1. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3n^2+1}$$

Does it absolutely converge? Does it converge? [5]

2. Consider the sequence $(a_n) \in \mathbb{R}^{\mathbb{N}}$. It satisfies $\limsup |a_n|^{1/n} < 1$

What can you say about $\lim_{n\to\infty} a_n$? [5]

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Solutions:

1. $\left\{\frac{n}{3n^2+1}\right\}$ is decreasing

$$\frac{n}{3n^2+1} \geq \frac{n+1}{3(n+1)^2+1}$$

$$\Leftrightarrow \quad 3n(n+1)^2+\cancel{n} \geq 3n^2(n+1) + \cancel{n}+1$$

$$\Leftrightarrow \quad 3n(n^2+2n+1) \geq 3n^3+3n^2+1$$

$$\Leftrightarrow \quad \cancel{3n^3}+6n^2+3n \geq \cancel{3n^3}+3n^2+1$$

$$\Leftrightarrow \quad 3n^2+3n \geq 1 \quad \Leftrightarrow \quad 3n(n+1) \geq 1$$ which is trivially true.

$$\lim_{n\to\infty} \frac{n}{3n^2+1} = 0.$$

By alternating series test, $\sum \frac{(-1)^n n}{3n^2+1}$ converges.

$$\sum_{n=1}^{k} \frac{n}{3n^2+1} \geq \sum_{n=1}^{k} \frac{n+1}{3(n+1)^2} = \sum_{n=2}^{k+1} \frac{1}{3n}$$

RHS diverges as $k \to \infty \Rightarrow \lim(\text{LHS}) = \infty$.

(2) $\limsup |a_n|^{1/n} < 1$

$\overset{\text{Root test}}{\Longrightarrow} \sum a_n \in \mathbb{R}$

$\Rightarrow \lim a_n = 0$