
$[0,1]$ not open in $\mathbb{R}$

but open in $[0,1]$.

(5) $K \subseteq \mathbb{R}$ compact $\Rightarrow$ K closed (in $\mathbb{R}$).

This result is true for all "spaces" having the Hausdorff property.

Pf: Let $x \in \mathbb{R} \setminus K$.



$\forall\, y \in K \quad \exists$ open $U_y, V_y \subseteq \mathbb{R}$

s.t. $y \in U_y$, $x \in V_y$,

$U_y \cap V_y = \phi$.

Notice $\mathcal{U} = \{ U_y : y \in K \}$ is an open cover for K.

(i.e., each U_y open & $K \subseteq \bigcup_{y \in K} U_y$).

K has a finite subcover, say $\{U_{y_1}, U_{y_2}, \ldots, U_{y_n}\} \subseteq \mathcal{U}$

(From now we simply write $U_i = U_{y_i}$, $V_i = V_{y_i}$.

$x \in V_i \quad \forall i$. So $x \in \bigcap_{i=1}^{n} V_i =: V$.

V_i's open $\Longrightarrow$ V open.

$V_i \cap U_i = \phi$

$$\Rightarrow \quad V \cap K \subseteq V \cap \left(\bigcup_{i=1}^{n} U_i \right) = \bigcup_{i=1}^{n} (V \cap U_i)$$
$$= \phi$$

$\Rightarrow V \cap K = \phi \Rightarrow V \subseteq \mathbb{R} \setminus K$

$\therefore \mathbb{R} \setminus K$ open. $\therefore K$ closed.

(6) Closed subsets of compact sets are compact.

(Let $K \subseteq \mathbb{R}$ be compact, $X \subseteq K$ closed. Then X is compact)

Pf: Let $\mathcal{U} = \{ V_\lambda \}_{\lambda \in \Lambda}$ is an open cover of X.

$$\therefore X \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda \Rightarrow K = X \cup (K \setminus X) \subseteq \left(\bigcup_{\lambda \in \Lambda} V_\lambda \right) \cup (K \setminus X)$$

$\therefore \mathcal{U} \cup \{ K \setminus X \}$ is an open cover of K.

K compact $\Rightarrow \exists\ \lambda_1, \lambda_2, \dots, \lambda_n$ s.t.

$$K \subseteq V_{\lambda_1} \cup V_{\lambda_2} \cup \dots \cup V_{\lambda_n} \cup (K \setminus X).$$

Clearly $\{V_{\lambda_i} : i = 1, \dots, n\}$ is a finite subcover for X.

(This is true in general).

(7) Let $a, b \in \mathbb{R}$ $(a \leq b)$. Then $[a, b]$ is compact.

Pf: $I = [a, b]$. Let $\mathcal{U} = \{V_\lambda\}_{\lambda \in \Lambda}$ be an open cover for I.

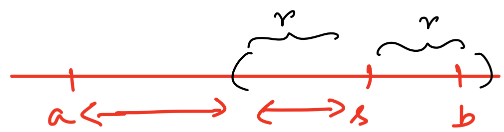
Let $T = \left\{ \alpha \in I : [a, \alpha] \text{ has a finite subcover in } \mathcal{U} \right\}$

$\rightarrow$ $a \in T \Rightarrow T \neq \phi$. Further $T \subseteq I$, hence bounded.

$s := \sup T \quad (\in I)$.

$\therefore \exists\ \lambda$ s.t. $s \in V_\lambda$. Further $\exists\ r > 0$ s.t.

$(s - r, s + r) \subseteq V_\lambda$.



$[a, s] = [a, s - r] \cup (s - r, s]$

Since s is the sup of T, $\exists\ t \in [s - r, s]$ s.t. $t \in T$

$\Rightarrow [a, t]$ has a finite subcover in $\mathcal{U}$

$\Rightarrow [a, s - r]$ " " " " in $\mathcal{U}$

So $s - r \in T$. This means $[a, s]$ has a finite subcover in $\mathcal{U}$. So $s \in T$.

$\rightarrow$ Let $y \in I \cap (s, s + r)$. $y \in V_\lambda$. Infact $(s, y] \subseteq V_\lambda$.

$\therefore [a, y]$ has a finite subcover in $\mathcal{U}$.

$[\because [a, y] = [a, s] \cup (s, y]]$.

$\therefore y \in T \Rightarrow y \leq s$ $[\because s$ supremum$]$

But by choice of y, $y > s$. This is a contradiction.

Hence $I \cap (s, s + r) = \phi$.

$\therefore s = b$.

From the above we can conclude that $b \in T$.

$\Rightarrow I = [a, b]$ has a finite subcover in $\mathcal{U}$.

(8) (Heine Borel theorem) Let $K \subseteq \mathbb{R}$. Then

K compact $\iff$ K closed and bounded.

Pf: ($\Rightarrow$) Proved in (1) and (5).

($\Leftarrow$) K bounded $\Rightarrow \exists\ a, b \in \mathbb{R}$ $(a \leq b)$ s.t. $K \subseteq [a,b]$.

But $[a, b]$ compact by (7). K closed.

By (6), K is compact.

(9) (Sequential compactness) Let $K \subseteq \mathbb{R}$.

K is compact $\iff$ every seq in K has a limit point in K.

Pf: ($\Rightarrow$) K compact $\Rightarrow$ K bounded.

Let $(x_n)_{n \in \mathbb{N}}$ be any K-seq. It has a convergent subseq, say $(x_{n_k})_{k \in \mathbb{N}}$.

Let $\lim\limits_{k \to \infty} x_{n_k} = x \in \mathbb{R}$.

K compact $\Rightarrow$ K closed $\Rightarrow$ $x \in K$.

($\Leftarrow$) Assume every seq in K has a limit pt in K.

Suppose K unbdd. $\forall n \in \mathbb{N}$ $\exists$ $x_n \in K$ s.t. $|x_n| > n$.

(x_n) is a seq in K that does not have a limit point. Contradiction! $\therefore$ K must be bdd.

Let $x \in K'$. $\exists$ a seq (x_n) in K s.t. $\lim x_n = x$, $x_n \neq x\ \forall n$. By hypothesis, there is a convegent subseq (x_{n_k}) whose limit is in K.

But the limit of any subseq of a convergent seq X is the limit of the original convergent seq X

$\therefore x = \lim_{k\to\infty} x_{n_k} \in K$. $\therefore K' \subseteq K$. $\therefore K$ closed.

By (8), K is compact.