Prof. P.A. Parrilo Spring 2024

MIT 6.7230
Algebraic techniques and semidefinite programming
Homework assignment # 3

Date Given: March 6th, 2024
Date Due: March 13th, 1PM

P1. [***15 pts***] Prove the following statement, that we used in the derivation of the Grothendieck-Krivine rounding.

Lemma 1 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that both $f+g$ and $f-g$ have nonnegative Taylor coefficients. Let*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} f[X_{11}] & g[X_{12}] \\ g[X_{12}^T] & f[X_{22}] \end{bmatrix},$$

where the functions are applied componentwise. Then, $X \succeq 0$ implies $Y \succeq 0$.

P2. [***10 pts***] Show, using either a direct argument or via induction, that the dimension of the vector space of polynomials in n variables of degree d is equal to $\binom{n+d}{d}$.

P3. [***6 pts***] Given two ideals I_1 and I_2 in a commutative ring, we define their *sum* and *intersection* as

$$I_1 + I_2 := \{f + g : f \in I_1, g \in I_2\},$$

and

$$I_1 \cap I_2 := \{f : f \in I_1, f \in I_2\},$$

respectively. Show that both the sum and the intersection of ideals are also ideals.

P4. [***10 pts***] Given a set $S \subseteq \mathbb{C}^n$, the *vanishing ideal* $\mathbf{I}(S)$ of the set S is defined as

$$\mathbf{I}(S) = \{f \in \mathbb{C}[\mathbf{x}] : f(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in S\}.$$

Similarly, given a set of polynomials $F \subseteq \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$, its *affine algebraic variety* or *zero locus* is the set

$$\mathbf{V}(F) = \{\mathbf{x} \in \mathbb{C}^n : f(\mathbf{x}) = 0 \quad \forall f \in F\}.$$

Prove the following statements:

(a) If $S_1 \subseteq S_2$, then $\mathbf{I}(S_1) \supseteq \mathbf{I}(S_2)$.
(b) If $F_1 \subseteq F_2$, then $\mathbf{V}(F_1) \supseteq \mathbf{V}(F_2)$.
(c) What is $\mathbf{I}(\emptyset)$? What is $\mathbf{I}(\mathbb{C}^n)$? What is $\mathbf{V}(\{0\})$? What is $\mathbf{V}(\mathbb{C}[\mathbf{x}])$?
(d) Show that $F \subseteq \mathbf{I}(\mathbf{V}(F))$.
(e) Show that $S \subseteq \mathbf{V}(\mathbf{I}(S))$.

P5. [***9 pts***] Prove the following statements (they're easier than they seem, hints below!):

(a) A field is formally real if and only if -1 is not a sum of squares.
(b) If a field is formally real, then it must have an infinite number of elements.
(c) If a field can be ordered, it is formally real.

Hint for (b): $1 + 1 + 1 + \cdots$.
Hint for (c): Try proving it by contradiction.