1) Linear Maps and Basis :-

- Suppose $W \subset V$, $\dim V = n$. $\dim W < n$

a) $T : V \to W$ be F-linear map. Then T is injective iff for any basis B of V, $T(B)$ is linearly independent.

b) $T : V \to W$ be F-linear map. Then T is surjective iff for any basis B of V, $T(B)$ is spanning.

$$T : V \to W$$

$$T(0) = 0$$

$$\underset{T(0)}{\overset{\shortparallel}{T(0+0)}} = T(0) + T(0) \implies T(0) = T(0) + T(0) \implies T(0) = 0$$

$$0 \in \ker T$$

a) T is injective. To prove that for <u>any</u> basis B, $T(B)$ is linearly independent.

B be a basis for V

Pick any $\{T(b_1), \dots, T(b_n)\} \subseteq T(B)$

$$\alpha_1 T(b_1) + \dots + \alpha_n T(b_n) = 0 \qquad \alpha_i \in F$$

$$\implies T(\alpha_1 b_1 + \dots + \alpha_n b_n) = 0 = T(0)$$

$$\implies \alpha_1 b_1 + \dots + \alpha_n b_n = 0$$

$\{b_1, \dots, b_n\} \subseteq B$

$$\implies \alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_n = 0$$

So $\{T(b_1), \dots, T(b_n)\}$ is L.I.

Let for any basis B, $T(B)$ be L.I.

To prove that T is injective.

$\boxed{T \text{ is injective iff } \ker T = \{0\}}$

$v \neq 0$, $v \in \ker T$ $\quad T(v) = 0 = T(0)$

$\Rightarrow v = 0$

$T(a) = T(b) \Rightarrow T(a-b) = 0$

$\Rightarrow a-b \in \ker T \Rightarrow a-b=0$

$\Rightarrow a = b$

* To prove $\ker T = \{0\}$

$v \in \ker T \Rightarrow T(v) = 0$

Fix a basis B, then $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$

$T(v) = 0 \Rightarrow \alpha_1 T(b_1) + \cdots + \alpha_n T(b_n) = 0$

$T(B)$ is L.I. $\Rightarrow \{T(b_1), \ldots, T(b_n)\} \subseteq T(B)$ is L.I.

$\Rightarrow \boxed{\alpha_i = 0}$

$\Rightarrow v = 0 \qquad \ker T = \{0\}$

2) Do it yourself.

T injective	iff	sends	Basis	to	L.I
T surjective	"	"	"	to	Spanning
T bijective	"	"	"	"	Basis

$B \qquad T(B) \to$ L.I

$\qquad\quad T(B) \to$ Spanning

V

$T: V \to W$

$T(v)$ for every $v \in V$

$B \subset V$

$b_1 \longrightarrow$

$b_2 \longrightarrow$

V , $\{b_1, \dots, b_n\}$

$V \longrightarrow \mathbb{R}^n$

$v \in V$

$T(b_1)$

$T(b_2) \quad \dots \quad T(b_n)$

$v \in V$, $v = \alpha_1 b_1 + \dots + \alpha_n b_n$ $\quad T(v) = \alpha_1 T(b_1) + \dots + \alpha_n T(b_n)$

V, W

$\downarrow \quad \downarrow$

$r \quad n$

$\{b_1, \dots, b_n\} \subset V$

$\{c_1, \dots, c_n\} \subset W$

$b_1 \longrightarrow c_1$

$\vdots$

$b_n \longrightarrow c_n$

__H.W:-__

$T: V \to W$, F-linear injective map. Let V be finite dimensional, and W is also finite dimensional, then

$$\dim V \leq \dim W$$

$T: V \to W$

$T: V \cong T(V) \subset W$

$\boxed{V \overset{S}{\cong} U \text{ then } \dim V = \dim U}$

Isomorphism Theorems :-

1) First Isomorphism :-

$T: V \to W$. Then $V/\ker T \cong T(V)$

$$V/\ker T := \{v + \ker T \mid v \in V\}$$

$$\phi : V/\ker T \longrightarrow T(v)$$

$$\phi(v + \ker T) = T(v)$$

a) Well defined ness :-

Let $v + \ker T = u + \ker T$

$\Rightarrow v - u \in \ker T$

$\Rightarrow T(v-u) = 0 \Rightarrow T(v) = T(u)$

$\Rightarrow \phi(v + \ker T) = \phi(u + \ker T)$

b) Linear Map :-

$$\phi((v + \ker T) + (u + \ker T))$$

$$= \phi((v+u) + \ker T) = T(v+u) = \underbrace{T(v)}_{\phi(v+\ker T)} + \underbrace{T(u)}_{\phi(u+\ker T)}$$

$$\phi(\alpha(v + \ker T))$$

$$= \phi(\alpha v + \ker T) = T(\alpha v) = \alpha T(v)$$

$$= \alpha \phi(v + \ker T)$$

c) Injective :-

$$\phi(v + \ker T) = \phi(u + \ker T)$$

$\Rightarrow T(v) = T(u) \Rightarrow T(v-u) = 0 \Rightarrow v - u \in \ker T$

$\Rightarrow v + \ker T = u + \ker T$

d) Surjective :-

$$T(v) \in \operatorname{Im} T$$

$$\phi(v + \ker T) = T(v)$$

Done!

$$\begin{tikzcd}
V \arrow[rr, "T"] \arrow[dr, "\pi"'] & & T(V) \\
& V/\ker T \arrow[ur, "\phi", "\sim"'] &
\end{tikzcd}$$

$T = \phi \circ \pi$

<u>H.W</u> :- Every subspace W of V is kernel of some linear map!