Real Analysis

Problem Set 6

June 21, 2021

1. Using the condensation test, determine whether $\sum x_n \in \mathbb{R}$, where x_n are as follows:

   (a) $x_n = \dfrac{1}{n}$

   (b) $x_n = \dfrac{1}{(n+1)\log(n+1)}$

   (c) $x_n = \dfrac{1}{n^2}$

   (d) $x_n \dfrac{1}{\left(\log(n+1)\right)^2}$

   (e) $x_n \dfrac{1}{(n+1)\left(\log(n+1)\right)^2}$

   (f) $x_n = \dfrac{\log n}{n^2}$

   (g) $x_{n-15} = \dfrac{1}{n\left(\log n\right)\left(\log\log n\right)\left(\log\log\log n\right)^p}$ if $p > 1$

   (h) $x_{n-15} = \dfrac{1}{n\left(\log n\right)\left(\log\log n\right)\left(\log\log\log n\right)^p}$ if $p \leq 1$

   (i) $x_n = \dfrac{1}{n^p}$ if $p > 1$

   (j) $x_n = \dfrac{1}{n^p}$ if $0 < p \leq 1$

2. Determine whether the following sequences converge in $\mathbb{R}$:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n^2+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{3n^2+1}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{2^n}{1+4^n}}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\left(\log n\right)^p}$ where $p > 0$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(\log n\right)^n}$

   (g) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\left(\log n\right)\left(\log\log n\right)^p}$ where $p > 0$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\sqrt{n}-(-1)^n\right)}{n}$

3. Let $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$, $A_n := \sum_{i=1}^{n} a_i$ and let $\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ be a function. Prove that

   (a) $\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{j} \left(\alpha\left(i,j\right)\right) = \sum_{i=1}^{n}\sum_{j=i}^{n} \left(\alpha\left(i,j\right)\right)$

   (b) $\displaystyle\sum_{i=1}^{n} a_i b_i = b_{n+1}A_n - \sum_{i=1}^{n} A_i\left(b_{i+1}-b_i\right)$

4. Let $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}, A_n := \sum_{i=1}^{n} a_i$. Suppose (A_n) is bounded. It is given that $\sum_{i=1}^{n} (b_{i+1} - b_i)$ converges absolutely and $\lim_{n\to\infty} b_n = 0$.

   (a) Show that $\lim_{n\to\infty} A_n b_{n+1} = 0$.

   (b) Show that $\displaystyle\sum_{i=1}^{n} A_i (b_{i+1} - b_i)$ is convergent.

   (c) Conclude that $\displaystyle\sum_{i=1}^{n} a_i b_i$ converges.

5. Prove using the above

   (a) If $(x_n) \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ is decreasing and $\lim x_n = 0$ then $\sum (-1)^n x_n < \infty$.

   (b) If $(x_n) \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ is such that $\exists B > 0$ satisfying $\sum_{i=1}^{n} x_i \leq B \, \forall n$, then $\displaystyle\sum \frac{a_n}{n} < \infty$.

6. Let $a > 0$. Prove $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(a+n+1)(a+n)} < \infty$. Find the limit.

7. Let $a > 0$ and $m \in \mathbb{N}$.

   (a) Show that $\displaystyle\sum_{k=1}^{n} \frac{m}{\prod_{j=0}^{m}(a+k+j)} = \frac{1}{\prod_{j=1}^{m}(a+j)} - \frac{1}{\prod_{j=1}^{m}(a+n+j)}$.
   Hint: Induct on n.

   (b) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\prod_{j=0}^{m}(a+n+j)} = \frac{1}{m\prod_{j=1}^{m}(a+j)}$

8. Let $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}, A_n := \sum_{i=1}^{n} a_i, A_n := \sum_{i=1}^{n} b_i$. Prove that [1]

$$\sum_{k=n+1}^{m} a_k B_k = A_m B_m - A_n B_{n+1} - \sum_{k=n+1}^{m-1} A_k b_{k+1}$$

9. *(Use your knowledge of high-school integration)* Let $(a_n) \in \mathbb{R}^{\mathbb{N}\cup\{0\}}$ be a sequence and let its partial sums be $A_n := \sum_{k=0}^{n} a_n$. Fix real numbers $x < y$. $\varphi : [x, y] \to \mathbb{R}$ is a continuously differentiable function. Show that

$$\sum_{n=\lfloor x\rfloor+1}^{\lfloor y\rfloor} a_n \varphi(n) = A\left(\lfloor y\rfloor\right)\varphi(y) - A\left(\lfloor x\rfloor\right)\varphi(x) - \int_x^y A\left(\lfloor t\rfloor\right)\varphi'(t)\,dt$$

[1] *(Maybe a hint)* One is tempted to recall the integration by parts formula. Let $F(x) := \int_a^x f(x)\,dx$, $G(x) := \int_a^x g(x)\,dx$. Then

$$\int_a^b f(x)G(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F(x)g(x)\,dx$$