We say $\sum x_n$ is absolutely convergent if $\sum |x_n| \in \mathbb{R}$

Lemma: Let $\sum x_n$ be absolutely convergent. Then $\sum x_n \in \mathbb{R}$.

Proof: $S_n = \sum_{i=1}^{n} x_i$.

$\sum x_n \in \mathbb{R} \iff S_n$ converges in $\mathbb{R} \iff S_n$ is Cauchy.

$\therefore$ Enough to show: S_n is a Cauchy seq

Let $\varepsilon > 0$ given. We know $\sum |x_n| \in \mathbb{R}$. So $\exists\ N \in \mathbb{N}$ s.t.

$$\left| \sum_{i=1}^{q} |x_i| - \sum_{i=1}^{p} |x_i| \right| < \varepsilon \quad \forall\ q > p \geq N.$$

Then $q > p \geq N \Rightarrow$
$$\begin{aligned} |S_q - S_p| &= \left| \sum_{i=p+1}^{q} x_i \right| \\ &\leq \sum_{i=p+1}^{q} |x_i| \\ &= \sum_{i=1}^{q} |x_i| - \sum_{i=1}^{p} |x_i| \\ &= \left| \sum_{i=1}^{q} |x_i| - \sum_{i=1}^{p} |x_i| \right| < \varepsilon \end{aligned}$$

$\because q > p \geq N$.

$\therefore S_n$ converges. ∎

Condensation test

Consider a sequence $X = (x_n)$ of reals s.t. $x_n \geq 0 \ \forall n$.

Also assume $x_n \downarrow$. Let $y_n := 2^n \cdot x_{2^n}$

Take $N \geq 1$. Then for a unique k we have $2^{k-1} \leq N < 2^k$.

$$\sum_{n=2^{k-1}}^{2^k-1} x_n$$

$$\geq \sum_{n=2^{k-1}}^{2^k-1} x_{2^k} = (2^k - 2^{k-1}) x_{2^k} = 2^{k-1} x_{2^k} = \tfrac{1}{2} y_k$$

$$\leq \sum_{n=2^{k-1}}^{2^k-1} x_{2^{k-1}} = 2^{k-1} \cdot x_{2^{k-1}} = y_{k-1}$$

$$x_1 + \cdots + x_n = x_1 + (x_2+x_3) + (x_4+x_5+x_6+x_7) + \cdots (+x_n)$$

$$\leq x_1 + 2x_2 + 4x_4 + 8x_8 + \cdots$$

$$= y_0 + y_1 + y_2 + \cdots$$

$$\geq x_2 + 2x_4 + 4x_8$$

$$= \tfrac{1}{2}(y_1 + y_2 + y_3 + \cdots)$$

⊙ Suppose $\Sigma x_n \in \mathbb{R}$.

$$\tfrac{1}{2}(y_1 + \cdots + y_{k-1}) \leq x_1 + \cdots + x_N \Rightarrow \Sigma y_n \text{ bounded}$$

$$\left(x_1 + \cdots + x_N \geq x_1 + \cdots + x_{2^{k-1}-1} \geq \tfrac{1}{2}(y_1 + \cdots + y_{k-1})\right)$$

Σy_n inc & bdd $\Rightarrow \Sigma y_n \in \mathbb{R}$.

⊙ Suppose $\Sigma x_n = \infty$. $x_1 + \cdots + x_N \leq x_1 + \cdots + x_{2^k-1} \leq y_0 + \cdots + y_{k-1}$

Then $\Sigma y_n = \infty$.

What we have established so far:

$$\sum x_n \in \mathbb{R} \iff \sum y_n \in \mathbb{R}.$$

My test is: Let (x_n) be a <u>decreasing</u> seq of <u>non-neg</u> reals

Define $y_n := 2^n x_{2^n}$. Then $\sum x_n \in \mathbb{R} \iff \sum y_n \in \mathbb{R}$.

Example:

(1) $x_n = \frac{1}{n}$. $x_n \downarrow$ & $x_n \geq 0$.

$\therefore y_n = 2^n \cdot x_{2^n} = 1$

$\sum y_n = \infty$. So $\sum x_n = \infty$.

(2) $x_n = \frac{1}{n}(\log n)^{-b}$, $b \geq 0$

$$= \frac{1}{n \; (\log n)^b}$$

(Just look at the tail)

$x_n \downarrow$. $x_n \geq 0$ $\forall n \geq N$ for some N.

$$y_n = 2^n \cdot x_{2^n} = \frac{2^n}{2^n \; (n \log 2)^b} = \frac{1}{n^b (\log 2)^b}$$

$\sum x_n \in \mathbb{R} \iff \sum y_n \in \mathbb{R} \iff b > 1$.

why?

(3) $x_n = \frac{1}{n^s}$. $s \geq 0$.

$x_n \downarrow$, $x_n \geq 0$. Then $y_n = 2^n \cdot x_{2^n} = \frac{2^n}{2^{ns}} = \frac{1}{2^{n(s-1)}}$

$\sum x_n \in \mathbb{R} \iff \sum y_n \in \mathbb{R} \iff n(s-1) > 0 \; \forall n \iff s > 1$.

ROOT TEST

Let (x_n) be a seq of reals. Let $\varepsilon > 0$

$\theta := \limsup |x_n|^{1/n}$.

Say $\theta < 1$. (This means $n \geq N \Rightarrow |x_n|^{1/n} < 1 - \varepsilon$ for some N)

$\Rightarrow |x_n|^{1/n} \leq \theta \quad \forall\ n \geq M$ (some M).

$\Rightarrow |x_n| \leq \theta^n \quad \forall\ n \geq M$

$\Rightarrow \Sigma |x_n| \leq \frac{1}{1-\theta} \in \mathbb{R}$

Say $\theta > 1$. $|x_n|^{1/n} > \theta - \varepsilon$ for inf many n.

So $\lim x_n \neq 0 \Rightarrow \Sigma x_n \notin \mathbb{R}$.

$(\Sigma x_n \in \mathbb{R} \Rightarrow x_n \to 0)$

Say $\theta = 1$? Can't say anything.

$x_n = \frac{1}{n} \Rightarrow \limsup |x_n|^{1/n} = 1$. But $\Sigma x_n \notin \mathbb{R}$

$x_n = \frac{1}{n^2} \Rightarrow \limsup |x_n|^{1/n} = 1$. But $\Sigma x_n \in \mathbb{R}$.

Test: $\theta := \limsup |x_n|^{1/n}$

$\theta > 1 \Rightarrow \Sigma x_n \notin \mathbb{R}$

$\theta < 1 \Rightarrow \Sigma |x_n| \in \mathbb{R}$

$\theta = 1 \Rightarrow$ Root test inconclusive.

RATIO TEST

Let $R = \limsup \left|\frac{x_{n+1}}{x_n}\right|$, $r = \liminf \left|\frac{x_{n+1}}{x_n}\right|$.

Clearly $r \leq R$.

Say $R < 1$: $\left|\frac{x_{n+1}}{x_n}\right| < 1-\varepsilon$ (for some $\varepsilon > 0$ s.t. $R < 1-\varepsilon < 1$ $\forall\, n \geq N$, some N).

$= \delta$ $\quad (\delta := 1-\varepsilon)$

$= \frac{\delta^{n+1}}{\delta^n}$

$$\Rightarrow \left|\frac{x_{n+1}}{\delta^{n+1}}\right| < \left|\frac{x_n}{\delta^n}\right| \qquad (n \geq N)$$

$$\Rightarrow \left|\frac{x_n}{\delta^n}\right| \leq \left|\frac{x_N}{\delta^N}\right| =: \lambda \qquad (n \geq N)$$

$$\Rightarrow |x_n| \leq \lambda \cdot \delta^n \qquad (n \geq N)$$

$$\Rightarrow \sum |x_n| \leq \sum_{i=1}^{N} x_i + \frac{\lambda}{1-\delta} < \infty$$

$\because \delta < 1$.

$$\Rightarrow \sum |x_n| < \infty$$

Say $r > 1$: $\left|\frac{x_{n+1}}{x_n}\right| > 1$ ($n \geq N$, some N)

$$\Rightarrow |x_{n+1}| > |x_n| \qquad (n \geq N)$$

$$\Rightarrow |x_n| > |x_N| \qquad (n > N)$$

$$\Rightarrow \lim_{n\to\infty} |x_n| \neq 0$$

$$\Rightarrow \sum x_n \notin \mathbb{R}$$

Test:

$R = \limsup \left|\frac{x_{n+1}}{x_n}\right|$ $\qquad r = \liminf \left|\frac{x_{n+1}}{x_n}\right|$

$R < 1 \Rightarrow \sum |x_n| \in \mathbb{R}$

$r > 1 \Rightarrow \sum x_n \notin \mathbb{R}$

$r \leq 1 \leq R \Rightarrow$ Inconclusive