Real Analysis

Problem Set 4

June 03, 2021

1. Denote $A \sim B$ iff A and B are equicontinuous. Prove the following:
   (a) $A \sim A$ for all sets A.
   (b) If $A \sim B$ then $B \sim A$, for all sets A, B.
   (c) If $A \sim B$ and $B \sim C$ then $A \sim C$, for all sets A, B, C.

2. Prove the following.
   (a) $\mathbb{N} \nsim \mathbb{R}$
   (b) $(0,1) \sim (0,1] \sim [0,1]$
   (c) $(0,1) \sim \mathbb{R}$
   (d) $2^{\mathbb{N}} \sim \mathbb{R}$
   (e) $\mathbb{R}^{\mathbb{N}} \nsim \mathbb{N}$
   (f) $\mathbb{R}^{(\mathbb{N})} \sim \mathbb{N}$
   (g) $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$
   (h) $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

3. Let S be any set. Show that f is not a surjection $\forall f \in \mathscr{P}(S)^S$.

4. Let $S \subseteq \mathbb{R}$ be a finite set. Show that $S' = \emptyset$.

5. Let $X = (x_n) \in \mathbb{R}^{\mathbb{N}}$. And let X' be the set of its limit points in $\mathbb{R}$. Let $q \in \mathbb{R}$ be a limit point of X' in $\mathbb{R}$. Show that $q \in X'$.

6. For any set $S \subseteq \mathbb{R}$ define the diameter of S as $\operatorname{diam}(S) \coloneqq \sup \left\{|x-y| : x, y \in S\right\}$.
   (a) Let $X = (x_n) \in \mathbb{R}^{\mathbb{N}}$. Show that X is Cauchy iff $\lim\limits_{n\to\infty} \operatorname{diam}\left(\{x_k : k \geq n\}\right) = 0$.
   (b) Let $S \subseteq \mathbb{R}$ be a bounded set. Show that $\operatorname{diam}\left(S' \cup S\right) = \operatorname{diam}(S)$.

7. Let $X \in \mathbb{R}^{\mathbb{N}}$ be monotone. Show that X converges iff it is bounded.

8. Let $x_n = \frac{(-1)^n}{1+\frac{1}{n}}$. Compute $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$.

9. Let $X = (x_n) \in \mathbb{R}^{\mathbb{N}}$. Show that $\lim\limits_{n\to\infty} x_n = a \in \overline{\mathbb{R}}$ iff $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n = a$.

10. For $a \in \mathbb{R}$ what is $\lim\limits_{n\to\infty} n^a$?

11. For $a \in \mathbb{R}$ what is $\lim\limits_{n\to\infty} a^{\frac{1}{n}}$?

12. Let $x_1 = \sqrt{2}$ and inductively define $x_{n+1} = \sqrt{2+\sqrt{x_n}}$. Show that $x_n < 2$. Show that x_n converges.

13. Let $x_1 = 0$ and inductively define

$$x_n = \begin{cases} \frac{x_{\frac{n}{2}-1}}{2} & \text{if } 2 \mid n \\ \frac{1}{2} + x_{n-1} & \text{otherwise} \end{cases}$$

Find $\limsup_{n\to\infty} x_n$, $\liminf_{n\to\infty} x_n$ converges.

14. Let J be an indexing set and $\{A_\alpha : \alpha \in J\}$ be a collection of sets, indexed by J. Describe the following sets:

$$\bigcap_{\alpha\in J} A_\alpha \qquad \bigcup_{\alpha\in J} A_\alpha$$

15. Let $f : X \to Y$ be a function and $A, B \subseteq X$, $C, D \subseteq Y$. Define $f^{-1}(Y') := \{x \in X : f(x) \in C\}$ for any $Y' \subseteq Y$. For $y \in Y$ define $f^{-1}(y) := f^{-1}(\{y\})$. Show that

   (a) $f(A \cup B) = f(A) \cup f(B)$
   (b) $f(A \cap B) \subseteq f(A) \cap f(B)$
   (c) $A \subseteq f^{-1}(f(A))$
   (d) $f(f^{-1}(C)) \subseteq C$
   (e) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
   (f) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
   (g) $f^{-1}(C^c) = (f^{-1}(C))^c$ where $C^c = Y \setminus C$, $(f^{-1}(C))^c = S \setminus f^{-1}(C)$

16. Let $x_1 = a$, $x_2 = b$. Inductively define $a_{n+1} = \dfrac{a_n + a_{n+1}}{2} \forall n \geq 1$. Show that x_n converges. Further show that the limit of this sequence is $\frac{a+2b}{3}$.

17. Let $a_1 = b_1 = 1$. Inductively define $a_{n+1}, b_{n+1} \in \mathbb{Z}$ to be such that $a_{n+1} + b_{n+1}\sqrt{2} = \left(a_n + b_n\sqrt{2}\right)^2$. Show that $a_n - 2b_n^2 = 1 \forall n$. Deduce that $\frac{a_n}{b_n}$ converges to $\sqrt{2}$.

18. Let $a \in (0, 1)$ and $x_1 = a$. Inductively define $x_{n+1} = \sqrt{1 - x_n^2}\ \forall n \geq 1$. Prove that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{2}$.

19. Let $x \in [0, 1]$. Define a sequence as $a_{n,m} = \cos^{2n}(m!\pi x)$. Show that $\lim_{m\to\infty} \lim_{n\to\infty} a_{n,m}$ exists in all cases.
Hint: Consider separate cases for $x \in [0, 1] \cap \mathbb{Q}$ and $x \in [0, 1] \setminus \mathbb{Q}$.

20. For some $x_1 \in \mathbb{R}$ define a sequence inductively by $x_{n+1} = x_n^3 + 6$. if $x_1 = \frac{1}{2}$, prove that the sequence converges. What happens if $x_1 = \frac{3}{2}$ or $x_2 = \frac{5}{2}$?