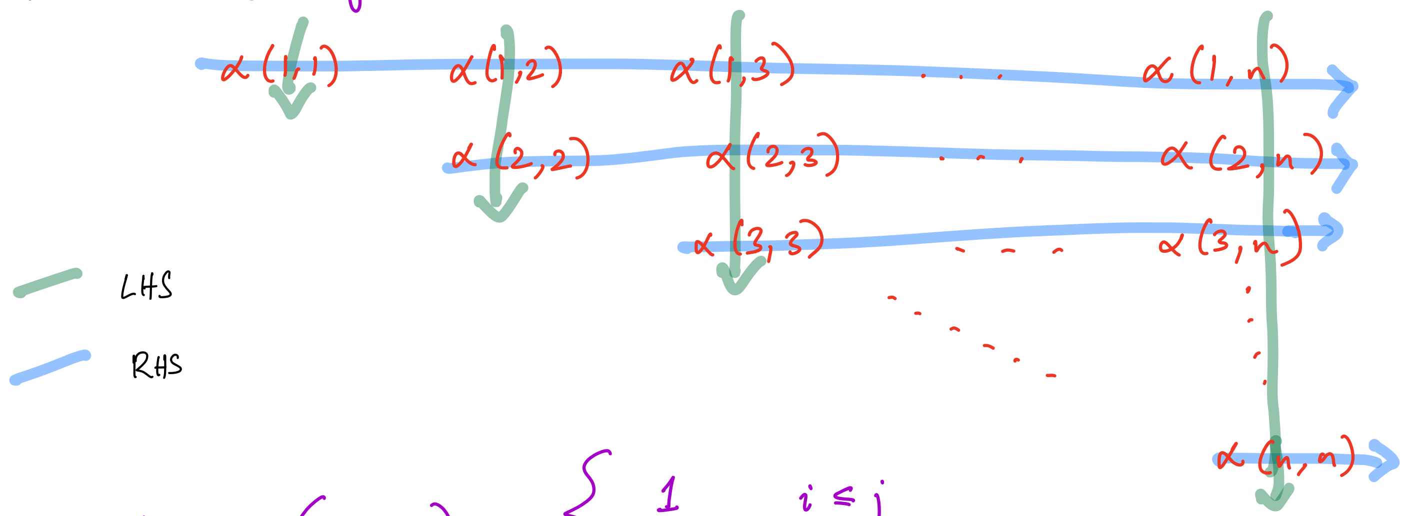
Problem 3(a) of PS 6 :



$$\mathbb{1}_{x \le y}(i, j) = \begin{cases} 1 & i \le j \\ 0 & \text{otherwise} \end{cases}$$

Write $\mathbb{1}$ for $\mathbb{1}_{x\le y}$ here.

$$\sum_{j=1}^{n} \sum_{i=1}^{j} \alpha(i,j) = \sum_{j=1}^{n} \left[\sum_{i=1}^{j} \alpha(i,j) \right]$$

$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{j} 1 \cdot \alpha(i,j) + \sum_{i=j+1}^{n} 0 \cdot \alpha(i,j) \right]$$

$$= \sum_{j} \left[\sum_{i=1}^{j} \mathbb{1}(i,j) \cdot \alpha(i,j) + \sum_{i=j+1}^{n} \mathbb{1}(i,j) \cdot \alpha(i,j) \right]$$

$$= \sum_{j} \left[\sum_{i=1}^{n} \mathbb{1}(i,j)\, \alpha(i,j) \right]$$

$$= \sum_{j} \sum_{i} \mathbb{1}(i,j)\, \alpha(i,j)$$

$$= \sum_{i} \sum_{j} \mathbb{1}(i,j)\, \alpha(i,j)$$

$$= \sum_{i} \left[\sum_{j=1}^{i-1} 0 \cdot \alpha(i,j) + \sum_{j=i}^{n} 1 \cdot \alpha(i,j) \right]$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha(i,j).$$

$$\boxed{\sum_{i=1}^{n} a_i b_i = b_{n+1} A_n - \sum_{i=1}^{n} A_i (b_{i+1} - b_i)}$$

Abel's summation Test

(1) $(a_n),(b_n) \in \mathbb{R}^{\mathbb{N}}$. Assume

(a) $\sum a_n \in \mathbb{R}$

(b) (b_n) monotone

(c) (b_n) bounded.

Then $\sum a_n b_n \in \mathbb{R}$

Pf: (b_n) converges. Say $b = \lim b_n$.

$A_n = \sum_{i=1}^{n} a_i$. Say $A = \lim A_n$.

Further $\{A_n\}$ is bdd. Say $|A_n| < \beta \ \forall n \in \mathbb{N}$.

$\lim (b_{n+1} \cdot A_n) = b \cdot A$.

Claim: $\sum A_i (b_{i+1} - b_i)$ cgs absolutely.

Pf: $\sum_{i=1}^{n} |A_i| |(b_{i+1} - b_i)| < \beta \sum_{i=1}^{n} |b_{i+1} - b_i|$

$\rightarrow$ Say $(b_n) \uparrow$: $\sum_{i=1}^{n} |b_{i+1} - b_i| = \sum_{i=1}^{n} (b_{i+1} - b_i) = b_{n+1} - b_1$

$\rightarrow$ Say $(b_n) \downarrow$: $\sum_{i=1}^{n} |b_{i+1} - b_i| = b_1 - b_{n+1}$

$\therefore \sum_{i=1}^{n} |A_i| \cdot |b_{i+1} - b_i| < \beta \cdot \sum_{i=1}^{n} |b_{i+1} - b_i|$

$= \beta \cdot |b_1 - b_{n+1}|$

(b_n) cgt $\Rightarrow (b_1 - b_n)$ cgt $\Rightarrow (|b_1 - b_n|)$ cgt

$\Rightarrow (|b_1 - b_n|)$ bounded.

(say $< \alpha$)

$\therefore \sum_{i=1}^{m} |A_i| \cdot |b_{i+1} - b_i| < \beta\alpha \quad (\forall n)$

$\therefore \sum_{i=1}^{n} A_i (b_{i+1} - b_i)$ abs cgt ($\Rightarrow$ cgt).

By Problem 3(b), $\sum a_n b_n$ cgt.

(2) $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$. Assume $\left(A_n = \sum_{i=1}^{n} a_i\right)$

(a) (A_n) bdd

(b) $(b_n) \downarrow$

(c) $\lim b_n = 0$

Then $\sum a_n b_n$ cgs.

Pf: Looking at P4 (PS 6), it is enough to show that $\sum_i |b_{i+1} - b_i| \in \mathbb{R}$.

$\therefore$ Enough to show: $\left(\sum_{i=1}^{n} |b_{i+1} - b_i|\right)$ is bdd.

Reason:
$$\sum_{i=1}^{n} |b_{i+1} - b_i| = \sum_{i=1}^{n} (b_i - b_{i+1}) = b_1 - b_{n+1}$$
$$\therefore \sum_{i=1}^{\infty} |b_{i+1} - b_i| = \lim_{n\to\infty} (b_1 - b_{n+1}) = b_1.$$

<u>Alternating series test</u>: $(x_n) \downarrow$, $\lim x_n = 0$. Then $\sum (-1)^n x_n \in \mathbb{R}$.

Pf: In (2) above, take $a_n = (-1)^n$ & $b_n = x_n$ $\forall n$.

Then $(b_n) \downarrow$, $\lim b_n = \lim x_n = 0$, (A_n) bdd.

Hypothesis is satisfied $\Rightarrow \sum a_n b_n = \sum (-1)^n x_n \in \mathbb{R}$. $\square$

<u>Cauchy Product</u>

Take two power series
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$g(x) = \sum_{n=0}^{\infty} b_n x^n.$$

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_n x^n.$$
$$(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots)$$
$$= x^0 (a_0 b_0) + x^1 (a_0 b_1 + b_0 a_1) + x^2 (a_0 b_2 + a_1 b_1 + a_2 b_0) + \dots$$

$c_0 = a_0 b_0$

$c_1 = a_0 b_1 + a_1 b_0$

$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$

$\vdots$

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

(c_n) defined above is said to be the Cauchy product of sequences (a_n) & (b_n).

Say $\sum a_n$ & $\sum b_n$ cgs (to a, b): $f(1) = \sum a_n$, $g(1) = \sum b_n$,

Then we can expect $\underbrace{f(1) \cdot g(1)}_{a \cdot b} = \sum c_n$

Thm: let $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$ and (c_n) be Cauchy product. let $a = \sum a_n$, $b = \sum b_n \in \mathbb{R}$ and atleast one of $\sum a_n$ or $\sum b_n$ converges absolutely. Then $\sum c_n = a \cdot b$.

Pf: Please have a look at Pg 74 of Rudin (Thm 3.50).