Real Analysis

Problem Set 7 (Solutions)

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In short, we will denote $X^c := \mathbb{R} \setminus X$.

1. Let $U \subseteq \mathbb{R}$ be nonempty and open. Show that $\exists r \in \mathbb{Q}, s \in \mathbb{R} \setminus \mathbb{Q}$ such that $r, s \in U$.

   Pick $x \in U \neq \varnothing$. $\exists \varepsilon > 0$ such that $\mathscr{B}_\varepsilon(x) \subseteq U$. We know $\exists r \in \mathbb{Q} \cap \mathscr{B}_\varepsilon(x), s \in (\mathbb{R} \setminus \mathbb{Q}) \cap \mathscr{B}_\varepsilon(x)$.

2. Let $U \subseteq \mathbb{R}$ be clopen (i.e., both open and closed). Show that U is either $\varnothing$ or $\mathbb{R}$.

   Let $V = \mathbb{R} \setminus U$. U closed $\implies$ V open. Further $U \cup V = \mathbb{R}, U \cap V = \varnothing$. Suppose U is neither $\varnothing$, nor $\mathbb{R}$. Pick $x \in U, y \in V$. WLOG, assume that $x < y$. Define $A := \{t \in \mathbb{R} : [x, t] \subseteq U\}$. Clearly $x \in A \implies A \neq \varnothing$. Further, $t \in A \implies t \leq y$. This means $s := \sup A \in \mathbb{R}$. Clearly, $s \in U$ or $s \in V$.
   Now, if $s \in U$ then $\exists r > 0$ such that $\mathscr{B}_r(s) \subseteq U$ whence $s + \frac{r}{2} \in U \implies s$ is not an upper bound of U.
   Similarly, if $s \in V$ then $\exists r > 0$ such that $\mathscr{B}_r(s) \subseteq V \implies \left(s - \frac{r}{2}, s\right) \cap U \cap V \neq \varnothing$.

3. Prove that every closed set in $\mathbb{R}$ is the intersection of a countable collection of open sets.

   Let $F \subseteq \mathbb{R}$ be closed. Then $U_n := \bigcup\limits_{x \in F} \mathscr{B}_{\frac{1}{n}}(x)$ is open $\forall n$. Now define $U := \bigcap\limits_{n \in \mathbb{N}} U_n$. Clearly, $F \subseteq U$. Let $a \in U$. This just means that $\exists$ a sequence (x_n) in F such that $a \in \mathscr{B}_{\frac{1}{n}}(x_n) \forall n \in \mathbb{N}$. Hence $x_n \in \mathscr{B}_{\frac{1}{n}}(a) \forall n$. In other words, $\lim\limits_{n \to \infty} a_n = a$. Now $F = \overline{F} \implies a \in F$.

4. Let $U, V \subseteq \mathbb{R}$. Show that $(U \cap V)^o = U^o \cap V^o, (U \cup V)^o \supseteq U^o \cup V^o$ and $(U \cup V)' = U' \cup V'$.

   $U \cap V \subseteq U \implies (U \cap V)^o \subseteq U^o$. Similarly $(U \cap V)^o \subseteq V^o$. $\therefore (U \cap V)^o \subseteq U^o \cap V^o$.
   Next note that $U^o \cap V^o$ is open. But $U^o \subseteq U, V^o \subseteq V \implies U^o \cap V^o \cap U \cap V \implies U^o \cap V^o \subseteq (U \cap V)^o$.

   Again, $U^o \cup V^o$ is open and $U^o \subseteq U, V^o \subseteq V \implies U^o \cup V^o \cap U \cup V \implies U^o \cup V^o \subseteq (U \cup V)^o$.

   Let $x \in U$. $\exists (x_n) \in U^{\mathbb{N}}$ such that $x_n \neq x \forall x, \lim x_n = x$. Then $(x_n) \in (U \cup V)^{\mathbb{N}}$ whence $x \in (U \cup V)'$. This just means $U' \subseteq (U \cup V)'$. Similarly $V' \subseteq (U \cup V)'$. So $U' \cup V' \subseteq (U \cup V)'$.
   Now say $x \in (U \cup V)'$. $\exists X = (x_n) \in (U \cup V)^{\mathbb{N}}$ such that $x_n \neq x \forall n, \lim x_n = x$. Now, infinitely many terms of X lie in either U or V (say, U). The subsequence obtained by deleting the terms of X not in U converges to x, and none of its terms equals x. Hence, $x \in U' \cup V'$. We thus have $(U \cup V)' \subseteq U' \cup V'$.

5. Show that S' is closed for any $S \subseteq \mathbb{R}$.

   Let $x \in (S')'$. Let $\varepsilon > 0$ be arbitrary. Then $\exists y \in \mathscr{B}_{\frac{\varepsilon}{2}}(x) \cap S' \setminus \{x\}$. Again, $\exists z \in \mathscr{B}_{\frac{\varepsilon}{2}}(y) \cap S \setminus \{x, y\}$. Which means $0 < |z - x| \leq |z - y| + |y - x| < \varepsilon$. That is $\mathscr{B}_\varepsilon(x) \cap S \setminus \{x\} \neq \varnothing$. This means $x \in S'$, i.e., $(S')' \subseteq S'$.

6. Let $S \subseteq \mathbb{R}$ be a bounded set containing infinitely many points.

   (a) Show that there must be reals $a, b \in \mathbb{R}$ such that $S \subseteq [a, b]$.

(b) Show that we can find an increasing sequence (a_n) and a decreasing sequence (b_n) such that

- $a \le a_1 \le b_1 \le b$
- $b_n - a_n = \dfrac{b-a}{2^n} \ \forall n$
- $[a_n, b_n] \cap S$ is an infinite set $\forall n$.

(c) Show that $\sup a_n = \inf b_n$. Call this l.

(d) Conclude that S has a limit point. (**Hint:** l will be a limit point of S).

Choose $a = \inf S - 1 \in \mathbb{R}, b = \sup S + 1 \in \mathbb{R}$ so that $S \subseteq (a,b) \subseteq [a,b]$. Let $a_0 := a, b_0 := b$. Look at $m_0 := \frac{a_0+b_0}{2}$. S is infinite, so either $S \cap [a_0, m_0]$ or $S \cap [m_0, b_0]$ is infinite. In the former case, take $a_1 := a_0, b_1 := m_0$, otherwise take $a_1 := m_0, b_1 := b_0$. Again, either $S \cap [a_1, m_1]$ or $S \cap [m_1, b_1]$ is infinite, where $m_1 := \frac{a_1+b_1}{2}$. Pick a_2, b_2 accordingly. Continue this way, to get sequences $(a_n), (b_n)$. By induction, it is clear that $(a_n) \uparrow$ and $(b_n) \downarrow$. By construction, we say $b_n - a_n = \frac{b-a}{2^n}$ and $S \cap [a_n, b_n]$ is infinite $\forall n$.
Now, each of these two sequences is clearly bounded (and monotone), thus convergent. In fact, $\lim a_n = \sup a_n, \lim b_n = \inf b_n$. Further, $\lim(b_n - a_n) = \lim \frac{b-a}{2^n} = 0 \implies \sup a_n = \lim a_n = \lim b_n = \inf b_n$.
Let $l := \lim a_n$. Note that $l \in S \cap (a_n, b_n) \forall n$. Further $S \cap (a_n, b_n)$ is infinite, whence $S \cap (a_n, b_n) \setminus \{l\}$ is also infinite. Now, just pick $x_n \in S \cap (a_n, b_n) \setminus \{l\}$ so that $|x_n - l| \le |b_n - a_n| = \frac{b-a}{2^n}$. This means (x_n) is a sequence in $S \setminus \{l\}$ such that $\lim x_n = l$.

7. Let $S \subseteq [a,b]$ be a set with no limit point.

(a) Let $x \in [a,b]$. Show that $\exists$ an open set $U_x \subseteq \mathbb{R}$ such that $x \in U_x$ and $U_x \cap S \subseteq \{x\}$.

(b) Conclude that S is finite. (**Hint:** Compactness of closed intervals).

$x \in I := [a,b] \implies x \notin S'$. $\exists$ an open ball $\mathscr{B} \subseteq \mathbb{R}$ around x such that $\mathscr{B} \cap S$ is finite, whence $\mathscr{B}_{r_x}(x) \cap S \subseteq \{x\}$ for some $r_x > 0$. Let $U_x := \mathscr{B}_{r_x}(x)$. Now, $\mathscr{U} := \{U_x\}_{x \in I}$ is an open cover for $[a,b]$. Pick a finite subcover $\left\{U_{x_i}\right\}_{i=1}^{n}$. Then, $S = S \cap [a,b] \subseteq S \cap \left(\bigcup_{i=1}^{n} U_{x_i}\right) \subseteq \bigcup_{i=1}^{n} (S \cap U_{x_i}) \subseteq \bigcup_{i=1}^{n} \{x_i\}$. So S is finite.

8. Let $S \subseteq [a,b]$ be an infinite set.

(a) Prove that there is a sequence in $[a,b]$, all of whose terms are in S with no repeated terms.

(b) Show that the above sequence has a limit point $l \in [a,b]$.

(c) Conclude that S has a limit point. (**Hint:** l will be a limit point of S).

We can define a sequence $X = (x_n)$ inductively. Take $x_1 \in S$ arbitrarily. Whenever we have picked up distinct $x_1, \cdots, x_n \in S$, we know $F_n := S \setminus \{x_1, \cdots, x_n\}$ is infinite, so take any $x_{n+1} \in F_n$. By construction, $x_i \neq x_j$ whenever $i \neq j$. By Bolzano-Weierstraß theorem, there is a convergent subsequence $\left(x_{n_k}\right)_{k \in \mathbb{N}}$. Let $l := \lim_{k \to \infty} x_{n_k} \in [a,b]$. Now, for any $\varepsilon > 0, \exists K \in \mathbb{N}$ such that $|l - x_k| < \varepsilon \forall k \ge K$ whence $\left\{x_{n_k} : k \ge K\right\} \subseteq \mathscr{B}_\varepsilon(l) \cap S$. Uniqueness of all terms of X guarantees that $\mathscr{B}_\varepsilon(l) \cap S$ is inifinite.

9. $S \subseteq \mathbb{R}$ is a bounded infinite set. Let $T := \{x \in \mathbb{R} : \text{there are infinitely many points in } S \text{ more than } x\}$.

(a) Show that $T \neq \varnothing$ and T is bounded above. Let $s := \sup T$. Clearly $s \in \mathbb{R}$.

(b) Let $a \in \mathbb{R} \setminus T$. Show that a is an upper bound of T.

(c) Show that s is a limit point of S.

$\exists u < v \in \mathbb{R}$ such that $S \subseteq [u, v]$. Clearly $u \in T$ so $T \neq \varnothing$. Also, $x \in T \implies \exists y \in S$ such that $y > x$. This means $v + 1$ is an upper bound of T. Now $s = \sup T \in \mathbb{R}$. Suppose a is not an upper bound of T. So $\exists x \in T \cap (a, \infty)$. By definition, $S \cap (x, \infty)$ is infinite $\implies$ $S \cap (a, \infty) (\supseteq S \cap (x, \infty))$ is infinite $\implies$ $a \in T$. We thus have $(-\infty, s) \subseteq T$. Let $r > 0$. Then $s - r \in T$. By definition, $S \cap (s - r, \infty)$ is infinte. Further, $s + \frac{r}{2} > s \implies s + \frac{r}{2} \in T^c \implies S \cap \left(s + \frac{r}{2}, \infty\right)$ is finite $\implies S \cap \mathscr{B}_r(s)$ is infinte. By definition, $s \in S'$.

10. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a decreasing (under containment) sequence of compact sets of $\mathbb{R}$. Suppose $\bigcap_{n \in \mathbb{N}} \mathfrak{C}_n = \varnothing$.

    (a) Show that $\mathscr{U} := \{\mathbb{R} \setminus \mathfrak{C}_n : n \in \mathbb{N}\}$ is an open cover of $\mathfrak{C}_1$.

    (b) Show that $\exists K \in \mathbb{N}$ such that $k \geq K \implies \mathfrak{C}_k = \varnothing$.

    $\bigcup_n \mathfrak{C}_n^c = \left(\bigcap_n \mathfrak{C}_n\right)^c = \mathbb{R} \supseteq \mathfrak{C}_1$. $\because \mathfrak{C}_1$ is compact, there is a finite subcover $\left\{\mathfrak{C}_n^c\right\}_{n \in A}$ where $A \subseteq \mathbb{N}$ is finite. Let $K \in A$ be largest. So $\mathfrak{C}_1 \subseteq \bigcup_{n \in A} \mathfrak{C}_n^c = \left(\bigcap_{n \in A} \mathfrak{C}_n\right)^c = \mathfrak{C}_K^c \implies \mathfrak{C}_K = \mathfrak{C}_1 \cap \mathfrak{C}_K = \varnothing \implies \mathfrak{C}_k = \varnothing \forall k \geq K$.

11. For a bounded set $S \subseteq \mathbb{R}$ define $\operatorname{diam} S := \sup_{x,y \in S} |x - y|$. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a decreasing sequence of nonempty compact sets of $\mathbb{R}$ such that $\lim_{n \to 0} (\operatorname{diam} \mathfrak{C}_n) = 0$. Show that $\bigcap_{n \in \mathbb{N}} \mathfrak{C}_n$ is a singleton.

    By the contrapositive of problem 10, conclude that $\mathfrak{C} := \bigcap_{n \in \mathbb{N}} \mathfrak{C}_n \neq \varnothing$. Let $x, y \in \mathfrak{C}$. So $x, y \in \mathfrak{C}_n \forall n$. $\forall r > 0, \exists n \in \mathbb{N}$ such that $\operatorname{diam} \mathfrak{C}_n < r$ whence $|x - y| < r$. $\therefore |x - y| = 0$ so that $x = y$, i.e., $\mathfrak{C} = \{x\}$.

12. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be a sequence of closed subsets of compact $\mathfrak{C} \subseteq \mathbb{R}$ such that $\bigcap_{i \in A} \mathfrak{C}_i \neq \varnothing$ for any finite $A \subseteq \mathbb{N}$. Show $\bigcap_{n \in \mathbb{N}} \mathfrak{C}_n \neq \varnothing$. (**Hint:** Use a similar construction as in problem 10).

    Let $\mathfrak{F}_i := \mathfrak{C}_i^c$. Each $\mathfrak{F}_i$ is open. Suppose $\bigcap_{n \in \mathbb{N}} \mathfrak{C}_n = \varnothing$. Then $\bigcup_{n \in \mathbb{N}} \mathfrak{F}_n \supseteq \mathfrak{C}$. Let $\{\mathfrak{F}_i : i \in A\}$ be a finite subcover of $\mathfrak{C}$, for some finite $A \subseteq \mathbb{N}$. Then $\mathfrak{C} \subseteq \bigcup_{i \in A}^{n} \mathfrak{F}_i \implies \bigcap_{i \in A} \mathfrak{C}_i = \varnothing$. This contradicts our hypothesis.

13. For $S \subseteq \mathbb{R}$, show that $\mathbb{R} \setminus (\overline{S}) = (\mathbb{R} \setminus S)^o$.

    We want to show $\left(\overline{S}\right)^c = (S^c)^o$. Let $x \in (S^c)^o$. $\exists r > 0$ such that $\mathscr{B}_r(x) \subseteq S^c$. Assume $\exists (x_n) \in S^{\mathbb{N}}$ such that $\lim x_n = x$ (i.e., assume $x \in \overline{S}$). $\exists N \in \mathbb{N}$ such that $x_N \in \mathscr{B}_{\frac{r}{2}}(x) \subseteq S^c \implies x_N \notin S$, a contradiction. So, $x \in \left(\overline{S}\right)^c$. Hence $(S^c)^o \subseteq \left(\overline{S}\right)^c$. But $S \subseteq \overline{S} \implies \left(\overline{S}\right)^c \subseteq S^c \overset{\because (\overline{S})^c \text{ open}}{\implies} \left(\overline{S}\right)^c \subseteq (S^c)^o$. Conclude $\left(\overline{S}\right)^c = (S^c)^o$.

14. (Something from sequences and series) Let (a_n) be a sequence of real numbers converging to a. Define a sequence (b_n) by $b_n := \dfrac{\sum_{i=1}^n i \cdot a_i}{n(n+1)}$. Prove that $\lim_{n \to \infty} b_n = \dfrac{a}{2}$.

    We equivalently $c_n := \dfrac{\sum_{i=1}^n i \cdot a_i}{n(n+1)/2} = \dfrac{\sum_{i=1}^n i \cdot a_i}{\sum_{i=1}^n i} \overset{n \to \infty}{\to} a$. Fix $\varepsilon > 0$. Let N be such that $|a_i - a| < \frac{\varepsilon}{2} \forall i \geq N$. For large n, $|c_n - a| = \dfrac{\sum_{i=1}^n i \cdot |a_i - a|}{\sum_{i=1}^n i} = \dfrac{\sum_{i=1}^N i \cdot |a_i - a| + \sum_{i=N+1}^n i \cdot |a_i - a|}{n(n+1)/2} < \dfrac{\sum_{i=1}^N i \cdot |a_i - a|}{n(n+1)/2} + \dfrac{\sum_{i=N+1}^n i \cdot |a_i - a|}{n(n+1)/2} < \dfrac{\sum_{i=1}^N i \cdot |a_i - a|}{n(n+1)/2} + \dfrac{\varepsilon}{2}$. We can always choose $M (> N)$ for which $\dfrac{\sum_{i=1}^N i \cdot |a_i - a|}{n(n+1)/2} < \dfrac{\varepsilon}{2} \forall n \geq M$. This proves that $|c_n - a| < \varepsilon \forall n \geq M$.