Algebraic Complement

Let V be a F-vector space.

Def^{n} :- Let W be a subspace of V. " $\boxed{\text{An}}$ algebraic complement of W in V " is a subspace U of V s.t

$$U \cap W = \{0\}$$

and

$$W + U = V$$

Note that algebraic complement is a subspace

Theorem :- For any subspace W of V, its algebraic complement exists. If V is finite dimensional then $\dim W_F^c = \dim V - \dim W$

Theorem :- If W is a subspace of V and let U be an algebraic complement of W. Then for every $v \in V$, $\exists$ unique $u \in U$ and $w \in W$ s.t $v = u + w$, i.e. if

$$v = u + w = u' + w'$$

$u, u' \in U$; $w, w' \in W$, then we have

$$u = u', \quad w = w'$$

Proof $\rightarrow$ If $v = u + w = u' + w' \Rightarrow u - u' = w' - w = v'$

Note that $v' \in U$, $v' \in W \Rightarrow v' \in U \cap W = \{0\} \Rightarrow v' = 0$

$\Rightarrow u = u'$
$w = w'$

Note that given a subspace W of V, there might exist more than one algebraic complements of W. But all these algebraic complements are "unique up to isomorphism"

Proposition :- Given a subspace W of V, all its algebraic complements in V are isomorphic to V/W. Thus every pair of algebraic complements are isomorphic (unique upto isomorphism)

Proof $\to$ Let U be an al ebraic com lement of W.

Consider the map

$$\left.\begin{aligned} \phi : U &\longrightarrow V/W \\ u &\longmapsto u + W \end{aligned}\right\} \quad \text{Note that } \phi = \pi|_U \quad \pi : V \to V/W, \ v \mapsto v + W$$

Check that this is a well defined, linear map

It is injective as $\phi(u) = \phi(u') \Rightarrow u + W = u' + W \Rightarrow u - u' \in W$

but $u - u' \in U \Rightarrow u - u' = 0$ as $U \cap W = \{0\}$

It is surjective as, take any typical element of V/W,

$$v + W$$

We know that $v = u + w$, $u \in U$, $w \in W$

Then $(u + w) + W = u + W$

Thus done!

Now we are going to enjoy the above propositions,

Theorem :-

1) Let $\varphi : V \to W$ be linear map. Then

$$(\ker \varphi)^c_F \cong \operatorname{Im} \varphi$$

Proof :- We have

$$(\ker \varphi)^c_F \xrightarrow[\pi]{\sim} V/\ker \varphi \xrightarrow{\sim} \operatorname{Im} \varphi$$

1st Isomorphism Thm.

However one might note that a possible isomorphism between $(\ker \varphi)^c_F$ and $\operatorname{Im} \varphi$ is $\varphi|_{(\ker \varphi)^c_F}$ itself.

2) Rank - Nullity :- Let V and W be finite dimensional

Then

$$\underbrace{\dim (\ker \varphi)}_{\text{Nullity}} + \underbrace{\dim (\operatorname{Im} \varphi)}_{\text{Rank}} = \dim V$$

Proof → $\dim (\ker \varphi) + \dim ((\ker \varphi)^c_F) = \dim V$

By the previous theorem done!

3) Let V be finite dimensional. W be subspace

Then

$$\dim V/W = \dim V - \dim W$$

Proof → Consider $\pi : V \longrightarrow V/W$. Now apply Rank-Nullity.