Today's Goal :-

i> Cayley Hamilton Proof

ii> Finishing the criterion of Diagonalizability

i> A Proof of Cayley Hamilton Theorem :-

We do this in several steps.

i> Given a matrix A, its char.poly annihilates it.
$\hookrightarrow n\times n$

$$p(x) = \det(A - xI) = \det(C)$$

where $C = A - xI$

$$C^a C = (\det C)\cdot I$$

adjoint matrix

$$C^a = C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0$$

constant Matrix of dim. $n\times n$

$$\det C = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \checkmark$$

$$(\det C) I = I\cdot x^n + (a_{n-1}\cdot I)\, x^{n-1} + \cdots + (a_1 I)\, x + a_0$$

$$C^a C = (C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0)\, C$$

$T: V \to V$

its characteristic polynomial annihilates

$A \qquad \det(xI - A) = p(x)$

$T: V \to V$

$A \quad B$

$M_B^B(T) = C \longrightarrow$

$B = P^{-1}AP$

$M_{B'}^{B'}(T) = C' \longrightarrow$

$T: V \to V \qquad q(x) = 3x^2 + 2x + 1$

$q(x) \cdot T = 3T^2 + 2T + 1 : V \to V$

$T^2 = T \circ T \qquad (2 \cdot T)(v) = 2 \cdot (T(v))$

$\text{char}_T(x) \cdot T = 0 : V \to V$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix} \qquad C^a := \begin{bmatrix} (x-3)(x+1) & 4 & 2(x+1) \\ -2(x-3) & (x-3)(x-1)-8 & -4 \\ 4(x+1)-4 & 2(x-1) & (x+1)(x-1) \end{bmatrix}^T$$

$$C = \begin{bmatrix} 1-x & -2 & 4 \\ 0 & -1-x & 2 \\ 2 & 0 & 3-x \end{bmatrix}$$

$(x-3)(x+1)$

$$= \begin{bmatrix} x^2-2x-3 & 6-2x & 4x \\ 4 & x^2-4x-5 & 2x-2 \\ 2x+2 & -4 & x^2-1 \end{bmatrix}$$

$$C^a C = I\cdot x^n + (a_{n-1}\cdot I)\cdot x^{n-1} + \cdots + (a_1 I)\,x + a_0$$

$$\Rightarrow C^a(A - xI) = I\cdot x^n + (a_{n-1}\cdot I)\cdot x^{n-1} + \cdots + (a_1 I)\,x + a_0$$

$$\Rightarrow C^a A - xC^a = I\cdot x^n + (a_{n-1}\cdot I)\cdot x^{n-1} + \cdots + (a_1 I)\,x + a_0$$

$$\Rightarrow (C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0)\cdot A - xC^a$$
$$= I\cdot x^n + (a_{n-1}\cdot I)\cdot x^{n-1} + \cdots + (a_1 I)\,x + a_0 I$$

$$\Rightarrow (C_{n-1}A)x^{n-1} + (C_{n-2}A)\,x^{n-2} + \cdots + (C_1 A)x + C_0 A$$
$$- x(C_{n-1}x^{n-1} + \cdots + C_1 x + C_0)$$

$$= -C_{n-1}x^n + (C_{n-1}A - C_{n-2})\,x^{n-1} + \cdots + (C_1 A - C_0)\,x + C_0 A$$

$$= I\cdot x^n + (a_{n-1}\cdot I)\,x^{n-1} + \cdots + (a_1 I)\,x + a_0 I$$

Compare coefficients

$$-C_{n-1} = I \longrightarrow A^n$$
$$C_{n-1}A - C_{n-2} = a_{n-1} I \longrightarrow A^{n-1}$$
$$C_{n-2}A - C_{n-3} = a_{n-2} I \longrightarrow A^{n-2}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$C_1 A - C_0 = a_1 I \longrightarrow A$$
$$C_0 A = a_0 I \longrightarrow A^0 = I$$

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I$$
$$= 0$$

(proved)

$$C^a C = (x^3 - 3x^2 - 9x + 3) \cdot I = \begin{bmatrix} x^3 - 3x^2 - 9x + 3 & 0 & 0 \\ 0 & x^3 - 3x^2 + 9x + 3 & 0 \\ 0 & 0 & x^3 - 3x^2 + 9x + 3 \end{bmatrix}$$

$$\begin{bmatrix} x^2 - 2x - 3 & 6 - 2x & 4x \\ 4 & x^2 - 4x - 5 & 2x - 2 \\ 2x + 2 & -4 & x^2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x^2$$

$$+ \begin{bmatrix} -2 & -2 & 4 \\ 0 & -4 & 2 \\ 2 & 0 & 0 \end{bmatrix} x$$

$$+ \begin{bmatrix} -3 & 6 & 0 \\ 4 & -5 & -2 \\ 2 & -4 & -1 \end{bmatrix}$$

$\mathbb{R}[x]$

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$\mathrm{char}_T(x)$

$= (x-3)\left[(x-1)^2+4\right] \rightarrow = (x-3)\underline{(x^2-2x+5)}$

$\min_T(x) = (x-3)(x^2-2x+5)$

$(x^2-2x+5) = (x-\alpha)(x-\beta)$

$\Rightarrow \alpha, \beta$ is root of x^2-2x+5

$\Rightarrow \alpha, \beta \notin \mathbb{R}$ $(\rightarrow\leftarrow)$

$\mathrm{char}_T(x) = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ $\qquad p_1^2 p_2^3 \underline{\underline{p_3^5}}$

$\min_T(x) = p_1^{\beta_1} \cdots p_k^{\beta_k}$ $\qquad \min \mid \mathrm{char}$ $\qquad p_1^2 p_2$

$\beta_1, \beta_2, \ldots, \beta_k$

ch. $(x-3)^2(x^2-2x+5)^3$

min $(x-3)^2 \; (x^2-2x+5)^{1/2/3}$

ch $\rightarrow$ $(x-1)^2(x^2+x+1)(x^2+3x+5)^3(x^2+5)^5(x-4)$

min $\rightarrow$ $(x-1)$ X

$(x-1)(x^2+3x+5)$ X

$(x-1)^2(x-4)(x^2+x+1)$ X

$(x^2+x+1)(x^2+3x+5)(x^2+5)$ X

$(x-1)(x^2+x+1)(x^2+3x+5)(x^2+5)(x-4)$ ✓

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Theorem :- Let $p(x)$ be a irreducible factor of $char_T(x)$. Then $p(x) \mid min_T(x)$

Defn :- Given a polynomial $p(x) \in F[x]$, it is defined to be "splliting over F" if

$$p(x) = (x-r_1)^{\alpha_1} \cdots (x-r_k)^{\alpha_k}$$

where $r_i \in F$ and $\alpha_1 + \cdots + \alpha_k = \deg(p(x))$

Thm :- $T: V \to V$ a linear map, V is F-vector space. T is diagonalizable iff the minimal poly of T "splits over F and has distinct roots"

$$\left.\begin{aligned} min_T(x) &= (x-\lambda_1) \cdots (x-\lambda_k) \\ \deg(min_T(x)) &= k \end{aligned}\right\}$$