1> What is meant by "cosets" :-

2> For any subgroup $H \leq G$, define G/H :-

3> G/H is a group under the operation

$$(aH)(bH) = (ab)H$$

iff H is Normal i.e. $gH = Hg \quad \forall g \in G$

4> $H \trianglelefteq G$ iff H is kernel of some homomorphism defined on G

- G, $H \leq G$, $\quad G/H := \{aH : a \in G\}$ $\qquad a \sim b \Leftrightarrow ab^{-1} \in H$

G is finite

$$|G/H| := \frac{|G|}{|H|}$$

If $H \leq G$, $|G| < \infty$, $\quad |H| \,\big|\, |G|$

Lagrange's Theorem

$H \leq G$, $|G/H|$

$$|G:H| = \frac{|G|}{|H|} := \text{index of } H$$

$|G| < \infty$, $\quad x \in G$. $\quad |x| \,\big|\, |G|$

$$|\langle x \rangle| = |x|$$

$|\langle x \rangle| = |x|$

2> $|G| = p$, a prime $\qquad G \cong \mathbb{Z}/p\mathbb{Z}$

3> $|G:H| \leq 2$, $\quad H \trianglelefteq G$

4) $H \leq G$, $K \leq G$

$HK := \{ hk \ \forall h \in H, k \in K \}$ is grp iff $HK = KH$

1) $|G| < \infty$, $x \in G$, then

$$|\langle x \rangle| = |x|$$

$|\langle x \rangle| \,\big|\, |G|$

$|x| \,\big|\, |G|$

$\rightarrow$ $|x| = n$

$$G_n := \{0, 1, \ldots, n-1\}$$

$$f : G_n \to \langle x \rangle$$

$$m \to x^m$$

$m \geq k$

$$x^m = x^k \Rightarrow x^{m-k} = 1$$

$$\Rightarrow m-k = 0 \qquad n > m-k > 0$$

$$x^l = x^{nq+r} \qquad 0 \leq r < n$$

$$= x^{nq} \cdot x^r$$

$$= (x^n)^q \cdot x^r$$

$$= 1^q \cdot x^r = x^r$$

$x^l = x^r$

$f(r) = x^r = x^l$

2) $|G| = p$, is a prime. $G \cong \mathbb{Z}/p\mathbb{Z}$

$G \cong \mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_1^2\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z}$

$\rightarrow$ To prove $\exists\, x \in G$ s.t. $\langle x \rangle = G$

To prove that any element $1 \neq x \in G$ satisfies $\langle x \rangle = G$

$x \in G$. $|\langle x \rangle| = p = |G|$

$|\langle x \rangle| \,\big|\, |G| = p \Rightarrow |\langle x \rangle| = p$ or $|\langle x \rangle| = 1$

$D_8 := \langle \{1, r, r^2, r^3, s\} \rangle$

$\langle x \rangle = G$

$\langle r^2 \rangle = D_8$

$G = \{1, x, x^2, \dots, x^{p-1}\}$

$\{1, r^2\}$

$\mathbb{Z}/p\mathbb{Z}$

$x^k \longrightarrow \bar{k}$

3) $|G:H| \leq 2 \Rightarrow H \trianglelefteq G$

$\longrightarrow \quad |G:H| = 1 \qquad \frac{|G|}{|H|} = 1 \Rightarrow |G| = |H| \Rightarrow H = G$

$|G:H| = 2 \qquad G:H := \{H, aH\}$

$gH = Hg \qquad \forall g \in G$

On cont. $\exists\, g \in G$ s.t. $gH \neq Hg$

$gH = H$ or $gH = aH$

$\Rightarrow g \in H \qquad \Rightarrow ga^{-1} \in H$

$\Rightarrow gH = H = Hg \qquad \Rightarrow ga^{-1}H = H$

$\Rightarrow gH = Hg$

4) $HK := \{hk : h \in H, k \in K\}$

Assume $HK = KH$. $\qquad HK \leq G$

$a \in HK, \; b \in HK$

$ab^{-1} \in HK$

$a = h_1 k_1, \; b = h_2 k_2 \Rightarrow ab^{-1} = h_1 k_1 k_2^{-1} h_2^{-1}$

$= h_1 k h_2^{-1}$

$= h_1 h_3 k_3 = (h_1 h_3) k_3$

$HK \leq G, \qquad HK = KH$

$hk \in HK$

$(hk)^{-1} = h_1 k_1 \Rightarrow hk = k_1^{-1} h_1^{-1} \in KH$

$KH \subseteq HK$

$HK \subseteq KH$

$KH \subseteq HK$

$k \in K, \; h \in H$

$$\left.\begin{matrix} k \in HK \\ h \in HK \end{matrix}\right\} \Rightarrow kh \in HK$$

$KH \subseteq HK$