Limit point

Let $S \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is a limit point of S if $\forall\, \varepsilon > 0$ $\exists$ $y \in S \setminus \{x\}$ s.t. $|y - x| < \varepsilon$

(in other words: $x \in \mathbb{R}$ is a limit pt of S if $\forall\, \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap (S \setminus \{x\}) \neq \phi$).

From now on we will denote $B(x, r) := (x-r, x+r) \subseteq \mathbb{R}$
$= \{y \in \mathbb{R} : |y - x| < r\}$.

$+\infty \in \overline{\mathbb{R}}$ is said to be a lim point of $S \subseteq \mathbb{R}$ if $\forall\, k \in \mathbb{R}$ $\exists$ $y \in S$ s.t. $y > k$

(in other words, $\underbrace{(k, \infty)}_{\{x \in \mathbb{R} : x > k\}} \cap S \neq \phi$ $\forall\, k \in \mathbb{R}$)

We can similarly define "$-\infty$ is a lim pt. of $S \subseteq \mathbb{R}$"

Let $X = (x_n)$ be a seq of real numbers. We say $x \in \overline{\mathbb{R}}$ is a limit point of X if $\exists$ a subseq $(x_{n_k})_{k \in \mathbb{N}}$ of X s.t. $\lim\limits_{k \to \infty} x_{n_k} = x$.

The set of limit pts of any seq is always non-empty (in $\overline{\mathbb{R}}$).

Motivation : $X = (x_n)$ a real seq. $x \in \mathbb{R}$ is a lim pt of X if $\forall\, \varepsilon > 0$, $\exists$ inf many $n \in \mathbb{N}$ s.t. $x_n \in (x - \varepsilon, x + \varepsilon)$.

$$\Updownarrow$$

$\exists$ a subseq $\{x_{n_k}\}_{k \in \mathbb{N}}$ s.t. $\lim\limits_{k \to \infty} x_{n_k} = x$.

$+\infty$: if $\forall\, a \in \mathbb{R}$ $\exists$ $n \in \mathbb{N}$ s.t. $x_n > a$.

$\Updownarrow$

$\exists$ a subseq $\{x_{n_k}\}$ s.t. $x_{n_k} \xrightarrow{k \to \infty} \infty$.

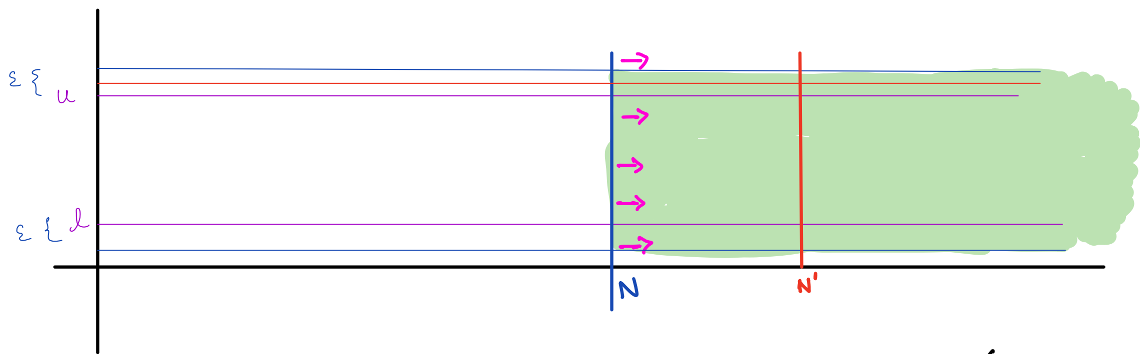
Limsup and Liminf

Let $X=(x_n)$ be a real valued sequence. Let X' be the set of limit points of the sequence X, in $\overline{\mathbb{R}}$.

Define

$$\left.\begin{aligned} \overline{\lim}\ x_n &= \limsup_{n\to\infty} x_n := \sup X' \\ \underline{\lim}\ x_n &= \liminf_{n\to\infty} x_n := \inf X' \end{aligned}\right\} \text{ in } \overline{\mathbb{R}}$$

- Clearly limsup & liminf exist $\because X' \neq \phi$.
- $+\infty \geq \limsup x_n \geq \liminf x_n \geq -\infty$
- Say $u = \limsup x_n$, $l = \liminf x_n \in \mathbb{R}$. $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. $x_n \in (l-\varepsilon, u+\varepsilon)$ $\forall\ n \geq N$.



$$\begin{cases} u = \inf\limits_{N} \sup\limits_{n\geq N} \{x_n\} \\ l = \sup\limits_{N} \inf\limits_{n\geq N} \{x_n\} \end{cases}$$

True even if $u, l \in \overline{\mathbb{R}}$.

Motivation: (Assume all limits finite; seq bdd)

$$\inf_N \sup_{n\geq N}\{x_n\} = \lim_{N\to\infty}\left(\sup_{n\geq N}\{x_n\}\right)$$

- Finally:

$u = \limsup x_n \iff$ $\forall\ \varepsilon>0$:
  - $\to u-\varepsilon < x_n$ for infinitely many n.
  - $\to x_n < u+\varepsilon$ $\forall\ n \geq N$ for some N.

$l = \liminf x_n \iff$ $\forall\ \varepsilon>0$:
  - $\to x_n < l+\varepsilon$ for infinitely many n.
  - $\to x_n > l-\varepsilon$ $\forall\ n \geq N$ for some N.

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$$M_n = \begin{bmatrix} r & s & s & \cdots & s \\ s & r & s & \cdots & s \\ s & s & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & s \\ s & s & \cdots & s & r \end{bmatrix}. \qquad \det M_n = ?$$

$$\det M_n = (r+(n-1)s) \begin{vmatrix} 1 & s & s & s & s & \cdots & s \\ 1 & r & s & s & \cdots & & s \\ 1 & s & r & s & \cdots & & s \\ \vdots & & & & & & \\ 1 & s & \cdots & & & s & r \end{vmatrix}$$

$$= (r+(n-1)s) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & r-s & 0 & & \vdots & \vdots \\ 1 & 0 & r-s & & \vdots & \vdots \\ \vdots & 0 & 0 & & & \vdots \\ \vdots & \vdots & \vdots & & 0 & \vdots \\ & & \cdots & 0 & r-s & 0 \\ 1 & 0 & 0 & \cdots & 0 & r-s \end{vmatrix}$$

$$= (r+(n-1)s) \begin{vmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & r-s & 0 & 0 & \cdots & 0 \\ 0 & 0 & r-s & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \\ & & & & & 0 \\ 0 & 0 & \cdots & & 0 & r-s \end{vmatrix}$$

$$= (r+(n-1)s)\,(r-s)^{n-1}$$