Proof that every vector space has a basis

Nilava Metya

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Definition 1. A *poset* (Partially ordered set) is a non-empty set X together with a relation $\leq$ on X such that the following hold $\forall a, b, c \in X$:

- (Reflexivity) $a \leq a$.
- (Anti-symmetry) $a \leq b, b \leq a \implies a = b$.
- (Transitivity) $a \leq b \leq c \implies a \leq c$.

2. $C \subseteq X$ is called a *chain* if $x, y \in X \implies x \leq y$ or $y \leq x$.

3. $x \in X$ is called an *upper bound* of $Y \subseteq X$ if $y \leq x \, \forall \, y \in Y$.

4. $x \in X$ is called a *maximal element* if $\big(y \in X\big) \wedge \big(x \leq y\big) \implies x = y$.

Lemma (Zorn's lemma)
Let $\langle X, \leq \rangle$ be a poset. Suppose every chain $C \subseteq X$ has an upper bound in X. Then X has a maximal element.

Theorem
Let V be a $k-$vector space. Let $S \subseteq V$ be a linearly independent subset. Then S is contained in a maximal linearly independent set.

Proof. Define

$$X := \{ \, T : S \subseteq T \subseteq V, T \text{ is linearly independent} \, \}$$

If $\mathcal{B}$ is a maximal element of X then $S \subseteq \mathcal{B}$ and $\mathcal{B}$ is a maximal linearly independent set. So $\mathcal{B}$ is a basis.
Define the partial ordering $\leq$ for sets $A, B \in X$ as: $A \leq B \iff A \subseteq B$
Let $C \subseteq X$ be a chain.
Let

$$M := \bigcup_{T \in C} T$$

Note that $S \subseteq M \subseteq V$.
To show that M is an upper bound of C, we further need to show that M is linearly independent, so that it is in X. Suppose not. Then by definition, there exists $v_1, v_2, \ldots, v_n \in M$ and $k_1, k_2, \ldots, k_n \in k$, not all 0, such that $k_1v_1 + k_2v_2 + \cdots + k_nv_n = 0$.
But, $\exists \, T_1, T_2, \ldots, T_n \in C$ (not necessarily distinct) such that $v_i \in T_i \, \forall \, i$. but T_i are all comparable. So $T_1 \cup T_2 \cup \cdots \cup T_n = T_j$ for some $1 \leq j \leq n$, which is the maximal element among T_i's.
This proves that every chain in X has an upper bound. Hence by Zorn's lemma, we conclude that X has a maximal element $\mathcal{B}$, which is indeed a basis. $\square$