Linear Algebra

Linear Independence, Spanning and Basis

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This note is a summary of what we did in the class for last few days. We start with the following definitions;

Let V be a F-Vector Space.

Definition 1: A subset $L \subset V$ is defined to be Linearly Independent if for any finite subset $\{v_1, ..., v_n\} \subset L$ we have the following property to hold

$$\alpha_1 v_1 + ... + \alpha_n v_n = 0 \implies \alpha_i = 0, \ \forall i \leq n$$

Note that the α_i's are elements of F.

Definition 2: A subset $S \subset V$ is defined to be a "Spanning Set for V" (or "generator of V") if any element of V can be written as F linear combination of finitely many elements of S, i.e. given $v \in V$, there exists $s_1, ...s_n \in S$ and $\alpha_1, ..., \alpha_n \in F$ such that

$$\sum_{i=1}^{n} \alpha_i s_i = v$$

Note that no L.I. set can contain $0 \in V$.
Then we immediately have the following corollary whose proof is left as exercise;

Corollary 1: Subset of a L.I. set is L.I. and superset of a spanning set is spanning.

Now it is time to define a basis;

Theorem 1: Let V be a F- Vector Space, then TFAE;

i. $B \subset V$ such that B is L.I. and Spanning.

ii. $B \subset V$ such that every element can be written uniquely as F-linear combination of elements of B.

iii. $B \subset V$ such that B is a Minimal Spanning Set.

iv. $B \subset V$ such that B is a Maximal L.I. Set.

I am attaching a handwritten page as the proof;

$\boxed{\text{i} \Rightarrow \text{ii}}$

On contrary let, $v \in V$ have two different representation i.e.

$$v = \alpha_1 b_1 + \cdots + \alpha_n b_n = \beta_1 c_1 + \cdots + \beta_m c_m$$

where $\{b_1, \ldots, b_n, c_1, \ldots, c_m\} \subseteq B$.

Note that such a representation always exist as B is spanning.

Now if $\{b_1, \ldots, b_n\} \cap \{c_1, \ldots, c_m\} = \{d_1, \ldots, d_k\}$

Then linear independence of $\{b_1, \ldots, b_n, c_1, \ldots, c_m\}$ implies that coefficients of $\{d_1, \ldots, d_k\}$ is same in both representation and the rest of the co-efficients are 0. So the representations are the same.

$\boxed{\text{ii} \Rightarrow \text{iii}}$

To prove

a) B is spanning

b) for any $C \subset B$, C is not spanning

Now a) is trivial as it is given to us every vector in V has a representation.

To prove b> assume on contrary that $\exists\ C \subset B$, (note the strict inclusion) is spanning. So $\exists\ x \in B$ s.t. $x \notin C$. But

$$x = \alpha_1 c_1 + \cdots + \alpha_n c_n \ ; \quad c_i \in C, \ \alpha_i \in F$$

as C is spanning. Then note that x has two different representation in elements of B $(\longrightarrow\longleftarrow)$

$\boxed{iii \Rightarrow iv}$

To prove a> B is L.I.

b> Any $B \subset C$ is not L.I.

First let's prove a>. Assume on contrary that B is not L.I. Then $\exists\ \{b_1, \ldots, b_n\} \subseteq B$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq F$ s.t

$$\alpha_1 b_1 + \cdots + \alpha_n b_n = 0$$

where not all α_i are 0. Then WLOG, let $\alpha_1 \neq 0$
Consider $B \setminus \{b_1\} = B_1$. We claim that B_1 is spanning
To see it consider any $v \in V$. As B is spanning
$\exists\ \{c_1, \ldots, c_m\} \subset B$ s.t.

$$v = \beta_1 c_1 + \cdots + \beta_m c_m$$

If none of $c_i = b_1$, we are done! WLOG be $c_m = b_1$
Then

$$\begin{aligned} v &= \beta_1 c_1 + \cdots + \beta_m b_1 \\ &= \beta_1 c_1 + \cdots + \beta_{m-1} b_{m-1} + \beta_m \left(-\frac{\alpha_2}{\alpha_1} b_2 - \cdots - \frac{\alpha_n}{\alpha_1} b_n \right) \end{aligned}$$

So, $v \in \langle B_1 \rangle$. Done!

Proof for b> is trivial

$\boxed{iv \Rightarrow i}$

B is by default L.I., just to prove spanning.
On contrary let $\langle B\rangle \subset V$. Then $\exists\, v \in V$ s.t $v \notin \langle B\rangle$.
Then consider $B \cup \{v\} = B_0$
Since $v \notin \langle B\rangle$, B_0 is still L.I. but $B \subset B_0$
which contradicts the maximality of B.

Then we have the definition of a basis;

Definition 3: Given a vector space V, any subset satisfying any of the above equivalent condition, is defined to be a *Basis*.

Now the question is given a Vector Space, whether there exists a basis or not? The answer is YES. Regarding that we have the following theorem.

Theorem 2.1: Given a F-Vector Space V and a Linearly Independent subset L of V, there exists a Basis B containing L.

Theorem 2.2: Given a F-Vector Space V and a Spanning subset S of V, there exists a Basis B contained in S.

Theorem 2.3: Given a F-Vector Space V and a Linearly Independent subset L and a Spanning Subset S such that $B \subset L$ of V, there exists a Basis B containing $L \subset B \subset S$.

Note that ϕ is a L.I. set and V is a spanning subset of V itself. So given any Vector space a L.I. set and a Spanning set always exists. Thus given a Vector Space a basis always exists.
The proofs of these theorems involves Zorn's Lemma and are not required. Anyway interested people can look at https://www.cmi.ac.in/~nilavam/RSM/ for the proofs. The next two theorems give us a clear view of how a Basis can be created.

V is a F- Vector Space;

Theorem 3.1: Given a L.I. subset L of V, and an element $v \notin \langle L\rangle$ the set $L \cup \{v\}$ is also L.I.

Theorem 3.2: Given a Spanning subset S of V and a finite subset $\{s_1, ..., s_n\} \subset S$ with $\alpha_1 s_1 + ... + \alpha_n s_n = 0$ where $\alpha_1 \neq 0$, the set $S \setminus \{\alpha_1\}$ is also Spanning.

The proofs of these theorem are very easy and left as exercise. So start with a L.I. set and keep adding elements outside of its span. Thus the L.I. set gradually acquires the Spanning property and become a basis. Similarly start with a Spanning set and keep deleting elements from it according the above rule, and thus gradually it becomes a L.I. set and hence a Basis.
Next we are going define Finite dimensional Vector space.

Theorem 4: Let V be a F- Vector Space. Let V have a finite basis B with $B = \{b_1, b_2, ..., b_n\}$. Then any basis of B has cardinality less than or equal to n.

Proof → Let's assume that there is a basis C such that $| C |$ is more than n. We will prove that the elements of B can be one by one *replaced* by elements of C so that at each step of the replacement, the modified B continues to remain a basis. Then gradually all the elements of B will be replaced by elements of C and we will get a subset of C which is a basis, that contradicts the fact that C is minimal spanning set. This is our strategy.

Pick up any element of C namely c_1. Then

$$c_1 = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n$$

for some $\alpha_i \in F$. Since c_1 is not zero as C is L.I. and no L.I. set can contain 0. Then at least one of α_i must be *non-zero*. WLOG let $\alpha_1 \neq 0$. Then Consider

$$B_1 = \{c_1, b_2, b_3, ..., b_n\}$$

Note that using **Theorem 3.2** we can see that B_1 is still spanning. Also B_1 is L.I. as note that $c_1 \notin \langle\{b_2, b_3, ..., b_n\}\rangle$ as if it were then let

$$\alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n = c_1 = \beta_2 b_2 + ... + \beta_n b_n \implies \alpha_1 b_1 + (\alpha_2 - \beta_2) b_2 + ... + (\alpha_n - \beta_n) b_n = 0 \implies \alpha_1 = 0$$

which is contradicting our assumption that $\alpha_1 \neq 0$. So we just proved that $c_1 \notin \langle\{b_2, b_3, ..., b_n\}\rangle$ and hence by **Theorem 3.1** we have that B_1 is L.I. Thus B_1 is a basis.

Now Pick up another $c_2 \in C$ and let

$$c_1 = \gamma_1 c_1 + \gamma_2 b_2 + ... + \gamma_n b_n$$

Note that at least one of γ_i is non-zero as again C is L.I. But note that it is not possible that $\gamma_i = 0$ for all $i \geq 2$ and $\gamma_1 = 0$ as then we would have $c_2 = \gamma_1 c_1$ which is impossible as again C is L.I. Thus some $\gamma_i \neq 0$ with $i \geq 2$. WLOG let $\gamma_2 \neq 0$. Then consider

$$B_2 = \{c_1, c_2, b_3, ..., b_n\}$$

Using **Theorem 3.2** B_2 is again spanning and Using **Theorem 3.1** as before B_2 is also L.I. Thus B_2 is still a basis. Continue this process inductively and after n steps we will get $B_n = \{c_1, ...c_n\}$ which will be still a basis but $B_n \subset C$; a desired contradiction achieved.

Q.E.D

The idea of replacement in the above proof is particularly important. Note that in the above Theorem we did use nothing but the L.I. independence of C. So the above theorem can be made even stronger. In that strong version we can say that <u>if a Vector Space admits a finite basis of cardinality n *then any* **L.I.** subset of V has cardinality $\leq n$.</u> The idea of the proof and the stronger version is written more elaborately below;

Let V be a F-vector space.

***Replacement* Theorem**: Let $B = \{b_1, ...b_n\}$ be a basis for V and C be a L.I. subset if V. Then there exists $\{c_1, ..., c_n\} \subset C$ such that one can replace elements of B after some ordering the elements of B, so that $B_k = \{c_1, ..., c_k, b_{k+1}, ..., b_n\}$ is a basis or V for every $k \leq n$.

Corollary 2: If a Vector Space admits a finite basis then any L.I. subset of V has cardinality lesser than or equal to that of B.

Now let's come to another important corollary of Replacement Theorem;

Let V be a F-vector space which admits a finite basis $B = \{b_1, ..., b_n\}$.

Corollary 3: Any basis $C = \{c_1, ..., c_k\}$ of V has Cardinality n.

Corollary 4: Any spanning subset S of V has cardinality $\geq n$.

Proof → We just proved that $| C | \leq n$ using Replacement Theorem. Now if $| C | \leq n$, then using Replacement Theorem we can replace elements of B by $C = \{c_1, ..., c_k\}$ and yet it will be a basis. So let that replaced version be $B' = \{c_1, ..., c_k, b_{k+1}, ..., b_n\}$. But as C spans V clearly B' is not L.I. that contradicts the fact that B' is a basis. So we must have $k = n$.

Replacement Theorem and this **Corollary 3** can be used to prove **Corollary 4**. Consider a Spanning set S, by **Theorem 2.2** it contains a basis B. But $\mid B \mid = n$ as per **Corollary 3**. So trivially $\mid S \mid \geq n$.

Q.E.D

We end this note by the definition of a Finite Dimensional Vector Space.

Definition: If a F-Vector Space V admits a finite basis of cardinality n, we define V to be a finite dimensional vector space and also define its dimension to be

$$\dim V = n$$

Note that since any basis has cardinality n, this definition of "dimension" is well defined.