Continued from yesterday.

(1) $\frac{6}{8} = \frac{3 \times 2}{2^3} = 3 \times 2^{-2}$

$\left|\frac{6}{8}\right|_3 = \frac{1}{3}$

(2) $\frac{6}{8} = 3 \times 2^{-2}$

$\left|\frac{6}{8}\right|_2 = 4$

$3/4 = 3 \times 2^{-2}$

$|3/4|_2 = 4$

(3) $5/2 = 5 \times 2^{-1}$

$|5/2|_5 = \frac{1}{5}$ $\quad$ $|5/2|_2 = 2$

(4) $|5/2|_7 = 1$

Fix prime $p = 3$.

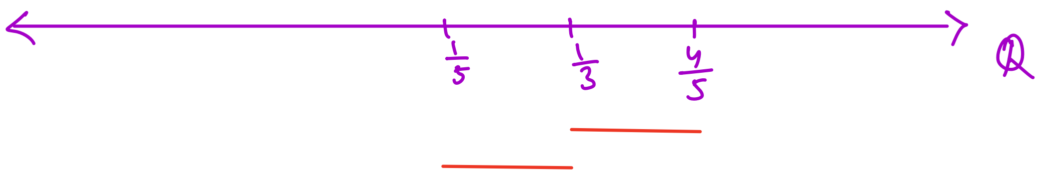
"Dist" between $1/5$ and $4/5$ $= \left|\frac{4}{5} - 1/5\right|_3$

$= |3/5|_3 = 1/3$

"3-adic distance"

$1/5$ and $1/3$ $= |1/3 - 1/5| = \left|\frac{2}{15}\right|_3 = 3$

$\frac{4}{5}$ and $1/3$ $= |4/5 - 1/3|_3 = \left|\frac{7}{15}\right|_3 = 3$



Why is $d(x,y) = |x-y|_p$ a metric?
(prime p).

Instead of triangle inequality we will show that

$|x+y|_p \leq \max\{|x|_p, |y|_p\}$. $\to$ $x, y \in \mathbb{Q}$.

Proof:

If $x=0$ or $y=0$ or $x+y=0$ then trivial.

$v_p(x+y) \geq \min\{v_p(x), v_p(y)\}$: Enough to show.

$x = a/b$, $y = c/d$. $\therefore x+y = \dfrac{ad+bc}{bd}$.

The highest power of $ad+bc$ is atleast $\min\{v_p(ad), v_p(bc)\}$.

$$
\begin{aligned}
v_p(x+y) &= v_p(ad+bc) - v_p(bd) \\
&= v_p(ad+bc) - v_p(b) - v_p(d) \\
&\geq \min\{v_p(ad), v_p(bc)\} - v_p(b) - v_p(d) \\
&= \min\{v_p(a)+v_p(d), v_p(b)+v_p(c)\} \\
&\qquad - v_p(b) - v_p(d)
\end{aligned}
$$

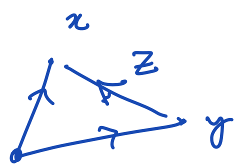
Say min is $v_p(ad)$, then $v_p(x+y) \geq v_p(a) + v_p(d) - v_p(b) - v_p(d)$

$= v_p(a/b) = v_p(x)$

Say min is $v_p(bc)$ then $v_p(x+y) \geq v_p(c/d) = v_p(y)$.

$$= \min\{v_p(x), v_p(y)\}$$

Let $a, b, c \in \mathbb{Q}$. Fix prime p.



$x = a - c$

$y = b - c$

$z = x - y \quad (= a - b)$

Say $|x|_p < |y|_p$

$|z|_p = |x-y|_p \leq \max\{|x|_p, |-y|_p\} = \max\{|x|_p, |y|_p\}.$

$= |y|_p$

$|y|_p = |x - z| \leq \max\{|x|_p, |z|_p\}$ $\quad$ (So $|z|_p > |x|_p$)

$X \leq |x|_p \qquad \leq |z|_p$

$\therefore |y|_p = |z|_p.$

We say $|\cdot|_p$ is a non-Archimedean norm.

& the corresponding metric is said to be an ultrametric.

non-Archimedean norm: A norm $\|\cdot\|$ is said to be non Archimedean if $\|x+y\| \leq \max\{\|x\|, \|y\|\}$.

METRIC SPACES

Let (X, d) be a metric space.

Proposition : (i) X, ϕ are open.

(ii) (Arbitrary) union of open sets is open.

(iii) Finite intersection of open sets is open.

Closed sets are just the complements (in X) of open sets.

Proposition : (i) X, ϕ are closed.

(ii) (Arbitrary) intersection of closed sets is closed.

(iii) Finite union of closed sets is closed.

Def : (i) Let $E \subseteq X$. $x \in X$ is said to be a limit point of E if

(either) • every open ball in X around x intersects E nontrivially

(or) • there is a seq in $E \setminus \{x\}$ converging to x.

(ii) Denote the set of lim pts of E by E'.

Define the closure of E as $\overline{E} := E \cup E'$.

(iii) A point $x \in E$ is said to be an interior pt of E if $\exists$ an open ball B s.t. $x \in B \subseteq E$.

(iv) The interior of E is $E^{\circ} = \text{int}(E) = \{x \in E : x \text{ is an int point of } E\}$.

(v) If $x \in E$ & $x \notin E'$ then x is an isolated pt of E.

Lemma: Let $E \subseteq X$.

(i) E closed $\iff$ $E' \subseteq E$

(ii) $\overline{E}$ closed.

(iii) $\overline{E}$ is the smallest closed set containing E.

Pf: (i) E closed $\Rightarrow$ E^c open.

$x \in E^c \Rightarrow \exists\, r > 0$ s.t. $B(x,r) \subseteq E^c \Rightarrow x \notin E' \Rightarrow x \in (E')^c$.
$\therefore E^c \subseteq (E')^c \Rightarrow E' \subseteq E$.

Say $E' \subseteq E$. Let $x \in E^c$. Then $x \notin E'$. $\therefore \exists$ an open ball B s.t. $x \in B \subseteq E^c$. $\therefore E^c$ open $\Rightarrow$ E closed.

(ii) $\overline{E} = E \cup E'$. Let $x \in (\overline{E})'$. Enough to show: $x \in \overline{E}$.

Suppose the contrary. $\therefore x \notin E$, $x \notin E'$.

$\exists\, r > 0$ s.t. $B(x,r) \subseteq E^c$.

But x is a lim pt of $\overline{E} \Rightarrow B(x,r) \cap \overline{E} \neq \phi$.

Let $y \in \overline{E} \cap B(x,r)$. Then y is a limit pt of E.

$\therefore \exists\, r' > 0$ $(r' < r)$, $a \in E$ s.t. $a \in B(y,r') \subseteq B(x,r)$.

But this contradicts $E \cap B(x,r) = \phi$.

Conclude by (i).

(iii) Let F be closed s.t. $E \subseteq F$. Then by (i),

$E' \subseteq F' \subseteq F$. $\therefore \overline{E} = E \cup E' \subseteq F$.

Lemma : Let $E \subseteq X$.

(i) E^0 is open.

(ii) $E = E^0 \Leftrightarrow E$ is open

(iii) E^0 is the largest open set contained in E.

Pf: (i) $x \in E^0 \Rightarrow \exists\, r>0$ s.t. $B(x,r) \subseteq E$

Let $y \in B(x,r) \Rightarrow \exists\, r'>0$ s.t. $B(y,r') \subseteq B(x,r) \subseteq E$.

Every pt of $B(x,r)$ is an int. pt. of E.

$x \in B(x,r) \subseteq E^0 \Rightarrow x \in (E^0)^0$.

$\Downarrow$

E^0 open.

(ii) E open $\Rightarrow$ $E \subseteq E^0$, $E^0 \subseteq E$ $\Rightarrow$ $E = E^0$

$E = E^0 \Rightarrow E$ open $\because E^0$ open.

(iii) Let $U (\subseteq E)$ be open. Let $x \in U = U^0$ then $\exists\, r>0$ s.t. $B(x,r) \subseteq U \subseteq E \Rightarrow x \in E^0$. $\therefore U \subseteq E^0$.

Definition (Convergence). $(x_n) \in X^{\mathbb{N}}$.

We say (x_n) converges if $\exists\, x \in X$ s.t. $\forall\, \varepsilon>0$ $\exists\, N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(x_n, x) < \varepsilon$.

$\Updownarrow$

$x_n \in B(x,\varepsilon)$.

In this case we say x is a limit of (x_n).

Lemma: (2) $(x_n) \in X^{\mathbb{N}}$. Let $x, y \in X^{\mathbb{N}}$ both be limits of (x_n). Then $x = y$.

(1) Let $x, y \in X$ $(x \neq y)$. $\exists\ r_1, r_2 > 0$ s.t.

(Hausdorff Property) $B(x, r_1) \cap B(y, r_2) = \phi$.

Pf: (1) $d = d(x, y) > 0$.

Take $r_1 = r_2 = d/10 = r$.

Let $z \in B(x, r) \cap B(y, r) \Rightarrow d(z, y) < d/10$

$$d(z, x) < d/10$$

$$\Rightarrow \quad d = d(x, y) \leq d(z, x) + d(z, y) < d/5$$