

Recall that

TFAE :- V be F -vector space.

a) $B \subset V$ s.t. $\langle B \rangle = V$ and B is L.I.

b) Every element of V can be uniquely written as linear combination of elements of B

c) B is minimal spanning

d) B is maximal L.I.

$B \subset C$ s.t. C is L.I.

$$x \in C \setminus B \quad x = \alpha_1 b_1 + \dots + \alpha_n b_n \quad b_i \in B$$

$$\{x, b_1, \dots, b_n\} \subset C \Rightarrow L.I.$$

$$\Rightarrow x - \alpha_1 b_1 - \dots - \alpha_n b_n = 0$$

$$\Rightarrow 1 = 0$$

$$\{b_1, b_2, \dots, b_n\} \subset B \Rightarrow \alpha_1 b_1 + \dots + \alpha_n b_n = 0 \quad \alpha_i \in F$$

$B \setminus \{b_1\}$ WLOG $\alpha_1 \neq 0$

$$b_1 = -\frac{\alpha_2}{\alpha_1} b_2 - \frac{\alpha_3}{\alpha_1} b_3 - \dots - \frac{\alpha_n}{\alpha_1} b_n$$

$$v \in V \quad v = \beta_1 b_1 + \dots + \beta_n b_n$$

Any $B \subset V$ satisfying any of the above equivalent of the statements is defined to be a **Basis** for V .

How do we create a basis, when a F-Vector space is given?

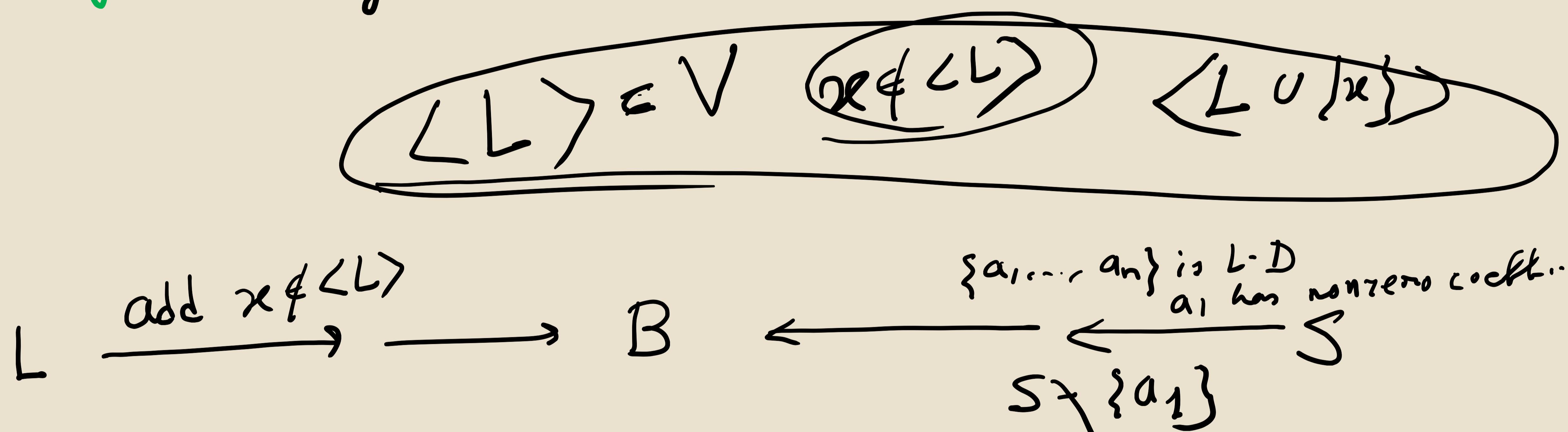
- Does a basis always exist?
- If so, then how do we create one such?

Theorem :- Given a F-vector space V , and a L.I. subset L of V , \exists a basis B containing L .

Theorem(2) :- Given a F-vector space V , and a spanning set S of V , \exists a basis B contained in S .

Theorem(3) :- Given a F-vector space V , and a L.I. subset L of V , and a spanning subset S of V s.t $L \subset S$, \exists a basis B s.t $L \subset B \subset S$

Corollary :- Every vector space has a basis



Theorem :- Given a vector space V , if V admits a finite basis then every basis of V is finite.

Proof → On contrary let V admit an infinite basis C and let B be a finite basis for V . $B = \{b_1, \dots, b_n\}$

$$c_1 \in C \quad c_1 = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n, \quad \alpha_i \in F$$

$c_1 \in C$ and no L.I. set contains 0.

$$\boxed{\alpha_1 \neq 0}$$

$$b_1 = \frac{1}{\alpha_1} c_1 - \frac{\alpha_2}{\alpha_1} b_2 - \dots - \frac{\alpha_n}{\alpha_1} b_n \quad b_1 \quad \{c_1, b_2, \dots, b_n\}$$

$$B_1 = \{c_1, b_2, b_3, \dots, b_n\}$$

$\langle B_1 \rangle = V$ also prove that B_1 is L.I.

$v \in V$ to show that $v \in \langle B_1 \rangle$

$$\langle B \rangle = V \Rightarrow v = \beta_1 b_1 + \dots + \beta_n b_n$$

$$= \beta_1 \left(\frac{1}{\alpha_1} c_1 - \frac{\alpha_2}{\alpha_1} b_2 - \dots - \frac{\alpha_n}{\alpha_1} b_n \right) + \beta_2 b_2 + \dots + \beta_n b_n$$

$$\underbrace{\beta_1 c_1 + \dots + \beta_n b_n}_{} = 0$$

$$\beta_1 (\alpha_1 b_1 + \dots + \alpha_n b_n) + \beta_2 b_2 + \dots + \beta_n b_n = 0$$

$$\Rightarrow \underbrace{\alpha_1 \beta_1 b_1}_{} + \underbrace{(\alpha_2 \beta_1 + \beta_2)}_{\beta_2} b_2 + \dots + (\alpha_n \beta_1 + \beta_n) b_n = 0$$

$$\Rightarrow \alpha_1 \beta_1 = 0 \Rightarrow \alpha_1 = 0 \quad \text{on } \boxed{\beta_1 = 0}$$

$$\beta_2 b_2 + \dots + \beta_n b_n = 0$$

$$\beta_2 = \beta_3 = \dots = \beta_n = 0$$

$\Rightarrow B_1$ is L.I.

\subset L.I. is L.I.

$$B = \{b_1, \dots, b_n\}$$



$$c_1 \in C$$

$$B_1 = \{c_1, b_2, b_3, \dots, b_n\} \quad \text{still a basis}$$

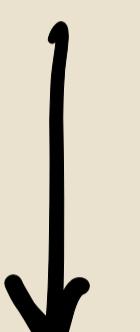


$$c_2 \in C$$

$$c_2 = \underline{\alpha_1} c_1 + \underline{\alpha_2} b_2 + \dots + \underline{\alpha_n} b_n$$

α_2

$$B_2 = \{c_1, c_2, b_3, \dots, b_n\} \quad \text{still a basis}$$



⋮

still basis

$$B_n = \{c_1, \dots, c_n\}$$

$$\subset C$$

$$\leq |B|$$

$$|C| = \infty$$

$$\underline{|C|} > |B|$$