METRIC SPACE

Def: Let X be a non-empty set. A function $d: X \times X \to \mathbb{R}^{\geq 0}$ is called a <u>metric</u> (or distance function) if

(i) $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x) \quad \forall\, x,y \in \mathbb{R}$

(iii) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\, x,y,z \in X$.

We say (X,d) is a metric space.

(For short-hand, we often write: X is a metric; whenever the metric is understood from the context).

Example

(1) $X = \mathbb{R}^n$. (a) $d_2((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sqrt{\Sigma (x_i - y_i)^2}$

(b) $d_\infty(\vec{x},\vec{y}) = \max\limits_{1 \leq i \leq n} |x_i - y_i|$

(c) $d_1(\vec{x},\vec{y}) = \sum\limits_{i=1}^{n} |x_i - y_i|$

(d) $d_p(\vec{x},\vec{y}) = \left[\sum\limits_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$

n fixed. $a_1,\dots,a_n \geq 0$ fixed. $\qquad \lim\limits_{n\to\infty} (5^n + 7^n)^{1/n} = 7$

$\alpha_p = \left[\Sigma\, a_i^p\right]^{1/p}$

$= \left(\max\limits_{1\leq i \leq n} a_i\right)\left(\dots \right)^{1/p}$

$\lim\limits_{p\to\infty} \alpha_p = (\max a_i)\underbrace{(\quad)}_{\text{cgs to } 1} = \max a_i$

(2) Let S be any set. Define

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \quad . \quad \boxed{\text{Discrete metric}}$$

Triangle ineq: $\quad d(x,y) \qquad d(x,z) + d(y,z)$

$\begin{cases} x = y : & \text{LHS} = 0, \text{ trivially true.} \\ x \neq y : & \text{LHS} = 1. \text{ Either } z \neq x, \text{ or } z \neq y \text{ so RHS} \geq 1. \end{cases}$

Topology (Fix a metric space X, d).

Open balls : $B_r(a) = \{ x \in X : d(x,a) < r \}$

Open set : (1) S is an open set if S can be written as a union of open balls

Equivalent definitions

(2) S is an open set if $\forall x \in S \ \exists r > 0$ s.t. $B_r(x) \subseteq S$.

Example: (1) $B_r(a)$ is an open set $\forall a \in S, r > 0$. ((S,d) is a metric space)

Pf : Take $\mathcal{U} = \{ B_r(a) \}$. Then $B_r(a) = \bigcup_{V \in \mathcal{U}} V$.

(i) union of sets inside $\mathcal{U}$ is $B_r(a)$.
(ii) $\mathcal{U}$ contains only open ball(s).

(2) S any non empty set, d is the discrete metric on S.
(Want to show that T is open for any $T \subseteq S$).
Since union of open sets is open, it is enough to show that $\{x\}$ is open $\forall x \in S$.

Let $x \in S$. Then $\{x\} = B_1(x)$ is an open set in S

Examples of metric spaces:

(1) Let $X = C[0,1] = \{ f : [0,1] \to \mathbb{R} : f \text{ continuous} \}$.

Define $d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$ given by

$d(f,g) := \sup\{ |f(x) - g(x)| : x \in [0,1] \}$.

Exercise: Show that for any $f \in X$, $f([0,1]) = \{ f(x) : x \in [0,1] \}$ is bounded.

$\therefore$ The RHS of the definition is a real number.

Show that d is a metric on X.

(i) $d(f,g) = 0 \iff \sup\{ |f(x) - g(x)| : x \in [0,1] \} = 0$

$\iff |f(x) - g(x)| = 0 \ \forall x \in [0,1] \iff f(x) = g(x) \ \forall x$

$\iff f = g$.

(ii) $d(f,g) = d(g,f) \quad \forall f,g \in X.$

(iii) $f, g, h \in X.$

$$|f(x) - g(x)| + |g(x) - h(x)| \geq |f(x) - h(x)|$$

$\Rightarrow d(f,g) + d(g,h)$

$= \sup\{|f(x)-g(x)| : x \in [0,1]\} + \sup\{|g(x)-h(x)| : x \in [0,1]\}$

$\geq \sup\{|f(x)-g(x)| + |g(x)-h(x)| : x \in [0,1]\}$

$\geq \sup\{|f(x)-h(x)| : x \in [0,1]\}$

$= d(f,h)$

What are the open balls?

(2) let $X = [0,1]^{\mathbb{N}} = \{f : \mathbb{N} \to [0,1]\}$

$= \{\text{sequences on } [0,1]\}$

let $(x_n), (y_n) \in X$, define

$d((x_n),(y_n)) := \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i - y_i|$ is a metric on X.

Question: let $P : \mathbb{N} \to [0,1]$ be a function s.t.

$$\sum_{i=1}^{\infty} P(i) = 1$$

Define $d_p((x_n),(y_n)) := \sum_{i=1}^{\infty} P(i) \cdot |x_i - y_i|$

When d_p a metric on X.

$\left\{ \sum_{i=1}^{n} \frac{1}{2^i} |x_i - y_i| \right\}_{n \in \mathbb{N}}$

(a_n)

$\sum a_n \in \mathbb{R} : \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i \right) \in \mathbb{R}$

$\sum a_n = \sum_{n=1}^{\infty} a_n$

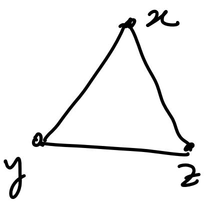
→ Monotone clear

→ $|x_i - y_i| \leq 1 \quad \forall i$

$\Rightarrow 0 \leq \sum_{i=1}^{n} \frac{1}{2^i} |x_i - y_i| \leq \sum_{i=1}^{n} \frac{1}{2^i} < 1$

"Weird" metric Spaces : There are certain metric spaces (X, d) where for any 3 distinct pts $x, y, z \in X$ we have

$$\left.\begin{array}{l} d(x,y) \\ d(y,z) \\ d(z,x) \end{array}\right\} \text{ 2 of these are same}$$



Example of an ultrametric.

$X = \mathbb{Q}$. Fix a prime $p \in \mathbb{N}$.

Fix $a/b \in \mathbb{Q}$ s.t. $\gcd(a,b) = 1$.

For any $n \in \mathbb{Z}$ define $v_p(n)$ to be the highest power of prime that divides n.

$v_2(2) = 1$, $v_2(3) = 0$, $v_3(27) = 3$,

$v_2(32) = 5$, $v_2(12) = 2$, $v_3(24) = 1$, $v_2(24) = 3$.

For $m, n \in \mathbb{Z}$, $v_p(m \cdot n) = v_p(m) + v_p(n)$.

$$\left.\begin{array}{l} m = p^{\square} \cdots \\ n = p^{\boxtimes} \cdots \end{array}\right\} m.n = p^{\square + \boxtimes} \cdots$$

Define $v_p(a/b)$ to be $v_p(a) - v_p(b)$

Using this extended definition of v_p (on $\mathbb{Q}$) one can check that $v_p(rs) = v_p(r) + v_p(s)$ $\forall$ $r, s \in \mathbb{Q}$.

Now define $(x \in \mathbb{Q})$

$$|x|_p = \begin{cases} \dfrac{1}{p^{v_p(x)}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Define $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}^{\geq 0}$ by

$$d(x,y) = |x-y|_p .$$

Verify :

(a) $d(x,y) = 0 \iff x = y$

(b) $d(x,y) = d(y,x) \quad \forall\, x,y \in \mathbb{Q}.$