I. LIMIT POINTS / Accummulation point.

Let $S \subseteq \mathbb{R}$ be a set. $x \in \mathbb{R}$ is said to be an accumulation / limit / cluster point of S if $\forall\ \varepsilon > 0$ $\exists\ y \in S \setminus \{x\}$ s.t. $|y - x| < \varepsilon$.

∞ is said to be a limit point of S if $\forall\ k > 0$ $\exists\ y \in S$ s.t. $y > k$.

Similarly we define "$-\infty$ is a limit pt of S".

Let $X \in \mathbb{R}^{\mathbb{N}}$ $(X = (x_n))$. We say $x \in \overline{\mathbb{R}}$ is a limit point of the sequence X if $\exists$ a subsequence (x_{n_k}) of X s.t. $\lim\limits_{k \to \infty} x_{n_k} = x$, $x_{n_k} \neq x\ \forall\ k$.

II.

Let $X = (x_n) \in \mathbb{R}^{\mathbb{N}}$. Define a new sequence $(s_n) \in \mathbb{R}^{\mathbb{N}}$ by

$$s_n = \sum_{i=1}^{n} x_i \qquad (n^{th} \text{ partial sum})$$

We say (write) $\sum\limits_{n=1}^{\infty} x_n = \sum x_n = x \iff \lim\limits_{n \to \infty} s_n = x$.

Note that we can have $x \in \overline{\mathbb{R}}$.

Example: (i) $x_n = 0\ \forall\ n$. $\sum x_n = 0$.

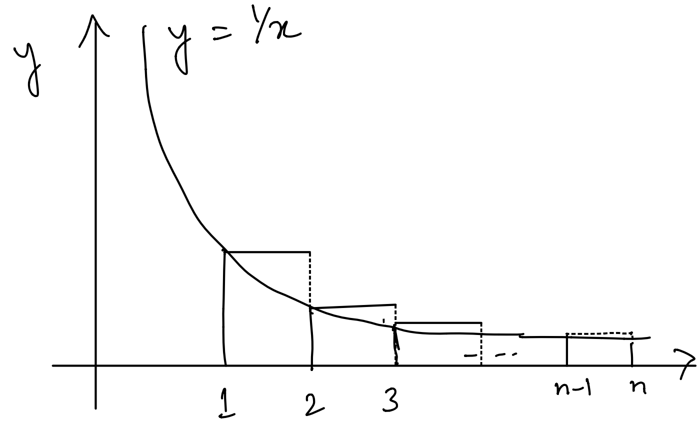
(ii) $x_n = \frac{1}{n^2}\ \forall\ n \geq 1$. $\sum x_n = \frac{\pi^2}{6}$ (Basel Problem)

(iii) $x_n = (-1)^n\ \forall\ n \geq 1$. $\sum x_n$ does not exist in $\overline{\mathbb{R}}$.

(iv) $x_n = 1\ \forall\ n \geq 1$. $\sum x_n = \infty$.

(v) $x_n = \frac{1}{n}\ \forall\ n \geq 1$. $\sum x_n = \infty$.

Proof that $\sum \frac{1}{n} = \infty$:



$$s_{n-1} = 1\cdot 1 + 1\cdot\frac{1}{2} + 1\cdot\frac{1}{3} + \cdots + 1\cdot\frac{1}{n-1} > \int_1^n \frac{1}{x}\,dx$$

$$= \ln n$$

$$s_n \geq \ln(n+1)$$

Let $k > 0$. Then take $N = \lceil e^k \rceil + 1000$

$$\Rightarrow \quad N > e^k - 1$$

If $n > N$ then $s_n \geq \ln(n+1) > \ln e^k = k$.

$\therefore \sum \frac{1}{n} = \infty$.

Fact : ① $\sum \left(\frac{1}{n} - \ln n\right) \in \mathbb{R}$

The limit of the above series is known as the Euler - Mascheroni constant.

② $\sum \frac{1}{n^3}$ converges to Apery constant.

- If $\sum x_n \notin \mathbb{R}$ we say the series does not converge in $\mathbb{R}$.
- If $\sum x_n \in \mathbb{R}$ then $\lim x_n = 0$.
- If $x_n \geq 0 \ \forall n$ then $\sum x_n$ converges in $\mathbb{R} \cup \{\infty\}$.
- Say $0 \leq x_n \leq y_n \quad \forall n \geq 1$
  - $\sum y_n \in \mathbb{R} \Rightarrow \sum x_n \in \mathbb{R}$
  - $\sum x_n = \infty \Rightarrow \sum y_n = \infty$.
- $x_n \geq 0$. $M \geq 1$.

  $\sum x_n \in \mathbb{R} \Rightarrow \sum (x_n)^M \in \mathbb{R}$

$\lim a_n = \infty$.

- Then a_n diverges.
- Then a_n converges in $\overline{\mathbb{R}}$.

- $x_n \geq 0$. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijection. Let $y_n = x_{\pi(n)}$. Then (y_n) is said to be a rearrangement of (x_n).

  Furthermore $\sum y_n = \sum x_n$
- We say $\sum x_n$ absolutely converges if $\sum |x_n| < \infty$