Continuous functions

Let (X, d) & (Y, d') be metric spaces. A function $f: X \to Y$ is said to be continuous at $x \in X$ if

$\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$.

f is said to be cont (on X) if f is cont at all $x \in X$.

Example: $\to X = Y = \mathbb{R}$. $f(x) = x^2$.

$\to X = \mathbb{R}^2,\ Y = \mathbb{R}^{\geq 0}$. $f((a,b)) = \sqrt{a^2+b^2}$.

$\to (X, d)$ metric space, $Y = \mathbb{R}^{\geq 0}$. Fix $a \in X$.

Then $f = d(a, \cdot): X \to \mathbb{R}^{\geq 0}$ is cont.

$x \in X$ fixed.
$\varepsilon > 0$ given. Take $\delta = \varepsilon$.

$$d(y,x) < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$$

$$|f(y)-f(x)| = |d(y,a) - d(x,a)| < \varepsilon$$

$$\left.\begin{aligned} &d(x,y) + d(y,a) \geq d(x,a) \\ \Rightarrow\ &d(x,y) \geq d(x,a) - d(y,a) \\ &d(x,y) + d(x,a) \geq d(y,a) \\ \Rightarrow\ &d(x,y) \geq d(y,a) - d(x,a) \end{aligned}\right\} \Rightarrow d(x,y) \geq |d(y,a) - d(x,a)|$$

$$d(x,y) < \delta$$

$$\Rightarrow |f(x) - f(y)| = |d(y,a) - d(x,a)| \leq d(y,x) < \delta = \varepsilon.$$

$\therefore d(a,\cdot)$ is cont at x. But $x \in X$ arbitrarily chosen.
Hence $d(a,\cdot)$ is cont on X.
$\therefore d(a,\cdot)$ is cont on X $\forall\ a \in X$.

Lemma: Let (X, d), (Y, d') be metric spaces & $f: X \to Y$ a function. Fix $x \in X$.

f is cont at x iff for each $\underbrace{\text{open nbd}}_{\text{an open set containing } f(x)}$ V of $f(x)$ (in Y), $\exists$ an open nbd U of x (in X) s.t. $f(U) \subseteq V$.

Theorem: (X, d), (Y, d') metric spaces. $f: X \to Y$ is a function.

The following are equivalent.

(i) f is continuous.

(ii) $f^{-1}(V) \subseteq X$ is open $\forall$ open $V \subseteq Y$.

(iii) $f^{-1}(F) \subseteq X$ is closed $\forall$ closed $F \subseteq Y$.

Lemma: (X, d), (Y, d') metric spaces. $f: X \to Y$ function. $x \in X$.

The following are equivalent:

(i) f cont at x.

(ii) if $(x_n) \in X^{\mathbb{N}}$ is a seq s.t. $\lim x_n \overset{X}{=} x$ then

$$f(x_n) \xrightarrow[Y]{n \to \infty} f(x).$$

Pf: Let $X, Y, d, d', f, x \in X$ be as given.

(i) $\Rightarrow$ (ii): Say f cont at x. Let $(x_n) \in X^{\mathbb{N}}$ s.t. $\lim x_n = x$. Let V be any open nbd of $f(x)$.

$\therefore \exists$ an open nbd U of x s.t. $f(U) \subseteq V$.

$\therefore \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n \in U \Rightarrow f(x_n) \in f(U) \subseteq V$.

$\therefore \lim f(x_n) = f(x)$.

(ii) $\Rightarrow$ (i): Suppose for every seq, $(x_n) \in X^{\mathbb{N}}$ converging to x we have $f(x_n)$ converges to $f(x)$.

Let V be an open nbd of $f(x)$.

Suppose $\nexists$ any open U around x s.t. $f(U) \subseteq V$.

$\therefore$ Pick $x_n \in B_{1/n}(x)$, $n \geq 1$ s.t. $f(x_n) \notin V$.

However $x_n \longrightarrow x$ but $\exists\ \delta > 0$ s.t. $d'(f(x_n), f(x)) > \delta$

which means $f(x_n)$ does not converge to $f(x)$ (in Y).

Contradiction!

$\therefore \exists$ open U around x s.t. $f(U) \subseteq V$.

This precisely means that f is cont at x.

<u>Example</u> Look at $X = \mathbb{R}^n$. Consider the following 3 metrics:

$$d_2(\vec{x}, \vec{y}) = \sqrt{\sum |x_i - y_i|^2}$$

$$d_1(\vec{x}, \vec{y}) = \sum |x_i - y_i|$$

$$d_\infty(\vec{x}, \vec{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$$

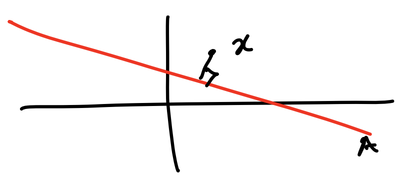
One can show that

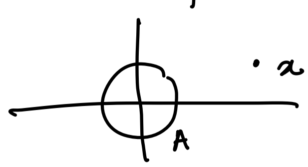
$$d_\infty(\vec{x}, \vec{y}) \leq d_2(\vec{x}, \vec{y}) \leq d_1(\vec{x}, \vec{y}) \leq n \cdot d_\infty(\vec{x}, \vec{y})$$

Let $\mathcal{T}_p$ be the collection of open sets determined by the metric d_p $(p \in \{1, 2, \infty\})$. Check that $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_\infty$.

<u>Example (Important)</u> (X, d) metric space. $\phi \neq A \subseteq X$

What is a way to define the dist of $x \in X$ from A?





We define (for $x \in X$)

$$d(x, A) = \inf_{a \in A} d(x,a)$$

Note that $d(\cdot,\cdot) \geq 0$ & $A \neq \phi \Rightarrow d(x,A) \in \mathbb{R}_{\geq 0}$.

Define $f(x) = f_A(x) = d(x, A)$ $\quad (f : X \to \mathbb{R})$

$$d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a)$$

Take inf over $a \in A$. Get : $d(x, A) \leq d(x,y) + d(y, A)$

$$\Rightarrow f(x) - f(y) \leq d(x,y)$$

In the same way : $f(y) - f(x) \leq d(x,y)$.

This tells $|f(x) - f(y)| \leq d(x,y)$.

Fix $x \in X$.

Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{2}$ s.t. $d(x, y) < \delta \Rightarrow |f(x) - f(y)| \leq \delta < \varepsilon$

$\therefore f$ cont at $x \in X$ (x was arbitrary)

$\Rightarrow f_A$ cont on X (for every $\phi \neq A \subseteq X$)

(Exercise) : $\phi \neq A \subseteq X$.

$$\overline{A} = \{x \in X : d(x, A) = 0\}$$

Theorem: ((X,d) metric space.) Let E, F be disjoint closed subsets of X. Then :

(i) $\exists$ a cont function $f : X \to [0,1]$ s.t. $E = f^{-1}(0)$ & $F = f^{-1}(1)$.

(ii) There are disjoint open sets U, V s.t. $E \subseteq U$, $F \subseteq V$.

(iii) Let $x \in X \setminus F$. $\exists$ disjoint open sets U, V s.t. $x \in U$, $F \subseteq V$.

Pf : (i) Define $\quad f(x) = \dfrac{d(x,E)}{d(x,E) + d(x,F)}$.

Denominator nonzero : $d(x,E) + d(x,F) = 0 \Leftrightarrow d(x,E) = d(x,F) = 0$

$$\Leftrightarrow x \in \overline{E} \cap \overline{F} = E \cap F = \phi.$$

Clearly $0 \le f(x) \le 1 \quad \forall x \in X$. f cont $\because$ num^r & deno^r cont.

$$f^{-1}(0) = \{x \in X : f(x) = 0\}$$
$$= \{x \in X : d(x, E) = 0\}$$
$$= \overline{E} = E$$

$$f^{-1}(1) = \{x \in X : f(x) = 1\}$$
$$= \{x \in X : d(x, E) = d(x, E) + d(x, F)\}$$
$$= \{x \in X : d(x, F) = 0\} = \overline{F} = F.$$

(ii) Take $U = f^{-1}([0, 1.1))$, $V = f^{-1}((0.9, 1])$.

(iii) Special case when $E = \{x\}$.