Topology on $\mathbb{R}$

1 Open and closed sets

For every point $x \in \mathbb{R}$ we have the notion of an r-ball around x which is usually denoted by $\mathcal{B}(x, r)$ and defined as the set $\{y \in \mathbb{R} : |x - y| < r\}$. r is said to be the radius and x is said to be the center of this ball. From experience, one might observe that $\mathcal{B}(x, r) = (x - r, x + r)$.
Let $U \subset \mathbb{R}$. We say that U is an *open set* (or, open set in $\mathbb{R}$) if there is a family of open balls $\{\mathcal{B}(x_i, r_i)\}_{i \in I}$ (for some indexing set I) such that $U = \bigcup_{i \in I} \mathcal{B}(x_i, r_i)$. In other words, we say U is open if it is a union of open balls. We will simply call such sets 'open', instead of using the phrase 'open set'.
We say U is a *closed set* (or, closed set in $\mathbb{R}$) if $\mathbb{R} \setminus U$ is open (in $\mathbb{R}$). We will simply call such sets 'closed', instead of using the phrase 'closed set'.
It is important to note the following:

1. The empty set $\emptyset$ is considered open. One thinks of it as the empty union $\bigcup_{i \in I} \mathcal{B}(x_i, r_i)$ where $I = \emptyset$.
2. $\mathbb{R}$ is open because $\mathbb{R} = \bigcup_{i \in R} \mathcal{B}(x_i, r_i)$ where $x_i = i, r_i = 1 \forall i \in \mathbb{R}$.
3. $\mathbb{R}$ is open $\implies$ $\emptyset$ is closed.
4. $\emptyset$ is open $\implies$ $\mathbb{R}$ is closed.

Since open sets are defined to be union of open balls, one observes immediately that an (arbitrary) union of open sets is also a union of open intervals (those that made up the open sets, that we started with). Indeed if we have a collection of open sets $\{U_i\}_{i \in I}$, then notice that for each $i \in I$, there is a collection of open balls $\{V_j\}_{J_i}$ such that $U_i = \bigcup_{j \in J_i} V_j$. This gives us $\bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{i \in J_i} V_j$. Since closed sets are complements of open sets, one recalls deMoivre's theorems and immediately concludes that an (arbitrary) intersection of closed sets is closed. I invite the reader to prove it as an exercise that the intersection of two open sets is open, and the the union of two closed sets is closed. We thus have the following

Theorem 1
1. Open sets are closed under arbitrary union and finite intersection
2. Closed sets are closed under arbitrary intersection and finite union.

Why aren't arbitrary intersection of open sets open? We produce a counterexample. Consider the collection of open sets $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right)\right\}_{n \in \mathbb{N}}$. Their intersection is $\{0\}$ which is not open. Can you show an example to demonstrate that an arbitrary union of closed sets is not necessarily closed?

We introduce an equivalent notion of open-ness.

Proposition 2
Let $X \subseteq \mathbb{R}$. X is open iff $\forall x \in \mathbb{R} \exists r > 0$ such that $\mathcal{B}(x, r) \subseteq X$.

Proof. Suppose X is open. Then there is a family of open balls $\{\mathcal{B}(x_i, r_i)\}_{i \in I}$ such that $U = \bigcup_{i \in I} \mathcal{B}(x_i, r_i)$. $x \in X \implies x \in \mathcal{B}(x_i, r_i)$ for some $i \in I$. Take $r = \min\{x_i + r_i - x, x - (x_i - r_i)\}/2$. Note that $x \in$

$\mathcal{B}(x_i, r_i) = (x_i - r_i, x_i + r_i) \implies x_i - r_i < x < x_i + r_i \implies x - x_i + r_i > 0, x - (x_i - r_i) > 0$ whence $r > 0$.
Therefore $\mathcal{B}(x, r) \subseteq \mathcal{B}(x_i, r_i) \subseteq X$.
Suppose $\forall x \in \mathbb{R} \exists r(x) = r > 0$ such that $\mathcal{B}(x, r) \subseteq X$. Clearly $X = \bigcup_{x \in X} \mathcal{B}(x, r(x))$. ■

Definition (Relatively open and closed sets)**.** Let $X \subseteq \mathbb{R}$ and A be a subset of X.
We say that A is *open in* X if there is an open set $U \subset \mathbb{R}$ satisfying $A = U \cap X$.
We say that A is *closed in* X if $X \setminus A$ is *open in* X.

Let $A \subseteq X \subseteq \mathbb{R}$. It is not hard to observe that A is closed in X iff there is some closed $F \subseteq \mathbb{R}$ such that $A = X \cap F$. Indeed if $X \setminus A = X \cap U$ then $A = X \setminus U = X \cap (\mathbb{R} \setminus U)$, and $F := \mathbb{R} \setminus U$ is closed.

Example 1.

- $(0, 1)$ is an open set. In fact any open interval is an open set.
- Any closed interval is a closed set.
- $[0, 1]$ is open in $[0, 1]$ but not open in $\mathbb{R}$. In fact for any $X \subseteq \mathbb{R}$, X is clopen in X.

Relative openness and closedness are fairly 'well-behaved" in the following sense.

Lemma 3
Let $X \subseteq \mathbb{R}$ and $W \subseteq X$.

1. If X is open and W is open in X, W is open.
2. If X is closed and W is closed in X, W is closed.

Proof. Let X, W be as given.

1. W open in $X \implies W = X \cap U$ for some open U. W is open as it is the intersection of two open sets.
2. W closed in $X \implies W = X \cap F$ for some closed F. W is closed as it is the intersection of closed sets.

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2 Look at points for openness

2.1 Interior points

For $X \subseteq \mathbb{R}$ we say that $x \in \mathbb{R}$ is *an interior point* of X if $\exists \varepsilon > 0$ such that $\mathcal{B}(x, \varepsilon) \subseteq X$. We denote the set of interior points of X by X^o or $int(X)$ and call it the *interior of* X. By definition, $X^o \subseteq X$. This is a very important set (for given $X \subseteq \mathbb{R}$) because of the following.

Proposition 4
Let $X \subseteq \mathbb{R}$. X is open $\iff X = X^o$.

Proof. Suppose X is open. Then every point in X is an interior point, by definition, so that $X \subseteq X^o$. The other inclusion is true for any X. So $X = X^o$.
Suppose $X = X^o$. Then for every x point in X, there is an open ball centered at x contained in X, because x is an interior point. By the characterization of open sets (proposition 2), X is open. ■

Corollary 5
X^o is open for every $X \subseteq \mathbb{R}$.

2.2 Interior of a set

Let $X \subseteq \mathbb{R}$. We have already seen what X^o means. The article 'the' makes this a special open set. Here's why. Notice that if $A \subseteq B$ and if $x \in A^o$, so that $\mathcal{B}(x, r) \subseteq A \subseteq B$ whence x is also an interior point of B. This just shows that $A \subseteq B \implies A^o \subseteq B^o$.

Proposition 6
Let $X \subseteq \mathbb{R}$. If U is any open set such that $U \subseteq X$, then $U \subseteq X^o$.

Proof. Let U, X be as given. Then $U = U^o \subseteq X^o$. ■

Corollary 7
For any $X \subseteq \mathbb{R}$ we have $X^o = \bigcup_{\substack{U \subseteq X \\ U \text{open}}} U$.

3 Look at points for closedness

3.1 Limit points

For $X \subseteq \mathbb{R}$ we say that $x \in \mathbb{R}$ is *a limit point* of X if $\forall \varepsilon > 0$, $(\mathcal{B}(x,\varepsilon) \cap X) \setminus \{x\} \neq \emptyset$. We denote the set of limit points of X as X' and define the closure of X as $\overline{X} := X \cup X'$.
Let $x \in \mathbb{R}$ be a limit point of X. We construct a sequence (a_n) as follows: Let a_n to be any element of $\left(\mathcal{B}\left(x, \frac{1}{n}\right) \cap X\right) \setminus \{x\} \neq \emptyset$. Clearly this satisfies that $a_n \neq x \forall n$ and $\lim a_n = x$.
Conversely say $x \in \mathbb{R}$ is such that we can find a sequence (a_n) in X satisfies $\lim\limits_{n\to\infty} a_n = x$ and $a_n \neq x \forall n$. Clearly for any $\varepsilon > 0 \exists n \in \mathbb{N}$ such that $x_n \in \mathcal{B}(x,\varepsilon)$. Further, $x_n \in X \setminus \{x\}$. It follows that $(\mathcal{B}(x,\varepsilon) \cap X) \setminus \{x\} \neq \emptyset$.
The above argument establishes the following lemma:

Lemma 8
Let $X \subseteq \mathbb{R}$. $x \in \mathbb{R}$ is a limit point of X iff there is a sequence $(a_n) \in X^{\mathbb{N}}$ such that $a_n \neq x \forall n$ and $\lim a_n = x$.

The above characterization enables us to prove the following

Proposition 9
Let $X \subseteq \mathbb{R}$. We then have the following

1. $\overline{X} = \left\{x \in \mathbb{R} : x \text{ is the limit of some sequence in } X^{\mathbb{N}}\right\}$.

2. $\overline{X} = \{x \in \mathbb{R} : X \cap \mathcal{B}(x,r) \neq \emptyset \forall r > 0\}$.

Proof.

1. Denote by $S := \left\{x \in \mathbb{R} : x \text{ is the limit of some sequence in } X^{\mathbb{N}}\right\}$. If $x \in X$, the constant sequence $(x) \in X^{\mathbb{N}}$ converges to X and does the job. If $x \in X'$, there is a sequence in $X^{\mathbb{N}}$ which converges to x, by the equivalent characterization of limit points in lemma 8. It follows that $\overline{X} = X \cup X' \subseteq S$.
   Now say $x \in S$. Then there is a sequence $(a_n) \in X^{\mathbb{N}}$ converging to x. If $a_n = x$ for some n, then $x = a_n \in X$. Otherwise, $x \in X'$ by the characterization of limit points. Therefore $S \subseteq X \cup X' = \overline{X}$.

2. Denote by $T := \{x \in \mathbb{R} : X \cap \mathcal{B}(x,r) \neq \emptyset \forall r > 0\}$. If $x \in X$ we have $x \in X \cap \mathcal{B}(x,r) \forall r > 0$ whence $X \cap \mathcal{B}(x,r) \neq \emptyset \forall r > 0$. If $x \in X'$ then $(\mathcal{B}(x,r) \cap X) \setminus \{x\} \neq \emptyset$ by definition whence $X \cap \mathcal{B}(x,r) \neq \emptyset \forall r > 0$. It follows that $\overline{X} = X \cup X' \subseteq T$.
   Say $x \in T$. If $x \in X$, we're done. Suppose $x \in T \setminus X$. Then $\emptyset \neq X \cap \mathcal{B}(x,r) = (X \cap \mathcal{B}(x,r)) \setminus \{x\}$ whence $x \in X'$ by definition. it follows that $T \subseteq X \cup X' = \overline{X}$

■

Limit points help in characterizing closed sets beautifully.

Proposition 10
Let $F \subseteq \mathbb{R}$. F is closed $\iff F' \subseteq F$.

Proof. Suppose F is closed. Let $x \in F' \setminus F$. Then $\exists r > 0$ such that $\mathcal{B}(x,r) \subseteq \mathbb{R} \setminus F \because F$ is closed. This means that $|x - y| \geq r \forall y \in F$. In other words, no F-valued sequence can converge to x. This means that $F' \setminus F = \emptyset$ whence $F' \subseteq F$.
Suppose $F' \subseteq F$. So $\overline{F} = F$. Let $x \in U := \mathbb{R} \setminus F$. Then $\exists r > 0$ such that $F \cap \mathcal{B}(x,r) = \emptyset$, by contrapositive of proposition 9. In other words, $\mathcal{B}(x,r) \subseteq U$. So F is closed. ■

Corollary 11
Let $F \subseteq \mathbb{R}$. F is closed $\iff F = \overline{F}$.

Corollary 12
Let $F \subseteq \mathbb{R}$. $\overline{F}$ is closed.

3.2 Closure of a set

Let $X \subseteq \mathbb{R}$. We have defined its closure $\overline{X}$ above (corollary 12). The term 'closure' is there for a reason. We have already seen why.
If you've read carefully, we used the article 'the' closure of a set X which makes it a special set, in some sense - a similar reason as why the interior is a special open set. What I mean is, given any set X, the closure of X is a special closed set containing X. This is established by the following proposition.

Proposition 13
Let $X \subseteq \mathbb{R}$. If F is any closed set such that $X \subseteq F$ then $\overline{X} \subseteq F$. In other words, $\overline{X}$ is the smallest closed set containing X.

Proof. Let X, F be as given. An X-valued sequence is also an F-valued sequence. So $X' \subseteq F'$. It follows that $\overline{X} = X \cup X' \subseteq F \cup F' = F$. ■

Corollary 14
For any $X \subseteq \mathbb{R}$ we have $\overline{X} = \bigcap_{\substack{F \supseteq X \\ F \text{closed}}} F$.