Quotienting :-

1) Equivalence Relation and partition :-

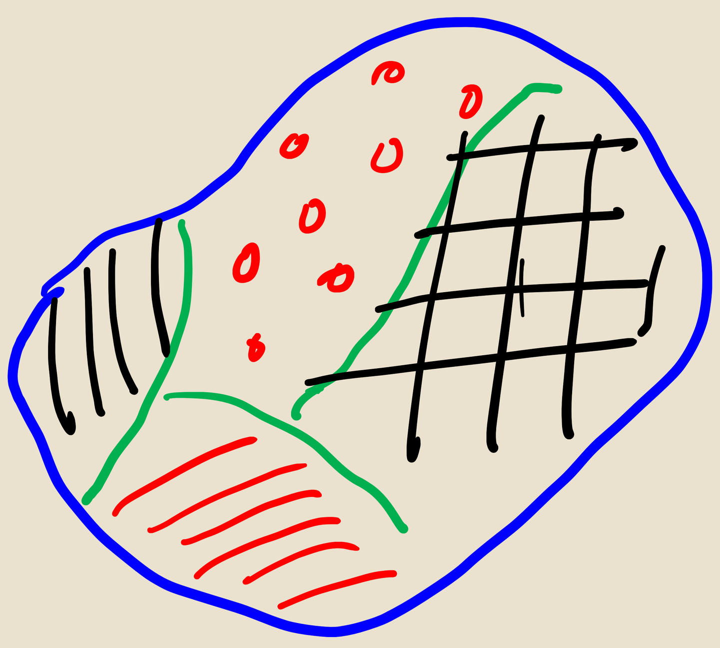
a) S is a set. An equivalence relation is a relation on S that is

Reflexive
Symmetric
Transitive

b) S is a set. A partition of S is a collection $\mathcal{C}$ of subsets of S s.t

i) $A \in \mathcal{C}$, and $B \in \mathcal{C}$, then $A \cap B = \phi$

ii) $\bigcup_{A \in \mathcal{C}} A = S$



Theorem :- Consider a set S. Then given an equivalence relation on S, we can generate a partition of S and vice-versa.

Proof :- Let '$\sim$' be an equivalence relation on S.
To generate a partition of S.

Let $\mathcal{C}$ be a partition of S s.t for any $C \in \mathcal{C}$, $a, b \in \mathcal{C}$ iff $a \sim b$. (Or in other word, pick an $a \in S$. Consider define $C_a = \{b \in S \mid a \sim b\}$)

Claim :- $\{C_a\}_{a\in S} \to S/\sim$ is a partition of S

→ Note that $a\sim b \Rightarrow C_a = C_b$

↳ ($x\in C_a \Rightarrow a\sim x$, but $a\sim b \Rightarrow b\sim a$

$\Rightarrow b\sim a,\ a\sim x \Rightarrow b\sim x \Rightarrow x\in C_b$

$C_a \subseteq C_b$. Similarly $C_b \subseteq C_a \Rightarrow C_a = C_b$)

Also note that if $C_a \cap C_b \neq \phi \Rightarrow C_a = C_b$

↳ ($x\in C_a\cap C_b \Rightarrow$ $\left.\begin{array}{l} a\sim x \\ b\sim x \Rightarrow x\sim b \end{array}\right\} \Rightarrow a\sim b$

$\Rightarrow C_a = C_b$)

Also note that $a\not\sim b \Rightarrow C_a\cap C_b = \phi$

↳ ($C_a\cap C_b \neq \phi \Rightarrow C_a = C_b \Rightarrow a\sim b$ ($\rightarrow\leftarrow$))

Thus for any two classes C_a and C_b

$C_a\cap C_b = \phi$ or $C_a = C_b$

$\Updownarrow$ $\Updownarrow$

$a\not\sim b$ $a\sim b$

Now $\bigcup_{a\in S} C_a = S$.

↳ ($a\in S$, $\underset{\substack{\downarrow \\ \text{Reflexive}}}{a\sim a} \Rightarrow a\in C_a$)

$\boxed{\{C_a\} \text{ is a partition of } S}$

Converse :- Given a partition $\mathcal{C}$ of S, to find a equivalence on S.

Proof → $a \sim b$ iff $\exists\, C \in \mathcal{C}$ s.t. $a, b \in C$.

(Check the rest yourself)

Quotienting :- Let V be a F-vector space. $W \subset V$ be a subspace.

$$\boxed{\text{Take any } a, b \in V. \text{ Define } a \sim b \text{ iff } a - b \in W}$$

→ $a - a \in W$, $\forall a \in V \Rightarrow a \sim a$

→ $a \sim b \Rightarrow a - b \in W \Rightarrow b - a \in W \Rightarrow b \sim a$

→ $a \sim b,\ b \sim c \Rightarrow \left.\begin{array}{l} a - b \in W \\ b - c \in W \end{array}\right\} \Rightarrow a - c \in W \Rightarrow a \sim c$

So this is an equivalence relation on V.

Hence we get a partition of V. This partition is denoted by $V/W = \{C_a\}_{a \in V} = \{a + W\}_{a \in V}$

$$\boxed{a + W = b + W \iff a - b \in W}$$

$$\boxed{\text{Goal :- is to give } V/W \text{ a } F\text{-vector space structure}}$$

$$V/W = \{ a + W : a \in V \}$$

Define $(a+W) + (b+W) := (a+b) + W$

$\lambda (a+W) := (\lambda a) + W, \quad \lambda \in F$

Are these operation well defined?

$$\left.\begin{array}{l} a+W = c+W \\ b+W = d+W \end{array}\right\} \begin{array}{l} \text{Check whether} \\ (a+W)+(b+W) = (c+W)+(d+W) \end{array}$$

$$\left(\begin{array}{l} (a+b)+W \\ \quad \shortparallel \\ (c+d)+W \end{array} \right. \Leftrightarrow \left. \begin{array}{l} a+b-(c+d) \in W \\ = (a-c)+(b-d) \end{array} \right)$$

$a-c \in W$ $\qquad (a-c)+(b-d) \in W$

$b-d \in W$

So V/W is a vector space

H.W. :- Do these quotienting for Rings, Modules and try to give then corresponding structure

Is this quotienting and giving the quotient a structure possible for groups.

Linear Maps :-

Let V and W be F- vector spaces.

$$T : V \longrightarrow W \quad s.t$$

$$\left.\begin{aligned} T(v+u) &= T(v) + T(u) \\ T(\lambda v) &= \lambda \cdot T(v) \end{aligned}\right\} \quad \begin{aligned} &v, u \in V \\ &\lambda \in F \end{aligned}$$

$T(V)$ is a vector space

- If T is bijective, it is defined to be an "isomorphism"
- kernel of T, $\ker T := \{v \in V : T(v) = 0\}$

$\boxed{\text{Note that } T \text{ is injective iff } \ker T = \{0\}}$

Examples :-

1) V is F-vector space

$$T : V \longrightarrow V$$

$$T(v) = v$$

Check that this is an isomorphism

2) Let V be an n-dim, F-vector space. Then

$$V \cong F^n$$

→ Let a basis for V be $\{b_1, \dots, b_n\}$

Then define $T : V \longrightarrow F^n$

$$T(v) = (\alpha_1, \alpha_2, \dots, \alpha_n) \quad \text{where } v = \alpha_1 b_1 + \dots + \alpha_n b_n$$

a) $\boxed{T \text{ is well defined}}$:-

$$v = w \Rightarrow v = w = \alpha_1 b_1 + \dots + \alpha_n b_n$$

$$\Rightarrow T(v) = T(w) = (\alpha_1, \dots, \alpha_n)$$

b) $\boxed{T \text{ is injective}}$:-

Let $v \in \ker T$

$\Rightarrow T(v) = 0$, $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$

$\Rightarrow (\alpha_1, \ldots, \alpha_n) = 0 = (0, 0, \ldots, 0)$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0 \Rightarrow v = 0$

c) $\boxed{T \text{ is surjective}}$

Take any $(\alpha_1, \ldots, \alpha_n) \in F^n$

$T(\alpha_1 b_1 + \cdots + \alpha_n b_n) = (\alpha_1, \ldots, \alpha_n)$

d) $\boxed{\begin{array}{l} T(v+w) = T(w) + T(v) \\ T(\lambda v) = \lambda T(v) \end{array}}$

$$V \cong F^n$$

Corollary :- Any two finite dimensional F vector spaces of same dimension are isomorphic.

$\left(\text{Note that } \begin{array}{l} V \overset{T}{\underset{S}{\cong}} W \\ W \cong U \end{array} \right\} \Rightarrow V \underset{S \circ T}{\cong} U \Bigg)$

2) V is F-vector space, $W < V$ subspace

$\pi : V \to V/W$

$\pi(v) = v + W$

Check that π is a well-defined linear map. Also π may not be injective. π is definitely surjective.