By $\mathbb{N}$ we mean +ve integers, $\mathbb{N} := \mathbb{Z}_{\geq 1}$.

I. Roots of positive reals

Let $a \in \mathbb{R}^+$, $a > 1$, and $n \in \mathbb{N}$

Lemma 1: If $x, y \in \mathbb{R}^+$ s.t. $x^n = y^n = a$ then $x = y$.

Proof: WLOG $x > y$. Then $x^n > y^n \Rightarrow a > a$ $(\Rightarrow\Leftarrow)$.

Theorem: $\exists\ x \in \mathbb{R}^+$ s.t. $x^n = a$.

Pf: $S = \{ t \in \mathbb{R}^+ : t^n < a \} \subseteq \mathbb{R}$

Goal: Show that $x = \sup S$ works

- $S \neq \phi$ (Why? $1 \in S$)
- S is bdd above (an ub is a)

By lub property of $\mathbb{R}$, $\sup S$ exists. Say $x := \sup S$.

$x^n < a$ $\qquad$ $x^n = a$ $\qquad$ $x^n > a$

Case 1: Suppose $x^n < a$. $\quad (a - x^n > 0)$

Strategy: to find some $\varepsilon > 0$ s.t. $x + \varepsilon \in S$
this will contradict that x is an u.b.

Pick $\varepsilon \in (0,1)$ s.t.

$$\varepsilon < \frac{a - x^n}{n(x+1)^{n-1}}$$

Take ε to be half of $\min\left\{1, \frac{a-x^n}{n(x+1)^{n-1}}\right\}$

Exercise: if $r > s$ then
$r^n - s^n < (r-s)\, n \cdot r^{n-1}$

$$(x+\varepsilon)^n - x^n < \varepsilon \cdot n \cdot (x+\varepsilon)^{n-1}$$

$$< \varepsilon\, n \cdot (x+1)^{n-1}$$

$$< a - x^n$$

$\Rightarrow (x+\varepsilon)^n < a \Rightarrow x + \varepsilon \in S$
contradicts maximality of x.

Case 2: $x^n - a > 0$

Strategy: To find $\varepsilon > 0$ s.t. $x - \varepsilon$ is an u.b. of S.

Choose $\varepsilon > 0$ s.t.

$$\varepsilon < \frac{x^n - a}{n x^{n-1}}$$

$$x^n - (x-\varepsilon)^n < n \cdot \varepsilon \cdot x^{n-1} < x^n - a$$

$\Rightarrow$ $a < (x-\varepsilon)^n$ $\Rightarrow$ $x - \varepsilon$ is an u.b. of S.

$\Rightarrow$ x is not <u>least</u> u.b. of S. $(\Rightarrow\Leftarrow)$

Only other possibility is $x^n = a$.

II. Extended real no. system

$\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ is said to be the extended real no. system

$+\infty, -\infty$ → Just symbols

These symbols must satisfy (defn): $(\forall\ x \in \mathbb{R})$

(i) $-\infty < x < +\infty$

(ii) $x + (+\infty) = +\infty = x - (-\infty)$

(iii) $x + (-\infty) = -\infty = x - (+\infty)$

(iv) $\dfrac{x}{-\infty} = 0 = \dfrac{x}{+\infty}$

(v) $x > 0 \Rightarrow x(+\infty) = +\infty$, $x(-\infty) = -\infty$

$x < 0 \Rightarrow x(+\infty) = -\infty$, $x(-\infty) = +\infty$

(vi) $(+\infty) + (+\infty) = +\infty = (+\infty)(+\infty) = (-\infty)(-\infty)$

$(+\infty)(-\infty) = (-\infty) + (-\infty) = -\infty$

If $S \subseteq \mathbb{R}$ is unbounded above then we say $\sup S = +\infty$

Similarly " below $\inf S = -\infty$.

Open nbds of $-\infty$ are of the form $(-\infty, x)$ $(x \in \mathbb{R})$

II. Normed field $\mathbb{Q}$.

We have the notion of an "absolute value" or a "norm"

$|\cdot| : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Verify:
- i) $|x| = 0 \Leftrightarrow x = 0$
- ii) $|xy| = |x| \cdot |y|$
- iii) $|x+y| \leq |x| + |y|$

$|\cdot|$ is also referred to as $|\cdot|_\infty$ or the ∞-norm.

It turns out that every norm on $\mathbb{Q}$ "looks like" $|\cdot|_p$ or $|\cdot|_\infty$ (Ostrowski's theorem).

Fix a prime p.

For an integer $x (\neq 0)$ we can define

$$v_p(x) = \text{highest power of } p \text{ divides } x$$
$$= \text{highest } n \in \mathbb{Z}_{\geq 0} \text{ s.t. } p^n \mid x$$
$$= \text{The unique } n \in \mathbb{Z}_{\geq 0} \text{ s.t. } p^n \mid x \text{ but } p^{n+1} \nmid x.$$

$v_5(50) = v_5(2 \times 5^2) = 2$
$v_5(3) = 0$
$v_p(x) = v_p(-x)$

Observe that $v_p(x \cdot y) = v_p(x) + v_p(y)$. $\forall x, y \in \mathbb{Z}$

This can be extended to rationals by saying that if $x = (a/b) \in \mathbb{Q}$ $(a, b \in \mathbb{Z})$ then define $v_p(x) = v_p(a) - v_p(b)$

($x = ca/cb$ then $v_p(x) = v_p(ca) - v_p(cb) = v_p(a) - v_p(b)$)

$|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$ by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-v_p(x)} & \end{cases}$$

It turns out that $|\cdot|_p$ is a norm $\forall$ primes p.
Stronger: $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ (proof not given here)