Theorem :-

Given a set S,

an equivalence relation $\sim$ $\Longleftrightarrow$ a partition of S

$$\sim_a := \{ b \in S : a \sim b \}$$

$$\sim_a \cap \sim_b = \begin{cases} \phi & \text{iff } a \nsim b \\ \sim_a = \sim_b & \text{iff } a \sim b \end{cases}$$

$$S/\sim := \{\sim_a\}_{a \in S}$$

Ex :-

i) $f : S \to T$. Define a relation on S as follows

$$a \in S,\ b \in S$$

$$a \sim b \quad \text{iff} \quad f(a) = f(b)$$

$$\sim_a := \{ b \in S \text{ s.t. } f(b) = f(a) \} = f^{-1}(f(a))$$

$$S/\sim := \{\sim_a\}_{a \in S} \quad \text{partition of } S$$

$$S/\sim \ \underset{\text{bijective}}{\cong} \ f(S)$$

Define

$$\phi : S/\sim \longrightarrow f(S)$$
$$\sim_a \longmapsto f(a)$$

ϕ is bijection

$$\pi : S \longrightarrow S/\sim$$
$$a \longrightarrow \sim_a$$

```latex
\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\pi"'] & f(S) \\
S/\sim \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}
```

$S \qquad f : S \to T$

$$\phi \circ \pi = f$$

ii) $\mathbb{Z} \setminus \{0\} = \mathbb{Z}^*$

$(\mathbb{Z} \times \mathbb{Z}^*)$ $\quad \sim$

$$(a,b) \sim (c,d) \text{ iff } ad = bc$$

· $\sim$ is an equivalence relation

Given an (a,b), find $\sim_{(a,b)}$

$$\sim_{(a,b)} := \{(na, nb) : n \in \mathbb{Z}^*\}$$

$$\mathbb{Z} \times \mathbb{Z}^* / \sim \; = \{\sim_{(a,b)}\}_{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*}$$

$\sim_{(a,b)} = \frac{a}{b}$

$\frac{2}{4}$

$\frac{1}{2} \in \mathbb{Q}$

$\frac{2}{4} \notin \mathbb{Q}$

$\frac{1}{2} = \frac{2}{4}$

$$\sim_{(a,b)} \cdot \sim_{(c,d)} = \sim_{(ac,bd)}$$

$$\sim_{(a,b)} = \sim_{(a',b')}$$

$$\sim_{(c,d)} = \sim_{(c',d')}$$

Example 3 :-

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & f(S) \\ S/\sim \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

Homomorphism

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & f(G) \\ G/\sim \arrow[ur, "\phi"'] & \end{tikzcd}$$

ϕ has to be group homomorphism

$\phi \circ \pi = f$

i) $G/\sim$ has to be a group!

$a, b \in G$

$$\sim_a \cdot \sim_b = \sim_{ab}$$

$\sim_a = \sim_c$

$\sim_b = \sim_d$

$\boxed{\sim_{ab} = \sim_{cd}}$?

$\sim_a = \sim_c \Rightarrow f(a) = f(c)$

$\sim_b = \sim_d \Rightarrow f(b) = f(d)$

$\Big\} f(ab) = f(cd)$

$x \in \sim_{ab} \Leftrightarrow f(x) = f(ab) = f(cd)$

$\Leftrightarrow x \sim cd \Leftrightarrow x \in \sim_{cd}$

identity $\sim_1$

$\sim_a$ inverse $\sim_{a^{-1}}$

$G/_{\sim}$ is a group

Is π a homomorphism

$\pi(a) \cdot \pi(b) = \sim_a \cdot \sim_b = \sim_{ab} = \pi(ab)$

Is ϕ a homomorphism ?

$\phi(\sim_a) \cdot \phi(\sim_b) = f(a) \cdot f(b) = f(ab) = \phi(\sim_{ab})$

Theorem :- Let S be a set/group. Let f be a function / g-homomorphism on S, then $\exists$ a bijection/isomorphism $\phi : S/_{\sim} \longrightarrow f(S)$, in other words, the following diagram commutes

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\pi"'] & & f(S) \\ & S/_{\sim} \arrow[ur, "\sim"', "\phi"] & \end{tikzcd}$$

$\phi \circ \pi = f$