

### Theorem :-

Given a set  $S$ ,

an equivalence relation  $\sim \Leftrightarrow$  a partition of  $S$

$$\sim_a := \{b \in S : a \sim b\}$$

$$S/\sim := \{\sim_a\}_{a \in S}$$

$$\sim_a \cap \sim_b = \begin{cases} \emptyset & \text{iff } a \not\sim b \\ \sim_a = \sim_b & \text{iff } a \sim b \end{cases}$$

Ex:-

i)  $f: S \rightarrow T$ . Define a relation on  $S$  as follows

$$a \in S, b \in S$$

$$a \sim b \text{ iff } f(a) = f(b)$$

$$\sim_a := \{b \in S : f(b) = f(a)\} = f^{-1}(f(a))$$

$$S/\sim := \{\sim_a\}_{a \in S} \text{ partition of } S$$

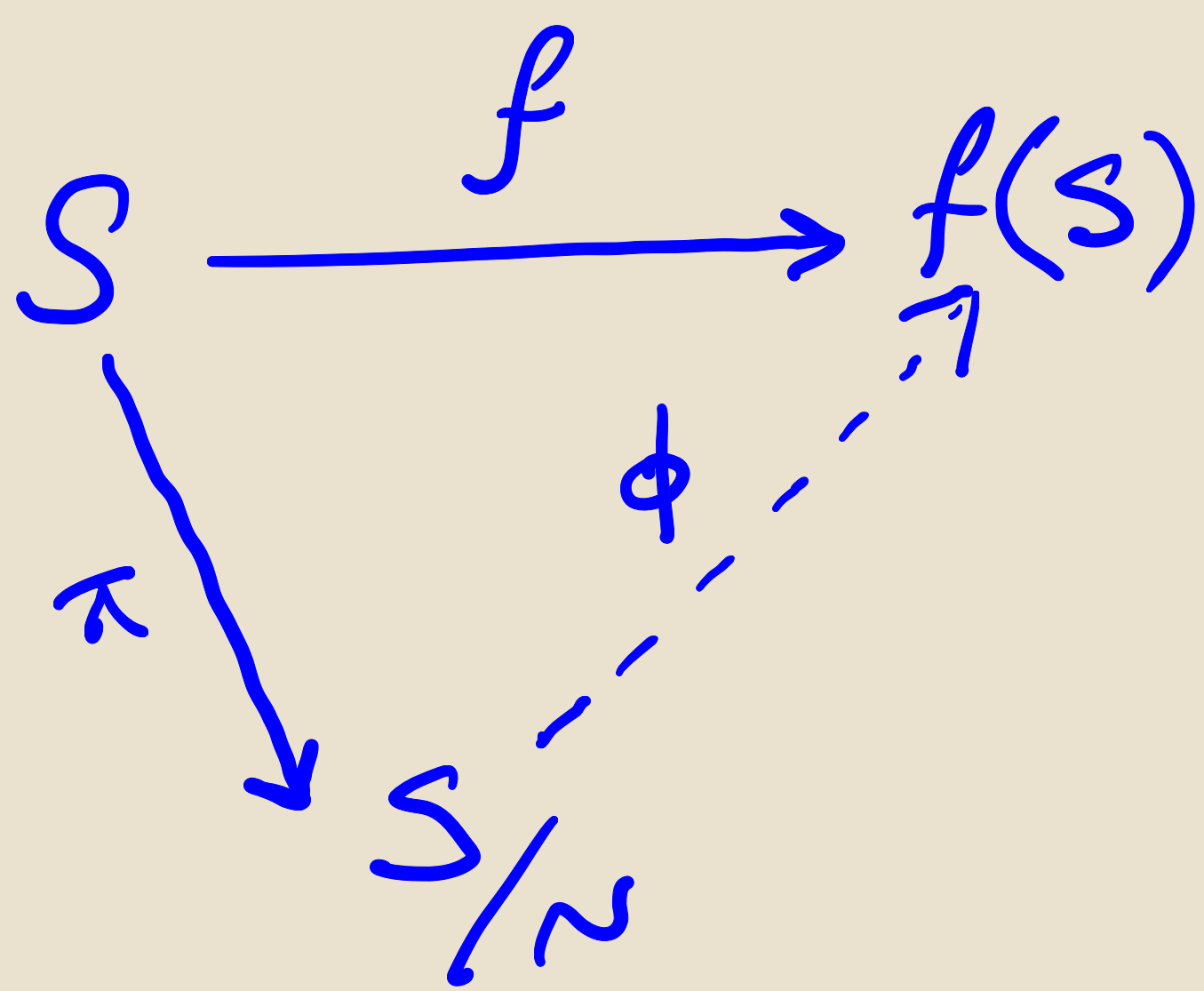
$$S/\sim \xrightarrow[\text{bijective}]{\cong} f(S)$$

Define

$$\phi: S/\sim \longrightarrow f(S)$$

$$\sim_a \longmapsto f(a)$$

$\phi$  is bijection



$$\begin{aligned} \pi: S &\longrightarrow S/\sim \\ a &\longrightarrow \sim_a \end{aligned}$$

$$S \quad f: S \rightarrow T$$

$$\phi \circ \pi = f$$

$$ii) \mathbb{Z} \setminus \{0\} = \mathbb{Z}^*$$

$$(\mathbb{Z} \times \mathbb{Z}^*) \sim$$

$$(a, b) \sim (c, d) \text{ iff } ad = bc$$

$\sim$  is an equivalence relation

Given an  $(a, b)$ , find  $\sim(a, b)$

$$\sim(a, b) := \{ (na, nb) : n \in \mathbb{Z}^* \}$$

$$\frac{\mathbb{Z} \times \mathbb{Z}^*}{\sim} = \{ \sim(a, b) \}_{(a, b) \in \mathbb{Z} \times \mathbb{Z}^*}$$

$$\sim(a, b) = \frac{a}{b}$$

$$\frac{2}{4}$$

$$\sim(a, b) \cdot \sim(c, d) = \sim(ac, bd)$$

$$\frac{1}{2} \in \mathbb{Q}$$

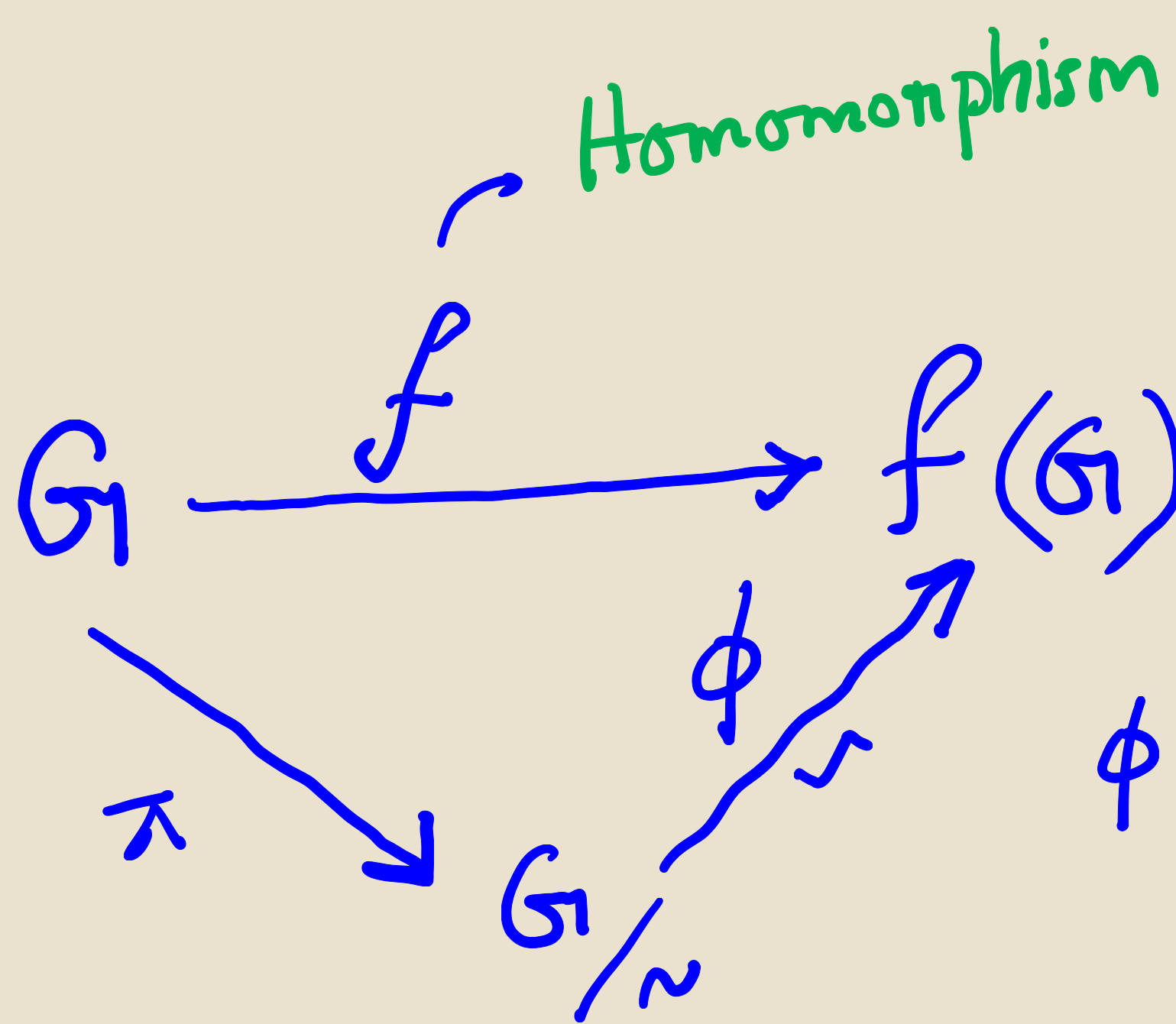
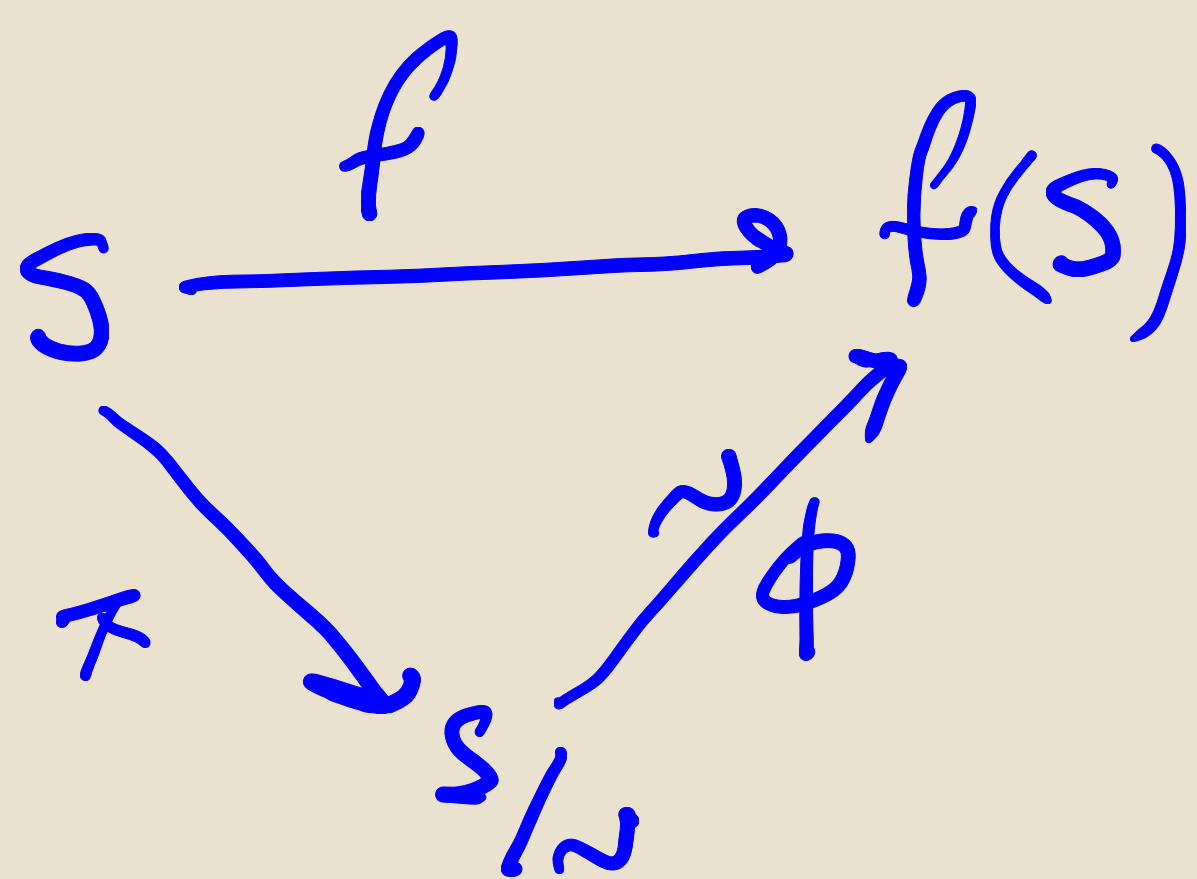
$$\frac{2}{4} \notin \mathbb{Q}$$

$$\sim(a, b) = \sim(a', b')$$

$$\sim(c, d) = \sim(c', d')$$

$$\frac{1}{2} = \frac{2}{4}$$

Example 3 :-



Homomorphism

$\phi$  has to be group homomorphism

$$\phi \circ \pi = f$$

i)  $G/\sim$  has to be a group!

$$a, b \in G$$

$$\sim_a \cdot \sim_b = \sim_{ab}$$

$$\sim_a = \sim_c$$

$$\sim_b = \sim_d$$

$$\sim_{ab} = \sim_{cd} ?$$

$$\begin{aligned} x \in \sim_{ab} &\Leftrightarrow f(x) = f(ab) = f(cd) \\ &\Leftrightarrow x \sim cd \Leftrightarrow x \in \sim_{cd} \end{aligned}$$

$$\begin{aligned} \sim_a = \sim_c &\Rightarrow f(a) = f(c) \\ \sim_b = \sim_d &\Rightarrow f(b) = f(d) \end{aligned} \left. \vphantom{\begin{aligned} \sim_a = \sim_c \\ \sim_b = \sim_d \end{aligned}} \right\} \begin{aligned} &f(ab) \\ &= f(cd) \end{aligned}$$



identity  $\sim_1$

$\sim_a$  inversive  $\sim_{a^{-1}}$

$G/\sim$  is a group

$\pi$  is a homomorphism

$$\pi(a) \cdot \pi(b) = \sim_a \cdot \sim_b = \sim_{ab} = \pi(ab)$$

$\pi$  is a homomorphism?

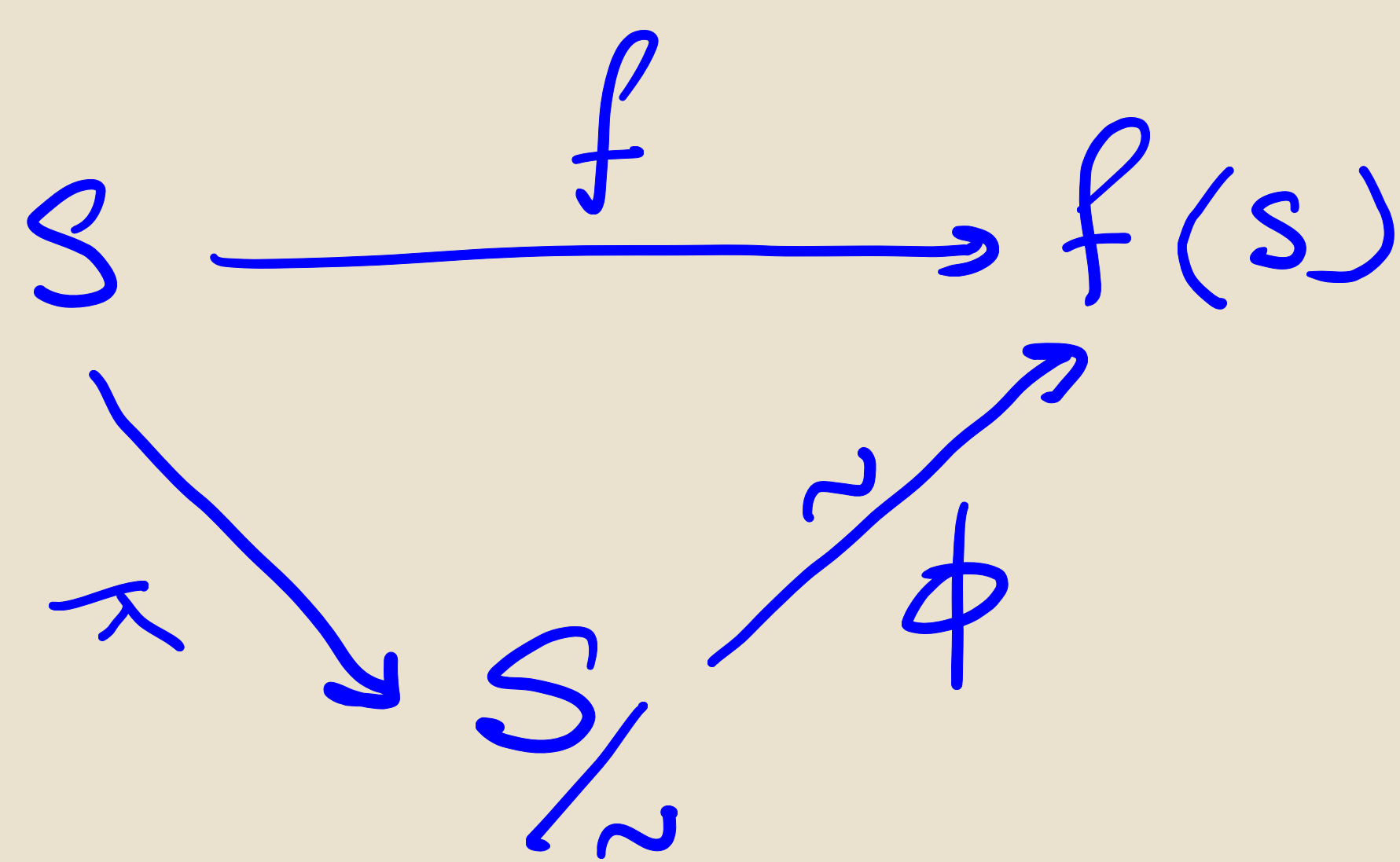
$$\phi(\sim_a) \cdot \phi(\sim_b) = f(a) \cdot f(b) = f(ab) = \phi(\sim_{ab})$$

**Theorem:** Let  $S$  be a set/group. Let  $f$  be a

function / g-homomorphism on  $S$ , then  $\exists$  a bijection / isomorphism

$\phi : S/\sim \rightarrow f(S)$ , in other words, the following diagram

commutes



$$\phi \circ \pi = f$$

