Problem Set - 1

1) $V = \mathbb{R}^n$, and $(a_1, a_2, \ldots, a_n)$ be a fixed vector in V. Prove that the collection of all vectors $(x_1, \ldots, x_n)$ such that

$$a_1 x_1 + \cdots + a_n x_n = 0$$

is a subspace of V and also find its dimension

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2) Prove that the space of real valued functions on $[a, b]$ is an infinite dimensional vector space over $\mathbb{R}$.

Prove that the space of real valued continuous functions on $[a, b]$ is an infinite dimensional vector space over $\mathbb{R}$.

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3) let $\varphi : V \longrightarrow W$ be linear map and V and W are finite dimensional vector space such that $\dim V = \dim W$

$$\varphi \text{ injective} \Rightarrow \varphi \text{ isomorphism}$$
$$\varphi \text{ surjective} \Rightarrow \varphi \text{ isomorphism}$$

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4) $\varphi : V \longrightarrow V$ be a linear map and V is finite dimensional vector space. Prove that $\exists\, m \in \mathbb{N}$ such that

$$\operatorname{Im}(\varphi^m) \cap \ker(\varphi^m) = \{0\}$$

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5) Let $\varphi : V \to V$ be F-linear map. A nonzero $v \in V$ is defined to be an "eigenvector of φ with eigenvalue λ" if

$$\varphi(v) = \lambda v$$

Prove that for given $\lambda \in F$, the set of eigenvectors with eigenvalue λ is a vector sub-space.

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6) Let $\varphi : V \longrightarrow V$ be a linear map and let $\{v_i\}_{i=1}^{k}$ be be eigenvectors of φ with corresponding eigenvalue $\{\lambda_i\}_{i=1}^{k}$. Also assume that $\lambda_i \neq \lambda_j$ for $1 \leq i \neq j \leq k$. Prove that $\{v_i\}_{i=1}^{k}$ is linearly independent.

Thus any $\varphi : V \to V$, $\dim V = n$ has atmost n distinct eigenvalues.