Theorem: Let (X,d) be a metric space. Endow $X \times X$ with the metric d_p for $p \in [1,\infty]$.
Then $f: X \times X \longrightarrow \mathbb{R}$ given by $f(x,y) = d(x,y)$ is uniformly continuous.

δ does not depend on x.

Proof: Enough to show for $p = \infty$.

Let $\varepsilon > 0$. Let $\delta = \varepsilon/2$.

$$|d(x,y) - d(x',y')| \le d(x,x') + d(y,y')$$
$$\le 2 \cdot \max\{d(x,x'), d(y,y')\}$$
$$= 2\, d_\infty((x,y),(x',y'))$$
$$< 2 \cdot \delta = 2 \cdot \varepsilon/2 \quad \blacksquare$$

$d_\infty((x,y),(x',y')) < \delta$

$\Rightarrow \varepsilon > |d(x,y) - d(x',y')|$

$d(x,y) \le d(x,x') + d(x',y)$
$\le d(x,x') + d(x',y') + d(y',y)$

X	$\times$	Y	$\mathbb{R} \times \mathbb{R}_{d'}$ (x,y) (x',y')
(d)		(d')	d $[d(x,x')^p + d'(y,y')^p]^{1/p}$

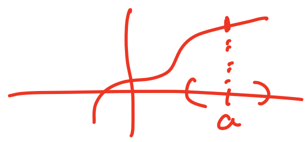
$$d_p((x,y),(x',y')) = \left[d(x,x')^p + d'(y,y')^p\right]^{1/p}$$

$$d_\infty((x,y),(x',y')) = \max\{d(x,x'), d'(y,y')\}.$$

- Class of all open sets in these metrics coincide
- Class of all cgt sequences in these metrics coincide
- Class of cont func in these metrics coincide.

BACK TO $\mathbb{R}$

Theorem: Let $S \subseteq \mathbb{R}$ be an interval, $f: S \to \mathbb{R}$ be a func cont at $a \in S$ s.t. $f(a) \neq 0$. Then $\exists\, r > 0$ s.t. $f(x) \cdot f(a) > 0 \ \forall x \in B_r(a) \cap S$.



Pf: WLOG, $f(a) > 0$. For $\varepsilon = \frac{f(a)}{2} > 0$ $\exists\, \delta > 0$ s.t.

$y \in S$ and $|y - a| < \delta \Rightarrow |f(y) - f(a)| < \varepsilon = f(a)/2$

$\Rightarrow 0 < f(a)/2 < f(y) < 3f(a)/2$

$\square$

Thm (IVP): Let $f: [a,b] \longrightarrow \mathbb{R}$ be cont with $f(a) \cdot f(b) < 0$.
Then $\exists\, c \in (a,b)$ s.t. $f(c) = 0$.

Pf: WLOG $f(a) > 0$, $f(b) < 0$. Let $S = \{x \in [a,b] : f([a,x]) \subseteq \mathbb{R}_{\ge 0}\}$.
$a \in S$ & b is an u.b. of S. Let $c = \sup(S) \in [a,b]$

Claim: $f(c) = 0$

Pf: ⊙ $f(c) > 0 \Rightarrow$ take $r > 0$ s.t. $f(x) > 0 \ \forall x \in B_r(c)$
$\Rightarrow c + r/2 \in S \Rightarrow c \neq \sup(S)$.

⊙ $f(c) < 0 \Rightarrow$ take $r > 0$ s.t. $f(x) < 0 \ \forall x \in B_r(c)$
$\Rightarrow c - \frac{r}{2}$ is an u.b. of $S \Rightarrow c \neq \sup(S)$. $\square$

$c \in (a,b)$ $\because$ $f(c) = 0$, $f(a) > 0$, $f(b) < 0$. $\blacksquare$

Cor: Say $f : S \to \mathbb{R}$ is cont with S a compact interval. If $f(a) \neq f(b)$ for some $a, b \in S$ then f attains every value b/w $f(a)$ & $f(b)$.

In General Metric Spaces

Continuous image of compact set is compact.

Thm: Let (X,d) & (Y,d') be metric spaces. $f : X \to Y$ continuous. If $C \subseteq X$ is compact then $f(C)$ is compact.

Pf: $X \xrightarrow{f} Y$

$\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ open cover of $f(C)$

$f^{-1}(U_\lambda)$ open $\longleftarrow U_\lambda$

Show: $\{f^{-1}(U_\lambda)\}$ form an open cover of C in X

$\downarrow$

finite cover for C $f^{-1}(U_{\lambda_1}), \dots, f^{-1}(U_{\lambda_n})$ $\longrightarrow$ $U_{\lambda_1}, \dots, U_{\lambda_n}$ finite cover for $f(C)$

Show that $f(C) \subseteq \bigcup_{i=1}^{n} U_{\lambda_i}$ assuming $C \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i})$ ∎

Thm: Let (X, d) be a compact metric space. Let $f : X \to \mathbb{R}$ be a continuous function. Then $\exists\, x, y \in X$ s.t. $\inf f(X) = f(x)$, $\sup f(X) = f(y)$.

Pf: $f(X)$ is compact. $\therefore f(X)$ closed and bounded in $\mathbb{R}$. Say $m = \inf f(S)$ and $M = \sup f(S)$. $m, M \in \overline{f(S)} = f(S)$. ∎

Cor (in $\mathbb{R}$): Let $S \subseteq \mathbb{R}$ be a compact interval. Say $f : S \to \mathbb{R}$ is continuous. Then $f(S) = [\inf f(S), \sup f(S)]$.

Remark: It may fail if S is not an interval in $\mathbb{R}$.

In fact the image of $id : [0,1] \cup [2,3] \longrightarrow \mathbb{R}$ is disconnected.

Uniform Continuity

Let $\underset{(d)}{f : X} \to \underset{(d')}{Y}$ be a function. f is said to be uniformly continuous if

$\forall\, \varepsilon > 0\ \exists\ \delta > 0$ s.t. $\forall\ x, y \in X$ with $d(x,y) < \delta$ we have $d'\big(f(x), f(y)\big) < \varepsilon$.

In continuity, δ depends both on ε and x (the point at which we check continuity)

In uniform continuity, δ depends only on ε.

Example:

(1) $f : (0, 1] \to \mathbb{R}$, $f(x) = 1/x$.

✓ Continuous
✗ Uniformly continuous

(2) $f : [1, \infty) \to \mathbb{R}$, $f(x) = 1/x$.

✓ Continuous
✓ Uniformly continuous ?
$\Updownarrow$
$(x_n \sim y_n \iff f(x_n) \sim f(y_n))$

$\frac{1}{x}$ cont. on $[a, \infty)$ $\forall\ a > 0$

Say $x, y \geq a$, $\varepsilon > 0$. $\delta = a^2 \varepsilon$.

$|x-y| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x-y|}{xy} \leq \frac{1}{a^2} |x-y| < \frac{\delta}{a^2} = \varepsilon$

A theorem by Heine: Let X, Y be compact metric spaces. Say $f : X \to Y$ is continuous. Then f is uniformly continuous.

Pf: Please look up yourself.

CONTRACTION (on $\mathbb{R}$)

Let $f: \mathbb{R} \to \mathbb{R}$.

→ $x \in \mathbb{R}$ is a fixed point (of f) if $f(x) = x$.

→ f is a contraction (of $\mathbb{R}$) if $\exists\ \alpha \in (0,1)$ s.t.

$$|f(x) - f(y)| \leq \alpha \cdot |x-y| \quad \forall\ x, y \in \mathbb{R}.$$

$f: X \to X \qquad d(f(x), f(y)) \leq \alpha\, d(x,y)$

Proposition: Every contraction is uniformly continuous.

Thm: Every contraction $\underset{\hookrightarrow f}{}$ (on $\mathbb{R}$) has a unique fixed point.

Pf: Uniqueness: x, y fixed $\Rightarrow |x-y| = |f(x) - f(y)| \leq \alpha |x-y| \underset{|x-y| \neq 0}{<} |x-y|$.

Existence: Take any $x \in \mathbb{R}$. Consider the seq of points $a_n = f^n(x)$.

composition

completeness

(a_n) is Cauchy due to contraction map property

$\therefore a_n \to l,\ l \in \mathbb{R}$. $f(l) = f(\lim a_n) = \lim f(a_n) = \lim a_{n+1} = l$.

$\therefore l$ is a fixed point.

$$|f^{k+1}(x) - f^k(x)| \leq \alpha^k \cdot |f(x) - x| \quad \forall\ k.$$

$$\begin{aligned}
|f^{m+n}(x) - f^m(x)| &\leq |f^{m+n}(x) - f^{m+n-1}(x)| + \dots + |f^{m+1}(x) - f^m(x)| \\
&\leq [\alpha^{m+n-1} + \dots + \alpha^m]\, |f(x) - x| \\
&= \alpha^m \underbrace{[1 + \dots + \alpha^{n-1}]}_{< B}\, |f(x) - x| \\
&< \alpha^m |f(x) - x| \cdot B \qquad < \varepsilon \quad \text{for large } m \text{ \& any } n.
\end{aligned}$$

DERIVATIVES ($\mathbb{R} \to \mathbb{R}$)

Let $f : (a,b) \to \mathbb{R}$. We say f is differentiable at $k \in (a,b)$ if $\exists\, l \in \mathbb{R}$ s.t.

$$\lim_{x \to k} \frac{f(x) - f(k) - l\cdot(x-k)}{x-k} = 0$$

Proposition: If f is diff at $k \in (a,b)$, & l, l' both satisfy the above then $l = l'$.

We usually denote the above l by $f'(k)$ or $\frac{df}{dx}(k)$.

Check: (1) f is diff at k iff $\displaystyle\lim_{t \to k} \frac{f(t) - f(k)}{t-k}$ exists.

(2) If f is diff at k then

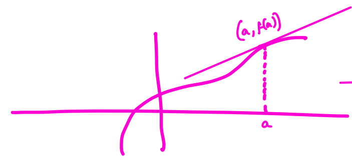
$$f'(k) = \lim_{t \to k} \frac{f(t) - f(k)}{t-k}.$$

by the way we defined

$\frac{\Delta y}{\Delta x} = \frac{f'(a)\cdot t}{t} = f'(a)$

$(a+t,\ f(a) + t\cdot f'(a))$

$\{(a, f(a)) + t\cdot(1, f'(a)) :\ t \in \mathbb{R}\}$



$\frac{y - f(a)}{x-a} = f'(a)$

$y = f(a) + (x-a) f'(a)$

We say $f : (a,b) \to \mathbb{R}$ is diff if $f'(k)$ exists $\forall\, k \in (a,b)$.

This gives a map $f' : (a,b) \to \mathbb{R}$. This is the first derivative.

Similarly we have n^{th} derivative $f^{(n)}(k) = \left(f^{(n-1)}\right)'(k)$.

Derivatives give you a local linearization.

Thm: Let $f : (a,b) \to \mathbb{R}$ be diff at $k \in (a,b)$. Then there is a function $f^* : (a,b) \to \mathbb{R}$, which is cont at k, $f'(k) = f^*(k)$ and satisfies $f(t) - f(k) = (t-k)\, f^*(t)$ $\forall\, t \in (a,b)$. — (#)

Conversely if $\exists$ a function f^*, cont at k satisfying (#) then f is diff at k and $f'(k) = f^*(k)$.

Pf: Say f is diff at k. Then define $f^*(t) := \begin{cases} \dfrac{f(t) - f(k)}{t-k}, & t \neq k \\ f'(k), & t = k \end{cases}$

Then f is cont at k and (#) holds.

For the other way, divide by $x-k$ and let $x \to k$ so that $f'(k)$ exists and equals $f^*(k)$. □

Cor: f diff at k $\Rightarrow$ f cont at k.