Real Analysis

Problem Set 6

June 21, 2021

1. Using the condensation test, determine whether $\sum x_n \in \mathbb{R}$, where x_n are as follows:

   (a) $x_n = \dfrac{1}{n}$

   $$\sum_n \frac{1}{n} < \infty \iff \sum_n \frac{2^n}{2^n} = \sum_n 1 < \infty. \text{ Therefore diverges.}$$

   (b) $x_n = \dfrac{1}{(n+1)\log(n+1)}$

   $$\sum_{n=2}^{\infty} \frac{1}{n\log n} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{2^n \log 2^n} = \sum_{n=2}^{\infty} \frac{1}{n} < \infty. \text{ Therefore diverges.}$$

   (c) $x_n = \dfrac{1}{n^2}$

   $$\sum_n \frac{1}{n^2} < \infty \iff \sum_n \frac{2^n}{2^n \times 2^2 n} = \sum_n \frac{1}{2^n} = 1 < \infty. \text{ Therefore converges.}$$

   (d) $x_n = \dfrac{1}{\left(\log(n+1)\right)^2}$

   $$\sum_{n=2}^{\infty} \frac{1}{\left(\log n\right)^2} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{n^2\left(\log 2\right)^2} < \infty. \text{ Therefore diverges.}$$

   (e) $x_n = \dfrac{1}{(n+1)\left(\log(n+1)\right)^2}$

   $$\sum_{n=2}^{\infty} \frac{1}{n\left(\log n\right)^2} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{2^n \cdot n^2\left(\log 2\right)^2} = \sum_{n=2}^{\infty} \frac{1}{n^2\left(\log 2\right)^2} < \infty. \text{ Therefore converges.}$$

   (f) $x_n = \dfrac{\log n}{n^2}$

   $$\sum_n \frac{\log n}{n^2} < \infty \iff \sum_n \frac{2^n \cdot n}{4^n} = \sum_n \frac{n}{2^n} < \infty. \text{ Therefore converges.}$$

   (g) $x_{n-15} = \dfrac{1}{n\left(\log n\right)\left(\log\log n\right)\left(\log\log\log n\right)^p}$ if $p > 1$

   (h) $x_{n-15} = \dfrac{1}{n\left(\log n\right)\left(\log\log n\right)\left(\log\log\log n\right)^p}$ if $p \leq 1$

   (i) $x_n = \dfrac{1}{n^p}$ if $p > 1$

   (j) $x_n = \dfrac{1}{n^p}$ if $0 < p \leq 1$

2. Determine whether the following sequences converge in $\mathbb{R}$:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n^2+1}$

Check that $\dfrac{n}{3n^2+1} \geq \dfrac{n+1}{3(n+1)^2+1}$ $\forall\, n \in \mathbb{N}$ and that $\lim_n \dfrac{n}{3n^2+1} = 0$. Conclude that the series converges, by alternating series test. The sequence does not absolutely converge because $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n^2+1} \geq \sum_{n=1}^{\infty} \frac{n+1}{3(n+1)^2} = \sum_{n=2}^{\infty} \frac{1}{3n} = \infty$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{3n^2+1}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2+1} < \infty \iff \sum_{n=1}^{\infty} \frac{2^n \cdot 2^{n/2}}{3 \cdot 2^{2n}+1} < \infty$. But $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \cdot 2^{n/2}}{3 \cdot 2^{2n}+1} \leq \sum_{n=1}^{\infty} \frac{2^{1.5n}}{2 \cdot 2^{2n}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}^n} < \infty$. The series is thus (absolutely) convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{2^n}{1+4^n}}$

Note that $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{2^n}{1+4^n}} \leq \sum_{n=1}^{\infty} \sqrt{\frac{2^n}{4^n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}^n} < \infty$. The series is thus absolutely convergent (so convergent).

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ where $p > 0$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^p} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{n^p (\log 2)^p} < \infty \iff \sum_{n=2}^{\infty} \frac{2^n}{n^p} < \infty$. The series diverges.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Note that $\displaystyle\limsup \frac{(n+1)! \cdot n^n}{(n+1)^{n+1} \cdot n!} = \limsup \left(\frac{n}{n+1}\right)^n = \lim_n \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$ whence by ratio test, the series (absolutely) converges.

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\log n)^n}$

$\displaystyle\limsup \left|\frac{1}{(\log n)^n}\right|^{\frac{1}{n}} = \limsup \frac{1}{(\log n)} = \lim \frac{1}{\log n} = 0 < 1$ whence by root test, the series (absolutely) converges.

(g) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\,(\log n)\,(\log\log n)^p}$ where $p > 0$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\sqrt{n}-(-1)^n\right)}{n}$

3. Let $(a_n),(b_n)\in\mathbb{R}^{\mathbb{N}}, A_n := \sum\limits_{i=1}^{n} a_i$ and let $\alpha:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ be a function. Prove that

   (a) $\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{j}(\alpha(i,j)) = \sum_{i=1}^{n}\sum_{j=i}^{n}(\alpha(i,j))$

   Define $\mathbf{1}_{i\le j} := \begin{cases} 0 & \text{if } i>j \\ 1 & \text{if } i\le j \end{cases}$. Now

   $$\sum_{j=1}^{n}\sum_{i=1}^{j}\alpha(i,j) = \sum_{j=1}^{n}\sum_{i=1}^{n}\mathbf{1}_{i\le j}\cdot\alpha(i,j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{1}_{i\le j}\cdot\alpha(i,j) = \sum_{i=1}^{n}\sum_{j=i}^{n}\mathbf{1}_{i\le j}\cdot\alpha(i,j) = \sum_{i=1}^{n}\sum_{j=i}^{n}\alpha(i,j).$$

   (b) $\displaystyle\sum_{i=1}^{n}a_ib_i = b_{n+1}A_n - \sum_{i=1}^{n}A_i(b_{i+1}-b_i)$

   $$\begin{aligned} b_{n+1}A_n - \sum_{i=1}^{n}A_i(b_{i+1}-b_i) &= b_{n+1}A_n - \sum_{i=1}^{n}\sum_{j=1}^{i}a_j(b_{i+1}-b_i) = b_{n+1}A_n - \sum_{j=1}^{n}\sum_{i=j}^{n}a_j(b_{i+1}-b_i) \\ &= b_{n+1}A_n - \sum_{j=1}^{n}a_j(b_{n+1}-b_j) = \sum_{j=1}^{n}b_{n+1}a_j - \sum_{j=1}^{n}a_jb_{n+1} + \sum_{j=1}^{n}a_jb_j \\ &= \sum_{j=1}^{n}a_jb_j. \end{aligned}$$

4. Let $(a_n),(b_n)\in\mathbb{R}^{\mathbb{N}}, A_n := \sum\limits_{i=1}^{n} a_i$. Suppose (A_n) is bounded. It is given that $\sum_{i=1}^{n}(b_{i+1}-b_i)$ converges absolutely and $\lim\limits_{n\to\infty} b_n = 0$.

   (a) Show that $\lim\limits_{n\to\infty} A_nb_{n+1} = 0$.

   (b) Show that $\displaystyle\sum_n A_n(b_{n+1}-b_n)$ is convergent.

   (c) Conclude that $\displaystyle\sum_n a_nb_n$ converges.

   Let $B>0$ be such that $|A_n|\le B\,\forall n$. For $\varepsilon>0\exists N>0$ such that $|b_n|<\frac{\varepsilon}{B}\,\forall n>N$ whence $|A_nb_{n+1}|<\varepsilon\,\forall n\ge N$. By definition, $\lim A_nb_{n+1}=0$.
   $\sum(b_{n+1}-b_n)$ converges absolutely, say the limit is L. Note that $\sum_{i=1}^{n}|A_i(b_{i+1}-b_i)|\le B\sum_{i=1}^{n}|(b_{i+1}-b_i)|\le BL$. It follows that the sequence $\left\{\sum_{i=1}^{n}|(b_{i+1}-b_i)|\right\}_n$ is monotonous and bounded, whence it converges. It follows that $\sum A_n(b_{n+1}-b_n)$ converges.
   Combining these along with the Abel summation formula, conclude that $\sum a_nb_n$ converges.

5. Prove using the above

   (a) If $(x_n)\in(\mathbb{R}_{\ge 0})^{\mathbb{N}}$ is decreasing and $\lim x_n = 0$ then $\sum(-1)^nx_n<\infty$.
   Use the above with $a_n=(-1)^n, b_n=x_n$.

   (b) If $(x_n)\in(\mathbb{R}_{\ge 0})^{\mathbb{N}}$ is such that $\exists B>0$ satisfying $\sum\limits_{i=1}^{n}x_i\le B\,\forall n$, then $\displaystyle\sum\frac{a_n}{n}<\infty$.
   Use the above with a_n as it is, and $b_n=1/n$.

6. Let $a > 0$. Prove $\sum_{n=1}^{\infty} \frac{1}{(a+n+1)(a+n)} < \infty$. Find the limit.

7. Let $a > 0$ and $m \in \mathbb{N}$.

   (a) Show that $\sum_{k=1}^{n} \frac{m}{\prod_{j=0}^{m}(a+k+j)} = \frac{1}{\prod_{j=1}^{m}(a+j)} - \frac{1}{\prod_{j=1}^{m}(a+n+j)}$.
   Hint: Induct on n.

   (b) Show that $\sum_{n=1}^{\infty} \frac{1}{\prod_{j=0}^{m}(a+n+j)} = \frac{1}{m\prod_{j=1}^{m}(a+j)}$

8. Let $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}, A_n := \sum_{i=1}^{n} a_i, A_n := \sum_{i=1}^{n} b_i$. Prove that [1]

$$\sum_{k=n+1}^{m} a_k B_k = A_m B_m - A_n B_{n+1} - \sum_{k=n+1}^{m-1} A_k b_{k+1}$$

9. *(Use your knowledge of high-school integration)* Let $(a_n) \in \mathbb{R}^{\mathbb{N}\cup\{0\}}$ be a sequence and let its partial sums be $A(n) := \sum_{k=0}^{n} a_n$. Fix real numbers $x < y$. $\varphi : [x, y] \to \mathbb{R}$ is a continuously differentiable function. Show that

$$\sum_{n=\lfloor x\rfloor+1}^{\lfloor y\rfloor} a_n\varphi(n) = A\left(\lfloor y\rfloor\right)\varphi(y) - A\left(\lfloor x\rfloor\right)\varphi(x) - \int_x^y A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt$$

Fix some $y \in \mathbb{R}_{\geq 0}$. Let $k = \lfloor y\rfloor$. Clearly $\sum_{n=0}^{k} a_n\varphi(n) = A(k)\varphi(k+1) - \sum_{n=0}^{k} A(n)\left(\varphi(n+1) - \varphi(n)\right) = A(k)\varphi(k) - \sum_{n=0}^{k-1} A(n)\left(\varphi(n+1) - \varphi(n)\right)$. But $\sum_{n=0}^{k-1} A(n)(\varphi(n+1) - \varphi(n)) = \sum_{n=0}^{k-1} A(n)\int_n^{n+1} \varphi'(t)\,dt = \sum_{n=0}^{k-1}\int_n^{n+1} A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt = \int_0^k A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt = \int_0^y A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt - \int_{\lfloor y\rfloor}^y A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt$
$= \int_0^y A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt - A\left(\lfloor y\rfloor\right)\int_{\lfloor y\rfloor}^y \varphi'(t)\,dt = \int_0^y A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt - A\left(\lfloor y\rfloor\right)\varphi(y) + A\left(\lfloor y\rfloor\right)\varphi\left(\lfloor y\rfloor\right)$.

Putting together, $\sum_{n=0}^{\lfloor y\rfloor} a_n\varphi(n) = A\left(\lfloor y\rfloor\right)\varphi(y) - \int_0^y A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt$.

The required identity trivially follows.

10. Let (a_n) be a sequence of non-negative reals such that $\sum a_n \in \mathbb{R}$. Let $p \geq 0$. Show that $\sum \sqrt{a_n}\cdot n^{-p} \in \mathbb{R}$ if $p > \frac{1}{2}$. Find a counterexample if $p = \frac{1}{2}$.

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[1] *(Maybe a hint)* One is tempted to recall the integration by parts formula. Let $F(x) := \int_a^x f(x)\,dx$, $G(x) := \int_a^x g(x)\,dx$. Then

$$\int_a^b f(x)G(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F(x)g(x)\,dx$$

$\sqrt{a_n} n^{-p} = \sqrt{a_n n^{-2p}} \leq \dfrac{a_n + n^{-2p}}{2}$. $\sum \dfrac{a_n + n^{-2p}}{2} \in \mathbb{R} \because 2p > 1$. It follows by comparison that $\sum \sqrt{a_n} n^{-p} \in \mathbb{R}$.

Suppose $p = \frac{1}{2}$. Consider $a_n = \dfrac{1}{n (\log n)^2}$. Note that $a_n \geq 0$ and $\sum a_n \in \mathbb{R}$ by Cauchy condensation test. But $\sum \sqrt{a_n} n^{-\frac{1}{2}}$ diverges by Cauchy condensation test.