

# Algebra Qualifying Exams

Rutgers - the State University of New Jersey

Syllabus

Nilava Metya

## Contents

<b>Spring 2023</b>	<b>2</b>
<b>Fall 2022</b>	<b>3</b>
<b>Spring 2022</b>	<b>4</b>
<b>Fall 2021</b>	<b>5</b>
<b>Spring 2021</b>	<b>6</b>
<b>Fall 2020</b>	<b>7</b>
<b>Spring 2020</b>	<b>8</b>

## Spring 2023

### Groups

Classify all groups of order 309, up to isomorphism.

### Groups

Let  $A$  be the abelian group with generators  $x, y, z$  and the relations

$$4x + 3y + z = 0, x + 2y + 3z = 0, 3x + 2y + 5z = 0$$

Show that  $A$  is a cyclic abelian group, and determine its order.

### Linear Algebra

Let  $A$  be a complex  $n \times n$  matrix. Prove that there is an invertible complex  $n \times n$  matrix  $B$  such that  $AB = BA^t$ . ( $A^t$  is the transpose of  $A$ .)

#### Solution

The given statement is equivalent to showing the existence of an invertible  $B$  such that  $A^t = B^{-1}AB$ . This is just saying that  $A, A^t$  are similar. Since we are working over  $\mathbb{C}$ , we can simply work with JCF. This suffices because if  $A = X^{-1}JX$  where  $J$  is the JCF of  $A$ , then  $A^t = B^{-1}AB$  is equivalent to saying that  $YJ^tY^{-1} = B^{-1}X^{-1}JXB$  where  $Y = X^t$ , which is equivalent to saying that  $J^t = (XBY)^{-1}X(XBY)$ . This is simply saying that  $J$  is similar to its transpose. Since  $J$  is made of block matrices, transpose treats every square block independently, and using the fact that  $\begin{bmatrix} P & \\ & Q \end{bmatrix} \sim \begin{bmatrix} U & \\ & V \end{bmatrix}$  if  $P \sim U$  and  $Q \sim V$ , it is enough to show that every Jordan block is similar to its transpose. (Here  $\sim$  stands for similarity of matrices.)

To see this, we start with a Jordan block  $J$  of size  $n \times n$  and eigenvalue  $\lambda$ . Let  $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a linear transformation whose matrix with respect to the basis  $\mathbf{e} = (e_1, \dots, e_n)$  is  $J$ . The action of  $T$  is given by  $Te_1 = \lambda e_1$  and  $Te_j = \lambda e_j + e_{j-1}$  for  $1 < j \leq n$ . Now we look at the matrix of  $T$  in the basis  $\mathbf{f} = (f_1, \dots, f_n)$  where  $f_i = e_{n-i+1} \forall 1 \leq i \leq n$ . Clearly the first column of  $T$  in this basis is determined by  $Jf_1 = \lambda e_n + e_{n-1} = \lambda f_1 + f_2$  which corresponds to the column matrix where first two entries are  $\lambda, 1$  respectively and everything else is 0. The  $j^{\text{th}}$  column ( $1 \leq j < n$ ) is given by  $Tf_j = Te_{n+1-j} = \lambda e_{n+1-j} + e_{n-j} = \lambda f_j + f_{j+1}$  which corresponds to the columns where the  $j^{\text{th}}, (j+1)^{\text{st}}$  entries are  $\lambda, 1$  respectively, and everything else is 0. This means that  $[T]_{\mathbf{e}} = [T]_{\mathbf{f}}^t$ . Since both the matrices  $[T]_{\mathbf{e}}, [T]_{\mathbf{f}}$  correspond to the same linear operator, but represented in different bases, they are similar. This proves that every Jordan block is similar to its transpose.

### Rings

Prove that the subring  $\mathbb{Z}[3i]$  of  $\mathbb{C}$  is not a Principal Ideal Domain.

### Rings

If  $R = \mathbb{Z}[x]$ , show that the sequence  $R \xrightarrow{f} R^2 \xrightarrow{g} R$  is exact, where  $f(a) = (ax, -2a)$  and  $g(c, d) = 2c + dx$ .

## Fall 2022

### Groups

Let  $G$  be a finite simple group. Prove that  $G \times G$  has exactly 4 normal subgroups (including  $G \times G$ ) if and only if  $G$  is non-abelian.

### Rings

Let  $R$  be a principal ideal domain and  $I, J$  be ideals of  $R$ . Show that  $I \cap J = IJ$  holds if and only if  $I = 0$  or  $J = 0$  or  $I + J = R$ .

### Linear Algebra

Let  $A \in M_n(\mathbb{R})$  be a symmetric matrix with real coefficients. Show that all eigenvalues of  $A$  are non-negative if and only if  $A = P^T P$  for some matrix  $P \in M_n(\mathbb{R})$ .

#### Solution

Suppose  $A = P^T P$ . Then  $P \in M_n(\mathbb{R}) \implies A = P^\dagger P$  where  $P^\dagger$  is the conjugate transpose. Let  $(\mathbf{x}, \lambda) \in \mathbb{C}^n \times \mathbb{C}$  be an eigenvector-eigenvalue pair for  $A$ . Clearly  $\mathbf{x}^\dagger A \mathbf{x} = (P \mathbf{x})^\dagger (P \mathbf{x}) = \|P \mathbf{x}\|^2 \geq 0$ . But also  $\mathbf{x}^\dagger A \mathbf{x} = \lambda \mathbf{x}^\dagger \mathbf{x} = \lambda \|\mathbf{x}\|^2$  and  $\|\mathbf{x}\|^2 > 0$ . This shows that  $\lambda \in \mathbb{R}_{\geq 0}$ .

Suppose  $A$  is symmetric real matrix with non-negative eigenvalues. So  $A$  is Hermitian, and by the spectral theorem of real symmetric matrices, we can write it as  $A = U D U^T$  where  $D$  comprises of eigenvalues of  $A$ , and  $U$  is orthogonal (comprising of an eigenbasis of  $A$ ). Since eigenvalues are non-negative,  $D$  has all non-negative entries  $\lambda_1, \dots, \lambda_n$  in its diagonal (0 elsewhere). Consider  $E = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$  so that  $D = E^2 = E E^T$ . Then  $A = U D U^T = (U E)(U E)^T$ . Taking  $P = (U E)^T \in M_n(\mathbb{R})$  gives  $A = P^T P$  as desired.

### Rings

Let  $R$  be an integral domain and  $R[x, y, z]$  the polynomial ring in three variables over  $R$ . Show that  $I = \langle x^3, y^2, y^3 - z^2 y \rangle \subseteq R[x, y, z]$  is a prime ideal.

Hint: Show that  $I$  is the kernel of a ring homomorphism  $R[x, y, z] \rightarrow R[t]$ .

### Linear Algebra

Let  $A$  and  $B$  be commuting complex matrices. Assume that  $B \notin \mathbb{C}[A]$ , that is,  $B$  cannot be written as a polynomial in  $A$ . Show that some eigenspace of  $A$  has dimension at least two.

## Spring 2022

### Rings

Prove that the rings  $\mathbb{Q}[x]/(x^2 - 1)$  and  $\mathbb{Q} \oplus \mathbb{Q}$  are isomorphic.

### Groups

Let  $p$  be a prime. Show that any element of order  $p$  in  $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$  can be conjugated to the matrix  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

### Fields

Let  $a$  and  $b$  be elements of a field of order  $2^n$  where  $n$  is odd. Prove that if  $a^2 + ab + b^2 = 0$  then  $a = b = 0$ .

#### Solution

Since  $F$  has order  $2^n$  (with  $n$  odd, say  $2k+1$ ), we have  $x^{2^n-1} = 1$  for  $x \in F^\times$  because  $F^\times$  is a multiplicative group. Further note that  $2^n - 1 = 2 \times 4^k - 1 \equiv 1 \pmod{3} \implies (3, 2^n - 1) = 1$ . There are integers  $u, v$  such that  $3u + (2^n - 1)v = 1$ . Note that

$$\begin{aligned} a^2 + ab + b^2 &= 0 \\ \implies a^3 - b^3 &= (a - b)(a^2 + ab + b^2) = 0 \\ &\implies a^3 = b^3 \\ \implies a &= (a)^{3u} \cdot (a)^{(2^n-1)v} = (b)^{3u} \cdot (b)^{(2^n-1)v} = b \\ &\implies a = b \end{aligned}$$

But  $0 = a^2 + ab + b^2 = 3a^2 \implies a^2 = 0$  as  $F$  has characteristic 2, whence 3 is invertible. Finally,  $a^2 = 0$  means  $a = 0$ .

### Linear Algebra

Let  $A, B$  be linear operators on a nonzero finite-dimensional vector space  $V$  over  $\mathbb{C}$  such that  $A^2 = B^2 = \text{Id}$ . Prove that there exists a nonzero subspace  $W$  of  $V$  which is invariant under  $A$  and  $B$  and  $\dim W \leq 2$ .

#### Solution

Consider  $S = AB, T = BA$ . Then  $ST = AB^2A = A^2 = \text{Id} = B^2 = BA^2B = TS$ . Thus  $S, T$  are commuting operators on finite dimensional vector spaces. This means they have a common eigenvector, say  $v$ . Then there are scalars  $\lambda_S, \lambda_T \in \mathbb{C}$  such that  $Sv = \lambda_S v, Tv = \lambda_T v$ . Consider  $W = \langle v, Av \rangle \subseteq V$ . We show  $W$  is stable under  $A, B$ :

- $Av \in W$  by definition.
- $Bv = A^2Bv = A(AB)v = AS \cdot v = \lambda_S Av \in W$ .
- $A(Av) = A^2v = v \in W$ .
- $B(Av) = BAv = Tv = \lambda_T v \in W$ .

### Linear Algebra

Let  $A$  be a complex  $n \times n$  matrix. Let  $a_k$  denote the dimension of the null space of  $A^k$  (in particular,  $a_0 = 0$ ). Prove that  $a_k + a_{k+2} \leq 2a_{k+1}$  for all  $k \geq 0$ .

## Fall 2021

### Groups

Let  $G$  be a group and  $Z(G)$  the center of  $G$ . Show that the group  $G/Z(G)$  does not have prime order. Find a group  $G$  such that  $G/Z(G)$  has 4 elements.

### Rings

Show that every prime ideal  $P$  in  $\mathbb{Z}[x]$  which is not principal contains a prime number.

### Groups

Show that every finite noncyclic group is a finite union of proper subgroups, and that if a group maps surjectively to a finite noncyclic group then it is a finite union of proper subgroups and use this to determine for which positive integers the product of  $n$  copies of the integers is a finite union of proper subgroups.

### Linear Algebra

Let  $A$  and  $B$  be two square matrices over a field  $F$ . Suppose  $\text{diag}(A, A)$  and  $\text{diag}(B, B)$  are similar. Show that  $A$  and  $B$  are similar.

### Groups

- (a) Suppose that  $p$  and  $q$  are distinct primes and a group  $G$  is generated by elements of order  $p$  and also by elements of order  $q$ . Show that any homomorphism of  $G$  to an abelian group is trivial.
- (b) Show that for  $n \geq 5$  the alternating group  $A_n$  of even permutations of  $n$  objects is generated by elements of order 2, and also by elements of order 3, so that for such  $n$  the only homomorphisms to abelian groups are trivial.

## Spring 2021

### Rings

The following are four classes of commutative rings, in alphabetical order:

- fields
- integral domains
- principal integral domains
- unique factorization domains

These are contained in one-another, in some order, so that  $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq A_4$ .

- Determine the order.
- Give an example in each class to show that the inclusions are proper.

### Rings

- If  $R$  is a commutative ring, define what it means for  $R$  to be Noetherian and state Hilbert's basis theorem.
- Give an example of a non-Noetherian commutative ring.

### Groups

Let  $G$  be a group of order 105 and let  $P_3$ ,  $P_5$ , and  $P_7$  be Sylow 3, 5, and 7 subgroups, respectively. Assuming the Sylow theorems, prove the following:

- At least one of  $P_5$  or  $P_7$  is normal in  $G$ .
- $G$  has a cyclic subgroup of order 35.
- Both  $P_5$  and  $P_7$  are normal in  $G$ .

### Linear Algebra

Find all similarity classes of  $2 \times 2$  matrices  $A$  with entries in  $\mathbb{Q}$  satisfying  $A^4 = I$ . What are the corresponding rational canonical forms?

### Linear Algebra

- Find the possible Jordan Canonical Forms of any matrix such that  $A^4 = I$  over  $F = \mathbb{F}_5$ .
- Give an example of a matrix  $B$  over  $F = \mathbb{F}_3$  satisfying  $B^4 = I$ , such that  $B$  is not diagonalizable.

## Fall 2020

### Linear Algebra

Prove that for any pair of commuting  $n \times n$ -matrices with complex entries there exists a common eigenvector.

### Groups

Prove that there exists no simple group of order 56.

### Rings

Prove that a ring which contains a principal ideal ring  $R$ , and which is contained in the field of fractions of  $R$ , is a principal ideal ring.

### Linear Algebra

Let  $A$  and  $B$  be two projection linear maps in a vector space over a field  $K$ . Prove that if  $A + B$  is a projection linear map and  $\text{char}K \neq 2$  then  $AB = BA = 0$ .

#### Solution

Given that  $A, B, A + B$  are projections. That is, they satisfy  $x^2 = x$ . Then  $A + B = A^2 + B^2 + AB + BA = A + B + AB + BA \implies AB = -BA$ . But  $AB = A^2B = -ABA = BAA = BA^2 = BA$ . It follows that  $AB = BA = -AB \implies AB = 0 = BA$ . (Where is  $\text{char}K \neq 2$  used?)

### Groups

Prove that in the group  $\mathbb{Q}/\mathbb{Z}$  for any natural number  $n$  there exists exactly one subgroup of order  $n$ .

## Spring 2020

### Algebra

Suppose that  $A$  is a not necessarily commutative, finite dimensional associative algebra with a unit over a field  $F$  and  $P \trianglelefteq A$  is a two-sided ideal such that for  $a, b \in A$ ,  $ab \in P \implies a \in P$  or  $bP \in P$ . Show that  $A/P$  must be a division algebra (i.e. every nonzero element has a multiplicative inverse).

### Groups

Show that every group of order 2020 contains a unique (and hence normal) subgroup of order 505.

### Linear Algebra

Let  $M$  be a matrix with integer entries.

- (a) Prove that the minimal polynomial of  $M$  over  $\mathbb{C}$

$$f_{\min}(t) = t^k + \sum_{i=0}^{k-1} a_i t^i$$

has integer coefficients.

- (b) Prove that if  $M$  is diagonalizable over  $\mathbb{Q}$  then there exists an integer  $N$  such that the matrix  $M \bmod p$  is diagonalizable over  $\mathbb{Z}/p\mathbb{Z}$  for all  $p > N$ .

### Rings

Let  $F$  be a field and let  $L$  be the ring of Laurent polynomials  $L = F[x, x^{-1}]$  (it is the subring of  $F(x)$  generated over  $F$  by  $x$  and  $x^{-1}$ ). We consider  $L$  as a module over the ring of polynomials  $R = F[x]$ . (a) Show that  $L$  is not a finitely generated module over  $R$ . (b) Show that every finitely generated submodule of  $L$  is free with a single generator.

### Rings

Let  $R$  be a commutative integral domain and let  $I \trianglelefteq R$  be an ideal.

- (a) Show that every alternating bilinear form

$$f : I \times I \rightarrow R$$

is zero.

- (b) Show that if  $R$  is a principal ideal domain, then every alternating bilinear form  $f : I \times I \rightarrow M$  to any  $R$ -module  $M$  is zero.