Real Analysis

Problem Set 7

August 16, 2021

1. Let $U \subseteq \mathbb{R}$ be nonempty and open. Show that $\exists r \in \mathbb{Q}, s \in \mathbb{R} \setminus \mathbb{Q}$ such that $r, s \in U$.
2. Let $U \subseteq \mathbb{R}$ be clopen (i.e., both open and closed). Show that U is either $\varnothing$ or $\mathbb{R}$.
3. Prove that every closed set in $\mathbb{R}$ is the intersection of a countable collection of open sets.
4. Let $U, V \subseteq \mathbb{R}$. Show that $(U \cap V)^o = U^o \cap V^o, (U \cup V)^o \supseteq U^o \cup V^o$ and $(U \cup V)' = U' \cup V'$.
5. Show that S' is closed for any $S \subseteq \mathbb{R}$.
6. Let $S \subseteq \mathbb{R}$ be a bounded set containing infinitely many points.
   (a) Show that there must be reals $a, b \in \mathbb{R}$ such that $S \subseteq [a, b]$.
   (b) Show that we can find an increasing sequence (a_n) and a decreasing sequence (b_n) such that
      - $a \leq a_1 \leq b_1 \leq b$
      - $b_n - a_n = \dfrac{b-a}{2^n} \; \forall n$
      - $[a_n, b_n] \cap S$ is an infinite set $\forall n$.

   (c) Show that $\sup a_n = \inf b_n$. Call this l.
   (d) Conclude that S has a limit point. (**Hint:** l will be a limit point of S).
7. Let $S \subseteq [a, b]$ be a set with no limit point.
   (a) Let $x \in [a, b]$. Show that $\exists$ an open set $U_x \subseteq \mathbb{R}$ such that $x \in U_x$ and $U_x \cap S \subseteq \{x\}$.
   (b) Conclude that S is finite. (**Hint:** Compactness of closed intervals).
8. Let $S \subseteq [a, b]$ be an infinite set.
   (a) Prove that there is a sequence in $[a, b]$, all of whose terms are in S with no repeated terms.
   (b) Show that the above sequence has a limit point $l \in [a, b]$.
   (c) Conclude that S has a limit point. (**Hint:** l will be a limit point of S).
9. $S \subseteq \mathbb{R}$ is a bounded infinite set. Let $T := \{x \in \mathbb{R} : \text{there are infinitely many points in } S \text{ more than } x\}$.
   (a) Show that $T \neq \varnothing$ and T is bounded above. Let $s := \sup T$. Clearly $s \in \mathbb{R}$.
   (b) Let $a \in \mathbb{R} \setminus T$. Show that a is an upper bound of T.
   (c) Show that s is a limit point of S.

10. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a decreasing (under containment) sequence of compact sets of $\mathbb{R}$. Suppose $\bigcap_{n\in\mathbb{N}} \mathfrak{C}_n = \varnothing$.

    (a) Show that $\mathscr{U} := \{\mathbb{R} \setminus \mathfrak{C}_n : n \in \mathbb{N}\}$ is an open cover of $\mathfrak{C}_1$.

    (b) Show that $\exists K \in \mathbb{N}$ such that $k \geq K \implies \mathfrak{C}_k = \varnothing$.

11. For a bounded set $S \subseteq \mathbb{R}$ define

$$\operatorname{diam} S := \sup_{x,y\in S} |x - y|.$$

Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a decreasing sequence of nonempty compact sets of $\mathbb{R}$ such that $\lim_{n\to 0} (\operatorname{diam} \mathfrak{C}_n) = 0$. Show that $\bigcap_{n\in\mathbb{N}} \mathfrak{C}_n$ is a singleton.

12. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a sequence of closed subsets of compact $\mathfrak{C} \subseteq \mathbb{R}$ such that $\bigcap_{i\in A} \mathfrak{C}_i \neq \varnothing$ for any finite $A \subseteq \mathbb{N}$. Show $\bigcap_{n\in\mathbb{N}} \mathfrak{C}_n \neq \varnothing$.
(**Hint:** Use a similar construction as in problem 10).

13. For $S \subseteq \mathbb{R}$, show that $\mathbb{R} \setminus \left(\overline{S}\right) = (\mathbb{R} \setminus S)^o$.

14. (Something from sequences and series) Let (a_n) be a sequence of real numbers converging to a. Define a sequence (b_n) by $b_n := \dfrac{\sum_{i=1}^{n} i \cdot a_i}{n(n+1)}$. Prove that $\lim_{n\to\infty} b_n = \dfrac{a}{2}$.