$$|B| = n, \quad |C| = m$$

$$T : \underset{B}{V} \longrightarrow \underset{C}{W}$$

$$M^C_B(T) = A$$

$$[v]_B = x, \qquad Ax = [T(v)]_C$$

$$M^B_{B'} = M^B_{B'}(id)$$

$$B' \qquad C' \qquad\qquad M^{C'}_{B'}(T) = A_0$$

$$\begin{tikzcd}
{[v]_{B'}} \arrow[r] & M^B_{B'} \cdot [v]_{B'} \arrow[r] & M^C_B(T)\left(M^B_{B'} \cdot [v]_{B'}\right) \\
v \arrow[r, "id"'] & v \arrow[r, "T"'] & T(v) \arrow[u]
\end{tikzcd}$$

$$(T \circ id)$$

$$M^{C'}_C(id) = M^{C'}_C$$

$$M^{C'}_{B'}(T) := \boxed{M^{C'}_C \cdot M^C_B(T) \cdot M^B_{B'}}$$

$T: V \longrightarrow V$, $\dim(V) = n$

B and C be basis for V

$$M_B^B(T) = M_c^B \cdot M_c^c(T) \cdot M_B^c$$

$$= (M_B^c)^{-1} M_c^c(T) \cdot M_B^c$$

$$= P^{-1} M_c^c(T) P$$

M and N are equivalent iff $M = PNP^{-1}$ for $P \in GL_n(F)$

Problem :- V be n-dim vector space, $T: V \to V$

a) Suppose that V has a basis consisted of the eigenvectors of T, call it B. Find $\{v_1, \dots, v_n\}$

$T(v_i) = \lambda_i v_i$

$M_B^B(T)$

b) If A is a $n \times n$ matrix representing T wrt. a given basis of V, prove that A is equivalent to a diagonal matrix iff V has an eigenbasis.

a) $B = \{v_1, \dots, v_n\}$

$\{T(v_1), T(v_2), \dots, T(v_n)\}$

$T(v_1) = \lambda_1 v_1$

$T(v_2) = \lambda_2 v_2$

$$\begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & & & \lambda_n \end{bmatrix} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

b) Let $A = M_B^B(T)$ and $P^{-1}AP = D$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Consider a basis C for which $M_C^B = P$

Then $D = M_C^C(T)$, Let $C = \{v_1, \dots, v_n\}$

$B := \{u_1, \dots, u_n\}$

$a_{11}u_1 + a_{21}u_2 + \dots + a_{n1}u_n$

Then $[T(v_1)]_C = M_C^C(T) \cdot [v_1]_C$

$$= \lambda_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 \cdot [v_1]_C$$

$\Rightarrow T(v_1) = \lambda_1 v_1 \quad \dots$

$$D = M_B^C \, M_B^B(T) \, M_C^B$$

$$= M_C^C(T)$$

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$T(v_1) = \lambda_1 v_1$$

$$T(v_i) = \lambda_i v_i$$

Conversely let V admit an eigenbasis $C = \{v_1, \dots, v_n\}$

$$M_C^C(T) = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

$M_B^B(T)$ equivalent to $M_C^C(T)$ for any basis B

$A \cong$