COMPONENT INTERVALS

The idea is that the "building blocks" of open $S \subseteq \mathbb{R}$ are the open intervals.

Definition: Let $S \subseteq \mathbb{R}$ be open. An interval $I \subseteq S$ is said to be a <u>component interval</u> of S if I is the maximal open interval contained in S.

(What maximal means: If $J \subseteq S$ is another open interval s.t. $I \subseteq J$ then $J \subseteq I$)

Observation: If I & J are (max'l) components of open S s.t. $I \cap J \neq \phi$ then $I = J$.

Pf: Say $x \in I \cap J$. Then $I \cup J$ is an open interval contained in S. $I \subseteq I \cup J \Rightarrow I \cup J \subseteq I \Rightarrow J \subseteq I$

Similarly, $J \subseteq I \cup J \Rightarrow I \cup J \subseteq J \Rightarrow I \subseteq J$.

$\therefore I = J$. $\square$

Lemma: Let $S \subseteq \mathbb{R}$ be open & $x \in S$. $\exists !$ component interval $I_x \subseteq S$ s.t. $x \in I_x$.

Proof: Let $U = \{ t > 0 : (x, x+t) \subseteq S \}$

$L = \{ t > 0 : (x-t, x) \subseteq S \}$.

S open $\Rightarrow \exists\ \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq S$.

$\therefore \varepsilon \in U, \quad \varepsilon \in L \Rightarrow U \neq \phi, \ L \neq \phi$.

Let $b = \sup U, \quad a = \sup L$. Take $I_x := (x-a, x+b)$.
(Note that b, a can as well be $+\infty$).

Claim: $I_x \subseteq S$.

Claim: I_x is a component interval of S.

Pf: Let $J = (x-\alpha, x+\beta) \subseteq S$ s.t. $I \subseteq J$.

$I \subseteq J \Rightarrow (x-a, x+b) \subseteq (x-\alpha, x+\beta)$

$\Rightarrow \alpha \geq a, \ \beta \geq b$.

Note that $(x-\alpha, x) \subseteq J \subseteq S \Rightarrow \alpha \in L \Rightarrow \alpha \leq a$.

Similarly $\beta \in U \Rightarrow \beta \leq b$.

It follows that $\alpha = a, \ b = \beta$. $\therefore I = J$. ∎

Let K be a component interval of S which contains x. $\therefore I_x \cap K \neq \phi \Rightarrow I_x = K$ (by previous observation) □

Theorem: Every non-empty open set $S \subseteq \mathbb{R}$ is the disjoint union of component intervals. Moreover, this union is countable.

Pf: For $x \in S$ let I_x be the ! component interval in S s.t. $x \in I_x$. $\mathcal{U} := \{ I_x : x \in S \}$.

Claim 1: $S = \bigsqcup_{U \in \mathcal{U}} U$

Claim 2: $\mathcal{U}$ is countable.

Pf of claim 1: Clearly $S = \bigcup_{U \in \mathcal{U}} U \quad \because x \in I_x \in \mathcal{U}$.

The union is disjoint $\because$ if $U \cap V \neq \phi$ for some $U, V \in \mathcal{U}$ then $U = V$.

Pf of claim 2: We know $\mathbb{Q}$ is countable. $\therefore \exists$ bij $f: \mathbb{N} \to \mathbb{Q}$.

For each $U \in \mathcal{U}$ we assign a natural number:

Let $U \in \mathcal{U}$. $\exists\ q \in \mathbb{Q}$ s.t. $q \in U$.
Say $i \in \mathbb{N}$ is s.t. $f(i) = q$. Let $i_0 \in \mathbb{N}$ be smallest s.t. $f(i_0) \in U$.
(why does such i_0 exist? WOP of $\mathbb{N}$)
Assign i_0 to U.

Call the above rule $g: \mathcal{U} \to \mathbb{N}$.

g is 1-1: Say $U, V \in \mathcal{U}$ s.t. $g(U) = g(V) = k$
$\Rightarrow f(k) \in U \cap V \Rightarrow U \cap V \neq \phi \Rightarrow U = V$.

$\therefore \mathcal{U}$ is countable.