Algebra Qualifying Exams

Rutgers - the State University of New Jersey

Syllabus

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Contents

Spring 2023

Groups

Classify all groups of order 309, up to isomorphism.

Groups

Let A be the abelian group with generators x, y, z and the relations

$$4x+3y+z=0, x+2y+3z=0, 3x+2y+5z=0$$

Show that A is a cyclic abelian group, and determine its order.

Linear Algebra

Let A be a complex $n\times n$ matrix. Prove that there is an invertible complex $n\times n$ matrix B such that $AB = BA^t$. (A^t is the transpose of A.)

Solution

The given statement is equivalent to showing the existence of an invertible B such that $A^t = B^{-1}AB$. This is just saying that A, A^t are similar. Since we are working over $\mathbb{C}$, we can simply work with JCF. This suffices because if $A = X^{-1}JX$ where J is the JCF of A, then $A^t = B^{-1}AB$ is equivalent to saying that $YJ^tY^{-1} = B^{-1}X^{-1}JXB$ where $Y = X^t$, which is equivalent to saying that $J^t = (XBY)^{-1}X(XBY)$. This is simply saying that J is similar to its transpose. Since J is made of block matrices, transpose treats every square block independently, and using the fact that $\begin{bmatrix} P & \\ & Q \end{bmatrix} \sim \begin{bmatrix} U & \\ & V \end{bmatrix}$ if $P\sim U$ and $Q\sim V$, it is enough to show that every Jordan block is similar to its transpose. (Here $\sim$ stands for similarity of matrices.)
To see this, we start with a Jordan block J of size $n\times n$ and eigenvalue λ. Let $T:\mathbb{C}^n\to\mathbb{C}^n$ be a linear transformation whose matrix with respect to the basis $\boldsymbol{e} = (e_1,\cdots,e_n)$ is J. The action of T is given by $Te_1 = \lambda e_1$ and $Te_j = \lambda e_j + e_{j-1}$ for $1 < j \le n$. Now we look at the matrix of T in the basis $\boldsymbol{f} = (f_1,\cdots,f_n)$ where $f_i = e_{n-i+1}\forall 1\le i\le n$. Clearly the first column of T in this basis is determined by $Jf_1 = \lambda e_n + e_{n-1} = \lambda f_1 + f_2$ which corresponds to the column matrix wehere first two entries are $\lambda, 1$ respectively and everything else is 0. The j^{th} column $(1\le j < n)$ is given by $Tf_j = Te_{n+1-j} = \lambda e_{n+1-j} + e_{n-j} = \lambda f_j + f + j + 1$ which corresponds to the columns where the $j^{\text{th}}, (j+1)^{\text{st}}$ entries are $\lambda, 1$ respectively, and everything else is 0. This means that $[T]_{\boldsymbol{e}} = [T]^t_{\boldsymbol{f}}$. Since both the matrices $[T]_{\boldsymbol{e}}, [T]_{\boldsymbol{f}}$ correspond to the same linear operator, but represented in different bases, they are similar. This proves that every Jordan block is similar to its transpose.

Rings

Prove that the subring $\mathbb{Z}[3i]$ of $\mathbb{C}$ is not a Principal Ideal Domain.

Rings

If $R = \mathbb{Z}[x]$, show that the sequence $R \xrightarrow{f} R^2 \xrightarrow{g} R$ is exact, where $f(a) = (ax, -2a)$ and $g(c,d) = 2c + dx$.

Fall 2022

Groups

Let G be a finite simple group. Prove that $G \times G$ has exactly 4 normal subgroups (including $G \times G$) if and only if G is non-abelian.

Rings

Let R be a principal ideal domain and I, J be ideals of R. Show that $I \cap J = IJ$ holds if and only if $I = 0$ or $J = 0$ or $I + J = R$.

Linear Algebra

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix with real coefficients. Show that all eigenvalues of A are non-negative if and only if $A = P^T P$ for some matrix $P \in M_n(\mathbb{R})$. .

Solution

Suppose $A = P^T P$. Then $P \in M_n(\mathbb{R}) \implies A = P^\dagger P$ where $P^\dagger$ is the conjugate transpose.. Let $(\boldsymbol{x}, \lambda) \in \mathbb{C}^n \times \mathbb{C}$ be an eigenvector-eigenvalue pair for A. Clearly $\boldsymbol{x}^\dagger A\boldsymbol{x} = (P\boldsymbol{x})^\dagger (P\boldsymbol{x}) = \|P\boldsymbol{x}\|^2 \geq 0$. But also $\boldsymbol{x}^\dagger A\boldsymbol{x} = \lambda \boldsymbol{x}^\dagger \boldsymbol{x} = \lambda \|\boldsymbol{x}\|^2$ and $\|\boldsymbol{x}\|^2 > 0$. This shows that $\lambda \in \mathbb{R}_{\geq 0}$.

Suppose A is symmetric real matrix with non-negative eigenvalues. So A is Hermitian, and by the spectral theorem of real symmetric matrices, we can write it as $A = UDU^T$ where D comprises of eigenvalues of A, and U is orthogonal (comprising of an eigenbasis of A). Since eigenvalues are non-negative, D has all non-negative entries $\lambda_1, \cdots, \lambda_n$ in its diagonal (0 elsewhere). Consider $E = \operatorname{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$ so that $D = E^2 = EE^T$. Then $A = A = (UE)(UE)^T$. Taking $P = (UE)^T \in M_n(\mathbb{R})$ gives $A = P^T P$ as desired.

Rings

Let R be an integral domain and $R[x, y, z]$ the polynomial ring in three variables over R. Show that $I = \langle x^3, y^2, y^3 - z^2 y \rangle \subseteq R[x, y, z]$ is a prime ideal.

Hint: Show that I is the kernel of a ring homomorphism $R[x, y, z] \to R[t]$.

Linear Algebra

Let A and B be commuting complex matrices. Assume that $B \notin \mathbb{C}[A]$, that is, B cannot be written as a polynomial in A. Show that some eigenspace of A has dimension at least two.

Spring 2022

Rings

Prove that the rings $\mathbb{Q}[x]/(x^2-1)$ and $\mathbb{Q}\oplus\mathbb{Q}$ are isomorphic.

Groups

Let p be a prime. Show that any element of order p in $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ can be conjugated to the matrix $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$.

Fields

Let a and b be elements of a field of order 2^n where n is odd. Prove that if $a^2+ab+b^2=0$ then $a=b=0$.

Solution

Since F has order 2^n (with n odd, say $2k+1$), we have $x^{2^n-1}=1$ for $x\in F^\times$ because $F^\times$ is a multiplicative group. Further note that $2^n-1=2\times 4^k-1\equiv 1 \pmod 3 \implies (3,2^n-1)=1$. There are integers u,v such that $3u+(2^n-1)v=1$. Note that

$$\begin{aligned} a^2+ab+b^2&=0\\ \implies a^3-b^3=(a-b)(a^2+ab+b^2)&=0\\ \implies a^3&=b^3\\ \implies a=(a)^{3u}\cdot(a)^{(2^n-1)v}=(b)^{3u}\cdot(b)^{(2^n-1)v}&=b\\ \implies a&=b \end{aligned}$$

But $0=a^2+ab+b^2=3a^2 \implies a^2=0$ as F has characteristic 2, whence 3 is invertible. Finally, $a^2=0$ means $a=0$.

Linear Algebra

Let A,B be linear operators on a nonzero finite-dimensional vector space V over $\mathbb{C}$ such that $A^2=B^2=\mathrm{Id}$. Prove that there exists a nonzero subspace W of V which is invariant under A and B and $\dim W\le 2$.

Solution

Consider $S=AB, T=BA$. Then $ST=AB^2A=A^2=\mathrm{Id}=B^2=BA^2B=TS$. Thus S,T are commuting operators on finite dimensional vector spaces. This means they have a common eigenvector, say v. Then there are scalars $\lambda_S,\lambda_T\in\mathbb{C}$ such that $Sv=\lambda_S v, Tv=\lambda_T v$. Consider $W=\langle v,Av\rangle\subseteq V$. We show W is stable under A,B:

- $Av\in W$ by definition.
- $Bv=A^2Bv=A(AB)v=AS\cdot v=\lambda_S Av\in W$.
- $A(Av)=A^2v=v\in W$.
- $B(Av)=BAv=Tv=\lambda_T v\in W$.

Linear Algebra

Let A be a complex $n \times n$ matrix. Let a_k denote the dimension of the null space of A^k (in particular, $a_0 = 0$). Prove that $a_k + a_{k+2} \le 2a_{k+1}$ for all $k \ge 0$.

Solution

Nullity of A is same as the same as the nullity of the matrix for the Jordan canonical form of A. Say the A looks like $\operatorname{diag}(J_1, \cdots, J_t)$, where J_i are the Jordan blocks, then A^k looks like $\operatorname{diag}\left(J_1^k, \cdots, J_t^k\right)$. So it is enough to show the problem for an irreducible Jordan block.

Say $A = J$ is a Jordan block of size $n \times n$ with eigenvalue λ. Then $a_0 = 0$.
Suppose $\lambda \ne 0$, then each A^k is non-singular. This means that each $a_k = 0$. So the result holds trivially.
Suppose $\lambda = 0$. So A looks like $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$. We can show that $\left(A^k\right)_{i,i+k} = 1 \forall i$ and 0 otherwise.
Clearly $a_i = i \forall i \le n$ and $a_i = n \forall i > n$. The rest follows clearly.

Fall 2021

Groups

Let G be a group and $Z(G)$ the center of G. Show that the group $G/Z(G)$ does not have prime order. Find a group G such that $G/Z(G)$ has 4 elements.

Rings

Show that every prime ideal P in $\mathbb{Z}[x]$ which is not principal contains a prime number.

Groups

Show that every finite noncyclic group is a finite union of proper subgroups, and that if a group maps surjectively to a finite noncyclic group then it is a finite union of proper subgroups and use this to determine for which positive integers the product of n copies of the integers is a finite union of proper subgroups.

Linear Algebra

Let A and B be two square matrices over a field F. Suppose $\operatorname{diag}(A, A)$ and $\operatorname{diag}(B, B)$ are similar. Show that A and B are similar.

Groups

(a) Suppose that p and q are distinct primes and a group G is generated by elements of order p and also by elements of order q. Show that any homomorphism of G to an abelian group is trivial.

(b) Show that for $n \geq 5$ the alternating group A_n of even permutations of n objects is generated by elements of order 2, and also by elements of order 3, so that for such n the only homomorphisms to abelian groups are trivial.

Spring 2021

Rings

The following are four classes of commutative rings, in alphabetical order:

- fields
- integral domains
- principal integral domains
- unique factorization domains

These are contained in one-another, in some order, so that $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq A_4$.

(a) Determine the order.
(b) Give an example in each class to show that the inclusions are proper.

Solution

$$\text{Fields} \subsetneq \text{Principal Ideal Domains} \subsetneq \text{Unique Factorization Domains} \subsetneq \text{Integral Domains}$$

Integral domain but not UFD: $\mathbb{Z}\left[\sqrt{-5}\right]$. This clearly has no zero divisors. But 6 can be factored as $2 \cdot 3$ and $\left(1+\sqrt{-5}\right)\left(1-\sqrt{-5}\right)$ and $2, 3, 1 \pm \sqrt{-5}$ are all primes.
UFD but not PID: $\mathbb{Z}[x]$. This is known to be a UFD, but the ideal $(2, x)$ is not generated by one element.
PID but not field: $\mathbb{Z}$. This is known to be a PID, but 2 doesn't have an inverse.

Rings

(a) If R is a commutative ring, define what it means for R to be Noetherian and state Hilbert's basis theorem.
(b) Give an example of a non-Noetherian commutative ring.

Solution (a) A commutative ring R is said to be Noetherian if R satisfies one of the folowing:

(i) R satisfies the *ascending chain condition* on ideals: Every increasing chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of R stabilizes, that is, there is some N such that $I_i = I_N \forall i \geq N$.
(ii) Every ideal of R is finitely generated.
(iii) Every set of ideals contains a maximal element.

Hilbert's basis theorem states that if R is a Noetherian (commutative) ring, so is the polynomial $R[x]$ in one variable.

(b) The polynomial ring $\mathbb{Z}[x_1, x_2, \cdots]$ in countably many variables with coefficients in $\mathbb{Z}$ is not Noetherian. We reason as follows, depending on the above three defintions:

(i) $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$ is a strictly increasing chain of ideals (hence never stabilizes).
(ii) The ideal $(x_1, x_2, \cdots)$ is not finitely generated.
(iii) The set of ideals $\{(x_1, \cdots, x_n) : n \geq 1\}$ has no maximal element.

Groups

Let G be a group of order 105 and let P_3, P_5, and P_7 be Sylow 3, 5, and 7 subgroups, respectively. Assuming the Sylow theorems, prove the following:

(a) At least one of P_5 or P_7 is normal in G.
(b) G has a cyclic subgroup of order 35.
(c) Both P_5 and P_7 are normal in G.

Linear Algebra

Find all similarity classes of 2×2 matrices A with entries in $\mathbb{Q}$ satisfying $A^4 = I$. What are the corresponding rational canonical forms?

Linear Algebra

(a) Find the possible Jordan Canonical Forms of any matrix such that $A^4 = I$ over $F = \mathbb{F}_5$.
(b) Give an example of a matrix B over $F = \mathbb{F}_3$ satisfying $B^4 = I$, such that B is not diagonalizable.

Fall 2020

Linear Algebra

Prove that for any pair of commuting $n \times n-$matrices with complex entries there exists a common eigenvector.

Groups

Prove that there exists no simple group of order 56.

Rings

Prove that a ring which contains a principal ideal ring R, and which is contained in the field of fractions of R, is a principal ideal ring.

Linear Algebra

Let A and B be two projection linear maps in a vector space over a field K. Prove that if $A+B$ is a projection linear map and $\operatorname{char} K \neq 2$ then $AB = BA = 0$.

Solution

Given that $A, B, A+B$ are projections. That is, they satisfy $x^2 = x$. Then $A+B = A^2 + B^2 + AB + BA = A+B+AB+BA \implies AB = -BA$. But $AB = A^2B = -ABA = BAA = BA^2 = BA$. It follows that $AB = BA = -AB \implies AB = 0 = BA$. (Where is $\operatorname{char} K \neq 2$ used?)

Groups

Prove that in the group $\mathbb{Q}/\mathbb{Z}$ for any natural number n there exists exactly one subgroup of order n.

Spring 2020

Algebra

Suppose that A is a not necessarily commutative, finite dimensional associative algebra with a unit over a field F and $P \trianglelefteq A$ is a two-sided ideal such that for $a, b \in A, ab \in P \implies a \in P$ or $bP \in P$. Show that A/P must be a division algebra (i.e. every nonzero element has a multiplicative inverse).

Groups

Show that every group of order 2020 contains a unique (and hence normal) subgroup of order 505.

Linear Algebra

Let M be a matrix with integer entries.

(a) Prove that the minimal polynomial of M over $\mathbb{C}$

$$f_{\min}(t) = t^k + \sum_{i=0}^{k-1} a_i t^i$$

has integer coefficients.

(b) Prove that if M is diagonalizable over $\mathbb{Q}$ then there exists an integer N such that the matrix M mod p is diagonalizable over $\mathbb{Z}/p\mathbb{Z}$ for all $p > N$.

Rings

Let F be a field and let L be the ring of Laurent polynomials $L = F[x, x^{-1}]$ (it is the subring of $F(x)$ generated over F by x and x^{-1}). We consider L as a module over the ring of polynomials $R = F[x]$. (a) Show that L is not a finitely generated module over R. (b) Show that every finitely generated submodule of L is free with a single generator.

Rings

Let R be a commutative integral domain and let $I \trianglelefteq R$ be an ideal.

(a) Show that every alternating bilinear form

$$f : I \times I \to R$$

is zero.

(b) Show that if R is a principal ideal domain, then every alternating bilinear form $f : I \times I \to M$ to any $R-$module M is zero.