Given a group G, and a homomorphism ϕ on G we have the following commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \phi(G) \\ & G/\sim \arrow[ur, "\sim", "\tilde{\phi}"'] & \end{tikzcd}$$

$G/\sim$, for $\sim_a \in G/\sim$, $\sim_b \in G/\sim$ $\qquad \sim_a \cdot \sim_b := \sim_{ab}$

$\sim_1$

$$G/\sim \; := \{\sim_a\} \quad \text{and} \quad \sim_a := \phi^{-1}(\phi(a)) = \{x \in G : \phi(x) = \phi(a)\}$$

$$\sim_a \subseteq G$$

$$\phi(x) = \phi(a)$$

$\Rightarrow \phi(a^{-1}x) = 1$

$\Rightarrow a^{-1}x \in \ker\phi$

$\Rightarrow a^{-1}x = k' \in \ker\phi \Rightarrow x = ak'$

$\Rightarrow \phi(x \cdot a^{-1}) = 1$

$\Rightarrow x \cdot a^{-1} \in \ker\phi$

$\Rightarrow x = k \cdot a \;,\; k \in \ker\phi$

$x \in \sim_a \quad \Rightarrow x = k \cdot a$

$$(\ker\phi)a := \{k \cdot a : k \in \ker\phi\}$$

$x \in \sim_a \quad \Rightarrow x \in (\ker\phi) \cdot a$

$$\sim_a \subseteq (\ker\phi) \cdot a$$

$k \cdot a \in (\ker\phi)a \;, \quad \phi(k \cdot a) = \phi(k) \cdot \phi(a) = \phi(a)$

$$\Rightarrow k \cdot a \in \sim_a$$

$(\ker\phi) \cdot a \subseteq \sim_a$

Summary :-

$\sim_a = (\ker \phi) \cdot a$ → The right coset of $(\ker \phi)$ generated by a

$a(\ker \phi) := \{ a \cdot k : k \in \ker \phi \}$

$\sim_a = a(\ker \phi)$

Corollary:-

$$\boxed{a(\ker \phi) = (\ker \phi) a} \quad \forall a \in G$$

$H \leq G$

$aH := \{ ah : \forall h \in H \}$

$Hb := \{ hb : \forall h \in H \}$

→ Def: Cosets

$H \leq G$

$aH = Ha$?

NO,

S_3 :-

$H := \{ 1, (1\ 2) \}$

$a = (2\ 3)$

$aH := \{ (2\ 3), (1\ 3\ 2) \}$

$(2\ 3)(1\ 2) = (1\ 3\ 2)$

$Ha := \{ (2\ 3), (1\ 2\ 3) \}$

$(1\ 2) \cdot (2\ 3) = (1\ 2\ 3)$

$aH \neq Ha$

Def^n :- Given a group G, and a subgroup $H \leq G$, H is Normal iff

$$aH = Ha \quad \forall\ a \in G$$

So $\ker \phi$ is normal.

$H \leq G$
$H \trianglelefteq G \to$ Normal

Now $H \trianglelefteq G$. Does there exists a homomorphism ϕ on G s.t. $\ker \phi = H$?

Lemma :- $\sim$ on G s.t. $a \sim b$ iff $ab^{-1} \in H$

$$\{\sim_a\} = G/H$$

G/H is a group iff $aH = Ha \quad \forall\ a \in G$

$\sim_a \cdot \sim_b = \sim_{ab}$

$\sim_a = \sim_c \Rightarrow ac^{-1} \in H$
$\sim_b = \sim_d \Rightarrow bd^{-1} \in H$

$ab(cd)^{-1} = abd^{-1}c^{-1}$

$bd^{-1} = h \in H$

ahc^{-1}
$= h'(ac^{-1}) \to \in H$
$= h'(ac^{-1}) \in H$

$aH = Ha$
$ah = h'a$
$= h'a^{-1}$

$(ab)(cd)^{-1} \in H \qquad \sim_a \cdot \sim_b = \sim_{ab}$ well defined

G/H is a group. H is normal ?

For some $a \in H$

$aH \neq Ha$

$h \in H$
$ah \notin Ha$

$(aH) \cdot (cH) = aH = (cH) \cdot (aH)$

G/H

$$\pi : G \longrightarrow G/H$$

$$\ker \pi = H$$

ker of any homomorphism is normal. $\Rightarrow H \trianglelefteq G$

So

Normal Subgroup $\Leftrightarrow$ ker of some homomorphism