ADVANCED ALGORITHM DESIGN

Homework 3

December 4, 2024

Problem 1

(a) Let A, B be symmetric, real matrices with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ (and similarly for B). Prove that for every k, $\lambda_k(A) + \lambda_n(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_1(B)$. Use this claim to establish that $|\lambda_k(A+B) - \lambda_k(A)| \leq \max\{\lambda_1(B), |\lambda_n(B)|\}$.

(b) Let A be the adjacency matrix of a not necessarily regular graph G with m edges and n vertices with eigenvalues $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$. Prove that $\lambda_1 \geq 2m/n$.

Solution

(a) Recall from the Courant-Fisher theorm that $\lambda_k(M) = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim S = k}} \min_{\boldsymbol{x} \in S \setminus \{0\}} \frac{\boldsymbol{x}^\top M \boldsymbol{x}}{\|\boldsymbol{x}\|_2^2}$ for any symmetric real symmetric matrix M. Define $f_M(\boldsymbol{x}) := \frac{\boldsymbol{x}^\top M \boldsymbol{x}}{\|\boldsymbol{x}\|_2^2}$ (where λ_i means as usual). In particular, $\lambda_1(M) \geq \boldsymbol{f}_M(\boldsymbol{x}) \geq \lambda_n(M)$. Note that $f_{A+B}(\boldsymbol{x}) = f_A(\boldsymbol{x}) + f_B(\boldsymbol{x})$. Therefore, $f_A(\boldsymbol{x}) + \lambda_1(B) \geq f_{A+B}(\boldsymbol{x}) \geq f_A(\boldsymbol{x}) + \lambda_n(B)$. Taking $\max\min$ with appropriate constraints preserves inequalities, whence $\lambda_k(A) + \lambda_1(B) \geq \lambda_k(A+B) \geq \lambda_k(A) + \lambda_n(B)$.

This gives $\lambda_1(B) \geq \lambda_k(A+B) - \lambda_k(A) \geq \lambda_n(B)$.

- If $\lambda_n(B) \geq 0$ then $|\lambda_k(A+B) - \lambda_k(A)| = \lambda_k(A+B) - \lambda_k(A) \leq \lambda_1(B) \leq \max\{\lambda_1(B), |\lambda_n(B)|\}$.
- If $\lambda_1(B) \geq 0 > \lambda_n(B)$ then $|\lambda_k(A+B) - \lambda_k(A)| = \max\{\lambda_k(A+B) - \lambda_k(A), \lambda_k(A) - \lambda_k(A+B)\} \leq \max\{\lambda_1(B), -\lambda_n(B)\} = \max\{\lambda_1(B), |\lambda_n(B)|\}$.
- If $0 > \lambda_1(B)$ then $|\lambda_k(A+B) - \lambda_k(A)| = \lambda_k(A) - \lambda_k(A+B) \leq -\lambda_n(B) \leq \max\{\lambda_1(B), |\lambda_n(B)|\}$.

(b) $\lambda_1(A) \overset{\text{Courant-Fisher}}{=} \max_{\boldsymbol{x} \neq 0} \frac{\boldsymbol{x}^\top A \boldsymbol{x}}{\|\boldsymbol{x}\|_2^2} \geq \left(\frac{\boldsymbol{x}^\top A \boldsymbol{x}}{\|\boldsymbol{x}\|_2^2} \right)_{\boldsymbol{x} = \mathbf{1}} = \frac{\sum_{1 \leq i,j \leq n} A_{ij}}{(\sqrt{n})^2} = \frac{2m}{n}$ where $\mathbf{1} \in \mathbb{R}^n$ has all 1's.

Problem 2

Let R be a random symmetric matrix with uniformly random ± 1 entries. In this problem, for any $\varepsilon \in (0,1)$, let S_ε be a finite set of N_ε unit vectors in $\mathbb{R}^n$ such that for every unit vector $u \in \mathbb{R}^n$, there is a vector $\boldsymbol{u}' \in S$ such that $\|\boldsymbol{u} - \boldsymbol{u}'\|_2 \leq \varepsilon$.

(a) Prove that for every unit vector $\boldsymbol{u}$ and $t \geq 0$, $\mathbb{P}\left[\left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq t\right] \leq 2\exp\left\{\frac{-t^2}{2}\right\}$.

(b) Prove that $\mathbb{P}\left[\exists \boldsymbol{u} \in S_\varepsilon \text{ s.t. } \left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq t\right] \leq 2N_\varepsilon \exp\left\{\frac{-t^2}{2}\right\}$.

(c) Prove that for every unit vector $\boldsymbol{u}$, and any ± 1-entry matrix B, $\left|\boldsymbol{u}^\top B\boldsymbol{u}\right| \leq n^C$ for some $C > 0$. What's the smallest C for which you can establish this claim?

(d) In this part, you can assume without proof that for every $\varepsilon > 0$, there is an S_ε of size $N_\varepsilon \leq \left(\frac{c}{\varepsilon}\right)^n$. Using this and the results of the previous parts, argue that $\mathbb{P}\left[\|R\|_2 \geq \mathcal{O}\left(\sqrt{n\log n}\right)\right] \leq \frac{1}{n}$.

(e) *(Extra credit)* Prove the assumption in part (d). That is, prove that there is an S_ε as described in part (2) of size $(c/\varepsilon)^n$ for some $c > 0$.

Solution

(a) Note that $\boldsymbol{u}^\top R\boldsymbol{u} = \sum\limits_{1\leq i,j\leq n} R_{ij}u_iu_j$. We will use Höffding bound[1] with the n^2 random variables $X_{ij} \coloneqq R_{ij}u_iu_j$. Note that $|X_{ij}| = |u_iu_j|$ because $R_{ij} \in \{\pm 1\}$. So $X_{ij} \in [a_{ij}, b_{ij}]$ where $a_{ij} \coloneqq -|u_iu_j|, b_{ij} \coloneqq |u_iu_j|$. The denominator in the exponential of our Höfding bound becomes $\sum\limits_{1\leq i,j\leq n}(b_{ij}-a_{ij})^2 = \sum\limits_{1\leq i,j\leq n} 4u_i^2u_j^2 = 4\sum_i u_i^2 \sum_j u_j^2 = 4$ because $\|\boldsymbol{u}\|_2^2 = 1$. Noting that $\sum\limits_{i,j}\mathbb{E}[X_{ij}] = \sum\limits_{i,j} u_iu_j\mathbb{E}[R_{ij}] = 0$ we get,

$$\mathbb{P}\left[\left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \leq t\right] = \mathbb{P}\left[\left|\sum_{i,j} X_{ij}\right| \leq t\right] \leq 2\exp\left\{\frac{-2t^2}{\sum\limits_{1\leq i,j\leq n}(b_{ij}-a_{ij})^2}\right\} = 2\exp\left\{\frac{-2t^2}{4}\right\} = 2\exp\left\{\frac{-t^2}{2}\right\}.$$

(b) Let the $(N =)N_\varepsilon$ vectors in S_ε be $u_1, \cdots, u_N$. Then $\mathbb{P}\left[\exists \boldsymbol{u} \in S_\varepsilon \text{ s.t. } \left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq t\right] =$

$$\mathbb{P}\left[\bigcup_{\boldsymbol{u}\in S_\varepsilon}\left\{\left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq t\right\}\right] \overset{\text{union bound}}{\leq} \sum_{\boldsymbol{u}\in S_\varepsilon}\mathbb{P}\left[\left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq t\right] \leq \sum_{\boldsymbol{u}\in S_\varepsilon} 2\exp\left\{\frac{-t^2}{2}\right\} = 2N_\varepsilon \exp\left\{\frac{-t^2}{2}\right\}.$$

(c) Recall that for any $\boldsymbol{x} \in \mathbb{R}^n$ we have $\|\boldsymbol{x}\|_1 \leq \sqrt{n}\,\|\boldsymbol{x}\|_2$. If $B \in \{\pm 1\}^{n\times n}$ and $\boldsymbol{u} \in \mathbb{R}^n$ has $\|\boldsymbol{u}\|_2 = 1$ then $\left|\boldsymbol{u}^\top B\boldsymbol{u}\right| = \left|\sum\limits_{i,j\in[n]} B_{ij}u_iu_j\right| \leq \sum\limits_{i,j\in[n]}]\,|B_{ij}u_iu_j| = \sum_i |u_i| \sum_j |u_j| = \|\boldsymbol{u}\|_1^2 \leq n\,\|\boldsymbol{u}\|_2^2 = n^1$. So $C = 1$ works.

We will show that this is the best C (by showing that $C = 1$ is attained). This is because when $B = \mathbf{1}\mathbf{1}^\top$ (which is the matrix of all 1) and $\boldsymbol{u} = \frac{\mathbf{1}}{\sqrt{n}}$ (which is the vector with each entry $1/\sqrt{n}$) then $\|\boldsymbol{u}\|_2 = n \cdot \frac{1}{n} = 1$ and $\left|\boldsymbol{u}^\top B\boldsymbol{u}\right| = \left|\sum\limits_{i,j} u_iu_j\right| = n^2 \cdot \frac{1}{n} = n$.

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[1] If $X_1, \cdots, X_n$ are independent with $X_i \in [a_i, b_i]$ then $\mathbb{P}\left[\sum\limits_{i\in[n]}(X_i - \mathbb{E}[X_i])\right] \leq 2\exp\left\{\frac{-2t^2}{\sum\limits_{i\in[n]}(b_i-a_i)^2}\right\}$

(d) Let $\mathbb{S}^{n-1}$ be the collection of all unit vectors in $\mathbb{R}^n$.

Lemma 1
$$\frac{\max\limits_{\boldsymbol{u}\in S_\varepsilon}\left|\boldsymbol{u}^\top R\boldsymbol{u}\right|}{1-2\varepsilon} \geq \max_{\boldsymbol{u}\in\mathbb{S}^{n-1}}\left|\boldsymbol{u}^\top R\boldsymbol{u}\right| = \|R\|_2.$$

Proof. Say $\boldsymbol{x}\in\mathbb{S}^{n-1}$ is an eigenvector for the largest (in magnitude) eigenvalue of R. Let this eigenvalue be λ. So $\|R\|_2 = |\lambda|$. But $\exists\, \boldsymbol{y}\in S_\varepsilon$ such that $\|\boldsymbol{y}-\boldsymbol{x}\|_2 \leq \varepsilon$. Then $\left|\boldsymbol{x}^\top R\boldsymbol{x} - \boldsymbol{y}^\top R\boldsymbol{y}\right| = \left|\boldsymbol{x}^\top R(\boldsymbol{x}-\boldsymbol{y}) - (\boldsymbol{y}-\boldsymbol{x})^\top R\boldsymbol{y}\right| \leq \left|\boldsymbol{x}^\top R(\boldsymbol{x}-\boldsymbol{y})\right| + \left|(\boldsymbol{y}-\boldsymbol{x})^\top R\boldsymbol{y}\right| \leq \|\boldsymbol{x}-\boldsymbol{y}\|_2\,\|R\|_2\,(\|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2) = 2\,\|R\|_2\,\|\boldsymbol{x}-\boldsymbol{y}\|_2 = 2\varepsilon\,|\lambda|$.
Thus $\max\limits_{\boldsymbol{u}\in S_\varepsilon}\left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq \left|\boldsymbol{y}^\top R\boldsymbol{y}\right| \geq \left|\boldsymbol{x}^\top R\boldsymbol{x}\right| - \left|\boldsymbol{x}^\top R\boldsymbol{x} - \boldsymbol{y}^\top R\boldsymbol{y}\right| \geq |\lambda| - 2\varepsilon\,|\lambda| = |\lambda|\,(1-2\varepsilon)$. ■

Then $\mathbb{P}\left[\|R\|_2 \geq t\right] \leq \mathbb{P}\left[\frac{\max\limits_{\boldsymbol{u}\in S_\varepsilon}\left|\boldsymbol{u}^\top R\boldsymbol{u}\right|}{1-2\varepsilon} \geq t\right] = \mathbb{P}\left[\max\limits_{\boldsymbol{u}\in S_\varepsilon}\left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq t(1-2\varepsilon)\right] =$
$\mathbb{P}\left[\exists u\in S_\varepsilon \text{ s.t. } \left|\boldsymbol{u}^\top R\boldsymbol{u}\right| \geq t(1-2\varepsilon)\right] \overset{\text{(b)}}{\leq} 2N_\varepsilon \exp\left\{\frac{-t^2(1-2\varepsilon)^2}{2}\right\} \leq 2\left(\frac{c}{\varepsilon}\right)^n \exp\left\{\frac{-t^2(1-2\varepsilon)^2}{2}\right\}$
where the first inequality is true because it's more probable for a larger quantity to be $\geq t$. Take $\varepsilon = \frac{\log n}{n}$ so that $1-2\varepsilon \geq \frac{1}{2}$ for large n. Take $t = \alpha\sqrt{n\log n}$. Then $2\left(\frac{c}{\varepsilon}\right)^n \exp\left\{\frac{-t^2(1-2\varepsilon)^2}{2}\right\} \leq 2\left(\frac{cn}{\log n}\right)^n \exp\left\{\frac{-\alpha^2 n\log n}{8}\right\} = 2\left(\frac{cn}{\log n}\right)^n n^{\frac{-\alpha^2 n}{8}} = 2\left(\frac{c}{n^{\frac{\alpha^2}{8}-1}\log n}\right)^n \leq \frac{1}{n}$ by choosing large constant α.

(e) Consider the following algorithm for any given input $\varepsilon > 0$ to find a set $S_\varepsilon \subseteq \mathbb{S}^{n-1}$.

Input: $\varepsilon > 0$, dimension n
Output: a number N and points $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_N \in \mathbb{S}^{n-1}$ such that every point in $\mathbb{S}^{n-1}$ is $\varepsilon-$close to some $\boldsymbol{v}_i$.

begin
    $N \leftarrow 1$
    $\boldsymbol{v}_1 \leftarrow (1,0,\cdots,0)\in\mathbb{R}^n$
    $S \leftarrow B^o_\varepsilon(\boldsymbol{v}_1)\cap\mathbb{S}^{n-1}$ ▷ points in S^{n-1} which are at distance $< \varepsilon$ from $\boldsymbol{v}_1$
    while $N \geq 1$ **do**
        $\boldsymbol{v}_N \leftarrow$any point in $\mathbb{S}^{n-1} \setminus S$
        $S \leftarrow S \cup (B_\varepsilon(\boldsymbol{v}_2)\cap\mathbb{S}^{n-1})$
        if $S = \mathbb{S}^{n-1}$ **then** ▷ check if $\mathbb{S}^{n-1}$ has been covered
            break
        else
            $N \leftarrow N+1$
        end if
    end while
    return $N, S_\varepsilon = \{\boldsymbol{v}_1,\cdots,\boldsymbol{v}_N\}$
end

Now we prove that this algorithm actually gives S_ε and N_ε as desired. In what follows, B, B^o

will respectively denote closed and open balls.

If the above algorithm terminates with answer N, S_ε, then $\mathbb{S}^{n-1} \subseteq \bigcup_{i=1}^{N} B^o_\varepsilon(\boldsymbol{v}_i) \subseteq \bigcup_{i=1}^{N} B_\varepsilon(\boldsymbol{v}_i)$.

Claim 2
The above algorithm terminates.

Proof. Suppose the algorithm goes on forever. So we get a sequence of points $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots$ such that $\mathbb{S}^{n-1} \subseteq \bigcup_{i\in\mathbb{N}} B^o_\varepsilon(\boldsymbol{v}_i)$. Since $\mathbb{S}^{n-1}$ is compact there is a finite N such that $\mathbb{S}^{n-1} \subseteq \bigcup_{i=1}^{N} B^o_\varepsilon(\boldsymbol{v}_i)$. This is a contradiction to our original assumption. ∎

Next we note that just by how our algorithm is designed, if $\boldsymbol{x}, \boldsymbol{y} \in S_\varepsilon$ then $\|\boldsymbol{x}-\boldsymbol{y}\|_2 \geq \varepsilon$. This is because a new point (line 6) is always chosen so that it is not in the $\varepsilon-$ball around any of the previously chosen points, and distance is symmetric.

Further S_ε is maximal in the sense that if $S' \supsetneq S_\varepsilon$ is a collection of points in $\mathbb{S}^{n-1}$, there will be two points in S' which are at most $\varepsilon-$close to each other. This is by our breaking criterion on line 8. Simply put, S_ε covers $\mathbb{S}^{n-1}$ with $\varepsilon-$balls.

Claim 3
If $\boldsymbol{x}, \boldsymbol{y} \in S_\varepsilon$ are distinct, then $B^o_{\frac{\varepsilon}{2}}(\boldsymbol{x}) \cap B^o_{\frac{\varepsilon}{2}}(\boldsymbol{y}) \cap \mathbb{S}^{n-1} = \varnothing$.

Proof. Suppose $\boldsymbol{p} \in \mathbb{S}^{n-1} \cap B^o_{\frac{\varepsilon}{2}}(\boldsymbol{x}) \cap B^o_{\frac{\varepsilon}{2}}(\boldsymbol{y})$ and say $\boldsymbol{y}$ was picked after $\boldsymbol{x}$ in the algorithm. Then $\|\boldsymbol{x}-\boldsymbol{y}\|_2 \leq \|\boldsymbol{x}-\boldsymbol{p}\|_2 + \|\boldsymbol{p}-\boldsymbol{y}\|_2 \leq \varepsilon$. Moreover equality here occurs only when $\|\boldsymbol{p}-\boldsymbol{x}\|_2 = \|\boldsymbol{p}-\boldsymbol{y}\|_2 = \frac{\varepsilon}{2}$ which means $p \notin B^o_{\frac{\varepsilon}{2}}(\boldsymbol{x})$ which is a contradiction. So it must happen that $\|\boldsymbol{x}-\boldsymbol{y}\|_2 < \varepsilon$ which contradicts the constructive step in line 6 because this indicated that $\boldsymbol{y}$ was picked in the $\varepsilon-$ball around $\boldsymbol{x}$. ∎

Claim 4
If $\boldsymbol{x} \in \bigcup_{i\in[N]} B_\varepsilon(\boldsymbol{v}_i) \subseteq \mathbb{R}^n$ then $\|\boldsymbol{x}\|_2 \leq 1 + \frac{\varepsilon}{2}$.

Proof. Say $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{v}_i)$ for some i. Then $\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{v}\|_2 + \|\boldsymbol{x}-\boldsymbol{v}\|_2 \leq 1 + \varepsilon$. ∎

Denote $V_n := \mathrm{vol}(\mathbb{D}^n), A_n := \mathrm{area}(\mathbb{S}^{n-1})$. These are respectively the $n-$dimensional volume of D^n (the solid unit ball with $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$) and the $(n-1)-$dimensional volume (area) of $\mathbb{S}^{n-1}$. By claim 4, $\bigcup_{i\in[N]} B_\varepsilon(\boldsymbol{v}_i) \subseteq \mathbb{R}^n \subseteq B_{1+\frac{\varepsilon}{2}}(0)$. $\mathrm{vol}(B_{1+\frac{\varepsilon}{2}}(0)) = \left(1+\frac{\varepsilon}{2}\right)^n V_n$. So $\mathrm{vol}\left(\bigcup_{i\in[N]} B_\varepsilon(\boldsymbol{v}_i)\right) \leq V_n \left(1+\frac{\varepsilon}{2}\right)^n$. Claim 3 says the above union is almost disjoint (the intersections form a set of measure 0). Hence $\mathrm{vol}\left(\bigcup_{i\in[N]} B_\varepsilon(\boldsymbol{v}_i)\right) = \sum_{i=1}^{N} \mathrm{vol}\left(B_\varepsilon(\boldsymbol{v}_i)\right) = \sum_{i=1}^{N} \varepsilon^n V(n) = NV_n\varepsilon^n$. These prove that $N\varepsilon^n \leq \left(1+\frac{\varepsilon}{2}\right)^n \implies N \leq \left(\frac{1}{\varepsilon}+\frac{1}{2}\right)^n = \left(\frac{(2+\varepsilon)/2}{\varepsilon}\right)^n \leq \left(\frac{2}{\varepsilon}\right)^n$.

Problem 3

Let G be a graph on n vertices (n is even) chosen as follows:

1. Pick an arbitrary S of size $n/2$,
2. For each pair i, j of vertices such that $i, j \in S$ or $i, j \notin S$, include $\{i, j\}$ in G with probability p,
3. For each pair i, j such that $i \in S, j \notin S$ or $i \notin S, j \in S$, include $\{i, j\}$ in G with probability q.

Suppose that $p - q > c$ for some fixed constant $c > 0$.

Consider the following algorithm:

1. pick a vertex v,
2. Output $\hat{S}$ obtained by including in $\hat{S}$ the $n/2$ vertices that have the fewest common neighbors with v.

Prove that for large n, with probability at least 0.99 over the draw of G, $\hat{S}$ either equals S or $V \setminus S$.

Solution

For any vertex v in $G = (V = [n], E)$, denote by $N(v)$ the neighbors of v in G. Say $n = 2k$. Denote $T \coloneqq V \setminus S$. Then $|T| = |S| = k$. Let's proceed as the hint suggests.

Assume $v \in S$ WLOG (otherwise replace S with T). Let u be an arbitrary vertex. Denote $X_u \coloneqq |N(u) \cap N(v)|$. Note that $X_u = \sum_{x \in V} \mathbf{1}[x \in N(u) \cap N(v)] = \sum_{x \in V} \mathbf{1}_{x \in N(u)} \cdot \mathbf{1}_{x \in N(v)}$ because the events $\{x \in N(u)\}, \{x \in N(v)\}$ are independent. This can be further split as $X_u = \sum_{x \in S \setminus \{u,v\}} \mathbf{1}_{x \in N(u)} \cdot \mathbf{1}_{x \in N(v)} + \sum_{x \in T \setminus \{u,v\}} \mathbf{1}_{x \in N(u)} \cdot \mathbf{1}_{x \in N(v)} \implies \mathbb{E}[X_u] = \sum_{x \in S \setminus \{u,v\}} \mathbb{P}[x \in N(u)] \cdot \mathbb{P}[x \in N(v)] + \sum_{x \in T \setminus \{u,v\}} \mathbb{P}[x \in N(u)] \cdot \mathbb{P}[x \in N(v)] = p \sum_{x \in S \setminus \{u,v\}} \mathbb{P}[x \in N(u)] + q \sum_{x \in T \setminus \{u,v\}} \mathbb{P}[x \in N(u)]$.

- Say $u \in S$. Then $\mathbb{E}[X_u] = (k-2)p^2 + kq^2$.
- Say $u \in T$. Then $\mathbb{E}[X_u] = (k-1)pq + (k-1)pq = 2(k-1)pq$.

Denote $b \coloneqq (k-2)p^2 + kq^2$ and $a \coloneqq 2(k-1)pq$. One observes that $d \coloneqq b - a = (k-1)(p-q)^2 + q^2 - p^2$. It is important to note that $b \leq k(p^2 + q^2) \leq 2k$ and $d \geq (k-1)c^2$.

So far we learnt that, in expectation, vertices in S has higher number of common neighbors with v than those in T. So if X_i's are well concentrated in (t, ∞) for $i \in S$, and X_j's are well concentrated in $(-\infty, t)$ where $t \in (a, b)$ (where $t \in (a, b)$) then $X_i \geq X_j \ \forall\, i \in S, j \in T$. So $\hat{S}$ would be T.

For $i \in S \setminus \{v\}$ let A_i be the event $\{X_i \leq t\}$. For $j \in T$ let B_j be the event $\{X_j \geq t\}$. We are interested in the event $E \coloneqq \left(\bigcup_{i \in S \setminus \{v\}} A_i \right) \bigcup \left(\bigcup_{j \in T} B_j \right)$. But $\mathbb{P}[E] = \sum_{i \in S \setminus \{v\}} \mathbb{P}[A_i] + \sum_{j \in T} \mathbb{P}[B_j] = \sum_{i \in S \setminus \{v\}} \mathbb{P}[A_i] + \sum_{j \in T} \mathbb{P}[B_j]$.

We will take $t = \frac{a+b}{2}$ so that $t = b\left(1 - \frac{b-a}{2b}\right) = a\left(1 + \frac{b-a}{2a}\right)$. Recall the lower and upper tail Chernoff bounds for a random variable X which is a sum of (finitely many) independent $0/1$ random variables:

- $\mathbb{P}\left[X \le \mathbb{E}\left[X\right](1-\varepsilon)\right] \le \exp\left\{-\frac{\varepsilon^2 \mathbb{E}[X]}{2}\right\}$. Using this for A_i (with $i \in S$) gives

$$\begin{aligned}\mathbb{P}\left[A_i\right] &= \mathbb{P}\left[X_i \le \frac{a+b}{2}\right] = \mathbb{P}\left[X_i \le b\left(1 - \frac{b-a}{2b}\right)\right] \\ &\le \exp\left\{-\frac{(b-a)^2}{8b}\right\} = \exp\left\{-\frac{d^2}{8b}\right\} \le \exp\left\{-\frac{d^2}{12b}\right\} \\ &\le \exp\left\{-\frac{(k-1)^2 c^4}{12 \cdot 2k}\right\} \overset{\because k \ge 2 \implies k-1 \ge \frac{2k}{3}}{\le} \exp\left\{\frac{-c^4 k}{54}\right\}.\end{aligned}$$

- $\mathbb{P}\left[X \ge \mathbb{E}\left[X\right](1+\varepsilon)\right] \le \exp\left\{-\frac{\varepsilon^2 \mathbb{E}[X]}{3(1+\varepsilon)}\right\}$. Using this for B_j (with $j \in T$) gives

$$\begin{aligned}\mathbb{P}\left[B_i\right] &= \mathbb{P}\left[X_i \ge \frac{a+b}{2}\right] = \mathbb{P}\left[X_i \le a\left(1 + \frac{b-a}{2a}\right)\right] \\ &\le \exp\left\{-\frac{(b-a)^2}{6(a+b)}\right\} \le \exp\left\{-\frac{d^2}{12b}\right\} \\ &\le \exp\left\{-\frac{(k-1)^2 c^4}{16 \cdot 2k}\right\} \overset{\because k \ge 2 \implies k-1 \ge \frac{2k}{3}}{\le} \exp\left\{\frac{-c^4 k}{54}\right\}.\end{aligned}$$

Therefore $\mathbb{P}\left[E\right] \le (n-1)\exp\left\{\frac{-c^4 n}{108}\right\} \le n\exp\left\{\frac{-c^4 n}{108}\right\}$. Let $C := \frac{c^4}{108}$. We recall that $e^x \ge 1 + x + x^2/2$ for $x \ge 0$. Hence $e^{-x} \le \frac{1}{1+x+x^2/2}$ on $\mathbb{R}_{\ge 0}$. Taking $x = Cn > 0$ and $n > \dfrac{200}{C^2}$ gives $n\exp\left\{-Cn\right\} \le \frac{n}{1+nC+n^2C^2/2} = \frac{1}{\frac{1}{n}+C+nC^2/2} \overset{\left[\because \frac{1}{n}+C>0\right]}{<} \frac{1}{nC^2/2}\frac{1}{100} = 0.01$, whence $PE^c \ge 0.99$. Recall that A_i^c (for $i \in S$) was the event that the number of common neighbors of i, v is $> \frac{a+b}{2}$, B_j^c (for $j \in T$) was the event that the number of common neighbors of j, v is $< \frac{a+b}{2}$. So E^c is the event that every vertex in S has $> \frac{a+b}{2}$ common neighbors with v and every vertex in T has $< \frac{a+b}{2}$ common neighbors with v, hence it's a special case with $\hat{S} = T$ which is a subset of the event that we are interested in. Therefore the chance that, over the draw of G, the $n/2$ vertices with fewest common neighbors of v is at least $\mathbb{P}\left[E^c\right] \ge 0.99$.

Problem 4

In the class, we saw that we can distinguish between a graph $G \sim G(n, 1/2)$ and $G \sim G(n, 1/2, k)$ (i.e., $G \sim G(n, 1/2)$ with an added $k-$clique) in polynomial time if $k \geq c\sqrt{n}$ for some $c > 0$.

Find an algorithm that for any $t \in \mathbb{N}$, runs in time $n^{\mathcal{O}(t)}$ and succeeds in the same goal for $k \geq \sqrt{n/2^t}$. (Hint: suppose you were given, in addition, a set S of t vertices in the planted clique if there was one. Can you now reduce the problem to graphs on a smaller number of vertices?)

Solution

Let's first present the algorithm to the input graph $G = (V = [n], E)$ which is as follows. We loop through all $\binom{n}{t+1+z} = n^{\mathcal{O}(t)}$ subsets of V which are of size $s := t + 1 + z$ and form a clique. For each such subset $S \subseteq V$, take the vertices in V connected to every vertex in S and call it $\mathcal{N}_S$ and take $\mathcal{A}_S$ to be the $\pm 1-$adjacency matrix of the subgraph of G induced by $\mathcal{N}_S$. We declare that "G has a $k-$planted clique" if $\|\mathcal{A}_S\|_2 > C\sqrt{|\mathcal{N}_S|}$ (this is the algorithm discussed in class). If this fails for every such subset S, we declare that there is "no planted clique".

In the above, we do $n^{\mathcal{O}(1)}$ (where the $\mathcal{O}$ is with respect to t) computations per $S \subseteq V$ of size t. The number of loops is $n^{\mathcal{O}(t)}$ and checking if S forms a clique takes time of order $t^2 < n^2$. So the total running time of the above algorithm is $n^{\mathcal{O}(t)}$.

Let's now analyze why this algorithm works.

Let's first do the analysis assuming that the graph $G = (V = [n], E)$ has a $t-$ clique $S \subseteq V$. Now let's look at all those vertices in $V \setminus S$ which are connected to each vertex in S and call it $\mathcal{N} = \mathcal{N}_S$. Let $\mathbf{1}_{v\in\mathcal{N}}$ be the indicator variable that is 1 if $v \in \mathcal{N}$, and 0 otherwise. So $|\mathcal{N}| = \sum_{v\in V\setminus S} \mathbf{1}_{v\in\mathcal{N}}$. Now $\mathbb{E}[|\mathcal{N}|] = \sum_{v\in V\setminus S} \mathbb{E}[\mathbf{1}_{v\in\mathcal{N}}] = \sum_{v\in V\setminus S} \mathbb{P}[v \in \mathcal{N}] = (1 + o(1))\dfrac{n-t}{2^t}$. We will see that the size of $\mathcal{N}$ is well-concentrated around its mean. Note that $\mathbf{1}_{v\in\mathcal{N}}$ (for $v \in V \setminus S$) are all independent. Chernoff gives $\mathbb{P}[|\mathcal{N}| \geq 2\mathbb{E}[|\mathcal{N}|]] \leq \exp\left\{-\mathcal{O}\left(\frac{n-t}{2^t}\right)\right\}$. So we can take $|\mathcal{N}| = \frac{2n}{2^t}$ with probability exponentially (in n) close to 1.

Recall that in class we have a constant C from lemma 2 in lecture 19 (second last page). Let $z \in \mathbb{N}$ be such that $2^z > C$.

- Say $G \sim G_{n,\frac{1}{2},k}$. Then the algorithm reaches some $(t+1+z)-$clique S for which $\mathcal{N}_S$ contains a $(k-s)-$clique (recall $s = t + 1 + z$). But $k - s > \sqrt{\frac{n}{2^t}} - s > 2C\sqrt{\frac{2(n-s)}{2^s}} = 2C\sqrt{|\mathcal{N}|_S}$ with probability $1 - \exp\left\{-\mathcal{O}\left(\frac{n-s}{2^s}\right)\right\}$. Therefore the failure chance in this case is inverse exponential in n.
- Say $G \sim G_{n,\frac{1}{2}}$. The chance that we get some S such that $|\mathcal{N}_S|$ is $> \frac{2n}{2^s}$ is inverse-exponential in $\frac{n}{2^s}$. Among the small $|\mathcal{N}_S|'s$, the success probability of the algorithm discussed in class is large because $k - s \geq 2C\sqrt{|\mathcal{N}_S|}$ By the lemma given on Ed discussion, $\mathbb{P}\left[\mathcal{A}_S \leq C\sqrt{|\mathcal{N}_S|}\right] \leq C'' \exp\{-C'n/2^s\}$. Therefore the chance that some S gives $\|\mathcal{A}_S\|_2 > C\sqrt{|\mathcal{N}|_S}$ is (by union bound) $\leq C''\binom{n}{s}\exp\{-C'n/2^s\}$.