MIT 6.7230/18.456
Algebraic techniques and semidefinite programming
Homework assignment # 2

Date Given: February 21st, 2024
Date Due: March 1st, 1:00PM

P1. [***20 pts***] Consider the following SDP in the variables (x, y):

$$\min x \qquad \text{s.t.} \quad \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix} \succeq 0.$$

- **(a)** Draw the feasible set. Is it convex?
- **(b)** Write the dual SDP.
- **(c)** Is the primal strictly feasible? Is the dual strictly feasible?
- **(d)** What can you say about strong duality? Are the results consistent with Theorem 3.1 in the Vandenberghe & Boyd paper?

P2. [***20 pts***] Given a set $\mathcal{S} \subseteq \mathbb{R}^n$ strictly containing the origin, its *polar set* $\mathcal{S}^o \subseteq \mathbb{R}^n$ is:

$$\mathcal{S}^o := \{y \in \mathbb{R}^n \,|\, y^T x \leq 1, \quad \forall x \in \mathcal{S}\}.$$

- **(a)** Let $\mathcal{S}$ be the polyhedron $\mathcal{S} = \text{conv}(\{v_1, \ldots, v_m\})$. What is its polar $\mathcal{S}^o$?
- **(b)** Let $\mathcal{S}$ be the (possibly off-center) circle $\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - a)^2 + x_2^2 \leq 1\}$, where $|a| < 1$. What is its polar $\mathcal{S}^o$?
- **(c)** Let $\mathcal{S}$ be the feasible set of an SDP, i.e.,

$$\mathcal{S} = \{x \in \mathbb{R}^m \,|\, \sum_{i=1}^{m} x_i A_i \preceq C\},$$

where A_i and C are given symmetric matrices, and $C \succ 0$. Find a convenient description of $\mathcal{S}^o$. Can you use it to optimize a linear function over $\mathcal{S}^o$?

P3. [***20 pts***] Consider the graph given in Figure 1, known as the Petersen graph.

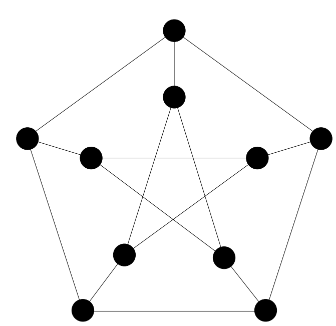

Figure 1: Petersen graph

- **(a)** Compute the SDP upper bound on the size of its largest stable set (i.e., the Lovász theta function).

(b) Compute the SDP upper bound on the value of its maximum cut.

(c) Are these bounds tight? Can you find the true optimal solutions?

We suggest to use a suitable parser (e.g., CVX, YALMIP, JuMP) for the formulation of the corresponding SDP.

P4. [***15 pts***] The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to color all vertices, in such a way that adjacent vertices receive distinct colors.

(a) Consider the SDP (5) from Lecture 2, used to compute the Lovász theta function of a graph G. Write the corresponding dual SDP.

(b) Show that the inequality

$$\vartheta(G) \leq \chi(\bar{G})$$

holds, where $\bar{G}$ is the complement of the graph G. Thus, we can use the theta function to produce lower bounds on the chromatic number.

Hint: Given a coloring of $\bar{G}$, construct a feasible solution of the dual SDP.

P5. [***10 pts***] Consider the Euclidean distance matrix characterization presented in the notes for Lecture 2 (Section 3.3). Show that the given characterization always implies the triangle inequality $d_{ik} \leq d_{ij} + d_{jk}$ for all triples (x_i, x_j, x_k) of points. Is the converse true?

P6. [***15 pts***] Consider the discrete-time dynamical system:

$$x[k+1] = A\,x[k] + B\,u[k], \qquad x[0] = x_0.$$

(a) Let

$$A = \begin{bmatrix} 0 & 0 & 3 & -2 & 0 \\ 4 & 1 & 3 & 0 & 2 \\ -3 & 5 & 2 & 0 & -4 \\ -1 & 1 & 6 & 2 & 1 \\ 1 & 2 & -3 & -3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \\ 0 & 1 \\ 0 & -2 \end{bmatrix}.$$

Find a control $u = Kx$ such that the matrix $A + BK$ is Schur stable.

(b) The system has a *non-stabilizable mode* if the matrix A has a nonzero left eigenvector w such that $w^T A = \lambda w^T$, $w^T B = 0$, and $|\lambda| \geq 1$. Show that if this is the case, then the final synthesis SDP in Section 3.1 cannot be feasible. Interpret this statement in terms of the eigenvalues of $A + BK$.

(c) Can you modify the given SDP formulation in such a way that all the eigenvalues of $A + BK$ satisfy $|\lambda_i| \leq \frac{1}{2}$? Test your proposed solution with the matrices A, B given above.

P7. [***25 pts***] Here we study numerically the standard SDP relaxation for the boolean optimization problem described in Lecture 3, i.e., the minimization of $x^T Q x$ over vectors x with components ± 1. For the matrix Q you will generate at the beginning a random 200×200 symmetric matrix (e.g., with entries that are i.i.d. unit Gaussians, `Q=randn(200,200); Q=Q+Q';` in MATLAB or Julia). The matrix Q will be kept fixed throughout the exercise (so save it!).

(a) Assume we use a naive randomization scheme, where the variables x_i are picked independently, and are equal to ± 1 with equal probability. What is the expected value of the objective?

(b) Assume we solve the SDP relaxation, and use Goemans-Williamson rounding instead. What is the expected value of the cost? How does this compare with the previous value? What is the best solution x you have found?

(c) Produce histograms of the achieved objective values for both procedures, for a large number of samples (e.g., 10000). What do you observe? Are the results consistent with the expected values you computed earlier, and with the SDP lower bound?

(d) (Optional) Experiment with other matrices Q. For instance, you can try using other distributions, adding sparsity, low-rank matrices, etc. Any interesting observations?

P8. [***15 pts***] Consider the primal-dual pair of relaxations for optimization problems presented in the notes for Lecture 3, Section 4 ("Linearly constrained problems").

(a) Verify that they indeed constitute a primal-dual pair of SDPs.

(b) Why does the solution of (12) provide a lower bound on the objective?

(c) What is the relationship between the matrix X and the variable of the original optimization problem?