# Problem

Let $S_1$ and $S_2$ be subsets of a $k-$vector space $V$ such that $S_1 \subseteq S_2$. If $S_1$ is linearly independent and $S_2$ generates $V$, then there exists a basis $\beta$ for $V$ such that $S_1 \subseteq \beta \subseteq S_2$.

# Solution

Given $S_1 \subseteq S_2 \subseteq V$ where $V$ is a $k-$vector space. $S_1$ is linearly independent. $S_2$ generates $V$. Define

$$X := \{\ T \mid S_1 \subseteq T \subseteq S_2,\ T \text{ is linearly independent }\}$$

Define the relation $\leq$ for $\mathcal{G}, \mathcal{H} \subseteq V$ as: $\mathcal{G} \leq \mathcal{H} \iff \mathcal{G} \subseteq \mathcal{H}$
Verify that $\langle X, \leq\rangle$ is a *poset*:

- $A \subseteq A\ \forall\ A \in X$
- For $A, B \in X$, if $A \subseteq B$ and $B \subseteq A$, then $A = B$
- For $A, B, C \in X$, if $A \subseteq B, B \subseteq C$, then $A \subseteq C$

Let $C \subseteq X$ be an arbitrary chain. Define

$$M := \bigcup_{T \in C} T$$

By definition, $\forall\ \mathcal{G}, \mathcal{H} \in C$, either $\mathcal{G} \leq \mathcal{H}$ or $\mathcal{H} \leq \mathcal{G}$.
$\forall\ T \in C$ we must have that $T \subseteq M$ by definition of $M$.
But $S_1 \subseteq T\ \forall\ T \in X \implies S_1 \subseteq T\ \forall\ T \in C \implies S_1 \subseteq \bigcup_{T \in C} T = M$
Also, $T \subseteq S_2\ \forall\ T \in X \implies T \subseteq S_2\ \forall\ T \in C \implies M = \bigcup_{T \in C} T \subseteq S_2$
Therefore, $S_1 \subseteq M \subseteq S_2$

**Claim**
$M$ is linearly independent.

*Proof.* Suppose $A = \{\ x_1, x_2, \ldots, x_n\ \}$ be an arbitrary finite subset of $M$ for some $n \in \mathbb{N}$.
Since $M = \bigcup_{T \in C} T$, so $\exists\ T_1, T_2, \ldots, T_n \in C$ (not necessarily distinct) such that
$x_1 \in T_1, x_2 \in T_2, \ldots, x_n \in T_n$. Since $T_i \in C\ \forall\ i$, so $T_i$'s are totally ordered, as $C$ is totally ordered.
$\therefore\ T_1 \cup T_2 \cup \cdots \cup T_n = T_j$ for some $j \in \{\ 1, 2, \ldots, n\ \} \implies x_1, x_2, \ldots, x_n \in T_j$
$\implies A \subseteq T_j \implies A$ is linearly independent.
Therefore, we have that any finite subset of $M$ is linearly independent.
By definition of linear independence, we conclude that $M$ is linearly independent. ■

Hence, $M \in X$. Also, by construction of $M$, $T \leq M\ \forall\ T \in C$. Thus any chain in $X$ is bounded above.
So, by Zorn's lemma, $X$ has a maximal element, say $\mathcal{B}$.

We now have to show that $\mathcal{B}$ spans $V$.
For this it is enough to show that $S_2 \subseteq \langle\mathcal{B}\rangle$.
This is because of the following reason: If $S_2 \subseteq \langle\mathcal{B}\rangle$ then every element of $S_2$ can be written as a linear combination of some elements of $\mathcal{B}$. But $S_2$ spans $V$. So every element of $V$ can be written as a linear combination of some elements of $S_2$. But since all elements of $S_2$ can be expressed as a linear combination of some elements of $\mathcal{B}$, so all elements of $V$ can be expressed as a linear combination of some elements of $\mathcal{B}$.
On the contrary, suppose that $\exists\ v \in S_2 \backslash \langle\mathcal{B}\rangle$. Now we let $B' = \mathcal{B} \cup \{\ v\ \}$. By construction, $S_1 \subseteq B' \subseteq S_2$.

**Claim**
$B'$ is linearly independent.

*Proof.* Say, $B'$ is linearly dependent. By definition, $\exists\ v_1, v_2, \ldots, v_n \in B' = \mathcal{B} \cup \{\ v\ \}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in k$ (not all 0) for some $n \in \mathbb{N}$, such that $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$.
If $v_i \neq v$ for any $i$ then $v_1, v_2, \ldots, v_n \in \mathcal{B}$ which is linearly independent $\implies \lambda_i = 0\ \forall\ i$, which contradicts our assumption that $\lambda_i$'s are not all 0. So, without loss of generality, we let $v_1 = v$.
Then $\lambda_1 \neq 0$. If $\lambda_1 = 0$, then $\lambda_2 v_2 + \cdots + \lambda_n v_n = 0$. But $v_2, \ldots, v_n \in \mathcal{B} \implies \lambda_2 = \cdots = \lambda_n = 0 = \lambda_1$, which contradicts our assumption that $\lambda_i$'s are not all 0.
$\therefore\ \lambda_1 v + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0 \implies v = \frac{-\lambda_2}{\lambda_1} v_2 + \cdots + \frac{-\lambda_n}{\lambda_1} v_n$.
But this is impossible as $v \notin \langle\mathcal{B}\rangle$. So, our assumption that $B'$ is linearly dependent is incorrect.
Hence, $B'$ must be linearly independent. ■

So we have that $S_1 \subseteq B' \subseteq S_2$ and $B'$ is linearly independent. So, $B' \in X$. But this contradicts the maximality of $\mathcal{B}$ in $X$, because by construction $\mathcal{B} \leq B'$. This suggests that $S_2 \backslash \langle\mathcal{B}\rangle = \phi \implies S_2 \subseteq \mathcal{B}$. But we have already argued that $S_2 \subseteq \mathcal{B} \implies V = \langle\mathcal{B}\rangle$. So, $\mathcal{B}$ is linearly independent and $V = \langle\mathcal{B}\rangle$.
By definition $\mathcal{B}$ is a basis of $V$, and by construction, $S_1 \subseteq \mathcal{B} \subseteq S_2$.