7. Let $A_n = \{ x \in [0,1] : f(x) > \frac{1}{n} \}$ $(n \in \mathbb{N})$.

A_n is finite.

Reason: No. of elts in A_n is $\leq Mn$.

Suppose not. Say $\exists$ $x_1, \ldots, x_m \in A_n$, $m > Mn$

and all x_i are distinct.

$x_i \in A_n \Rightarrow f(x_i) > \frac{1}{n}$ $(\forall i)$

Given $f(x_1) + \cdots + f(x_m) \leq M$.

$$\Rightarrow \frac{m}{n} < \sum_{i=1}^{m} f(x_i) \leq M$$

$\Rightarrow$ $m < Mn$ (But this is a contradiction

because $m > Mn$ by hypothesis).

$\therefore$ A_n has atmost $M \cdot n$ elements.

Let $S = \{ x \in [0,1] : f(x) \neq 0 \}$.

It is given that $f(x) \geq 0$ $\forall x$.

$\therefore S = \{ x \in [0,1] : f(x) > 0 \}$.

Claim:
$$S = \bigcup_{n \in \mathbb{N}} A_n .$$

Proof:

$S \subseteq \bigcup_{n \in \mathbb{N}} A_n$: Let $x \in S$. $\therefore f(x) > 0 \Rightarrow f(x) > \frac{1}{k}$ for

some $k \in \mathbb{N} \Rightarrow x \in A_k \Rightarrow x \in \bigcup_{n \in \mathbb{N}} A_n$.

$\bigcup_{n \in \mathbb{N}} A_n \subseteq S$: $x \in \bigcup_{n \in \mathbb{N}} A_n \Rightarrow x \in A_k$ for some $k \in \mathbb{N}$

$\Rightarrow f(x) > \frac{1}{k} > 0 \Rightarrow x \in S$. $\square$

Countable union of countable sets (A_n, which are finite) is always countable. $\therefore$ S is countable.

8. G has 3 subgroups: $\{1\}$, G, $\{1\} \subsetneq H \subsetneq G$.

$\therefore G$ has exactly one nontrivial proper subgp H.

Let $g \in G \setminus H$. Consider $G' = \langle g \rangle \underset{\text{subgp}}{\subseteq} G$.

Clearly $G' \neq \{1\}$. Also $g \notin H \Rightarrow G' \neq H$.

This forces $G' = G$. $\therefore G$ is cyclic.

G is finite: If G were infinite then

$\langle g^4 \rangle \underset{\text{Properly contained}}{\subsetneq} \langle g^2 \rangle \subsetneq G$. $\therefore \langle g^4 \rangle = \{1\} \Rightarrow g^4 = 1$

$\Rightarrow G = \{1, g, g^2, g^3\}$

This is a contradiction.

Let $n = \text{ord}(G)$.

Let $d \mid n$ $(d > 0)$. Consider $G_1 = \langle g^d \rangle$. $\therefore \text{ord}(G_1) = \frac{n}{d}$.

Acc to what is given, G_1 is exactly one of G, H or $\{1\}$. This, respectively, means $\frac{n}{d}$ is n, $|H|$, 1. This respectively implies, d is 1, $\frac{n}{|H|}$, n.

$\therefore n$ has exactly one nontrivial proper +ve divisor.

If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ $(\alpha_i > 0)$, then no. of +ve divisors of n is $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$.

$\therefore n$ is s.t. $(\alpha_1+1) \cdots (\alpha_k+1) = 3$

$\Rightarrow k = 1$ & $\alpha_1 + 1 = 3$

$\therefore n = p^{\alpha_1} = p^2$ (for some prime p).

Exercise: If the prime factorization of n is $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ when $\alpha_i > 0$, p_i are distinct primes, then no. of +ve divisors of n is exactly $(\alpha_1+1)\cdots(\alpha_k+1)$.

Hints: Let $\tau(m) :=$ no. of +ve divisors of m.

Show that (i) $\tau(p^k) = k+1$ if p is prime,

(ii) $\tau(ab) = \tau(a)\tau(b)$ if a, b are coprime.

③ Show that C is closed & bounded.
(in $\mathbb{R}$)

$$\downarrow$$

$$\left(\exists B > 0 \text{ s.t. } |x| < B \ \ \forall x \in C \right)$$

Let $x \in C' \setminus C$. $\exists$ a seq in C (x_n) s.t. $x_n \to x$. $x - x_i \neq 0$ $\forall i$.
Define $f : C \to \mathbb{R}$ by $f(z) = \frac{1}{z - x}$.
Then f is cont & unbdd.
$\therefore$ Such x does not exist $\Rightarrow C' \setminus C = \phi$
$\Rightarrow C' \subseteq C \Rightarrow C$ closed.

This trick will be used to prove that a cont $\mathbb{R}$-func on a compact set will attain extrema.

①

$$M_n = \begin{bmatrix} r & & s \\ & \ddots & \\ s & & r \end{bmatrix}_{n \times n}. \quad \det M_n = ?$$

$$e_i = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \end{array} i$$

$$M_n \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} i = (r - s) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} i$$

Let $P_n(x)$ be the char poly of M_n.

$\therefore$ $e_i - e_{i+1}$ is an eigenvector of M_n with eigenvalue $r - s$ $(\forall i = 1, \dots, n-1)$.

Note: $M_n \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = ((n-1)s + r) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

$\therefore$ $e_1 + \cdots + e_n$ is an e.vector of M_n with e.value $r + (n-1)s$.

$\therefore$ $(x - r - (n-1)s) \mid P(x) \Rightarrow$ alg mult of $r - s$ is $\leq n-1$

But already seen geo mult of $r - s \geq n-1$.

$\therefore$ $n - 1 \leq$ geo mult $\leq$ alg mult $\leq n-1$

$\Rightarrow$ alg mult $= n-1 =$ geo mult.

$$\therefore P(x) = (x - (r-s))^{n-1} \cdot (x - (r + (n-1)s)) \cdot Q(x)$$

$\therefore$ $Q(x) = 1$ $\because$ $\deg P = n$.

$$\det(M_n) = (r-s)^{n-1} (r + (n-1)s).$$

If we let $V_\lambda = \{\vec{v} \in \mathbb{R}^n : M_n \cdot \vec{v} = \lambda \cdot \vec{v}\}$, then V_λ is a subspace of $\mathbb{R}^n$: $\forall \lambda \in \mathbb{R}$. (Prove!)

We say that λ has geometric multiplicity $= \dim V_\lambda$.

Look at the characteristic poly $P(x)$ of M_n. Let $k \geq 0$ be highest s.t. $(x-\lambda)^k \mid P(x)$. This k is said to be the algebraic multiplicity of $P(x)$.

Fact: Geo mult $\leq$ Alg mult.

Note: Whenever someone says "ring" assume it is a commutative ring with 1.

6. Randomly choose $j \in \{1, \dots, n\}$

A_d is the event that $d \mid j$.

A, B indep
$$\Updownarrow$$
$$P(A \cap B) = P(A) \cdot P(B)$$
$$\Updownarrow$$
$$P(A \mid B) = P(A)$$

$A_{k_1} \cap A_{k_2}$.

How many fav events? $\left\lfloor \frac{n}{k_1 k_2} \right\rfloor = \frac{n}{k_1 k_2}$

$j \in A_{k_1} \cap A_{k_2} \Leftrightarrow k_1 \mid j \;\&\; k_2 \mid j \Leftrightarrow k_1 k_2 \mid j$

A_{k_i}: How many fav events? $\left\lfloor \frac{n}{k_i} \right\rfloor = \frac{n}{k_i}$

$$\left.\begin{aligned} P(A_{k_1} \cap A_{k_2}) &= \frac{n}{k_1 k_2} \times \frac{1}{n} = \frac{1}{k_1 k_2} \\ P(A_{k_1}) \cdot P(A_{k_2}) &= \left(\frac{n}{k_1} \times \frac{1}{n}\right)\left(\frac{n}{k_2} \times \frac{1}{n}\right) = \frac{1}{k_1 k_2} \end{aligned}\right\} \text{Equal} \Downarrow \text{Indep.}$$

Similarly for other pairs.