Class - 1:

Recall the Def$^{\underline{n}}$ of a Vector Space :-

Let F be a field. $(V,+,\cdot)$ is defined to be a F-Vector space if

i) $(V,+)$ is an abelian group

ii) $\cdot : F \times V \longrightarrow V$ such that

a) $1 \cdot v = v$

b) $\alpha(v+u) = \alpha v + \alpha u$

c) $(\alpha+\beta)v = \alpha v + \beta v$

d) $(\alpha\beta)v = \alpha(\beta v)$

Examples :-

i) $\{0\}$ over F, F is any field.

ii) F over F.

$E \supset F$ fields, E is F-vector space, and dim E define $[E:F]$

iii) F^n is F-vector space

iv) $F[x]$ is a F-vector space

v) $M_n(F)$ is a vector space

vi) $\mathcal{C}((0,1),\mathbb{R})$ is a $\mathbb{R}$ vector space.

vii) $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y + a_0 = 0$

V is F-vector Space. $B \subset V$. The TFAE

i) B spans V and B is L.I. over F

ii) Every element in V can be uniquely expressed as finite F-linear combinations of elements of B

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \cdots + \alpha_n b_n$$

iii) B is minimal spanning set, $\{C \subset B,\ C \text{ is not spanning}\}$

iv) B is maximal L.I. set, $\{B \subset C,\ C \text{ is not L.I.}\}$

i $\Rightarrow$ ii)

Note that every element in V can be represented as F linear combination (as $\langle B \rangle = V$)

$$v = \underline{\alpha_1 b_1 + \cdots + \alpha_n b_n} = \beta_1 b_1 + \underline{\beta_2 c_2 + \cdots + \beta_m c_m}$$

$$\{b_1, \ldots, b_n\} \cap \{c_1, \ldots, c_m\} = \phi$$

$$\{b_1, \ldots, b_n\} \underset{b_k}{\cap} \{c_1, \ldots, c_m\} \neq \phi$$

ii $\Rightarrow$ iii

$C \subset B$ $\quad$ C is not spanning

$b \in B$ s.t $b \notin C$ $\quad$ $b = \alpha_1 c_1 + \cdots + \alpha_n c_n$ $\quad$ $\{c_1, \ldots, c_n\} \subset C$

$$\Rightarrow 1 \cdot b - \alpha_1 c_1 - \cdots - \alpha_n c_n = 0$$

$\{b, c_1, \ldots, c_n\}$ it is L. I. as B is L. I.

$\Rightarrow \boxed{1 = 0}$ $(\rightarrow\leftarrow)$