From last day

(1) Find a Cauchy seq in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.

A: Let x be an irrational number. $x_n = \frac{\lfloor nx \rfloor}{n}$ cgs to x.

$\therefore$ x_n does not converge in $\mathbb{Q}$.

But $x_n \in \mathbb{Q}$ $\forall n$ and (x_n) is Cauchy.

(2) Let $(x_n) \in \mathbb{R}^{\mathbb{N}}$ s.t. x_n cgs. Show x_n is Cauchy.

A: Say $\lim x_n = x$. Let $\varepsilon > 0$ be given.

$\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |x_n - x| < \varepsilon/2$.

$\therefore$ $p > q \geq N \Rightarrow |x_p - x_q| = |x_p - x + x - x_q|$

$\leq |x_p - x| + |x - x_q| < \varepsilon$.

Countability of sets

Let $\mathcal{C}$ be the collection of all sets.

Define an equivalence relation $\sim$ on $\mathcal{C}$ as follows:

$$A \sim B \overset{\text{def}}{\iff} \exists \text{ a bijection } f : A \to B.$$

Recall: Say $f : X \to Y$ is a function.

(injective) f is said to be 1-1 if $a \neq b$ then $f(a) \neq f(b)$.

(Surjective) f is said to onto if $\forall y \in Y, \exists x \in X$ s.t. $y = f(x)$.

f is said to be bijective if f is 1-1 & onto

Defn: (1) For $n \in \mathbb{N}$, define $[n] = \{1, 2, \dots, n\}$.

Proposition: $\sim$ is an equivalence relation on $\mathscr{C}$.

Pf: Reflexive: Let $A \in \mathscr{C}$. Consider $f: A \to A$ given by $f(a) = a$ $\forall a \in A$. This is a bijection.
$\therefore A \sim A$.

Symmetric: Suppose $A, B \in \mathscr{C}$ s.t. $A \sim B$, i.e., $\exists$ a bij $f: A \to B$. There is a bij $g: B \to A$ (why?) s.t. $f \circ g = id_B$ & $g \circ f = id_A$. $\therefore B \sim A$

Transitive: Suppose $A, B, D \in \mathscr{C}$ s.t. $A \sim B$ & $B \sim D$, ie, $\exists$ bij $f: A \to B$, $g: B \to D$. Define $h: A \to D$ given by $h(x) = g(f(x))$ (i.e., $h = g \circ f$).
Check: h is a bij.

The above together show that $\sim$ is an equivalence relation.

Prop: Let A be a set. If $\exists\, m, n \in \mathbb{N}$ s.t. $A \sim [n]$ and $A \sim [m]$, then $m = n$.

Defn: ① $A \in \mathscr{C}$ is said to be finite if either of the following is true:
- $A = \phi$. In this case we say $|A| = 0$.
- $\exists\, n \in \mathbb{N}$ s.t. $A \sim [n]$. In this case, we say $|A| = n$.

② $A \in \mathscr{C}$ is said to be infinite if A is not finite.

③ $A \in \mathscr{C}$ is said to be countably infinite if $A \sim \mathbb{N}$.

④ $A \in \mathscr{C}$ is said to be countable if A is either finite or countably infinite.

⑤ $A \in \mathscr{C}$ is said to be uncountable if it is not countable.

Q: Let $A = \{0,1\}^{\mathbb{N}}$. Then A is uncountable.

(Fact: There is a function $f : \mathbb{N} \to \mathbb{N}$ which cannot be realized via a C++ program).

F: { C++ programs that realize some func $\mathbb{N} \to \mathbb{N}$ } ↳ Countable

S

$\downarrow$ ← ∴ This is not bij

{ func $f : \mathbb{N} \to \mathbb{N}$ } → Uncountable

$A_n = \{$Programs with n characters$\}$. A_n is finite.

$$S = \bigcup_{n=1}^{\infty} A_n$$