


## Sequences

**Definition:** Let  $X$  be a set. A sequence is simply a function  $f: \mathbb{N} \rightarrow X$ . Such a sequence will be called an  $X$ -valued sequence.

$a_1, a_2, a_3, a_4, \dots$

Eg:  $X = \{\text{green}, \text{blue}, \text{red}\}$

An  $X$ -valued sequence is


$$f(n) = \begin{cases} \text{green} & \text{if } n \equiv 0, -1, -2 \pmod{9} \\ \text{red} & \text{if } n \equiv 1, 2, 3 \pmod{9} \\ \text{blue} & \text{if } n \equiv 4, 5, 6 \pmod{9} \end{cases}$$

Can also view it as

$$f = \{(1, \text{red}), (2, \text{red}), (3, \text{red}), (4, \text{blue}), \dots\}$$

$$f: \mathbb{N} \rightarrow X$$

$$f(1), f(2), f(3), \dots$$

$$a_1, a_2, a_3, \dots$$

We will be interested in  $X = \mathbb{R}$

(later  $\mathbb{R}^n$ , and even later, any metric space  $X$ , finally  $X$  will be a topological space).

The set of all functions  $f: Y \rightarrow X$  is denoted by  $X^Y$ .  
In other words,  $X^Y = \{ f: Y \rightarrow X \mid f \text{ is a function} \}$ .

Exercise: let  $S$  be a finite set &  $n \in \mathbb{N}$ . Denote

$$[n] = \{1, \dots, n\}.$$

Show that  $|S^{[n]}| = |S|^n$

[Number of functions  $f: \{1, \dots, n\} \rightarrow S$  is  $|S|^n$ ]

So the set of all  $X$ -valued sequences is  $X^{\mathbb{N}}$ .  
 " " "  $\mathbb{R}$ - " " is  $\mathbb{R}^{\mathbb{N}}$ .

Prop: let  $(V, +, \cdot)$  be a vector space over  $F$ . Define

$$\oplus : V^N \times V^N \rightarrow V^N \quad \text{as}$$

$$f \oplus g := \left( n \longmapsto f(n) + g(n) \right)$$

and  $\odot : F \times V^{\mathbb{N}} \rightarrow V^{\mathbb{N}}$  as

$$c \odot f := \left( n \mapsto c \cdot f(n) \right)$$

I need to define  $g = c \circ f$ . Then  $g: \mathbb{N} \rightarrow V$ .

$$g(n) = c \cdot f(n)$$

$$n \mapsto c \cdot f(n)$$

Then  $(V^N, \oplus, \odot)$  is a vector space over  $F$ .

With the above proposition,  $V = \mathbb{R}(=F)$  gives that

$\mathbb{R}^N$  is an  $\mathbb{R}$ -vector space.

Exercise: let  $F$  be a field. Then  $F$  is an  $F$ -vector space.

## Exercise

## Subsequences :

$(a_n) =: A$  is an  $X$ -valued seq.

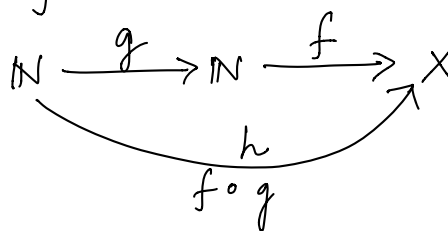
$a_1, a_2, a_3, \dots$

$f \in X^{\mathbb{N}}$

Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be a strictly increasing function  
(i.e.,  $m > n \Rightarrow g(m) > g(n)$ )

$(a_{g(n)})$  is a subseq. of  $A$ .

$h := f \circ g \in X^{\mathbb{N}}$  is a subseq  
of  $f$ .



Example: ①  $g(x) = 2x$ .

$(a_1, a_2, \dots, a_n, \dots) \leftarrow \text{Seq}$

$(a_2, a_4, a_6, a_8, \dots) \leftarrow \text{subseq}$

②  $g(x) = x^2$

seq  $\rightarrow (1, 2, 3, 4, 5, 6, \dots)$

subseq  $\rightarrow (1, 4, 9, 16, 25, \dots)$

converges in  $\mathbb{R}$

$$a_n = \frac{\lfloor \sqrt{2} n \rfloor}{n} \in \mathbb{Q}$$

Cauchy seq  
(in  $\mathbb{Q}$ )

diverge in  $\mathbb{Q}$

$\rightarrow$  Proves that  $\mathbb{Q}$  is  
not a "complete" metric  
space.

For those who know metric spaces :

Let  $X$  be a set ( $X \neq \emptyset$ ). Show that  $\exists d: X \times X \rightarrow \mathbb{R}$   
s.t.  $d$  is a metric on  $X$ .

Defn: We say  $(X, d)$  is a metric space, where  $X$  is a set and  $d: X \times X \rightarrow \mathbb{R}_{\geq 0}$  is a function, if the following hold ( $\forall x, y, z \in X$ ):

- (i)  $d(x, y) \geq 0$  with equality iff  $x = y$ .
- (ii)  $d(x, y) = d(y, x)$
- (iii)  $d(x, y) \leq d(x, z) + d(z, y)$

Example:  $(\mathbb{C}, d)$  where  $d(z_1, z_2) = |z_1 - z_2|$

Exercise: Consider  $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{o/w.} \end{cases}$$

Verify that  $(\mathbb{R}, d)$  is a metric space.

"Discrete metric"