Real Analysis

Problem Set 5

June 10, 2021

1. Show that the $\sum x_n \notin \mathbb{R}$ where x_n are as follows:

   (a) $x_n = 1$ (b) $x_n = (-1)^n$

   Does $\sum x_n \in \overline{R}$ in the each of the above?

2. Let $x_n \geq 0$ such that $\sum x_n \in \mathbb{R}$. If $m \geq 1$, show that $\sum (x_n)^m \in \mathbb{R}$. What happens if $m \in (0, 1)$?

3. Let $f \in \mathbb{R}^{\mathbb{N}}$ be a sequence. Let $i, j \in \mathbb{N}^{\mathbb{N}}$ be strictly increasing functions such that $i(\mathbb{N}) \bigcup j(\mathbb{N}) = \mathbb{N}$ and $i(\mathbb{N}) \bigcap j(\mathbb{N}) = \emptyset$. Now look at the subsequences $\alpha := f \circ i, \beta := f \circ j \in \mathbb{R}^{\mathbb{N}}$. In simpler terms, we are looking at two disjoint subsequences which exhaust the original sequence. If α, β converge to the same limit $L \in \mathbb{R}$, show that f converges to $L \in \mathbb{R}$.

4. Let $x_n \geq 0$ be a decreasing sequence such that $\lim x_n = 0$. Consider $s_n = \sum_{i=1}^{n} (-1)^n x_i$.

   (a) Show that each subsequence $\{s_{2n}\}$ and $\{s_{2n-1}\}$ is monotonic and bounded.

   (b) Show that $\lim s_{2n} = \lim s_{2n-1}$. Call this limit L.

   (c) Conclude that $\lim s_n = L$.

5. Show that $\sum \frac{(-1)^n}{n} \in \mathbb{R}$.