[I]t is impossible to be a mathematician without being a poet in soul... [T]he poet has only to perceive that which others do not perceive, to look deeper than others look. And the mathematician must do the same thing.
—Sonya Kovalevskaya

PROMYS Number Theory **Problem Set #1** *Boston University, July 5, 2021*

Reading Search

Q1: What is a perfect number? Give four examples of perfect numbers.

Q2: What is a Mersenne prime? Give four examples of Mersenne primes.

Q3: What is the meaning of $\sum_{\substack{d|N \\ d>0}} \frac{1}{d}$? Compute the value of this sum for some small values of N.

Asking Good Questions

P1: Compare the following mathematical systems with each other: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $2\mathbb{Z}$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_{11}$. Each of these systems has two operations (addition and multiplication). Which of these systems resemble each other in regard to the essential properties of the operations?

Exploration

P2: Consider the set of all polynomials in x with coefficients in $\mathbb{Z}_3$. Denote this set of polynomials by $\mathbb{Z}_3[x]$. Consider $f(x) = x^2 + 2x + 1 \in \mathbb{Z}_3[x]$ and $g(x) = 2x^2 + x + 1 \in \mathbb{Z}_3[x]$. Calculate $f(x) + g(x)$ and $f(x) \cdot g(x)$. Factor $x^4 - 1$ into linear factors in $\mathbb{Z}_5[x]$ and $x^6 - 1$ into linear factors in $\mathbb{Z}_7[x]$.

Prove or Disprove and Salvage if Possible

In resolving questions and problems posed in our problem sets, one has to make use of many properties which one associates with integers. Make a list (inventory) of such properties and see if this inventory suffices for every discussion of questions of arithmetic which we undertake. If this inventory proves to be adequate for such discussion, then we shall consider it as an acceptible description of the integers.

P3: $a|a$ for every $a \in \mathbb{Z}$.

P4: $a|b \Rightarrow b|a$ for all a and b in $\mathbb{Z}$.

P5: For all $a, b, c \in \mathbb{Z}$, if $a|b$ and $b|c$ then $a|c$.

P6: $a|b \Rightarrow a|bc$ for all integers a, b, c.

P7: For all a, b, c in $\mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b + c)$.

P8: For all $a, b, c \in \mathbb{Z}$, $a|bc \Rightarrow a|b$ or $a|c$.

Numerical Problems (*Some food for thought*)

P9: Using division to base seven, write $N = (34652)_7$ to base two, to base five, to base eleven.

P10: Write each of the following numbers to base three: 3, 9, 27, 243, 1/3, 1/9, 1/27. Write each of these numbers to base two. Any conjectures?

P11: Find the following elements in $\mathbb{Z}_5$: -1, $1/2$, $2/3$, $\sqrt{-1}$. How many of these elements can you find in $\mathbb{Z}_6$? in $\mathbb{Z}_{10}$? in $\mathbb{Z}_{11}$? in $\mathbb{Z}_{13}$?

P12: Make a list of the perfect squares in $\mathbb{Z}_5$, in $\mathbb{Z}_{17}$, in $\mathbb{Z}_{19}$, in $\mathbb{Z}_{21}$. How many squares are there in each case? Any conjectures?

P13: Calculate the sums: $\sum_{\substack{d|6 \\ d>0}} \frac{1}{d}$, $\sum_{\substack{d|28 \\ d>0}} \frac{1}{d}$. Can you make an interesting conjecture? Perhaps another example may help: $\sum_{\substack{d|496 \\ d>0}} \frac{1}{d}$. Calculate this sum and compare its value to the values of the first two sums.

The Art of Counting

P14: Consider the set $S = \{a, b, c, d\}$. How many distinct subsets of S are there? How many distinct subsets of the set $T = \{c, d, e, f, g, h\}$ are there? The sets S and T have 4 and 6 elements respectively. How many elements are there in $S \cap T$? in $S \cup T$? In general, if you know the number of elements in each one of two given sets S and T, can you say how many elements there are in $S \cup T$?