1) Given a vector space V, $\varphi : V \to V$ a linear map if $\{\lambda_i\}_{i=1}^{k}$ are distinct eigenvalues with corresponding eigenvectors $\{v_i\}_{i=1}^{k}$, then

$$\{v_i\}_{i=1}^{k} \text{ is L.I.}$$

Thus if $\dim V = n$, atmost n distinct eigenvalue of φ exists

2) i) If V has an eigenbasis $\mathcal{B}$ for some linear map φ, then

$$M_{\mathcal{B}}^{\mathcal{B}}(\varphi) = \operatorname{diag}(\underbrace{\lambda_i}_{\text{eigenvalues}})$$

ii) Let $M_{\mathcal{C}}^{\mathcal{C}}(\varphi) = A$. Then V has an eigenbasis wrt. φ iff $A = PBP^{-1}$ for some $B = \operatorname{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$

Lemma :- The following statements are equivalent. $T : V \to V$, linear, $\dim V = n$

i) λ is an eigenvalue of T

ii) $\lambda I_n - T$ is singular / non-invertible

iii) $\det(\lambda I_n - T) = 0$

A is inv. iff $\det A \neq 0$

$A = [a_{ij}]_{i=1, j=1}^{n}$ $\quad \det(A) \quad \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right)$

$$M_{\mathcal{B}}^{\mathcal{B}}(T) \circlearrowright M_{\mathcal{C}}^{\mathcal{C}}(T)$$

$$\sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \left[a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \dots \cdot a_{n\sigma(n)} \right]$$

i $\Leftrightarrow$ ii

λ, $v \neq 0$ $\quad Tv = \lambda v$ $\qquad I_n : V \longrightarrow V$, $\; v \longrightarrow v$

$\Rightarrow (\lambda \cdot I_n - T) v = 0$

$(\lambda I_n - T)$ not inventible $\qquad \lambda I_n - T : V \longrightarrow V$

$(\lambda I_n - T)$ not injective

$(\lambda I_n - T) v = 0$, $\quad v \neq 0$

$\Rightarrow T(v) = \lambda v$

ii $\Leftrightarrow$ iii

A is singular $\Leftrightarrow \det(A) = 0$

$\det(x I_n - T) = p(x) \in F[x]$

Characteristic polynomial.

$p(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k}$ $\qquad \lambda_1, \dots, \lambda_k$ distinct eigenvalues

n

Algebraic multiplicity of λ_1

Eigenspace :-

$T : V \to V$, λ is an eigenvalue of T

$E_\lambda = \{ v \in V \text{ s.t. } T(v) = \lambda v \}$ $\qquad v \to \lambda$, $\; \mu v \to \lambda$

The eigenspace of λ.

$\dim(E_\lambda) =$ Geometric multiplicity of λ.

Algebraic multiplicity : as a root of char poly.

$GM \leq AM$

Theorem :- Algebraic Multiplicity $\geq$ Geometric Multiplicity :-

Proof → λ. $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$E_\lambda = \langle \{x_1, x_2, \dots, x_r\} \rangle$$

B

r

$$(x-\lambda)^r \mid p(x)$$

$$\{x_1, \dots, x_r, x_{r+1}, \dots, x_n\} = B' \text{ of } \mathbb{R}^n$$

$$M_{B'}^{B'}(T) = \left[(Tx_1)_{B'}, (Tx_2)_{B'} \cdots (Tx_r)_{B'}, (Tx_{r+1})_{B'} \cdots (Tx_n)_{B'} \right]$$

$$Tx_1 = \lambda x_1 \quad \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda - x & 0 & \cdots & 0 & \\ 0 & \lambda - x & & 0 & \\ 0 & 0 & & 0 & \\ \vdots & \vdots & & \lambda - x & \\ & & & 0 & \\ 0 & 0 & & 0 & \end{bmatrix} = Y$$

$$(\lambda - x)^r \mid \text{char poly of } T$$

$$M_{st}^{st}(T) = A \qquad M_{st}^{st}(T) = A$$

$$\det(xI - A) = p(x) \qquad M_{B'}^{B'}(T)$$

$$M_{B'}^{B'}(T) = S^{-1} A S$$

$$S = M_{B'}^{std}(id)$$

$$\begin{aligned} M_{B'}^{B'}(T) &= \det(x I_n - S^{-1} A S) \\ &= \det(S^{-1}(x I_n) S - S^{-1} A S) \\ &= \det(S^{-1}(x I_n - A) S) \\ &= \det(S^{-1}) \det(x I_n - A) \det(S) \\ &= \det(x I_n - A) \end{aligned}$$

Lemma :- If $A = PBP^{-1}$, $\det(xI_n - A) = \det(xI_n - B)$

chan poly of Y = $\det(xI - A)$

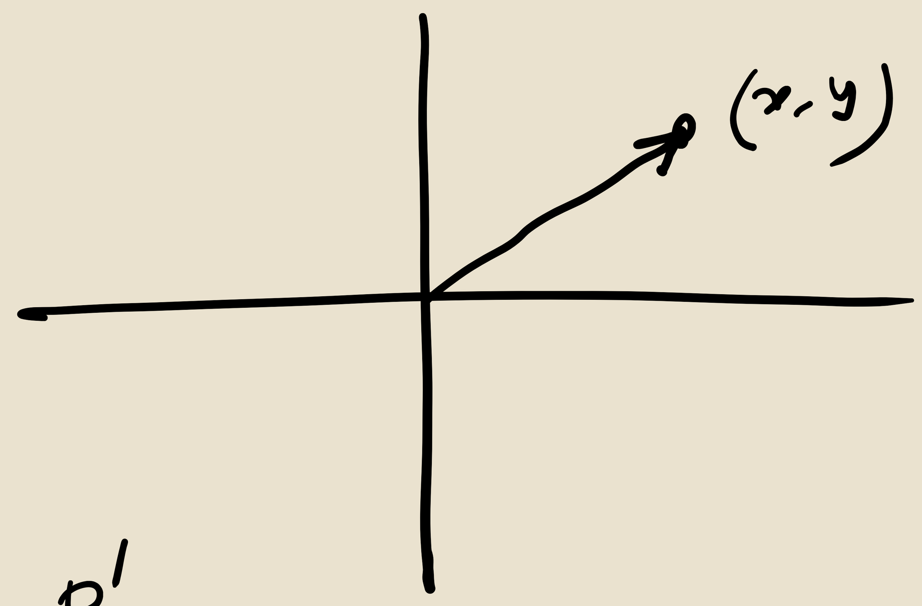
$T(x, y) = x$

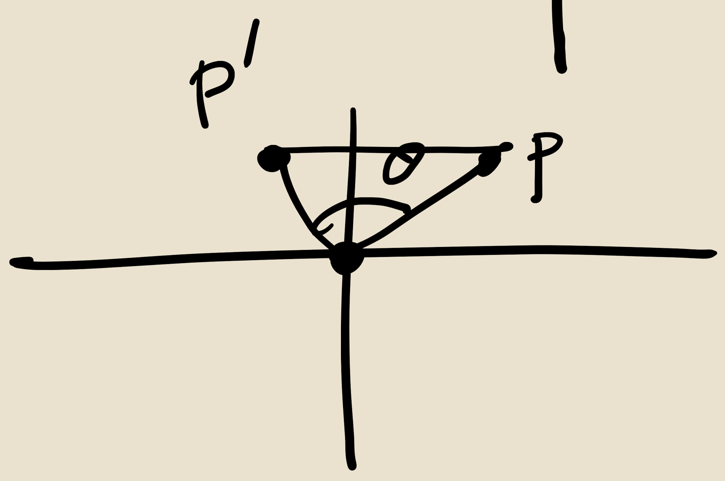
$T : \mathbb{R}^2 \to \mathbb{R}^2$

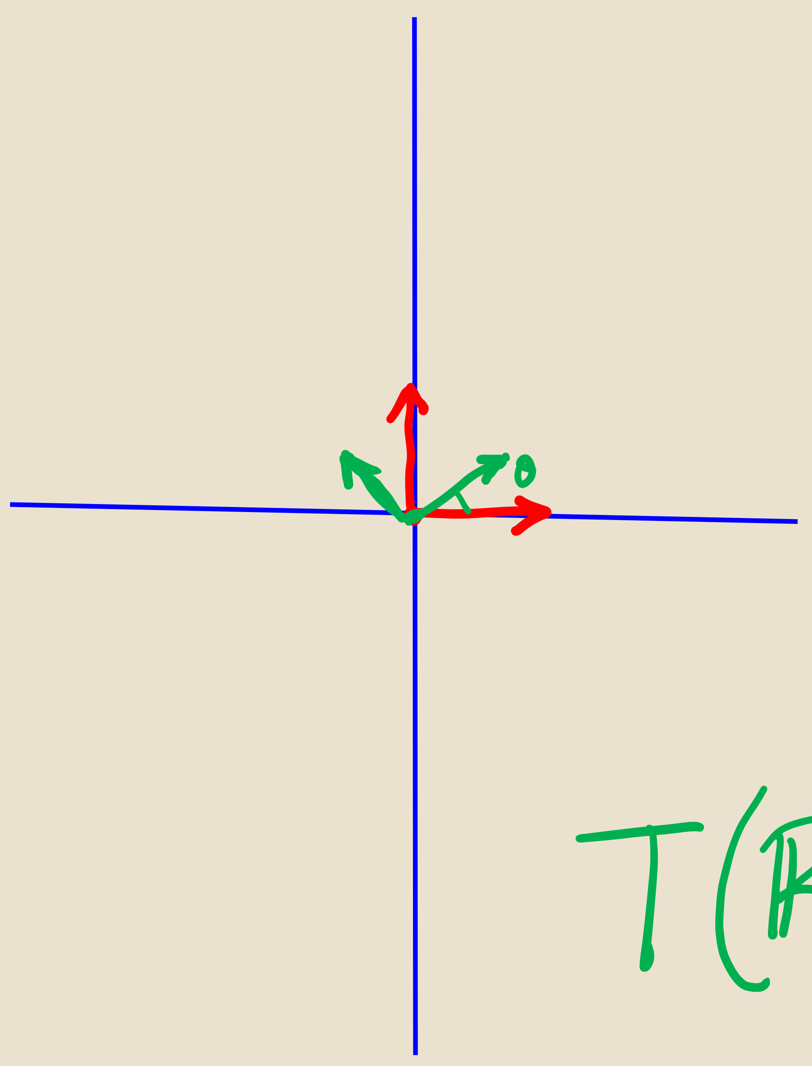
$\mathbb{R}^2 \to \mathbb{R}$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\underset{P}{T\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{P'}{\begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}}$$



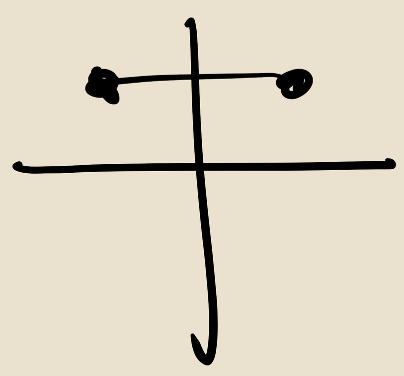




$T(\mathbb{R}^2)$

$T(x,y) = (kx, ky)$ $k > 1$

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$



< >

$T(x,y) = (-x, y)$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T(v) = \lambda v$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$p(t) = (1 - t)(t^2 - (2\cos\theta)t + 1) = 0.$$

$$1, \cos\theta \pm i\sin\theta.$$