Algebra Qualifying Exams

Rutgers - the State University of New Jersey

Syllabus

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Contents

Spring 2023

Groups

Classify all groups of order 309, up to isomorphism.

Groups

Let A be the abelian group with generators x, y, z and the relations

$$4x + 3y + z = 0, x + 2y + 3z = 0, 3x + 2y + 5z = 0$$

Show that A is a cyclic abelian group, and determine its order.

Linear Algebra

Let A be a complex $n \times n$ matrix. Prove that there is an invertible complex $n \times n$ matrix B such that $AB = BA^t$. (A^t is the transpose of A.)

Rings

Prove that the subring $\mathbb{Z}[3i]$ of $\mathbb{C}$ is not a Principal Ideal Domain.

Rings

If $R = \mathbb{Z}[x]$, show that the sequence $R \xrightarrow{f} R^2 \xrightarrow{g} R$ is exact, where $f(a) = (ax, -2a)$ and $g(c, d) = 2c + dx$.

Fall 2022

Groups

Let G be a finite simple group. Prove that $G \times G$ has exactly 4 normal subgroups (including $G \times G$) if and only if G is non-abelian.

Rings

Let R be a principal ideal domain and I, J be ideals of R. Show that $I \cap J = IJ$ holds if and only if $I = 0$ or $J = 0$ or $I + J = R$.

Linear Algebra

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix with real coefficients. Show that all eigenvalues of A are non-negative if and only if $A = P^T P$ for some matrix $P \in M_n(\mathbb{R})$.

Rings

Let R be an integral domain and $R[x, y, z]$ the polynomial ring in three variables over R. Show that $I = \langle x^3, y^2, y^3 - z^2 y\rangle \subseteq R[x, y, z]$ is a prime ideal.
Hint: Show that I is the kernel of a ring homomorphism $R[x, y, z] \to R[t]$.

Linear Algebra

Let A and B be commuting complex matrices. Assume that $B \notin \mathbb{C}[A]$, that is, B cannot be written as a polynomial in A. Show that some eigenspace of A has dimension at least two.

Spring 2022

Rings

Prove that the rings $\mathbb{Q}[x]/(x^2-1)$ and $\mathbb{Q}\oplus\mathbb{Q}$ are isomorphic.

Groups

Let p be a prime. Show that any element of order p in $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ can be conjugated to the matrix $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$.

Fields

Let a and b be elements of a field of order 2^n where n is odd. Prove that if $a^2+ab+b^2=0$ then $a=b=0$.

Linear Algebra

Let A, B be linear operators on a nonzero finite-dimensional vector space V over $\mathbb{C}$ such that $A^2=B^2=\mathrm{Id}$. Prove that there exists a nonzero subspace W of V which is invariant under A and B and $\dim W\le 2$.

Linear Algebra

Let A be a complex $n\times n$ matrix. Let a_k denote the dimension of the null space of A^k (in particular, $a_0=0$). Prove that $a_k+a_{k+2}\le 2a_{k+1}$ for all $k\ge 0$.

Fall 2021

Groups

Let G be a group and $Z(G)$ the center of G. Show that the group $G/Z(G)$ does not have prime order. Find a group G such that $G/Z(G)$ has 4 elements.

Rings

Show that every prime ideal P in $\mathbb{Z}[x]$ which is not principal contains a prime number.

Groups

Show that every finite noncyclic group is a finite union of proper subgroups, and that if a group maps surjectively to a finite noncyclic group then it is a finite union of proper subgroups and use this to determine for which positive integers the product of n copies of the integers is a finite union of proper subgroups.

Linear Algebra

Let A and B be two square matrices over a field F. Suppose $\operatorname{diag}(A, A)$ and $\operatorname{diag}(B, B)$ are similar. Show that A and B are similar.

Groups

(a) Suppose that p and q are distinct primes and a group G is generated by elements of order p and also by elements of order q. Show that any homomorphism of G to an abelian group is trivial.

(b) Show that for $n \geq 5$ the alternating group A_n of even permutations of n objects is generated by elements of order 2, and also by elements of order 3, so that for such n the only homomorphisms to abelian groups are trivial.

Spring 2021

Rings

The following are four classes of commutative rings, in alphabetical order:

- fields
- integral domains
- principal integral domains
- unique factorization domains

These are contained in one-another, in some order, so that $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq A_4$.

(a) Determine the order.

(b) Give an example in each class to show that the inclusions are proper.

Rings

(a) If R is a commutative ring, define what it means for R to be Noetherian and state Hilbert's basis theorem.

(b) Give an example of a non-Noetherian commutative ring.

Groups

Let G be a group of order 105 and let P_3, P_5, and P_7 be Sylow 3, 5, and 7 subgroups, respectively. Assuming the Sylow theorems, prove the following:

(a) At least one of P_5 or P_7 is normal in G.

(b) G has a cyclic subgroup of order 35.

(c) Both P_5 and P_7 are normal in G.

Linear Algebra

Find all similarity classes of 2×2 matrices A with entries in $\mathbb{Q}$ satisfying $A^4 = I$. What are the corresponding rational canonical forms?

Linear Algebra

(a) Find the possible Jordan Canonical Forms of any matrix such that $A^4 = I$ over $F = \mathbb{F}_5$.

(b) Give an example of a matrix B over $F = \mathbb{F}_3$ satisfying $B^4 = I$, such that B is not diagonalizable.

Fall 2020

Linear Algebra

Prove that for any pair of commuting $n \times n-$matrices with complex entries there exists a common eigenvector.

Groups

Prove that there exists no simple group of order 56.

Rings

Prove that a ring which contains a principal ideal ring R, and which is contained in the field of fractions of R, is a principal ideal ring.

Linear Algebra

Let A and B be two projection linear maps in a vector space over a field K. Prove that if $A+B$ is a projection linear map and $\mathrm{char} K \neq 2$ then $AB = BA = 0$.

Groups

Prove that in the group $\mathbb{Q}/\mathbb{Z}$ for any natural number n there exists exactly one subgroup of order n.

Spring 2020

Algebra

Suppose that A is a not necessarily commutative, finite dimensional associative algebra with a unit over a field F and $P \trianglelefteq A$ is a two-sided ideal such that for $a, b \in A, ab \in P \implies a \in P$ or $bP \in P$. Show that A/P must be a division algebra (i.e. every nonzero element has a multiplicative inverse).

Groups

Show that every group of order 2020 contains a unique (and hence normal) subgroup of order 505.

Linear Algebra

Let M be a matrix with integer entries.

(a) Prove that the minimal polynomial of M over $\mathbb{C}$

$$f_{\min}(t) = t^k + \sum_{i=0}^{k-1} a_i t^i$$

has integer coefficients.

(b) Prove that if M is diagonalizable over $\mathbb{Q}$ then there exists an integer N such that the matrix M mod p is diagonalizable over Z/pZ for all $p > N$.

Rings

Let F be a field and let L be the ring of Laurent polynomials $L = F[x, x^{-1}]$ (it is the subring of $F(x)$ generated over F by x and x^{-1}). We consider L as a module over the ring of polynomials $R = F[x]$. (a) Show that L is not a finitely generated module over R. (b) Show that every finitely generated submodule of L is free with a single generator.

Rings

Let R be a commutative integral domain and let $I \trianglelefteq R$ be an ideal.

(a) Show that every alternating bilinear form

$$f : I \times I \to R$$

is zero.

(b) Show that if R is a principal ideal domain, then every alternating bilinear form $f : I \times I \to M$ to any $R-$module M is zero.