# Problem Set 2

## Linear Algebra

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**1.** Given a field $K$ and a subfield $F$ of $K$, note that a $K-$ Vector Space $V$ is also a $F$- Vector space obtained by restricting the action of $K$ over $V$ to action of $F$.

Having the above idea in mind, is it possible to find a subset of $\mathbb{R}^3$ that is independent over $\mathbb{Q}$ but dependent over $\mathbb{R}$. What about the converse?

(**Remark:** Given a a field $K$ and its subfield $F$ we saw that a vector space on $K$ can be converted to a vector space on $F$. What about the converse? i.e. given a vector space over $F$ can you extend it to a vector space over $K$. This question is quite hard and requires some non-trivial (possibly) construction. I don't compel you to think about it, but if you want you can)

**2.** Let $V$ be a finite dimensional $\mathbb{C}$-Vector Space, with dim $V = n$. Convert it to a real vector space $V^{\text{real}}$ as in problem 1. Is $V^{\text{real}}$ finite dimensional. If so, what is its dimension as $\mathbb{R}$- Vector Space?

**3.** Given a finite dimensional $F$- Vector space $V$ and a strict subspace $W$, prove that $W$ is finite dimensional and dim $W <$ dim $V$.

**4.** A subset $\mathbb{E}$ of $\mathbb{R}^n$ such that $\mid E \mid \geq n$ is defined to be relatively independent if every $n$ element subset of $\mathbb{E}$ is L.I. For example consider three vectors in $\mathbb{R}^2$ such that no two of them are collinear with the origin. This set of 3 vectors is an example of Relatively Independent subset of $\mathbb{R}^2$. Note that one can always find a subset of $\mathbb{R}^n$ consisted of $n+1$ vectors

Find the largest possible number of vectors in a Relatively Independent subset of $\mathbb{R}^n$. State with proof. What about an arbitrary $n-$ dimensional vector space over an arbitrary *infinite* field instead of $\mathbb{R}$?

(**Hint:** Prove that if $\{x_1, ..., x_k\}$ is a relatively independent subset, where $k \geq n$, then there exists $x_{k+1}$ such that $\{x_1, ..., x_k, x_{k+1}\}$ is Relatively Independent)

### 5. Finite Fields:

Here you will learn about basic facts on Finite fields. We could have done it in much simpler way, but that will require First Isomorphism Theorem.

First, note that given a field $K$ with a subfield $F$, we can consider $K$ as a $F-$ Vector Space.

**Definition:** Given a field $F$, consider the additive subgroup pf $F$ generated by $\{1\}$. Order of this additive subgroup is defined to be the *Characteristic* of the field $F$.

In simpler word, if characteristic is $n$, then $n$ is the smallest natural number such that

$$1 + 1 + ... + 1 \text{ (Added } n \text{ times)} = 0$$

**Problem 5.1:** Given a field $F$ prove that its cardinality is either Infinite or a Prime number. In fact if the field $F$ is finite then certainly the cardinality is a Prime number.

**Problem 5.2:** Prove that given any field $K$, the additive subgroup generated by $\{1\}$ is actually a subfield of $K$. When $K$ is of finite characteristic, this subfield is defined as the *Prime Subfield* of $F$.

**Problem 5.3:** Any finite field $K$ contains $\mathbb{F}_p$ as a subfield (up to isomorphism) where $p$ is the characteristic of $K$. Here $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. And by the word "up to isomorphism" we mean that $K$ contains some isomorphic copy of $\mathbb{F}_p$. More explicitly, $K$ contains $F$ as a subfield where $F \cong \mathbb{F}_p$.

[**Remark and Hint:** You might not be familiar with Field Isomorphism. So here is the definition. Given two fields $E$ and $F$, a map $\phi : E \to F$ is a field isomorphism if and only if $\phi$ is a set bijection and $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$.

Also prove that any two fields of prime order $p$ are isomorphic]

**Problem 5.3:** Prove that any finite field $K$ has cardinality $p^q$ where $p$ is a prime number.

**6.** Let $F$ be a field with $q$ elements. How may elements are there in $F^n$? How may bases are there in $F^n$?

**7.** Given two $F-$Vector Subspaces $U$ and $W$ of a Vector Space $V$ prove that $U+W = \{u+w : \forall\, u \in U, w \in W\}$ is also a vector subspace of $V$.

**8.** Given a vector Space $V$ suppose there exists two subspaces $U$ and $W$ such that $U+W = V$ and $U \cap W = \{0\}$. Then $W$ is defined to be the *"Algebraic Complement"* of the subspace $U$ in $V$ and vice-versa.

Prove that given a subspace $U$ of $V$, one can **always** find an Algebraic Complement of $U$ in $V$.

**9.** Given a vector space $V$ and a subset $W$ of $V$, convince yourself that checking the facts on closure property only, i.e. $w_1 \in W, w_2 \in W \implies w_1 + w_2 \in W$ and $\in W, \alpha \in F \implies \alpha w \in W$ are enough to conclude that $W$ is a subspace of $V$.

Consider similar problems for groups/rings/modules and answer them too. Do the groups especially!