

# NOTES ON SYMPLECTIC GEOMETRY

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These notes were written for a reading course with Professor Eric Zaslow on the basics of symplectic geometry. These notes are quite rough: *caveat lector*.<sup>1</sup>

## 1. WEEK 1

### 1.1. The cotangent bundle.

**Definition 1.** Let  $X$  be a smooth  $n$ -manifold and  $\pi : M = T^*X \rightarrow X$  be its cotangent bundle. We define the **canonical one-form**  $\theta \in \Omega^1(M)$  as follows. For any  $p = (x, \xi) \in M$ , set

$$\theta_p(v) = \xi(d_x \pi(v)).$$

The one-form  $\theta$  is canonical (or tautological) in the sense that its value at a point is simply given by the covector determined by that point. More precisely, we have the following characterization.

**Proposition 2.** *The canonical one-form  $\theta$  is the (unique) one-form such that for every  $\lambda \in \Omega^1(X)$ ,  $\lambda^* \theta = \lambda$ .*

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*Date:* Fall 2015.

<sup>1</sup>add references!

*Proof.* We compute, for  $v \in T_p X$ ,

$$\begin{aligned} (\lambda^* \theta)_p(v) &= \theta_{\lambda(p)}(d_p \lambda(v)) \\ &= \lambda_p(d_p(\pi \circ \lambda)(v)) \\ &= \lambda_p(v), \end{aligned}$$

where we have used the fact that  $\lambda$  is a section of  $\pi$ , i.e.  $\pi \circ \lambda = \text{id}_X$ . Uniqueness is easily checked.  $\square$

**Definition 3.** The **canonical symplectic form**  $\omega \in \Omega^2(M)$  is now defined to be the exterior derivative

$$\omega = -d\theta,$$

of the canonical one-form. To be symplectic,  $\omega$  must be closed and nondegenerate. That it is closed is obvious.

**Proposition 4.** *The form  $\omega \in \Omega^2(M)$  is nondegenerate and thus defines a symplectic structure on  $M = T^*X$ .<sup>2</sup>*

*Proof.* For  $\omega$  to be non-degenerate, it must be nondegenerate at each point  $p \in M$ . Given coordinates  $p = (x, \xi) = (x^1, \dots, x^n, \xi_1, \dots, \xi_n)$  in a neighborhood of  $p$ , we can compute

$$\begin{aligned} \theta_{(x, \xi)} \left( v^i \frac{\partial}{\partial x^i} + \nu^i \frac{\partial}{\partial \xi^i} \right) &= \xi \left( v^i \frac{\partial}{\partial x^i} \right) \\ &= \xi_i v^i \end{aligned}$$

and hence

$$\theta = \xi_i dx^i.$$

Taking an exterior derivative, we find that

$$\begin{aligned} \omega &= -d\theta \\ &= dx^i \wedge d\xi_i. \end{aligned}$$

Fix  $v \in T_p M$  and suppose that  $\iota_v \omega_p = 0$ , i.e.  $\omega_p(v, w) = 0$  for all  $w \in T_p M$ . In coordinates, this implies that

$$\begin{aligned} \iota_{v^j \frac{\partial}{\partial x^j} + \nu^j \frac{\partial}{\partial \xi^j}} (dx^i \wedge d\xi_i) &= v^i d\xi_i - \nu^i dx^i \\ &= 0, \end{aligned}$$

and hence that  $v^i = \nu^i = 0$ , i.e.  $v = 0$ . We conclude that  $\omega_p$  is nondegenerate at each  $p \in M$ .  $\square$

*Remark 5.* Note that a 2-form  $\omega$  on a manifold  $M$  is nondegenerate if and only if  $\omega^n$  is nowhere vanishing. Fix  $p \in M$  and consider the vector space  $(T_p M, \omega_p)$ . If  $\omega_p$  is nondegenerate, we can find a symplectic basis for  $T_p M$ , and so  $\omega_p^n$  evaluated on  $(u_1, \dots, u_n, v_1, \dots, v_n)$  is nonzero, whence  $\omega_p^n$  is not zero on  $V$ . On the other hand, suppose  $\omega_p$  is degenerate, i.e. there is a  $v \neq 0$  such that  $\omega_p(v, w) = 0$  for all  $w \in V$ . Choosing a basis  $v_1, \dots, v_{2n}$  for  $V$  such that  $v_1 = v$ , we find that  $\omega_p(v_1, \dots, v_{2n}) = 0$  and hence  $\omega_p = 0$  on  $V$ .

We conclude that every symplectic manifold is orientable.

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<sup>2</sup>Is there a coordinate invariant proof?

It is easy to see that  $\omega$  provides an isomorphism  $\iota : T_x X \xrightarrow{\sim} T_x^* X$  between tangent and cotangent spaces at each point  $x \in X$ : since  $\omega_x$  is nondegenerate, the linear map  $\iota : v \mapsto \omega_x(v, -)$  is injective and hence bijective. In fact, we can say more.

**Proposition 6.** *The metric  $\omega$  induces an isomorphism of vector bundles  $\iota : TX \xrightarrow{\sim} T^*X = M$ .*

*Proof.* Recall that an isomorphism in the category of smooth vector bundles is a smooth bijection<sup>3</sup>  $\iota$  such that the diagram

$$\begin{array}{ccc} TX & \xrightarrow{\quad \iota \quad} & T^*X \\ & \searrow \pi_1 & \swarrow \pi_2 \\ & X & \end{array}$$

commutes and for each  $x \in X$ , the restriction  $\iota_x : T_x X \rightarrow T_x^* X$  is linear. The map  $\iota : TX \rightarrow T^*X$  taking  $(x, v) \mapsto (x, \omega(v, -))$  fits into the diagram above and is bijective and fiberwise linear. Moreover,  $\iota$  is a smooth map, as is seen by its coordinate description computed above.  $\square$

**Definition 7.** A **Hamiltonian** is a smooth function  $H : M = T^*X \rightarrow \mathbb{R}$ . we define the **Hamiltonian vector field**  $v_H$  associated to  $H$  to be the vector field on  $M$  satisfying

$$\iota_{v_H} \omega = dH.$$

The (local) flow  $F : (-\varepsilon, \varepsilon) \times M \rightarrow M$  determined by  $v_H$  is called the **Hamiltonian flow**.<sup>4</sup>

Note that an integral curve  $\gamma_{v_H} : (-\varepsilon, \varepsilon) \rightarrow M$  of  $v_H$  can be thought of as the trajectory of a physical state in phase space. Indeed, Hamilton's equations are given

$$\begin{aligned} \frac{\partial x^i}{\partial t} &= \frac{\partial H}{\partial \xi_i} \\ \frac{\partial \xi_i}{\partial t} &= -\frac{\partial H}{\partial x^i}, \end{aligned}$$

which is precisely the condition that  $\gamma'_{v_H}(t) = (v_H)_{\gamma(t)}$ . Moreover,  $H$  is constant along the Hamiltonian flow, as

$$dH(v_H) = (\iota_{v_H} \omega)(v_H) = \omega(v_H, v_H) = 0,$$

i.e.  $v_H$  is perpendicular to the level sets of  $H$ . In a physical system, where  $H$  is the energy functional on phase space, this phenomenon is the law of conservation of energy.

**Proposition 8.** *The Hamiltonian flow is a symplectomorphism, i.e.  $F_t^* \omega = \omega$ .*<sup>5</sup>

*Proof.* We use the following trick:

$$\int_0^t \frac{d}{dt} F_t^* \omega \, dt = F_t^* \omega - \omega$$

<sup>3</sup>Existence of a smooth inverse is automatic (reference?).

<sup>4</sup>Is this a global flow? Does it depend on  $X$ ?

<sup>5</sup>Is there a better proof?

since  $F_0 = \text{id}_M$ , and hence  $F_t$  is a symplectomorphism if and only if the integrand is zero. But

$$\begin{aligned} \frac{d}{dt} F_t^* \omega &= \frac{d}{ds} \Big|_{s=0} F_{t+s}^* \omega = F_t^* \frac{d}{ds} \Big|_{s=0} F_s^* \omega \\ &= F_t^* \mathcal{L}_{v_H} \omega, \end{aligned}$$

and Cartan's magic formula,

$$\mathcal{L}_{v_H} \omega = d\iota_{v_H} \omega + \iota_{v_H} d\omega,$$

tells us that  $\mathcal{L}_{v_H} \omega = 0$  since  $\iota_{v_H} \omega = dH$  is closed, as is  $\omega$ .  $\square$

**Corollary 9** (Liouville's Theorem). *The volume form  $\omega^n$  on  $M = T^*X$  is preserved by the Hamiltonian flow.*

**1.2. Geodesic flow as Hamiltonian flow.** We wish to discuss geodesics and geodesic flow. For this, we need the concept of connections and covariant derivatives.<sup>6</sup>

**Definition 10.** A **connection** on a vector bundle  $E \rightarrow X$  is an  $\mathbb{R}$ -linear map  $\nabla : \Gamma(X, E) \rightarrow \Gamma(X, E \otimes T^*X)$  such that the Leibniz rule

$$\nabla(f\sigma) = (\nabla\sigma)f + \sigma \otimes df,$$

for all  $f \in C^\infty(X)$  and  $\sigma \in \Gamma(X, E)$ .

**Theorem 11.** *Given a Riemannian manifold  $(X, g)$ , there exists a unique connection on  $\pi : TX \rightarrow X$ , known as the **Levi-Civita connection**, satisfying*

(i) *symmetry:*

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

*for  $X, Y \in \Gamma(X, TX)$ ;*

(ii) *compatibility with  $g$ :*

$$Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,$$

*for  $X, Y, Z \in \Gamma(X, TX)$ .*

**Definition 12.** Let  $v$  be a vector field on  $(X, g)$ ; we define the **covariant derivative** of  $v$  along a smooth curve  $c : I \rightarrow X$  to be the vector field

$$\frac{Dv}{dt} = \nabla_{dc/dt} v,$$

where  $\nabla$  is the Levi-Civita connection. Explicitly, if we write  $v = v^i \partial/\partial x^i$  and  $c(t) = (c_1(t), \dots, c_n(t))$ ,

$$\frac{Dv}{dt} = \sum_i \frac{dv^i}{dt} \frac{\partial}{\partial x^i} + \sum_{ijk} \frac{dc_i}{dt} v^i \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Here  $\Gamma_{ij}^k$  are the Christoffel symbols of  $\nabla$ , determined by

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \sum_{ijk} \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

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<sup>6</sup>Reference do Carmo.

We say that  $c$  is **geodesic** at some  $t \in I$  if  $D/dt(dc/dt) = 0$  at  $t$ , and that  $c$  is geodesic if it is geodesic at all  $t \in I$ . In coordinates, the condition for  $c$  to be geodesic is given by a system of second-order differential equations:

$$\frac{d^2 c^i}{dt^2} + \sum_{jk} \Gamma_{jk}^i \frac{dc^j}{dt} \frac{dc^k}{dt} = 0,$$

for  $i = 1, \dots, n$ .

For the rest of the section, assume  $(X, g)$  is Riemannian and we fix the Hamiltonian  $H : M = T^*X \rightarrow \mathbb{R}$  as

$$H(x, \xi) = \frac{1}{2} |\xi_x|_g^2,$$

i.e. consisting of only a kinetic term. Here we are implicitly using the nondegeneracy of  $g$  to associate  $\xi_x$  with its corresponding vector (or, equivalently, using  $g^{-1}$ ).

**Proposition 13.** *The Hamiltonian flow on  $M = T^*X$  is dual to the geodesic flow on  $TX$ . In other words, the integral curves of the Hamiltonian vector field  $v_H$  associated to the Hamiltonian above project to geodesics of  $g$  on  $X$ .<sup>7</sup>*

*Proof.* It suffices to show, in coordinates, that Hamilton's equations (i.e. the condition for being on the integral curve) yield the geodesic equations above after the necessary dualization. Note first that in coordinates the Hamiltonian becomes

$$H(x, \xi) = \frac{1}{2} g^{ij} \xi_i \xi_j.$$

For convenience we will denote the components of an integral curve as  $x^i(t)$ . Hamilton's equations yield

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial}{\partial \xi_i} \left( \frac{1}{2} g^{jk} \xi_j \xi_k \right) \\ &= \frac{1}{2} g^{jk} \delta_{ij} \xi_k + \frac{1}{2} g^{jk} \xi_j \delta_{ik} \\ &= g^{ij} \xi_j \\ \frac{d\xi_i}{dt} &= -\frac{\partial}{\partial x^i} \left( \frac{1}{2} g^{jk} \xi_j \xi_k \right) \\ &= -\frac{1}{2} \frac{\partial g^{jk}}{\partial x^i} \xi_j \xi_k. \end{aligned}$$

Differentiating the first equation with respect to  $t$  and using both of Hamilton's equations yields

$$\begin{aligned} \frac{d^2 x^i}{dt^2} &= \frac{\partial g^{ij}}{\partial x^k} \frac{dx^k}{dt} \xi_j + g^{im} \frac{d\xi_m}{dt} \\ &= g^{kl} \left( \frac{\partial}{\partial x^k} g^{ij} \right) \xi_l \xi_j - \frac{1}{2} g^{im} \left( \frac{\partial}{\partial x^m} g^{nr} \right) \xi_n \xi_r. \end{aligned}$$

Next, differentiating the identity  $g^{ij} g_{jk} = \delta_k^i$ , it easy to see that

$$\frac{\partial}{\partial x^i} g^{kl} = -g^{la} g^{kb} \frac{\partial}{\partial x^i} g_{ab}.$$

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<sup>7</sup>Is there a coordinate-free proof? See Paternain's book.

Using this, contracting indices, and using the first Hamilton's equation to dualize  $\xi$ 's into  $dx/dt$ 's, we find

$$\begin{aligned}
\frac{d^2 x^i}{dt^2} &= -g^{ib} \left( \frac{\partial}{\partial x^k} g_{lb} \right) \frac{dx^k}{dt} \frac{dx^l}{dt} + \frac{1}{2} g^{im} \left( \frac{\partial}{\partial x^m} g_{ts} \right) \frac{dx^s}{dt} \frac{dx^t}{dt} \\
&= -\frac{1}{2} g^{ib} \left( \frac{\partial}{\partial x^k} g_{lb} \right) \frac{dx^k}{dt} \frac{dx^l}{dt} - \frac{1}{2} g^{ib} \left( \frac{\partial}{\partial x^l} g_{kb} \right) \frac{dx^k}{dt} \frac{dx^l}{dt} \\
&\quad + \frac{1}{2} g^{im} \left( \frac{\partial}{\partial x^m} g_{ts} \right) \frac{dx^s}{dt} \frac{dx^t}{dt} \\
&= -\Gamma_{kl}^i \frac{dx^k}{dt} \frac{dx^l}{dt},
\end{aligned}$$

as desired. □

## 2. WEEK 2

## 2.1. Darboux's theorem.

**Theorem 14** (Darboux). *Let  $(M, \omega)$  be a symplectic  $2n$ -manifold. Then  $M$  is locally symplectomorphic to  $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ .*

We prove Darboux's theorem using the following stronger statement.

**Theorem 15** (Moser's trick). *Let  $M$  be a  $2n$ -dimensional manifold and  $Q \subset M$  be a compact submanifold. Suppose that  $\omega_1, \omega_2 \in \Omega^2(M)$  are closed 2-forms such that at each point  $q$  of  $Q$  the forms  $\omega_0$  and  $\omega_1$  are equal and nondegenerate on  $T_q M$ . Then there exist neighborhoods  $N_0$  and  $N_1$  of  $Q$  and a diffeomorphism  $\psi : N_0 \rightarrow N_1$  such that  $\psi|_Q = \text{id}_Q$  and  $\psi^* \omega_1 = \omega_0$ .*

*Proof.* Consider the family of closed two-forms

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0)$$

on  $M$  for  $t \in [0, 1]$ . Note that  $\omega_t|_Q = \omega_0|_Q$  is nondegenerate and hence there exists an open neighborhood  $N_0$  of  $Q$  such that  $\omega_t|_{N_0}$  is nondegenerate.<sup>8</sup> Suppose, for now, that there is a one-form  $\sigma \in \Omega^1(N_0)$  (possibly shrinking  $N_0$ ), such that  $\sigma|_{T_Q M} = 0$  and  $d\sigma = \omega_1 - \omega_0$  on  $N_0$ . Then

$$\omega_t = \omega_0 + t d\sigma$$

and we obtain by nondegeneracy a smooth vector field  $X_t$  on  $N_0$  characterized by

$$\iota_{X_t} \omega_t = -\sigma.$$

The condition  $\sigma|_{T_Q M} = 0$  implies, again by nondegeneracy of  $\omega_t$ , that  $X_t|_Q = 0$ . Now consider the initial value problem for the flow  $\psi_t$  of  $X_t$ ,

$$\begin{aligned} \frac{d}{dt} \psi_t &= X_t \circ \psi_t \\ \psi_0 &= \text{id}. \end{aligned}$$

This differential equation can be solved uniquely for  $t \in [0, 1]$  on some open neighborhood of  $Q$  contained in  $N_0$ , call it again  $N_0$ .<sup>9</sup> Note that  $\psi_t|_Q = \text{id}_Q$  since  $X_t|_Q = 0$ . We compute now that

$$\begin{aligned} \frac{d}{dt} \psi_t^* \omega_t &= \psi_t^* \left( \frac{d}{dt} \omega_t + \mathcal{L}_{X_t} \omega_t \right) \\ &= \psi_t^* (d\sigma + d\iota_{X_t} \omega_t) \\ &= 0. \end{aligned}$$

Hence  $\psi_1^* \omega_1 = \psi_0^* \omega_0 = \omega_0$ . Thus the desired diffeomorphism is  $\psi_1$  and the desired neighborhoods are  $N_0$  and  $N_1$ . The above argument is known as **Moser's trick**, and is extremely useful in symplectic geometry.

It remains to construct a smooth one-form  $\sigma$  satisfying  $\sigma|_{T_Q M} = 0$  and  $d\sigma = \omega_1 - \omega_0$ . If  $Q$  were a point (or more generally, diffeomorphic to a star-shaped subset of Euclidean space), we could simply use the Poincaré lemma; in general, however the construction is as follows. Fix any Riemannian metric on  $M$  and consider the

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<sup>8</sup>Why?

<sup>9</sup>Why?

restriction of the exponential map  $\exp : TM \rightarrow M$  to a neighborhood  $U_\varepsilon$  of the zero section of the normal bundle  $TQ^\perp \rightarrow M$ :

$$U_\varepsilon = \{(q, v) \in TM \mid q \in Q, v \in T_q Q^\perp, |v| < \varepsilon\}.$$

Recall that  $\exp$  becomes a diffeomorphism for  $\varepsilon$  sufficiently small, so we choose  $\varepsilon$  such that  $N_0 = \exp(U_\varepsilon)$  is contained in the neighborhood of  $Q$  above on which  $\omega_t$  is nondegenerate. Define now a family of maps  $\phi_t : N_0 \rightarrow N_0$  for  $t \in [0, 1]$  by

$$\phi_t(\exp(q, v)) = \exp(q, tv).$$

Note that  $\phi_t$  is a diffeomorphism onto its image for  $t \neq 0$ . Moreover,  $\phi_t|_Q = \text{id}_Q$ ,  $\phi_0(N_0)$ , and  $\phi_1 = \text{id}_{N_0}$ . If we now write  $\tau = \omega_1 - \omega_0$ , we find that

$$\begin{aligned}\phi_0^* \tau &= 0 \\ \phi_1^* \tau &= \tau,\end{aligned}$$

since  $\tau = 0$  on  $T_Q M$ . Now, for  $t \in (0, 1]$ , we define a family of vector fields,

$$Y_t = \left( \frac{d}{dt} \phi_t \right) \circ \phi_t^{-1}.$$

Then for any  $\delta > 0$ ,

$$\begin{aligned}\phi_1^* \tau - \phi_\delta^* \tau &= \int_\delta^1 \frac{d}{dt} \phi_t^* \tau dt = \int_\delta^1 \phi_t^* \mathcal{L}_{Y_t} \tau dt \\ &= \int_\delta^1 \phi_t^* (d\iota_{Y_t} \tau) dt \\ &= d \int_\delta^1 \phi_t^* (\iota_{Y_t} \tau) dt\end{aligned}$$

Clearly  $\phi_1^* \tau - \phi_\delta^* \tau = \tau - \phi_\delta^* \tau$  approaches  $\tau$  as  $\delta \rightarrow 0^+$ , so we find that

$$\tau = d \int_0^1 \phi_t^* (\iota_{Y_t} \tau) dt.$$

Defining

$$\sigma = \int_0^1 \phi_t^* (\iota_{Y_t} \tau) dt,$$

we find that  $\tau = \omega_1 - \omega_0 = d\sigma$  and  $\sigma|_{T_Q M} = 0$  because  $\phi_t|_Q = \text{id}_Q$  and  $\tau = 0$  on  $Q$ , forcing the integrand to vanish on  $T_Q M$ . Hence  $\sigma$  is the one-form required above for Moser's trick, and we are done.<sup>10</sup>  $\square$

The proof of Darboux's theorem is now straightforward: we choose a coordinate chart  $\phi$  so that  $\phi^* \omega$  is equal to the standard form on a subset of  $\mathbb{R}^{2n}$  at a single point, and then apply Moser's theorem with  $Q$  equal to the chosen point.

*Proof of Darboux's theorem.* Let  $q \in M$  and fix a symplectic basis  $\{u_i, v_i\}$  for the symplectic vector space  $(T_q M, \omega_q)$ . Fix any Riemannian metric on  $M$  and pick an open  $U \ni 0$  small enough such that  $\exp$  restricted to  $U \subset T_q M$  is a diffeomorphism

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<sup>10</sup>Why is  $\sigma$  smooth?



and hence a chart  $(x^i, y_i) = \exp : U \subset \mathbb{R}^{2n} \rightarrow M$  ( $i = 1, \dots, n$ ) such that  $x^i(p) = y_i(p) = 0$ . Now we can compute, for example,

$$\begin{aligned} \exp^* \omega_p \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k} \right) &= \omega_p \left( \exp_* \frac{\partial}{\partial x^j}, \exp_* \frac{\partial}{\partial y^k} \right) \\ &= \omega_p(u_j, v_k) = \delta_{jk}, \end{aligned}$$

to check that  $\exp^* \omega_p = (\omega_0)_0$  where  $\omega_0$  is the standard form on  $T_0 U$ . Here we have used the fact that  $\exp_* = \text{id}$  at  $0 \in U$ . Applying Theorem 2.1 to  $U$  with  $Q = 0 \in U$ , we obtain a diffeomorphism  $\psi$  of (some possibly smaller)  $U$  such that  $\psi^* \exp^* \omega = \omega_0$  on  $U$ . But now  $\exp \circ \psi$  provides a symplectomorphism in a neighborhood of  $q$  to a neighborhood of  $\mathbb{R}^{2n}$  pulling  $\omega$  back to the standard form  $\omega_0$ .  $\square$

## 3. WEEK 3

## 3.1. Submanifolds of symplectic manifolds.

**Definition 16.** Let  $(V, \omega)$  be a symplectic vector space. We define the **symplectic complement**  $U^\omega$  of a subspace  $U \subset V$  as

$$U^\omega = \{v \in V \mid \omega(v, u) = 0 \text{ for all } u \in U\}.$$

**Lemma 17.** For any subspace  $U \subset V$ ,  $U^{\omega\omega} = U$  and

$$\dim U + \dim U^\omega = \dim V.$$

*Proof.* Nondegeneracy of  $\omega$  yields an isomorphism  $\iota_\omega : V \rightarrow V^*$  which identifies  $U^\omega$  with  $U^\perp \equiv \{\nu \in V^* \mid \nu(u) = 0 \text{ for all } u \in U\}$ . The result now follows from the fact that  $\dim U + \dim U^\perp = \dim V$ .  $\square$

**Definition 18.** Let  $(M, \omega)$  be a symplectic manifold. A submanifold  $Q \subset M$  is called **symplectic, isotropic, coisotropic, or Lagrangian** if for each  $q \in Q$ , the linear subspace  $T_q Q \equiv V_q$  of  $(T_q M, \omega_q)$  is

- (a) symplectic:  $V_q \cap V_q^{\omega_q} = 0$ ,
- (b) isotropic:  $V_q \subset V_q^{\omega_q}$ ,
- (c) coisotropic:  $V_q^{\omega_q} \subset V_q$ ,
- (d) Lagrangian:  $V_q = V_q^{\omega_q}$ ,

respectively.

*Remark 19.* Note that  $Q \subset M$  is Lagrangian if and only if the restriction of  $\omega$  to  $Q$  is zero and  $\dim Q = \dim M/2$ .

**Example 20.** Let  $X$  be any manifold, and  $(M = T^*X, \omega)$  be its cotangent bundle with the usual symplectic structure. Recall that  $\omega = -d\theta$ , where  $\theta_\xi(v) = \xi(d_x \pi(v))$ .<sup>11</sup> In coordinates, if  $(x^i, \xi^i)$  are coordinates for  $M$ , we can write  $\omega = dx^i \wedge d\xi^i$ .

It is then easy to see that the fibre  $T_x^*X \subset M$  is Lagrangian, as

$$\begin{aligned} 0 &= (dx^i \wedge d\xi^i) \left( a_j \frac{\partial}{\partial \xi^j}, b_k \frac{\partial}{\partial \xi^k} + c_l \frac{\partial}{\partial x^l} \right) \\ &= (dx^i \wedge d\xi^i) \left( a_j \frac{\partial}{\partial \xi^j}, c_l \frac{\partial}{\partial x^l} \right) \\ &= a_i c_i, \end{aligned}$$

forces  $c_i = 0$ .

Similarly, the zero section  $\Gamma_0 \subset M$  is Lagrangian, as

$$\begin{aligned} 0 &= (dx^i \wedge d\xi^i) \left( a_j \frac{\partial}{\partial x^j}, b_k \frac{\partial}{\partial \xi^k} + c_l \frac{\partial}{\partial x^l} \right) \\ &= (dx^i \wedge d\xi^i) \left( a_j \frac{\partial}{\partial x^j}, b_k \frac{\partial}{\partial \xi^k} \right) \\ &= a_i b_i, \end{aligned}$$

forces  $b_i = 0$ .

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<sup>11</sup>Can we do this coordinate-invariantly?

More generally, given a submanifold  $Q \subset L$ , the annihilator

$$TQ^\perp = \{(q, \nu) \in T^*L \mid q \in Q, \nu|_{T_q Q} = 0\}$$

is Lagrangian.

**Example 21.** Let  $(M, \omega)$  be a symplectic manifold. The product  $M \times M$  can be given a symplectic structure  $\omega' = \alpha\pi_1^*\omega + \beta\pi_2^*\omega$  for  $\alpha, \beta \in \mathbb{R}$ . Consider in particular the case of  $\alpha = 1, \beta = -1$ . Then it is clear that  $M \times \{m\}$  and  $\{m\} \times M$  are symplectic submanifolds. Moreover, the diagonal  $\Delta \subset M \times M$  is Lagrangian, as

$$\begin{aligned} 0 &= \omega'((u, u), (v, w)) \\ &= \omega(u, v) - \omega(u, w) \\ &= \omega(u, v - w) \end{aligned}$$

and hence  $v = w$ , as desired.

**Example 22.** Let  $S \subset (M, \omega)$  be a codimension 1 submanifold. Then  $S$  is coisotropic. Indeed, fix  $s \in S$ , and note that  $T_s S \subset T_s M$  is codimension one. By Lemma 17,  $T_s S^{\omega_s}$  is a one-dimensional subspace. Pick any vector  $v \in T_s S^{\omega_s}$ ;  $v$  spans the entire symplectic complement, and hence if  $v$  is not in  $T_s S^{\omega_s}$ ,  $T_s S \cap T_s S^{\omega_s} = 0$  and  $T_s S$  is symplectic and thus even-dimensional. This is a contradiction, and hence  $T_s S$  must be coisotropic.

**Proposition 23.** *The graph  $\Gamma_\sigma \subset T^*X$  of a one-form is Lagrangian if and only if  $\sigma$  is closed.*

*Proof.* Note that  $\Gamma_\sigma$  is defined to be the image of the embedding  $\sigma : X \rightarrow T^*X$ . Then  $\dim \Gamma_\sigma = n$ , so it remains to show that  $\omega$  restricts to zero on  $\Gamma_\sigma$  if and only if  $\sigma$  is closed. Using Proposition 2, we compute

$$d\sigma = d\sigma^*\theta = \sigma^*d\theta = -\sigma^*\omega,$$

which yields the desired statement, as  $\sigma^*\omega = 0$  on  $X$  if and only if  $\omega = 0$  on  $\Gamma_\sigma$ , by virtue of  $\sigma$  being an embedding.  $\square$

With these definitions out of the way, we present a number of theorems characterizing neighborhoods of special submanifolds of symplectic manifolds.

**Theorem 24** (Symplectic neighborhood theorem). *Let  $(M_0, \omega_0), (M_1, \omega_1)$  be symplectic manifolds with compact symplectic submanifolds  $Q_0, Q_1$  respectively. Suppose there is an isomorphism  $\Phi : TQ_0^\omega \rightarrow TQ_1^\omega$  of symplectic normal bundles covering a symplectomorphism  $\phi : (Q_0, \omega_0) \rightarrow (Q_1, \omega_1)$ . Then  $\phi$  extends to a symplectomorphism  $\psi : (N(Q_0), \omega_0) \rightarrow (N(Q_1), \omega_1)$  such that  $d\psi$  induces the map  $\Phi$  on  $TQ_0^\omega$ .*

*Proof.* We use implicitly throughout that since  $Q$  is symplectic, there is an isomorphism  $TQ^\omega \rightarrow TQ^\perp$ . Let  $\exp_0, \exp_1$  be diffeomorphisms mapping neighborhoods of the zero section in the normal bundle to neighborhoods of  $Q_0, Q_1$  in  $X$ , respectively. Then we obtain

$$\phi' = \exp_1 \circ \Phi \circ \exp_0^{-1},$$

a diffeomorphism between these neighborhoods of  $Q_0$  and  $Q_1$ . Now  $\phi'^*\omega_1$  and  $\omega_0$  are two symplectic forms on  $M_0$  whose restrictions to  $Q_0$  agree. Now  $\phi'$  extends to the desired  $\psi$  by Theorem 2.1.  $\square$

**Theorem 25** (Lagrangian neighborhood theorem). *Let  $(M, \omega)$  be a symplectic manifold and let  $L \subset M$  be a compact Lagrangian submanifold. Then there exists a neighborhood  $N(\Gamma_0) \subset T^*L$  of the zero section  $\Gamma_0$ , a neighborhood  $U \subset M$  of  $L$ , and a diffeomorphism  $\phi : N(\Gamma_0) \rightarrow U$  such that  $\phi^*\omega = -d\theta$  and  $\phi|_L = \text{id}$ , where  $\theta$  is the canonical one-form on  $T^*L$ .*

We postpone the proof of this theorem until after the discussion of complex structures.

**3.2. Contact manifolds.** Let  $X$  be a differential manifold and  $H \subset TX$  be a smooth hyperplane field, i.e. a smooth subbundle of codimension one. Then, locally on some open  $U$ , we can write  $H = \ker \alpha$ , for  $\alpha \in \Omega_1(U)$ . In fact, if we assume that  $H$  is **coorientable**, we can extend  $U$  to all of  $X$ .<sup>12</sup> We will assume for what follows that  $H$  is coorientable.

**Definition 26.** Let  $X$  be a manifold of odd dimension  $2n+1$ . A **contact structure** on  $X$  is a hyperplane field  $H = \ker \alpha$  where the top-dimensional form  $\alpha \wedge (d\alpha)^n$  is nowhere vanishing. We call  $\alpha$  a **contact form**, and the pair  $(X, H)$  a **contact manifold**.

*Remark 27.* Suppose we have  $\alpha, \alpha' \in \Omega^1(X)$  such that  $H = \ker \alpha = \ker \alpha'$ . Then  $\alpha$  is a contact form if and only if  $\alpha'$  is. This is because the condition that  $\alpha, \alpha'$  cut out  $H$  requires  $\alpha' = f\alpha$  for some nonzero  $f : X \rightarrow \mathbb{R}$ .

*Remark 28.* In the language of distributions,  $H$  can be described as a codimension one distribution that is maximally non-integrable in the following sense. Recall that a distribution on  $X$  is said to be integrable if every point  $p$  of  $X$  is contained in a integral manifold of  $H$ , i.e. in a nonempty immersed submanifold  $N \subset X$  such that  $T_p N = H_p$ . The Frobenius theorem tells us that  $H$  is integrable if and only if  $H$  is involutive, i.e.  $H$  is closed under the Lie bracket of local sections. Now, since

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X, Y],$$

we find that  $H$  is integrable if and only if  $d\alpha = 0$  on  $H$ . Thus asking for  $d\alpha$  to be nondegenerate on  $H$  forces the distribution to be “as non-integrable as possible.”

Indeed, we obtain the above definition of a contact structure by noting that  $d\alpha$  is nondegenerate on  $H$  if and only if  $\alpha \wedge (d\alpha)^n$  is nowhere vanishing, as follows. By remark 5,  $d\alpha$  is nondegenerate on  $H$  if and only if  $(d\alpha)^n$  is nowhere vanishing, but this is simply equivalent to asking that  $\alpha \wedge (d\alpha)^n$  be nowhere vanishing.

Armed simply with the definition of a contact manifold, one might think that contact geometry is somewhat obscure. We provide the following list of examples as evidence that contact manifolds are actually quite common.

**Example 29.** Let  $X = \mathbb{R}^{2n+1}$  with coordinates  $(x^1, \dots, x^n, y^1, \dots, y^n, z)$ . The one-form

$$\alpha = dz + x^i dy^i$$

is a contact form, as

$$\alpha \wedge (d\alpha)^n = dz \wedge dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n,$$

which is nowhere vanishing. We define the standard contact structure on  $\mathbb{R}^{2n+1}$  to be  $H = \ker \alpha$ .

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<sup>12</sup>Why?

For the next few examples the following lemma will be useful.

**Lemma 30.** *Let  $(M, \omega)$  be a symplectic manifold of dimension  $2n$ . A vector field  $Y$  on  $M$  satisfying  $\mathcal{L}_Y \omega = \omega$  is called a **Liouville vector field**. In this case,  $\alpha = \iota_Y \omega$  is a contact form on any hypersurface  $Q \subset M$  transverse to  $Y$  (i.e. at any point  $p$ ,  $T_p Q$  and  $Y_p$  span  $T_p M$ ).*

*Proof.* Cartan's magic formula in this case tells us that  $\omega = d\iota_Y \omega$ , and hence

$$\begin{aligned} \alpha \wedge (d\alpha)^{n-1} &= \iota_Y \omega \wedge \omega^{n-1} \\ &= \iota_Y (\omega^n) / n. \end{aligned}$$

Now, since  $\omega^n$  is a volume form on  $M$ , we find that  $\alpha \wedge (d\alpha)^{n-1}$  is a volume form when restricted to the tangent bundle of any hypersurface transverse to  $Y$ .  $\square$

**Example 31.** Consider  $M = \mathbb{R}^4$  with its usual symplectic form  $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$ . The vector field

$$Y = \frac{1}{2} \left( x_1 \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial y^1} + x^2 \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^2} \right)$$

is clearly transverse to the sphere  $S^3$  given by  $(x^1)^2 + (y^1)^2 + (x^2)^2 + (y^2)^2 = 1$ . It is a straightforward computation to check that  $Y$  is Liouville, using the identity

$$(\mathcal{L}_Y \omega)(v, w) = \mathcal{L}_Y(\omega(v, w)) - \omega([Y, v], w) - \omega(v, [Y, w]).$$

We conclude, using the previous lemma, that  $S^3$  is a contact manifold, with a contact structure  $\ker \iota_Y \omega$ . This example is easily extended to show that  $S^{2n+1}$  has a contact structure.

**Example 32.** Let  $(M, g)$  be a Riemannian  $n$ -manifold. We define the **unit cotangent bundle**

$$ST^*M = \{(p, \xi) \in T^*M \mid |\xi_p|_g^2 = 1\} \subset T^*M.$$

The unit cotangent bundle is a manifold of dimension  $2n - 1$  as it can be written as the level set of a Hamiltonian  $H(p, \xi) = |\xi_p|_g^2 / 2$ . Moreover, it is a sub-fiber bundle of the cotangent bundle, with fiber  $S^{n-1}$ . We claim that the canonical one-form on  $T^*M$  is a contact form for  $ST^*M$ . Indeed, let  $Y$  be a vector field on  $T^*M$  given by  $\iota_Y \omega = \theta$ . Then  $Y$  is Liouville:  $d(\iota_Y \omega) = d\theta = \omega$ . In coordinates,  $Y = p^i \partial / \partial p^i$ , and hence is transverse to  $ST^*M$ . Note that if  $M$  is compact, so is  $SY^*M$  and in this case  $ST^*M$  is an example of a compact contact manifold.

**Example 33.** Let  $(M, H = \ker \alpha)$  be a contact manifold. Then, if  $\pi_M : M \times \mathbb{R} \rightarrow M$  is the projection onto the second factor, we claim that  $(M \times \mathbb{R}, \omega = d(e^t \pi_M^* \alpha))$  is a symplectic manifold. Indeed, if  $M$  has dimension  $2n - 1$ , we compute

$$\begin{aligned} \omega^n &= (e^t dt \wedge \pi_M^* \alpha + \pi_M^* d\alpha)^n \\ &= ne^{nt} dt \wedge \pi_M^* \alpha \wedge \pi_M^* (d\alpha)^{n-1} \\ &= ne^{nt} dt \wedge \pi_M^* (\alpha \wedge (d\alpha)^{n-1}) \\ &\neq 0. \end{aligned}$$

We call  $(M \times \mathbb{R}, d(e^t \pi_M^* \alpha))$  the **symplectization** of  $(M, \alpha)$ . Note that  $\partial / \partial t$  is a Liouville vector field for  $\omega$ <sup>13</sup> and  $M \subset M \times \mathbb{R}$  is a hypersurface transverse to  $\partial / \partial t$ .

---

<sup>13</sup>compute!

**Definition 34.** A **contactomorphism** from  $(M_1, H_1)$  to  $(M_2, H_2)$  is a diffeomorphism  $f : M_1 \rightarrow M_2$  such that  $df(H_1) = H_2$ . Equivalently, if  $H_1 = \ker \alpha_1$  and  $H_2 = \ker \alpha_2$  then we require  $f^* \alpha_2 = g \alpha_1$  for some nowhere vanishing function  $g : M_1 \rightarrow \mathbb{R} \setminus \{0\}$ .

## 4. WEEK 4

## 4.1. Symplectic linear group and linear complex structures.

**Definition 35.** Let  $(V, \omega)$  be a symplectic vector space. We denote the group of symplectomorphisms from  $V$  to itself as  $\text{Sp}(V, \omega)$ , the **symplectic linear group**. In the case of the standard symplectic structure on  $\mathbb{R}^{2n}$  we write the group as  $\text{Sp}(2n)$ .

**Lemma 36.** A real  $2n \times 2n$  matrix  $\Psi$  is in  $\text{Sp}(2n)$  if and only if

$$\Psi^\top J_0 \Psi = J_0,$$

where

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \text{Sp}(2n).$$

*Proof.* Let  $u_i, v_i$  be a symplectic basis for  $V$ . For  $x, y \in V$  write  $x = (a, b), y = (c, d)$  for  $a, b, c, d \in \mathbb{R}^n$ . Then

$$\omega(x, y) = a^i d^i - b^i c^i = -x^\top J_0 y.$$

Clearly  $\Psi^* \omega = \omega$  if and only if  $\Psi^\top J_0 \Psi = J_0$ . □

**Definition 37.** Let  $V$  be a vector space. A **complex structure** on  $V$  is an automorphism  $J : V \rightarrow V$  such that  $J^2 = -\text{id}_V$ . We denote the set of all complex structures on  $V$  by  $\mathcal{J}(V)$ . Now suppose  $(V, \omega)$  is a symplectic vector space. We say that a complex structure  $J$  is **compatible** with  $\omega$  if

$$\omega(Jv, Jw) = \omega(v, w)$$

for all  $v, w \in V$ , and

$$\omega(v, Jv) > 0$$

for all nonzero  $v \in V$ . We denote the set of all compatible complex structures on  $(V, \omega)$  by  $\mathcal{J}(V, \omega)$ .

**Lemma 38.** Let  $J \in \mathcal{J}(V, \omega)$  be a compatible complex structure on  $(V, \omega)$ . Then

$$g_J(v, w) = \omega(v, Jw)$$

defines an inner product on  $V$ .

**Lemma 39.** Let  $(V, \omega)$  be a symplectic vector space and  $J$  be a complex structure on  $V$ . Then the following are equivalent:

- (a)  $J$  is compatible with  $\omega$ ;
- (b) the bilinear form  $g_J : V \times V \rightarrow \mathbb{R}$  defined by

$$g_J(v, w) = \omega(v, Jw)$$

is symmetric, positive-definite, and  $J$ -invariant.

- (c) if we view  $V$  as a complex vector space with  $J$  as its complex structure, the form  $H : V \times V \rightarrow \mathbb{C}$  defined by

$$H(v, w) = \omega(v, Jw) + i\omega(v, w)$$

is complex linear in  $w$ , complex antilinear in  $v$ , satisfies  $H(w, v) = \overline{H(v, w)}$ , and has a positive-definite real part. Such a form is called a **Hermitian inner product** on  $(V, J)$ .

*Proof.* That (a) implies (b) is clear from Lemma 38. For (b) implies (c), note first that the real part of  $H$  is simply  $g_J$  and hence is positive-definite. For linearity, we compute

$$\begin{aligned} H(Jv, w) &= \omega(Jv, Jw) + i\omega(Jv, w) \\ &= g_J(Jv, w) - ig_J(w, v) \\ &= g_J(w, Jv) - ig_J(v, w) \\ &= -iH(v, w), \end{aligned}$$

and

$$\begin{aligned} H(v, Jw) &= -\omega(v, w) + i\omega(Jv, Jw) \\ &= -\omega(v, w) + ig_J(Jv, w) \\ &= -\omega(v, w) + i\omega(v, w) \\ &= iH(v, w), \end{aligned}$$

as desired. Finally, note that

$$\begin{aligned} H(w, v) &= \omega(w, Ju) + i\omega(w, v) \\ &= \omega(v, Jw) - i\omega(v, w) \\ &= \overline{H(v, w)}. \end{aligned}$$

For (c) implies (a),  $\omega(v, Jv) > 0$  because the real part  $\omega(v, Jw)$  is by hypothesis positive-definite. Moreover,  $\omega(Jv, Jw) = \operatorname{im} H(Jv, Jw) = \operatorname{im} H(v, w) = \omega(v, w)$ .  $\square$

The following result shows that all linear complex structures are isomorphic to the standard complex structure.

**Proposition 40.** *Let  $V$  be a  $2n$ -dimensional real vector space and let  $J \in \mathcal{J}(V)$ . Then there exists a vector space isomorphism  $\Phi : \mathbb{R}^{2n} \rightarrow V$  such that*

$$J\Phi = \Phi J_0.$$

*Proof.* Consider the extension  $J^\mathbb{C}$  of  $J$  to the complexification  $V^\mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C} \cong V$  given by  $J \otimes 1$ . Clearly  $J^\mathbb{C}$  is a complex structure on  $V^\mathbb{C}$  and thus has eigenvalues  $\pm i$ . We obtain a direct sum decomposition  $V^\mathbb{C} \cong E^+ \oplus E^-$  of the  $\pm i$  eigenspaces respectively, i.e.  $J^\mathbb{C}|_{E^\pm} = \pm iI$ . Clearly  $\dim_{\mathbb{C}} E^\pm = n$ . We claim that a basis  $w_j = u_j + iv_j$  for  $E^+$  yields a basis  $u_j, v_j$  for  $V$ . It suffices to show that these vectors are linearly independent. Since  $w_j$  is a basis for  $E^+$ ,

$$\sum_{j=1}^n (a_j + ib_j)(u_j \otimes 1 + v_j \otimes i) = 0$$

for  $a_j, b_j \in \mathbb{R}$  implies that  $a_j = b_j = 0$  for all  $j$ . Suppose there exist  $\alpha_j, \beta_j \in \mathbb{R}$  such that

$$\sum_{j=1}^n \alpha_j u_j + \beta_j v_j = 0.$$



Now since  $w_j \in \ker(I - iJ)$ , a straightforward computation reveals that  $Ju_j = -v_j$  and  $Jv_j = u_j$ . Applying  $J$  to the above equation, we obtain

$$\sum_{j=1}^n \beta_j u_j - \alpha_j v_j = 0.$$

Then, taking  $a_j = \beta_j, b_j = \alpha_j$ , we find that

$$\begin{aligned} \sum_{j=1}^n (\beta_j + i\alpha_j)(u_j \otimes 1 + v_j \otimes i) &= \left( \sum_{j=1}^n \beta_j u_j - \alpha_j v_j \right) \otimes 1 + \left( \sum_{j=1}^n \beta_j v_j + \alpha_j u_j \right) \otimes i \\ &= 0. \end{aligned}$$

Linear independence of the  $w_j$  now forces  $\alpha_j = \beta_j = 0$ . Hence  $u_j, v_j$  forms a basis for  $V$ .

The required  $\Phi : \mathbb{R}^{2n} \rightarrow V$  can now be written explicitly as

$$\Phi(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{j=1}^n (x_j u_j - y_j v_j).$$

This map is clearly an isomorphism; moreover, if  $x = (r_1, \dots, r_n, s_1, \dots, s_n) \in \mathbb{R}^{2n}$  then

$$J\Phi x = -s_1 u_1 - r_1 v_1 - \dots - s_n u_n - r_n v_n = \Phi J_0 x,$$

as desired.  $\square$

*Remark 41.* Define an action of  $\mathrm{GL}(2n, \mathbb{R})$  on the set  $\mathcal{J}(V)$  by  $g \cdot J = g^{-1} J g$ . By Lemma 40,  $\mathrm{GL}(2n, \mathbb{R}) \cdot J_0 = \mathcal{J}(V)$ , i.e. the orbit of  $J_0$  is the entire set. Moreover, since  $\mathrm{GL}(n, \mathbb{C})$  is naturally embedded (as a Lie subgroup) in  $\mathrm{GL}(2n, \mathbb{R})$  as  $\{A \in \mathrm{GL}(2n, \mathbb{R}) \mid J_0 A = A J_0\}$ , the stabilizer of  $J_0$  is  $\mathrm{GL}(n, \mathbb{C})$ .<sup>14</sup> We conclude that  $\mathcal{J}(V)$  can be given the structure of a smooth manifold such that  $\mathcal{J}(V) \cong \mathrm{GL}(2n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{C})$ .

The following result shows that the choice of complex structure compatible with a fixed symplectic form on  $V$  is canonical up to homotopy.

**Proposition 42.** *The set  $\mathcal{J}(V, \omega)$  of compatible complex structures is naturally identified with the space  $\mathcal{P}$  of symmetric positive-definite symplectic matrices. In particular,  $\mathcal{J}(V, \omega)$  is contractible.*

*Proof.* By fixing a symplectic basis for  $V$  we may assume that  $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$ . By the proof of Lemma 36, we note that  $J \in \mathrm{Aut}(\mathbb{R}^{2n})$  is a compatible complex structure if and only if the conditions

$$\begin{aligned} J^2 &= -\mathrm{id}_{\mathbb{R}^{2n}}, \\ J_0 &= J^\top J_0 J, \\ 0 &< -v^\top J_0 J v, \end{aligned}$$

hold (for  $v \neq 0$ ). Set  $P = J_0 J$ .  $P$  is symmetric, since

$$(J_0 J)^\top = -J^\top J_0 = J^\top J_0 J^2 = J_0 J,$$

<sup>14</sup>The embedding is given by replacing each entry  $a + bi$  with a block of the form  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ .

as well as positive-definite, and symplectic. Moreover, it is easy to check that if any matrix  $P$  has these three properties, then  $J = -J_0P$  is a compatible complex structure. Hence  $\mathcal{J}(V, \omega)$  is in bijective correspondence with the space  $\mathcal{P}$  of symmetric positive-definite symplectic matrices. It remains to show that  $\mathcal{P}$  is contractible. Suppose, for now, that if  $P \in \mathcal{P}$  then  $P^\alpha \in \mathcal{P}$  for all  $\alpha > 0$ ,  $\alpha \in \mathbb{R}$ . Then the map  $h : [0, 1] \times \mathcal{P} \rightarrow \mathcal{P}$  given by  $h(t, P) = P^{1-t}$  is a homotopy from  $\text{id}_{\mathcal{P}}$  to the constant map  $P \mapsto \text{id}_V$ , and we are done.

We now show that if  $P \in \mathcal{P}$  then  $P^\alpha \in \mathcal{P}$  for all  $\alpha > 0$ . It is easy to see that  $P^\alpha$  is symmetric and positive-definite. It remains to show that  $\omega_0(P^\alpha v, P^\alpha w) = \omega_0(v, w)$  for all  $\alpha > 0$ . Decompose  $\mathbb{R}^{2n}$  into eigenspaces  $V_\lambda$  for eigenvalues  $\lambda$  of  $P$ . Note that for a symplectic matrix  $P$ , if  $\lambda, \lambda'$  are eigenvalues such that  $\lambda\lambda' \neq 1$  then  $\omega_0(z, z') = 0$ , where  $z, z'$  are the eigenvectors of  $\lambda, \lambda'$ , respectively:

$$\lambda\lambda'\omega_0(z, z') = \omega_0(Pz, Pz') = \omega_0(z, z').$$

Now, since  $V_\lambda$  is also the eigenspace for the eigenvalue  $\lambda^\alpha$  for  $P^\alpha$ , if  $z \in V_\lambda, z' \in V_{\lambda'}$ ,

$$\omega_0(P^\alpha z, P^\alpha z') = (\lambda\lambda')^\alpha \omega_0(z, z').$$

Writing any  $v, w \in \mathbb{R}^{2n}$  in the basis of eigenvectors for  $P^\alpha$ , we find by linearity, and the remarks about  $\lambda, \lambda'$  above, that  $\omega_0(P^\alpha v, P^\alpha w) = \omega_0(v, w)$  for all  $\alpha > 0$ .  $\square$

Often it is enough to consider a slightly weaker notion of compatibility.

**Definition 43.** A complex structure  $J \in \mathcal{J}(V)$  is called  $\omega$ -**tame** if  $\omega(v, Jv) > 0$  for all nonzero  $v \in V$ . The set of all  $\omega$ -tame complex structures on  $V$  is written  $\mathcal{J}_\tau(V, \omega)$ . Note that  $\mathcal{J}_\tau(V, \omega)$  is an open subset of  $\mathcal{J}(V) \cong \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$  (as per Remark 41).

In this case, we note that  $g_J(v, w) = (\omega(v, Jw) + \omega(w, Jv))/2$  defines an inner product on  $V$ , for all  $J \in \mathcal{J}_\tau(V, \omega)$ . We note that there is an analog of Proposition 42 for  $\omega$ -tame complex structures.

**Proposition 44.** *The space  $\mathcal{J}_\tau(V, \omega)$  is contractible.*

*Proof.* See, for instance, McDuff/Salamon or Gromov.  $\square$

#### 4.2. Symplectic vector bundles.

**Definition 45.** A **symplectic vector bundle**  $(E, \omega)$  over  $X$  is a real vector bundle  $\pi : E \rightarrow X$  together with a smooth symplectic bilinear form  $\omega \in \Gamma(X, E^* \wedge E^*)$ , i.e. a symplectic bilinear form on each  $E_x$  that varies smoothly with  $x$ . A **complex structure** on  $\pi : E \rightarrow M$  is a bundle automorphism  $J : E \rightarrow E$  such that  $J^2 = -\text{id}_E$ . We say  $J$  is **compatible** with  $\omega$  if the induced complex structure on  $E_x$  is compatible with  $\omega_x$  for all  $x \in X$ . We thus obtain a symmetric, positive-definite bilinear form  $g_J \in \Gamma(X, \text{Sym}^2 E^*)$ , and we call the triple  $(E, \omega, g_J)$  a **Hermitian structure** on  $E$ .

**Theorem 46.** *Let  $E \rightarrow X$  be a  $2n$ -dimensional vector bundle. For any symplectic structure  $\omega$  on  $E$ , the space of compatible complex structures is nonempty and contractible. For any complex structure  $J$  on  $E$ , the space of symplectic structures compatible with  $J$  is nonempty and contractible.*

*Proof.* See McDuff/Salamon.<sup>15</sup>  $\square$

<sup>15</sup>Understand this!

We now prove the Theorem 25, the Lagrangian neighborhood theorem, with the help of the following lemma.

**Lemma 47.** *Let  $J \in \mathcal{J}(V, \omega)$ . Then a subspace  $\Lambda \subset V$  is Lagrangian if and only if  $J\Lambda^\perp = \Lambda$  with respect to  $g_J$ .*

*Proof.* For  $v \in \Lambda, w \in V$ , the assertion that

$$g_J(Jv, w) = \omega(Jv, Jw) = \omega(v, w) = 0$$

implies that  $\Lambda$  is Lagrangian if and only if  $J\Lambda^\perp = \Lambda$ .  $\square$

**Theorem 48** (Lagrangian neighborhood theorem). *Let  $(M, \omega)$  be a symplectic manifold and let  $L \subset M$  be a compact Lagrangian submanifold. Then there exists a neighborhood  $N(\Gamma_0) \subset T^*L$  of the zero section  $\Gamma_0$ , a neighborhood  $U \subset M$  of  $L$ , and a diffeomorphism  $\phi : N(\Gamma_0) \rightarrow U$  such that  $\phi^*\omega = -d\theta$  and  $\phi|_L = \text{id}$ , where  $\theta$  is the canonical one-form on  $T^*L$ .*

*Proof.* By Theorem 46, we can fix an arbitrary complex structure  $J$  on the tangent bundle  $TM$  and denote the associated metric by  $g_J$ . Note that the metric yields a diffeomorphism of bundles  $\Phi : T^*L \rightarrow TL$  given by

$$g_J(\Phi_q(v^*), v) = v^*(v)$$

for  $v \in T_qL, v^* \in T_q^*L$ . Now the map  $\phi : T^*L \rightarrow M$  defined by

$$\phi(q, v^*) = \exp_q(J_q\Phi_q v^*)$$

is a diffeomorphism from some neighborhood  $N(\Gamma_0)$  of  $\Gamma_0$  onto its image  $U$ , where  $\exp$  is the exponential map on  $M$  corresponding to  $g_J$ .

Now if  $v = (v_0, v_1^*) \in T_{(q,0)}T^*L = T_qL \oplus T_q^*L$ , we claim that

$$d\phi_{(q,0)}(v) = v_0 + J_q\Phi_q v_1^*.$$

By linearity, it suffices to compute  $d\phi_{(q,0)}$  on  $T_qL$  and  $T_q^*L$  separately. In particular, let  $c : [0, 1] \rightarrow TM$  be a curve given by  $c(t) = (a(t), 0)$ , with  $c'(0) = (v_0, 0)$ . Then

$$\begin{aligned} d\phi_{(q,0)}(v_0, 0) &= \left. \frac{d}{dt} \right|_{t=0} \exp_{a(t)}(J_{a(t)}\Phi_{a(t)}0) \\ &= \left. \frac{d}{dt} \right|_{t=0} a(t) \\ &= v_0. \end{aligned}$$

Next take  $c(t) = (q, tv_1^*)$ . Clearly  $c'(0) = (0, v_1^*)$ . Then

$$\begin{aligned} d\phi_{(q,0)}(0, v_1^*) &= \left. \frac{d}{dt} \right|_{t=0} \exp_p(J_p\Phi_p tv_1^*) \\ &= J_p\Phi_p v_1^*, \end{aligned}$$

as desired.

We can now compute, for  $v = (v_0, v_1^*), w = (w_0, w_1^*) \in T_{(q,0)}T^*L$ ,

$$\begin{aligned}
\phi^* \omega_{(q,0)}(v, w) &= \omega_q(v_0 + J_q \Phi_q v_1^*, w_0 + J_q \Phi_q w_1^*) \\
&= \omega_q(v_0, J_q \Phi_q w_1^*) - \omega_q(w_0, J_q \Phi_q v_1^*) \\
&= g_J(v_0, \Phi_q w_1^*) - g_J(w_0, \Phi_q v_1^*) \\
&= w_1^*(v_0) - v_1^*(w_0) \\
&= -d\theta_{(q,0)}(v, w).
\end{aligned}$$

This shows that  $\phi^* \omega = -d\theta$  on the zero section. Now the result follows from Moser's trick, Theorem [2.1](#).  $\square$

## 5. WEEK 5

## 5.1. Almost complex manifolds.

**Definition 49.** Let  $M$  be a  $2n$ -dimensional real manifold. An **almost complex structure** on  $M$  is a complex structure  $J$  on the tangent bundle  $TM$ . In this situation we say that  $(M, J)$  is an almost complex manifold. The almost complex structure is **compatible** with a nondegenerate two-form  $\omega$  on  $M$  if  $J$  is compatible with  $\omega$ .

**Theorem 50.** *For each nondegenerate two-form  $\omega$  on  $M$  the space of almost complex structures compatible with  $\omega$  is nonempty and contractible. Conversely, for every almost complex structure on  $M$  the space of compatible nondegenerate two-forms is nonempty and contractible.*

*Proof.* See Theorem 46. □

**Example 51.** Let  $X \subset \mathbb{R}^3$  be an oriented hypersurface. Let  $\nu : X \rightarrow S^2$  be the Gauss map, which assigns to each point  $x \in X$  the outward-pointing normal vector  $\nu(x) \perp T_x X$ . Define, for  $u \in T_x X$ ,

$$J_x u = \nu(x) \times u,$$

where the product is the vector (cross) product on  $\mathbb{R}^3$ . It follows from the vector triple product identity  $a \times (b \times c) = b(g(a, c)) - c(g(a, b))$ , where  $g$  is the standard metric on  $\mathbb{R}^3$ , that  $J_x^2 = -\text{id}_{T_x X}$ . Define a two-form  $\omega$  on  $X$  by

$$\begin{aligned} \omega(v, w) &= \iota(\nu(x))\Omega \\ &= g(\nu(x), v \times w), \end{aligned}$$

where  $\Omega(u, v, w)$  is the determinant of the matrix whose columns are  $u, v, w$ . It is straightforward to check that  $J$  is compatible with  $\omega$ : for  $v, w \in T_x X$ ,

$$\begin{aligned} \omega(J_x v, J_x w) &= g(\nu(x), (\nu(x) \times v) \times (\nu(x) \times w)) \\ &= g(\nu(x), \nu(x)g(\nu(x) \times v, w)) \\ &= g(w, \nu(x) \times v) \\ &= g(\nu(x), v \times w) \\ &= \omega(v, w) \\ \omega(v, J_x v) &= g(\nu(x), v \times (\nu(x) \times v)) \\ &= g(\nu(x), g(v, v)\nu(x)) \\ &= g(v, v) \\ &> 0, \end{aligned}$$

where we have used the vector triple product identity as well as the cyclic property of the scalar triple product.

**Example 52.** Consider  $S^2 \subset \mathbb{R}^3$  with the almost complex structure  $J$  from the previous example. We compute the expression of  $J$  in stereographic coordinates. Recall we have  $\phi : S^2 - (0, 0, 1) \rightarrow \mathbb{R}^2$  given by

$$\phi(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right)$$

and inverse

$$\psi(X, Y) = \left( \frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{X^2 + Y^2 - 1}{X^2 + Y^2 + 1} \right).$$

For a point  $p = (x, y, z) \in S^2$  and a vector  $u = (v, w) \in T_p S^2$ , some computation reveals that

$$\begin{aligned} J_p(v, w) &= d\phi((x, y, z) \times d\psi(v, w)) \\ &= (w, -v). \end{aligned}$$

**Definition 53.** Let  $(X, J)$  be an almost complex manifold. We define the **Nijenhuis tensor**  $N_J$  by

$$N_J(v, w) = [v, w] + J[Jv, w] + J[v, Jw] - [Jv, Jw]$$

for  $v, w$  vector fields on  $X$ .

**Lemma 54.** *The Nijenhuis tensor is a skew-symmetric covariant  $(2,0)$ -tensor on  $X$  satisfying*

- (a)  $N_J(v, Jv) = 0$  for all vector fields  $v$ ;
- (b)  $N_{J_0} = 0$ ;
- (c) If  $\phi \in \text{Diff}(M)$  and  $v, w$  are vector fields then

$$N_{\phi^* J}(\phi^* v, \phi^* w) = \phi^* N_J(v, w).$$

*Proof.* Writing  $v = v^i \partial / \partial x^i, w = w^j \partial / \partial x^j$  in local coordinates, the Lie bracket  $[v, w]$  becomes<sup>16</sup>

$$[v, w] = \left( w^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial w^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

Finish this. □

Suppose now that  $(X, J)$  is an almost complex manifold. Denote by  $T_{\mathbb{C}}X$  the complexification of the real vector bundle  $TX$ , i.e.  $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$ . We note that the complexified tangent bundle splits into  $\pm i$   $J$ -eigenbundles  $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ , respectively.<sup>17</sup> These are often referred to as the holomorphic and antiholomorphic tangent bundles of  $X$ .

**Definition 55.** Let  $X$  be an almost complex manifold. We define the vector bundles

$$\begin{aligned} \bigwedge_{\mathbb{C}}^k X &\equiv \bigwedge^k (T_{\mathbb{C}}X)^* \\ \bigwedge^{p,q} X &\equiv \bigwedge^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^q (T^{0,1}X)^*. \end{aligned}$$

and write  $\mathcal{A}_{X, \mathbb{C}}^k$  and  $\mathcal{A}_X^{p,q}$  for their sheaves of sections, respectively. We denote the projections  $\mathcal{A}^{\bullet} \rightarrow \mathcal{A}^k$  and  $\mathcal{A}^{\bullet} \rightarrow \mathcal{A}^{p,q}$  by  $\Pi^k$  and  $\Pi^{p,q}$  respectively. It is not hard to show that

$$\begin{aligned} \bigwedge_{\mathbb{C}}^k X &= \bigoplus_{p+q=k} \bigwedge^{p,q} X \\ \mathcal{A}_{\mathbb{C}}^k &= \bigoplus_{p+q=k} \mathcal{A}^{p,q} \end{aligned}$$

<sup>16</sup>We follow McDuff/Salamon in the convention that  $[v, w] \equiv -\mathcal{L}_v w$ .

<sup>17</sup>Here,  $J$  is really  $J \otimes \mathbb{C}$ .

and additionally, that  $\overline{\bigwedge^{p,q} X} = \bigwedge^{q,p} X$  and  $\overline{\mathcal{A}^{p,q}} = \mathcal{A}^{q,p}$ . Now if  $d : \mathcal{A}_{\mathbb{C}}^k \rightarrow \mathcal{A}_{\mathbb{C}}^{k+1}$  is the exterior derivative<sup>18</sup>, we write

$$\begin{aligned}\partial &\equiv \Pi^{p+1,q} \circ d \\ \bar{\partial} &\equiv \Pi^{p,q+1} \circ d,\end{aligned}$$

and  $\partial, \bar{\partial}$  satisfy the appropriate graded Leibniz rule.

With this notation now set, we come to the key definition.

**Proposition 56.** *Let  $(X, J)$  be an almost complex manifold. Then the following conditions are equivalent:*

- (a)  $d = \partial + \bar{\partial}$  on  $\mathcal{A}^\bullet$ ;
- (b)  $\Pi^{0,2} \circ d = 0$  on  $\mathcal{A}^{1,0}$ ;
- (c)  $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$ ;
- (d)  $N_J = 0$ .

If  $X$  satisfies one of these equivalent conditions then  $J$  is said to be an *integrable* almost complex structure.

*Proof.* We show that (a) is equivalent to (b), (b) is equivalent to (c), and that (c) is equivalent to (d).

For (a) $\leftrightarrow$ (b), suppose first that  $d = \partial + \bar{\partial}$  and  $\alpha \in \mathcal{A}^{1,0}$ . Then

$$\begin{aligned}\Pi^{0,2}d\alpha &= \Pi^{0,2}(\partial + \bar{\partial})\alpha \\ &= \Pi^{0,2}(\Pi^{2,0} + \Pi^{1,1})d\alpha \\ &= 0.\end{aligned}$$

Conversely, suppose  $\Pi^{0,2}d = 0$  on  $\mathcal{A}^{1,0}$ . Clearly  $d = \partial + \bar{\partial}$  if and only if  $d\alpha \in \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}$  for all  $\alpha \in \mathcal{A}^{p,q}$ . Now any  $\alpha \in \mathcal{A}^{p,q}$  can locally be written as a linear combination of terms of the form  $f_{IJ}w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge w'_{j_1} \wedge \cdots \wedge w'_{j_q}$ , with the  $w \in \mathcal{A}^{1,0}$  and  $w' \in \mathcal{A}^{0,1}$ . Then  $d\alpha$  is expressed as a linear combination of terms involving  $df_{IJ}$ ,  $dw_i$ , and  $dw'_j$ . We have that  $df \in \mathcal{A}_{\mathbb{C}}^2 = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$ , which takes care of the terms containing  $df_{IJ}$ . Similarly, since  $\Pi^{0,2}d = 0$  on  $\mathcal{A}^{1,0}$  by assumption,  $dw_i \in \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1}$ , which takes care of the terms containing the  $dw_i$ . Finally, we have that  $dw'_j \in \mathcal{A}^{1,1} \oplus \mathcal{A}^{0,2}$  since  $\Pi^{2,0}d = 0$  on  $\mathcal{A}^{0,1}$  (seen by conjugating (b)), which takes care of the terms containing the  $dw'_j$ . We conclude that  $d\alpha \in \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}$ , as desired.

We now prove (b) $\leftrightarrow$ (c). Fix any  $\alpha \in \mathcal{A}^{1,0}$  and  $v, w$  sections of  $T^{0,1}$ . Then, by definition of  $d\alpha$ , and since  $\alpha$  vanishes on  $T^{0,1}$ , we find that

$$\begin{aligned}(d\alpha)(v, w) &= v\alpha(w) - w\alpha(v) - \alpha[v, w] \\ &= -\alpha[v, w].\end{aligned}$$

We conclude that  $\Pi^{0,2}d = 0$  if and only if  $[v, w] \in T^{0,1}$ .

We now prove (c)  $\leftrightarrow$  (d). Suppose for now that any section of  $T^{0,1}$  can be written as  $v + iJv$  for  $v$  a section of  $TX \otimes \mathbb{C}$ . Then

$$[v + iJv, w + iJw] = [v, w] - [Jv, Jw] - i([Jv, w] + [v, Jw]).$$

This is of the form  $u + iJu$  if and only if

$$J([v, w] - [Jv, Jw]) = [Jv, w] + [v, Jw],$$

<sup>18</sup>Here,  $d$  is really  $d \otimes \mathbb{C}$ .

which is equivalent to  $N_J(v, w) = 0$ . It remains to show that any section of  $T^{0,1}$  can be written as  $v + iJv$ . Finish this.  $\square$

**Example 57.** Let  $X$  be a complex manifold. Then we have local coordinates  $z_i, \bar{z}_i$  for  $i = 1, \dots, n$  and the standard almost complex structure  $J_0$  acting as  $i$  on  $\partial/\partial z_i$  and  $-i$  on  $\partial/\partial \bar{z}_i$ . Now we note that for  $\alpha \in \mathcal{A}^{p,q}$  written  $\alpha = \alpha_{IJ} dz^I \wedge d\bar{z}^J$ , we have

$$d\alpha = \left( \frac{\partial \alpha_{IJ}}{\partial z^k} dz^k + \frac{\partial \alpha_{IJ}}{\partial \bar{z}^k} d\bar{z}^k \right) \wedge dz^I \wedge d\bar{z}^J.$$

Clearly then  $d = \partial + \bar{\partial}$ , as  $\partial = \Pi^{p+1,q}d$  and  $\bar{\partial} = \Pi^{p,q+1}$ . Hence, by Proposition 56(a),  $J_0$  is integrable.

The above example shows that complex manifolds induce integrable almost complex structures on their underlying real manifolds in a natural way. It is a highly nontrivial fact that the converse is also true.

**Theorem 58** (Newlander-Nirenberg, 1957). *Let  $(X, J)$  be an almost complex manifold. Then  $J$  is integrable if and only if  $X$  has a holomorphic atlas (making it a complex manifold) such that the induced almost complex structure is  $J$ .*

**Example 59.** Let  $(X, J)$  be a two-dimensional almost complex manifold. In this case  $\mathcal{A}_{\mathbb{C}}^2 = \mathcal{A}^{1,1}$  and hence by Proposition 56(b), we find that  $J$  is integrable. We conclude using the Newlander-Nirenberg theorem that every two-dimensional almost complex manifold is in fact a complex manifold.

**Example 60.** It turns out that there exists a vector product on  $\mathbb{R}^7$  that is bilinear and skew-symmetric, and hence it follows along the lines of Example 51 that every oriented hypersurface  $X \subset \mathbb{R}^7$  carries an almost complex structure. This argument shows, in particular, that  $S^6$  is an almost complex manifold. It was shown by Calabi, however, that this almost complex structure is not integrable. Indeed, the existence of an integrable almost complex structure on  $S^6$  is still an open problem.

## 5.2. Kähler manifolds.

**Definition 61.** A **Kähler** manifold is a symplectic manifold  $(M, \omega)$  equipped with an integrable almost complex structure  $J \in \mathcal{J}(M, \omega)$ .

**Example 62.** The most basic example of a Kähler manifold is  $(\mathbb{R}^{2n}, \omega_0, J_0)$ . Indeed, viewing  $\mathbb{R}^{2n}$  as  $\mathbb{C}^n$  we can introduce coordinates  $z^i = x^i + iy^i, \bar{z}^i = x^i - iy^i$  with respect to which  $T^{1,0}\mathbb{C}^n$  and  $T^{0,1}\mathbb{C}^n$  are trivialized by the frames  $\partial/\partial z^i$  and  $\partial/\partial \bar{z}^i$ , respectively. Then it is straightforward to check that  $d = \partial + \bar{\partial}$  on  $\mathcal{A}_{\mathbb{C}}^\bullet$ . In these coordinates,

$$\begin{aligned} dz^i &= dx^i + idy^i \\ d\bar{z}^i &= dx^i - idy^i. \end{aligned}$$

and a easy computation reveals that the symplectic form  $\omega_0$  can be written

$$\omega_0 = \frac{i}{2} \sum_{i=1}^n dz^i \wedge d\bar{z}^i.$$

In fact, if we let  $f = \sum_i^n \bar{z}^i z^i$ , we can write  $\omega_0 = i\partial\bar{\partial}f/2$ .

**Example 63.** Every two-dimensional symplectic manifold is Kähler with respect to any compatible almost complex structure.



**Example 64** (Complex projective space). Let  $\mathbb{P}^n$  denote the complex projective space, which is a complex manifold of dimension  $n$ . Let  $J$  be the induced integrable almost complex structure.