① (b) $\frac{1}{n^2+1} \longrightarrow 0$

Let $\varepsilon > 0$ given. $\exists\ N \in \mathbb{N}$ s.t. $N\varepsilon > 1$.
(A. Property of reals)

$$\therefore n \in \mathbb{N},\ n \geqslant N \Rightarrow n\varepsilon \geqslant N\varepsilon > 1$$

$$\Rightarrow (n^2+1)\varepsilon > n\varepsilon > 1 \quad \left[\because n^2+1 > n \ \forall\, n \in \mathbb{N}\right]$$

$$\Rightarrow \varepsilon > \frac{1}{n^2+1} = \left|\frac{1}{n^2+1} - 0\right|$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n^2+1}\right) = 0$$

(c) $\frac{6n+5}{n+1} \longrightarrow 6$

$$a_n = \frac{6n+5}{n+1}$$

$$\left|\frac{6n+5}{n+1} - 6\right| = \left|\frac{6n+5-6n-6}{n+1}\right| = \frac{1}{n+1}$$

Let $\varepsilon > 0$ given. $\exists\ N \in \mathbb{N}$ s.t. $N \cdot \varepsilon > 1$
$\left[\text{or take } N = 1 + \max\{100, \lceil 1/\varepsilon \rceil\}\right]$

$$\therefore n \geqslant N \Rightarrow (n+1)\varepsilon > n\varepsilon \geqslant N\varepsilon > 1$$

$$\Rightarrow \varepsilon > \frac{1}{n+1} = |a_n - 6|$$

$$\Rightarrow \lim_{n\to\infty} \frac{6n+5}{n+1} = 6$$

(i) $\frac{(-1)^n}{n} \longrightarrow 0$

Let $\varepsilon > 0$ given. Take $N = \max\left\{\left\lceil \frac{1}{\varepsilon} \right\rceil, 100\right\} + 1$

$$\therefore n \geqslant N \Rightarrow n \geqslant N > \frac{1}{\varepsilon} \Rightarrow \varepsilon > \frac{1}{n} = \left|\frac{(-1)^n}{n}\right|$$

$$= \left|\frac{(-1)^n}{n} - 0\right|$$

$$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

(h) $n^{\frac{1}{n}} \geq 1$

$\quad \rotatebox{90}{=}$

$1 + x_n$ (Note $x_n \geq 0$)

$$n = (1+x_n)^n = \sum_{j=0}^{n} \binom{n}{j} x_n^j \geq 1 + \binom{n}{2} x_n^2$$

$$\Rightarrow n-1 \geq \frac{n(n-1)}{2} x_n^2 \qquad (n>1)$$

$$\Rightarrow \frac{2}{n} \geq x_n^2$$

(e) $\frac{pn+q}{n^2-101} \longrightarrow 0$. Assume $p \neq 0$.

$$|a_n| = \frac{|pn+q|}{n^2-101} < \frac{|pn+q|}{(n-11)(n+11)}$$

$$< \frac{|pn+q|}{(n-11)^2} = \frac{|p|\cdot|n-11+\overbrace{(11+\frac{q}{p})}^{d}|}{(n-11)^2}$$

$$= |p| \times \left| \frac{1}{n-11} + \frac{d}{(n-11)^2} \right|$$

$$\leq |p| \times \left[\frac{1}{n-11} + \frac{|d|}{(n-11)^2} \right]$$

Pick $N_1 > 11$ s.t. $(N_1 - 11)\cdot\varepsilon > 2|p|$

Pick $N_2 > 11$ s.t. $(N_2-11)^2\,\varepsilon > 2|p|\cdot|d|$

$n \geq \max(N_1, N_2) \Rightarrow \frac{1}{n-11} < \frac{\varepsilon}{2|p|}$

and $\frac{|d|}{(n-11)^2} < \frac{\varepsilon}{2|p|}$

$$\Rightarrow |a_n - 0| = |a_n| \leq \left[\frac{1}{n-11} + \frac{|d|}{(n-11)^2} \right] |p| < \varepsilon$$

$$\Rightarrow \lim a_n = 0$$

② (a) $a_n = 100 + (-1)^n$ does not converge.

Say a_n converges to $a \in \mathbb{R}$.

But $a_{2n} = 101 \quad \forall n$ So $a_{2n} \to 101 = a$

$a_{2n+1} = 99 \quad \forall n$ so $a_{2n-1} \to 99 = a$

This is a contradiction as $101 \neq 99$.

(b) Suppose $a_n = (-1)^n \cdot n$ converges to a.

$\therefore a_{2n} = 2n \longrightarrow a$

$\therefore$ For $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t. $n \geqslant N \Rightarrow |a_{2n} - a| < 1$

$\Rightarrow |2n - a| < 1$

$\Rightarrow -1 < 2n - a < 1$

$\Rightarrow a - 1 < 2n < a + 1$ (also violates Archimedean property)

$\therefore a - 1 < 2N < a + 1$

$$\Rightarrow a + 1 \underset{\substack{\uparrow \\ \text{from previous inequality}}}{<} 2(N+1) \underset{\substack{\uparrow \\ \because N+1 > N}}{<} a + 1$$

$\Rightarrow a + 1 < a + 1$ (contradiction).

Lemma: Let (a_n) be a cgt $(\mathbb{R})$ seq. Then $\exists B > 0$ s.t. $|a_n| < B \ \forall n \in \mathbb{N}$.

Pf: For $\varepsilon = 1$ $\exists N \in \mathbb{N}$ s.t. $|a_n - a| < 1$
(Say $\lim a_n = a$)

$\Rightarrow a - \varepsilon < a_n < a + 1$

$\Rightarrow |a_n| < \underbrace{\max\{|a \pm 1|\}}_{C} \quad \forall n \geqslant N$

Choose $B = \max\{C, |a_1|, |a_2|, \dots, |a_{N-1}|\} + 100$

Then $|a_n| < B \quad \forall n \in \mathbb{N}$.