Eigen-Stuffs

Let's start with the basic definitions;

Definition :- Let $T : V \to V$ be a linear operator where $\dim V = n$. A nonzero $v \in V$ is defined to be an "eigenvector" of T with "corresponding eigenvalue $\lambda \in F$" iff

$$T(v) = \lambda v$$

Now we know that given a linear operator and given a basis, $\exists$ a matrix representating the operator. Also given a matrix and given a basis we can always find a linear map represented by same basis. In other words

$$\mathrm{End}(V) \cong M_n(F)$$

as linear space

So "Eigenvector / Eigenvalue of a linear map" and "Eigenvector / Eigenvalue of a matrix" can be invariably used.

In the class we have seen the following basic properties of Eigenvalues & Eigenvectors

Proposition 1 :-

Let $\lambda_1, \dots, \lambda_k \in F$ be distinct eigenvalues, and let $v_1, \dots, v_k \in V$ be eigenvectors corresponding to them. The

$$\{v_1, \dots, v_k\}$$

is linearly independent.

This proposition tells us that given a vector space of dimension n it can not have more than n distinct eigenvalues.

Def$\underline{\underline{n}}$:- $T: V \to V$ be linear map. An eigenbasis B of V is a basis of V consisted of eigenvectors of T

Proposition 2 :-

Let $T: V \to V$. For some basis B let $M_B^B(T) = A$. Then A is similar to a diagonal matrix iff V has an eigenbasis wrt. T

Here recall that two matrices A and B are similar iff $\exists$ invertible matrix P s.t

$$P^{-1} A P = B$$

Also $M_B^B(T)$ is the matrix of the linear map T wrt. the basis B.

Also recall that if B and C are two bases then

$$M_B^B(T) = M_B^C(id)^{-1} M_C^C(T) \; M_B^C(id)$$

Characteristic Polynomial:

Let $T: V \to V$ be a linear map.

Proposition: Let A be a matrix and B be a matrix similar to A. Then

$$\det(A-\lambda I) = \det(B-\lambda I)$$

for any $\lambda \in F$

Proof → Let $B = P^{-1}AP$

$$\det(B-\lambda I) = \det\left(P^{-1}AP - P^{-1}(\lambda I)P\right)$$

$$= \det\left(P^{-1}(A-\lambda I)P\right)$$

$$= \det(A-\lambda I)$$

■

This motivates us to define characteristic polynomial of T.

Defn: Let $M_{\varepsilon}^{\varepsilon}(T) = A$ for some basis ε of V.

Then characteristic polynomial of T is

$$\mathrm{char}_T(x) = \det(A - xI)$$

Note that due to the above proposition, this characteristic polynomial is well defined.

Now the giant comes!

Theorem :- λ is an eigenvalue of T iff $char_T(\lambda) = 0$

It is left to reader to prove

Since $F[x]$ is a ED we can write

$$char_T(x) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k} \cdot p(x) \cdots \quad (i)$$

where λ_i's are distinct eigenvalues of T.

However note that degree of char. poly is $n = \dim V$

Look at (i) once again. The natural number n_i is called as Algebraic Multiplicity of the eigenvalue λ_i.

On the other hand, for given an eigenvalue λ of T, define

$$E_\lambda := \{ v \in V : T(v) = \lambda v \}$$

and then note that E_λ is a subspace of V. Now we define

$$\dim E_\lambda$$

to be the Geometric Multiplicity of λ.

Now comes the theorem that we proved in the last class

Thm :- Let $T: V \to V$. λ be an eigenvalue. Then

Alg. Multiplicity of $\lambda \geq$ Geo. Multiplicity of λ

Proof → Let Geo. Multiplicity of λ be m.

Then let

$\mathcal{E}' = \{v_1, \dots, v_m\}$ be a basis for E_λ

Extend $\mathcal{E}'$ to a basis $\mathcal{E}$ of V.

Let $M_{\mathcal{E}}^{\mathcal{E}}(T) = B$

But how does $M_{\mathcal{E}}^{\mathcal{E}}(T)$ looks like?

Note that $T(v_i) = \lambda v_i$, $i \leq m$

So

$$[T(v_i)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \\ \end{array}$$

So the first m- coloumns of B will be like this

In block diagonal form B looks like

$$\begin{bmatrix} \lambda I_m & X \\ 0 & Y \end{bmatrix}$$

Now prove that characteristic polynomial of B (aka char poly. of T) is divisibly by $(x-\lambda)^m$

So $(x-\lambda)^m \mid \text{char}_T(x)$

$\Rightarrow$ Alg Mult. $\geq$ Geo. Multiplicity