Problem 5 of Activity 1

What if I allow $a_i = b_i$?

$I_1 = [0, \varepsilon/2]$

$S \setminus I_1$ is a finite set, say $\{x_1, x_2, \dots, x_m\}$

Why finite? $S \setminus I_1 = \{\frac{1}{n} : n \in \mathbb{N}, \frac{1}{n} > \varepsilon/2\}$

$= \{\frac{1}{n} : n \in \mathbb{N}, n < \frac{2}{\varepsilon}\}$

This is finite.

$I_2 = [x_1, x_1]$, $I_3 = [x_2, x_2], \dots, I_{m+1} = [x_m, x_m]$

(Here $k = m+1$)

$\Sigma(b_i - a_i) = \frac{\varepsilon}{2} < \varepsilon$.

What if I ask for $a_i < b_i$?

$I_1 = [0, \varepsilon/2]$

$S \setminus I_1$ finite, say, $\{x_2, x_3, \dots, x_m\}$

$I_k = \left[x_k, x_k + \frac{\varepsilon}{2m}\right] \quad \forall k \geq 2$.

$\Sigma(b_i - a_i) = \frac{\varepsilon}{2} + (m-1) \times \frac{\varepsilon}{2m}$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

The above was an example of a "measure 0" set or a "content 0" set.

Exercise: You pick a number from $\mathbb{Z}$. Denote this by X.

What is $\mathbb{P}(X = 0)$? 0

What is $\mathbb{P}(X \text{ is even})$? $\frac{1}{2}$

What is $\mathbb{P}(X \text{ is a multiple of } 4)$? $\frac{1}{4}$

OPEN SETS AND INTERIOR POINTS

$X \subseteq \mathbb{R}$ be subset.

We say $x \in X$ is an interior point of X if there is an open interval U s.t. $x \in U \subseteq X$.

Example: ① $X = [0,1]$. $x = \frac{1}{2} \in (0,1) \subseteq X$.
x is an int pt

② $X = [0,1]$. $x = 1$. This is not an int pt.
Exercise: Prove this.

The set of all interior points of X is called the interior of X & denoted by X^0 or $\mathrm{int}(X)$.

Fact: (1) X finite (or countable) $\implies X^0 = \phi$

(2) X open $\iff X^0 = X$

(3) X any subset of $\mathbb{R} \implies X^0 \subseteq X$.

(4) X^0 is the largest open set contained in X.

(5) $U \subseteq X$ is open then $U \subseteq X^0$.

(6) $$X^0 = \bigcup_{\substack{U \subseteq X \\ U \text{ open}}} U$$

(7) $(X^0)^0 = X^0$ $\because$ X^0 is open.

What does "largest" mean?

Def: Let $X \subseteq \mathbb{R}$. We say $A \subseteq X$ is <u>open in X</u> if there is some open $V \subseteq \mathbb{R}$ s.t.
$A = X \cap V$.

We say $A \subseteq X$ is <u>closed in X</u> if $X \setminus A$ is open in X.

Example : (1) $X = (0,1)$, $A = (0,1)$

A is open in X [$V = (0,1)$]

A is closed in X [ϕ is open in X]

(2) $X = [0,1]$. $A = [0,1]$

A is open in X [$V = \mathbb{R}$]

A is closed in X [ϕ is open in X]

$B = [0, \frac{1}{2})$. B open in X [$V = (-\frac{1}{2}, \frac{1}{2})$]

B not closed in X [$X - B = [\frac{1}{2}, 1]$ is not open in X]

(Exercise: why not closed)

(3) Let $X \subseteq \mathbb{R}$. $A = X$

A is open in X [$V = \mathbb{R}$]

A is closed in X [ϕ is open in X]

Theorem: (1) Open sets are closed under arbitrary union & finite intersection.

(2) Closed sets are closed under arbitrary intersection & finite union.

[The above statements are equivalent because of deMoivre's law]

Theorem: Let $X \subseteq \mathbb{R}$ be open. Let $A \subseteq X$ be open in X. Then A is open.

Pf: $A = X \cap V$ for some open set V.

Open sets closed under finite union $\Rightarrow$ A open.

Let $X \subseteq \mathbb{R}$. Let X' be the set of all limit points of X in $\mathbb{R}$. Define $\overline{X} := X \cup X'$.

Thm: (1) Let $X \subseteq \mathbb{R}$, $\overline{X}$ be its closure in $\mathbb{R}$. Then

$$\overline{X} = \left\{ x \in \mathbb{R} \;\middle|\; x = \lim_{n \to \infty} a_n \text{ for some seq } \{a_n\} \text{ in } X \right\}$$

(2) $$\overline{X} = \left\{ x \in \mathbb{R} \;\middle|\; U \cap X \neq \phi \text{ for every open nbd } U \text{ of } x \right\}$$