Real Analysis

Topology on Metric Spaces

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Fix a metric space (X, d).

We have already seen aspects of openness and closedness and their characterizations using interior points and limit points.

We might note that the open balls, sort of, 'build up' the open sets. In other words, they form a 'base' for the open sets.

Definition 1 *A collection $\mathscr{U} = \{U_i\}_{i\in I}$ of open sets of X is said to be a base if every open set in X can be expressed as a union of members of $\mathscr{U}$.*

If we fix a base, any member of the base can be called a *basic open set*. The above definition suggests that $\{\mathscr{B}_r(x) : x \in X, r > 0\}$ is a base. We look at an alternate characterization:

Lemma 2 *Let $\mathscr{U}$ be a collection of open sets in X. $\mathscr{U}$ is a base iff for each $x \in X$ and each open set V containing $x, \exists U \in \mathscr{U}$ such that $x \in U \subseteq B$.*

Proof Suppose $\mathscr{U} := \{U_i\}_{i\in I}$ is a base. Let $x \in X$ and V be an open set containing x. Then $\exists J \subseteq I$ such that $x \in V = \bigcup_{j\in J} U_j$ whence $x \in U_\alpha$ for some $\alpha \in J \subseteq I$. That is, $x \in U_\alpha \in \mathscr{U}$.

Now suppose for each $x \in X$ and each open set V containing x, we can find an open set $U_{x,V} \in \mathscr{U}$ such that $x \in U_{x,V} \subseteq V$.
Now fix an open set V. Then $U_x = U_{x,V} \in \mathscr{U} \forall x \in V$. It is clear that $V = \bigcup_{x\in V} U_x$. ■

Now we recall the Lindelöf covering theorem. Notice that while proving the theorem for $\mathbb{R}$, we had mostly relied upon the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. In fact, if you look at the first line on the third page of the notes on Lindelöf covering theorem in $\mathbb{R}$, it just mentions a base. Due to the dense nature and countability of $\mathbb{Q}$, we can extract a countable base from the usual base $\{\mathscr{B}_r(x) : x \in \mathbb{R}, r > 0\}$ for $\mathbb{R}$.
We recall that D being dense in X means that every open set in X nontrivially intersects D. Alternately, this also means that $\overline{D} = X$. A third way to look is: $\forall x \in X, r \in \mathbb{R}^+, \exists q \in D$ such that $q \in \mathscr{B}_r(x)$, that is, every element of X can be approximated using elements of D upto arbitrary precision.
This and the above discussion should give a hint as to where we are going.

Lemma 3 *The following are equivalent for a metric space (X, d):*

(a) X has a countable dense set (i.e., X is separable)

(b) X has a countable base

(c) Every open cover of X has a countable subcover (i.e., X is a Lindelöf space)

Proof $(a) \implies (b)$: Let $D = \{x_n : n \in \mathbb{N}\}$ be a countable dense set of X. Define $\mathscr{U} := \{\mathscr{B}_r(x_n) : n \in \mathbb{N}, r \in \mathbb{Q}^+\}$ (relate with the construction used in case of $\mathbb{R}$). Clearly $\mathscr{U}$ is countable and contains open sets. Since D is dense, $\mathscr{U}$ is a base for X.

$(b) \implies (c)$: Say $\mathscr{U} = \{U_n\}_{n\in\mathbb{N}}$ is a countable base for X. Let $\mathscr{V} = \{V_i\}_{i\in I}$ be an open cover for X. Fix $x \in X$. Then $\exists V_x \in \mathscr{V}$ such that $x \in V_x$. Since $\mathscr{U}$ is a base, $\exists n \in \mathbb{N}$ such that $x \in U_n \subseteq V_x$ (take the smallest such $n = n(x) \in \mathbb{N}$).
Clearly $J := \{n(x) : x \in X\} \subseteq \mathbb{N}$. So, we can take a countable set $S = \{x_n : n \in \mathbb{N}\} \subseteq X$ such that $\{n(x) : x \in S\} = J$. Further, $X = \bigcup_{U\in\mathscr{U}'} U \subseteq \bigcup_{x\in S} V_x \subseteq X$.

$(c) \implies (a)$: Fix $n \in \mathbb{N}$. Note that $\left\{\mathscr{B}_{\frac{1}{n}}(x) : x \in X\right\}$ is an open cover of X. By hypothesis, $\exists$ a countable subcover $\left\{\mathscr{B}_{\frac{1}{n}}(x_{n,i})\right\}_{i\in\mathbb{N}}$.
Take $D := \left\{x_{n,m} : m \in \mathbb{N}, n \in \mathbb{N}\right\}$. Clearly D is countable. We'll show that D is dense in X. In fact, let $x \in X, r > 0$. Then $\exists n \in \mathbb{N}$ such that $r > \frac{1}{n}$. Now, $\left\{\mathscr{B}_{\frac{1}{n}}(x_{n,i})\right\}_{i\in\mathbb{N}}$ is a countable cover for X whence $x \in \mathscr{B}_{\frac{1}{n}}(x_{n,m}) \subseteq \mathscr{B}_r(x_{n,m})$ for some $m \in \mathbb{N}$. It follows that D is dense. ■

We give an alternate proof for $(b) \implies (c)$: Let $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable base for X and $\mathscr{V}$ an open cover for X. Then $V = \bigcup_{B\in\mathscr{U}, B\subseteq V} B$ for each $V \in \mathscr{V}$. Let $\mathscr{B} = \{B \in \mathscr{U} : B \subseteq V \text{ for some } V \in \mathscr{V}\} \subseteq \mathscr{U}$. Now $\mathscr{B}$ is also an open cover for X. For each $B_i \in \mathscr{B}$ pick some $V_i \in \mathscr{V}$ such that $B \subseteq V$. Then $X = \bigcup_i V_i$.