Preset vs poset.

Set P and a relation $\leq$ on P satisfying:

(Reflexive) $a \leq a \quad \forall\ a \in P$
(Transitive) $(a \leq b) \wedge (b \leq c) \implies a \leq c \quad \forall\ a, b, c \in P$.

Then $(P, \leq)$ is a preordered set (preset). $\leq$ is called preorder.

In addition if we have anti-symmetry $((a \leq b) \wedge (b \leq a) \Rightarrow a = b)$ then $(P, \leq)$ is a partially ordered set (Poset). $\leq$ is a partial order.

⊛ $\mathbb{R}$ with usual $\leq$. This is a poset.

⊛ P = set of integer factors of 6
$= \{1, 2, 3, 6, -6, -3, -2, -1\}$
define $\leq$ as: $a \leq b \iff a \mid b$

Is P a preset? Yes
Is P a poset? No. $-6 \leq 6$ and $6 \leq -6$ but $6 \neq -6$

⊛ P = set of +ve int factors of 6 $= \{1, 2, 3, 6\}$

Is P a preset? Yes
Is P a poset? Yes

Say $(P, \leq)$ is a preset. Define a new relation $\sim$ as:

$a \sim b \iff a \leq b$ and $b \leq a$. $\sim$ is an eq relation.

Consider $\tilde{P} = P/\sim$.

Give the partial order $\tilde{\leq}$ on $\tilde{P}$ as follows:

$[a] \ \tilde{\leq}\ [b] \iff a \leq b$ (guess)

Is this well defined? $[a] = [a']$, $[b] = [b']$, $a \leq b$.
Need to show: $a' \leq b'$.
Why? $[a] = [a'] \iff a \leq a'$ & $a' \leq a$ } $a' \leq a \leq b \leq b'$
$[b] = [b'] \iff b \leq b'$ & $b' \leq b$ }

$\Rightarrow a' \leq b'$. ✓

$\tilde{\leq}$ is transitive? Yes
$\tilde{\leq}$ is reflexive? Yes
$\tilde{\leq}$ is antisymmetric?

Say $[a] \,\tilde{\leq}\, [b]$ & $[b] \,\tilde{\leq}\, [a]$

$$\underbrace{\begin{array}{ccc} [a] \,\tilde{\leq}\, [b] & & [b] \,\tilde{\leq}\, [a] \\ \Updownarrow & & \Updownarrow \\ a \leq b & & b \leq a \end{array}}$$

$$\Downarrow$$

$$a \sim b$$

$$\Updownarrow$$

$$[a] = [b]$$

Yes, antisymmetric

$\therefore \left(\tilde{P} = P/\sim,\ \tilde{\leq}\right)$ is a poset.

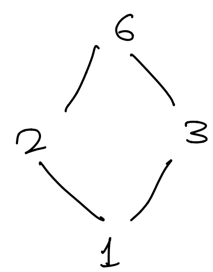
Say $(P, \leq)$ is a finite poset. A chain C is a subset of P s.t. $\underbrace{\forall\, a, b \in C \text{ either } a \leq b \text{ or } b \leq a.}_{\text{any two elements of } C \text{ are comparable.}}$

An antichain C is a subset of P s.t. no two elements of C are comparable.

A chain cover is a collection $C_1, \dots, C_n$ of chains in P s.t.

$$P = \bigcup_{i=1}^{n} C_i$$

Similarly define antichain cover.



$P = \{1, 2, 3, 6\}$, $\leq$ = "divides"

$C = \{1, 2, 6\}$ is a chain
$A = \{2, 3\}$ is an antichain

$C_1 = \{1, 2, 6\}$, $C_2 = \{1, 3, 6\}$. Then $P = C_1 \cup C_2$. This is a min'l chain cover.
A is max'l antichain

Say P is a poset with $mn+1$ elements. Show there is a chain with $m+1$ elts or an antichain with $n+1$ elts.

50 distinct line segments are given on a line. Prove that some 8 of them have a common point or 8 of them are pairwise disjoint.

Define $<$: $[a, b] < [c, d] \iff b < c$

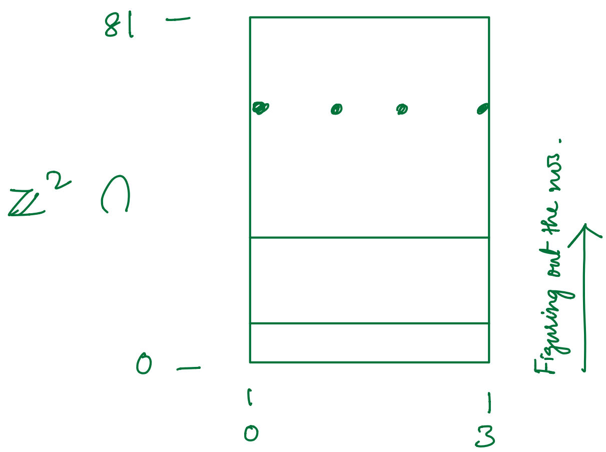
Define $\leq$: $[a, b] \leq [c, d] \iff (a = c,\ b = d)$ or $([a,b] < [c,d])$

Verify $\leq$ is a partial order.

A chain in this poset has no common pt, pairwise.

An antichain in this poset contains elements which all share a common pt.

Now apply Dilworth's theorem.



82 rows

Colors for each row: $3^4 = 81$

So 2 rows same colour seq

x x x x

x x x x

$C_1\ C_2\ C_3\ C_4$

4 colours, 3 possibilities

$C_i = C_j \quad (i \neq j)$