Last day we saw the meaning of $\lim_{n\to\infty} x_n = \infty$

and $\lim_{n\to\infty} x_n = -\infty$.

Remarks / Examples :

1. $\lim_{n\to\infty} (n) = \infty$

2. let $x > 1$. $\lim x^n = \infty$

$x = 1$. $\lim x^n = 1$

$-1 < x < 1$: $\lim x^n = 0$

$x < -1$: $\lim x^n$ does not exist.

I. <u>Countability of sets</u> :

We know what is a finite set, infinite.

$[n] := \{1, \dots, n\} = \mathbb{Z} \cap [1, n]$.

Def: Let A, B be sets. We say A, B are equinumerous if $\exists$ a bijection $f: A \to B$.

If A, B are equinumerous we denote that by $A \sim B$.

Rudin

→ S is finite if $S \sim [n]$ for some $n \in \mathbb{Z}^{\geq 0}$.

→ If S is not finite, S is said to be infinite.

countable → S is countably infinite if $S \sim \mathbb{N}$.

almost countable → S is countable if $S \sim \mathbb{N}$ or S is finite.

→ S is uncountable if S is not countable.

Hilbert's hotel : You are the manager of a hotel which has rooms marked 1, 2, 3, ... The hotel is full.

You have an announcement system through which you can ask the residents to shift rooms.

⊙ One person comes to your hotel and asks for a room.

What do you do? Ask everyone to shift by 1 room, i.e., ask the person in room n to move to room $n+1$. So room 1 is now empty.

$\mathbb{N} \sim \mathbb{N} \cup \{0\}$

⊙ A bus comes with infinitely many passengers named as 1, 2, 3, 4, ... They ask for rooms.

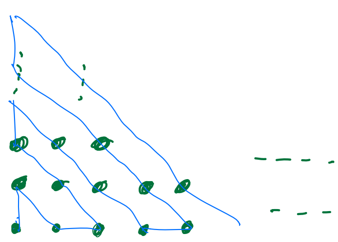
What do you do? Ask the person in room n to shift to room $2n$.
Ask the passenger named j to live in room $2j-1$.

$\mathbb{N} \sim A \cup B$
where $A, B \sim \mathbb{N}$, $A \cap B = \phi$

⊙ Infinitely many such buses come now, these buses are named $B_1, B_2, B_3, \ldots$

What do you do? Name the j^{th} person in B_i as (i, j)

$\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$



⊙ A bus comes with infinitely many people where the name of person is a $\underbrace{\text{binary string}}_{\text{sequence of 0's \& 1's}}$ of infinite length.

What do you do? There is no way to accommodate all people

$\mathbb{N} \nsim 2^{\mathbb{N}}$ ↪ all functions $\mathbb{N} \to \{0, 1\}$

Why? Suppose you can accommodate all of them.

$$(n = 110\ldots) \qquad \begin{array}{ll} a_1 & 0\ 1\ 1\ 1\ 0\ 0\ 0\ \ldots \\ a_2 & 1\ 0\ 0\ 1\ 0\ 1\ \ldots \\ a_3 & 1\ 1\ 1\ 1\ \ldots\ 0\ 1\ \ldots \\ \vdots & \end{array}$$

Let b_i be the name of the person in room a_i.

Consider the name n whose j^{th} digit is 0 if j^{th} digit of b_j is 1, and 1 o/w.

Notice that n has not been enlisted, i.e., $\nexists\, j$ s.t. $n = b_j$.

Suppose n got room a_k. b_k resides in a_k. So $n = b_k$.

Let $x = k^{th}$ digit of b_k, $y = k^{th}$ digit of n. $\therefore x = y$

By construction of n, $\left.\begin{array}{l} x = 0 \iff y = 1 \\ x = 1 \iff y = 0 \end{array}\right\}$ So $x \neq y$.

Exercise: (1) Show that $\nexists$ a surjection $f : A \to \underbrace{\mathcal{P}(A)}_{\text{power set}}$ for any set A.

(2) $\mathbb{N} \not\sim \mathbb{R}$. (Prove)

Interesting Corollary: $\exists\, f \in \mathbb{N}^{\mathbb{N}}$ which cannot be realized by any C++ program.

Prop: (1) Every subset of a countably infinite set is countable.

(2) Countable union of countable sets is countable.

(3) $\mathbb{Q}$ is countable.

(4) $\{0,1\}^{\mathbb{N}}$ is uncountable

(5) $\{0,1\}^{(\mathbb{N})} = \{$ all 0,1 sequences s.t. all but finitely many are zero $\}$ is countable.

II. Sequences in $\mathbb{R}$

Definition: A sequence $X = (x_n) \in \mathbb{R}^{\mathbb{N}}$ is said to be Cauchy if

$$\forall \varepsilon > 0 \quad \exists \; N \in \mathbb{N} \text{ s.t. } m \geq n \geq N \Rightarrow |a_n - a_m| < \varepsilon.$$

The space of all Cauchy sequences in $\mathbb{R}$ is an $\mathbb{R}$-VS

Every Cauchy seq is bounded but not conversely.

We write $X \uparrow$ iff $x_n \leq x_{n+1} \; \forall n$.
$X \downarrow$ iff $x_n \geq x_{n+1} \; \forall n$.

We say X is monotonic iff X is either $\uparrow$ or $\downarrow$.

Theorem: Let X be a sequence in $\mathbb{R}$, bounded + monotonic. Then it converges in $\mathbb{R}$.

(If $X \downarrow$ then $x_n \longrightarrow \inf X$
$X \uparrow$ then $x_n \longrightarrow \sup X$.

Note: This is not true in $\mathbb{Q}$. $\left\{\frac{1}{10^n} \lfloor \sqrt{2} \times 10^n \rfloor\right\} = \{1.4, 1.41, 1.414, ..\}$

This cgs to $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Remarks: (1) If (x_n) is Cauchy and has a convergent subseq (x_{n_k}) then (x_n) is convergent, and it converges to $\lim x_{n_k}$.

Is it true for $\mathbb{Q}$? Yes

(2) Every bounded seq in $\mathbb{R}$ has a monotonic subseq.

True in $\mathbb{Q}$? Yes.

(3) Every Cauchy seq in $\mathbb{R}$ is convergent.

X cauchy

$\downarrow$

X bounded

$\downarrow$

$\exists$ a monotone sub seq

$\downarrow$ (This seq bdd)

This seq converges

$\downarrow$

X convergent

(4) Every convergent seq is Cauchy. (in $\mathbb{R}$)

X converges (to a).

$\forall\ \varepsilon > 0 \quad \exists\ N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |x_n - a| < \varepsilon/2$

$\therefore\ n, m \geq N \Rightarrow |a_n - a_m| \leq |a_n - a| + |a - a_m|$

$< \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon$

$\therefore$ Cauchy.

Theorem: Let X be a sequence in $\mathbb{R}$. X is Cauchy iff X is convergent in $\mathbb{R}$.

Let $S \subseteq \mathbb{R}$ be a sequence. Denote S' to be the set of all limit points in S.

Defn: $x \in \mathbb{R}$ is said to be a limit point of S if $\forall\ \varepsilon > 0 \quad \exists\ y \in S \setminus \{x\}$ s.t. $|y - x| < \varepsilon$

Let X be a seq and consider the set X' of limit points of X in $\mathbb{R}$

Def: $x \in \mathbb{R}$ is said to be a lim point of X if

$\forall\ \varepsilon > 0 \quad \exists\ n \in \mathbb{N}$ s.t. $x_n \neq x,\ |x_n - x| < \varepsilon$.

For a seq X, $\overline{X} := X' \cup X$.

- $X' \subseteq \mathbb{R}$
- If X is bounded then $X' \neq \phi$.

Def: Let $X \in \mathbb{R}^{\mathbb{N}}$, We say $\alpha \in \overline{\mathbb{R}}$ is said to be a limit point of X if $\exists$ a subseq $\{x_{n_k}\}$ of X s.t. $\lim\limits_{k\to\infty} x_{n_k} = \alpha$.

Now if X' is the set of all limit points of X in $\overline{\mathbb{R}}$ then:

- $X' \subseteq \overline{\mathbb{R}}$
- $X' \neq \phi$

Definition: For a sequence $(x_n) = X \in \mathbb{R}^{\mathbb{N}}$, let X' be the set of limit points in $\overline{\mathbb{R}}$. Then

$$\overline{\lim}\, x_n = \limsup_{n\to\infty} (x_n) := \sup X' \quad \text{(in } \overline{\mathbb{R}}\text{)}$$

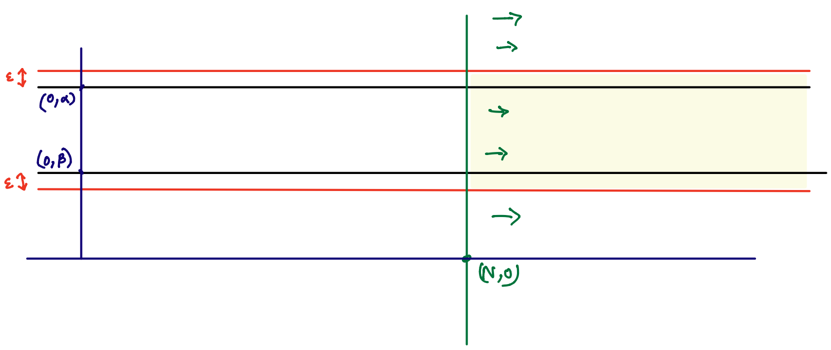
$$\underline{\lim}\, x_n = \liminf_{n\to\infty} (x_n) := \inf X' \quad \text{(in } \overline{\mathbb{R}}\text{)}$$

$\rightarrow$ $\overline{\lim}\, x_n$, $\underline{\lim}\, x_n \in \overline{\mathbb{R}}$

$\rightarrow$ $-\infty \leq \underline{\lim}\, x_n \leq \overline{\lim}\, x_n \leq \infty$

$\rightarrow$ Say $\alpha = \overline{\lim}\, x_n$, $\beta = \underline{\lim}\, x_n \in \mathbb{R}$. Let $\varepsilon > 0$.

$\exists\ N \in \mathbb{N}$ s.t. $\beta - \varepsilon < x_n < \alpha + \varepsilon \quad \forall\ n \geq N$



$\rightarrow$ $\limsup x_n = -\liminf(-x_n)$

Theorem: $\limsup x_n = \inf_{n} \sup_{k \geq n} \{x_k\}$

$= \inf_{n} \sup \{x_n, x_{n+1}, x_{n+2}, \ldots\}$

Summary: $\rightarrow$ $\limsup X$, $\liminf X$ always in $\overline{\mathbb{R}}$

$\rightarrow$ $\limsup x_n = \alpha \iff$ (for any $\varepsilon > 0$) $\begin{cases} \alpha - \varepsilon < x_n & \text{for inf many } x_n \\ \alpha + \varepsilon > x_n & \text{for almost all } n. \end{cases}$

Similar for lim inf.

Example: $(-1)^n$: $\limsup = 1$

$\liminf = -1$