Mathematical proofs, like diamonds, are hard as well as clear, and will be touched with nothing but strict reasoning.
—John Locke

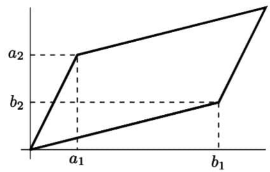
PROMYS Number Theory **Problem Set #6** *Boston University, July 12, 2021*

Reading Search

Q1: What is the least common multiple of two integers?

Exploration

P1: The parallelogram below is determined by the vector $\mathbf{a} = (a_1, a_2)$ with components a_1 and a_2, and the vector $\mathbf{b} = (b_1, b_2)$ with components b_1 and b_2. The area of this parallelogram is equal to the absolute value of the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$.



Prove or Disprove and Salvage if Possible

P2: If d is the smallest positive value of $ax + by$ for integral values of x and y, then $d = (a, b)$. True in $\mathbb{Z}$.

P3: Given integers a, b, and d, then d is a common divisor of a and $b \Leftrightarrow d$ is a divisor of (a, b).

P4: If $a|c$ and $b|c$ then $ab|c$. True in $\mathbb{Z}$.

P5: If p is a prime and $p|ab$ then $p|a$ or $p|b$. True in $\mathbb{Z}$.

P6: Let m be the least common multiple of two integers a and b. Then every common multiple of a and b is divisible by m.

P7: If (x_0, y_0) is an integral solution of the Diophantine equation $ax + by = c$, where a, b, and c are integers, then all the integral solutions (x, y) of this equation are given by the formulas $x = x_0 + bt,\ y = y_0 - at$ for $t \in \mathbb{Z}$.

P8: Every rational integer > 1 is a product of positive primes.

Numerical Problems (*Some food for thought*)

P9: Find an integral solution (x, y) of the Diophantine equation $5391x + 3976y = 11$.

P10: Expand $\frac{5391}{3976}$ into a simple continued fraction and make use of the process given in P11, Set #3 (and also in P1, Set #5) for calculating the values of the successive convergents. Compare the actual value of the difference between each convergent $\frac{P_n}{Q_n}$ and the given fraction, with $\frac{1}{Q_n^2}$, with $\frac{1}{Q_n Q_{n+1}}$. Any conjectures?

P11: Find all the positive integral solutions (x, y) of the Diophantine equation $5391x + 3976y = 4,000,000$.

P12: Multiply $2x^3 + 3x^2 + x + 4$ by $3x^2 + 2x + 2$ in $\mathbb{Z}_5[x]$. Multiply the same two polynomials in $\mathbb{Z}_6[x]$. In each case compare the degrees of the factors with the degree of the product. Explain.

P13: Compute the following expressions. What conjectures are you prepared to make?

$$\frac{\sigma(5)}{5},\quad \frac{\sigma(6)}{6},\quad \frac{\sigma(28)}{28},\quad \varphi(7),\quad \varphi(8),\quad \varphi(45),\quad \sum_{\substack{d|30\\d>0}} \varphi(d),\quad \sum_{\substack{d|36\\d>0}} \varphi(d),\quad \sum_{\substack{d|35\\d>0}} \varphi(d),$$

$$\sum_{\substack{d|5\\d>0}} \frac{1}{d},\quad \sum_{\substack{d|6\\d>0}} \frac{1}{d},\quad \sum_{\substack{d|28\\d>0}} \frac{1}{d},\quad \sum_{\substack{d|4\\d>0}} \mu(d),\quad \sum_{\substack{d|15\\d>0}} \mu(d),\quad \sum_{\substack{d|225\\d>0}} \mu(d).$$