I. Space of sequences

Classically we know that sequences look like

$a_1, a_2, a_3, \dots$

What are these a_i? $a_i \in X$ where X is any set.

This can be regarded as a function $f: \mathbb{N} \to X$, given by $f(i) = a_i$.

The space of X-valued sequences is just $\{f : \mathbb{N} \to X\}$.

Notation: $X^Y := \{ f : Y \to X \mid f \text{ function} \}$

The space of X-valued is just $X^{\mathbb{N}}$.

Intuition: $\underset{(n \in \mathbb{Z}_{\geq 0})}{S^n}$

$$
\begin{aligned}
S^n &= S \times S \times S \times \cdots \times S \\
&= \{ (s_1, s_2, \dots, s_n) \mid s_i \in S \} \\
&\cong \{ f : \{1, \dots, n\} \to S \} \\
&= \{ f : [n] \to S \} = S^{[n]}
\end{aligned}
$$

If Y is finite then $X^Y \cong X^{|Y|}$

If X, Y are finite then $|X^Y| = |X|^{|Y|}$.

Let's focus on $\mathbb{R}^{\mathbb{N}}$: real valued sequences.

$\mathbb{R}^{\mathbb{N}}$ is an $\mathbb{R}$-vector space.

Addition: Termwise $f, g \in \mathbb{R}^{\mathbb{N}}$

$$(f+g) := (n \mapsto f(n) + g(n))$$

Scalar mult: $c \in \mathbb{R}$, $f \in \mathbb{R}^{\mathbb{N}}$ then

$$(c \cdot f) = (n \mapsto c \cdot f(n))$$

Zero: $f = (n \mapsto 0)$

Basis ? $\mathcal{B} := \{ (1,0,\ldots), (0,1,0,\ldots), (0,0,1,0,\ldots) \ldots \}$

HARD

Not a basis

Why is this not a basis ? It does not span $\mathbb{R}^{\mathbb{N}}$.

Does $f = (n \to 1)$ lie in the span of $\mathcal{B}$?

However $\mathcal{B}$ spans $\mathbb{R}^{(\mathbb{N})}$

$\mathbb{R}^{(\mathbb{N})} = \{$ all sequences in $\mathbb{R}^{\mathbb{N}}$ which are eventually $0 \}$.

Multiplication on $\mathbb{R}^{\mathbb{N}}$: $(f \cdot g) = (n \mapsto f(n) \cdot g(n))$

Multiplicative id : $f = (n \mapsto 1)$

$\therefore \mathbb{R}^{\mathbb{N}}$ is a ring.

$\therefore \mathbb{R}^{\mathbb{N}}$ is an $\mathbb{R}$-algebra (VS + ring)

Q: If I have seq f and g in $\mathbb{R}^{\mathbb{N}}$, s.t. $f \cdot g$ is the zero seq. Is it true that either f or g must be the zero seq ?

Ans. No. $f = (1,0,1,0,\ldots)$
$g = (0,1,0,1,\ldots)$

<u>Subsequence</u> : let $f \in \mathbb{R}^{\mathbb{N}}$. A subseq of f is some $h \in \mathbb{R}^{\mathbb{N}}$ s.t. $h = f \circ g$ for some $g : \mathbb{N} \to \mathbb{N}$ strictly increasing.
(Clearly g is 1-1)

Subseq is just the main seq with some terms ignored.

II. (Sub) space of convergent real seq

Let $f \in \mathbb{R}^{\mathbb{N}}$. We say that f converges to $x \in \mathbb{R}$ if

$\forall\ \varepsilon > 0 \quad \exists\ N \in \mathbb{N} \ni |f(n) - x| < \varepsilon \quad \forall n \geq N.$

We say f converges if $\exists\ x \in \mathbb{R}$ s.t. f converges to x.

If we write f as (x_n) then
notation: $x_n \to x$ for (x_n) cgs to $x \in \mathbb{R}$.
$\to \lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} f(n) = x$ (if $x_n \to x$.)

Note: N depends on ε.

1. $\frac{1}{n} \to 0$
2. (n) does not converge
3. $(1) \to 1$

Properties

1. $x \in \mathbb{R}$ s.t. $|x| \leq \varepsilon \quad \forall\ \varepsilon \in \mathbb{R}^+$ then $x = 0$.

   (Same is true for $|x| < \varepsilon$

2. If $x_n \to a$ & $x_n \to b$ then $a = b$.

Pf: Let $\varepsilon > 0$ given.

$$
\begin{aligned}
|a-b| &= |a - x_n + x_n - b| \\
&\leq |a - x_n| + |x_n - b| \\
&\leq |x_n - a| + |x_n - b| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } n \geq N
\end{aligned}
$$

$\Rightarrow a - b = 0 \Rightarrow a = b.$

③ Say $x_n \to x$, $y_n \to y$. Let $a, b \in \mathbb{R}$. Then:

(i) $a x_n + b y_n \longrightarrow ax + by$ $\longrightarrow$ Space of cgt seq is a vector subspace of $\mathbb{R}^{\mathbb{N}}$

(ii) $x_n \cdot y_n \longrightarrow xy$ $\longrightarrow$ subring of $\mathbb{R}^{\mathbb{N}}$

Subalgebra of $\mathbb{R}^{\mathbb{N}}$.

(iii) If $x_n \neq 0 \ \forall n$, $x \neq 0$ then $\frac{1}{x_n} \longrightarrow \frac{1}{x}$.

④ $x_n \to x \iff (x_n - x) \longrightarrow 0$

⑤ $x_n \xrightarrow{n\to\infty} x$. Let (x_{n_k}) be a subseq. Then $x_{n_k} \xrightarrow{k\to\infty} x$.

⑥ Say $x_n \to x$. $x_n \geq 0 \ \forall n$. Then $x \geq 0$,

⑦ $x_n \to x$, $y_n \to y$. $x_n \geq y_n \ \forall n$. Then $x \geq y$.

Q. Say $x_n \to x$, $y_n \to y$ s.t. $x_n > y_n \ \forall n$.
Can we say $x > y$?

Ans. No. $\{\frac{1}{n}\} \to 0$, $\frac{1}{n} > 0$.

Sandwich Thm: x_n, y_n, z_n are s.t. $x_n \to a$, $z_n \to a$, and $x_n \leq y_n \leq z_n \ \forall n$. Then $y_n \to a$.

Definition: (1) We say $\lim_{n\to\infty} x_n = \infty$ if
$\forall g \in \mathbb{R}^+ \ \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n > g$.

(2) We say $\lim_{n\to\infty} x_n = -\infty$ if $\lim_{n\to\infty} (-x_n) = \infty$.