CHARACTERISTIC CLASSES II

NILAY KUMAR

1. Stiefel-Whitney classes

Recall from last time the axiomatic definition of the Stiefel-Whitney classes of a vector bundle $\xi : E \to B$:[1]

Theorem 1. *Let $\xi : E \to B$ be a real vector bundle. Then there exists a unique sequence of cohomology classes*

$$w_i(\xi) \in \mathrm{H}^i(B, \mathbb{F}_2)$$

for $i = 0, 1, 2, \dots$ called the Stiefel-Whitney classes of ξ satisfying the following properties:

(I) the class $w_0(\xi)$ is equal to the unit element $1 \in \mathrm{H}^0(B, \mathbb{F}_2)$ and $w_i(\xi) = 0$ for $i > \operatorname{rk} \xi$;

(II) if $f^*\xi$ is the pullback of E along $f : A \to B$ then $w_i(f^*\xi) = f^* w_i(\xi)$;

(III) if $\eta : E' \to B$ is another real vector bundle then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta);$$

(IV) if γ_1^1 is the tautological line bundle over $\mathbb{R}P^1$ then $w_1(\gamma_1^1) \in \mathrm{H}^1(\mathbb{R}P^1, \mathbb{F}_2) = \mathbb{F}_2$ is the unique nonzero element.

The total (inhomogeneous) Stiefel-Whitney class of ξ is the sum

$$w(\xi) = \sum_{i=0}^{\operatorname{rk} \xi} w_i(\xi) = 1 + w_1(\xi) + \cdots + w_{\operatorname{rk} \xi}(\xi) \in \mathrm{H}^\bullet(B, \mathbb{F}_2).$$

Thus given, the Stiefel-Whitney classes allowed us to make some strong statements about parallelizability and cobordisms. The goal of the first half of this talk is to sketch a proof of the above theorem, i.e. show that Stiefel-Whitney classes do indeed exist. We will first prove the existence of certain analogous classes:

Theorem 2. *Let $\iota_n : \mathrm{O}(n-1) \to \mathrm{O}(n)$ and $p_{ij} : \mathrm{O}(i) \times \mathrm{O}(j) \to \mathrm{O}(i+j)$ be the obvious inclusions. Denote by the same symbols the induced maps on classifying spaces. Then there are unique classes $w_i \in \mathrm{H}^i(\mathscr{B}\,\mathrm{O}(n), \mathbb{F}_2)$ satisfying:*

(I) $w_0 = 1$ and $w_i = 0$ if $i > n$;

(II) $\iota_n^* w_i = w_i$ (and hence $\iota_n^* w_n = 0$);

(III) $p_{ij}^* w_k = \sum_{a+b=k} w_a \otimes w_b$;

(IV) $w_1 \in \mathrm{H}^1(\mathscr{B}\,\mathrm{O}(1), \mathbb{F}_2) = \mathrm{H}^1(\mathbb{R}P^\infty, \mathbb{F}_2)$ is the unique nonzero element.

Date: January 20, 2016.

[1] Throughout, all base spaces are assumed to be connected and paracompact.

Here, as usual,

$$\mathscr{B} : \textsc{Grp} \to \textsc{Top}$$

denotes the classifying space functor. Almost all the work lies in computing the cohomology of $\mathscr{B}\,\mathrm{O}(n)$. In an earlier talk, Guchuan mentioned that – via a number of computations using spectral sequences – one can show the following:

Lemma 3. *Let Σ_n be the symmetric group on n letters. Then there is a map $\Psi_n : (\mathbb{R}P^\infty)^n \to \mathscr{B}\,\mathrm{O}(n)$ inducing an isomorphism*

$$\Psi_n^* : \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n), \mathbb{F}_2) \cong \mathrm{H}^\bullet((\mathbb{R}P^\infty)^n, \mathbb{F}_2)^{\Sigma_n} \cong \mathbb{F}_2[\sigma_1, \ldots, \sigma_n],$$

where Σ_n acts on $(\mathbb{R}P^\infty)^n$ by permutation and hence σ_i are the symmetric polynomials in n variables with $\deg \sigma_i = i$.

Proof. This is rather technical and requires more wizardry with spectral sequences than I am familiar with. Reference unfinished May. ☐

With this in hand, the proof of Theorem 2 is now a straightforward diagram chase with symmetric polynomials.

Proof of Theorem 2. We begin by proving existence of the classes w_i. Define the Stiefel-Whitney classes as

$$w_i \equiv (\Psi_n^*)^{-1}\sigma_i$$

where Ψ_n^* is the isomorphism from Lemma 3 above, and where

$$\begin{aligned}
\sigma_1 &= x_1 + \cdots + x_n \\
\sigma_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\
&\ \ \vdots \\
\sigma_n &= x_1 \cdots x_n,
\end{aligned}$$

are the symmetric polynomials on the generators of $\mathrm{H}^\bullet((\mathbb{R}P^\infty)^n, \mathbb{F}_2) \cong \otimes_i^n \mathbb{F}_2[x_i]$. Set $\sigma_0 = 1$ and $\sigma_i = 0$ for $i > n$. Under these definitions, axiom I is immediate.

Next, denoting $h_n : \mathscr{B}\,\mathrm{O}(1)^{n-1} \to \mathscr{B}\,\mathrm{O}(1)^n$ the map induced by the inclusion, we have a commutative diagram of inclusions

$$\begin{tikzcd}
\mathscr{B}\,\mathrm{O}(1)^{n-1} \arrow[r, "\Psi_{n-1}"] \arrow[d, "h_n"] & \mathscr{B}\,\mathrm{O}(n-1) \arrow[d, "\iota_n"] \\
\mathscr{B}\,\mathrm{O}(1)^n \arrow[r, "\Psi_n"] & \mathscr{B}\,\mathrm{O}(n)
\end{tikzcd}$$

and taking cohomology,

$$\begin{tikzcd}
\mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n)) \arrow[r, "\Psi_n^*"] \arrow[d, "\iota_n^*"] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1)^n) \arrow[d, "h_n^*"] \\
\mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n-1)) \arrow[r, "\Psi_{n-1}^*"] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1)^{n-1})
\end{tikzcd}$$

Clearly $h_n^* x_i = x_i$ for $i < n$ and $h_n^* x_n = 0$. Therefore $h_n^* \sigma_i = \sigma_i$, which implies – by the diagram above – that $\iota_n^* w_i = w_i$. So much for axiom II.

Consider now the commutative diagram of inclusions

$$\begin{tikzcd} \mathscr{B}\,\mathrm{O}(1)^i \times \mathscr{B}\,\mathrm{O}(1)^j \arrow[r, "\Psi_i \times \Psi_j"] \arrow[d, equal] & \mathscr{B}\,\mathrm{O}(i) \times \mathscr{B}\,\mathrm{O}(j) \arrow[d, "p_{ij}"] \\ \mathscr{B}\,\mathrm{O}(1)^{i+j} \arrow[r, "\Psi_{i+j}"] & \mathscr{B}\,\mathrm{O}(i+j) \end{tikzcd}$$

which after taking cohomology and applying Künneth becomes

$$\begin{tikzcd} \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(i+j)) \arrow[r, "\Psi_{i+j}^*"] \arrow[d, "p_{ij}^*"] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1)^{i+j}) \arrow[d] \\ \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(i)) \otimes \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(j)) \arrow[r, "\Psi_i^* \otimes \Psi_j^*"] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1)^i) \otimes \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1)^j) \end{tikzcd}$$

In this diagram, all the arrows are injective except for p_{ij}^* and hence p_{ij}^* is injective. Moreover,

$$(\Psi_i^* \otimes \Psi_j^*) p_{ij}^* w_k = \Psi_{i+j}^* w_k = \sigma_k(x_1, \ldots, x_{i+j}).$$

Some algebra with symmetric polynomials reveals that

$$\sigma_k(x_1, \ldots, x_{i+j}) = \sum_{a+b=k} \sigma_a(x_1, \ldots, x_i)\sigma_b(x_{i+1}, \ldots, x_{i+j}),$$

whence

$$(\Psi_i \otimes \Psi_j^*) p_{ij}^* w_k = \sum_{a+b=k} \Psi_i^* w_a \otimes \Psi_j^* w_b = (\Psi_i^* \otimes \Psi_j^*) \sum_{a+b=k} w_a \otimes w_b,$$

proving axiom III by injectivity of $\Psi_i \otimes \Psi_j$.

Axiom IV is clear: $w_1 = (\Psi_1^*)^{-1}\sigma_1(x_1) = x_1$, the nonzero element in $\mathrm{H}^1(\mathbb{R}P^\infty, \mathbb{F}_2)$. Finally, we prove uniqueness by induction on n. The base case $n = 1$ is trivial. Assume uniqueness of the w_i in $\mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(m), \mathbb{F}_2)$ for $m < n$. Then for $i < n$ the $w_i \in \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n), \mathbb{F}_2)$ are uniquely determined by axiom II and the fact that ι_n is an isomorphism in degrees smaller than n. For $i = n$ we note that $p_{1,n-1}^* w_n \in \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(1), \mathbb{F}_2) \otimes \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(n-1), \mathbb{F}_2)$ and hence w_n is determined by the induction hypothesis since $p_{1,n-1}^*$ is injective. This completes the proof. $\square$

To relate these classes sitting in the cohomology of $\mathscr{B}\,\mathrm{O}(n)$ to the previous axiomatic definition of classes sitting in the cohomology of the base B, we need a classification theorem for vector bundles on B. Before we start, we note that

$$\mathrm{Vect}_{\mathbb{R}}^n : \textsc{Top}^{\mathrm{op}} \to \textsc{Set}$$

will denote the contravariant functor taking B to the set (isomorphism classes) of real vector bundles over B and taking $f : B \to B'$ to the pullback $f^* : \mathrm{Vect}_{\mathbb{R}}^n B' \to \mathrm{Vect}_{\mathbb{R}}^n B$. Recall that two vector bundles over B are isomorphic if there is a map lifting id_B that is a fiberwise linear isomorphism.

Theorem 4. *The space $\mathscr{B}\,\mathrm{O}(n) \cong \mathrm{Gr}_n\,\mathbb{R}^\infty$ classifies rank n real vector bundles, i.e. the natural transformation*

$$\Phi : [-, \mathscr{B}\,\mathrm{O}(n)] \overset{\cdot}{\longrightarrow} \mathrm{Vect}_{\mathbb{R}}^n -,$$

given by pullback $[f] \mapsto f^*\gamma_\infty^n$ of the tautological bundle, is a natural isomorphism.

We first check that Φ is well-defined:

Lemma 5. *The pullbacks of a vector bundle along homotopic maps are isomorphic, i.e. the functor* $\mathrm{Vect}_{\mathbb{R}}$ *factors through the homotopy category.*

Proof sketch. Let $\xi : E \to B$ be a rank n vector bundle and let $f, g : A \to B$ be two maps homotopic via $h : A \times I \to B$. Note first that $h^*E|_{A\times\{0\}} = f^*E$ and $h^*E|_{A\times\{1\}} = g^*E$. Thus it suffices to prove that for a vector bundle $\eta : F \to A \times I$, there is an isomorphism $F|_{A\times\{0\}} \cong F|_{A\times\{1\}}$. The idea, roughly, is to find countably many local trivializations over $U_i \subset B$ for E and to then locally push $F|_{A\times\{0\}}$ to the right along $U_i \times I$. For details, see Hatcher, VBKT. $\square$

Proof of Theorem 4. Naturality of Φ follows immediately from the fact that if $\alpha : A \to B$ and $f \in [B, \mathscr{B}\mathrm{O}(n)]$ then $(f \circ \alpha)^*\gamma_n^\infty = \alpha^* f^* \gamma_n^\infty$. We now prove that $\Phi_B : [B, \mathscr{O}(n)] \to \mathrm{Vect}_{\mathbb{R}} B$ is a bijection.

The key observation is as follows. Let $\xi : E \to B$ be a rank n vector bundle. Then an isomorphism $E \cong f^*\gamma_n^\infty$ (for some map $f : B \to \mathrm{Gr}_n \mathbb{R}^\infty$) is equivalent to a map $g : E \to \mathbb{R}^\infty$ that is a linear injection on each fiber. To see this, suppose first that we have such an isomorphism. Then we have a commutative diagram

$$\begin{tikzcd} E \cong f^*\gamma_n^\infty \arrow[r, "\tilde{f}"] \arrow[d, "\xi"] & \gamma_n^\infty \arrow[r, "\pi"] \arrow[d] & \mathbb{R}^\infty \\ B \arrow[r, "f"] & \mathrm{Gr}_n \mathbb{R}^\infty & \end{tikzcd}$$

where π is the obvious projection. Now $\pi \circ \tilde{f} : E \to \mathbb{R}^\infty$ is a fiberwise linear injection as both f and π are. Conversely, given $g : E \to \mathbb{R}^\infty$ a fiberwise linear injection, we can define a map $f : B \to \mathrm{Gr}_n \mathbb{R}^\infty$ given by $x \mapsto [g(\xi^{-1}(x))]$. Then $E \cong f^*\gamma_n^\infty$ because we have fiberwise linear isomorphisms

$$f^*\gamma_n^\infty|_b \cong \gamma_n^\infty|_{f(b)} \cong E|_b.$$

Now, for surjectivity of Φ_B, given $\xi : E \to B$ it suffices by the previous paragraph to construct a map $E \to \mathbb{R}^\infty$ a linear injection on each fiber. To do this, we fix countably many local trivializations over $U_i \subset B$ of E together with partitions of unity ϕ_i subordinate to the U_i. Then for each i we obtain a map $g_i : E \to \mathbb{R}^n$ that is zero outside $\xi^{-1}U_i$ and the composition $E \to U_i \times \mathbb{R}^n \to \mathbb{R}^n$ otherwise. Summing $g = \sum g_i$, we obtain a map $g : E \to (\mathbb{R}^n)^\infty \cong \mathbb{R}^\infty$ that is obviously a linear injection on fibers.

For injectivity, suppose we have isomorphisms $E \cong f_0^*\gamma_n^\infty$ and $E \cong f_1^*\gamma_n^\infty$ for two maps $f_0, f_1 : B \to \mathrm{Gr}_n \mathbb{R}^\infty$. By arguments above, we obtain maps $g_0, g_1 : E \to \mathbb{R}^\infty$ that are fiberwise linear injections. We claim that g_0 and g_1 are homotopic through maps g_t that are fiberwise linear injections; this implies that f_0 and f_1 are homotopic via $f_t(x) = g_t(\xi^{-1}x)$. To do this, we first homotope g_0 so that it takes values only in odd coordinates via

$$(x_1, x_2, \ldots) \mapsto (1-t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, 0, \ldots)$$

and homotope g_1 so that it takes values only in even coordinates similarly. Now $g_t = (1-t)g_0 + tg_1$ provides the necessary homotopy and it is clearly linear and injective on fibers. $\square$

We can finally prove the existence of Stiefel-Whitney classes for vector bundles.

Proof of Theorem 9. Let $\xi : E \to B$ be a real vector bundle of rank n. By Theorem 4 there exists a unique map $\Phi_B : B \to \mathscr{B}O(n)$ such that $E \cong f^*\gamma_n^\infty$. Define

$$w_i(\xi) \equiv \Phi_B^* w_i.$$

Axiom I now follows immediately from Theorem 2. Now suppose we have a pullback diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\xi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Then

$$w_i(f^*\xi) = \Phi_A^* w_i = f^*\Phi_B^* w_i = f^* w_i(\xi),$$

which proves axiom II.

Invoking Künneth and axiom III of Theorem 2, the diagram

$$\begin{tikzcd} B \arrow[r, "\xi\times\eta"] \arrow[d, "\Delta"] & \mathscr{B}\mathrm{O}(i)\times\mathscr{B}\mathrm{O}(j) \arrow[r, "p_{ij}"] & \mathscr{B}\mathrm{O}(i+j) \\ B\times B, \arrow[ur, "f_\xi\times f_\eta"'] & & \end{tikzcd}$$

proves axiom III.

The tautological line bundle on $\mathbb{R}P^1$ is given by the pullback of γ_1^∞ along the inclusion $j : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$ so $w_1(\gamma_1^1) = j^* w_1$ is the unique nonzero element in $\mathrm{H}^1(\mathbb{R}P^1, \mathbb{Z}/2)$ by axiom IV of Theorem 2 and because j^* is an isomorphism in degrees $\leqslant 1$.

The proof of uniqueness is essentially identical to the proof in Theorem 2. $\square$

Recall that a vector bundle is called *orientable* if there is an assignment of orientation to each fiber as well as orientation-preserving local trivializations.

Proposition 6. *Let B be a connected CW complex and let $\xi : E \to B$ be a real vector bundle. Then E is orientable if and only if $w_1(\xi) = 0$.*

Proof. Let $f : B \to \mathscr{B}\mathrm{O}(n)$ be the map such that $\xi = f^*\gamma_n^\infty$. Then, by the universal coefficient theorem and the fact that $\mathrm{H}_1(B)$ is the abelianization of $\pi_1(B)$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathscr{B}\mathrm{O}(n),\mathbb{Z}_2) \arrow[r, "\sim"] \arrow[d, "f^*"] & \mathrm{Hom}(\mathrm{H}_1(\mathscr{B}\mathrm{O}(n)),\mathbb{Z}_2) \arrow[r, "\sim"] \arrow[d, "\circ f_*"] & \mathrm{Hom}(\pi_1(\mathscr{B}\mathrm{O}(n)),\mathbb{Z}_2) \arrow[d, "\circ f_*"] \\ \mathrm{H}^1(B,\mathbb{Z}_2) \arrow[r, "\sim"] & \mathrm{Hom}(\mathrm{H}_1(B),\mathbb{Z}_2) \arrow[r, "\sim"] & \mathrm{Hom}(\pi_1(B),\mathbb{Z}_2) \end{tikzcd}$$

where the horizontal arrows are isomorphisms. Note, now that $\pi_1(\mathscr{B}\mathrm{O}(n)) \cong \mathbb{Z}_2$, and so $w_1 \in \mathrm{H}^1(\mathscr{B}\mathrm{O}(n), \mathbb{Z}_2)$ corresponds to $\mathrm{id}_{\mathbb{Z}_2} \in \mathrm{Hom}(\pi(\mathscr{B}\mathrm{O}(n)), \mathbb{Z}_2)$. Hence $w_1(\xi) = f^* w_1$ corresponds to a map $\pi_1(B) \to \mathbb{Z}_2$ that is trivial if and only if $f_* = 0$. This is precisely the condition for f to lift to the universal cover $\mathscr{B}\mathrm{SO}(n) = \widetilde{\mathscr{B}\mathrm{O}(n)}$. We conclude that, since $\mathscr{B}\mathrm{SO}(n)$ is the classifying space for orientable rank n bundles, $w_1(\xi) = 0$ if and only if E is orientable. $\square$

2. Chern classes

Chern classes are the complex analog of Stiefel-Whitney classes, but with integral coefficients.

Theorem 7. *Let $\iota_n : \mathrm{U}(n-1) \to \mathrm{U}(n)$ and $p_{ij} : \mathrm{U}(i) \times \mathrm{U}(j) \to \mathrm{U}(i+j)$ be the obvious inclusions. Denote by the same symbols the induced maps on classifying spaces. Then there are unique Chern classes $c_i \in \mathrm{H}^{2i}(\mathscr{B}\,\mathrm{U}(n), \mathbb{Z})$ satisfying:*

(I) $c_0 = 1$ *and* $c_i = 0$ *if* $i > n$*;*
(II) $\iota_n^* c_i = c_i$ *(and hence* $\iota_n^* c_n = 0$*);*
(III) $p_{ij}^* c_k = \sum_{a+b=k} c_a \otimes c_b$*;*
(IV) $c_1 \in \mathrm{H}^2(\mathscr{B}\,\mathrm{U}(1), \mathbb{Z}) = \mathrm{H}^2(\mathbb{C}P^\infty, \mathbb{Z})$ *is the canonical generator.*

Proof. The proof is exactly the same as in the case of Stiefel-Whitney classes. We note, in particular, that $\mathrm{H}^\bullet(\mathscr{B}\,\mathrm{U}(n), \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$. $\square$

Chern classes can be related to complex vector bundles via the following analog of Theorem 4. Notice that here $\mathrm{Gr}_n V$ denotes the space of complex n-planes in V.

Theorem 8. *The space $\mathscr{B}\,\mathrm{U}(n) \cong \mathrm{Gr}_n \mathbb{C}^\infty$ classifies complex vector bundles, i.e. the natural transformation*

$$\Phi : [-, \mathscr{B}\,\mathrm{U}(n)] \stackrel{\cdot}{\longrightarrow} \mathrm{Vect}_{\mathbb{C}} -,$$

given by pullback $[f] \mapsto f^* \gamma_\infty^n$ of the tautological bundle, is a natural isomorphism.

We can now define the Chern class of a complex vector bundle.

Theorem 9. *Let $\xi : E \to B$ be a complex vector bundle. Then there exists a unique sequence of cohomology classes*

$$c_i(\xi) \in \mathrm{H}^{2i}(B, \mathbb{Z})$$

for $i = 0, 1, 2, \ldots$ called the Chern classes of ξ satisfying the following properties:

(I) *the class $c_0(\xi)$ is equal to the generator $1 \in \mathrm{H}^0(B, \mathbb{Z})$ and $c_i(\xi) = 0$ for $i > \mathrm{rk}\,\xi$;*
(II) *if $f^*\xi$ is the pullback of E along $f : A \to B$ then $c_i(f^*\xi) = f^* c_i(\xi)$;*
(III) *if $\eta : E' \to B$ is another complex vector bundle then*

$$c_k(\xi \oplus \eta) = \sum_{i=0}^{k} c_i(\xi) \smile c_{k-i}(\eta);$$

(IV) *if γ_1^1 is the tautological complex line bundle over $\mathbb{C}P^1$ then $c_1(\gamma_1^1) \in \mathrm{H}^2(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}$ is the canonical generator.*

The total (inhomogeneous) Chern class of ξ is the sum

$$c(\xi) = \sum_{i=0}^{\mathrm{rk}\,\xi} c_i(\xi) = 1 + c_1(\xi) + \cdots + c_{\mathrm{rk}\,\xi}(\xi) \in \mathrm{H}^\bullet(B, \mathbb{Z}).$$

Last time, Yajit computed the total Stiefel-Whitney class of the tangent bundle $T\mathbb{R}P^n$ to be $w(\tau) = (1+a)^{n+1}$, where a is the generator of $\mathrm{H}^\bullet(\mathbb{R}P^n, \mathbb{Z}_2)$. This computation extends almost identically to the complex case, but we first make a remark about conjugate bundles.

Remark 10. If ξ is a complex vector bundle then there is a *conjugate bundle* $\bar{\xi}$, which is the the underlying real vector bundle of ξ equipped with the opposite complex structure. Swapping the complex structure is a nontrivial operation! In particular, if τ is the tangent bundle of $\mathbb{C}P^1$ then an isomorphism $\tau \to \bar{\tau}$ of complex vector bundles would consist of a reflection across a line at each tangent space. This implies the existence of a continuous nonvanishing vector field on S^2, so we conclude that $\tau \ncong \bar{\tau}$.

We note, in particular, that

$$c_k(\bar{\xi}) = (-1)^k c_k(\xi)$$

and that in the presence of a Hermitian metric, $\bar{\xi}$ is canonically isomorphic to the dual bundle $\mathrm{Hom}(\xi, \underline{\mathbb{C}})$ (under the assignment $v \mapsto \langle -, v\rangle$).

Example 11 (The total Chern class of $T\mathbb{C}P^n$)**.** Recall that the tangent bundle $\tau : T\mathbb{C}P^n \to \mathbb{C}P^n$ is identified with the bundle $\mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp)$, and so

$$\begin{aligned} \tau \oplus \underline{\mathbb{C}} &\cong \tau \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \\ &\cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp \oplus \gamma_n^1) \\ &\cong \mathrm{Hom}(\gamma_n^1, \underline{\mathbb{C}}^{n+1}) \\ &\cong \oplus^{n+1}\overline{\gamma_n^1} \end{aligned}$$

We conclude that

$$c(\tau) = c(\tau \oplus \underline{\mathbb{C}}) = c(\overline{\gamma_n^1})^{n+1} = (1 - c_1(\gamma_n^1))^{n+1} = (1+a)^{n+1},$$

where we have taken $a = -c_1(\gamma_n^1)$.

Taking the underlying bundle of any complex rank n vector bundle yields a real rank $2n$ vector bundle. Hence it is natural to ask how Stiefel-Whitney classes are related to Chern classes.

Proposition 12. *Let $\mu_n : \mathscr{B}\,\mathrm{U}(n) \to \mathscr{B}\,\mathrm{O}(2n)$ be the map induced by the natural inclusion. Then $\mu_n^* w_{2i+1} = 0$ and $\mu_n^* w_{2i} = c_i$.*

Proof. That $\mu_n^* w_{2i+1} = 0$ is immediate from axiom II together with the fact that $\mathscr{B}\,\mathrm{U}(n)$ has cells only in even dimensions. For the other case, we recall that

$$\begin{aligned} \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(2n), \mathbb{Z}_2) &\cong \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(2)^n, \mathbb{Z}_2)^{\Sigma_n} \cong \mathbb{Z}_2[x_1, \ldots, x_{2n}]^{\Sigma_n} \\ \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{U}(n), \mathbb{Z}_2) &\cong \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{U}(1)^n, \mathbb{Z}_2)^{\Sigma_n} \cong \mathbb{Z}_2[y_1, \ldots, y_n]^{\Sigma_n}, \end{aligned}$$

where $\deg x_{2i+1} = 1, \deg x_{2i} = 2$ and $\deg y_i = 2$. We note that $\mu_1^* x_1 = 0$ since $\mathscr{B}\,\mathrm{U}(1)$ has cells only in even degrees and $\mu_1^* x_2 = y_1$ (why?). Write σ_i for the symmetric polynomials in x_i and ρ_i the symmetric polynomials in y_i. Using the commutativity of the diagram,

$$\begin{tikzcd} \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(2n), \mathbb{Z}_2) \arrow[r, "\mu_n^*"] \arrow[d] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{U}(n), \mathbb{Z}_2) \arrow[d] \\ \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{O}(2)^n, \mathbb{Z}_2)^{\Sigma_n} \arrow[r, "(\mu_1^*)^n"] & \mathrm{H}^\bullet(\mathscr{B}\,\mathrm{U}(1)^n, \mathbb{Z}_2)^{\Sigma_n} \end{tikzcd}$$

it now follows that $\mu_n^* \sigma_i = \rho_i$. ∎