Real Analysis

Baire's theorem

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Definition 1 (Nowhere dense set) *A subset A of a metric space (X,d) is <u>nowhere dense</u> or <u>rare</u> if $\left(\overline{A}\right)^o = \varnothing$.*

In other words, A is rare iff it is contained in a closed set with empty interior. In fact, if A is rare then A is contained in $F = \overline{A}$ which has empty interior. Conversely if A is contained in closed F with $F^o = \varnothing$ then $\left(\overline{A}\right)^o \subseteq \left(\overline{F}\right)^o = F^o = \varnothing$.

We recall what *dense* means.

Definition 2 (Dense set) *A subset A of a metric space (X,d) is said to be <u>dense</u> if $\overline{A} = X$.*

In case of subsets of $\mathbb{R}$, we can equivalently say that A is dense iff $\forall x \in \mathbb{R}, r > 0, \exists a \in A$ such that $a \in \mathscr{B}_r(x)$.

We might guess, from the terminology, that the complement of a nowhere dense set might be dense. This is true, as we shall see in the next paragraph. One might get more bold and claim that A is rare iff A^c is dense. Well, not quite. Think about $A = \mathbb{R} \setminus \mathbb{Q}$ which is dense in $\mathbb{R}$. But the closure of $A^c = \mathbb{Q}$ has nonempty interior, hence not rare.

It turns out that A is rare iff $\left(\overline{A}\right)^c$ is dense. Indeed recall that $\overline{S} = ((S^c)^o)^c$ for any set S. Take $S = \left(\overline{A}\right)^c$. This gives $(S^c)^o = \left(\overline{A}\right)^o = \varnothing \iff \overline{S} = ((S^c)^o)^c = X \iff S$ is dense $\iff \left(\overline{A}\right)^c$ is dense.
Clearly $A \subseteq \overline{A} \iff \left(\overline{A}\right)^c \subseteq A^c$. It thus stands that A is rare $\iff X = \overline{\left(\overline{A}\right)^c} \subseteq \overline{A^c} \implies \overline{A^c} = X \iff A^c$ is dense.

Proposition 3 *(a) Any subset of a rare set is rare.*

(b) A finite union rare sets is rare.

(c) The closure of a nowhere dense set is nowhere dense.

Proof (a) Let $A \subseteq B$ where B is rare. Then $\overline{A} \subseteq \overline{B}$ whence $\left(\overline{A}\right)^o \subseteq \left(\overline{B}\right)^o = \varnothing$.

(b) Let A, B be rare sets. Equivalently, $\left(\overline{A}\right)^c, \left(\overline{B}\right)^c$ are dense. Let $S := A \cup B$. Let $T \neq \varnothing$ be open. $\left(\overline{A}\right)^c$ dense $\implies T \cap \left(\overline{A}\right)^c \neq \varnothing$. Further, $T \cap \left(\overline{A}\right)^c$ is a nonempty open set whence $\varnothing \neq T \cap \left(\overline{A}\right)^c \cap \left(\overline{B}\right)^c = T \cap \left(\overline{A} \cup \overline{B}\right)^c = T \cap \left(\overline{A \cup B}\right)^c = T \cap \left(\overline{S}\right)^c$ whence S is rare.

(c) A rare $\iff \left(\overline{A}\right)^o = \varnothing \implies \left(\overline{\left(\overline{A}\right)}\right)^o = \left(\overline{A}\right)^o = \varnothing$.

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Exercise *Let A, B be closed sets such that $(A \cup B)^o \neq \varnothing$. Show that either $A^o \neq \varnothing$ or $B^o \neq \varnothing$.*

Exercise *Give examples of two sets $A, B \subseteq \mathbb{R}$ such that $(A \cup B)^o \neq \varnothing$ but $A^o = B^o = \varnothing$.*

Exercise *Show that either $\mathbb{Q}$ can be written as a countable union of rare sets in $\mathbb{R}$.*

The above proposition must ring a bell in your mind and raise a question like "What about the *countable* union of rare sets?" One recalls the example that $\mathbb{Q}$ is a countable union of rare sets in $\mathbb{R}$ but $\mathbb{Q}$ is not itself rare - $\overline{\mathbb{Q}} = \mathbb{R}$ whence $\left(\overline{\mathbb{Q}}\right)^o = \mathbb{R}$. Such countable unions are not dense and mathematicians gave a name for it.

Definition 4 *Let A be a subset of a metric space (X, d).*
A is said to be meagre or of the first category if A can be written as a countable union of rare sets in X.
If A is not meagre, it is said to be nonmeagre or of the second category.
A is said to be residual if its complement is meagre.

We further see another small, but useful result.

Proposition 5 *The following are equivalent for a metric space (X, d). Note that we are not yet claiming about their truth or falsity.*

(a) A meagre set has empty interior.

(b) A countable intersection of open dense sets is dense.

(c) A residual set is dense.

PROOF We prove them in a circular way as follows.

$(a) \implies (b)$: Note that the complement of an open dense set is a closed rare set. Let $\mathscr{U}$ be a countable collection of open dense sets in X and consider $S := \bigcap_{U \in \mathscr{U}} U$. Then $S^c := \bigcup_{U \in \mathscr{U}} U^c$ is a countable union of closed rare sets. By definition, S^c is meagre, whence by hypothesis, $(S^c)^o = \varnothing$. But $(S^c)^o = \left(\overline{S}\right)^c$ so that $\overline{S} = X$.

$(b) \implies (c)$: By definition, a residual set is the complement of a meagre set whence it is a countable intersection of some sets with dense interiors. In other words, a rare set contains a countable intersection of open dense sets, which is dense by hypothesis. Since any superset of a dense set must be dense, conclude that a residual set is dense.

$(c) \implies (a)$: Let S be meagre. Then S^c is residual. By hypothesis, $\overline{S^c} = X \implies S^o = \left(\overline{S^c}\right)^c = \varnothing$.

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We have built up to an important result known as the Baire category theorem.

Theorem 6 (Baire category theorem) *Let (X, d) be a complete metric space. A countable intersection of open dense sets in a complete metric space (X, d) is dense.*

PROOF Let $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable collection of open dense sets in X. Let $V \neq \varnothing$ be any open set. Clearly $V \cap U_1 \neq \varnothing$. Pick a closed disc $\overline{\mathscr{B}_{r_1}(x_1)} \subset V \cap U_1$ with $r_1 < 1$. Since U_2 is dense, $\mathscr{B}_{r_1}(x_1) \cap U_2 \neq \varnothing$ (also open). So pick a closed disc $\overline{\mathscr{B}_{r_2}(x_2)} \subset \mathscr{B}_{r_1}(x_1) \cap U_2$ such that $r_2 < \frac{1}{2}$. Continuing this process will give us a decreasing sequence of closed balls $\mathscr{B}_{r_n}(x_n)$ with $0 < r_n < \frac{1}{n}$. Further notice that the sequence (x_n) is Cauchy in X: for any $n \in \mathbb{N}$, we can pick $N = n$ so that $p \geq q \geq N \implies d(x_p, x_q) \leq \frac{1}{q} \leq \frac{1}{n}$. By completeness, X converges to a point, say x, in X. By definition, for any $n \in \mathbb{N}, \exists N \geq n \in \mathbb{N}$ such that $x \in \mathscr{B}_{\frac{1}{n}}(x_k) \forall k \geq N$; but $k \geq N \geq n \implies x \in \mathscr{B}_{\frac{1}{n}}(x_k) \subseteq \overline{\mathscr{B}_{\frac{1}{n}}(x_n)} \subseteq V \cap \left(\bigcap_{i=1}^{n} U_i \right)$. This means $x \in U_i \forall i$ and $x \in V$ whence $V \cap \left(\bigcap_{i \in \mathbb{N}} U_i \right) \neq \varnothing$. Since V was an arbitrary open set to start with, we conclude that $\bigcap_{i \in \mathbb{N}} U_i$ is dense in X. ■

Corollary 7 *One cannot write a complete metric space (X, d) as a countable union of rare sets. In other words, a complete metric space (X, d) is not meagre.*

Corollary 8 *A residual set in a complete metric space (X, d) is not meagre.*

PROOF We can note that a countable union of meagre sets is meagre ($\because$ a countable union of countable sets is countable). Let $A \subseteq X$ be residual, whence A^c is meagre. If A were meagre, so would be $X = A \cup A^c$. But A^c is meagre, so that X is meagre which is clearly false. ■

Corollary 9 $\mathbb{R} \setminus \mathbb{Q}$ *is not meagre in* $\mathbb{R}$.

PROOF $\mathbb{Q}$ is meagre $\implies$ $\mathbb{R} \setminus \mathbb{Q}$ is residual and thus, by the previous corollary, not meagre. ■

Corollary 10 $\mathbb{Q}$ *cannot be written as the intersection of countably many open sets in* $\mathbb{R}$.

PROOF Say $\mathbb{Q} = \bigcap_{n \in \mathbb{N}} U_n$ for some collection of open sets $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ in $\mathbb{R}$. Note that $U_n \supseteq \mathbb{Q} \implies \overline{U_n} = \overline{\mathbb{Q}} = \mathbb{R} \forall n$ whence each U_n is an open dense set in $\mathbb{R}$. Also note that $\mathscr{V} = \{\mathbb{R} \setminus \{q\} : q \in \mathbb{Q}\}$ is a countable collection of open dense sets in $\mathbb{R}$. Further $\bigcap_{V \in \mathscr{V}} V = \varnothing$ whence $\bigcap_{S \in \mathscr{U} \cup \mathscr{V}} S = \varnothing$ which contradicts theorem 6. ■