SERIES

Let $X = (x_n)$ be an $\mathbb{R}$-seq. Define a new sequence:
$S_n = \sum_{i=1}^{n} x_i$. This is called the n^{th} partial sum.

Def: We write $\Sigma x_k = x$ if S_n converges to $x \in \overline{\mathbb{R}}$.

Examples:

1. $x_n = \frac{1}{n}$. $\Sigma \frac{1}{n} = \infty$.

   When do we say $\lim a_n = \infty$?

   We say $\lim a_n = \infty$ if $\forall\ k \in \mathbb{R}$ $\exists\ N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow a_n > k$.

2. $x_n = \begin{cases} 1 & \text{if } n \equiv 1,2 \pmod 3 \\ -1 & \text{otherwise} \end{cases}$

   S_n : 1, 2, 1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 5, 6, 5, 6, 7, 6, ...

   $\begin{cases} S_{3k} = k \\ S_{3k+1} = k+1 \\ S_{3k+2} = k+2 \end{cases}$ We write $\Sigma x_n = \infty$.

   $\lim S_n = \infty$

   Pf: Let $k \in \mathbb{R}$ given. Pick $a = \max\{\lceil k+1 \rceil, 1\}$

   $S_{3a+1} = a+1 > a$
   $S_{3a+2} = a+2 > a$
   $S_{3a+3} = a+1 > a$
   $S_{3a+4} = a+2 > a$

   Claim: $n \geq 3a+1 \Rightarrow S_n > a$

   || Do the proof yourself.

   So we pick $N = 3a+1$. Then $n \geq N \Rightarrow S_n > a > k$.

   $\therefore \lim S_n = \infty$.

(3) $x_n = 0$. $\quad \sum x_n = 0$

(4) $x_n = c \ (<0)$. $\quad \sum x_n = -\infty$.

(5) $x_n = (-1)^n$. $\quad \sum x_n$ diverges in $\overline{\mathbb{R}}$.

$S_n: -1, 0, -1, 0, \dots$

(6) $x_n = \frac{1}{n^2}$. $\quad \sum x_n = \frac{\pi^2}{6}$.

$x_n = \frac{1}{n^4}$. $\quad \sum x_n = \pi^4/90$.

Facts: (1) Say $\sum x_n \in \mathbb{R}$. Then $\lim\limits_{n\to\infty} x_n = 0$.

Remark: $A: \sum x_n \in \mathbb{R}$. $\quad B: \lim\limits_{n\to\infty} x_n = 0$.

(Contrapositivity) $(A \Rightarrow B) \iff (\neg B \Rightarrow \neg A)$

$\therefore$ We can conclude: $\lim x_n \neq 0 \Rightarrow \sum x_n \notin \mathbb{R}$.

(2) If all $x_n \geq 0$ then $\sum x_n \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Reason: look at $S_n = \sum\limits_{k=1}^{n} x_k$. $S_n \uparrow$. Then S_n cgs (in $\overline{\mathbb{R}}$) to its sup. But its supremum is either a real or $+\infty$. $\therefore \lim S_n \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$.

(3) $0 \leq x_n \leq y_n$.

- $\sum y_n \in \mathbb{R} \Rightarrow \sum x_n \in \mathbb{R}$.
- $\sum x_n = \infty \Rightarrow \sum y_n = \infty$

(4) $x_n \geq 0$, $m \geq 1$. $\quad \sum x_n \in \mathbb{R} \Rightarrow \sum x_n^m \in \mathbb{R}$

(5) (Def) We say $\sum x_n$ converges absolutely if $\sum |x_n| \in \mathbb{R}$.

(6) $x_n \geq 0$, $\pi : \mathbb{N} \xrightarrow{\text{bij}} \mathbb{N}$. $\quad y_n = x_{\pi(n)}$.

Then $\sum y_n = \sum x_n$.

Idea: $\underset{y_3}{x_1} \quad x_2 \cdot \underset{y_n}{\cdot} \cdot \underset{y_2}{\cdot} - \ - \underset{y_2}{\cdot} - \ - \ - \ \overset{\text{Rightmost}}{x_k} -$

I circled n numbers

$\boxed{y_1 \quad y_2 \quad y_3 \quad \dots \quad y_n}$

Proof: Let $S_n = \sum_{i=1}^{n} x_i$

$S_n' = \sum_{i=1}^{n} y_i$

$\forall n \in \mathbb{N}, \ \exists\, m(n) \in \mathbb{N}$ s.t.

$$y_1 + \dots + y_n \leq x_1 + \dots + x_{m(n)+n}$$

$$\Leftrightarrow \quad S_n' \leq S_{m(n)+n}$$

Case: $\sum x_n = x \in \mathbb{R}$. Then $S_{m(n)+n} \leq x \ \forall n$.

$\therefore \ 0 \leq S_n' \leq S_{m(n)+n} \leq x \quad \forall n$.

$$\left.\begin{array}{l} S_n' \text{ inc } \because y_n \geq 0 \\ S_n' \text{ bdd} \end{array}\right\} \Rightarrow S_n' \text{ cgs.}$$

Say $y = \sum y_n$. $\quad S_n' \leq x \quad (\forall n)$

$\Rightarrow y \leq x$ (Taking $n \to \infty$).

Observe that (x_n) is a rearrangement of (y_n) (i.e., $x_n := y_{\sigma(n)}$, where $\sigma = \pi^{-1}$ is a bijection $\mathbb{N} \to \mathbb{N}$).

By the same reasoning,

$$\sum x_n \leq \sum y_n \Rightarrow x \leq y.$$

Conclude $x = y$.

Case: $\sum x_n = \infty$. $\quad x_n = y_{\sigma(n)}$

So $\forall n \in \mathbb{N} \ \exists\, k(n) \in \mathbb{N}$ s.t.

$$x_1 + \dots + x_n \leq y_1 + \dots + y_{n+k(n)}$$

$$\Rightarrow \quad S_n \leq S'_{n+k(n)}$$

Let $\alpha \in \mathbb{R}$ given. Then (by def & hypothesis), $\exists\ N \in \mathbb{N}$ s.t. $S_n > \alpha \ \forall\ n \geq N$.

Take $N' = N + k(N)$.

Then $n \geq N' \implies S_n' \underset{\substack{\uparrow \\ \because\, y_n \geq 0 \\ \forall n}}{\geq} S'_{N'} \geq S_N > \alpha$.

By def, $\lim\limits_{n\to\infty} S_n' = \infty$, i.e., $\sum y_n = \sum x_n = \infty$.