The circle of Basis

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Theorem

Let V be a $k-$vector space and $X \subseteq V$. The following are equivalent:

1. X is a *Maximal Linearly Independent* set
2. X is a *Minimal Spanning* set
3. X is Linearly Independent and $\langle X \rangle = V$
4. Every $v \in V$ is uniquely expressible as $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$ for $v_i \in X, \lambda_i \in k$

Proof

$\underline{1 \implies 2}$

Suppose that X is a *Maximal Linearly Independent* subset of V.
Note that if $v \in X$, then $v = \lambda v$ where $\lambda = 1$.
Say $v \in V$ but $v \notin X$. Then, the set $\mathcal{A} = X \cup \{ v \}$ must be Linearly Dependent, due to maximality of X. Hence, $\exists\, v_1, \ldots, v_n \in \mathcal{A}$ and $\lambda_1, \ldots, \lambda_n \in k$ (not all 0) such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$

Claim
$v \in \{ v_1, v_2, \ldots, v_n \}$

Proof. Suppose that $v \neq v_i$ for any i. Therefore, $\{ v_1, v_2, \ldots, v_n \} \subseteq X \implies \{ v_1, v_2, \ldots, v_n \}$ is linearly independent. Therefore, if we have that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then $\lambda_i = 0\ \forall\, i$, which contradicts our choice of λ_i's. Thus, $v \in \{ v_1, v_2, \ldots, v_n \}$. □

Without loss of generality, let $v = v_1$. Hence, $\lambda_1 v + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$

Claim
$\lambda_1 \neq 0$

Proof. All v_i are distinct in $\mathcal{A}$. Therefore $\{ v_2, \ldots, v_n \} \subseteq X \implies \{ v_2, \ldots, v_n \}$ is linearly independent. Suppose that $\lambda_1 = 0$. Then, $\lambda_2 v_2 + \cdots + \lambda_n v_n = 0$. Due to linear independence, $\lambda_2 = \cdots = \lambda_n = 0 = \lambda_1$ which contradicts our choice of λ_i's. Thus, $\lambda_1 \neq 0$. □

So, $v = \frac{-\lambda_2}{\lambda_1} v_2 + \frac{-\lambda_3}{\lambda_1} v_3 + \cdots + \frac{-\lambda_n}{\lambda_1} v_n$.
Thus we can write any element of V as a linear combination of the elements of $X \implies V = \langle X \rangle$
To prove the minimality of X as a spanning subset of V, we suppose that $\exists\, Y \subset X \neq \phi$ (proper subset) such that $V = \langle Y \rangle$. Let $v \in X \backslash Y$. Since Y spans V, so $\exists\, \lambda - i \in k, v_i \in Y$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$ for some $n \in \mathbb{N}$. But this means that $\lambda_1 v_1 + \cdots + \lambda_n v_n - v = 0$ with $\{ v, v_1, \ldots, v_n \} \subseteq X$. This is impossible as v has coefficient $-1 \neq 0$ and $\{ v_1, \ldots, v_n, v \}$ is linearly independent (since, it is a subset of X). Thus, such a proper subset Y does not exist. This proves that X is a *minimal spanning* subset of V. ■

$\underline{2 \implies 3}$

Suppose that X is a *Minimal Spanning* subset of V. Since X spans V, we directly have that $\langle X \rangle = V$.
For the sake of contradiction, suppose that X is linearly dependent. So, $\exists\, v_i \in X, \lambda_i \in k$ (not all 0) such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. Without loss of generallity, suppose that $\lambda_1 \neq 0$. Thus, $v_1 = \frac{-\lambda_2}{\lambda_1} v_2 + \frac{-\lambda_3}{\lambda_1} v_3 + \cdots + \frac{-\lambda_n}{\lambda_1} v_n$. This means that $Y = X \setminus \{ v_1 \}$ also spans V which contradicts the minimality of X as a spanning set as $Y \subset X$. Hence, X is linearly independent. ■

$\underline{3 \implies 4}$

Suppose that X is a linearly independent subset of V such that $\langle X \rangle = V$.
Let $v \in V$. Since X spans V, so $\exists\, \lambda_1, \ldots, \lambda_n \in k$ and $v_1, v_2, \ldots, v_n \in X$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$.
If possible, suppose that $u \in V$ is such that it can be represented as two different linear combinations of vectors of X, that is, $\exists\, \lambda_1, \ldots, \lambda_n \in k$ and $\lambda_1', \ldots, \lambda_n' \in k$ with vectors $u_1, u_2, \ldots, u_n \in X$ such that
$u = \lambda_1 u_1 + \cdots + \lambda_n u_n = \lambda_1' u_1 + \cdots + \lambda_n' u_n$
$\implies (\lambda_1 - \lambda_1')u_1 + \cdots + (\lambda_n - \lambda_n')u_n = 0$
Due to linear independence of $\{ u_1, \ldots, u_n \} \subset X$, we have that $\lambda_i - \lambda_i' = 0 \iff \lambda_i = \lambda_i'$. Thus the representation is unique. ■

$\underline{4 \implies 1}$

Let $Y \subset X$ be a finite subset such that $Y = \{ v_1, \ldots, v_n \}$ for some $n \in \mathbb{N}$. Consider the equation

$$\sum_{i=1}^{n} \lambda_i v_i = 0$$

for some $\lambda_i \in k$ and we want to solve for λ_i's.
First we notice that $\lambda_i = 0\ \forall\, i$ is a valid solution. But by our hypothesis, $0 = \lambda_1 v_1 + \cdots + \lambda_n v_n$ is uniquely expressible $\implies \lambda_i = 0\ \forall\, i$ is the only solution. Thus we have that: $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0 \implies \lambda_i = 0\ \forall\, i$. So, X is linearly independent.
Now, suppose there is a proper superset $Z \supset X$ ($Z \subseteq V$) such that Z is linearly independent. Choose some $v \in Z \setminus X \neq \phi$. But $v \in V$, so $\exists\, v_1, \ldots, v_n \in X$ and $\lambda_1, \ldots, \lambda_n \in k$ such that
$v = \lambda_1 v_1 + \cdots + \lambda_n v_n$
$\implies \lambda_1 v_1 + \cdots + \lambda_n v_n - v = 0$
This contradicts the fact thst Z is linearly independent because $\{ v, v_1, \ldots, v_n \} \subseteq Z$ is linearly independent (as it is a subset of Z) but coefficient of v in the equation is $-1 \neq 0$. Hence, such s proper superset Z does not exist.
This proves that X is a *maximal linearly independent* set. ■