Last day

Root test: $\sum x_n$ given

$\theta = \limsup |x_n|^{1/n}$. $\quad \theta < 1 \Rightarrow \sum |x_n|$ cgt

$\theta > 1 \Rightarrow \sum x_n$ diverge

$\theta = 1 \Rightarrow$ inconclusive

Ratio test: $\sum x_n$ given

$R = \limsup \left|\frac{x_{n+1}}{x_n}\right| \qquad R < 1 \Rightarrow \sum |a_n| < \infty$

$r = \liminf \left|\frac{x_{n+1}}{x_n}\right| \qquad r > 1 \Rightarrow \sum a_n$ diverges

$r \leq 1 \leq R \Rightarrow$ inconclusive

Problem Set 3, Problem 3 (a)

$$\sum_{j=1}^{n} \sum_{i=1}^{j} \alpha(i,j) = \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha(i,j)$$

Define

$$\mathbb{1}_{x \leq y}(i,j) := \begin{cases} 0 & i > j \\ 1 & i \leq j \end{cases}$$

$$\sum_{j=1}^{n} \sum_{i=1}^{j} \alpha(i,j) = \sum_{j=1}^{n} \left[\sum_{i=1}^{j} 1 \cdot \alpha(i,j) + \sum_{i=j+1}^{n} 0 \cdot \alpha(i,j) \right]$$

$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{j} \mathbb{1}_{x \leq y}(i,j) \cdot \alpha(i,j) + \sum_{i=j+1}^{n} \mathbb{1}_{x \leq y}(i,j) \alpha(i,j) \right]$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \mathbb{1}_{x \leq y}(i,j) \cdot \alpha(i,j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{1}_{x \leq y}(i,j) \cdot \alpha(i,j)$$

$$= \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} \mathbb{1}_{x \le y}(i,j) \cdot \alpha(i,j) + \sum_{j=i}^{n} \mathbb{1}_{x \le y}(i,j) \cdot \alpha(i,j) \right]$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha(i,j)$$

<u>Abel's test</u>

(1) ⊙ $\sum a_n \in \mathbb{R}$ ⊙ $\{b_n\}$ monotone ⊙ $\{b_n\}$ bdd.

Then $\sum a_i b_i \in \mathbb{R}$

Pf: $\lim b_n = b$ $\left(A_n = \sum_{k=1}^{n} a_k \right)$

$\sum a_n = A$ ($\{|A_n|\}$ bdd above by $\beta > 0$)

$\therefore \lim (b_{n+1} \cdot A_n) = \lim b_{n+1} \cdot \lim A_n = bA$

$\{b_n\} \downarrow$: $\sum_{i=1}^{n} |b_{i+1} - b_i|$

$= \sum_{i=1}^{n} (b_i - b_{i+1})$

$= b_1 - b_{n+1} = |b_1 - b_{n+1}|$

In view of Problem Set 6 (Problem 3b, 4)

$\{b_n\} \uparrow$: $\sum_{i=1}^{n} |b_{i+1} - b_i|$

$= \sum_{i=1}^{n} (b_{i+1} - b_i) = b_{n+1} - b_1 = |b_1 - b_{n+1}|$

$\therefore \lim_n \sum_{i=1}^{n} |b_{i+1} - b_i| = \lim_n |b_1 - b_{n+1}| = |b_1 - b| \; (= \alpha)$

$\sum |A_n| \cdot |b_{n+1} - b_n| \le \beta \sum |b_{n+1} - b_n| = \beta\alpha$

$\therefore \sum A_n (b_{n+1} - b_n)$ abs cgs $\Rightarrow$ converges.

$\sum_{n=1}^{k} a_n b_n = b_{k+1} A_k - \sum_{n=1}^{k} (b_{n+1} - b_n) A_n$

is convergent as both terms converge when $k \to \infty$.

② (i) $\left\{\sum_{k=1}^{n} a_k\right\}_n$ bdd (ii) $\{b_n\}$ dec (iii) $\lim b_n = 0$

Then $\sum a_n b_n \in \mathbb{R}$.

Pf: $\lim_n \sum_{i=1}^{n} |b_{i+1} - b_i| = \lim_n (b_1 - b_{n+1}) = b_1$

$\Rightarrow \sum (b_{n+1} - b_n)$ cgs abs

Now conclude by P4 of PS 6.

<u>Example</u>: $a_n = (-1)^n$, $b_n = \frac{1}{2n+1}$

(i) $(A_n) = (-1, 0, -1, 0, -1, 0, \ldots)$

$\therefore$ Bounded.

(ii) (b_n) dec

(iii) $\lim b_n = 0$

Then $4 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \in \mathbb{R}$

The above real no. is knows as π.

<u>Product of sequences</u>

Consider the **formal** power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

x is an indeterminate

$R[x]$

(R is a ring)

$f(x), g(x) \in R[x]$ where R a ring

$f(a) = g(a) \ \forall a \in R$ | Coeff of x^n in f = coeff of x^n in g $\forall n \geq 0$.

This is the defn of equality

I can find an example of a ring R and polynomials $f, g \in R[x]$ s.t. $g(a) = f(a) \ \forall a \in R$ but $f \neq g$.

Discuss this in the WhatsApp group!

$$h(x) = f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$\vdots$

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

$\vdots$

If we are given seq $(a_n), (b_n)$ then (c_n) defined as above is said to be the Cauchy product.

Thm: let $(a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$ & (c_n) be their Cauchy product. Say $\Sigma a_n = \alpha$, $\Sigma b_n = \beta$, atleast of Σa_n or Σb_n cgs absolutely. Then $\Sigma c_n = \alpha \cdot \beta$

Pf: Rudin Page 74 (Thm 3.50)

An important (cont group homo) morphism:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let $x \in \mathbb{R}$

$$a_n = \frac{x^n}{n!}$$

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \frac{|x|}{n+1} = 0 < 1$$

Ratio test $\Rightarrow$ $\sum_{n=0}^{\infty} \left| \frac{x^n}{n!} \right| < \infty$ (also convergent)

$\therefore$ We a define a function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$x, y \in \mathbb{R}$

i) $f(x) \cdot f(y) = f(x+y)$

ii) $f(0) = 1$

iii) $f(x) = f(x/2 + x/2) = \{f(x/2)\}^2 \geq 0$

$\forall x \in \mathbb{R}$

iv) $f(x) \cdot f(-x) = f(x - x) = f(0) = 1$

$\Rightarrow f(x) \neq 0 \quad \forall x$

Fact: $\lim_n \left(1 + \frac{1}{n}\right)^n = e$

$\therefore f : \mathbb{R} \to (0, \infty)$

cont group homomorphism

$\downarrow$

$f(x+y) = f(x) \cdot f(y)$

We define $e := f(1)$.

By induction $f(n) = f(1+\cdots+1) = f(1)^n = e^n$ $(n \in \mathbb{Z}_{\geq 0})$

Try to show $f(n) = e^n$ for $n \in \mathbb{Z}_{\leq 0}$

So $f(n) = e^n \quad \forall\, n \in \mathbb{Z}$

From here we can extend f to $\mathbb{Q}$: $f(p/q) = e^{p/q}$

I can further extend f to $\mathbb{R}$ because f is cont

& $\mathbb{Q}$ is "dense" in $\mathbb{R}$.