2) $\mathcal{C}([a,b],\mathbb{R}) \longrightarrow$ is a vector space

$f: [a,b] \to \mathbb{R}$ $\quad (f+g)(x) = f(x) + g(x), \ \forall x \in [a,b]$

$g: [a,b] \to \mathbb{R}$

$\lambda \in \mathbb{R}$ $\quad (\lambda f): [a,b] \to \mathbb{R}$

$(\lambda f)(x) = \lambda f(x) \quad \forall x \in [a,b]$

Consider all polynomial functions on $[a,b]$

This forms a subspace of $\mathcal{C}([a,b],\mathbb{R})$

but it is infinite dimensional with a basis $\{1, x, x^2, \dots\}$

1) $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\varphi(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$$

$\mathbb{R}^n = \{e_i\}_{i=1}^{n}$ $\quad e_i = (0, \dots, 0, 1, 0, \dots, 0)$

↳ std. basis for $\mathbb{R}^n$ $\quad$ i_{th} position

$$\varphi(e_i) = a_i$$

ker φ is the subset asked in the question, so subspace

$$\dim(\ker \varphi) + \underbrace{\dim(\operatorname{Im}\varphi)}_{=1} = \dim(\mathbb{R}^n) = n$$

$$\dim(\ker \varphi) = n - 1$$

$\dim V = \dim W = n$

$\{b_i\}_{i=1}^n < V$ this set has cardinality n

φ injective $\to \{\varphi(b_i)\}_{i=1}^n \subset W$ is L-I. as φ injective

Since $\dim W = n$, as $\left|\{\varphi(b_i)\}_{i=1}^n\right| = n$ due to injective φ,

this is a basis

φ surjective $\to \{\varphi(b_i)\}_{i=1}^n$ spans W

but $\dim W = n \Rightarrow$ any spanning subset of W has cardinality $\geq n$

$\varphi : V \longrightarrow W$ φ injective

$$\dim(\ker\varphi) + \dim(\operatorname{Im}\varphi) = \dim V = n$$

$\ker\varphi = \{0\}$ as φ injective

$\dim(\operatorname{Im}\varphi) = n = \dim W$

$\operatorname{Im}\varphi$ subspace of $W \Rightarrow \operatorname{Im}\varphi = W$

$\Rightarrow \varphi$ surjective

$\left(\dim(\text{subspace}) = \dim(\text{space}) \Rightarrow \text{subspace} = \text{space}\right)$

$U \subset W$ strict subspace iff $\dim U < \dim W$

φ surjective

$$\dim(\ker) + \dim(\operatorname{Im}) = \dim V = n$$

$\operatorname{Im}\varphi = W \Rightarrow \dim(\operatorname{Im}\varphi) = \dim(W) = n$

$\Rightarrow \dim(\ker\varphi) = 0$

$\Rightarrow \varphi$ injective

4) Note that

$v \in \ker \varphi^2$
$\varphi^2(v) = 0 \Rightarrow \varphi^3(v) = \varphi(0) = 0 \Rightarrow v \in \ker \varphi^3$

$$\ker \varphi \subseteq \ker \varphi^2 \subseteq \ker \varphi^3 \subseteq \cdots$$

$$\operatorname{Im} \varphi \supseteq \operatorname{Im} \varphi^2 \supseteq \operatorname{Im} \varphi^3 \supseteq \cdots$$

Since V is finite dimensional. Then $\exists\, m_0 \in \mathbb{N}$ s.t. $\forall\, n \geq m_0$

$$\ker \varphi^n = \ker \varphi^{m_0}$$

and there exists n_0 s.t. $\forall\, n \geq n_0$

$$\operatorname{Im} \varphi^n = \operatorname{Im} \varphi^{n_0}$$

Take $N = \max(m_0, n_0)$, then $\forall\, n \geq N$

$$\boxed{\begin{aligned} \ker \varphi^n &= \ker \varphi^N \\ \operatorname{Im} \varphi^n &= \operatorname{Im} \varphi^N \end{aligned}}$$

Let $x \in \ker \varphi^N \cap \operatorname{Im} \varphi^N$

$\Rightarrow \varphi^N(x) = 0 \Rightarrow \varphi^{2N}(y) = 0$ as $x \in \operatorname{Im} \varphi^N$, $\varphi^N(y) = x$

$\Rightarrow y \in \ker \varphi^{2N} = \ker \varphi^N$

$\Rightarrow \varphi^N(y) = 0 \Rightarrow x = 0$

6) We use induction.
Consider two eigenvectors $\{v_i\}_{i=1}^{2}$ corresponding to eigenvalues $\{\lambda_i\}_{i=1}^{2}$. Let on contrary $\alpha_1 v_1 + \alpha_2 v_2 = 0$, $\alpha_2 \neq 0$

$\Rightarrow \lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 v_2 = 0$

$\Rightarrow (\lambda_1 - \lambda_2)\alpha_2 \cdot v_2 = 0$

$\Rightarrow (\lambda_1 - \lambda_2)\alpha_2 = 0$ as $v_2 \neq 0 \in V$

$\Rightarrow \lambda_1 = \lambda_2$ as $\alpha_2 \neq 0$

Assume Induction hypothesis for any $\{v_i\}_{i=1}^{m}$, $\forall\, m \le k$

Consider $\{v_i\}_{i=1}^{k+1}$. Let on contrary

$$\alpha_1 v_1 + \cdots + \alpha_{k+1} v_{k+1} = 0$$

Let $m \le k+1$ be the smallest natural number s.t $\alpha_m \neq 0$

So $\alpha_m v_m + \alpha_{m+1} v_{m+1} + \cdots + \alpha_{k+1} v_{k+1} = 0$

$$\Rightarrow \lambda_m \alpha_m v_m + \lambda_{m+1} \alpha_{m+1} v_{m+1} + \cdots + \lambda_{k+1} \alpha_{k+1} v_{k+1} = 0$$

$$\Rightarrow \lambda_{m+1} \alpha_m v_m + \lambda_{m+1} \alpha_{m+1} v_{m+1} + \cdots + \lambda_{m+1} \alpha_{k+1} v_{k+1} = 0$$

$$\Rightarrow (\lambda_{m+1} - \lambda_m) \alpha_m v_m + (\lambda_{m+2} - \lambda_m) \alpha_{m+2} v_{m+2} + \cdots = 0$$

by induction hypothesis

$\{v_i\}_{i=m}^{k+1}$ is L.I. as $k-m \le k$

So $(\lambda_{m+1} - \lambda_m)\alpha_m = 0$ as $v_m \neq 0$

$\Rightarrow \lambda_{m+1} = \lambda_m$ as $\alpha_m \neq 0$

Done!