. . . Pooh was saying to himself, "If only I could think *of something!" For he felt sure that a Very Clever Brain could catch a Heffalump if only he knew the right way to go about it.* —A. A. Milne

PROMYS Number Theory **Problem Set #5** *Boston University, July 9, 2021*

Reading Search

Q1: What are the arithmetic functions $\varphi(n), \mu(n), \tau(n), \sigma(n)$? Find the following values of these functions: $\varphi(7)$, $\varphi(75)$, $\varphi(105)$, $\mu(7)$, $\mu(75)$, $\mu(105)$, $\tau(7)$, $\tau(75)$, $\tau(105)$, $\sigma(7)$, $\sigma(75)$, $\sigma(105)$.

Exploration

P1: We have considered a rational function of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

of the n variables $a_1, a_2, \ldots, a_n$. We have called this function a *finite continued fraction*. For this continued fraction we use the convenient notation $[a_1, a_2, \ldots, a_n]$. Let

$$\begin{array}{llll} P_1 = a_1, & P_2 = a_2 a_1 + 1, & \ldots, & P_k = a_k P_{k-1} + P_{k-2}, \quad \ldots \\ Q_1 = 1, & Q_2 = a_2, & \ldots, & Q_k = a_k Q_{k-1} + Q_{k-2}, \quad \ldots \end{array}$$

Then we assert $[a_1, a_2, \ldots, a_k] = \frac{P_k}{Q_k}$ (the kth convergent).

P2: Using P1, show that (1) $P_{k-1}Q_k - Q_{k-1}P_k = (-1)^{k-1}$ for $k > 1$, and that (2) $P_{k-2}Q_k - P_k Q_{k-2} = (-1)^k a_k$ for $k > 2$.

A Reminder

P3: Make an inventory of properties of $\mathbb{Z}$ which you think gives a good description of this mathematical system. Check to see that your inventory suffices to derive everything about $\mathbb{Z}$ which you have proved so far.

Prove or Disprove and Salvage if Possible

P4: Given integers $a, b, c \in \mathbb{Z}$, then $a > b \Leftrightarrow ac > bc$.

P5: Let a, b be integers with $a > 0$. Then $ab > 0 \Leftrightarrow b > 0$.

P6: Given positive integers $a, b > 0$, then $a > b \Leftrightarrow a^2 > b^2$.

P7: If $a, b \in \mathbb{Z}$ and $(a, b) = 1$ then $ax + by = 1$ has a solution in integers x and y.

P8: $a|bc$ and $(a, b) = 1 \Rightarrow a|c$. True in Z.

P9: p is prime $\Leftrightarrow 2^p - 1$ is prime.

P10: Let $u_1, u_2, \ldots, u_r$ be all the units in $\mathbb{Z}_m$. Let $u = u_1 \cdot u_2 \cdot \ldots \cdot u_r$ be the product of all the units. Then $u^2 = 1$.

Numerical Problems (*Some food for thought*)

P11: Find all the positive integral solutions (x, y) of the Diophantine equation $158x + 57y = 2000$.

P12: Find an integral solution (x, y) of the equation $2689x + 4001y = 17$.

P13: Find all of the solutions of $2017x = 532$ in $\mathbb{Z}_{4001}$. Explain.

P14: What is the order of 28 in U_{29}? of 16 in U_{29}? of $28 \cdot 16$ in U_{29}? Now consider U_{71}. What is the order of 7, of 2, of $7 \cdot 2 = 14$, of 54, of 51, of $54 \cdot 51$? Any conjectures?

P15: Expand $\sqrt{3}$ into a simple continued fraction using the results of P14, Set #4. How much do you know about the continued fraction for $\sqrt{3}$ after calculating, say, the first five partial quotients? Any conjectures?

Ingenuity

P16: Suppose $\mathcal{P}$ is a simple polygon all of whose vertices are lattice points. Let I be the number of lattice points in the interior of $\mathcal{P}$, and let B be the number of lattice points on the boundary of $\mathcal{P}$. Find a simple formula in terms of I and B for the area enclosed by $\mathcal{P}$. Justify your answer.

P17: Consider a rectangular Cartesian coordinate system in a plane. Points with integral coordinates we call *lattice points*. Show that there is no regular pentagon all of whose vertices are lattice points.