Tutorial Problems

(a)

$$3 \times 6!$$

$\uparrow$ even nos $\uparrow$ rest of them

(b)

Total $= \dfrac{8!}{2!\,2!}$

First & last no. are same:

- 1 .... 1 $\dfrac{6!}{2!}$
- 2 ..... 2 $\dfrac{6!}{2!}$

$$\left.\begin{matrix}\frac{6!}{2!}\\ \frac{6!}{2!}\end{matrix}\right\} \frac{6!}{2} + \frac{6!}{2} = 6!$$

Ans $= \dfrac{8!}{4} - 6! = 6!\,[14 - 1] = 13 \times 6!$

(c) $\binom{n}{k} \in \mathbb{Z}$ and the result follows.

$\ast$

$$\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$$

Pf: $S = \{1, 2, \ldots, n+1\}$.
How many subsets of size $r+1$ are there?

One way: $\binom{n+1}{r+1}$

Other way:

How many of these $r+1$-sized subsets have 1? $\binom{n}{r}$

don't have 1? $\binom{n}{r+1}$ $\square$

$$\binom{n}{2} = \binom{k}{2} + k \cdot (n-k) + \binom{n-k}{2}$$

One way: Directly choose 2 from those n objects. (LHS)

Other way: Split n objects into two groups of k obj (A) and $n-k$ objects (B).

Either $\quad 2$ from A, 0 from B $\rightarrow \binom{k}{2}$

$\quad\quad\quad 1$ from A, 1 from B $\rightarrow k(n-k)$

$\quad\quad\quad 0$ from A, 2 from B $\rightarrow \binom{n-k}{2}$

$\Big\}$ RHS

(e) $1 \leq k < p$ then $\binom{p}{k} \equiv 0 \pmod{p}$, p prime.

If p is not prime then we have a counterexample : $4 \nmid \binom{4}{2} = 6$

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = p \times \frac{(p-1)!}{k!\,(p-k)!} \in \mathbb{Z}$$

But $\overbrace{(k!, p)}^{\text{gcd}} = 1$, $(p, (p-k)!) = 1 \Rightarrow (k!\cdot(p-k)!, p) = 1$

$k! \times (p-k)! \mid p \times (p-1)!$

$\Rightarrow \quad k!\,(p-k)! \mid (p-1)!$

$\Rightarrow \quad p \mid \binom{p}{k}$.

$a \mid bc$ & $(a, b) = 1$ then $a \mid c$

(f) $k \geq 2n$. How many ways to distribute k sweets to n children so that each child gets at least 2 ?

Children are distinct.
Chocolates are not!

$x_1 + x_2 \ldots + x_n = k$ $(x_i \in \mathbb{Z},\ x_i \geq 2) \rightarrow$ Substitute $y_i = x_i - 2$

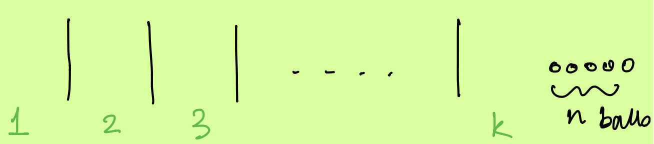
$\Leftrightarrow \sum_{i=1}^{k} y_i = k - 2n \rightsquigarrow \binom{k-2n+n-1}{n-1}$

$x_1 + x_2 + \cdots + x_k = n \quad$ $x_i \in \mathbb{Z}$, $x_i \geq 0$

How many solutions?

$$\binom{n+k-1}{k-1}$$

$k-1$ partitions/sticks



n balls to distribute into k boxes.

$$\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{(k-1)!\; n!}$$

Rearrangement of the $\underbrace{\text{balls}}_{\text{not distinct}}$ & $\underbrace{\text{sticks}}_{\text{not distinct}}$

$\Updownarrow$

Distributing balls into boxes

g

$f : [n] \to [n]$

$\exists$ exactly one i for which $f(i) = i$.

n choices

$$\underbrace{n}_{\substack{\text{Which } i \\ \text{to fix?}}} \times \underbrace{(n-1)^{n-1}}_{\substack{\text{The others} \\ \text{not mapping to} \\ \text{themselves.}}}$$

$1 \longrightarrow$	1
2	2
3	3