MIT 6.7230/18.456
Algebraic techniques and semidefinite programming
Homework assignment # 7

Date Given: May 8, 2024
Date Due:

P1. [***30 pts***] Recall the relaxations for linearly and quadratically constrained quadratic programming we have seen earlier (concretely, equation (9) in Lecture 3). Explain how these can be interpreted as a special case of Positivstellensatz-based relaxations (and more specifically, a Schmüdgen-type certificate).

P2. [***20 pts***] Consider the matrices

$$H = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 5 & 3 & 0 & 0 & 3 \\ 3 & 5 & 3 & 0 & 0 \\ 0 & 3 & 5 & 3 & 0 \\ 0 & 0 & 3 & 5 & 3 \\ 3 & 0 & 0 & 3 & 5 \end{bmatrix}.$$

(a) Show that the matrix H is copositive, but cannot be decomposed as the sum of a positive semidefinite and a nonnegative matrix.

(b) Show that the matrix G is doubly nonnegative (psd and nonnegative), but is not completely positive.

P3. [***35 pts***] In this problem, we analyze symmetry reduction in the case of sum of squares decompositions of univariate even polynomials. Let $p(x)$ be a univariate polynomial that satisfies $p(x) = p(-x)$ (i.e., it is even).

(a) Write down the "standard" SDP formulation for checking whether $p(x)$ is SOS.

(b) Is this SDP invariant under the action of a group?

(c) Restrict the feasible set to the fixed-point subspace. How does the problem simplify? (Hint: you may want to group the monomials depending on whether the exponents are even or odd).

(d) Explain why the new formulation is computationally better.

(e) Compare the results with making the substitution $y = x^2$ in the original polynomial, and imposing the constraint $y \geq 0$. How do they differ (if they do)?

P4. [***20pts***] Consider the following sextic form, known as the *Robinson form*:

$$R(x, y, z) = x^6 + y^6 + z^6 - x^4y^2 - y^4x^2 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2.$$

(a) Show that $R(x, y, z)$ is invariant under the symmetric group S_3.

(b) Show that R is not a sum of squares.

(c) Write a sum of squares decomposition of $(x^2 + y^2 + z^2) \cdot R(x, y, z)$, by solving an SDP where you exploit the symmetry. Compare the SDP size and running time of the two approaches (with/without exploiting symmetry).

P5. [***35 pts***] Let $M \in \mathcal{S}^n$, and let $z = [x_1^2, \ldots, x_n^2]^T$. As we have seen, M is copositive if and only if the homogeneous quartic polynomial $p(x) = z^T M z$ is nonnegative.

(a) Plot the region of $(a, b) \in \mathbb{R}^2$ for which the matrix

$$\begin{bmatrix} a & b \\ b & 1 \end{bmatrix}$$

is copositive.

(b) Prove that $p(x)$ is a sum of squares if and only if $M = P + N$, where P is positive semidefinite and N is componentwise nonnegative. (Hint: you may want to use the symmetry $p(x_1, \dots, x_n) = p(\pm x_1, \dots, \pm x_n)$).