Recap

$(x_n) \in \mathbb{R}^{\mathbb{N}}$. Define $s_n := \sum_{i=1}^{n} x_i$.

$\sum x_n = x \iff \lim s_n = x \qquad (x \in \overline{\mathbb{R}})$

If $\sum x_n \in \mathbb{R}$ then $x_n \to 0$.

If $x_n \geq 0$ then $\sum x_n \in \mathbb{R} \cup \{\infty\}$

Fact: $x_n \geq 0$. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijection. Let $y_n = x_{\pi(n)}$. (y_n is called a rearrangement of x_n).

Then we have $\sum x_n = \sum y_n$.

$s_n' := \sum_{i=1}^{n} y_i$

$s_n := \sum_{i=1}^{n} x_i$

Pf: For all $n \geq 1$, $\exists\, m \geq 1$ s.t.

$$0 \leq s_n' = y_1 + \cdots + y_n$$
$$\leq x_1 + \cdots + x_{n+m}$$
$$= s_{n+m}$$

Case: $\sum x_n = x \in \mathbb{R}$. Let $n \geq 1$. Then $\exists\, m \geq 1$ satisfying

$$0 \leq s_n' \leq s_{n+m} \leq x$$

$\therefore \{s_n'\}$ is inc + bdd

$\Rightarrow s_n'$ converges (to, say, y).

Moreover $y \leq x$.

Notice that (x_n) is a rearrangement of (y_n)

Why? $x_n = y_{\pi^{-1}(n)}$ & $\pi^{-1} : \mathbb{N} \to \mathbb{N}$ is a bij.

From above ($\because \sum y_n$ converges to y) we can conclude (by exchanging roles of x_n & y_n) that $x \leq y$.

Combining the above we have: $x = y$.

Case: $\sum x_n = \infty$. Note: $\forall\ n > 0\ \exists\ m_n > 0$ s.t.

$$0 \leqslant s_n \leqslant s'_{n+m_n}. \quad [\because (x_i) \text{ is a rearr of } (y_i)]$$

Let $k > 0$ given.

$\sum x_n = \infty \Rightarrow \exists\ N \in \mathbb{N}$ for which $k < s_N$.

$\therefore \exists\ M \in \mathbb{N}$ for which

$$k < s_N \leqslant s'_{N+M}$$

$\therefore n \geqslant N+M \Rightarrow s'_n \geqslant s'_{N+M} > k$

$\therefore \lim s'_n = \infty \Rightarrow \sum y_n = \infty.$

Definition: We say $\sum x_n$ is absolutely convergent if $\sum |x_n| < \infty$.

Proposition: Abs convergence $\Rightarrow$ convergence.

Pf: $s_n := \sum_{i=1}^{n} x_i$.

$$|s_{m+n} - s_n| = \left| \sum_{i=n+1}^{n+m} x_i \right|$$

$$\leq \sum_{i=n+1}^{n+m} |x_i| = \sum_{i=1}^{m+n} |x_i| - \sum_{i=1}^{n} |x_i|$$

$$= \left| \sum_{i=1}^{m+n} |x_i| - \sum_{i=1}^{n} |x_i| \right| < \varepsilon$$

for large enough n.

($< \varepsilon$ for all $m \in \mathbb{N}$, $n > N$ for some $N \in \mathbb{N}$)

$\therefore \{s_n\}$ Cauchy $\Rightarrow \{s_n\}$ converges.

Condensation test

let $(x_n) \in \mathbb{R}^{\mathbb{N}}$, $x_n \geq 0 \ \forall n$
x_n decreasing (i.e., $x_{n+1} \leq x_n \ \forall n$) $\left((x_n) \in (\mathbb{R}_{\geq 0})^{\mathbb{N}} \right)$

let $y_n = 2^n x_{2^n}$

Take $n \geq 1$ then $\exists!\ k \geq 1$ s.t. $2^{k-1} \leq n < 2^k$

$$\sum_{2^{k-1} \leq n < 2^k} x_n$$

$$\geq \sum_{2^{k-1} \leq n < 2^k} x_{2^k} = (2^k - 2^{k-1})\, x_{2^k} = 2^{k-1} \cdot x_{2^k} = \frac{1}{2} y_k$$

$$\leq \sum_{n=2^{k-1}}^{2^k - 1} x_{2^{k-1}} = 2^{k-1} \cdot x_{2^{k-1}} = y_{k-1}$$

Fix $N \in \mathbb{N}$. $\exists!\ m$ s.t $2^{m-1} \leq N < N$ | $s_n' = \sum_{i=1}^{n} y_i$

$$\frac{1}{2}(y_1 + \dots + y_m) \leq \sum_{i=1}^{N} x_i = s_N \leq y_1 + \dots + y_{m-1} + x_1$$

$$\frac{1}{2}(y_1 + \dots + y_m) \geq \frac{s'_{m-1}}{2}$$

$$y_1 + \dots + y_{m-1} + x_1 = s'_{m-1} + x_1$$

(s_n') converges $\Rightarrow$ (s_n) converges
(s_n) converges $\Rightarrow$ (s_n') converges.
$\therefore$ $\sum x_n$ converges $\Leftrightarrow$ $\sum y_n$ converges.

$$|||$$

$$\sum x_n < \infty \iff \sum y_n < \infty$$

Example :

① $x_n = \frac{1}{n} \Rightarrow y_n = 2^n \cdot x_{2^n} = 1$

Σy_n diverges $\Rightarrow$ Σx_n diverges.

② $x_n = n^{-a} (\log n)^{-b}$

$a = 1$: $x_n = \frac{1}{n} (\log n)^{-b}$

$y_n = \frac{1}{2^n} (n \log 2)^{-b} \times 2^n$

$y_n (\log 2)^b = n^{-b}$

$z_n = 2^n y_{2^n} (\log 2)^b = \frac{2^n}{(2^n)^b} = \frac{1}{(2^n)^{b-1}}$

$= \frac{1}{(2^{b-1})^n}$

$\sum_{n=1}^{\infty} \frac{1}{(2^{b-1})^n} < \infty \Leftrightarrow 2^{b-1} > 1 \Leftrightarrow b-1 > 0$

$\Leftrightarrow b > 1$

Fix s.

$x_n = n^{-s}$. $y_n = 2^n \cdot (2^n)^{-s} = (2^n)^{1-s} = (2^{1-s})^n$

$\zeta(s) = \Sigma x_n$ converges $\Leftrightarrow$ Σy_n converges

$\Leftrightarrow 2^{1-s} < 1$

$\Leftrightarrow 1-s < 0 \Leftrightarrow s > 1$

Digression : a^r has been defined for $a > 0$, $r \in \mathbb{R}$.

We can now define "logarithm".

$\log_a y = x \Leftrightarrow a^x = y \quad (a > 0)$

$\log_a(\cdot) : (0, \infty) \longrightarrow \mathbb{R}$

$y \longmapsto \log_a y$

Property : $\log_a (yz) = \log_a y + \log_a z$

This is a group homomorphism.

$\log_a$ is also continuous.

We say $\log_a$ is a continuous group homomorphism.

<u>Root test</u> : $(x_n) \in (\mathbb{R})^{\mathbb{N}}$. $\varepsilon > 0$

If $|x_n|^{1/n} < 1 - \varepsilon \quad \forall\, n \geq M \quad (\exists\, M)$

$\Rightarrow (\theta =) \limsup |x_n|^{1/n} < 1$

$\Rightarrow \quad |x_n| \leq \theta^n \qquad$ for $n > M'$.

$\Rightarrow \quad \sum |x_n| \leq \frac{1}{1-\theta} + (\text{some const}) < \infty$

$\Rightarrow \quad \sum x_n < \infty$

Else if $\theta = \limsup |x_n|^{1/n} > 1$

$\Rightarrow \quad |x_n| > (\theta - \varepsilon)^n \qquad$ for infinitely many n.

$\Rightarrow \quad \lim x_n \neq 0$

$\Rightarrow \sum x_n$ diverierges.

What if $\theta = 1$?

$x_n = 1 \Rightarrow \limsup |x_n|^{1/n} = 1. \quad \sum x_n = \infty.$

$x_n = \frac{1}{n^2} \Rightarrow \limsup |x_n|^{1/n} = 1. \quad \sum x_n < \infty$

(Check)

So root test is inconclusive for $\theta = 1$.

In a nutshell : $\left((x_n) \in (\mathbb{R})^{\mathbb{N}}\right)$

$\theta := \limsup |x_n|^{1/n}$.

$\theta > 1 \Rightarrow \sum x_n$ diverges (in $\mathbb{R}$)

$\theta < 1 \Rightarrow \sum x_n$ converges (in $\mathbb{R}$)

$\theta = 1 \Rightarrow$ inconclusive.

Proof: $\theta < 1$. Then $\exists\ \rho \in (\theta, 1)$

$\therefore\ |a_n|^{1/n} < \rho$ for almost all n (all but finitely many)

$\Rightarrow |a_n| < \rho^n$

$\Rightarrow \sum |a_n| < \infty \Rightarrow \sum a_n < \infty$

$\theta > 1 \Rightarrow |a_n|^{1/n} > 1$ for infinitely many n

$\Rightarrow |a_n| > 1$ - - - -

$\Rightarrow \lim a_n \neq 0$

$\Rightarrow \sum a_n$ diverges.

Ratio Test $(x_n) \in \mathbb{R}^{\mathbb{N}}$.

$$R = \limsup \left|\frac{x_{n+1}}{x_n}\right| \qquad r = \liminf \left|\frac{x_{n+1}}{x_n}\right|$$

Test: $R < 1 \Rightarrow \sum |a_n| < \infty \Rightarrow \sum a_n < \infty$

$r > 1 \Rightarrow \sum a_n$ diverges (in $\mathbb{R}$)

$r \leq 1 \leq R \Rightarrow$ inconclusive

$R<1$:

$$\left|\frac{x_{n+1}}{x_n}\right| < \rho = \frac{\rho^{n+1}}{\rho^n} \qquad \left(\begin{array}{l}\text{when } R<\rho<1 \\ \forall\ n \geq N\end{array}\right)$$

$$\Rightarrow \left|\frac{x_{n+1}}{\rho^{n+1}}\right| < \left|\frac{x_n}{\rho^n}\right| \qquad \forall\ n \geq N$$

$$\Rightarrow \left|\frac{x_n}{\rho^n}\right| < \left|\frac{x_N}{\rho^N}\right| =: k \qquad \forall\ n \geq N$$

$$\Rightarrow |x_n| < \rho^n k \qquad \forall\ n \geq N$$

$$\Rightarrow \sum |x_n| < k \cdot \sum \rho^n + (\text{some const}) < \infty$$

If $r > 1 \Rightarrow |a_{n+1}| > |a_n|$

$\Rightarrow |a_{n+1}| > |a_N| \quad \forall \; n \geq N$

$\Rightarrow \lim_{n \to \infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges.