CONVEX AND CONIC OPTIMIZATION

Homework 6

NILAVA METYA
nilava.metya@rutgers.edu
nm8188@princeton.edu

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Problem 1

1. Suppose you had a blackbox that given a 3SAT instance would tell you whether it is satisfiable or not. How can you make polynomially many calls to this blackbox to find a satisfying assignment to any satisfiable instance of 3SAT?
2. Suppose you had a blackbox that given a graph G and an integer k would tell you whether G has a stable set of size larger or equal to k. How can you make polynomially many calls to this blackbox to find a maximum stable set of a given graph?

Solution

1. Denote the blackbox by f. So f takes in a formula (in three variables) and outputs 1 if satisfiable, and 0 otherwise.
   Let S be a formula in CNF, with variables $x_1, \cdots, x_n$. Treat S as a polynomial in x_i's. Assume S is satisfiable, i.e., there are values $a_1, \cdots, a_n \in \{0,1\}$ such that $S(a_1, \cdots, a_n) = 1$. Often we denote $\boldsymbol{a} = (a_1, \cdots, a_n)$. We will make n calls to the blackbox.

   Claim 1
   For each $i \in [n]$, at least one of $f(x_i \wedge S)$ or $f(\overline{x_i} \wedge S)$ is 1.

   Proof. If $a_i = 1$ then $(x_i \wedge S)(\boldsymbol{a}) = 1 \cdot S(\boldsymbol{a}) = 1$. If $a_i = 0$ then $(\overline{x_i} \wedge S)(\boldsymbol{a}) = 1 \cdot S(\boldsymbol{a}) = 1$. ■

   We find each $f(x_i \wedge S)$ with the blackbox. For each i, if $f(x_i \wedge S) = 1$ then set $a_i = 1$, otherwise set $a_i = 0$. Denote $y_i := \begin{cases} x_i & \text{if } a_i = 1 \\ \overline{x_i} & \text{otherwise} \end{cases}$. $\boldsymbol{a}$ satisfies every $y_i S$ by the above. We'll show that $\boldsymbol{a}$ satisfies S.

$$\begin{aligned}
\left(\bigwedge_{i=1}^{n} (y_i S)\right)(\boldsymbol{a}) &= \left[\left(\bigwedge_{i=1}^{n} y_i\right) \wedge S\right](\boldsymbol{a}) && [\because \alpha \wedge \alpha = \alpha] \\
\implies \prod_{i=1}^{n} (y_i \wedge S)(\boldsymbol{a}) &= \left(\prod_{i=1}^{n} y_i(\boldsymbol{a})\right) \cdot S(\boldsymbol{a}) && [\because (\alpha \wedge \beta)(\boldsymbol{a}) = \alpha(\boldsymbol{a}) \cdot \beta(\boldsymbol{a})] \\
\implies 1 &= \left(\prod_{i=1}^{n} y_i(\boldsymbol{a})\right) \cdot S(\boldsymbol{a}) && [\because (y_i S)(a) = 1 \forall i] \\
\implies 1 &= S(\boldsymbol{a}) && [\because y_i(\boldsymbol{a}) = 1 \forall i \text{ by construction}]
\end{aligned}$$

2.

Problem 2

Consider a family of decision problems indexed by a positive integer k:

RANK-k-SDP

Input: Symmetric $n \times n$ matrices $A_1, \cdots, A_m$ with entries in $\mathbb{Q}$, scalars $b_1, \cdots, b_m \in \mathbb{Q}$.
Question: Is there a real symmetric matrix X that satisfies the constraints

$$\mathrm{Tr}(A_i X) = b_i, i \in [m], X \succeq 0, rank(X) = k?$$