Lecture 17

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Today we continue with some additional aspects of quantifier elimination. We will then recall the Positivstellensatz and its relations with semidefinite programming. After introducing copositive matrices, we present Pólya's theorem on positive forms on the simplex, and the associated relaxations. Finally, we conclude with an important result due to Schmüdgen about representation of positive polynomials on compact sets.

1 Certificates

Quantifier elimination and decision methods are extremely powerful, since in principle they can handle formulas with arbitrary (finite) quantifier alternations and general semialgebraic expressions. Contrast this with the case of the Psatz, which applies only to the case of existential quantifiers on conjunctions of polynomial equations/inequalities (emptiness of a basic semialgebraic set). Nevertheless, a very important practical advantage of Psatz techniques is that they provide *certificates* of infeasibility, that can be checked in a completely independent fashion, regardless of what process was used to obtain them. For quantifier elimination, typically the only way of certifying that the answer is correct is by ensuring the correctness of the design and implementation of the QE method itself.

2 Psatz revisited

Recall the statement of the Positivstellensatz.

Theorem 1 (Positivstellensatz)**.** *Consider the set* $S = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0,\, h_i(x) = 0\}$*. Then,*

$$S = \emptyset \qquad \Leftrightarrow \qquad \exists f, h \in \mathbb{R}[x] \; s.t. \begin{cases} f + h &= -1 \\ f &\in \textbf{cone}\{f_1, \ldots, f_s\} \\ h &\in \textbf{ideal}\{h_1, \ldots, h_t\} \end{cases}$$

Once again, since the conditions on the polynomials f, h are convex and affine, respectively, by restricting their degree to be less than or equal to a given bound d we have a finite-dimensional semidefinite programming problem.

2.1 Hilbert 17th problem

As we have seen, in the general case nonnegative multivariate polynomials can fail to be a sum of squares (the Motzkin polynomial being the classical counterxample). As part of his famous list of twenty-three problems that he presented at the International Congress of Mathematicians in 1900, David Hilbert asked the following[1]:

[1]This text was obtained from `http://mathcs.clarku.edu/~djoyce/hilbert/`, and corresponds to Newson's translation of Hilbert's original German address. In that website you will also find links to the current status of the problems, as well as the original German text.

> **17. Expression of definite forms by squares.** A rational integral function or form in any number of variables with real coefficient such that it becomes negative for no real values of these variables, is said to be definite. The system of all definite forms is invariant with respect to the operations of addition and multiplication, but the quotient of two definite forms in case it should be an integral function of the variables is also a definite form. The square of any form is evidently always a definite form. But since, as I have shown, not every definite form can be compounded by addition from squares of forms, the question arises which I have answered affirmatively for ternary forms whether every definite form may not be expressed as a quotient of sums of squares of forms. At the same time it is desirable, for certain questions as to the possibility of certain geometrical constructions, to know whether the coefficients of the forms to be used in the expression may always be taken from the realm of rationality given by the coefficients of the form represented.

In other words, can we write every nonnegative polynomial as a sum of squares of *rational functions*? As we we show next, this is a rather direct consequence of the Psatz. Of course, it should be clear (and goes without saying) that we are (badly) inverting the historical order! In fact, much of the motivation for the development of real algebra came from Hilbert's question.

How can we use the Psatz to prove that a polynomial $p(x)$ is nonnegative? Clearly, p is nonnegative if and only if the set $\{x \in \mathbb{R}^n \,|\, p(x) < 0\}$ is empty. Since our version of the Psatz does not allow for strict inequalities (there are slightly more general, though equivalent, formulations that do), we'll need a useful trick discussed earlier ("Rabinowitch's trick"). Introducing a new variable z, the nonnegativity of $p(x)$ is equivalent to the emptiness of the set described by

$$-p(x) \geq 0, \qquad 1 - zp(x) = 0.$$

The Psatz can be used to show that this holds if and only if there exist polynomials $s_0, s_1, t \in \mathbb{R}[x, z]$ such that

$$s_0(x, z) - s_1(x, z) \cdot p + t(x, z) \cdot (1 - zp) = -1,$$

where s_0, s_1 are sums of squares. Replace now $z \to 1/p(x)$, and multiply by p^{2k} (where k is sufficiently large) to obtain

$$\tilde{s}_0(x) - \tilde{s}_1(x) \cdot p(x) = -p(x)^{2k},$$

where $\tilde{s}_0, \tilde{s}_1$ are sums of squares in $\mathbb{R}[x]$. Solving now for p, we have:

$$p(x) = \frac{\tilde{s}_0(x) + p(x)^{2k}}{\tilde{s}_1(x)} = \frac{\tilde{s}_1(x)(\tilde{s}_0(x) + p(x)^{2k})}{\tilde{s}_1^2(x)},$$

and since the numerator is a sum of squares, it follows that $p(x)$ is indeed a sum of squares of rational functions.

3 Copositive matrices and Pólya's theorem

An interesting class of matrices are the *copositive matrices*, which are those for which the associated quadratic form is nonnegative on the nonnegative orthant.

Definition 2. *A matrix $M \in \mathcal{S}^n$ is* copositive *if it satisfies*

$$x^T M x \geq 0, \qquad \textit{for all} \quad x_i \geq 0.$$

As opposed to positive semidefiniteness, which can be checked in polynomial time, the recognition problem for copositive matrices is an NP-hard problem [MK87]. The set of copositive matrices is a proper cone, which we will call $\mathcal{C}$. By the remark above, checking membership to the cone $\mathcal{C}$ is a difficult problem, even though it is convex. Its dual cone $\mathcal{C}^*$ also has a nice characterization, since it corresponds to the set of *completely positive* matrices:

Definition 3. *A matrix $W \in \mathcal{S}^n$ is* completely positive *if it is the sum of outer products of nonnegative vectors, i.e.,*

$$W = \sum_i^m x_i x_i^T, \qquad x_i \geq 0.$$

Alternatively, the matrix W factors as $W = FF^T$, where F is a nonnegative matrix (i.e., $F = [x_1, \ldots, x_m] \in \mathbb{R}^{m \times n}$).

Let $\mathcal{B} = \mathcal{C}^*$ be the set of copositive matrices. A natural sufficient condition for a matrix M to be copositive is if it can be expressed as the sum of a positive semidefinite matrix and a nonnegative matrix, i.e.,

$$M = P + N, \qquad P \succeq 0, \quad N_{ij} \geq 0.$$

This gives the containments

$$\mathcal{C} \supseteq \mathcal{S}_+^n + \mathcal{P}_+^n, \qquad \mathcal{B} \subseteq \mathcal{S}_+^n \cap \mathcal{P}_+^n,$$

where $\mathcal{P}_+^n \approxeq \mathbb{R}_+^{\binom{n+1}{2}}$ is the (self-dual) cone of nonnegative matrices. The containments are strict for $n \geq 5$; specific counterexamples will be discussed in the homework set. It should be clear that these conditions can be checked via SDP.

A good reference on completely positive matrices is [BSM03].

Applications There are many interesting applications of copositive and completely positive matrices. Among others, we mention:

- Consider a graph G, with A being its the adjacency matrix. The stability number α of the graph G is equal to the cardinality of its largest stable set. By a result of Motzkin and Straus, it is known that it can be obtained as:

  $$\frac{1}{\alpha(G)} = \min_{x_i \geq 0, \sum_i x_i = 1} x^T (I + A) x$$

  This implies that $\alpha(G) \leq \gamma$ if and only if the matrix $\gamma \cdot (I + A) - ee^T$ is copositive.

- In the analysis of linear dynamical systems with piecewise affine dynamics, it is often convenient to use piecewise-quadratic Lyapunov functions. In this case, we need to verify positivity conditions of an indefinite quadratic on a polyhedron. To make this precise, consider an affine dynamical system $\dot{x} = Ax + b$, a polyhedron $\mathcal{S}$ and a Lyapunov function $V(x)$ defined by:

  $$\mathcal{S} := \left\{ x \in \mathbb{R}^n \mid L \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \right\}, \qquad V(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T P \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

  Then, conditions for V and $-\dot{V}$ to be nonnegative on the set are:

  $$P \succeq L^T C_1 L, \qquad P \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix}^T P \preceq -L^T C_2 L,$$

  with C_1, C_2 copositive.

- Another interesting application of copositive matrices is in the performance analysis of queueing networks; see e.g. [KM96]. Modulo some (important) details, the basic idea is to use a quadratic function $x^T M x$ as a Lyapunov function, where the matrix M is copositive and x represents the lengths of the queues.

Pólya's theorem and copositive hierarchies Matrix copositivity can be easily interpreted in terms of polynomial nonnegativitiy. Indeed, it exactly corresponds to the condition that the polynomial $p_M(z_1, \ldots, z_n) := \mathbf{z}^T M \mathbf{z}$ be nonnegative, where $\mathbf{z} := [z_1^2, \ldots, z_n^2]^T$. Geometrically, this means that the copositive cone is an affine slice of the cone of nonnegative quartic forms.

It can be shown that the natural SOS relaxation of the nonnegativity of $p_M(\mathbf{z})$ yields the $P + N$ condition described earlier. We can strengthen this result to produce a hierarchy of SDP-representable cones that approximate $\mathcal{B}$ and $\mathcal{C}$. To do this, we use a well-known result by Pólya on positive forms on the simplex:

Theorem 4 (Pólya)**.** *Consider a homogeneous polynomial in n variables of degree d, that is strictly positive on the unit simplex $\Delta_n := \{x \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^n x_i = 1\}$. Then, for large enough k, the polynomial $(x_1 + \cdots + x_n)^k p(x)$ has nonnegative coefficients.*

It is possible to formulate a natural hierarchy of sufficient conditions for a matrix to be copositive, by considering a sum of squares condition on the polynomial $(\mathbf{z}^T \mathbf{z})^k (\mathbf{z}^T M \mathbf{z}) = (z_1^2 + \cdots + z_n^2)^k p_M(z_1, \ldots, z_n)$. Completeness of this hierarchy follows directly from Pólya's theorem [Par00].

Furthermore, there are interesting connections between Pólya's result and a foundational theorem in probability known as de Finetti's exchangeability theorem. We explore some of these links in the homework problems.

4 Positive polynomials

The Positivstellensatz allows us to obtain certificates of the emptiness of a basic semialgebraic set, explicitly given by polynomials.

What if we want to apply this for optimization? As we have seen, it is relatively straightforward to convert an optimization problem to a family of feasibility problems, by considering the sublevel sets, i.e., the sets $\{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}$.

In the case of constrained problems, however, using the fully general Psatz would yield conditions that are not linear in the unknown parameter γ (because we need products between the constraints), and this presents a difficulty to the direct use of SDP. Notice nevertheless, that the problem is certainly an SDP for any fixed value of γ, and it thus quasiconvex (which is almost as good, except for the fact that we cannot use "standard" SDP solvers to solve it directly, but rather rely on methods such as bisection).

A possible approach, however, is to use certain "distinguished" representations of nonnegative polynomials over semialgebraic sets. Typically, these require some mild assumptions, such as compactness. A good example, which we will discuss later, is the celebrated theorem by Schmüdgen:

Theorem 5 ([Sch91])**.** *If $p(x)$ is strictly positive on the set $K = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0\}$, and K is compact, then $p(x) \in \mathbf{cone}\{f_1, \ldots, f_s\}$.*

In the next lecture we will describe the basic elements of Schmüdgen's proof. His approach combines both algebraic tools (using the Positivstellensatz to prove the boundedness of certain operators) and functional analysis (spectral measures of commuting families of operators and the

Hahn-Banach theorem). We will also describe some alternative versions due to Putinar, as well as a related purely functional-analytic result due to Megretski.

For a comprehensive treatment and additional references, we mention [BCR98, Mar00, PD01] among others.

References

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