MIT 6.7230/18.456
Algebraic techniques and semidefinite programming
Homework assignment # 1

Date Given: February 9th, 2024
Date Due: February 16th, 1:00PM

P1. [***20 pts***] Classify the following statements as true of false. A proof or counterexample is required.

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping, and $K \subset \mathbb{R}^n$ a cone.

(a) If K is convex, then $\mathcal{A}(K)$ is convex.

(b) If K is solid, then $\mathcal{A}(K)$ is solid.

(c) If K is pointed, then $\mathcal{A}(K)$ is pointed.

(d) If K is closed, then $\mathcal{A}(K)$ is closed.

(Optional) Do the answers change if $\mathcal{A}$ is injective and/or surjective? How?

P2. [***20 pts***] Consider the set of $n \times n$ nonnegative matrices, with all rows and column sums equal to 1 (i.e., the *doubly stochastic* matrices).

(a) Write explicitly the equations and inequalities describing this set for $n = 2, 3, 4$.

(b) Compute (using `CDD`, `lrs`, `MPT`, or some other software) all the extreme points of these polytopes.

(c) How many extreme points did you find? What is the structure of the extreme points? Can you conjecture what happens for arbitrary values of n?

(d) Google "Birkhoff - Von Neumann theorem," and check your guess.

P3. [***20 pts***] Given a graph $G = (V, E)$, we would like to describe the convex hull of the incidence vectors of all the stable sets of G (i.e., the *stable set polytope*, usually denoted $\mathrm{STAB}(G)$). This is a 0-1 polytope in $\mathbb{R}^{|V|}$. It is easy to see that the following are valid inequalities for this polytope:

$$\begin{aligned} 0 \leq x_i \leq 1 \qquad & i \in V \\ x_i + x_j \leq 1 \qquad & (i,j) \in E. \end{aligned} \tag{1}$$

These are usually known as the *nonnegativity* and *edge* inequalities, respectively. The first set of constraints is trivially true, and the second one says that for every edge, there is at most one vertex in a stable set. The polytope defined by the inequalities (1) is called $\mathrm{FRAC}(G)$ (the *fractional stable set* polytope). The discussion above implies the containment $\mathrm{STAB}(G) \subseteq \mathrm{FRAC}(G)$.

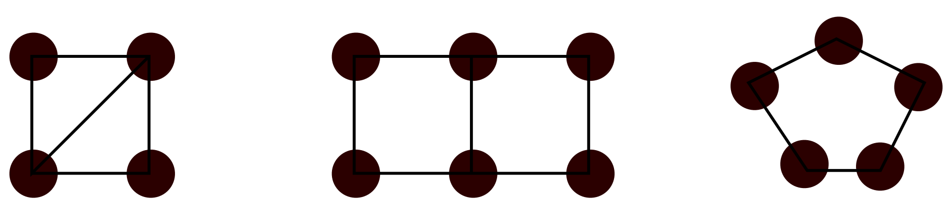

Figure 1: Three graphs

(a) Consider the graphs in Figure 1. Compute (either by hand, or using suitable software) all the extreme points of the corresponding polytopes FRAC(G). Do all the extreme points correspond to stable sets of G?

(b) (Optional) Conjecture (and prove?) conditions on the graph G under which STAB(G) = FRAC(G).

P4. [***20 pts***] Let K be a convex cone. An *extreme ray* of K is a subset $R = \{\alpha x \mid \alpha \geq 0,\, x \neq 0\}$, such that having $x = \lambda x_1 + (1-\lambda)x_2$, with $x_1, x_2 \in K$ and $\lambda \in (0,1)$ implies that x_1 and x_2 are both in R. (This is the natural analogue of extreme points, for the case of cones).

Find the extreme rays of:

(a) The nonnegative orthant $\mathbb{R}^n_+$.

(b) The Lorentz (or second order) cone.

(c) The PSD cone $\mathcal{S}^n_+$.

Provide a suitable justification (proof) of your answers.

P5. [***20pts***] Given a fixed vector $\omega \in \mathbb{R}^n$ with $\|\omega\| = 1$, consider the parametrized family of convex cones

$$K_a = \{x \in \mathbb{R}^n \,:\, a\|x\| \leq \omega^T x\},$$

where $0 < a < 1$. Here $\|\cdot\|$ is the standard Euclidean norm.

(a) Sketch the cone K_a in the cases where $n = 2$ and $n = 3$. Give a suitable geometric interpretation of the parameter a.

(b) Show that K_a is a proper cone.

(c) Show that the dual cone is given by $K_a^* = K_b$, where $b > 0$ is such that $a^2 + b^2 = 1$. Interpret this result geometrically.

(d) Give a description of K_a in terms of the standard Lorentz cone.

P6. [***15 pts***] Let K be a closed convex cone.

(a) Show that $K^{**} = K$.

(b) What can you say in the case when K is convex, but not closed?

Hint: you may want to apply the separating hyperplane theorem.

P7. [***15 pts***] Consider the polyhedral cone $K \subset \mathbb{R}^n$, where $K = \{x : a_i^T x \geq 0, \quad i = 1, \ldots, n\}$ and the a_i are linearly independent. Notice that there are exactly n inequalities defining the cone.

(a) Give an expression for the extreme rays of K and those of the dual cone K^*. (Hint: you may find it useful to consider the square matrix $A = [a_1, \ldots, a_n]$).

(b) As a simple application of these results, show the following statement in plane geometry, known as "Ravi substitution": There exists a triangle with sides (a, b, c) if and only if there exist nonnegative scalars x, y, z such that

$$a = x + y, \quad b = y + z, \quad c = x + z.$$

(c) Consider the cone K_{m} of "monotone nonnegative vectors", defined by

$$K_{\mathrm{m}} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

What are the extreme rays of K_{m}? What is the dual cone K_{m}^*?