CONVEX AND CONIC OPTIMIZATION

Homework 1

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February 27, 2024

Problem 1

Let $S \subseteq \mathbb{R}^n$.

1. The convex hull of S is the intersection of all convex sets that contain S.
2. If S is closed, then the convex hull of S is closed.
3. If S is bounded, then the convex hull of S is bounded.
4. If S is compact, then the convex hull of S is compact.
5. The sum of two quasiconvex functions is quasiconvex.
6. A quadratic function $f(x) = x^\top Qx + b^\top x + c$ is convex if and only if it is quasiconvex.
7. Any closed convex set $\Omega \subseteq \mathbb{R}^n$ can be written as $\Omega = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ for some convex function $g : \mathbb{R}^n \to \mathbb{R}$.
8. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex on a convex set $S \subseteq \mathbb{R}^n$, then f is continuous on S.
9. Suppose $P \in \mathbb{R}^{n\times n}$ is a matrix with nonnegative entries whose columns each sum up to one. Then, there exists $x \in \mathbb{R}^n$ such that $Px = x, x \geq 0$,and $\sum_i x_i = 1$.
10. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the midpoint convexity property
$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \forall x, y \in \mathbb{R}^n$$
is convex.

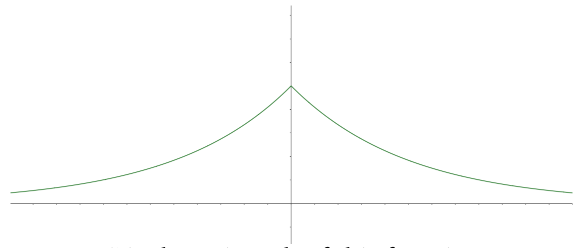
Solution

1. **True**.

   Let $\mathcal{C} = \text{conv}(S)$ as defined in class (collection of convex combinations) and let $\mathcal{X}$ be the intersection of all convex sets that contain S.
   It is clear that $S \subseteq \mathcal{C}$ whence $\boxed{\mathcal{X} \subseteq \mathcal{C}}$ by the description of $\mathcal{X}$.
   Now, if T is a convex set containing S, then any convex combination x of points in S is a convex combination of points in T, so $x \in T$. This shows that $\mathcal{C} \subseteq T$. Again by description of $\mathcal{X}$, we have $\boxed{\mathcal{C} \subseteq \mathcal{X}}$.

2. **False**.

Consider $S = \{(x,y) \in \mathbb{R}^2 \mid y \geq e^{-|x|}\}$. We'll show that $\mathcal{C} := \operatorname{conv}(S)$ is $\mathcal{X} := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. Clearly $S \subseteq \mathcal{X}$ because exponential takes only positive values, so $\boxed{\mathcal{C} = \operatorname{conv}(S) \subseteq \operatorname{conv}(\mathcal{X}) = \mathcal{X}}$, where the last equality is true because $\mathcal{X}$ is convex (if two points have positive $y-$coordinate, so do their convex combinations).
On the other hand, say $(a,b) \in \mathcal{X}$, pick a large enough θ such that $\min(|a+\theta|, |a-\theta|) > \ln \frac{1}{b}$. This ensures that $e^{-|a\pm\theta|} < b$ so that $(a\pm\theta, b) \in S$ and thus their average $(a,b) \in \mathcal{C}$. This proves that $\boxed{\mathcal{X} \subseteq \mathcal{C}}$.
So $\mathcal{X} = \operatorname{conv}(S)$ where S is closed but $\mathcal{X}$ (the strict upper half plane) is not.



S is the epigraph of this function

3. **True**.

   S is bounded, so $\exists \delta > 0$ such that $S \subseteq B(0,\delta)$. Recall that $B(0,\delta)$ is convex. This implies that $\operatorname{conv}(S) \subseteq \operatorname{conv}(B(0,\delta)) = B(0,\delta)$, whence $\operatorname{conv}(S)$ is bounded.

4. **True**.

   We'll use the fact that a set in Euclidean space is compact iff it is sequentially compact. We say that a set T is sequentially compact if every sequence of points in T has a convergent subsequence (with limit in T). $\boxed{\text{We will use a } \bullet \text{ in subscript to suppress the lower index of a sequence.}}$
   Assume S is compact (hence sequentially compact). It is also important to note that $[0,1]$ is (sequentially) compact. Consider a sequence of points $\{a_k\}_{k\in\mathbb{N}}$ in $\operatorname{conv}(S)$. By Caratheodory's theorem, for each a_k, there are $n+1$ non-negative reals $\lambda_1^{(k)}, \cdots, \lambda_{n+1}^{(k)} \in [0,1]$ adding to 1 and $n+1$ points $x_1^{(k)}, \cdots, x_1^{(k)} \in S$ such that $a_k = \sum_{i=1}^{n} \lambda_i^{(k)} x_i^{(k)}$ (I want to emphasize again that this is for each a_k).
   Consider the sequence $\left\{\lambda_k^{(1)}\right\}_{k\in\mathbb{N}}$. This is a sequence in the compact unit interval and thus has a convergent subsequence, say $\left\{\lambda_{d_k^{(1)}}^{(1)}\right\}_{k\in\mathbb{N}}$. Let the limit of this subsequence be $\lambda^{(1)}$. Next note that $\left\{x_{d_k^{(1)}}^{(1)}\right\}_{k\in\mathbb{N}}$ is a sequence in the compact set S and thus has a convergent subsequence, say $\left\{x_{\tilde{d}_k^{(1)}}^{(1)}\right\}_{k\in\mathbb{N}}$. So $\left\{\tilde{d}_k^{(1)}\right\}_{k\in\mathbb{N}}$ is a subsequence of $\left\{d_k^{(1)}\right\}_{k\in\mathbb{N}}$, and thus does not affect the convergence of $\lambda_{\tilde{d}_k^{(1)}}^{(1)}$. Let the limit of $\left\{x_{\tilde{d}_k^{(1)}}^{(1)}\right\}_{k\in\mathbb{N}}$ be $x^{(1)} \in S$. Notice that $\lim_{k\to\infty} \lambda_{\tilde{d}_k^{(1)}}^{(1)} x_{\tilde{d}_k^{(1)}}^{(1)} = \lambda^{(1)} x^{(1)}$, which happens because

$$\begin{aligned} \left\| \lambda_{\tilde{d}_k^{(1)}}^{(1)} x_{\tilde{d}_k^{(1)}}^{(1)} - \lambda^{(1)} x^{(1)} \right\| &\leq \left\| \lambda_{\tilde{d}_k^{(1)}}^{(1)} x_{\tilde{d}_k^{(1)}}^{(1)} - \lambda_{\tilde{d}_k^{(1)}}^{(1)} x^{(1)} \right\| + \left\| \lambda_{\tilde{d}_k^{(1)}}^{(1)} x^{(1)} - \lambda^{(1)} x^{(1)} \right\| \\ &= \lambda_{\tilde{d}_k^{(1)}}^{(1)} \underbrace{\left\| x_{\tilde{d}_k^{(1)}}^{(1)} - x^{(1)} \right\|}_{\text{can be made arbitrarily small}} + \underbrace{\left| \lambda_{\tilde{d}_k^{(1)}}^{(1)} - \lambda^{(1)} \right|}_{\text{can be made arbitrarily small}} \left\| x^{(1)} \right\|. \end{aligned}$$

   By following a similar procedure, one can further extract a subsequence $\left\{d_k^{(2)}\right\}_{k\in\mathbb{N}}$, which makes $\lambda_\bullet^{(2)}$ converge to $\lambda^{(2)}$, and a further sub-subquence $\left\{\tilde{d}_k^{(2)}\right\}_{k\in\mathbb{N}}$ which makes $x_\bullet^{(2)}$ converge to $x^{(2)}$. Note that this smaller sequence does not affect the convergence of $\lambda_\bullet^{(1)}$ and $x_\bullet^{(1)}$ (and also their product). By the same argument as above, $\lim_{k\to\infty} \lambda_{\tilde{d}_k^{(2)}}^{(2)} x_{\tilde{d}_k^{(2)}}^{(2)} = \lambda^{(2)} x^{(2)}$.

We do this (extracting subsequences in turn for convergence of subsequences of $\lambda_\bullet^{(1)}, x_\bullet^{(1)}, \lambda_\bullet^{(2)}, x_\bullet^{(2)}, \lambda_\bullet^{(3)}, \cdots$) for a total of $n+1$ times, and finally get a subsequence $\left\{\tilde{d}_k^{(n+1)}\right\}_{k\in\mathbb{N}}$ which ensures convergence of $\lambda^{(i)}_{\tilde{d}_k^{(n+1)}} \xrightarrow{k\to\infty} \lambda^{(i)}$ and $x^{(i)}_{\tilde{d}_k^{(n+1)}} \xrightarrow{k\to\infty} x^{(i)}$ for every $i = 1, \cdots, n+1$. It thus stands that $\lambda^{(i)} \in [0,1]$, $x^{(i)} \in S \forall i$ and $\sum_{i=1}^{n+1} \lambda^{(i)} = \sum_{i=1}^{n+1} \lim_{k\to\infty} \lambda^{(i)}_{\tilde{d}_k^{(n+1)}} = \lim_{k\to\infty} \sum_{i=1}^{n+1} \lambda^{(i)}_{\tilde{d}_k^{(n+1)}} = \lim_{k\to\infty} 1 = 1$ where the sum and limit could be exchanged because each limit exists. Thus $\left\{a_{\tilde{d}_k^{(n+1)}}\right\}_{k\in\mathbb{N}}$ is a subsequence of $\{a_k\}_{k\in\mathbb{N}}$ which converges to $a := \sum_{i=1}^{n+1} \lambda^{(i)} x^{(i)}$. $a \in \text{conv}(S)$ because each $x^{(i)} \in S$ by construction and the $\lambda^{(i)}$'s form a convex weight for the $x^{(i)}$'s.

5. **False**.

   Consider $f(x) = x^3, g(x) = x^2$. Then f is quasiconvex with sublevel sets $S_\alpha(f) = \left(-\infty, \alpha^{\frac{1}{3}}\right]$ and g is quasiconvex with sublevel sets $S_\alpha(g) = \begin{cases} [-\sqrt{\alpha}, \sqrt{\alpha}] & \text{if } \alpha \geq 0 \\ \varnothing & \text{otherwise} \end{cases}$. But $(f+g)(x) = x^3 + x^2$ is not quasiconvex. Indeed, $f + g \leq 0 \iff x^2(x+1) \leq 0 \iff x = 0$ or $x \leq -1$, so $S := S_0(f+g) = (-\infty, -1] \cup \{0\}$ which is not convex because $\frac{-1+0}{2} = \frac{-1}{2} \notin S$ even though $0, -1 \in S$.

6.

7.

8. **False**.

   The counterexample is obtained by taking $S = [0,1] \subseteq \mathbb{R}$ and $f(x) = \mathbf{1}_{x=1} = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{otherwise} \end{cases}$. In each dimension, there is a counterexample, namely $S = [0,1]^n \subseteq \mathbb{R}^n$ and $f = \mathbf{1}_{x=(1,\cdots,1)}$. S is clearly convex. The function is convex because it is constant 0 except for one point and if $p = (1, \cdots, 1)$ and $q \in S \setminus \{p\}$ then for any $\lambda \in (0,1]$ we have $\lambda p + (1-\lambda) q \neq (1, \cdots, 1)$ whence $f(\lambda p + (1-\lambda)q) = 0 \leq \lambda = \lambda f(p) + (1-\lambda) f(q)$. But clearly f is not continuous.

9.

10. **True**.

    Say f is midpoint convex (that is, satisfies the given condition).
    Consider epi$(f) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq t\}$. Consider a sequence $\{(x_n, t_n)\}_{n\in\mathbb{N}}$ of points in epi(f) that converge to $(a,s) \in \mathbb{R}^n \times \mathbb{R}$. This implies $\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} t_n = s$. By choice of points, $t_n \geq f(x_n)$ whence $\lim_{n\to\infty} t_n \geq \lim_{n\to\infty} f(x_n) \overset{f \text{ continuous}}{=} f(a)$. By definition, $(a,s) \in$ epi(f). Thus, epi(f) is closed.
    Say $(x,t), (y,u) \in$ epi(f). So $u \geq f(y), t \geq f(x)$. Then by midpoint convexity of f, we have $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \leq \frac{u+t}{2}$ and, by definition of epi(f), this implies $\left(\frac{x+y}{2}, \frac{u+t}{2}\right) \in$ epi(f). So epi(f) is mid-point convex.
    Since epi(f) is closed and midpoint convex, epi(f) is convex. This implies f is convex.