Integral numbers are the fountainhead of all mathematics. —Hermann Minkowski

PROMYS Number Theory **Problem Set #2** *Boston University, July 6, 2021*

Reading Search

Q1: What is the "greatest common divisor" (GCD) of two integers?

Q2: What is the Farey sequence of order 3? of order 7? of order n? How many elements are there in the Farey sequence of order 3? of order 7? of order 11? Any conjectures? Represent the fraction $\frac{m}{n}$ by the point (m, n) in the plane. Draw all entries of the Farey sequence of order 7.

Exploration

P1: Find all the roots of the equation $x^2-6x+8=0$ in $\mathbb{Z}_{15}$. Find all the roots in $\mathbb{Z}_{15}$ of the equation $x^2-6x+10=0$. Find all the roots in $\mathbb{Z}_{105}$ of the equation $x^2-6x+8=0$. Any conjectures?

Prove or Disprove and Salvage if Possible

P2: If $d|a$, $d|b$ then $d|(ax+by)$ for all integers x and y.

P3: If $a|b$ then $a \leq b$.

P4: If $a \nmid b$ and $a \nmid c$ then $a \nmid bc$.

P5: If $2 \nmid n$ then $8|(n^2-1)$.

Numerical Problems (*Some food for thought*)

P6: Use Euclid's algorithm to find the GCD of 29 and 11.

P7: Use your work in P6 to show that

$$\frac{29}{11} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}$$

This is the *simple continued fraction* for $\frac{29}{11}$.

P8: Any fixed real number may be approximated by a rational fraction $\frac{a}{b}$ with any desired degree of accuracy if we allow the denominator b to become large enough. The size of the denominator is the price we pay for a good approximation. In the light of these observations, how high is the price (i.e. the size of the denominator vs. the accuracy) which we pay as we approximate $\frac{29}{11}$ by the fractions

$$2, \quad 2+\frac{1}{1}, \quad 2+\cfrac{1}{1+\cfrac{1}{1}}, \quad 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}} ?$$

These are the *convergents* of the simple continued fraction in P7.

P9: Write down all the elements of U_7, of U_{15}, of U_{18}, of U_{21}. Which of these are cyclic? Any conjectures?

P10: Construct a table of "logarithms" (indices) for U_{17}.

P11: Use the logarithm table in P10 to find all the solutions of each of the following equations in $\mathbb{Z}_{17}$. (a) $x^2 = 2$; (b) $7x^2 = 6$; (c) $x^3 = 3$.

P12: Find the GCD of each of the following pairs of integers: (a) 6, 15; (b) 36, 49; (c) 483, 291; (d) 11413, 11289. Compare the efficiency of each of the following three methods when the integers involved become large.
(1) Write down all the divisors of each integer in a given pair, then pick out among these all the common divisors and select the largest one among the common divisors. (2) Write down the canonical decomposition into primes of each one of the given pair of integers. Using these canonical decompositions calculate the GCD. (3) Use Euclid's Algorithm.

Ingenuity

P13: Find three non-zero digits a_0, a_1, and a_2 and a base b such that $(a_0a_1a_2)_b = 2(a_0a_1a_2)_{10}$.

P14: The sum $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$ for $n > 1$ is never an integer.