Consider a finite dimensional vector space V with $\dim V = n$

$B = \{v_1, v_2, \dots, v_n\}$ be a basis

$v \in V$, $\exists$ unique $\{\alpha_i\}_{i=1}^{n}$, $\alpha_i \in F$

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

$$v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_B \rightarrow n \times 1 \text{ matrix}$$

1) Basis Transformation :-

$\mathbb{R}^3$ as $\mathbb{R}$ vector space. $B := \{(1,0,0), (0,1,0), (0,0,1)\}$

$B' := \{(1,1,0), (0,1,0), (0,0,1)\}$

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B \leftarrow (1,2,3) \longrightarrow \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}_{B'}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{B'} = (1,1,0) + (0,2,0) + (0,0,3) = (1,3,3)$$

$$(1,2,3) \in \mathbb{R}^3 \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{B'} \qquad x(1,1,0) + y(0,1,0) + z(0,0,1) = (1,2,3)$$

$$x = 1$$
$$x + y = 2 \Rightarrow y = 1$$
$$z = 3$$

$$(1,2,3) \longrightarrow \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}_{B'}$$

V be F- vector space, $\dim V = n$,

$$B = \{v_1, \dots, v_n\}, \quad B' = \{v_1', \dots, v_n'\}$$

$v = C_1 \longrightarrow n \times 1$ matrix

C_2

$$v_1 = a_{11} v_1' + a_{21} v_2' + \dots + a_{n1} v_n'$$

$$v_2 = a_{12} v_1' + a_{22} v_2' + \dots + a_{n2} v_2'$$

$$\vdots$$

$$v_n = a_{1n} v_1' + a_{2n} v_2' + \dots + a_{nn} v_n'$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} = A,$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_B$$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$$

$$Ax = \sum_{i=1}^{n} x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$$

Note that $v_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}_{B'}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_B$$

$$X = x_1 v_1 + \cdots + x_n v_n = x_1 (a_{11} v_1' + \cdots + a_{n1} v_n') + \cdots + x_n (a_{1n} v_1' + \cdots + a_{nn} v_n')$$

$$= \sum_{i=1}^{n} (x_1 a_{i1} + x_2 a_{i2} + \cdots + x_n a_{in}) v_i'$$

$$\begin{bmatrix} x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \cdots + x_n a_{2n} \\ \vdots \\ x_1 a_{n1} + \cdots + x_n a_{nn} \end{bmatrix} = \sum_{i=1}^{n} x_i \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = Ax$$

Theorem :- Let $B = \{v_1, \ldots, v_n\}$ and $B' = \{v_1', \ldots, v_n'\}$ be basis for V.

Let $v_i = \sum_{j=1}^{n} a_{ji} v_j'$, $1 \le i \le n$

(i.e. co-ordinate of v_i wrt. B' is

$$\begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = C_i$$

Then the matrix $A = [C_1 \; C_2 \; \cdots \; C_n] = [a_{ij}]_{i=1}^{n}{}_{j=1}^{n}$

is defined to be the "matrix of B wrt. B'"

and if a vector v has co-ordinate $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ wrt. B

then the column

$X' = Ax$ is the representation of v wrt B'

Remarks :-

1) Given two bases B and B' the matrix of B wrt. B' is unique.

2) Let A denote the matrix of B wrt. B'. Then A is invertible and matrix of B' wrt. B is A^{-1}

A'

$v \in V$, x, x'

$x' = Ax$ $\qquad$ $AA'x' = x'$, $\forall x'$

$x = A'x'$ $\Rightarrow$ $A'Ax = x$

$\Rightarrow$ $Bx = x$ $\quad \forall$ coloumn vector x

$x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\Rightarrow$ 1st of coloumn B is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

i_{th} coloumn of B is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i_{th}$ position

$\Rightarrow B = Id_{n \times n} = 1_n$

$A'A = 1_n$, $AA' = 1_n$ $\Rightarrow$ $A' = A^{-1}$

Matrix of a linear map :-

Let $T : V \rightarrow W$ be a linear map, where $\dim V = n$, $\dim W = m$. Fix a basis $B = \{v_1, \ldots, v_n\}$ for V and $C = \{w_1, \ldots, w_m\}$ " W.

Let the co-ordinate of $T(v_i)$ wrt. C be

$$\begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} = C_i$$

Then define " the matrix of T wrt. B and C " or in notation $\mathcal{M}_B^C(T)$ to be the $m \times n$ matrix

$$[C_1 \;\; C_2 \;\; \cdots \;\; C_n] = [a_{ij}]_{i=1}^{m} {}_{j=1}^{n}$$

Example :- Let $V = W = \mathbb{Q}^3$ as $\mathbb{Q}$ vector space

Let $T : V \to W$, $T(x,y,z) = (9x+4y+5z,\ -4x-3z,\ -6x-4y-2z)$

Then $M_B^C(T)$ where $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$C = \{(1,0,0), (0,1,0), (0,0,1)\}$

we see
$$T(1,0,0) = (9,-4,-6)$$
$$T(0,1,0) = (4,0,-4) \quad \Rightarrow \quad M_B^C(T) = \begin{bmatrix} 9 & 4 & 5 \\ -4 & 0 & -3 \\ -6 & -4 & -2 \end{bmatrix}$$
$$T(0,0,1) = (5,-3,-2)$$

Had we taken $C' = \{(1,1,0), (0,1,0), (0,0,1)\}$

Co-ordinate of $(9,-4,-6)$ wrt C' is $\begin{bmatrix} 9 \\ -13 \\ -6 \end{bmatrix}$

" " $(4,0,-4)$ " C' " $\begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}$

" " $(5,-3,-2)$ " C' " $\begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$

So, matrix of T wrt B and C' is

$$M_B^{C'}(T) = \begin{bmatrix} 9 & 4 & 5 \\ -13 & -4 & -8 \\ -6 & -4 & -2 \end{bmatrix}$$

$T : \underset{B}{V} \to \underset{C}{W}$

$v \in V$ $\quad M_B^C(T)$

$\downarrow$

x $\quad T(v)_C \quad M_B^C(T) \cdot x$

So different basis $\Rightarrow$ different matrix

Recall that $\mathrm{Hom}(V,W)$ is F- vector space

Theorem :- Thus, we get an one-to-one correspondence between $M_{m\times n}(F)$ and $\mathrm{Hom}(V,W)$, $(\dim V = n,\ \dim W = m)$.

More explicity, fix any bases B and C for V and W. The map

$$\varphi : \mathrm{Hom}(V,W) \longrightarrow M_{m\times n}(F)$$
$$T \longmapsto M_B^C(T)$$

is a $\boxed{\text{F- linear isomorphism}}$

Remark :- Note that $M_{m\times n}(F) \cong F^{mn} \Rightarrow \mathrm{Hom}(V,W)$ is finite dimensional vector space over F with dim. mn

Remark :- Note that for any isomorphism T, $M_B^C(T)$ is invertible and $M_B^C(T^{-1}) = \left(M_B^C(T)\right)^{-1}$

In that case we must have $m=n$

Theorem :- Let $T : V \to W$, $S : W \to U$ be linear maps.

Fix bases B, C, D for V, W, U respectively, the

$$M_B^D(S\cdot T) = M_C^D(S)\cdot M_B^C(T)$$

Corollary :-

Let V be a n-dim. vector space. Then the space of all isomorphisms on V, $GL(V)$ is isomorphic to $GL_n(F)$

Linear maps under change of Basis :-

Given $\dim V = n$, $\dim W = m$ we saw that for a linear map $T: V \to W$

$M_B^C(T)$ might vary as we vary the bases B and C

Suppose that C' is another basis of W. Then what is $M_B^{C'}(T)$?

So if we take any vector in V whose coordinate is x wrt. B, T will send that vector to a vector in W whose co-ordinate wrt. C is

$$M_B^C(T) \cdot x$$

matrix multiplication

$v \to x \cdots B$

$T(v) \to M_B^C(T) \cdot x \cdots C \cdots$

$\downarrow$

$T(v) = M_C^{C'} \cdot M_B^C(T) \cdot x$

Now wrt C' this vector has co-ordinate

$$M_C^{C'} \cdot \left(M_B^C(T) \cdot x\right)$$

where $M_C^{C'}$ denotes the matrix for C wrt. C'

So clearly

$$M_B^{C'}(T) = M_C^{C'} \cdot M_B^C(T)$$

Similarly if we want to find out

$$M^{C}_{B'}(T)$$

where B' is another basis for V,

Take any vector whose coordinate is x wrt. B'

$\downarrow$

convert its co-ordinate wrt. B, i.e. consider

$$M^{B}_{B'} \cdot x$$

$\downarrow$

Apply T on it ...

$$M^{C}_{B}(T) \cdot \left(M^{B}_{B'} \cdot x \right)$$

So

$$M^{C}_{B'}(T) = M^{C}_{B}(T) \; M^{B}_{B'}$$

You might use another viewpoint. Note that matrix of ε wrt. ε', where ε and ε' are basis of a vector space V, is nothing but

$$M^{\varepsilon'}_{\varepsilon}(Id) = M^{\varepsilon'}_{\varepsilon}$$

$$\underset{\varepsilon}{V} \longrightarrow \underset{\varepsilon'}{V}$$

So to find $M^{C'}_{B}(T)$, note that

$$\underset{B}{V} \xrightarrow{T} \underset{C}{W} \xrightarrow{Id} \underset{C'}{W}$$

So we are looking for

$$M_B^{C'}(T) = M_B^{C'}(Id \circ T) = M_C^{C'}(Id) \cdot M_B^{C}(T)$$

$$= M_C^{C'} \cdot M_B^{C}(T)$$

Agreeing to the previous derivation.

Now the ultimate touch; Given $M_B^C(T)$, find $M_{B'}^{C'}(T)$

So

$$M_{B'}^{C'}(T) = M_C^{C'} \cdot M_{B'}^{C}(T)$$

$$= M_C^{C'} \cdot \left(M_B^{C}(T) \cdot M_{B'}^{B} \right)$$

$$= M_C^{C'} \cdot M_B^{C}(T) \cdot M_{B'}^{B}$$

That makes sense;

Given a vector v with co-ordinate x in B'

$\downarrow$

Convert it to B, i.e. $M_{B'}^{B} \cdot x$

$\downarrow$

Apply T, i.e. $M_B^C(T) \cdot (M_{B'}^{B} \cdot x)$ (Coordinate of $T(v)$ wrt C)

Get co-ordinate wrt. C

$\downarrow$

Convert it to co-ordinate wrt. C', i.e.

$$M_C^{C'} \cdot \left(M_B^{C}(T) \cdot \left(M_{B'}^{B} \cdot x \right) \right)$$