$X^Y = \{\text{functions } f : Y \to X\}$

X-valued sequences : $X^{\mathbb{N}}$

$\mathbb{R}^{\mathbb{N}}$: all seq

Space of all convergent seq in $\mathbb{R}^{\mathbb{N}}$ – is subspace of $\mathbb{R}^{\mathbb{N}}$.

- If $(x_n), (y_n) \in \mathbb{R}^{\mathbb{N}}$ are convergent (to $x, y \in \mathbb{R}$), then $(z_n) \in \mathbb{R}^{\mathbb{N}}$ defined by $z_n = x_n + y_n \ \forall n$ is also convergent (it converges to $x+y$).
- $(x_n) \in \mathbb{R}^{\mathbb{N}}$, $c \in \mathbb{R}$ Then $(z_n) = (c \cdot x_n)$ converges to $c \cdot x$.
  ((x_n) cgs to $x \in \mathbb{R}$)

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

<u>Properties</u> :

→ if $x \in \mathbb{R}$ satisfies $|x| \leq \varepsilon \ \forall \ \varepsilon > 0$, then $x = 0$.

→ if $x_n \to a$ & $x_n \to b$ in $\mathbb{R}$ then $a = b$.

→ $(x_n) \to x$, $y_n \to y$. $a, b \in \mathbb{R}$. Then :

$(a x_n + b y_n) \to ax + by$

$(x_n \cdot y_n) \to xy$

if $x_n \neq 0$ for all but finitely many n & $x \neq 0$ then $\frac{1}{x_n} \to x$

→ $\lim x_n = x \iff \lim (x_n - x) = 0$

Pf: $\varepsilon > 0$ given.

$x_n \to x \iff |x_n - x| < \varepsilon \ \ \forall n \geq N$ ($\& \ \exists N \in \mathbb{N}$)

$\iff \lim_{n \to \infty} (x_n - x) = 0$ □

→ Let $(x_n)_n$ be a convergent seq (in $\mathbb{R}$).

Let $(x_{n_k})_k$ be a subseq. Then $\lim_{k \to \infty} x_{n_k} = \lim_{n \to \infty} x_n$

Pf: $x_n \to x$. Let $\varepsilon > 0$ given. $\exists N$ s.t. $|x_n - x| < \varepsilon$ $\forall n \geq N$. Also $n_k \geq k$.

$\therefore$ Pick $K = N$ then $k \geq K \Rightarrow n_k \geq k \geq N$

$$\Rightarrow |x_{n_k} - x| < \varepsilon$$

$$\Rightarrow \lim_{k \to \infty} x_{n_k} = x$$

$\to$ x_n cgs to $x \in \mathbb{R}$, $x_n \geq 0$ $\forall n$. Then $x \geq 0$.

Pf: Suppose $x < 0$. Choose $\varepsilon = \frac{|x|}{2} = -\frac{x}{2}$

Then $\exists$ n s.t. $|x_n - x| < \varepsilon$

$$\Rightarrow x_n \in (x - \varepsilon, x + \varepsilon)$$

$$\Rightarrow x_n \leq x + \varepsilon = x - \frac{x}{2} = \frac{x}{2} < 0$$

(contradiction).

$\to$ $x_n \to x$, $y_n \to y$, $x_n \geq y_n$ $\forall n$. Then $x \geq y$.

$\to$ $x_n \to x$, $y_n \to y$, $z_n \to z$ s.t. $x_n \geq y_n \geq z_n$ $\forall n$

Then $x \geq y \geq z$.

If $x = z$ then $y = x$ (Sandwich thm).

If $x_n \to x$, $y_n \to y$, $x_n > y_n$ $\forall n$. Then $x > y$.

This is false. Why? $x_n = \frac{1}{n}$, $y_n = 0$

$x_n > y_n$ $\forall n$

But $x = 0 = y$ so $x \not> y$.

Definition: (1) $(x_n) \in \mathbb{R}^{\mathbb{N}}$. We say $\lim x_n = \infty$ if

$\forall K \in \mathbb{R}$ $\exists N \in \mathbb{N}$ s.t. $x_n > k$ $\forall n \geq N$.

(2) $(x_n) \in \mathbb{R}^{\mathbb{N}}$.

We say $\lim x_n = -\infty$ if $\lim(-x_n) = \infty$.

Example: $\lim_{n\to\infty} 2^n = \infty$

$\lim_{n\to\infty} (-2)^n$ diverges in $\overline{\mathbb{R}}$

$\lim_{n\to\infty} n = \infty$

$x_n = (-1)^n \cdot n$ $\qquad\qquad$ $y_n = n$

<u>Cauchy Sequences</u>: A seq $X = (x_n) \in \mathbb{R}^{\mathbb{N}}$ is said to be Cauchy if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ s.t. $p > q \geq N \Rightarrow |x_p - x_q| < \varepsilon$.

Note: Set of all Cauchy sequences in $\mathbb{R}^{\mathbb{N}}$ forms an $\mathbb{R}$- V.S.

$X \in \mathbb{R}^{\mathbb{N}}$ Cauchy $\Rightarrow$ X bdd

Pf: For $\varepsilon = 1$ $\exists$ N s.t. $|x_N - x_p| < 1$ $\forall\ p \geq N$.

$$\Rightarrow |x_p| \leq |x_p - x_N| + |x_N|$$
$$\leq 1 + |x_N|$$

$\therefore$ a bound is $B = \max\{|x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\} + 2$

i.e., $|x_n| \leq B\ \forall\ n$. $\square$

Note: Similarly a Cauchy Seq can be defined in $\mathbb{Q}$ (in fact, for any metric space).

A key diff b/w $\mathbb{R}$ & $\mathbb{Q}$ is that every Cauchy seq must converge in $\mathbb{R}$ (will see in some time) but this fails in $\mathbb{Q}$.

Def: A metric space, where all Cauchy sequences converge, is said to be complete.

Monotone Convergence theorem : $(x_n) \in \mathbb{R}^{\mathbb{N}}$.

① (x_n) inc & bdd above $\Rightarrow \lim_n x_n = \sup\{x_n\}$

① (x_n) dec & bdd below $\Rightarrow \lim_n x_n = \inf\{x_n\}$

Pf: Read from Bartle Sherbert.

Bolzano Weierstrass: Every bounded (in $\mathbb{R}$) seq has a cgt subsequence.

Pf: Read from Bartle Sherbert.

Thm: Every Cauchy seq in $\mathbb{R}$ cgs.

Pf: X in $\mathbb{R}^{\mathbb{N}}$ Cauchy $\Rightarrow X$ bounded $\Rightarrow X$ has a cgt subseq

$\Downarrow$ Cauchy-ness

X converge

HW: ① Find a Cauchy Seq in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.

② P.T. Convergent (in $\mathbb{R}$) $\Rightarrow$ Cauchy