CHARACTERISTIC CLASSES II

NILAY KUMAR

1. Stiefel-Whitney classes

Recall from last time the axiomatic definition of the Stiefel-Whitney classes of a vector bundle $\xi : E \to B$:[1]

Theorem 1. *Let $\xi : E \to B$ be a real vector bundle. Then there exists a unique sequence of cohomology classes*

$$w_i(\xi) \in \mathrm{H}^i(B, \mathbb{F}_2)$$

for $i = 0, 1, 2, \ldots$ called the Stiefel-Whitney classes of ξ satisfying the following properties:

(I) the class $w_0(\xi)$ is equal to the unit element $1 \in \mathrm{H}^0(B, \mathbb{F}_2)$ and $w_i(\xi) = 0$ for $i > \operatorname{rk} \xi$;
(II) if $f^*\xi$ is the pullback of E along $f : A \to B$ then $w_i(f^*\xi) = f^* w_i(\xi)$;
(III) if $\eta : E' \to B$ is another real vector bundle then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta);$$

(IV) if γ_1^1 is the tautological line bundle over $\mathbb{R}P^1$ then $w_1(\gamma_1^1) \in \mathrm{H}^1(\mathbb{R}P^1, \mathbb{F}_2) = \mathbb{F}_2$ is the unique nonzero element.

make a remark about bundle map diagrams just being pullback diagrams, as Dylan pointed out last time. so we don't lose anything on II

Thus given, the Stiefel-Whitney classes allowed us to make some strong statements about parallelizability and cobordisms. The goal of the first half of this talk is to sketch a proof of the above theorem, i.e. show that Stiefel-Whitney classes do indeed exist. We will first prove the existence of certain analogous classes:

Theorem 2. *Let $\iota_n : \mathrm{O}(n-1) \to \mathrm{O}(n)$ and $p_{ij} : \mathrm{O}(i) \times \mathrm{O}(j) \to \mathrm{O}(i+j)$ be the obvious inclusions. Denote by the same symbols the induced maps on classifying spaces. Then there are unique classes $w_i \in \mathrm{H}^i(\mathscr{B}\,\mathrm{O}(n), \mathbb{F}_2)$ satisfying:*

(I) $w_0 = 1$ and $w_i = 0$ if $i > n$;
(II) $\iota_n^* w_i = w_i$ (and hence $\iota_n^* w_n = 0$);
(III) $p_{ij}^* w_k = \sum_{a+b=k} w_a \otimes w_b$;
(IV) $w_1 \in \mathrm{H}^1(\mathscr{B}\,\mathrm{O}(1), \mathbb{F}_2) = \mathrm{H}^1(\mathbb{R}P^\infty, \mathbb{F}_2)$ is the unique nonzero element.

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[1] Throughout, all base spaces are assumed to be connected and paracompact.

Here, as usual,

$$\mathscr{B} : \textsc{Grp} \to \textsc{Top}$$

denotes the classifying space functor. Almost all the work lies in computing the cohomology of $\mathscr{B}\,\mathrm{O}(n)$. In an earlier talk, Guchuan mentioned that – via a number of computations using spectral sequences – one can show the following:

Lemma 3. *Let Σ_n be the symmetric group on n letters. Then there is a map $\Psi_n : (\mathbb{R}P^\infty)^n \to \mathscr{B}\,\mathrm{O}(n)$ inducing an isomorphism*

$$\Psi_n^* : \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n), \mathbb{F}_2) \cong \mathrm{H}^\bullet((\mathbb{R}P^\infty)^n, \mathbb{F}_2)^{\Sigma_n} \cong \mathbb{F}_2[\sigma_1, \dots, \sigma_n],$$

where Σ_n acts on $(\mathbb{R}P^\infty)^n$ by permutation and hence σ_i are the symmetric polynomials in n variables with $\deg \sigma_i = i$.

Proof. This is rather technical and requires more wizardry with spectral sequences than I am familiar with. Reference unfinished May. □

With this in hand, the proof of Theorem 2 is now a straightforward diagram chase with symmetric polynomials.

Proof of Theorem 2. We begin by proving existence of the classes w_i. Define the Stiefel-Whitney classes as

$$w_i \equiv (\Psi_n^*)^{-1} \sigma_i$$

where Ψ_n^* is the isomorphism from Lemma 3 above, and where

$$\begin{aligned}
\sigma_1 &= x_1 + \cdots + x_n \\
\sigma_2 &= x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n \\
&\vdots \\
\sigma_n &= x_1 \cdots x_n,
\end{aligned}$$

are the symmetric polynomials on the generators of $\mathrm{H}^\bullet((\mathbb{R}P^\infty)^n, \mathbb{F}_2) \cong \otimes_i^n \mathbb{F}_2[x_i]$. Set $\sigma_0 = 1$ and $\sigma_i = 0$ for $i > n$. Under these definitions, axiom I is immediate.

Next, denoting $h_n : \mathscr{B}\,\mathrm{O}(1)^{n-1} \to \mathscr{B}\,\mathrm{O}(1)^n$ the map induced by the inclusion, we have a commutative diagram of inclusions

$$\begin{tikzcd}
\mathscr{B}\,\mathrm{O}(1)^{n-1} \arrow[r, "\Psi_{n-1}"] \arrow[d, "h_n"] & \mathscr{B}\,\mathrm{O}(n-1) \arrow[d, "\iota_n"] \\
\mathscr{B}\,\mathrm{O}(1)^n \arrow[r, "\Psi_n"] & \mathscr{B}\,\mathrm{O}(n)
\end{tikzcd}$$

and taking cohomology,

$$\begin{tikzcd}
\mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n)) \arrow[r, "\Psi_n^*"] \arrow[d, "\iota_n^*"] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1)^n) \arrow[d, "h_n^*"] \\
\mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n-1)) \arrow[r, "\Psi_{n-1}^*"] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1)^{n-1})
\end{tikzcd}$$

Clearly $h_n^* x_i = x_i$ for $i < n$ and $h_n^* x_n = 0$. Therefore $h_n^* \sigma_i = \sigma_i$, which implies – by the diagram above – that $\iota_n^* w_i = w_i$. So much for axiom II.

Consider now the commutative diagram of inclusions

$$\begin{tikzcd}
\mathscr{B}\mathrm{O}(1)^i \times \mathscr{B}\mathrm{O}(1)^j \arrow[r, "\Psi_i \times \Psi_j"] \arrow[d, equal] & \mathscr{B}\mathrm{O}(i) \times \mathscr{B}\mathrm{O}(j) \arrow[d, "p_{ij}"] \\
\mathscr{B}\mathrm{O}(1)^{i+j} \arrow[r, "\Psi_{i+j}"] & \mathscr{B}\mathrm{O}(i+j)
\end{tikzcd}$$

which after taking cohomology and applying Künneth becomes

$$\begin{tikzcd}
\mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(i+j)) \arrow[r, "\Psi_{i+j}^*"] \arrow[d, "p_{ij}^*"] & \mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(1)^{i+j}) \arrow[d] \\
\mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(i)) \otimes \mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(j)) \arrow[r, "\Psi_i^* \otimes \Psi_j^*"] & \mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(1)^i) \otimes \mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(1)^j)
\end{tikzcd}$$

In this diagram, all the arrows are injective except for p_{ij}^* and hence p_{ij}^* is injective. Moreover,

$$(\Psi_i^* \otimes \Psi_j^*)p_{ij}^* w_k = \Psi_{i+j}^* w_k = \sigma_k(x_1, \ldots, x_{i+j}).$$

Some algebra with symmetric polynomials reveals that

$$\sigma_k(x_1, \ldots, x_{i+j}) = \sum_{a+b=k} \sigma_a(x_1, \ldots, x_i)\sigma_b(x_{i+1}, \ldots, x_{i+j}),$$

whence

$$(\Psi_i \otimes \Psi_j^*)p_{ij}^* w_k = \sum_{a+b=k} \Psi_i^* w_a \otimes \Psi_j^* w_b = (\Psi_i^* \otimes \Psi_j^*) \sum_{a+b=k} w_a \otimes w_b,$$

proving axiom III by injectivity of $\Psi_i \otimes \Psi_j$.

Axiom IV is clear: $w_1 = (\Psi_1^*)^{-1}\sigma_1(x_1) = x_1$, the nonzero element in $\mathrm{H}^1(\mathbb{R}P^\infty, \mathbb{F}_2)$. Finally, we prove uniqueness by induction on n. The base case $n = 1$ is trivial. Assume uniqueness of the w_i in $\mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(m), \mathbb{F}_2)$ for $m < n$. Then for $i < n$ the $w_i \in \mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(n), \mathbb{F}_2)$ are uniquely determined by axiom II and the fact that ι_n is an isomorphism in degrees smaller than n. For $i = n$ we note that $p_{1,n-1}^* w_n \in \mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(1), \mathbb{F}_2) \otimes \mathrm{H}^\bullet(\mathscr{B}\mathrm{O}(n-1), \mathbb{F}_2)$ and hence w_n is determined by the induction hypothesis since $p_{1,n-1}^*$ is injective. This completes the proof. □

To relate these classes sitting in the cohomology of $\mathscr{B}\mathrm{O}(n)$ to the previous axiomatic definition of classes sitting in the cohomology of the base B, we need a classification theorem for vector bundles on B. Before we start, we note that

$$\mathrm{Vect}_{\mathbb{R}} : \textsc{Top}^{\mathrm{op}} \to \textsc{Set}$$

will denote the contravariant functor taking B to the set (isomorphism classes) of real vector bundles over B and taking $f : B \to B'$ to the pullback $f^* : \mathrm{Vect}_{\mathbb{R}} B' \to \mathrm{Vect}_{\mathbb{R}} B$. Recall that two vector bundles over B are isomorphic if there is a map lifting id_B that is a fiberwise linear isomorphism.

Theorem 4. *The space $\mathscr{B}\mathrm{O}(n) \cong \mathrm{Gr}_n \mathbb{R}^\infty$ classifies vector bundles, i.e. the natural transformation*

$$\Phi : [-, \mathscr{B}\mathrm{O}(n)] \xrightarrow{\cdot} \mathrm{Vect}_{\mathbb{R}} -,$$

given by pullback $[f] \mapsto f^*\gamma_\infty^n$ of the tautological bundle, is a natural isomorphism.

We first check that Φ is well-defined:

Lemma 5. *The pullbacks of a vector bundle along homotopic maps are isomorphic, i.e. the functor* $\mathrm{Vect}_{\mathbb{R}}$ *factors through the homotopy category.*

Proof sketch. Let $\xi : E \to B$ be a rank n vector bundle and let $f, g : A \to B$ be two maps homotopic via $h : A \times I \to B$. Note first that $h^*E|_{A\times\{0\}} = f^*E$ and $h^*E|_{A\times\{1\}} = g^*E$. Thus it suffices to prove that for a vector bundle $\eta : F \to A \times I$, there is an isomorphism $F|_{A\times\{0\}} \cong F|_{A\times\{1\}}$. The idea, roughly, is to find countably many local trivializations over $U_i \subset B$ for E and to then locally push $F|_{A\times\{0\}}$ to the right along $U_i \times I$. For details, see Hatcher, VBKT. □

Proof of Theorem 4. Naturality of Φ follows immediately from the fact that if $\alpha : A \to B$ and $f \in [B, \mathscr{B}\mathrm{O}(n)]$ then $(f \circ \alpha)^*\gamma_n^\infty = \alpha^* f^* \gamma_n^\infty$. We now prove that $\Phi_B : [B, \mathscr{O}(n)] \to \mathrm{Vect}_{\mathbb{R}} B$ is a bijection.

The key observation is as follows. Let $\xi : E \to B$ be a rank n vector bundle. Then an isomorphism $E \cong f^*\gamma_n^\infty$ (for some map $f : B \to \mathrm{Gr}_n \mathbb{R}^\infty$) is equivalent to a map $g : E \to \mathbb{R}^\infty$ that is a linear injection on each fiber. To see this, suppose first that we have such an isomorphism. Then we have a commutative diagram

$$\begin{tikzcd} E \cong f^*\gamma_n^\infty \arrow[r, "\tilde{f}"] \arrow[d, "\xi"] & \gamma_n^\infty \arrow[r, "\pi"] \arrow[d] & \mathbb{R}^\infty \\ B \arrow[r, "f"] & \mathrm{Gr}_n \mathbb{R}^\infty & \end{tikzcd}$$

where π is the obvious projection. Now $\pi \circ \tilde{f} : E \to \mathbb{R}^\infty$ is a fiberwise linear injection as both f and π are. Conversely, given $g : E \to \mathbb{R}^\infty$ a fiberwise linear injection, we can define a map $f : B \to \mathrm{Gr}_n \mathbb{R}^\infty$ given by $x \mapsto [g(\xi^{-1}(x))]$. Then $E \cong f^*\gamma_n^\infty$ because we have fiberwise linear isomorphisms

$$f^*\gamma_n^\infty|_b \cong \gamma_n^\infty|_{f(b)} \cong E|_b.$$

Now, for surjectivity of Φ_B, given $\xi : E \to B$ it suffices by the previous paragraph to construct a map $E \to \mathbb{R}^\infty$ a linear injection on each fiber. To do this, we fix countably many local trivializations over $U_i \subset B$ of E together with partitions of unity ϕ_i subordinate to the U_i. Then for each i we obtain a map $g_i : E \to \mathbb{R}^n$ that is zero outside $\xi^{-1}U_i$ and the composition $E \to U_i \times \mathbb{R}^n \to \mathbb{R}^n$ otherwise. Summing $g = \sum g_i$, we obtain a map $g : E \to (\mathbb{R}^n)^\infty \cong \mathbb{R}^\infty$ that is obviously a linear injection on fibers.

For injectivity, suppose we have isomorphisms $E \cong f_0^*\gamma_n^\infty$ and $E \cong f_1^*\gamma_n^\infty$ for two maps $f_0, f_1 : B \to \mathrm{Gr}_n \mathbb{R}^\infty$. By arguments above, we obtain maps $g_0, g_1 : E \to \mathbb{R}^\infty$ that are fiberwise linear injections. We claim that g_0 and g_1 are homotopic through maps g_t that are fiberwise linear injections; this implies that f_0 and f_1 are homotopic via $f_t(x) = g_t(\xi^{-1}x)$. To do this, we first homotope g_0 so that it takes values only in odd coordinates via

$$(x_1, x_2, \ldots) \mapsto (1-t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, 0, \ldots)$$

and homotope g_1 so that it takes values only in even coordinates similarly. Now $g_t = (1-t)g_0 + tg_1$ provides the necessary homotopy and it is clearly linear and injective on fibers. □

We can finally prove the existence of Stiefel-Whitney classes for vector bundles.

Proof of Theorem 8. Let $\xi : E \to B$ be a real vector bundle of rank n. By Theorem 4 there exists a unique map $\Phi_B : B \to \mathscr{B}O(n)$ such that $E \cong f^*\gamma_n^\infty$. Define

$$w_i(\xi) \equiv \Phi_B^* w_i.$$

Axiom I now follows immediately from Theorem 2. Now suppose we have a pullback diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\xi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Then

$$w_i(f^*\xi) = \Phi_A^* w_i = f^*\Phi_B^* w_i = f^* w_i(\xi),$$

which proves axiom II.

To prove axiom III we recall that the direct sum of vector bundles can be described as a pullback

$$\begin{tikzcd} E \oplus E' \cong \Delta^*(E \times E') \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

where Δ is the diagonal map. Then

$$w_k(\xi \oplus \eta) = w_k(\Delta^*(\xi \times \eta)) = \Phi_{B\times B}^* w_k$$

where $\Phi_{\xi\times\eta} : B \times B \to \mathscr{B}\,\mathrm{O}(n+m)$ is the unique classifying map. Figure out how to prove axiom III.

The tautological line bundle on $\mathbb{R}P^1$ is given by the pullback of γ_1^∞ along the inclusion $j : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ so $w_1(\gamma_1^1) = j^*w_1$ is the unique nonzero element in $\mathrm{H}^1(\mathbb{R}P^1, \mathbb{Z}/2)$ by axiom IV of Theorem 2 and because j^* is an isomorphism in degrees $\leqslant 1$.

Uniqueness blah blah blah. $\square$

Application: $w_1(\xi) = 0$ if and only if E is orientable? Why?

2. Chern classes

Chern classes are the complex analog of Stiefel-Whitney classes, but with integral coefficients.

Theorem 6. *Let $\iota_n : \mathrm{U}(n-1) \to \mathrm{U}(n)$ and $p_{ij} : \mathrm{U}(i) \times \mathrm{U}(j) \to \mathrm{U}(i+j)$ be the obvious inclusions. Denote by the same symbols the induced maps on classifying spaces. Then there are unique Chern classes $c_i \in \mathrm{H}^{2i}(\mathscr{B}\,\mathrm{U}(n), \mathbb{Z})$ satisfying:*

- *(I) $c_0 = 1$ and $c_i = 0$ if $i > n$;*
- *(II) $\iota_n^* c_i = c_i$ (and hence $\iota_n^* c_n = 0$);*
- *(III) $p_{ij}^* c_k = \sum_{a+b=k} c_a \otimes c_b$;*
- *(IV) $c_1 \in \mathrm{H}^2(\mathscr{B}\,\mathrm{U}(1), \mathbb{Z}) = \mathrm{H}^2(\mathbb{C}P^\infty, \mathbb{Z})$ is the canonical generator.*

Proof. The proof is exactly the same as in the case of Stiefel-Whitney classes. We note, in particular, that $\mathrm{H}^\bullet(\mathscr{B}\,\mathrm{U}(n), \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$. $\square$

Chern classes can be related to complex vector bundles via the following analog of Theorem 4. Notice that here $\mathrm{Gr}_n V$ denotes the space of complex n-planes in V.

Theorem 7. *The space $\mathscr{B}\,\mathrm{U}(n) \cong \mathrm{Gr}_n \mathbb{C}^\infty$ classifies complex vector bundles, i.e. the natural transformation*

$$\Phi : [-, \mathscr{B}\,\mathrm{U}(n)] \stackrel{\cdot}{\longrightarrow} \mathrm{Vect}_{\mathbb{C}} -,$$

given by pullback $[f] \mapsto f^*\gamma^n_\infty$ of the tautological bundle, is a natural isomorphism.

We can now define the Chern class of a complex vector bundle.

Theorem 8. *Let $\xi : E \to B$ be a complex vector bundle. Then there exists a unique sequence of cohomology classes*

$$c_i(\xi) \in \mathrm{H}^{2i}(B, \mathbb{Z})$$

for $i = 0, 1, 2, \ldots$ called the Chern classes of ξ satisfying the following properties:

(I) the class $c_0(\xi)$ is equal to the generator $1 \in \mathrm{H}^0(B, \mathbb{Z})$ and $c_i(\xi) = 0$ for $i > \mathrm{rk}\,\xi$;

(II) if $f^*\xi$ is the pullback of E along $f : A \to B$ then $c_i(f^*\xi) = f^*c_i(\xi)$;

(III) if $\eta : E' \to B$ is another complex vector bundle then

$$c_k(\xi \oplus \eta) = \sum_{i=0}^{k} c_i(\xi) \smile c_{k-i}(\eta);$$

(IV) if γ^1_1 is the tautological complex line bundle over $\mathbb{C}P^1$ then $c_1(\gamma^1_1) \in \mathrm{H}^2(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}$ is the canonical generator.

How does the Chern class restrict to the Stiefel-Whitney class? See p.47 of May unfinished

What are some applications of this machinery different than the ones Yajit gave?