Heuristics:

open
not closed

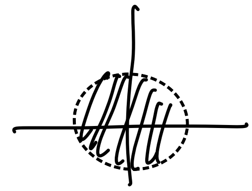
What is the boundary of $\{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ in $\mathbb{R}^2$?

$\{(x,y) : x^2+y^2=1\}$

closed
not open

What is the boundary of $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ in $\mathbb{R}^2$?

$\{(x,y) : x^2+y^2=1\}$



open
not closed

What is the boundary of $\{x \in \mathbb{R} : 0<x<1\}$ in $\mathbb{R}$? $\{0,1\}$

not closed
not open

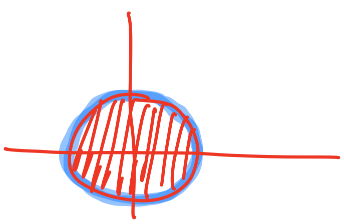
What is the boundary of $\{(x,0) \in \mathbb{R}^2 : 0<x<1\}$ in $\mathbb{R}$? $\{(x,0) : x \in [0,1]\}$

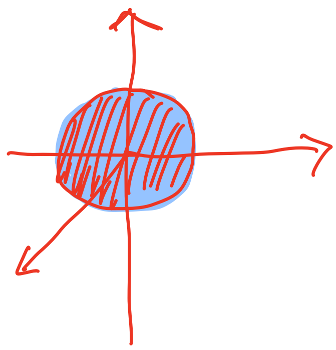
closed
not open

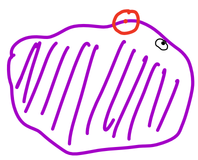
What is the boundary of $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$ in $\mathbb{R}$? $\{0,1\}$

closed
not open

What is the boundary of $\{(x,0) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ in $\mathbb{R}^2$? $\{(x,0) : x \in [0,1]\}$







closed
open

What is the boundary of $\mathbb{R}$ in $\mathbb{R}$? ϕ

closed + open

ϕ in $\mathbb{R}$? ϕ

Strategy: S a given set inside metric space X.
let ∂S be the boundary of S in X.

1) Is $\partial S \subseteq S$? Yes $\Rightarrow$ S is closed.
No $\Rightarrow$ S is not closed.

2) Is $\partial S \subseteq X \setminus S$? Yes $\Rightarrow$ S is open.
No $\Rightarrow$ S is not open.

Back to math:

Def: A set $S \subseteq \mathbb{R}$ is said to be closed in $\mathbb{R}$ (or simply "closed", when understood from context) is said to be closed if $\mathbb{R} \setminus S$ is open.

Example:

(1) $\mathbb{R}$ open $\Rightarrow$ ϕ closed

(2) ϕ open $\Rightarrow$ $\mathbb{R}$ closed

Lemma: Let $S \subseteq \mathbb{R}$. Then S is open iff $\forall\, x \in S$ $\exists\, r > 0$ s.t. $B_r(x) \subseteq S$.

Pf: ($\Rightarrow$) Say S open. Let $x \in S$. We know S is a union of open balls $\{ U_\lambda : \lambda \in \Lambda \}$. $S = \bigcup_{\lambda \in \Lambda} U_\lambda$.

$\exists\, \lambda_0 \in \Lambda$ s.t. $x \in U_{\lambda_0} = (a, b)$.

Let $r = \min\{|x-a|, |x-b|\} > 0$

Then $B_r(x) \subseteq U_{\lambda_0} \subseteq S$.

$U_{\lambda_0} = B_t(a)$.
Take $r = t - d(x,a)$
Then $B_r(x) \subseteq U_{\lambda_0}$
verify

($\Leftarrow$) Now say $\forall\, x \in S$, $\exists\, r_x > 0$ s.t. $B_{r_x}(x) \subseteq S$.

Then $S = \bigcup_{x \in S} B_{r_x}(x)$.

Lemma: (1) Arbitrary union of open sets is open. (in $\mathbb{R}$)

(2) If U, V are open (in $\mathbb{R}$), then $U \cap V$ open.

Proof of (ii): (works in general metric space)

Let $x \in U \cap V$.

$x \in U \Rightarrow \exists\, r_1 > 0$ s.t. $B_{r_1}(x) \subseteq U$.

$x \in V \Rightarrow \exists\, r_2 > 0$ s.t. $B_{r_2}(x) \subseteq V$.

Take $r = \min\{r_1, r_2\} > 0$. Then

$$\left.\begin{aligned} B_r(x) &\subseteq B_{r_1}(x) \subseteq U \\ B_r(x) &\subseteq B_{r_2}(x) \subseteq V \end{aligned}\right\} \Rightarrow B_r(x) \subseteq U \cap V.$$

Find an infinite collection of open sets s.t. their intersection is not open. (Exercise).