Topology on $\mathbb{R}$.

Open intervals : $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

$(-\infty \leq a \leq b \leq \infty)$

Open balls around $z \in \mathbb{R}$ (of radius r) :

$B(z,r) = (z-r,\ z+r) = \{y \in \mathbb{R} : |y-z| < r\}$

Definition : A set $U \subseteq \mathbb{R}$ is said to be an <u>open set</u> if it is a union (arbitrary) of open intervals.

Examples of open sets :

i) $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$

$\phi = \bigcup_{X \in \mathcal{I}} X$ where $\mathcal{I} = \{(n,n)\}$

ii) All open intervals.

$(a,b) = \bigcup_{X \in \mathcal{I}} X$ $\quad \mathcal{I} = \{(a,b)\}$

Lemma :

We say $X \subseteq \mathbb{R}$ has property (P) if $\forall\, x \in X$ $\exists\ r > 0$ s.t. $B(x,r) \subseteq X$.

Let $U \subseteq \mathbb{R}$. U is open iff U has property (P)

Pf : Suppose $U \subseteq \mathbb{R}$ is open.

By defn $\exists$ a collection $\mathcal{I}$ of open intervals in $\mathbb{R}$, say $\mathcal{I} = \{(a_\lambda, b_\lambda)\}_{\lambda \in \Lambda}$ (Λ is our indexing set), s.t. $U = \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda)$.

Let $x \in U$. Then $\exists\ \mu \in \Lambda$ s.t. $x \in (a_\mu, b_\mu)$.

Take $r = \frac{1}{2}\min\{|x-a_\mu|, |x-b_\mu|\} > 0$.

Clearly $B(x,r) \subseteq (a_\mu, b_\mu) \subseteq U$.

Now suppose U has property (P).

So $\forall x \in U \ \exists r_x > 0$ s.t. $B(x, r_x) \subseteq U$.

Clearly $U = \bigcup_{x \in U} B(x, r_x)$. $\square$

Lemma: (Arbitrary) union of open sets is open.

Pf: Let $\{U_\gamma\}_{\gamma \in \Gamma}$ be a collection of open sets in $\mathbb{R}$.

$\exists \ \mathcal{I}_\gamma = \{(a_\lambda, b_\lambda)\}_{\lambda \in \Lambda_\gamma}$ $(\forall \gamma \in \Gamma)$

s.t.

$$U_\gamma = \bigcup_{\lambda \in \Lambda_\gamma} (a_\lambda, b_\lambda)$$

Take $\mathcal{I} = \bigcup_{\gamma \in \Gamma} \mathcal{I}_\gamma$ which gives that

$$\bigcup_{\gamma \in \Gamma} U_\gamma = \bigcup_{\gamma \in \Gamma} \bigcup_{\lambda \in \Lambda_\gamma} (a_\lambda, b_\lambda) = \bigcup_{X \in \mathcal{I}} X .$$

Since $\mathcal{I}$ contains only open intervals in $\mathbb{R}$,

$\bigcup_{\gamma \in \Gamma} U_\gamma$ is open set. $\square$

Lemma: (i) If $U, V \subseteq \mathbb{R}$ are open, so is $U \cap V$.

(ii) Finite intersection of open sets is open.

Exercise: Provide a counterexample to the above if the intersection is for arbitrary no. of open sets. $\left(-\frac{1}{n}, \frac{1}{n}\right) \to$ Intersection $= \{0\}$.

Definition: A set $F \subseteq \mathbb{R}$ is said to closed if $F^c (= \mathbb{R} \setminus F)$ is open.

Exercise: Let $S \subseteq \mathbb{R}$. Can S be both open & closed?

Yes, ϕ & $\mathbb{R}$ are both clopen.

Exercise: The only clopen subsets of $\mathbb{R}$ are $\mathbb{R}$ & ϕ.

A look into the future: A topological space X is connected iff the only clopen sets (of X) are ϕ & X.

Exercise: Let $a, b \in \mathbb{R}$ $(a \neq b)$. Then $\exists$ open sets U & V s.t.

- $a \in U$, $b \in V$
- $U \cap V = \phi$

why: $r = \dfrac{|b-a|}{4}$.

Choose $U = B(a, r)$, $V = B(b, r)$.

This is called Hausdorff property.

Let $X \subseteq \mathbb{R}$ and $\mathcal{J} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a collection of open sets in $\mathbb{R}$. We say $\mathcal{J}$ is an open cover of X if $X \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$.

Example: $X = \mathbb{R}$, $\mathcal{J} = \{(-n, n)\}_{n \in \mathbb{N}}$. $\mathcal{J}$ is an open cover of X.

Exercise: Show that $\nexists$ a finite subset $\mathcal{S} \subseteq \mathcal{J}$ s.t. $\mathcal{S}$ is an open cover of $\mathbb{R}$.

Sol: Say we can find finite such $\mathcal{S} = \{(-n_i, n_i)\}_{i=1}^{k}$.

Let $n := \max\{n_1, \dots, n_k\}$

Then $\bigcup_{U \in \mathcal{S}} U = (-n, n) \not\supseteq \mathbb{R}$.

This is an open cover of $\mathbb{R}$ with no "finite subcover"

Let $a \in \mathbb{R}$. We say $X \subseteq \mathbb{R}$ is called a "neighbourhood" of a if $\exists\, r > 0$ s.t. $B(a, r) \subseteq X$.