Sequences

Definition: Let X be a set. A sequence is simply a function $f : \mathbb{N} \longrightarrow X$. Such a sequence will be called an X-valued sequence.

$$a_1, a_2, a_3, a_4, \ldots$$

Eg: $X = \{ \textcolor{green}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{red}{\bullet} \}$

An X valued sequence is

$$\underset{a_1}{\textcolor{red}{\bullet}}, \underset{a_2}{\textcolor{red}{\bullet}}, \underset{a_3}{\textcolor{red}{\bullet}}, \underset{a_4}{\textcolor{blue}{\bullet}}, \underset{a_5}{\textcolor{blue}{\bullet}}, \ldots \textcolor{blue}{\bullet}, \textcolor{green}{\bullet}, \textcolor{green}{\bullet}, \textcolor{green}{\bullet}, \textcolor{red}{\bullet}, \textcolor{red}{\bullet}, \textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{green}{\bullet}, \textcolor{green}{\bullet} \ldots$$

$$f(n) = \begin{cases} \textcolor{green}{\bullet} & \text{if } n \equiv 0, -1, -2 \pmod 9 \\ \textcolor{red}{\bullet} & \text{if } n \equiv 1, 2, 3 \pmod 9 \\ \textcolor{blue}{\bullet} & \text{if } n \equiv 4, 5, 6 \pmod 9 \end{cases}$$

Can also view it as

$$f = \{ (1, \textcolor{red}{\bullet}), (2, \textcolor{red}{\bullet}), (3, \textcolor{red}{\bullet}), (4, \textcolor{blue}{\bullet}), \ldots \}$$

$$f : \mathbb{N} \longrightarrow X$$

$$f(1), f(2), f(3), \ldots$$

$$a_1, a_2, a_3, \ldots$$

We will be interested in $X = \mathbb{R}$ (later $\mathbb{R}^n$, and even later, any metric space X, finally X will be a topological space).

The set of all functions $f : Y \to X$ is denoted by X^Y.
In other words, $X^Y = \{ f : Y \to X \mid f \text{ is a function} \}$.

Exercise: let S be a finite set & $n \in \mathbb{N}$. Denote

$[n] = \{1, \ldots, n\}$.

Show that $\left| S^{[n]} \right| = |S|^n$

[Number of functions $f : \{1, \ldots, n\} \to S$ is $|S|^n$]

So the set of all X-valued sequences is $X^{\mathbb{N}}$.
" " " " $\mathbb{R}$- " " is $\mathbb{R}^{\mathbb{N}}$.

Prop: let $(V, +, \cdot)$ be a vector space over F. Define
(F: field)

$\oplus : V^{\mathbb{N}} \times V^{\mathbb{N}} \to V^{\mathbb{N}}$ as

$$f \oplus g := \left(n \longmapsto f(n) + g(n) \right)$$

and $\odot : F \times V^{\mathbb{N}} \to V^{\mathbb{N}}$ as

$$c \odot f := \left(n \longmapsto c \cdot f(n) \right)$$

(I need to define $g = c \odot f$. Then $g : \mathbb{N} \to V$.

$$g(n) = c \cdot f(n)$$

$$n \mapsto c \cdot f(n)$$
)

Exercise

Then $(V^{\mathbb{N}}, \oplus, \odot)$ is a vector space over F.

With the above proposition, $V = \mathbb{R} (= F)$ gives that $\mathbb{R}^{\mathbb{N}}$ is an $\mathbb{R}$-vector space.

Exercise: let F be a field. Then F is an F-vector space.

Subsequences :

$(a_n) =: A$ is an X-valued seq.

$a_1, a_2, a_3, \ldots$

$f \in X^{\mathbb{N}}$

Let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function

(i.e., $m > n \Rightarrow g(m) > g(n)$)

$(a_{g(n)})$ is a subseq of A.

$h := f \circ g \in X^{\mathbb{N}}$ is a subseq of f.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "g"] \arrow[rr, bend right, "h"', "f \circ g"'] & \mathbb{N} \arrow[r, "f"] & X \end{tikzcd}$$

Example: ① $g(x) = 2x$.

$(a_1, a_2, \ldots, a_n, \ldots) \leftarrow$ Seq

$(a_2, a_4, a_6, a_8, \ldots) \leftarrow$ subseq

② $g(x) = x^2$

Seq $\to (1, 2, 3, 4, 5, 6, \ldots)$

Subseq $\to (1, 4, 9, 16, 25, \ldots)$

converges in $\mathbb{R}$

$a_n = \dfrac{\lfloor \sqrt{2}\, n \rfloor}{n} \in \mathbb{Q}$ → $\sqrt{2} \notin \mathbb{Q}$

Cauchy seq (in $\mathbb{Q}$)

diverge in $\mathbb{Q}$

→ Proves that $\mathbb{Q}$ is not a "complete" metric space.

For those who know metric spaces:

Let X be a set $(X \neq \phi)$. Show that $\exists$ $d : X \times X \to \mathbb{R}$ s.t. d is a metric on X.

Defn: We say (X, d) is a metric space, where X is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ is a function, if the following hold $(\forall\, x, y, z \in X)$:

(i) $d(x, y) \geq 0$ with equality iff $x = y$.

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

Example: $(\mathbb{C}, d)$ where $d(z_1, z_2) = |z_1 - z_2|$

Exercise: Consider $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$ given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{o/w.} \end{cases}$$

Verify that $(\mathbb{R}, d)$ is a metric space.

"Discrete metric"