(a) $a_0 = 1$
$a_{n+1} = 4a_n + 1$

Claim: $a_n = \dfrac{4^{n+1} - 1}{3} \quad \forall n \geq 0$

Base case clear.

Ind step.

Hyp — assume $a_k = \dfrac{4^{k+1}-1}{3}$ for some $k \geq 0$

So,

$$a_{k+1} = 4a_k + 1$$
$$= \frac{4}{3}(4^{k+1} - 1) + 1$$
$$= \frac{4^{k+2} - 4 + 3}{3}$$
$$= \frac{4^{k+2} - 1}{3}$$

Conclude by ind: true $\forall k$.

$$a_{n+1} = 4a_n + 1$$
$$= 4(4a_{n-1} + 1) + 1$$
$$= 4^2 a_{n-1} + 4 + 1$$
$$= 4^3 a_{n-2} + 4^2 + 4 + 1$$
$$\vdots$$
$$= 4^{n+1} a_0 + 4^n + 4^{n-1} + \dots + 1$$
$$= 4^{n+1} + 4^n + \dots + 1$$
$$= \frac{4^{n+2} - 1}{3}$$

$$a_n = \frac{4^{n+1} - 1}{3}$$

(b) $a_0 = 1$, $a_1 = 2$ $\qquad a_{n+1} = a_{n-1} + 2a_n \quad (n \geq 1)$

Store data in the form of vectors : $\underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{v_0}, \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{v_1}, \dots$

Similarity of matrices :
$A \sim B$ if $\exists P$ s.t. $A = PBP^{-1}$

Express v_{n+1} as the image of a linear map acting on v_n.
(this lin map should not depend on n)

See what you can do!

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix}$$

$$v_n = Av_{n-1} = A^2 v_{n-2} = \dots = A^n v_0$$

Conseq from Alg I :
A symmetric $\Rightarrow A \sim D$ (D is diag)

A is symmetric
$\Rightarrow A = PDP^{-1}$ for some P, diagonal D.

$$A = P D P^{-1}$$

$$D = \begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix}$$

$$P = \begin{bmatrix} -\sqrt{2}-1 & \sqrt{2}-1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \frac{1}{-2\sqrt{2}} \begin{bmatrix} 1 & 1-\sqrt{2} \\ -1 & -1-\sqrt{2} \end{bmatrix}$$

(matrix mult is assoc)

$$A^n = [PDP^{-1}]^n = PDP^{-1}\,PDP^{-1} \cdots PDP^{-1}$$

$$= P D^n P^{-1}$$

$$v_n = P D^n P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = P \begin{bmatrix} (1-\sqrt{2})^n & 0 \\ 0 & (1+\sqrt{2})^n \end{bmatrix} P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Another possible clever attempt:

$$A = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{C}$$

$$A^n = (B+C)^n$$

because B^k is diagonal and C^k is easy (nice form) and using binomial thm.

This fails. Why? — because $BC \neq CB$

(d) 1 variable: trivial

Assume true for n variables. (IH)

Let $a_1, \ldots, a_{n+1}$ (in descending order) be non-neg.

$\bar{a} = \frac{1}{n+1}\left(\sum a_i\right)$.

If $\bar{a} = a_i \ \forall i$ we are done
Suppose not. Then $\bar{a} < a_1$ and $a_{n+1} < \bar{a}$

Name a new quantity $y = a_1 + a_{n+1} - \bar{a}$ (≥ 0)

Average of $a_2, a_3, \ldots, a_n, y$ $=$ $\bar{a}$
(AM)

By IH,

$$a_2 \cdots a_n \times y \leq (\bar{a})^n$$

$$\Rightarrow a_2 \cdots a_n \times y\bar{a} \leq (\bar{a})^{n+1}$$

But $y\bar{a} = (a_{n+1} + a_1 - \bar{a})\,\bar{a}$

$$= (\bar{a} - a_{n+1})(a_1 - \bar{a}) + a_1 a_{n+1}$$

$$> 0 + a_1 a_{n+1}$$

$$\Rightarrow a_2 \cdots a_n \times a_1 a_{n+1} \leq (\bar{a})^{n+1}$$

$$\Rightarrow \bar{a} \geq \left[a_1 \cdots a_n a_{n+1}\right]^{\frac{1}{n+1}}$$

This completes the ind step.