1) Vector Space, Subspace etc.

$$W \subset V \qquad \begin{cases} \text{i) } a \in W, b \in W \Rightarrow a+b \in W \\ \text{ii) } \lambda \in F \quad a \in W \Rightarrow \lambda a \in W \end{cases}$$

2) L. I. $\longrightarrow$ Basis $\longrightarrow$ Spanning

3) Linear Maps, kernel, Isomorphism Theorems :-

4) Operation on Vector Spaces :-

i) V is a F-vector space. W and U be subspaces of V. Then we can add W and U

$$W+U := \{ \omega + u \; ; \; \forall \omega \in W, \; \forall u \in U \}$$

$$W \subseteq W+U, \quad U \subseteq W+U$$

$\omega_1 + u_1 \qquad \omega_2 + u_2 \qquad c \in F$

$$(\omega_1+u_1) + c(\omega_2+u_2) = \underbrace{(\omega_1 + c\omega_2)}_{W} + \underbrace{(u_1 + cu_2)}_{U}$$

So $W+U$ is a subspace.

ii) V is F-vector space. W and U are subspaces. and $W \cap U = \{0\}$.

Look at the elements of $W+U$.

$v \in W+U$

$v = \omega_1 + u_1 \qquad v = \omega_2 + u_2 \quad \Rightarrow \quad \omega_1 + u_1 = \omega_2 + u_2$

$$\Rightarrow (\omega_1 - \omega_2) = (u_2 - u_1) = v_0$$

$$v_0 \in W \qquad v_0 \in U$$

$$\Rightarrow v_0 \in W \cap U \quad \Rightarrow v_0 = 0$$

iii) V is F-vector space. W and U are subspace such that

1) $W \cap U = \{0\}$

2) $W + U = V$

Then W is defined to be the "algebraic complement" of U in V.

$$W + U_1 = V, \quad W \cap U_1 = \{0\}$$

$$W + U_2 = V, \quad W \cap U_2 = \{0\}$$

$\dim U_1 = \dim U_2 \Rightarrow U_1 \cong U_2$

iv) $W \subset V$, W is a subspace. Does W always have an algebraic complement?

Ans $\rightarrow$ $W \subset V$, $W \rightarrow B$ be a basis for W.

B is L.I. subset of V. So $\exists$ basis $\mathcal{B}$ of V s.t

$$B \subset \mathcal{B}$$

$\mathcal{B} \setminus B = C$

$\langle C \rangle = U$

$$W + U = V$$
$$W \cap U = \{0\}$$

iv) Corollary :- Let V be finite dim. let $\dim V = n$

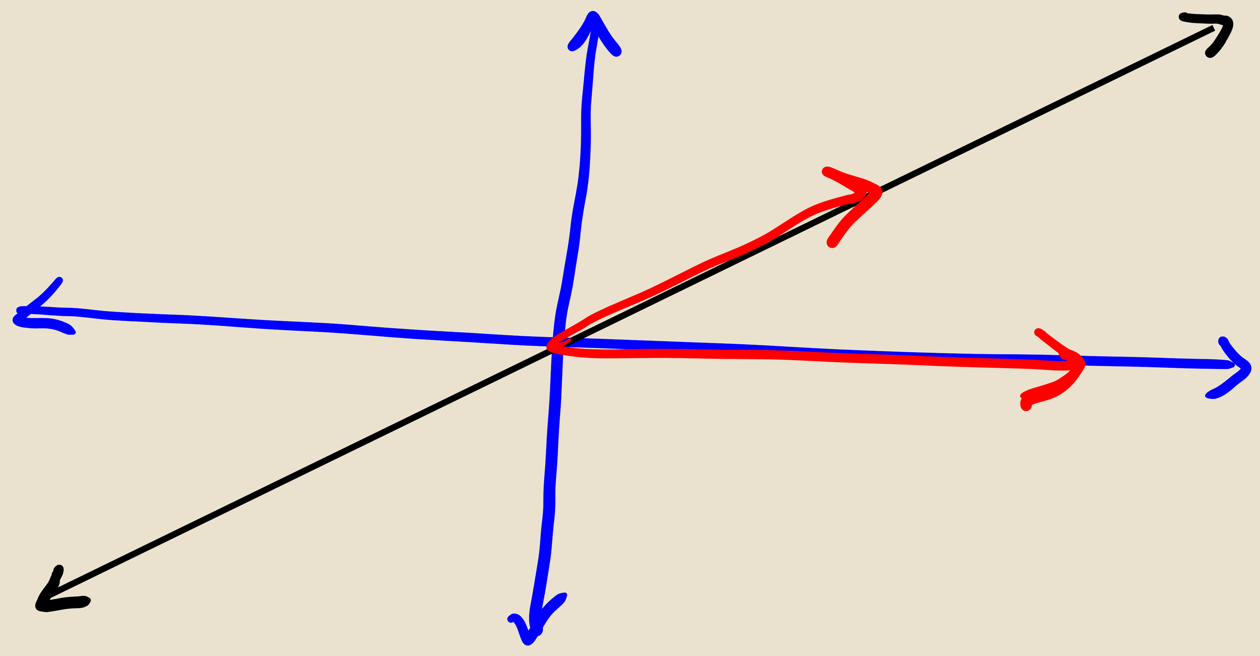
Let W be a subspace of V let $\dim W = k$

$$\dim (W_F^c) = n - k$$

Thus given a subspace W of V, every of its complement have same dimension.

$$V = \mathbb{R}^2 \qquad \overset{W}{\mathbb{R}\times\{0\}} \qquad \overset{U}{\{0\}\times\mathbb{R}}$$

$$W_F^c = U$$



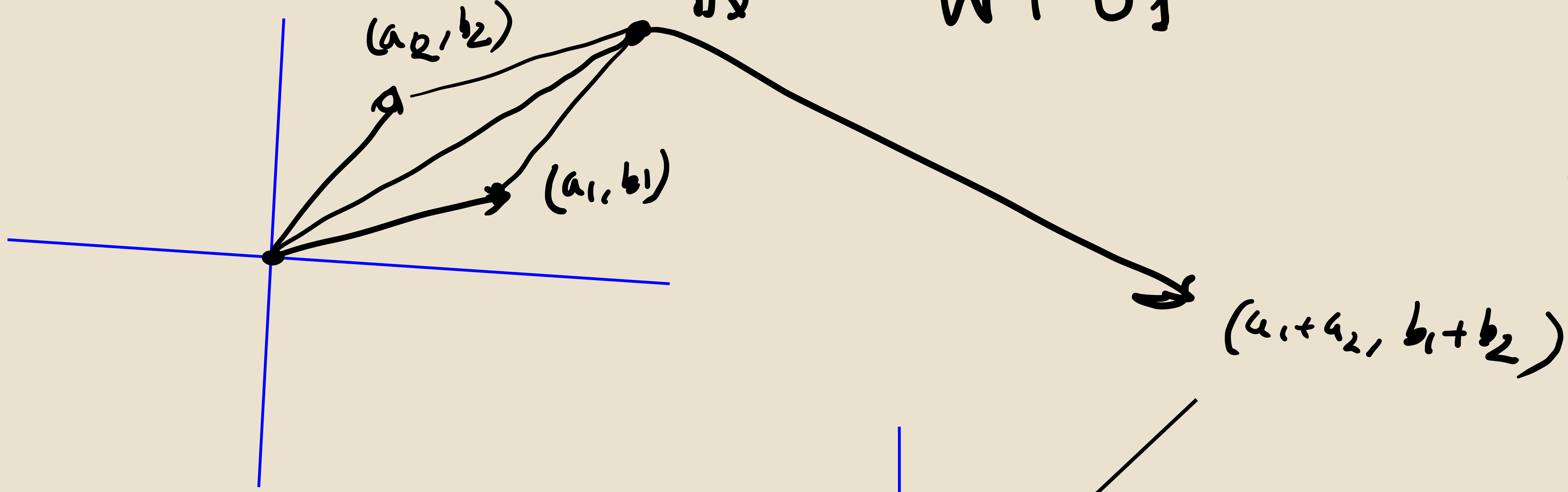
$$(a,b) = (a,0) + (0,b)$$

$$(0,0)$$

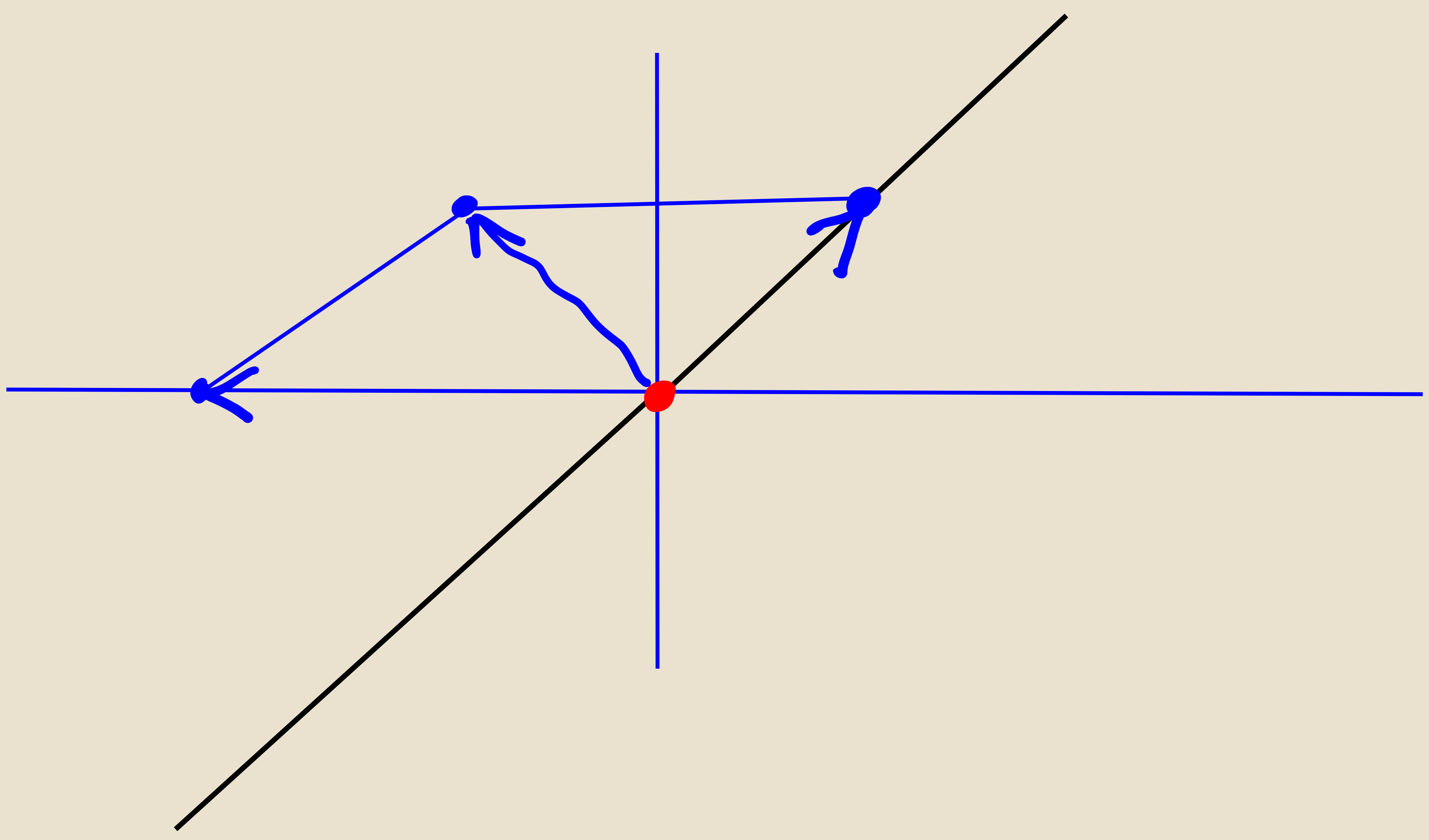
$$U_1 = \{(x,x) : x \in \mathbb{R}\}$$

$$(a,b) = (a-b,0) + (b,b)$$

$$\mathbb{R}^2 = W + U_1$$



$(a_1$



A. Complement of each vector space exists and it is unique upto isomorphism.

4) $T : V \longrightarrow W$, then $(\ker T)_F^c \cong \operatorname{Im}(T)$

$\rightarrow$ $\phi : (\ker T)_F^c \longrightarrow \operatorname{Im} T$

$v \longmapsto T(v)$

ϕ is linear map.

$$\left.\begin{array}{l} \phi(v) = \phi(u) \Rightarrow v-u \in \ker T \\ v-u \in (\ker T)_F^c \end{array}\right\} \begin{array}{l} \Rightarrow v-u = 0 \\ \Rightarrow v = u \end{array}$$

$T(v) \in \operatorname{Im} T$ $\qquad V = \ker T + (\ker T)_F^c \Rightarrow v = \underset{\ker T}{u} + \underset{(\ker T)_F^c}{w}$

$$T(v) = T(u+w) = T(u) + T(w) = T(w)$$

$T(v) \in \operatorname{Im} T$ $\qquad T(v) = 0$ $\qquad \phi(0) = T(0) = 0$

$T(v) \neq 0$

5) $\pi : V \longrightarrow V/W$

$\ker \pi = W$

$(\ker \pi)_F^c \cong V/W$

$\boxed{W_F^c \cong V/W}$

$\dim V = n$

$\dim W = k$

$$\boxed{\dim V/W = \dim(W_F^c) = n-k}$$

6) $(\ker T)_F^c \cong (\operatorname{Im} T)$

$\dim V = n$, $\dim(\ker T) = k$ $\qquad \dim(\ker T_F^c) = n-k$

$$\dim(\ker T) + \dim(\operatorname{Im} T) = k + n - k = \dim V$$

nullity $\qquad$ Rank