Algebra Qualifying Exams

Rutgers - the State University of New Jersey

Syllabus

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Contents

Spring 2023

Groups

Classify all groups of order 309, up to isomorphism.

Groups

Let A be the abelian group with generators x, y, z and the relations

$$4x + 3y + z = 0, x + 2y + 3z = 0, 3x + 2y + 5z = 0$$

Show that A is a cyclic abelian group, and determine its order.

Linear Algebra

Let A be a complex $n \times n$ matrix. Prove that there is an invertible complex $n \times n$ matrix B such that $AB = BA^t$. (A^t is the transpose of A.)

Rings

Prove that the subring $\mathbb{Z}[3i]$ of $\mathbb{C}$ is not a Principal Ideal Domain.

Rings

If $R = \mathbb{Z}[x]$, show that the sequence $R \xrightarrow{f} R^2 \xrightarrow{g} R$ is exact, where $f(a) = (ax, -2a)$ and $g(c, d) = 2c + dx$.

Fall 2022

Groups

Let G be a finite simple group. Prove that $G \times G$ has exactly 4 normal subgroups (including $G \times G$) if and only if G is non-abelian.

Rings

Let R be a principal ideal domain and I, J be ideals of R. Show that $I \cap J = IJ$ holds if and only if $I = 0$ or $J = 0$ or $I + J = R$.

Linear Algebra

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix with real coefficients. Show that all eigenvalues of A are non-negative if and only if $A = P^T P$ for some matrix $P \in M_n(\mathbb{R})$.

Rings

Let R be an integral domain and $R[x, y, z]$ the polynomial ring in three variables over R. Show that $I = \langle x^3, y^2, y^3 - z^2 y \rangle \subseteq R[x, y, z]$ is a prime ideal.
Hint: Show that I is the kernel of a ring homomorphism $R[x, y, z] \to R[t]$.

Linear Algebra

Let A and B be commuting complex matrices. Assume that $B \notin \mathbb{C}[A]$, that is, B cannot be written as a polynomial in A. Show that some eigenspace of A has dimension at least two.