

# Algebra Qualifying Exams

Rutgers - the State University of New Jersey

Syllabus

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## Spring 2023

### Groups

Classify all groups of order 309, up to isomorphism.

### Groups

Let  $A$  be the abelian group with generators  $x, y, z$  and the relations

$$4x + 3y + z = 0, x + 2y + 3z = 0, 3x + 2y + 5z = 0$$

Show that  $A$  is a cyclic abelian group, and determine its order.

### Linear Algebra

Let  $A$  be a complex  $n \times n$  matrix. Prove that there is an invertible complex  $n \times n$  matrix  $B$  such that  $AB = BA^t$ . ( $A^t$  is the transpose of  $A$ .)

#### Solution

The given statement is equivalent to showing the existence of an invertible  $B$  such that  $A^t = B^{-1}AB$ . This is just saying that  $A, A^t$  are similar. Since we are working over  $\mathbb{C}$ , we can simply work with JCF. This suffices because if  $A = X^{-1}JX$  where  $J$  is the JCF of  $A$ , then  $A^t = B^{-1}AB$  is equivalent to saying that  $YJ^tY^{-1} = B^{-1}X^{-1}JXB$  where  $Y = X^t$ , which is equivalent to saying that  $J^t = (XBY)^{-1}X(XBY)$ . This is simply saying that  $J$  is similar to its transpose. Since  $J$  is made of block matrices, and transpose treats every square block independently, it is enough to show that every Jordan block is similar to its transpose.

To see this, we start with a Jordan block  $J$  of size  $n \times n$  and eigenvalue  $\lambda$ . In the standard basis  $\mathbf{e} = (e_1, \dots, e_n)$ , the action of  $J$  is given by  $Je_1 = \lambda e_1$  and  $Je_j = \lambda e_j + e_{j-1}$  for  $1 < j \leq n$ . Now we look at the matrix in the basis  $\mathbf{f} = (f_1, \dots, f_n)$  where  $f_i = e_{n-i+1} \forall 1 \leq i \leq n$ . Clearly the first columns of  $J$  in this basis is given by  $Jf_1 = \lambda e_n + e_{n-1} = \lambda f_1 + f_2$  which corresponds to the column matrix with first two entries being  $\lambda, 1$  respectively and everything else is 0. The  $j^{\text{th}}$  column ( $1 \leq j < n$ ) is given by  $Jf_j = Je_{n+1-j} = \lambda e_{n+1-j} + e_{n-j} = \lambda f_j + f_{j+1}$  which corresponds to the columns where the  $j^{\text{th}}, (j+1)^{\text{st}}$  entries are  $\lambda, 1$  respectively, and everything else is 0. This means that  $[J]_{\mathbf{e}} = [J]_{\mathbf{f}}^t$ . Since both the matrices  $[J]_{\mathbf{e}}, [J]_{\mathbf{f}}$  correspond to the same linear operator, but represented in different bases, they are similar. This proves that every Jordan block is similar to its transpose.

### Rings

Prove that the subring  $\mathbb{Z}[3i]$  of  $\mathbb{C}$  is not a Principal Ideal Domain.

### Rings

If  $R = \mathbb{Z}[x]$ , show that the sequence  $R \xrightarrow{f} R^2 \xrightarrow{g} R$  is exact, where  $f(a) = (ax, -2a)$  and  $g(c, d) = 2c + dx$ .

## Fall 2022

### Groups

Let  $G$  be a finite simple group. Prove that  $G \times G$  has exactly 4 normal subgroups (including  $G \times G$ ) if and only if  $G$  is non-abelian.

### Rings

Let  $R$  be a principal ideal domain and  $I, J$  be ideals of  $R$ . Show that  $I \cap J = IJ$  holds if and only if  $I = 0$  or  $J = 0$  or  $I + J = R$ .

### Linear Algebra

Let  $A \in M_n(\mathbb{R})$  be a symmetric matrix with real coefficients. Show that all eigenvalues of  $A$  are non-negative if and only if  $A = P^T P$  for some matrix  $P \in M_n(\mathbb{R})$ .

### Rings

Let  $R$  be an integral domain and  $R[x, y, z]$  the polynomial ring in three variables over  $R$ . Show that  $I = \langle x^3, y^2, y^3 - z^2 y \rangle \subseteq R[x, y, z]$  is a prime ideal.

Hint: Show that  $I$  is the kernel of a ring homomorphism  $R[x, y, z] \rightarrow R[t]$ .

### Linear Algebra

Let  $A$  and  $B$  be commuting complex matrices. Assume that  $B \notin \mathbb{C}[A]$ , that is,  $B$  cannot be written as a polynomial in  $A$ . Show that some eigenspace of  $A$  has dimension at least two.

## Spring 2022

### Rings

Prove that the rings  $\mathbb{Q}[x]/(x^2 - 1)$  and  $\mathbb{Q} \oplus \mathbb{Q}$  are isomorphic.

### Groups

Let  $p$  be a prime. Show that any element of order  $p$  in  $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$  can be conjugated to the matrix  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

### Fields

Let  $a$  and  $b$  be elements of a field of order  $2^n$  where  $n$  is odd. Prove that if  $a^2 + ab + b^2 = 0$  then  $a = b = 0$ .

### Linear Algebra

Let  $A, B$  be linear operators on a nonzero finite-dimensional vector space  $V$  over  $\mathbb{C}$  such that  $A^2 = B^2 = \text{Id}$ . Prove that there exists a nonzero subspace  $W$  of  $V$  which is invariant under  $A$  and  $B$  and  $\dim W \leq 2$ .

#### Solution

Consider  $S = AB, T = BA$ . Then  $ST = AB^2A = A^2 = \text{Id} = B^2 = BA^2B = TS$ . Thus  $S, T$  are commuting operators on finite dimensional vector spaces. This means they have a common eigenvector, say  $v$ . Then there are scalars  $\lambda_S, \lambda_T \in \mathbb{C}$  such that  $Sv = \lambda_S v, Tv = \lambda_T v$ . Consider  $W = \langle v, Av \rangle \subseteq V$ . We show  $W$  is stable under  $A, B$ :

- $Av \in W$  by definition.
- $Bv = A^2Bv = A(AB)v = AS \cdot v = \lambda_S Av \in W$ .
- $A(Av) = A^2v = v \in W$ .
- $B(Av) = BAv = Tv = \lambda_T v \in W$ .

### Linear Algebra

Let  $A$  be a complex  $n \times n$  matrix. Let  $a_k$  denote the dimension of the null space of  $A^k$  (in particular,  $a_0 = 0$ ). Prove that  $a_k + a_{k+2} \leq 2a_{k+1}$  for all  $k \geq 0$ .

## Fall 2021

### Groups

Let  $G$  be a group and  $Z(G)$  the center of  $G$ . Show that the group  $G/Z(G)$  does not have prime order. Find a group  $G$  such that  $G/Z(G)$  has 4 elements.

### Rings

Show that every prime ideal  $P$  in  $\mathbb{Z}[x]$  which is not principal contains a prime number.

### Groups

Show that every finite noncyclic group is a finite union of proper subgroups, and that if a group maps surjectively to a finite noncyclic group then it is a finite union of proper subgroups and use this to determine for which positive integers the product of  $n$  copies of the integers is a finite union of proper subgroups.

### Linear Algebra

Let  $A$  and  $B$  be two square matrices over a field  $F$ . Suppose  $\text{diag}(A, A)$  and  $\text{diag}(B, B)$  are similar. Show that  $A$  and  $B$  are similar.

### Groups

- (a) Suppose that  $p$  and  $q$  are distinct primes and a group  $G$  is generated by elements of order  $p$  and also by elements of order  $q$ . Show that any homomorphism of  $G$  to an abelian group is trivial.
- (b) Show that for  $n \geq 5$  the alternating group  $A_n$  of even permutations of  $n$  objects is generated by elements of order 2, and also by elements of order 3, so that for such  $n$  the only homomorphisms to abelian groups are trivial.

## Spring 2021

### Rings

The following are four classes of commutative rings, in alphabetical order:

- fields
- integral domains
- principal integral domains
- unique factorization domains

These are contained in one-another, in some order, so that  $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq A_4$ .

- Determine the order.
- Give an example in each class to show that the inclusions are proper.

### Rings

- If  $R$  is a commutative ring, define what it means for  $R$  to be Noetherian and state Hilbert's basis theorem.
- Give an example of a non-Noetherian commutative ring.

### Groups

Let  $G$  be a group of order 105 and let  $P_3$ ,  $P_5$ , and  $P_7$  be Sylow 3, 5, and 7 subgroups, respectively. Assuming the Sylow theorems, prove the following:

- At least one of  $P_5$  or  $P_7$  is normal in  $G$ .
- $G$  has a cyclic subgroup of order 35.
- Both  $P_5$  and  $P_7$  are normal in  $G$ .

### Linear Algebra

Find all similarity classes of  $2 \times 2$  matrices  $A$  with entries in  $\mathbb{Q}$  satisfying  $A^4 = I$ . What are the corresponding rational canonical forms?

### Linear Algebra

- Find the possible Jordan Canonical Forms of any matrix such that  $A^4 = I$  over  $F = \mathbb{F}_5$ .
- Give an example of a matrix  $B$  over  $F = \mathbb{F}_3$  satisfying  $B^4 = I$ , such that  $B$  is not diagonalizable.

## Fall 2020

### Linear Algebra

Prove that for any pair of commuting  $n \times n$ -matrices with complex entries there exists a common eigenvector.

### Groups

Prove that there exists no simple group of order 56.

### Rings

Prove that a ring which contains a principal ideal ring  $R$ , and which is contained in the field of fractions of  $R$ , is a principal ideal ring.

### Linear Algebra

Let  $A$  and  $B$  be two projection linear maps in a vector space over a field  $K$ . Prove that if  $A + B$  is a projection linear map and  $\text{char} K \neq 2$  then  $AB = BA = 0$ .

#### Solution

Given that  $A, B, A + B$  are projections. That is, they satisfy  $x^2 = x$ . Then  $A + B = A^2 + B^2 + AB + BA = A + B + AB + BA \implies AB = -BA$ . But  $AB = A^2B = -ABA = BAA = BA^2 = BA$ . It follows that  $AB = BA = -AB \implies AB = 0 = BA$ . (Where is  $\text{char} K \neq 2$  used?)

### Groups

Prove that in the group  $\mathbb{Q}/\mathbb{Z}$  for any natural number  $n$  there exists exactly one subgroup of order  $n$ .

## Spring 2020

### Algebra

Suppose that  $A$  is a not necessarily commutative, finite dimensional associative algebra with a unit over a field  $F$  and  $P \trianglelefteq A$  is a two-sided ideal such that for  $a, b \in A$ ,  $ab \in P \implies a \in P$  or  $bP \in P$ . Show that  $A/P$  must be a division algebra (i.e. every nonzero element has a multiplicative inverse).

### Groups

Show that every group of order 2020 contains a unique (and hence normal) subgroup of order 505.

### Linear Algebra

Let  $M$  be a matrix with integer entries.

- (a) Prove that the minimal polynomial of  $M$  over  $\mathbb{C}$

$$f_{\min}(t) = t^k + \sum_{i=0}^{k-1} a_i t^i$$

has integer coefficients.

- (b) Prove that if  $M$  is diagonalizable over  $\mathbb{Q}$  then there exists an integer  $N$  such that the matrix  $M \bmod p$  is diagonalizable over  $\mathbb{Z}/p\mathbb{Z}$  for all  $p > N$ .

### Rings

Let  $F$  be a field and let  $L$  be the ring of Laurent polynomials  $L = F[x, x^{-1}]$  (it is the subring of  $F(x)$  generated over  $F$  by  $x$  and  $x^{-1}$ ). We consider  $L$  as a module over the ring of polynomials  $R = F[x]$ . (a) Show that  $L$  is not a finitely generated module over  $R$ . (b) Show that every finitely generated submodule of  $L$  is free with a single generator.

### Rings

Let  $R$  be a commutative integral domain and let  $I \trianglelefteq R$  be an ideal.

- (a) Show that every alternating bilinear form

$$f : I \times I \rightarrow R$$

is zero.

- (b) Show that if  $R$  is a principal ideal domain, then every alternating bilinear form  $f : I \times I \rightarrow M$  to any  $R$ -module  $M$  is zero.