Real Analysis

Problem Set 2

May 13, 2021

- *Assume that* $\mathbb{N}$ *does not contain* 0.
- *Denote* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

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1. Let $S \subseteq N$ with the following properties:
   - $2^k \in S \,\forall\, k \in \mathbb{N}$.
   - If $k \in S$ and $k \geq 2$ then $k - 1 \in S$.

   Prove that $S = \mathbb{N}$.

2. Let S be an ordered set with ordering $<$. Show that S has the *lub* property iff S has the *glb* property. **Hint:** For $E \subseteq S$ bounded below, consider $F := \left\{y \in S : y \leq x \,\forall\, x \in E\right\}$.

3. Find (with proof) the infimum and supremum (state clearly if does not exist) of $S := \left\{\frac{1}{a} + \frac{1}{b} : a, b \in \mathbb{N}\right\}$.

4. Let $S \subseteq \mathbb{R}$ be non-empty and bounded below by 0. Let $T := \left\{x^2 : x \in S\right\}$. Show that $\inf T = (\inf S)^2$.

5. Let $A, B \subseteq \mathbb{R}$ which are nonempty. Prove or disprove the following statements and salvage if possible:
   (a) Let $D := \{a - b : a \in A, b \in B\}$. Then $\sup D = \sup A - \sup B$ and $\inf D = \inf A - \inf B$.
   (b) Let $P := \{ab : a \in A, b \in B\}$. Then $\sup D = (\sup A) \cdot (\sup B)$ and $\inf D = (\inf A) \cdot (\inf B)$.
   (c) Let $S := \{a + b : a \in A, b \in B\}$. Then $\sup D = \sup A + \sup B$ and $\inf D = \inf A + \inf B$.
   (d) If $A \subseteq B$ then $\sup A \leq \sup B$ and $\inf A \geq \sup B$.
   (e) If $A \subsetneq B$ (strict containment) then either $\sup A < \sup B$ or $\inf A > B$ or both.
   (f) If $A \subsetneq B$ (strict containment) then $\sup A < \sup B$ and $\inf A > B$.

6. ***(Important)*** Fix $b > 1$. Take rationals to have positive denominators.
   (a) Let $m, n, p, q \in \mathbb{Z}$ such that $q, n > 0$ and $r := \frac{p}{q} = \frac{m}{n}$. Show that $(b^m)^{\frac{1}{n}} = \left(b^p\right)^{\frac{1}{q}}$. So define $b^r := (b^m)^{\frac{1}{n}}$.
   (b) Let $u, v \in \mathbb{Q}$. Show that $b^{u+v} = b^u \cdot b^v$.
   (c) Consider the set $\mathscr{B}(x) := \{b^t : t \in \mathbb{Q}, t \leq x\}$. Prove that if $u \in \mathbb{Q}$ then $\sup\left(\mathscr{B}(u)\right) = u$. So define $b^x := \sup\left(\mathscr{B}(x)\right)$.
   (d) Let $u, v \in \mathbb{R}$. Show that $b^{u+v} = b^u \cdot b^v$.

7. Let $\mathbb{K} := \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$. Prove the following $\forall\, x, y \in \mathbb{K}$:

(a) $x + y \in \mathbb{K}$.

(b) $x \cdot y \in \mathbb{K}$.

(c) $x \neq 0 \implies \frac{1}{x} \in \mathbb{K}$.

So $\mathbb{K}$ is a subfield of $\mathbb{Q}$ lying between $\mathbb{Q}$ and $\mathbb{R}$, that is, $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$.
Show that $\mathbb{M} := \left\{a + b\sqrt[3]{2} : a, b \in \mathbb{Q}\right\}$ does not satisfy at least one of the properties (a)-(c).

8. ***(Important)*** Let $|\cdot| : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ be as defined in class. Note that this can be extended to $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$, with the same definition. Prove the following $\forall x, y \in \mathbb{R}$.

   (a) $|x| = 0 \iff x = 0$

   (b) $|xy| = |x| \cdot |y|$

   (c) $|x + y| \leq |x| + |y|$

9. Let $x, y, z \in \mathbb{R}$ such that $x \leq z$. Prove the following.

   (a) $|x + y| = |x| + |y| \iff xy \geq 0$.

   (b) $|x - y| + |y - z| = |x - z| \iff x \leq y \leq z$.

10. For any $x, y \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x' - x| < \delta$ and $|y' - y| < \delta$ imply $|(x' + y') - (x + y)| < \varepsilon$. Prove this statement.

11. Let $n \in \mathbb{N}$. Prove the following.

    (a) If $a, b \in K$, where K is a field, then $a^n - b^n = (a - b)\left(\sum_{i=0}^{n} a^i b^{n-1-i}\right)$.

    (b) If $a > b$ are reals then $a^n - b^n < na^{n-1}(a - b)$.

12. Let $a, b \in \mathbb{R}$. Show that $|a| \leq b \iff -b \leq a \leq b$.

13. *(You may use anything you know about limits)* Let $a, b \in \mathbb{R}^+$. Show that $\lim_{n\to\infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}$.

14. *(Optional)* Let $p > 0$ be an integer prime. For an integer $n \neq 0$ define $v_p(n)$ to be the largest $k \in \mathbb{Z}$ such that $p^k \mid n$. In other words $v_p(n)$ is the unique integer k such that $p^k \mid n$ but $p^{k+1} \nmid n$. If $n = 0$ then we define $v_p(0) = \infty$. Show that $v_p(mn) = v_p(m) + v_p(n) \forall m, n \in \mathbb{Z}$.