Recall that

TFAE :- V be F-vector space.

a) $B \subset V$ s.t $\langle B \rangle = V$ and B is L.I

b) Every element of V can be uniquely written as Linear combination of elements of B

c) B is minimal spanning

d) B is maximal L.I.

$$B \subset C \quad \text{s.t } C \text{ is L.I.}$$

$$x \in C \setminus B \qquad x = \alpha_1 b_1 + \cdots + \alpha_n b_n \qquad b_i \in B$$

$$\{x, b_1, \ldots, b_n\} \subset C \Rightarrow \text{L.I.}$$

$$\Rightarrow x - \alpha_1 b_1 - \cdots - \alpha_n b_n = 0$$

$$\Rightarrow 1 = 0$$

$$\{b_1, b_2, \ldots, b_n\} \subset B \quad \text{s.t} \quad \alpha_1 b_1 + \cdots + \alpha_n b_n = 0 \quad \alpha_i \in F$$

$$B \setminus \{b_1\} \qquad \text{WLOG } \alpha_1 \neq 0$$

$$b_1 = \frac{-\alpha_2}{\alpha_1} b_2 - \frac{\alpha_3}{\alpha_1} b_3 - \cdots - \frac{\alpha_n}{\alpha_1} b_n$$

$$v \in V \qquad v = \beta_1 b_1 + \cdots + \beta_n b_n$$

Any $B \subset V$ satisfying any of the above equivalent of the statements is defined to be a **Basis** for V.

How do we create a basis, when a F-Vector space is given?

a) Does a basis always exist?

b) If so, then how do we create one such?

Theorem :- Given a F-vector space V, and a L.I. subset L of V, $\exists$ a basis B containing L.

Theorem (2) :- Given a F-vector space V, and a spanning set S of V, $\exists$ a basis B **contained in** S

Theorem (3) :- Given a F-vector space V, and a L.I. subset L of V, and a spanning subset S of V s.t $L \subset S$, $\exists$ a basis B s.t $L \subset B \subset S$

Corollary :- Every vector space has a basis

$\langle L \rangle = V$ $\quad x \notin \langle L \rangle$ $\quad \langle L \cup \{x\} \rangle$

$\{a_1, \ldots, a_n\}$ is L.D
a_1 has nonzero coeff..

$$L \xrightarrow{\text{add } x \notin \langle L \rangle} \longrightarrow B \longleftarrow \underset{S \setminus \{a_1\}}{\longleftarrow} S$$

Theorem :- Given a vector space V, if V admits a finite basis then every basis of V is finite.

Proof → On contrary let V admit a infinite basis C and let B be a finite basis for V. $B = \{b_1, \dots, b_n\}$

$c_1 \in C$ $\qquad c_1 = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$, $\alpha_i \in F$

$c_1 \in C$ and no L.I. set contains 0.

$$\boxed{\alpha_1 \neq 0}$$

$$b_1 = \frac{1}{\alpha_1} c_1 - \frac{\alpha_2}{\alpha_1} b_2 - \dots - \frac{\alpha_n}{\alpha_1} b_n \qquad b_1 \quad \{c_1, b_2, \dots, b_n\}$$

$$B_1 = \{c_1, b_2, b_3, \dots, b_n\}$$

$\langle B_1 \rangle = V$ $\qquad$ also prove that B_1 is L.I.

$v \in V$ $\quad$ to show that $v \in \langle B_1 \rangle$

$\langle B \rangle = V \Rightarrow v = \beta_1 b_1 + \dots + \beta_n b_n$

$$= \beta_1 \left(\frac{1}{\alpha_1} c_1 - \frac{\alpha_2}{\alpha_1} b_2 - \dots - \frac{\alpha_n}{\alpha_1} b_n \right) + \beta_2 b_2 + \dots + \beta_n b_n$$

$$\beta_1 c_1 + \dots + \beta_n b_n = 0$$

$$\beta_1(\alpha_1 b_1 + \dots + \alpha_n b_n) + \beta_2 b_2 + \dots + \beta_n b_n = 0$$

$$\Rightarrow \alpha_1 \beta_1 b_1 + (\alpha_2 \beta_1 + \beta_2) b_2 + \dots + (\alpha_n \beta_1 + \beta_n) b_n = 0$$

$$\Rightarrow \alpha_1 \beta_1 = 0 \Rightarrow \alpha_1 = 0 \text{ or } \boxed{\beta_1 = 0}$$

C L.I. is L.I.

$$\beta_2 b_2 + \dots + \beta_n b_n = 0$$

$$\beta_2 = \beta_3 = \dots = \beta_n = 0$$

$\Rightarrow B_1$ is L.I.

$B = \{b_1, \dots, b_n\}$ $\qquad c_1 \in C$

$\downarrow$

$B_1 = \{c_1, b_2, b_3, \dots, b_n\}$ still a basis $\qquad c_2 = \alpha_1 c_1$

$\downarrow$

$c_2 \in C \qquad c_2 = \alpha_1 c_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$

$\alpha_2 \dots \alpha_n$

$B_2 = \{c_1, c_2, b_3, \dots, b_n\}$ still a basis

$\downarrow$

$\vdots$

still basis

$B_n = \{c_1, \dots, c_n\} \subset C$

$|C| = \infty$

$|C| > |B|$

$\leq |B|$