NOTES ON SYMPLECTIC GEOMETRY

NILAY KUMAR

Contents

These notes were written for a reading course with Professor Eric Zaslow on the basics of symplectic geometry. They follow Mcduff/Salamon quite closely. These notes are rather rough, and in several places woefully incomplete: *caveat lector.*[1]

1. Week 1

1.1. **The cotangent bundle.**

Definition 1. Let X be a smooth n-manifold and $\pi : M = T^*X \to X$ be its cotangent bundle. We define the **canonical one-form** $\theta \in \Omega^1(M)$ as follows. For any $p = (x, \xi) \in M$, set

$$\theta_p(v) = \xi(d_x\pi(v)).$$

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[1] add references!

The one-form θ is canonical (or tautological) in the sense that its value at a point is simply given by the covector determined by that point. More precisely, we have the following characterization.

Proposition 2. *The canonical one-form θ is the (unique) one-form such that for every $\lambda \in \Omega^1(X)$, $\lambda^*\theta = \lambda$.*

Proof. We compute, for $v \in T_pX$,

$$\begin{aligned}(\lambda^*\theta)_p(v) &= \theta_{\lambda(p)}(d_p\lambda(v)) \\ &= \lambda_p(d_p(\pi \circ \lambda)(v)) \\ &= \lambda_p(v),\end{aligned}$$

where we have used the fact that λ is a section of π, i.e. $\pi \circ \lambda = \mathrm{id}_X$. Uniqueness is easily checked. $\square$

Definition 3. The **canonical symplectic form** $\omega \in \Omega^2(M)$ is now defined to be the exterior derivative

$$\omega = -d\theta,$$

of the canonical one-form. To be symplectic, ω must be closed and nondegenerate. That it is closed is obvious.

Proposition 4. *The form $\omega \in \Omega^2(M)$ is nondegenerate and thus defines a symplectic structure on $M = T^*X$.*[2]

Proof. For ω to be non-degenerate, it must be nondegenerate at each point $p \in M$. Given coordinates $p = (x, \xi) = (x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ in a neighborhood of p, we can compute

$$\begin{aligned}\theta_{(x,\xi)}\left(v^i \frac{\partial}{\partial x^i} + \nu^i \frac{\partial}{\partial \xi^i}\right) &= \xi\left(v^i \frac{\partial}{\partial x^i}\right) \\ &= \xi_i v^i\end{aligned}$$

and hence

$$\theta = \xi_i dx^i.$$

Taking an exterior derivative, we find that

$$\begin{aligned}\omega &= -d\theta \\ &= dx^i \wedge d\xi_i.\end{aligned}$$

Fix $v \in T_pM$ and suppose that $\iota_v\omega_p = 0$, i.e. $\omega_p(v, w) = 0$ for all $w \in T_pM$. In coordinates, this implies that

$$\begin{aligned}\iota_{v^j \frac{\partial}{\partial x^j} + \nu^j \frac{\partial}{\partial \xi^j}}(dx^i \wedge d\xi_i) &= v^i d\xi_i - \nu^i dx^i \\ &= 0,\end{aligned}$$

and hence that $v^i = \nu^i = 0$, i.e. $v = 0$. We conclude that ω_p is nondegenerate at each $p \in M$. $\square$

[2] Is there a coordinate invariant proof?

Remark 5. Note that a 2-form ω on a manifold M is nondegenerate if and only if ω^n is nowhere vanishing. Fix $p \in M$ and consider the vector space (T_pM, ω_p). If ω_p is nondegenerate, we can find a symplectic basis for T_pM, and so ω_p^n evaluated on $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ is nonzero, whence ω_p^n is not zero on V. On the other hand, suppose ω_p is degenerate, i.e. there is a $v \neq 0$ such that $\omega_p(v, w) = 0$ for all $w \in V$. Choosing a basis $v_1, \ldots, v_{2n}$ for V such that $v_1 = v$, we find that $\omega_p(v_1, \ldots, v_{2n}) = 0$ and hence $\omega_p = 0$ on V.

We conclude that every symplectic manifold is orientable.

It is easy to see that ω provides an isomorphism $\iota : T_xX \xrightarrow{\sim} T_x^*X$ between tangent and cotangent spaces at each point $x \in X$: since ω_x is nondegenerate, the linear map $\iota : v \mapsto \omega_x(v, -)$ is injective and hence bijective. In fact, we can say more.

Proposition 6. *The metric ω induces an isomorphism of vector bundles $\iota : TX \xrightarrow{\sim} T^*X = M$.*

Proof. Recall that an isomorphism in the category of smooth vector bundles is a smooth bijection[3] ι such that the diagram

$$\begin{tikzcd} TX \arrow[rr, "\iota"] \arrow[dr, "\pi_1"'] & & T^*X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

commutes and for each $x \in X$, the restriction $\iota_x : T_xX \to T_x^*X$ is linear. The map $\iota : TX \to T^*X$ taking $(x, v) \mapsto (x, \omega(v, -))$ fits into the diagram above and is bijective and fiberwise linear. Moreover, ι is a smooth map, as is seen by its coordinate description computed above. □

Definition 7. A **Hamiltonian** is a smooth function $H : M = T^*X \to \mathbb{R}$. we define the **Hamiltonian vector field** v_H associated to H to be the vector field on M satisfying

$$\iota_{v_H}\omega = dH.$$

The (local) flow $F : (-\varepsilon, \varepsilon) \times M \to M$ determined by v_H is called the **Hamiltonian flow**.[4]

Note that an integral curve $\gamma_{v_H} : (-\varepsilon, \varepsilon) \to M$ of v_H can be thought of as the trajectory of a physical state in phase space. Indeed, Hamilton's equations are given

$$\frac{\partial x^i}{\partial t} = \frac{\partial H}{\partial \xi_i}$$
$$\frac{\partial \xi_i}{\partial t} = -\frac{\partial H}{\partial x^i},$$

which is precisely the condition that $\gamma'_{v_H}(t) = (v_H)_{\gamma(t)}$. Moreover, H is constant along the Hamiltonian flow, as

$$dH(v_H) = (\iota_{v_H}\omega)(v_H) = \omega(v_H, v_H) = 0,$$

[3] Existence of a smooth inverse is automatic (reference?).

[4] Is this a global flow? Does it depend on X?

i.e. v_H is tangent to the level sets of H. In a physical system, where H is the energy functional on phase space, this phenomenon is the law of conservation of energy.

Proposition 8. *The Hamiltonian flow is a symplectomorphism, i.e. $F_t^*\omega = \omega$.*[5]

Proof. We use the following trick:

$$\int_0^t \frac{d}{dt} F_t^*\omega \; dt = F_t^*\omega - \omega$$

since $F_0 = \mathrm{id}_M$, and hence F_t is a symplectomorphism if and only if the integrand is zero. But

$$\begin{aligned} \frac{d}{dt} F_t^*\omega &= \left.\frac{d}{ds}\right|_{s=0} F_{t+s}^*\omega = F_t^* \left.\frac{d}{ds}\right|_{s=0} F_s^*\omega \\ &= F_t^* \mathcal{L}_{v_H}\omega, \end{aligned}$$

and Cartan's magic formula,

$$\mathcal{L}_{v_H}\omega = d\iota_{v_H}\omega + \iota_{v_H} d\omega,$$

tells us that $\mathcal{L}_{v_H}\omega = 0$ since $\iota_{v_H}\omega = dH$ is closed, as is ω. $\square$

Corollary 9 (Liouville's Theorem)**.** *The volume form ω^n on $M = T^*X$ is preserved by the Hamiltonian flow.*

1.2. **Geodesic flow as Hamiltonian flow.** We wish to discuss geodesics and geodesic flow. For this, we need the concept of connections and covariant derivatives.[6]

Definition 10. A **connection** on a vector bundle $E \to X$ is an $\mathbb{R}$-linear map $\nabla : \Gamma(X, E) \to \Gamma(X, E \otimes T^*X)$ such that the Leibniz rule

$$\nabla(f\sigma) = (\nabla\sigma)f + \sigma \otimes df,$$

for all $f \in C^\infty(X)$ and $\sigma \in \Gamma(X, E)$.

Theorem 11. *Given a Riemannian manifold (X, g), there exists a unique connection on $\pi : TX \to X$, known as the **Levi-Civita connection**, satisfying*

(i) symmetry:

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

for $X, Y \in \Gamma(X, TX)$;

(ii) compatibility with g:

$$Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,$$

for $X, Y, Z \in \Gamma(X, TX)$.

Definition 12. Let v be a vector field on (X, g); we define the **covariant derivative** of v along a smooth curve $c : I \to X$ to be the vector field

$$\frac{Dv}{dt} = \nabla_{dc/dt} v,$$

where ∇ is the Levi-Civita connection. Explicitly, if we write $v = v^i \partial/\partial x^i$ and $c(t) = (c_1(t), \ldots, c_n(t))$,

$$\frac{Dv}{dt} = \sum_i \frac{dv^i}{dt} \frac{\partial}{\partial x^i} + \sum_{ijk} \frac{dc_i}{dt} v^i \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

[5] Is there a better proof?

[6] Reference do Carmo.

Here Γ_{ij}^k are the Christoffel symbols of ∇, determined by

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \sum_{ijk} \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

We say that c is **geodesic** at some $t \in I$ if $D/dt(dc/dt) = 0$ at t, and that c is geodesic if it is geodesic at all $t \in I$. In coordinates, the condition for c to be geodesic is given by a system of second-order differential equations:

$$\frac{d^2 c^i}{dt^2} + \sum_{jk} \Gamma_{jk}^i \frac{dc^j}{dt} \frac{dc^k}{dt} = 0,$$

for $i = 1, \dots, n$.

For the rest of the section, assume (X, g) is Riemannian and we fix the Hamiltonian $H : M = T^*X \to \mathbb{R}$ as

$$H(x, \xi) = \frac{1}{2} \left|\xi_x\right|_g^2,$$

i.e. consisting of only a kinetic term. Here we are implicitly using the nondegeneracy of g to associate ξ_x with its corresponding vector (or, equivalently, using g^{-1}).

Proposition 13. *The Hamiltonian flow on $M = T^*X$ is dual to the geodesic flow on TX. In other words, the integral curves of the Hamiltonian vector field v_H associated to the Hamiltonian above project to geodesics of g on X.*[7]

Proof. It suffices to show, in coordinates, that Hamilton's equations (i.e. the condition for being on the integral curve) yield the geodesic equations above after the necessary dualization. Note first that in coordinates the Hamiltonian becomes

$$H(x, \xi) = \frac{1}{2} g^{ij} \xi_i \xi_j.$$

For convenience we will denote the components of an integral curve as $x^i(t)$. Hamilton's equations yield

$$\begin{aligned}
\frac{dx^i}{dt} &= \frac{\partial}{\partial \xi_i} \left(\frac{1}{2} g^{jk} \xi_j \xi_k \right) \\
&= \frac{1}{2} g^{jk} \delta_{ij} \xi_k + \frac{1}{2} g^{jk} \xi_j \delta_{ik} \\
&= g^{ij} \xi_j \\
\frac{d\xi_i}{dt} &= -\frac{\partial}{\partial x^i} \left(\frac{1}{2} g^{jk} \xi_j \xi_k \right) \\
&= -\frac{1}{2} \frac{\partial g^{jk}}{\partial x^i} \xi_j \xi_k.
\end{aligned}$$

Differentiating the first equation with respect to t and using both of Hamilton's equations yields

$$\begin{aligned}
\frac{d^2 x^i}{dt^2} &= \frac{\partial g^{ij}}{\partial x^k} \frac{dx^k}{dt} \xi_j + g^{im} \frac{d\xi_m}{dt} \\
&= g^{kl} \left(\frac{\partial}{\partial x^k} g^{ij} \right) \xi_l \xi_j - \frac{1}{2} g^{im} \left(\frac{\partial}{\partial x^m} g^{nr} \right) \xi_n \xi_r.
\end{aligned}$$

[7] Is there a coordinate-free proof? See Paternain's book.

Next, differentiating the identity $g^{ij}g_{jk} = \delta^i_k$, it easy to see that

$$\frac{\partial}{\partial x^i} g^{kl} = -g^{la} g^{kb} \frac{\partial}{\partial x^i} g_{ab}.$$

Using this, contracting indices, and using the first Hamilton's equation to dualize ξ's into dx/dt's, we find

$$\begin{aligned}
\frac{d^2 x^i}{dt^2} &= -g^{ib}\left(\frac{\partial}{\partial x^k} g_{lb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt} + \frac{1}{2} g^{im}\left(\frac{\partial}{\partial x^m} g_{ts}\right)\frac{dx^s}{dt}\frac{dx^t}{dt} \\
&= -\frac{1}{2} g^{ib}\left(\frac{\partial}{\partial x^k} g_{lb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt} - \frac{1}{2} g^{ib}\left(\frac{\partial}{\partial x^l} g_{kb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt} \\
&+ \frac{1}{2} g^{im}\left(\frac{\partial}{\partial x^m} g_{ts}\right)\frac{dx^s}{dt}\frac{dx^t}{dt} \\
&= -\Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt},
\end{aligned}$$

as desired. □

2. Week 2

2.1. Darboux's theorem.

Theorem 14 (Darboux). *Let (M, ω) be a symplectic $2n$-manifold. Then M is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.*

We prove Darboux's theorem using the following stronger statement.

Theorem 15 (Moser's trick). *Let M be a $2n$-dimensional manifold and $Q \subset M$ be a compact submanifold. Suppose that $\omega_1, \omega_2 \in \Omega^2(M)$ are closed 2-forms such that at each point q of Q the forms ω_0 and ω_1 are equal and nondegenerate on T_qM. Then there exist neighborhoods N_0 and N_1 of Q and a diffeomorphism $\psi : N_0 \to N_1$ such that $\psi|_Q = \mathrm{id}_Q$ and $\psi^*\omega_1 = \omega_0$.*

Proof. Consider the family of closed two-forms

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0)$$

on M for $t \in [0, 1]$. Note that $\omega_t|_Q = \omega_0|_Q$ is nondegenerate and hence there exists an open neighborhood N_0 of Q such that $\omega_t|_{N_0}$ is nondegenerate.[8] Suppose, for now, that there is a one-form $\sigma \in \Omega^1(N_0)$ (possibly shrinking N_0), such that $\sigma|_{T_QM} = 0$ and $d\sigma = \omega_1 - \omega_0$ on N_0. Then

$$\omega_t = \omega_0 + td\sigma$$

and we obtain by nondegeneracy a smooth vector field X_t on N_0 characterized by

$$\iota_{X_t}\omega_t = -\sigma.$$

The condition $\sigma|_{T_QM} = 0$ implies, again by nondegeneracy of ω_t, that $X_t|_Q = 0$. Now consider the initial value problem for the flow ψ_t of X_t,

$$\begin{aligned} \frac{d}{dt}\psi_t &= X_t \circ \psi_t \\ \psi_0 &= \mathrm{id}\,. \end{aligned}$$

This differential equation can be solved uniquely for $t \in [0, 1]$ on some open neighborhood of Q contained in N_0, call it again N_0.[9] Note that $\psi_t|_Q = \mathrm{id}_Q$ since $X_t|_Q = 0$. We compute now that

$$\begin{aligned} \frac{d}{dt}\psi_t^*\omega_t &= \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) \\ &= \psi_t^*\left(d\sigma + d\iota_{X_t}\omega_t\right) \\ &= 0. \end{aligned}$$

Hence $\psi_1^*\omega_1 = \psi_0^*\omega_0 = \omega_0$. Thus the desired diffeomorphism is ψ_1 and the desired neighborhoods are N_0 and N_1. The above argument is known as **Moser's trick**, and is extremely useful in symplectic geometry.

It remains to construct a smooth one-form σ satisfying $\sigma|_{T_QM} = 0$ and $d\sigma = \omega_1 - \omega_0$. If Q were a point (or more generally, diffeomorphic to a star-shaped subset of Euclidean space), we could simply use the Poincaré lemma; in general, however the construction is as follows. Fix any Riemannian metric on M and consider the

[8] Why?

[9] Why?

restriction of the exponential map $\exp : TM \to M$ to a neighbhorhood U_ε of the zero section of the normal bundle $TQ^\perp \to M$:

$$U_\varepsilon = \{(q, v) \in TM \mid q \in Q, v \in T_q Q^\perp, |v| < \varepsilon\}.$$

Recall that exp becomes a diffeomorphism for ε sufficiently small, so we choose ε such that $N_0 = \exp(U_\varepsilon)$ is contained in the neighborhood of Q above on which ω_t is nondegenerate. Define now a family of maps $\phi_t : N_0 \to N_0$ for $t \in [0, 1]$ by

$$\phi_t(\exp(q, v)) = \exp(q, tv).$$

Note that ϕ_t is a diffeomorphism onto its image for $t \neq 0$. Moreover, $\phi_t|_Q = \mathrm{id}_Q$, $\phi_0(N_0)$, and $\phi_1 = \mathrm{id}_{N_0}$. If we now write $\tau = \omega_1 - \omega_0$, we find that

$$\begin{aligned} \phi_0^*\tau &= 0 \\ \phi_1^*\tau &= \tau, \end{aligned}$$

since $\tau = 0$ on $T_Q M$. Now, for $t \in (0, 1]$, we define a family of vector fields,

$$Y_t = \left(\frac{d}{dt}\phi_t\right) \circ \phi_t^{-1}.$$

Then for any $\delta > 0$,

$$\begin{aligned} \phi_1^*\tau - \phi_\delta^*\tau &= \int_\delta^1 \frac{d}{dt}\phi_t^*\tau dt = \int_\delta \phi_t^* \mathcal{L}_{Y_t}\tau dt \\ &= \int_\delta^1 \phi_t^*(d\iota_{Y_t}\tau) dt \\ &= d\int_\delta^1 \phi_t^*(\iota_{Y_t}\tau) dt \end{aligned}$$

Clearly $\phi_1^*\tau - \phi_\delta^*\tau = \tau - \phi_\delta^*\tau$ approaches τ as $\delta \to 0^+$, so we find that

$$\tau = d\int_0^1 \phi_t^*(\iota_{Y_t}\tau) dt.$$

Defining

$$\sigma = \int_0^1 \phi_t^*(\iota_{Y_t}\tau) dt,$$

we find that $\tau = \omega_1 - \omega_0 = d\sigma$ and $\sigma|_{T_Q M} = 0$ because $\phi_t|_Q = \mathrm{id}_Q$ and $\tau = 0$ on Q, forcing the integrand to vanish on $T_Q M$. Hence σ is the one-form required above for Moser's trick, and we are done.[10] □

The proof of Darboux's theorem is now straightforward: we choose a coordinate chart ϕ so that $\phi^*\omega$ is equal to the standard form on a subset of $\mathbb{R}^{2n}$ at a single point, and then apply Moser's theorem with Q equal to the chosen point.

Proof of Darboux's theorem. Let $q \in M$ and fix a symplectic basis $\{u_i, v_i\}$ for the symplectic vector space $(T_q M, \omega_q)$. Fix any Riemannian metric on M and pick an open $U \ni 0$ small enough such that exp restricted to $U \subset T_q M$ is a diffeomorphism

[10] Why is σ smooth?

and hence a chart $(x^i, y_i) = \exp : U \subset \mathbb{R}^{2n} \to M$ $(i = 1, \ldots, n)$ such that $x^i(p) = y_i(p) = 0$. Now we can compute, for example,

$$\begin{aligned} \exp^* \omega_p \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k} \right) &= \omega_p \left(\exp_* \frac{\partial}{\partial x^j}, \exp_* \frac{\partial}{\partial y^k} \right) \\ &= \omega_p \left(u_j, v_k \right) = \delta_{jk}, \end{aligned}$$

to check that $\exp^* \omega_p = (\omega_0)_0$ where ω_0 is the standard form on $T_0 U$. Here we have used the fact that $\exp_* = \mathrm{id}$ at $0 \in U$. Applying Theorem 2.1 to U with $Q = 0 \in U$, we obtain a diffeomorphism ψ of (some possibly smaller) U such that $\psi^* \exp^* \omega = \omega_0$ on U. But now $\exp \circ \psi$ provides a symplectomorphism in a neighborhood of q to a neighborhood of $\mathbb{R}^{2n}$ pulling ω back to the standard form ω_0. $\square$

3. Week 3

3.1. Submanifolds of symplectic manifolds.

Definition 16. Let (V, ω) be a symplectic vector space. We define the **symplectic complement** U^ω of a subspace $U \subset V$ as

$$U^\omega = \{v \in V \mid \omega(v, u) = 0 \text{ for all } u \in U\}.$$

Lemma 17. *For any subspace $U \subset V$, $U^{\omega\omega} = U$ and*

$$\dim U + \dim U^\omega = \dim V.$$

Proof. Nondegeneracy of ω yields an isomorphism $\iota_\omega : V \to V^*$ which identifies U^ω with $U^\perp \equiv \{\nu \in V^* \mid \nu(u) = 0 \text{ for all } u \in U\}$. The result now follows from the fact that $\dim U + \dim U^\perp = \dim V$. $\square$

Definition 18. Let (M, ω) be a symplectic manifold. A submanifold $Q \subset M$ is called **symplectic, isotropic, coisotropic, or Lagrangian** if for each $q \in Q$, the linear subspace $T_qQ \equiv V_q$ of (T_qM, ω_q) is

(a) symplectic: $V_q \cap V_q^{\omega_q} = 0$,
(b) isotropic: $V_q \subset V_q^{\omega_q}$,
(c) coisotropic: $V_q^{\omega_q} \subset V_q$,
(d) Lagrangian: $V_q = V_q^{\omega_q}$,

respectively.

Remark 19. Note that $Q \subset M$ is Lagrangian if and only if the restriction of ω to Q is zero and $\dim Q = \dim M/2$.

Example 20. Let X be any manifold, and $(M = T^*X, \omega)$ be its cotangent bundle with the usual symplectic structure. Recall that $\omega = -d\theta$, where $\theta_\xi(v) = \xi(d_x\pi(v))$.[11] In coordinates, if (x^i, ξ^i) are coordinates for M, we can write $\omega = dx^i \wedge d\xi^i$.

It is then easy to see that the fibre $T_x^*X \subset M$ is Lagrangian, as

$$\begin{aligned} 0 &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial \xi^j}, b_k \frac{\partial}{\partial \xi^k} + c_l \frac{\partial}{\partial x^l}\right) \\ &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial \xi^j}, c_l \frac{\partial}{\partial x^l}\right) \\ &= a_i c_i, \end{aligned}$$

forces $c_i = 0$.

Similarly, the zero section $\Gamma_0 \subset M$ is Lagrangian, as

$$\begin{aligned} 0 &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial x^j}, b_k \frac{\partial}{\partial \xi^k} + c_l \frac{\partial}{\partial x^l}\right) \\ &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial x^j}, b_k \frac{\partial}{\partial \xi^k}\right) \\ &= a_i b_i, \end{aligned}$$

forces $b_i = 0$.

[11] Can we do this coordinate-invariantly?

More generally, given a submanifold $Q \subset L$, the annihilator

$$TQ^{\perp} = \{(q, \nu) \in T^*L \mid q \in Q, \nu|_{T_qQ} = 0\}$$

is Lagrangian.

Example 21. Let (M, ω) be a symplectic manifold. The product $M \times M$ can be given a symplectic structure $\omega' = \alpha\pi_1^*\omega + \beta\pi_2^*\omega$ for $\alpha, \beta \in \mathbb{R}$. Consider in particular the case of $\alpha = 1, \beta = -1$. Then it is clear that $M \times \{m\}$ and $\{m\} \times M$ are symplectic submanifolds. Moreover, the diagonal $\Delta \subset M \times M$ is Lagrangian, as

$$\begin{aligned} 0 &= \omega'\left((u, u), (v, w)\right) \\ &= \omega(u, v) - \omega(u, w) \\ &= \omega(u, v - w) \end{aligned}$$

and hence $v = w$, as desired.

Example 22. Let $S \subset (M, \omega)$ be a codimension 1 submanifold. Then S is coisotropic. Indeed, fix $s \in S$, and note that $T_sS \subset T_sM$ is codimension one. By Lemma 17, $T_sS^{\omega_s}$ is a one-dimensional subspace. Pick any vector $v \in T_sS^{\omega_s}$; v spans the entire symplectic complement, and hence if v is not in $T_sS^{\omega_s}$, $T_sS \cap T_sS^{\omega_s} = 0$ and T_sS is symplectic and thus even-dimensional. This is a contradiction, and hence T_sS must be coisotropic.

Proposition 23. *The graph $\Gamma_\sigma \subset T^*X$ of a one-form is Lagrangian if and only if σ is closed.*

Proof. Note that Γ_σ is defined to be the image of the embedding $\sigma : X \to T^*X$. Then $\dim \Gamma_\sigma = n$, so it remains to show that ω restricts to zero on Γ_σ if and only if σ is closed. Using Proposition 2, we compute

$$d\sigma = d\sigma^*\theta = \sigma^*d\theta = -\sigma^*\omega,$$

which yields the desired statement, as $\sigma^*\omega = 0$ on X if and only if $\omega = 0$ on Γ_σ, by virtue of σ being an embedding. $\square$

With these definitions out of the way, we present a number of theorems characterizing neighbhorhoods of special submanifolds of symplectic manifolds.

Theorem 24 (Symplectic neighborhood theorem)**.** *Let $(M_0, \omega_0), (M_1, \omega_1)$ be symplectic manifolds with compact symplectic submanifolds Q_0, Q_1 respectively. Suppose there is an isomorphism $\Phi : TQ_0^\omega \to TQ_1^\omega$ of symplectic normal bundles covering a symplectomorphism $\phi : (Q_0, \omega_0) \to (Q_1, \omega_1)$. Then ϕ extends to a symplectomorphism $\psi : (N(Q_0), \omega_0) \to (N(Q_1), \omega_1)$ such that $d\psi$ induces the map Φ on TQ_0^ω.*

Proof. We use implicitly throughout that since Q is symplectic, there is an isomorphism $TQ^\omega \to TQ^\perp$. Let $\exp_0, \exp_1$ be diffeomorphisms mapping neighborhoods of the zero section in the normal bundle to neighborhoods of Q_0, Q_1 in X, respectively. Then we obtain

$$\phi' = \exp_1 \circ \Phi \circ \exp_0^{-1},$$

a diffeomorphism between these neighborhoods of Q_0 and Q_1. Now $\phi'^*\omega_1$ and ω_0 are two symplectic forms on M_0 whose restrictions to Q_0 agree. Now ϕ' extends to the desired ψ by Theorem 2.1. $\square$

Theorem 25 (Lagrangian neighborhood theorem)**.** *Let (M, ω) be a symplectic manifold and let $L \subset M$ be a compact Lagrangian submanifold. Then there exists a neighborhood $N(\Gamma_0) \subset T^*L$ of the zero section Γ_0, a neighborhood $U \subset M$ of L, and a diffeomorphism $\phi : N(\Gamma_0) \to U$ such that $\phi^*\omega = -d\theta$ and $\phi|_L = \mathrm{id}$, where θ is the canonical one-form on T^*L.*

We postpone the proof of this theorem until after the discussion of complex structures.

3.2. **Contact manifolds.** Let X be a differential manifold and $H \subset TX$ be a smooth hyperplane field, i.e. a smooth subbundle of codimension one. Then, locally on some open U, we can write $H = \ker \alpha$, for $\alpha \in \Omega_1(U)$. In fact, if we assume that H is **coorientable**, we can extend U to all of X.[12] We will assume for what follows that H is coorientable.

Definition 26. Let X be a manifold of odd dimension $2n+1$. A **contact structure** on X is a hyperplane field $H = \ker \alpha$ where the top-dimensional form $\alpha \wedge (d\alpha)^n$ is nowhere vanishing. We call α a **contact form**, and the pair (X, H) a **contact manifold**.

Remark 27. Suppose we have $\alpha, \alpha' \in \Omega^1(X)$ such that $H = \ker \alpha = \ker \alpha'$. Then α is a contact form if and only if α' is. This is because the condition that α, α' cut out H requires $\alpha' = f\alpha$ for some nonzero $f : X \to \mathbb{R}$.

Remark 28. In the language of distributions, H can be described as a codimension one distribution that is maximally non-integrable in the following sense. Recall that a distribution on X is said to be integrable if every point p of X is contained in a integral manifold of H, i.e. in a nonempty immersed submanifold $N \subset X$ such that $T_pN = H_p$. The Frobenius theorem tells us that H is integrable if and only if H is involutive, i.e. H is closed under the Lie bracket of local sections. Now, since

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X, Y],$$

we find that H is integrable if and only if $d\alpha = 0$ on H. Thus asking for $d\alpha$ to be nondegenerate on H forces the distribution to be "as non-integrable as possible."

Indeed, we obtain the above definition of a contact structure by noting that $d\alpha$ is nondegenerate on H if and only if $\alpha \wedge (d\alpha)^n$ is nowhere vanishing, as follows. By remark 5, $d\alpha$ is nondegenerate on H if and only if $(d\alpha)^n$ is nowhere vanishing, but this is simply equivalent to asking that $\alpha \wedge (d\alpha)^n$ be nowhere vanishing.

Armed simply with the definition of a contact manifold, one might think that contact geometry is somewhat obscure. We provide the following list of examples as evidence that contact manifolds are actually quite common.

Example 29. Let $X = \mathbb{R}^{2n+1}$ with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n, z)$. The one-form

$$\alpha = dz + x^i dy^i$$

is a contact form, as

$$\alpha \wedge (d\alpha)^n = dz \wedge dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n,$$

which is nowhere vanishing. We define the standard contact structure on $\mathbb{R}^{2n+1}$ to be $H = \ker \alpha$.

[12] Why?

For the next few examples the following lemma will be useful.

Lemma 30. *Let (M,ω) be a symplectic manifold of dimension $2n$. A vector field Y on M satisfying $\mathcal{L}_Y\omega = \omega$ is called a* ***Liouville vector field****. In this case, $\alpha = \iota_Y\omega$ is a contact form on any hypersurface $Q \subset M$ transverse to Y (i.e. at any point p, T_pQ and Y_p span T_pM).*

Proof. Cartan's magic formula in this case tells us that $\omega = d\iota_Y\omega$, and hence

$$\begin{aligned}\alpha \wedge (d\alpha)^{n-1} &= \iota_Y\omega \wedge \omega^{n-1} \\ &= \iota_Y(\omega^n)/n.\end{aligned}$$

Now, since ω^n is a volume form on M, we find that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form when restricted to the tangent bundle of any hypersurface transverse to Y. □

Example 31. Consider $M = \mathbb{R}^4$ with its usual symplectic form $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. The vector field

$$Y = \frac{1}{2}\left(x_1\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial y^1} + x^2\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial y^2}\right)$$

is clearly transverse to the sphere S^3 given by $(x^1)^2 + (y^1)^2 + (x^2)^2 + (y^2)^2 = 1$. It is a straightforward computation to check that Y is Liouville, using the identity

$$(\mathcal{L}_Y\omega)(v,w) = \mathcal{L}_Y(\omega(v,w)) - \omega([Y,v],w) - \omega(v,[Y,w]).$$

We conclude, using the previous lemma, that S^3 is a contact manifold, with a contact structure $\ker \iota_Y\omega$. This example is easily extended to show that S^{2n+1} has a contact structure.

Example 32. Let (M,g) be a Riemannian n-manifold. We define the **unit cotangent bundle**

$$ST^*M = \{(p,\xi) \in T^*M \mid |\xi_p|_g^2 = 1\} \subset T^*M.$$

The unit cotangent bundle is a manifold of dimension $2n-1$ as it can be written as the level set of a Hamiltonian $H(p,\xi) = |\xi_p|_g^2/2$. Moreover, it is a sub-fiber bundle of the cotangent bundle, with fiber S^{n-1}. We claim that the canonical one-form on T^*M is a contact form for ST^*M. Indeed, let Y be a vector field on T^*M given by $\iota_Y\omega = \theta$. Then Y is Liouville: $d(\iota_Y\omega) = d\theta = \omega$. In coordinates, $Y = p^i\partial/\partial p^i$, and hence is transverse to ST^*M. Note that if M is compact, so is SY^*M and in this case ST^*M is an example of a compact contact manifold.

Example 33. Let $(M, H = \ker\alpha)$ be a contact manifold. Then, if $\pi_M : M \times \mathbb{R} \to M$ is the projection onto the second factor, we claim that $(M \times \mathbb{R}, \omega = d(e^t\pi_M^*\alpha))$ is a symplectic manifold. Indeed, if M has dimension $2n-1$, we compute

$$\begin{aligned}\omega^n &= (e^t dt \wedge \pi_M^*\alpha + \pi_M^* d\alpha)^n \\ &= ne^{nt} dt \wedge \pi_M^*\alpha \wedge \pi_M^*(d\alpha)^{n-1} \\ &= ne^{nt} dt \wedge \pi_M^*\left(\alpha \wedge (d\alpha)^{n-1}\right) \\ &\neq 0.\end{aligned}$$

We call $(M \times \mathbb{R}, d(e^t\pi_M^*\alpha))$ the **symplectization** of (M,α). Note that $\partial/\partial t$ is a Liouville vector field for ω[13] and $M \subset M \times \mathbb{R}$ is a hypersurface transverse to $\partial/\partial t$.

[13]compute!

Definition 34. A **contactomorphism** from (M_1, H_1) to (M_2, H_2) is a diffeomorphism $f : M_1 \to M_2$ such that $df(H_1) = H_2$. Equivalently, if $H_1 = \ker \alpha_1$ and $H_2 = \ker \alpha_2$ then we require $f^*\alpha_2 = g\alpha_1$ for some nowhere vanishing function $g : M_1 \to \mathbb{R} \setminus \{0\}$.

4. Week 4

4.1. Symplectic linear group and linear complex structures.

Definition 35. Let (V, ω) be a symplectic vector space. We denote the group of symplectomorphisms from V to itself as $\mathrm{Sp}(V, \omega)$, the **symplectic linear group**. In the case of the standard symplectic structure on $\mathbb{R}^{2n}$ we write the group as $\mathrm{Sp}(2n)$.

Lemma 36. *A real $2n \times 2n$ matrix Ψ is in $\mathrm{Sp}(2n)$ if and only if*

$$\Psi^\top J_0 \Psi = J_0,$$

where

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathrm{Sp}(2n).$$

Proof. Let u_i, v_i be a symplectic basis for V. For $x, y \in V$ write $x = (a, b), y = (c, d)$ for $a, b, c, d \in \mathbb{R}^n$. Then

$$\omega(x, y) = a^i d^i - b^i c^i = -x^\top J_0 y.$$

Clearly $\Psi^* \omega = \omega$ if and only if $\Psi^\top J_0 \Psi = J_0$. □

Definition 37. Let V be a vector space. A **complex structure** on V is an automorphism $J : V \to V$ such that $J^2 = -\,\mathrm{id}_V$. We denote the set of all complex structures on V by $\mathcal{J}(V)$. Now suppose (V, ω) is a symplectic vector space. We say that a complex structure J is **compatible** with ω if

$$\omega(Jv, Jw) = \omega(v, w)$$

for all $v, w \in V$, and

$$\omega(v, Jv) > 0$$

for all nonzero $v \in V$. We denote the set of all compatible complex structures on (V, ω) by $\mathcal{J}(V, \omega)$.

Lemma 38. *Let $J \in \mathcal{J}(V, \omega)$ be a compatible complex structure on (V, ω). Then*

$$g_J(v, w) = \omega(v, Jw)$$

defines an inner product on V.

Lemma 39. *Let (V, ω) be a symplectic vector space and J be a complex structure on V. Then the following are equivalent:*

(a) J is compatible with ω;

(b) the bilinear form $g_J : V \times V \to \mathbb{R}$ defined by

$$g_J(v, w) = \omega(v, Jw)$$

is symmetric, positive-definite, and J-invariant.

(c) if we view V as a complex vector space with J as its complex structure, the form $H : V \times V \to \mathbb{C}$ defined by

$$H(v, w) = \omega(v, Jw) + i\omega(v, w)$$

*is complex linear in w, complex antilinear in v, satisfies $H(w, v) = \overline{H(v, w)}$, and has a positive-definite real part. Such a form is called a **Hermitian inner product** on (V, J).*

Proof. That (a) implies (b) is clear from Lemma 38. For (b) implies (c), note first that the real part of H is simply g_J and hence is positive-definite. For linearity, we compute

$$\begin{aligned} H(Jv, w) &= \omega(Jv, Jw) + i\omega(Jv, w) \\ &= g_J(Jv, w) - ig_J(w, v) \\ &= g_J(w, Jv) - ig_J(v, w) \\ &= -iH(v, w), \end{aligned}$$

and

$$\begin{aligned} H(v, Jw) &= -\omega(v, w) + i\omega(Jv, Jw) \\ &= -\omega(v, w) + ig_J(Jv, w) \\ &= -\omega(v, w) + i\omega(v, w) \\ &= iH(v, w), \end{aligned}$$

as desired. Finally, note that

$$\begin{aligned} H(w, v) &= \omega(w, Ju) + i\omega(w, v) \\ &= \omega(v, Jw) - i\omega(v, w) \\ &= \overline{H(v, w)}. \end{aligned}$$

For (c) implies (a), $\omega(v, Jv) > 0$ because the real part $\omega(v, Jw)$ is by hypothesis positive-definite. Moreover, $\omega(Jv, Jw) = \operatorname{im} H(Jv, Jw) = \operatorname{im} H(v, w) = \omega(v, w)$. □

The following result shows that all linear complex structures are isomorphic to the standard complex structure.

Proposition 40. *Let V be a $2n$-dimensional real vector space and let $J \in \mathcal{J}(V)$. Then there exists a vector space isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that*

$$J\Phi = \Phi J_0.$$

Proof. Consider the extension $J^{\mathbb{C}}$ of J to the complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \cong V$ given by $J \otimes 1$. Clearly $J^{\mathbb{C}}$ is a complex structure on $V^{\mathbb{C}}$ and thus has eigenvalues $\pm i$. We obtain a direct sum decomposition $V^{\mathbb{C}} \cong E^+ \oplus E^-$ of the $\pm i$ eigenspaces respectively, i.e. $J^{\mathbb{C}}|_{E^\pm} = \pm iI$. Clearly $\dim_{\mathbb{C}} E^\pm = n$. We claim that a basis $w_j = u_j + iv_j$ for E^+ yields a basis u_j, v_j for V. It suffices to show that these vectors are linearly independent. Since w_j is a basis for E^+,

$$\sum_{j=1}^{n} (a_j + ib_j)(u_j \otimes 1 + v_j \otimes i) = 0$$

for $a_j, b_j \in \mathbb{R}$ implies that $a_j = b_j = 0$ for all j. Suppose there exist $\alpha_j, \beta_j \in \mathbb{R}$ such that

$$\sum_{j=1}^{n} \alpha_j u_j + \beta_j v_j = 0.$$

Now since $w_j \in \ker(I - iJ)$, a straightforward computation reveals that $Ju_j = -v_j$ and $Jv_j = u_j$. Applying J to the above equation, we obtain

$$\sum_{j=1}^{n} \beta_j u_j - \alpha_j v_j = 0.$$

Then, taking $a_j = \beta_j, b_j = \alpha_j$, we find that

$$\begin{aligned}\sum_{j=1}^{n} (\beta_j + i\alpha_j)(u_j \otimes 1 + v_j \otimes i) &= \left(\sum_{j=1}^{n} \beta_j u_j - \alpha_j v_j\right) \otimes 1 + \left(\sum_{j=1} \beta_j v_j + \alpha_j u_j\right) \otimes i \\ &= 0.\end{aligned}$$

Linear independence of the w_j now forces $\alpha_j = \beta_j = 0$. Hence u_j, v_j forms a basis for V.

The required $\Phi : \mathbb{R}^{2n} \to V$ can now be written explicitly as

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{j=1}^{n} (x_j u_j - y_j v_j).$$

This map is clearly an isomorphism; moreover, if $x = (r_1, \ldots, r_n, s_1, \ldots, s_n) \in \mathbb{R}^{2n}$ then

$$J\Phi x = -s_1 u_1 - r_1 v_1 - \cdots - s_n u_n - r_n v_n = \Phi J_0 x,$$

as desired. $\square$

Remark 41. Define an action of $\mathrm{GL}(2n, \mathbb{R})$ on the set $\mathcal{J}(V)$ by $g \cdot J = g^{-1} J g$. By Lemma 40, $\mathrm{GL}(2n, \mathbb{R}) \cdot J_0 = \mathcal{J}(V)$, i.e. the orbit of J_0 is the entire set. Moreover, since $\mathrm{GL}(n, \mathbb{C})$ is naturally embedded (as a Lie subgroup) in $\mathrm{GL}(2n, \mathbb{R})$ as $\{A \in \mathrm{GL}(2n, \mathbb{R}) \mid J_0 A = A J_0\}$, the stabilizer of J_0 is $\mathrm{GL}(n, \mathbb{C})$.[14] We conclude that $\mathcal{J}(V)$ can be given the structure of a smooth manifold such that $\mathcal{J}(V) \cong \mathrm{GL}(2n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{C})$.

The following result shows that the choice of complex structure compatible with a fixed symplectic form on V is canonical up to homotopy.

Proposition 42. *The set $\mathcal{J}(V, \omega)$ of compatible complex structures is naturally identified with the space $\mathcal{P}$ of symmetric positive-definite symplectic matrices. In particular, $\mathcal{J}(V, \omega)$ is contractible.*

Proof. By fixing a symplectic basis for V we may assume that $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$. By the proof of Lemma 36, we note that $J \in \mathrm{Aut}(\mathbb{R}^{2n})$ is a compatible complex structure if and only if the conditions

$$\begin{aligned} J^2 &= -\mathrm{id}_{\mathbb{R}^{2n}}, \\ J_0 &= J^\top J_0 J, \\ 0 &< -v^\top J_0 J v, \end{aligned}$$

hold (for $v \neq 0$). Set $P = J_0 J$. P is symmetric, since

$$(J_0 J)^\top = -J^\top J_0 = J^\top J_0 J^2 = J_0 J,$$

[14]The embedding is given by replacing each entry $a + bi$ with a block of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

as well as positive-definite, and symplectic. Moreover, it is easy to check that if any matrix P has these three properties, then $J = -J_0 P$ is a compatible complex structure. Hence $\mathcal{J}(V,\omega)$ is in bijective correspondence with the space $\mathcal{P}$ of symmetric positive-definite symplectic matrices. It remains to show that $\mathcal{P}$ is contractible. Suppose, for now, that if $P \in \mathcal{P}$ then $P^\alpha \in \mathcal{P}$ for all $\alpha > 0$, $\alpha \in \mathbb{R}$. Then the map $h : [0,1] \times \mathcal{P} \to \mathcal{P}$ given by $h(t,P) = P^{1-t}$ is a homotopy from $\mathrm{id}_{\mathcal{P}}$ to the constant map $P \mapsto \mathrm{id}_V$, and we are done.

We now show that if $P \in \mathcal{P}$ then $P^\alpha \in \mathcal{P}$ for all $\alpha > 0$. It is easy to see that P^α is symmetric and positive-definite. It remains to show that $\omega_0(P^\alpha v, P^\alpha w) = \omega_0(v,w)$ for all $\alpha > 0$. Decompose $\mathbb{R}^{2n}$ into eigenspaces V_λ for eigenvalues λ of P. Note that for a symplectic matrix P, if λ, λ' are eigenvalues such that $\lambda\lambda' \neq 1$ then $\omega_0(z,z') = 0$, where z, z' are the eigenvectors of λ, λ', respectively:

$$\lambda\lambda'\omega_0(z,z') = \omega_0(Pz, Pz') = \omega_0(z,z').$$

Now, since V_λ is also the eigenspace for the eigenvalue λ^α for P^α, if $z \in V_\lambda, z' \in V_{\lambda'}$,

$$\omega_0(P^\alpha z, P^\alpha z') = (\lambda\lambda')^\alpha \omega_0(z,z').$$

Writing any $v, w \in \mathbb{R}^{2n}$ in the basis of eigenvectors for P^α, we find by linearity, and the remarks about λ, λ' above, that $\omega_0(P^\alpha v, P^\alpha w) = \omega_0(v,w)$ for all $\alpha > 0$. □

Often it is enough to consider a slightly weaker notion of compatibility.

Definition 43. A complex structure $J \in \mathcal{J}(V)$ is called **ω-tame** if $\omega(v, Jv) > 0$ for all nonzero $v \in V$. The set of all ω-tame complex structures on V is written $\mathcal{J}_\tau(V,\omega)$. Note that $\mathcal{J}_\tau(V,\omega)$ is an open subset of $\mathcal{J}(V) \cong \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ (as per Remark 41).

In this case, we note that $g_J(v,w) = (\omega(v,Jw) + \omega(w,Jv))/2$ defines an inner product on V, for all $J \in \mathcal{J}_\tau(V,\omega)$. We note that there is an analog of Proposition 42 for ω-tame complex structures.

Proposition 44. *The space $\mathcal{J}_\tau(V,\omega)$ is contractible.*

Proof. See, for instance, McDuff/Salamon or Gromov. □

4.2. **Symplectic vector bundles.**

Definition 45. A **symplectic vector bundle** (E,ω) over X is a real vector bundle $\pi : E \to X$ together with a smooth symplectic bilinear form $\omega \in \Gamma(X, E^* \wedge E^*)$, i.e. a symplectic bilinear form on each E_x that varies smoothly with x. A **complex structure** on $\pi : E \to M$ is a bundle automorphism $J : E \to E$ such that $J^2 = -\mathrm{id}_E$. We say J is **compatible** with ω if the induced complex structure on E_x is compatible with ω_x for all $x \in X$. We thus obtain a symmetric, positive-definite bilinear form $g_J \in \Gamma(X, \mathrm{Sym}^2 E^*)$, and we call the triple (E,ω,g_J) a **Hermitian structure** on E.

Theorem 46. *Let $E \to X$ be a $2n$-dimensional vector bundle. For any symplectic structure ω on E, the space of compatible complex structures is nonempty and contractible. For any complex structure J on E, the space of symplectic structures compatible with J is nonempty and contractible.*

Proof. See McDuff/Salamon.[15] □

[15]Understand this!

We now prove the Theorem 25, the Lagrangian neighborhood theorem, with the help of the following lemma.

Lemma 47. *Let $J \in \mathcal{J}(V,\omega)$. Then a subspace $\Lambda \subset V$ is Lagrangian if and only if $J\Lambda^\perp = \Lambda$ with respect to g_J.*

Proof. For $v \in \Lambda, w \in V$, the assertion that

$$g_J(Jv, w) = \omega(Jv, Jw) = \omega(v, w) = 0$$

implies that Λ is Lagrangian if and only if $J\Lambda^\perp = \Lambda$. □

Theorem 48 (Lagrangian neighborhood theorem)**.** *Let (M,ω) be a symplectic manifold and let $L \subset M$ be a compact Lagrangian submanifold. Then there exists a neighborhood $N(\Gamma_0) \subset T^*L$ of the zero section Γ_0, a neighborhood $U \subset M$ of L, and a diffeomorphism $\phi : N(\Gamma_0) \to U$ such that $\phi^*\omega = -d\theta$ and $\phi|_L = \mathrm{id}$, where θ is the canonical one-form on T^*L.*

Proof. By Theorem 46, we can fix an arbitrary complex structure J on the tangent bundle TM and denote the associated metric by g_J. Note that the metric yields a diffeomorphism of bundles $\Phi : T^*L \to TL$ given by

$$g_J(\Phi_q(v^*), v) = v^*(v)$$

for $v \in T_qL, v^* \in T_q^*L$. Now the map $\phi : T^*L \to M$ defined by

$$\phi(q, v^*) = \exp_q(J_q\Phi_q v^*)$$

is a diffeomorphism from some neighborhood $N(\Gamma_0)$ of Γ_0 onto its image U, where exp is the exponential map on M corresponding to g_J.

Now if $v = (v_0, v_1^*) \in T_{(q,0)}T^*L = T_qL \oplus T_q^*L$, we claim that

$$d\phi_{(q,0)}(v) = v_0 + J_q\Phi_q v_1^*.$$

By linearity, it suffices to compute $d\phi_{(q,0)}$ on T_qL and T_q^*L separately. In particular, let $c : [0,1] \to TM$ be a curve given by $c(t) = (a(t), 0)$, with $c'(0) = (v_0, 0)$. Then

$$\begin{aligned} d\phi_{(q,0)}(v_0, 0) &= \frac{d}{dt}\bigg|_{t=0} \exp_{a(t)}\left(J_{a(t)}\Phi_{a(t)}0\right) \\ &= \frac{d}{dt}\bigg|_{t=0} a(t) \\ &= v_0. \end{aligned}$$

Next take $c(t) = (q, tv_1^*)$. Clearly $c'(0) = (0, v_1^*)$. Then

$$\begin{aligned} d\phi_{(q,0)}(0, v_1^*) &= \frac{d}{dt}\bigg|_{t=0} \exp_p(J_p\Phi_p t v_1^*) \\ &= J_p\Phi_p v_1^*, \end{aligned}$$

as desired.

We can now compute, for $v = (v_0, v_1^*), w = (w_0, w_1^*) \in T_{(q,0)}T^*L$,

$$\begin{aligned}
\phi^*\omega_{(q,0)}(v,w) &= \omega_q\left(v_0 + J_q\Phi_q v_1^*, w_0 + J_q\Phi_q w_1^*\right) \\
&= \omega_q(v_0, J_q\Phi_q w_1^*) - \omega_q(w_0, J_q\Phi_q v_1^*) \\
&= g_J(v_0, \Phi_q w_1^*) - g_J(w_0, \Phi_q v_1^*) \\
&= w_1^*(v_0) - v_1^*(w_0) \\
&= -d\theta_{(q,0)}(v,w).
\end{aligned}$$

This shows that $\phi^*\omega = -d\theta$ on the zero section. Now the result follows from Moser's trick, Theorem 2.1. $\square$

5. Week 5

5.1. Almost complex manifolds.

Definition 49. Let M be a $2n$-dimensional real manifold. An **almost complex structure** on M is a complex structure J on the tangent bundle TM. In this situation we say that (M, J) is an almost complex manifold. The almost complex structure is **compatible** with a nondegenerate two-form ω on M if J is compatible with ω.

Theorem 50. *For each nondegenerate two-form ω on M the space of almost complex structures compatible with ω is nonempty and contractible. Conversely, for every almost complex structure on M the space of compatible nondegenerate two-forms is nonempty and contractible.*

Proof. See Theorem 46. $\square$

Example 51. Let $X \subset \mathbb{R}^3$ be an oriented hypersurface. Let $\nu : X \to S^2$ be the Gauss map, which assigns to each point $x \in X$ the outward-pointing normal vector $\nu(x) \perp T_x X$. Define, for $u \in T_x X$,

$$J_x u = \nu(x) \times u,$$

where the product is the vector (cross) product on $\mathbb{R}^3$. It follows from the vector triple product identity $a \times (b \times c) = b(g(a,c)) - c(g(a,b))$, where g is the standard metric on $\mathbb{R}^3$, that $J_x^2 = -\operatorname{id}_{T_x X}$. Define a two-form ω on X by

$$\begin{aligned} \omega(v,w) &= \iota(\nu(x))\Omega \\ &= g(\nu(x), v \times w), \end{aligned}$$

where $\Omega(u,v,w)$ is the determinant of the matrix whose columns are u, v, w. It is straightforward to check that J is compatible with ω: for $v, w \in T_x X$,

$$\begin{aligned} \omega(J_x v, J_x w) &= g\left(\nu(x), (\nu(x) \times v) \times (\nu(x) \times w)\right) \\ &= g\left(\nu(x), \nu(x) g(\nu(x) \times v, w)\right) \\ &= g(w, \nu(x) \times v) \\ &= g(\nu(x), v \times w) \\ &= \omega(v,w) \\ \omega(v, J_x v) &= g(\nu(x), v \times (\nu(x) \times v)) \\ &= g(\nu(x), g(v,v)\nu(x)) \\ &= g(v,v) \\ &> 0, \end{aligned}$$

where we have used the vector triple product identity as well as the cyclic property of the scalar triple product.

Example 52. Consider $S^2 \subset \mathbb{R}^3$ with the almost complex structure J from the previous example. We compute the expression of J in stereographic coordinates. Recall we have $\phi : S^2 - (0,0,1) \to \mathbb{R}^2$ given by

$$\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

and inverse

$$\psi(X,Y) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right).$$

For a point $p = (x, y, z) \in S^2$ and a vector $u = (v, w) \in T_pS^2$, some computation reveals that

$$\begin{aligned} J_p(v,w) &= d\phi\left((x,y,z) \times d\psi(v,w)\right) \\ &= (w, -v). \end{aligned}$$

Definition 53. Let (X, J) be an almost complex manifold. We define the **Nijenhuis tensor** N_J by

$$N_J(v,w) = [v,w] + J[Jv,w] + J[v,Jw] - [Jv,Jw]$$

for v, w vector fields on X.

Lemma 54. *The Nijenhuis tensor is a skew-symmetric covariant (2,0)-tensor on X satisfying*
(a) $N_J(v, Jv) = 0$ *for all vector fields* v*;*
(b) $N_{J_0} = 0$*;*
(c) If $\phi \in \mathrm{Diff}(M)$ *and* v, w *are vector fields then*

$$N_{\phi^*J}(\phi^*v, \phi^*w) = \phi^*N_J(v,w).$$

Proof. Writing $v = v^i\partial/\partial x^i, w = w^i\partial/\partial x^i$ in local coordinates, the Lie bracket $[v, w]$ becomes[16]

$$[v,w] = \left(w^j\frac{\partial v^i}{\partial x^j} - v^j\frac{\partial w^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}.$$

Finish this. □

Suppose now that (X, J) is an almost complex manifold. Denote by $T_{\mathbb{C}}X$ the complexification of the real vector bundle TX, i.e. $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$. We note that the complexified tangent bundle splits into $\pm i$ J-eigenbundles $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$, respectively.[17] These are often referred to as the holomorphic and antiholomorphic tangent bundles of X.

Definition 55. Let X be an almost complex manifold. We define the vector bundles

$$\begin{aligned} \bigwedge\nolimits_{\mathbb{C}}^k X &\equiv \bigwedge\nolimits^k (T_{\mathbb{C}}X)^* \\ \bigwedge\nolimits^{p,q} X &\equiv \bigwedge\nolimits^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge\nolimits^q (T^{0,1}X)^*. \end{aligned}$$

and write $\mathcal{A}^k_{X,\mathbb{C}}$ and $\mathcal{A}^{p,q}_X$ for their sheaves of sections, respectively. We denote the projections $\mathcal{A}^\bullet \to \mathcal{A}^k$ and $\mathcal{A}^\bullet \to \mathcal{A}^{p,q}$ by Π^k and $\Pi^{p,q}$ respectively. It is not hard to show that

$$\begin{aligned} \bigwedge\nolimits_{\mathbb{C}}^k X &= \bigoplus_{p+q=k} \bigwedge\nolimits^{p,q} X \\ \mathcal{A}^k_{\mathbb{C}} &= \bigoplus_{p+q=k} \mathcal{A}^{p,q} \end{aligned}$$

[16] We follow McDuff/Salamon in the convention that $[v, w] \equiv -\mathcal{L}_v w$.
[17] Here, J is really $J \otimes \mathbb{C}$.

and additionally, that $\overline{\bigwedge^{p,q} X} = \bigwedge^{q,p} X$ and $\overline{\mathcal{A}^{p,q}} = \mathcal{A}^{q,p}$. Now if $d : \mathcal{A}^k_{\mathbb{C}} \to \mathcal{A}^{k+1}_{\mathbb{C}}$ is the exterior derivative[18], we write

$$\begin{aligned}\partial &\equiv \Pi^{p+1,q} \circ d\\ \bar\partial &\equiv \Pi^{p,q+1} \circ d,\end{aligned}$$

and $\partial, \bar\partial$ satisfy the appropriate graded Leibniz rule.

With this notation now set, we come to the key definition.

Proposition 56. *Let (X, J) be an almost complex manifold. Then the following conditions are equivalent:*
(a) $d = \partial + \bar\partial$ *on* $\mathcal{A}^\bullet$*;*
(b) $\Pi^{0,2} \circ d = 0$ *on* $\mathcal{A}^{1,0}$*;*
(c) $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$*;*
(d) $N_J = 0$*.*
If X satisfies one of these equivalent conditions then J is said to be an ***integrable*** *almost complex structure.*

Proof. We show that (a) is equivalent to (b), (b) is equivalent to (c), and that (c) is equivalent to (d).

For (a)↔(b), suppose first that $d = \partial + \bar\partial$ and $\alpha \in \mathcal{A}^{1,0}$. Then

$$\begin{aligned}\Pi^{0,2} d\alpha &= \Pi^{0,2}(\partial + \bar\partial)\alpha\\ &= \Pi^{0,2}(\Pi^{2,0} + \Pi^{1,1})d\alpha\\ &= 0.\end{aligned}$$

Conversely, suppose $\Pi^{0,2}d = 0$ on $\mathcal{A}^{1,0}$. Clearly $d = \partial + \bar\partial$ if and only if $d\alpha \in \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}$ for all $\alpha \in \mathcal{A}^{p,q}$. Now any $\alpha \in \mathcal{A}^{p,q}$ can locally be written as a linear combination of terms of the form $f_{IJ} w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge w'_{j_1} \wedge \cdots \wedge w'_{j_q}$, with the $w \in \mathcal{A}^{1,0}$ and $w' \in \mathcal{A}^{0,1}$. Then $d\alpha$ is expressed as a linear combination of terms involving df_{IJ}, dw_i, and dw'_j. We have that $df \in \mathcal{A}^2_{\mathbb{C}} = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$, which takes care of the terms containing df_{IJ}. Similarly, since $\Pi^{0,2}d = 0$ on $\mathcal{A}^{1,0}$ by assumption, $dw_i \in \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1}$, which takes care of the terms containing the dw_i. Finally, we have that $dw'_j \in \mathcal{A}^{1,1} \oplus \mathcal{A}^{0,2}$ since $\Pi^{2,0}d = 0$ on $\mathcal{A}^{0,1}$ (seen by conjugating (b)), which takes care of the terms containing the dw'_j. We conclude that $d\alpha \in \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}$, as desired.

We now prove (b)↔(c). Fix any $\alpha \in \mathcal{A}^{1,0}$ and v, w sections of $T^{0,1}$. Then, by definition of $d\alpha$, and since α vanishes on $T^{0,1}$, we find that

$$\begin{aligned}(d\alpha)(v, w) &= v\alpha(w) - w\alpha(v) - \alpha[v, w]\\ &= -\alpha[v, w].\end{aligned}$$

We conclude that $\Pi^{0,2}d = 0$ if and only if $[v, w] \in T^{0,1}$.

We now prove $(c) \leftrightarrow (d)$. Suppose for now that any section of $T^{0,1}$ can be written as $v + iJv$ for v a section of $TX \otimes \mathbb{C}$. Then

$$[v + iJv, w + iJw] = [v, w] - [Jv, Jw] - i\left([Jv, w] + [v, Jw]\right).$$

This is of the form $u + iJu$ if and only if

$$J\left([v, w] - [Jv, Jw]\right) = [Jv, w] + [v, Jw],$$

[18] Here, d is really $d \otimes \mathbb{C}$.

which is equivalent to $N_J(v,w)=0$. It remains to show that any section of $T^{0,1}$ can be written as $v+iJv$. Finish this. □

Example 57. Let X be a complex manifold. Then we have local coordinates $z_i, \bar{z}_i$ for $i=1,\dots,n$ and the standard almost complex structure J_0 acting as i on $\partial/\partial z_i$ and $-i$ on $\partial/\partial \bar{z}_i$. Now we note that for $\alpha \in \mathcal{A}^{p,q}$ written $\alpha = \alpha_{IJ} dz^I \wedge d\bar{z}^J$, we have

$$d\alpha = \left(\frac{\partial \alpha_{IJ}}{\partial z^k} dz^k + \frac{\partial \alpha_{IJ}}{\partial \bar{z}^k} d\bar{z}^k\right) \wedge dz^I \wedge d\bar{z}^J.$$

Clearly then $d=\partial+\bar{\partial}$, as $\partial = \Pi^{p+1,q}d$ and $\bar{\partial} = \Pi^{p,q+1}$. Hence, by Proposition 56(a), J_0 is integrable.

The above example shows that complex manifolds induce integrable almost complex structures on their underyling real manifolds in a natural way. It is a highly nontrivial fact that the converse is also true.

Theorem 58 (Newlander-Nirenberg, 1957)**.** *Let (X,J) be an almost complex manifold. Then J is integrable if and only if X has a holomorphic atlas (making it a complex manifold) such that the induced almost complex structure is J.*

Example 59. Let (X,J) be a two-dimensional almost complex manifold. In this case $\mathcal{A}^2_{\mathbb{C}} = \mathcal{A}^{1,1}$ and hence by Proposition 56(b), we find that J is integrable. We conclude using the Newlander-Nirenberg theorem that every two-dimensional almost complex manifold is in fact a complex manifold.

Example 60. It turns out that there exists a vector product on $\mathbb{R}^7$ that is bilinear and skew-symmetric, and hence it follows along the lines of Example 51 that every oriented hypersurface $X \subset \mathbb{R}^7$ carries an almost complex structure. This argument shows, in particular, that S^6 is an almost complex manifold. It was shown by Calabi, however, that this almost complex structure is not integrable. Indeed, the existence of an integrable almost complex structure on S^6 is still an open problem.

5.2. **Kähler manifolds.**

Definition 61. A **Kähler** manifold is a symplectic manifold (M,ω) equipped with an integrable almost complex structure $J \in \mathcal{J}(M,\omega)$.

Example 62. The most basic example of a Kähler manifold is $(\mathbb{R}^{2n},\omega_0,J_0)$. Indeed, viewing $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ we can introduce coordinates $z^i = x^i + iy^i, \bar{z}^i = x^i - iy^i$ with respect to which $T^{1,0}\mathbb{C}^n$ and $T^{0,1}\mathbb{C}^n$ are trivialized by the frames $\partial/\partial z^i$ and $\partial/\partial \bar{z}^i$, respectively. Then it is straightforward to check that $d=\partial+\bar{\partial}$ on $\mathcal{A}^\bullet_{\mathbb{C}}$. In these coordinates,

$$dz^i = dx^i + idy^i$$
$$d\bar{z}^i = dx^i - idy^i.$$

and a easy computation reveals that the symplectic form ω_0 can be written

$$\omega_0 = \frac{i}{2}\sum_{i=1}^n dz^i \wedge d\bar{z}^i.$$

In fact, if we let $f=\sum_i^n \bar{z}^i z^i$, we can write $\omega_0 = i\partial\bar{\partial} f/2$.

Example 63. Every two-dimensional symplectic manifold is Kähler with respect to any compatible almost complex structure.

Example 64 (Complex projective space)**.** Let $\mathbb{P}^n$ denote the complex projective space, which is a complex manifold of dimension n. Let J be the induced integrable almost complex structure.

6. Week 6

6.1. Poisson brackets.

Definition 65. Let (M, ω) be a symplectic manifold. We say that a vector field $X \in \mathcal{X}(M)$ is **symplectic** if

$$d(\iota(X)\omega) = 0,$$

or equivalently,

$$\mathcal{L}_X \omega = 0.$$

We denote the Lie algebra of symplectic vector fields by $\mathcal{X}(M, \omega)$.

Proposition 66. *Let M be closed and let $X \in \mathcal{X}(M)$ be a smooth vector field with flow $F : I \times M \to M$. Then F_t is a symplectomorphism for all t if and only if X is symplectic.*

Proof. Note that $F_t^*\omega : I \to \Gamma(M, \bigwedge^2 T^*M)$ gives us a smooth curve in the vector space $\Gamma(M, \bigwedge^2 T^*M)$. Then

$$\begin{aligned}
\frac{d}{dt}\left(F_t^*\omega\right) &= \left.\frac{d}{ds}\right|_{s=0}\left(F_{s+t}^*\omega\right) \\
&= F_t^*\mathcal{L}_X\omega \\
&= F_t^* d(\iota_X \omega)
\end{aligned}$$

and we see that the curve is constant at ω if and only if $X \in \mathcal{X}(M, \omega)$.[19] $\square$

For the most part, we will focus on a subset of symplectic vector fields known as Hamiltonian vector fields (also introduced in section 1).

Definition 67. Let $H : M \to \mathbb{R}$ be a smooth function and let X_H be the vector field determined uniquely by

$$\iota_{X_H}\omega = dH.$$

We say that X_H is a **Hamiltonian vector field** for the **Hamiltonian** H. If M is closed, X_H generates a smooth one-parameter group of symplectomorphisms F_H^t as its flow. We call this the **Hamiltonian flow** associated to H. Computing as in the proof of the proposition above, we find that

$$\begin{aligned}
\frac{d}{dt}\left((F_H^t)^* H\right) &= X_H H = dH(X_H) \\
&= (\iota_{X_H}\omega)(X_H) \\
&= \omega(X_H, X_H) \\
&= 0.
\end{aligned}$$

We conclude that H is constant along the Hamiltonian flow.

Example 68. Sphere with cylindrical polar coordinates and H the height function.

Definition 69. Let k be a field. A **Poisson algebra** A over k is an k-vector space equipped with bilinear products $\cdot$ and $\{\cdot, \cdot\}$ such that

(a) the product $\cdot$ gives A the structure of an associative k-algebra;
(b) the bracket $\{\cdot, \cdot\}$ gives A the structure of a Lie algebra;
(c) the bracket $\{\cdot, \cdot\}$ is a k-derivation over the product $\cdot$.

[19] Understand this computation better.

Proposition 70. *Let (M,ω) be a symplectic manifold. Define a product on $C^\infty(M)$ as*
$$\{f,g\} \equiv \omega(X_f, X_g).$$
Then $C^\infty(M)$ forms a real Poisson algebra.

Proof. That $C^\infty(M)$ is an associative $\mathbb{R}$-algebra under multiplication is clear (in fact, it is even commutative). Now, since
$$\iota_{X_{f_1}+X_{f_2}}\omega = \iota_{X_{f_1}}\omega + \iota_{X_{f_2}}\omega = df_1 + df_2 = d(f_1+f_2) = \iota_{X_{f_1+f_2}}\omega.$$
uniqueness forces $X_{f_1} + X_{f_2} = X_{f_1+f_2}$. It follows immediately that the Poisson bracket is bilinear. That the bracket is alternating follows from the fact that ω is. Similarly, since
$$\iota_{gX_h+hX_g}\omega = g\iota_{X_h}\omega + h\iota_{X_g}\omega = gdh + hdg = d(gh) = \iota_{X_{gh}\omega},$$
we conclude that $X_{gh} = gX_h + hX_g$, and hence
$$\{f,gh\} = \omega(X_f, X_{gh}) = g\omega(X_f,X_h) + h\omega(X_f,X_g) = g\{f,h\} + h\{f,g\},$$
which proves the derivation property (that the bracket is zero on a constant in $\mathbb{R}$ is easy to check).

It remains to check the Jacobi identity
$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$$
Using anticommutativity and the fact that
$$\{f,g\} = (\iota_{X_f}\omega)(X_g) = df(X_g) = X_g f,$$
we can rewrite the left-hand side as
$$X_fX_gh - X_gX_fh + X_{\{f,g\}}h = -[X_f,X_g]h + X_{\{f,g\}}h.$$
Hence it suffices to show that $X_{\{f,g\}} = [X_f, X_g]$[20] To see this, note that
$$\mathcal{L}_{X_f}\iota_{X_g}\omega = d\iota_{X_f}\iota_{X_g}\omega = d\{g,f\} = \iota_{X_{\{g,f\}}}\omega$$
and, using Cartan's (second magic) formula,[21]
$$\mathcal{L}_{X_f}\iota_{X_g}\omega = \iota_{\mathcal{L}_{X_f}X_g}\omega + \iota_{X_g}\mathcal{L}_{X_f}\omega = \iota_{[X_g,X_f]}\omega$$
(since $\mathcal{L}_{X_f}\omega = 0$), so
$$\iota_{X_{\{g,f\}}}\omega = \iota_{[X_f,X_g]}\omega.$$
Now uniqueness implies that $X_{\{f,g\}} = [X_f, X_g]$, as desired. □

A manifold equipped with a Poisson algebra structure on its smooth functions is called a Poisson manifold. The previous proposition shows that every symplectic manifold is a Poisson manifold. The following example shows that the converse is not true, as a Poisson manifold can have arbitrary dimension.

Example 71 (Lie-Poisson structure)**.** Let $\mathfrak{g}$ be a real Lie algebra. Denote by $\mathfrak{g}^*$ the dual vector space. Treating $\mathfrak{g}^*$ as a manifold, we note that the de Rham differential of $f \in C^\infty(\mathfrak{g}^*)$ is $df_\alpha : T_\alpha\mathfrak{g}^* = \mathfrak{g}^* \to \mathbb{R}$ for $\alpha \in \mathfrak{g}^*$. Since $\mathfrak{g}^{**}$ is naturally identified with $\mathfrak{g}$, it is easy to check that
$$\{f,g\}(\alpha) = \alpha[dg_\alpha, df_\alpha].$$
provides a Poisson structure on $\mathfrak{g}^*$.

[20] We follow Mcduff/Salamon in the convention that $[X,Y] = -\mathcal{L}_X Y$.

[21] See Morita's Geometry of Differential Forms, Theorem 2.11(1).

Note that the Poisson algebras in the two examples above are commutative in the product $\cdot$, but these need not be the case in general.

Morphisms in the category of Poisson manifolds? (see Wikipedia)

What happens if $H : M \to \mathbb{R}$ is Morse? This implies that $dH : M \hookrightarrow T^*M$ intersects the zero section of T^*M transversely. What does this give us?

Can we extend the Poisson structure to the exterior algebra of forms?

6.2. **Group actions.** Before discussing group actions on symplectic manifolds, we review some basic notions from Lie theory. Let G be a Lie group and $\mathfrak{g} = T_eG$ be its Lie algebra, and denote left (right) multiplication by g as L_g (R_g).

Lemma 72. *There is a Lie algebra isomorphism between the Lie algebra $\mathfrak{g}$ of G and the space of left-invariant vector fields on G. In particular $X \in \mathfrak{g}$ is sent to the vector field $\tilde{X}$ satisfying $(L_g^*\tilde{X})_h = \tilde{X}_{gh}$ for all $g \in G$ such that $\tilde{X}_e = X$.*

Lemma 73. *The left-invariant vector fields on G are complete, i.e. their flows define diffeomorphisms of G.*

Proof. By the uniqueness of integral curves, it suffices to show that if $\gamma : I \to G$ is an integral curve then $L_g \circ \gamma$ is as well. This is a straightforward computation:

$$\begin{aligned} \frac{d}{dt}(L_g \circ \gamma) &= (dL_g \circ d\gamma)\left(\frac{d}{dt}\right) \\ &= dL_g(X_{\gamma(t)}) \\ &= X_{L_g \circ \gamma(t)}, \end{aligned}$$

as desired. □

Definition 74. The **exponential map** is the smooth map $\exp : \mathfrak{g} \to G$ given by

$$\exp(\xi) = \phi_\xi^1(e),$$

where $\phi_\xi^1 : G \to G$ is the time 1 flow associated to the left-invariant vector field $\tilde{\xi}$. It is easy to see that $\exp(t\xi) = \phi_\xi^t(e)$. Moreover, if $[\xi, \eta] = 0$ then $\exp(\xi + \eta) = \exp(\xi)\exp(\eta)$. Finally, for a morphism $f : G \to H$ of Lie groups, we obtain a commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "df_e"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

which we will refer to as the naturality of exp. Note that the differential at e of a Lie group homomorphism is a Lie algebra homomorphism, as is df_e here.

We now consider the symplectic case.

Proposition 75. *The Lie algebra of the Lie group of symplectomorphisms* $\mathrm{Symp}(M, \omega)$ *is the space of symplectic vector fields $\mathcal{X}(M, \omega)$.*

Proof. This involves dealing with time-dependent vector fields, so I'll work through it later.[22] □

[22] Finish

Now suppose G acts on (M,ω) symplectomorphically, i.e. there is a group homomorphism $\psi : G \to \mathrm{Symp}(M,\omega)$ taking $g \mapsto \psi_g$. Differentiating this map at the identity yields a Lie algebra homomorphism $d\psi_e : \mathfrak{g} \to \mathcal{X}(M,\omega)$. We denote the image of $\xi \in \mathfrak{g}$ under $d\psi_e$ by ξ_M. Now let $c : G \to \mathrm{Aut}(G)$ be the conjugation homomorphism $c_g h = ghg^{-1}$ and denote by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ the homomorphism taking g to $(dc_g)_e$.

Lemma 76. *In the notation above, we have an equality of vector fields*

$$(\mathrm{Ad}_g \xi)_M = \psi^*_{g^{-1}} \xi_M,$$

for $\xi \in \mathfrak{g}$.

Proof. For $p \in M$ we can write, using the naturality of exp and the chain rule,[23]

$$\begin{aligned}(\mathrm{Ad}_g \xi)_M(p) &= \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\,\mathrm{Ad}_g \xi)}(p)\\ &= \frac{d}{dt}\bigg|_{t=0} \psi_{g\exp(t\xi)g^{-1}}(p)\\ &= \frac{d}{dt}\bigg|_{t=0} \psi_g \psi_{\exp(t\xi)}(\psi_{g^{-1}}(p))\\ &= d\psi_g|_{\psi_{g^{-1}}(p)} \xi_M(\psi_{g^{-1}}(p))\\ &= (\psi^*_{g^{-1}} d\psi|_e \xi)(p).\end{aligned}$$

∎

Definition 77. An action ψ of G on (M,ω) is **weakly Hamiltonian** if the vector field ξ_M is Hamiltonian for each $\xi \in \mathfrak{g}$, i.e.

$$\iota_{\xi_M}\omega = dH_\xi$$

for some $H_\xi \in C^\infty(M)$. For a weakly Hamiltonian action, then, we obtain a map $\mathfrak{g} \to C^\infty(M)$ taking $\xi \mapsto H_\xi$. This map is *a priori* not even linear. However, since each H_ξ is defined only up to a constant, we can choose the H_ξ to make $\mathfrak{g} \to C^\infty(M)$ linear.

We say the action ψ of G on (M,ω) is **Hamiltonian** if the map $\mathfrak{g} \to C^\infty(M)$ can be chosen to be a Lie algebra homomorphism (with respect to the Poisson structure on $C^\infty(M)$).

Definition 78. Suppose ψ is a Hamiltonian action of G on (M,ω). We say that a map $\mu : M \to \mathfrak{g}^*$ is a **moment map** for the action if

$$H_\xi(p) = \langle \mu(p), \xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Example 79. Consider the action of S^1 on the sphere (with its usual symplectic structure) that rotates the sphere about its vertical axis. More precisely, using cylindrical coordinates θ, z away from the poles, the action is given by $\psi : S^1 \times S^2 \to S^2$ as $(\rho,(\theta,z)) \mapsto (\theta+\rho, z)$. The associated Lie algebra action is then $d\psi_e : \mathfrak{u}(1) \cong \mathbb{R} \to \mathcal{X}(S^2,\omega)$ given by $\xi \mapsto \xi\partial/\partial\theta$. In the notation above,

$$\xi_M = \xi \frac{\partial}{\partial \theta}.$$

[23]Review this.

Now, since $\omega = d\theta \wedge dz$ away from the poles, we find that $\iota_{\xi_M}\omega = \xi dz$. Hence the action is Hamiltonian (the Poisson condition is trivial as $\mathfrak{u}(1)$ is one-dimensional) since

$$H_\xi = \xi z.$$

We obtain a moment map $\mu : S^2 \to \mathfrak{u}(1)^* \cong \mathbb{R}$ given simply by

$$\mu(\theta, z) = z.$$

This is simply the height function on the sphere, whose fibers are precisely the orbits of the S^1 action.

Definition 80. Suppose $(M, \omega = -d\lambda)$ be an exact symplectic manifold. We say that the action ψ of G on M is **exact** if $\psi_g^*\lambda = \lambda$ for each $g \in G$.

Remark 81. Recall that a closed symplectic $2n$-manifold (M, ω) cannot be exact. Indeed, if it were, the volume form ω^n would be exact and Stokes' theorem would imply that $\int_M \omega^n = 0$, which is not possible. Hence for M closed, ω must represent a nontrivial class in $\mathrm{H}^2(M; \mathbb{R})$.

Proposition 82. *Let $(M, \omega = -d\lambda)$ be an exact symplectic manifold. Then every exact action of G on M is Hamiltonian with*

$$H_\xi = \iota_{X_\xi}\lambda$$

for $\xi \in \mathfrak{g}$.

6.3. **Cohomological obstructions.** In general, weakly Hamiltonian actions need not be Hamiltonian. In this section, we digress briefly to derive sufficient conditions for an action to be weakly Hamiltonian, and a necessary condition for a weakly Hamiltonian action to be Hamiltonian. For this, we quickly present Lie algebra cohomology, following Ortega/Ratiu.[24]

Let G be a real Lie group of dimension n. Similarly to the case of vector fields, we say that a differential k-form $\omega \in \Omega^k(G)$ is left invariant if $L_g^*\omega = \omega$ for each $g \in G$. Note that left invariant k-forms can be identified with the k-forms $\Lambda^k\mathfrak{g}^*$, since they are determined by their action at the identity. We now obtain a chain complex of left-invariant forms

$$0 \longrightarrow \Lambda^0\mathfrak{g}^* \cong \mathbb{R} \longrightarrow \Lambda^1\mathfrak{g} \cong \mathfrak{g}^* \longrightarrow \cdots \longrightarrow \Lambda^n\mathfrak{g}^* \cong \mathbb{R} \longrightarrow 0,$$

where the differentials are given by the expected formula: for $\omega \in \Lambda^k\mathfrak{g}^*$,

$$d\omega(\xi_0, \ldots, \xi_k) = \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j}\omega([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_k).$$

We note the following low-dimensional cases, which will be the ones of interest to us. For $\omega \in \Lambda^0\mathfrak{g}^*$, clearly $d\omega = 0$. For $\omega \in \Lambda^1\mathfrak{g}^* \cong \mathfrak{g}^*$,

$$d\omega(\xi_1, \xi_2) = -\omega[\xi_1, \xi_2]$$

and for $\omega \in \Lambda^2\mathfrak{g}^*$,

$$d\omega(\xi_1, \xi_2, \xi_3) = -\omega([\xi_1, \xi_2], \xi_3) + \omega([\xi_3, \xi_1], \xi_2) + \omega([\xi_2, \xi_3], \xi_1).$$

We now define the **Lie algebra cohomology** $\mathrm{H}^\bullet(\mathfrak{g}, \mathbb{R})$ to be the cohomology of the above complex.

[24] add citation!

Remark 83. More generally, let $\mathfrak{g}$ be a Lie algebra over k and let M be a $\mathfrak{g}$-module. Denote by $-^{\mathfrak{g}} : \mathfrak{g}\text{-}\textsc{Mod} \to \mathfrak{g}\text{-}\textsc{Mod}$ the invariants functor. Then one defines $\mathrm{H}^\bullet(\mathfrak{g}, M)$, the cohomology groups of $\mathfrak{g}$ with coefficients in M, as the right derived functors $R^\bullet(-^{\mathfrak{g}})(M)$. Of course, this is much more generality than we will need; the formulation above is computing the cohomology of the Chevalley-Eilenberg resolution of $\mathbb{R}$.[25]

The first cohomology group is quite easily computed. Indeed,

$$\mathrm{H}^1(\mathfrak{g}, \mathbb{R}) = \{\omega \in \mathfrak{g}^* \mid \omega[\xi_1, \xi_2] = 0\}.$$

Noting that $\omega \in \mathfrak{g}^*$ is a map $\omega : \mathfrak{g} \to \mathbb{R}$ annihilating precisely $[\mathfrak{g}, \mathfrak{g}]$, and that such maps are in correspondence with maps $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathbb{R}$, we conclude that

$$\mathrm{H}^1(\mathfrak{g}, \mathbb{R}) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*.$$

In particular, we note that if $\mathfrak{g}$ is semisimple, $\mathrm{H}^1(\mathfrak{g}, \mathbb{R}) = 0$. We now relate these cohomology groups back to Hamiltonian actions.

Proposition 84. *The commutator of two symplectic vector fields on (M, ω) is Hamiltonian.*

Proof. Let $X, Y \in \mathcal{X}(M, \omega)$, i.e. $d\iota_X\omega = d\iota_Y\omega = 0$ or equivalently $\mathcal{L}_X\omega = \mathcal{L}_Y\omega = 0$. Now, using both of Cartan's magic formulas, we find that

$$\begin{aligned}\iota_{[X,Y]}\omega &= [\mathcal{L}_X, \iota_Y]\omega \\ &= \mathcal{L}_X\iota_Y\omega \\ &= d\iota_X\iota_Y\omega.\end{aligned}$$

We conclude that $[X, Y]$ is Hamiltonian with $H_{[X,Y]} = \omega(X, Y)$. □

Corollary 85. *Suppose G acts on (M, ω) through symplectomorphisms and that $\mathrm{H}^1(\mathfrak{g}, \mathbb{R}) = 0$ or $\mathrm{H}^1_{\mathrm{dR}}(M, \mathbb{R}) = 0$. Then the action is weakly Hamiltonian.*

Proof. If the first Lie algebra cohomology vanishes, we must have that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. In particular, the induced symplectic vector fields on M must be Hamiltonian by the previous proposition. If the first de Rham cohomology vanishes, every closed one-form on M is exact and thus, by definition, every symplectic vector field is Hamiltonian. □

Less trivial are the obstructions for a weakly Hamiltonian action to be Hamiltonian.

Proposition 86. *Suppose the action of G on (M, ω) is weakly Hamiltonian, where M is connected. Then the action determines a cocycle $[\tau] \in \mathrm{H}^2(\mathfrak{g}, \mathbb{R})$ which vanishes if and only if the action is Hamiltonian.*

Proof. Since the action is weakly Hamiltonian we may choose a linear map $\mathfrak{g} \to C^\infty(M)$ sending $\xi \mapsto H_\xi$ such that $\iota_{\xi_M}\omega = dH_\xi$. For each pair $\xi, \eta \in \mathfrak{g}$, define a function on M

$$\tau(\xi, \eta) = \{H_\xi, H_\eta\} - H_{[\xi,\eta]}.$$

Since

$$X_{H_{[\xi,\eta]}} = [\xi, \eta]_M = [\xi_M, \eta_M] = [X_{H_\xi}, X_{H_\eta}] = X_{\{H_\xi, H_\eta\}},$$

[25] Cite Weibel

we find that
$$d\left(H_{[\xi,\eta]} - \{H_\xi, H_\eta\}\right) = 0$$
so τ is locally constant, hence constant. Clearly then $\tau \in \Lambda^2\mathfrak{g}^*$.

We now claim that $d\tau = 0$, i.e.
$$\tau([\xi,\eta],\zeta) + \tau([\eta,\zeta],\xi) + \tau([\zeta,\xi],\eta) = 0.$$
Reasoning as in the previous paragraph, we find that
$$\{H_{[\xi,\eta]}, H_\zeta\} = \{\{H_\xi, H_\eta\}, H_\zeta\},$$
so by the Jacobi identity for the Poisson bracket,
$$d\tau(\xi,\eta,\zeta) = -\left(H_{[[\xi,\eta],\zeta]} + H_{[[\eta,\zeta],\xi]} + H_{[[\zeta,\xi],\eta]}\right) = 0,$$
by linearity of the map $\xi \mapsto H_\xi$ and the Jacobi identity for $\mathfrak{g}$.

Hence τ represents a cocycle $[\tau] \in \mathrm{H}^2(\mathfrak{g},\mathbb{R})$. If the action is Hamiltonian to begin with, obviously $\tau = 0$, since $\xi \mapsto H_\xi$ is a Lie algebra homomorphism. Conversely, suppose $[\tau] = 0$. This is equivalent to asking that τ be a coboundary
$$\tau(\xi,\eta) = \sigma[\xi,\eta]$$
for some $\sigma \in \mathfrak{g}^*$. Modifying the given map $\xi \mapsto H_\xi$ to $\xi \mapsto H_\xi + \sigma(\xi)$, we find that
$$[\xi,\eta] \mapsto H_{[\xi,\eta]} + \sigma[\xi,\eta] = \{H_\xi, H_\eta\},$$
and we conclude that the action is Hamiltonian. $\square$

Example 87. The second Whitehead lemma states that for $\mathfrak{g}$ semisimple, $\mathrm{H}^2(\mathfrak{g},\mathbb{R}) = 0$.[26] Thus, if the Lie algebra of G is semisimple, every weakly Hamiltonian G-action on (M,ω) is Hamiltonian.

[26]Reference Weibel

7. Week 7

The following section is essentially a rewrite of the previous section in order to clear things up in my head.[27]

7.1. Group actions on symplectic manifolds.

Definition 88. Suppose G acts on (M, ω) through symplectomorphisms. The G-action induces a $\mathfrak{g}$-action on (M, ω), i.e. a symplectic vector field $\xi_M \in \Gamma(TM, \omega)$ associated to each $\xi \in \mathfrak{g}$. We say that the action is **weakly Hamiltonian** if there exists a linear **comoment map** $\kappa : \mathfrak{g} \to C^\infty(M)$ such that

$$\iota_{\xi_M}\omega = d\kappa(\xi)$$

for all $\xi \in \mathfrak{g}$. Furthermore, we say that the action is **Hamiltonian** if κ is a Lie algebra homomorphism with respect to the Poisson structure on M, i.e.

$$\kappa([\xi, \eta]) = \{\kappa(\xi), \kappa(\eta)\}.$$

Remark 89. Recall that a symplectic vector field ξ_M on (M, ω) satisfies

$$\mathcal{L}_{\xi_M}\omega = d\iota_{\xi_M}\omega = 0.$$

Note that since every closed form is locally exact, every G-action through symplectomorphisms is locally weakly Hamiltonian.

Example 90. Consider S^2 with its usual symplectic structure written in cylindrical coordinates (away from the poles) as

$$\omega = d\theta \wedge dz.$$

Let S^1 act on the sphere by rotating it about its vertical axis. More precisely, the action is given by

$$\begin{aligned} \psi : S^1 \times S^2 &\to S^2 \\ (t, \theta, z) &\mapsto (\theta + t, z). \end{aligned}$$

It is easy to see that S^1 acts through symplectomorphisms. The associated Lie algebra action is then

$$\begin{aligned} d\psi_e : \mathfrak{u}(1) \cong \mathbb{R} &\to \Gamma(TS^2, \omega) \\ \xi &\mapsto \xi_{S^2} = \xi \frac{\partial}{\partial \theta}. \end{aligned}$$

We find that

$$\iota_{\xi_{S^2}}\omega = \xi dz.$$

Since $\mathfrak{u}(1)$ is an abelian Lie algebra and $\{z, z\} = 0$, we conclude that the action is Hamiltonian with comoment map $\kappa : \mathbb{R} \to C^\infty(S^2)$ given

$$\kappa(\xi) = \xi z.$$

Example 91. Consider T^*S^1 with its usual symplectic structure written in local cylindrical coordinates as

$$\omega = d\theta \wedge dz.$$

[27] Add pictures.

Consider the action of $\mathbb{R}$ on $T^*S^1 = S^1 \times \mathbb{R}$ that translates the fiber direction:

$$\begin{aligned} \psi : \mathbb{R} \times S^1 \times \mathbb{R} &\to S^1 \times \mathbb{R} \\ (t, \theta, z) &\mapsto (\theta, z + t). \end{aligned}$$

It is easy to see that $\mathbb{R}$ acts through symplectomorphisms. The associated Lie algebra action is

$$\begin{aligned} d\psi_e : \operatorname{Lie} \mathbb{R} \cong \mathbb{R} &\to \Gamma(TT^*S^1, \omega) \\ \xi &\mapsto \xi_{T^*S^1} = \xi \frac{\partial}{\partial z}. \end{aligned}$$

We find that

$$\iota_{\xi_{T^*S^1}} \omega = -\xi d\theta.$$

Now, since the one-form $d\theta$ is not exact on T^*S^1, we conclude that the action is not even weakly Hamiltonian. Of course, if $U \subset S^1$ is any proper open subset, then $d\theta$ is indeed exact on T^*U, as alluded to in Remark 89.

Proposition 92. *Consider an exact symplectic manifold $(M, \omega = -d\lambda)$ and let G act on M through symplectomorphisms. Suppose $\psi_g^* \lambda = \lambda$ for every $g \in G$, i.e. the G-action is **exact**. Then the G-action is Hamiltonian with a comoment map*

$$\kappa(\xi) = \iota_{\xi_M} \lambda.$$

Proof. We first verify that the action is weakly Hamiltonian, i.e. $\iota_{\xi_M}\omega = d\kappa(\xi)$:

$$\begin{aligned} d\left(\iota_{\xi_m} \lambda\right) &= \mathcal{L}_{\xi_M} \lambda - \iota_{\xi_M} d\lambda \\ &= \frac{d}{dt}\bigg|_{t=0} \psi^*_{\exp(t\xi)} \lambda + \iota_{\xi_M} \omega \\ &= \frac{d}{dt}\bigg|_{t=0} \lambda + \iota_{\xi_M} \omega \\ &= \iota_{\xi_M} \omega. \end{aligned}$$

To see that it is in fact Hamiltonian, we check that κ is a Lie algebra homomorphism:

$$\begin{aligned} \{\kappa(\xi), \kappa(\eta)\} &= \{\iota_{\xi_M} \lambda, \iota_{\eta_M} \lambda\} \\ &= d\lambda(\eta_M, \xi_M) \\ &= \eta_M \lambda(\xi_M) - \xi_M \lambda(\eta_M) - \lambda[\eta_M, \xi_M] \\ &= \iota_{[\xi_M, \eta_M]} \lambda = \kappa([\xi_M, \eta_M]). \end{aligned}$$

□

Lemma 93. *Let G act through diffeomorphisms ψ_g on a manifold X. Then the action lifts to symplectomorphisms $\tilde{\psi}_g : T^*X \to T^*X$ by*

$$\tilde{\psi}_g(\alpha_x)(v) = \alpha_x(d\psi_g v)$$

for $v \in T_{\psi_g^{-1} x} X$. In particular, the lifted action is exact with respect to the canonical one-form on T^*X.

Proof. More generally, we show that any diffeomorphism $f : X \to X$ lifts to a diffeomorphism $\tilde{f} : T^*X \to T^*X$ preserving the canonical one-form on T^*X. Define

$$\tilde{f}(\alpha_x)v = \alpha_x(dfv)$$

for $v \in T_{f^{-1}x}$. Fix a vector $w \in T_{\alpha_x}T^*X$. Then, invoking the definitions of θ and $\tilde{f}$ respectively,

$$\begin{aligned}(\tilde{f}^*\theta)_{\alpha_x}w &= \theta_{\tilde{f}(\alpha_x)}(d\tilde{f}w)\\ &= \tilde{f}(\alpha_x)(d(\pi\circ\tilde{f})w)\\ &= \alpha_x(d(f\circ\pi\circ\tilde{f})w)\\ &= \alpha_x(d\pi w)\\ &= \theta_{\alpha_x}w,\end{aligned}$$

as desired. We have used that $f \circ \pi \circ \tilde{f} = \pi : T^*X \to X$. □

The cotangent bundle of a manifold is the prototypical exact symplectic manifold, and hence every exact action on a cotangent bundle yields a Hamiltonian action. Thus the previous two results generate a large class of Hamiltonian actions on the cotangent bundle. We will see some examples shortly.

Note that Example 91 – though an action on the cotangent bundle – is not an example of an exact action, as $\lambda = zd\theta$ is not preserved under $z \mapsto z + t$.

7.2. **Cohomological obstructions.** In the sequel we will concern ourselves primarily with Hamiltonian actions. It is then natural to ask what the obstructions are to a G-action by symplectomorphisms being Hamiltonian. In this section we digress briefly to show that such obstructions are cohomological in nature. For this we quickly introduce the machinery of Lie algebra cohomology.

Definition 94. Let V be a left $\mathfrak{g}$-module, i.e. a real vector space V equipped with a Lie algebra homomorphism $\mathfrak{g} \to \operatorname{End} V$. A map of $\mathfrak{g}$-modules is a linear map commuting with the $\mathfrak{g}$-action. There is a functor $-^{\mathfrak{g}} : \textsc{Mod}(\mathfrak{g}) \to \textsc{Mod}(\mathbb{R})$ taking V to $V^{\mathfrak{g}} = \{v \in V \mid \xi v = 0, \xi \in \mathfrak{g}\}$, the invariant submodule. This functor is only left-exact in general, so if V is a $\mathfrak{g}$-module, we define the **Lie algebra cohomology of $\mathfrak{g}$ with coefficients in** V to be

$$\mathrm{H}^\bullet(\mathfrak{g}, V) \equiv R^\bullet(-)^{\mathfrak{g}}(V)$$

the associated right derived functors.

In practice, we use the following standard resolution.

Definition 95. The **Chevalley-Eilenberg resolution** of $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module V is the cochain complex of linear maps $(\operatorname{Hom}(\Lambda^\bullet\mathfrak{g}, V), d_{\mathrm{CE}})$, where the differential is given by

$$\begin{aligned}d^n_{\mathrm{CE}}\sigma(\xi_1,\ldots,\xi_{n+1}) &= \sum_i(-1)^{i+1}\xi_i\cdot\sigma(\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_n)\\ &\quad+\sum_{i<j}(-1)^{i+j}\sigma([\xi_i,\xi_j],\xi_1,\ldots,\hat{\xi}_i,\ldots,\hat{\xi}_j,\ldots,\xi_n),\end{aligned}$$

for $\sigma \in \mathrm{Hom}(\Lambda^n \mathfrak{g}, V)$.[28] Note that, by convention, we take $\Lambda^0\mathfrak{g}$ to be the trivial $\mathfrak{g}$-module $\mathbb{R}$ and $d^0_{\mathrm{CE}} = 0$.

We will use the following result in the sequel without further mention.

Theorem 96. *The Chevalley-Eilenberg resolution of $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module V computes the Lie algebra cohomology $\mathrm{H}^\bullet(\mathfrak{g}, V)$ of $\mathfrak{g}$.*

Proof. See, for instance, Weibel's book. □

Remark 97. The Chevalley-Eilenberg resolution of $\mathfrak{g}$ is closely tied to the de Rham cohomology of any compact connected Lie group G with the given Lie algebra, as we now sketch in some detail. For simplicity, we fix our coefficients to be the the trivial $\mathfrak{g}$-module $\mathbb{R}$. Recall that the de Rham complex on G is $(\Omega^\bullet(G), d_{\mathrm{dR}})$, the cochain complex of differential forms on G together with the usual exterior derivative. On the other hand, one can consider the subcomplex of left-invariant differential forms $(\Omega^\bullet_G(G), d_{\mathrm{dR}})$. Identifying the space of left-invariant vector fields on G with the Lie algebra $\mathfrak{g}$ as usual, we find by dualizing that $\Omega^\bullet_G \cong \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, \mathbb{R})$. Under this identification, the exterior derivative becomes d_{CE} by virtue of our definition above, so we conclude that the Chevalley-Eilenberg complex of $\mathfrak{g}$ is naturally identified with the subcomplex of left-invariant differential forms on G.

Now we claim that the inclusion ι

$$0 \longrightarrow \Omega^\bullet_G(G) \xrightarrow{\ \iota\ } \Omega^\bullet(G)$$

is a quasi-isomorphism, i.e. $\Omega^\bullet_G(G)$ computes the de Rham cohomology of G. We construct the quasi-inverse explicitly using the existence of the left-invariant Haar measure on G: if $\alpha \in \Omega^k(G)$, the average

$$\alpha^{\mathrm{L}} = \int_G L_g^*\alpha \; \mathrm{d}g$$

is now left-invariant. Since d commutes with pullbacks and the integral over $g \in G$, we find that $-^{\mathrm{L}} : \Omega^\bullet(G) \to \Omega^\bullet_G(G)$ is a map of cochain complexes. Clearly the composition $-^{\mathrm{L}} \circ \iota$ yields the identity on $\Omega^\bullet_G(G)$. It remains to show that $\iota \circ -^{\mathrm{L}}$ is cochain-homotopic to the identity on $\Omega^\bullet(G)$, i.e. there exists a linear map $h : \Omega^k(G) \to \Omega^{k-1}(G)$ such that

$$\alpha - \iota(\alpha^{\mathrm{L}}) = d_{\mathrm{dR}}h + hd_{\mathrm{dR}}$$

for $\alpha \in \Omega^k(G)$.

The construction of h is somewhat involved.[29]

This argument can be generalized by replacing $\mathbb{R}$ with an arbitrary $\mathfrak{g}$-module and G by any G-homogeneous space.

The first result characterizes when an action is weakly Hamiltonian.

Proposition 98. *Let G act on (M, ω) through symplectomorphisms. Then we have a linear map $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathrm{H}^1_{dR}(M, \mathbb{R})$ taking $[\xi] \mapsto [\iota_{\xi_M}\omega]$. The action of G is weakly Hamiltonian if and only if this map is identically zero.*

[28] That this is a complex relies nontrivially on the Jacobi identity and the fact that V is a $\mathfrak{g}$-module.

[29] Work through the argument via integration over fibers of $G \times G \to G$.

By the remark above, $\mathrm{H}^1_{\mathrm{dR}}(M,\mathbb{R}) \cong \mathrm{H}^1(\mathfrak{g};\mathbb{R})$ if G is compact and connected. In general, the left hand side is the Lie algebra *homology* of $\mathfrak{g}$, because $\mathrm{H}^1(\mathfrak{g},\mathbb{R}) \cong (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$, as is checked by an explicit computation.

Proof. To show that the map is well-defined, it suffices to show that the commutator of two symplectic vector fields is Hamiltonian (as elements of $[\mathfrak{g},\mathfrak{g}]$ are linear combinations of commutators):

$$\iota_{[\xi_M,\eta_M]}\omega = (\mathcal{L}_{\xi_M} - \iota_{\eta_M}\mathcal{L}_{\xi_M})\omega = \mathcal{L}_{\xi_M}\iota_{\eta_M}\omega = d\iota_{\xi_M}\iota_{\eta_M}\omega.$$

Here we have used both of Cartan's magic formulae. Now, by definition, the action is weakly Hamiltonian if and only if $\iota_{\xi_M}\omega$ is exact for all ξ. $\square$

In what follows, we will focus on Hamiltonian actions over those that are only weakly Hamiltonian. The next proposition, together with Whitehead's lemmas below show that in many cases of practical interest, where $\mathfrak{g}$ is semisimple, actions through symplectomorphisms are automatically Hamiltonian.

Proposition 99. *Let (M,ω) be connected and equipped with a weakly Hamiltonian G-action. Then the action determines a cocycle $[\tau] \in \mathrm{H}^2(\mathfrak{g},\mathbb{R})$ which vanishes if and only if the G-action is Hamiltonian.*

Proof. Let $\kappa : \mathfrak{g} \to C^\infty(M)$ be the comoment map for the action such that $\iota_{\xi_M} = d\kappa(\xi)$ for all $\xi \in \mathfrak{g}$. For each pair $\xi,\eta \in \mathfrak{g}$ the function $\tau_{\xi,\eta} \in C^\infty(M)$ given by

$$\tau_{\xi,\eta}(p) = \{\kappa(\xi),\kappa(\eta)\}(p) - \kappa([\xi,\eta])(p)$$

measures the failure of the comoment map to be a Lie algebra homomorphism. This function is locally constant and hence constant, since

$$\begin{aligned} d\tau_{\xi,\eta} &= \iota_{X_{\{\kappa(\xi),\kappa(\eta)\}}}\omega - \iota_{[\xi,\eta]_M}\omega \\ &= \iota_{[X_{\kappa(\xi)},X_{\kappa(\eta)}]}\omega - \iota_{[\xi_M,\eta_M]}\omega \\ &= 0, \end{aligned}$$

where X_f represents the Hamiltonian vector field of f, i.e. the uniquely determined vector field satisfying $\iota_{X_f}\omega = df$. As such, we may view τ as an element of $\mathrm{Hom}(\Lambda^2\mathfrak{g},\mathbb{R})$.

We now claim that $d_{\mathrm{CE}}\tau = 0$, or:

$$\tau([\xi,\eta],\zeta) + \tau([\eta,\zeta],\xi) + \tau([\zeta,\xi],\eta) = 0,$$

for $\xi,\eta,\zeta \in \mathfrak{g}$. This is immediate in view of

$$\begin{aligned} \{\{\kappa(\xi),\kappa(\eta)\},\kappa(\zeta)\} &= \iota_{X_{\{\kappa(\xi),\kappa(\eta)\}}}\iota_{X_{\kappa(\zeta)}}\omega = \iota_{[X_{\kappa(\xi)},X_{\kappa(\eta)}]}\iota_{X_{\kappa(\zeta)}}\omega \\ &= \iota_{[\xi_M,\eta_M]}\iota_{\zeta_M}\omega = \{\kappa([\xi,\eta]),\kappa(\zeta)\} \end{aligned}$$

because

$$\begin{aligned} \tau([\xi,\eta],\zeta) + \tau([\eta,\zeta],\xi) + \tau([\zeta,\xi],\eta) = {} & \{\{\kappa(\xi),\kappa(\eta)\},\kappa(\zeta)\} - \kappa([[\xi,\eta],\zeta]) \\ & + \{\{\kappa(\eta),\kappa(\zeta)\},\kappa(\xi)\} - \kappa([[\eta,\zeta],\xi]) \\ & + \{\{\kappa(\zeta),\kappa(\xi)\},\kappa(\eta)\} - \kappa([[\zeta,\xi],\eta]), \end{aligned}$$

which is zero by the Jacobi identity for $\{-,-\}$ and $[-,-]$ (and linearity of κ).

Hence τ represents a cocycle $[\tau] \in \mathrm{H}^2(\mathfrak{g}, \mathbb{R})$. If the action is Hamiltonian to begin with, obviously $\tau = 0$, since κ is a Lie algebra homomorphism by construction. Conversely, suppose $[\tau] = 0$. This is equivalent to asking that τ be a coboundary

$$\tau(\xi, \eta) = \sigma[\xi, \eta]$$

for some $\sigma \in \mathrm{Hom}(g, \mathbb{R}) = \mathfrak{g}^*$. Now define $\tilde{\kappa} : \mathfrak{g} \to C^\infty(M)$ as

$$\tilde{\kappa} = \kappa + \sigma$$

and note that

$$\begin{aligned}\tilde{\kappa}[\xi, \eta] &= \kappa[\xi, \eta] + \sigma[\xi, \eta] = \kappa[\xi, \eta] + \{\kappa(\xi), \kappa(\eta)\} - \kappa[\xi, \eta] \\ &= \{\kappa(\xi), \kappa(\eta)\}.\end{aligned}$$

We conclude that the action is Hamiltonian with comoment map $\tilde{\kappa}$. □

Theorem 100 (Whitehead's lemmas)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. If V is any finite-dimensional $\mathfrak{g}$-module then* $\mathrm{H}^1(\mathfrak{g}, V) = \mathrm{H}^2(\mathfrak{g}, V) = 0$.

Proof. See, for instance, Weibel's book. □

7.3. **Moment maps.** The moment map repackages the data of the comoment map as follows.

Definition 101. Suppose (M, ω) has a weakly Hamiltonian G-action with comoment map $\kappa : \mathfrak{g} \to C^\infty(M)$. Then we say that a map $\mu : M \to \mathfrak{g}^*$ is a **moment map** for the action if for $p \in M$

$$\kappa(\xi)(p) = \langle \mu(p), \xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the evaluation pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.[30]

Theorem 102 (Noether, Souriau, Smale)**.** *Let $\mu : M \to \mathfrak{g}^*$ be a moment map for a weakly Hamiltonian G-action on (M, ω) with comoment map $\kappa : \mathfrak{g} \to C^\infty(M)$. Then μ is constant along the flow of the Hamiltonian vector field associated to any G-invariant function $H \in C^\infty(M)^G$.*[31]

Thus, from the perspective of physical systems, the moment map carries the data of the comoment map, but now as a sort of generalized Hamiltonian that respects the symmetries of the system.

Proof. Since $H \in C^\infty(M)^G$, we have

$$H = \psi^*_{\exp(t\xi)} H$$

for each $\xi \in \mathfrak{g}$. Differentiating this identity at $t = 0$, we find that

$$\begin{aligned}0 &= \left.\frac{d}{dt}\right|_{t=0} \psi^*_{\exp(t\xi)} H \\ &= \mathcal{L}_{\xi_M} H = dH(\xi_M) \\ &= \omega(X_H, \xi_M) = \{H, \kappa(\xi)\} \\ &= X_H \kappa(\xi),\end{aligned}$$

[30] Uniqueness of moment maps?

[31] Find references in O-R for N,S,S.

and hence the function $\kappa(\xi)$ is constant along the Hamiltonian flow of H for all $\xi \in \mathfrak{g}$. Now since $\kappa(\xi) = \langle \mu, \xi \rangle$, we find that μ must be constant along the Hamiltonian flow of H. □

Example 103 (Linear momentum)**.** Consider the phase space $(T^*\mathbb{R}^3, \omega)$ of a 1-particle system in $\mathbb{R}^3$. The N-particle case is a straightforward but index-heavy extension. Translation provides an action of $(\mathbb{R}^3, +)$ on $\mathbb{R}^3$ as $q \mapsto q + v$ for $v \in \mathbb{R}^3$. An easy computation reveals that this lifts (in the sense of Lemma 93) to an action ψ of $\mathbb{R}^3$ on $T^*\mathbb{R}^3$ as $\psi_v : (q,p) \mapsto (q+v, p)$. This action is exact by Lemma 93 and hence Hamiltonian by Proposition 92. The associated Lie algebra action by $\operatorname{Lie}\mathbb{R}^3 \cong \mathbb{R}^3$ is

$$\xi \mapsto \xi_{T^*\mathbb{R}^3} = \xi^i \frac{\partial}{\partial q^i}.$$

Then

$$\iota_{\xi_{T^*\mathbb{R}^3}} \omega = \left(\sum_k dq^k \wedge dp^k \right) \xi_{T^*\mathbb{R}^3} = \sum_i \xi^i dp^i$$

and we obtain a comoment map $\kappa : \operatorname{Lie}\mathbb{R}^3 \cong \mathbb{R}^3 \to C^\infty(T^*\mathbb{R}^3)$ given by

$$\kappa(\xi) = \sum_i \xi^i p_i.$$

The moment map $\mu : T^*\mathbb{R}^3 \to (\mathbb{R}^3)^*$ is therefore given by

$$\sum_i \xi^i p_i = \langle \mu(q,p), \xi \rangle.$$

Identifying $(\mathbb{R}^3)^*$ with $\mathbb{R}^3$ via the standard Euclidean metric, we have that

$$\mu(q,p) = p,$$

the linear momentum of the particle.

Example 104 (Angular momentum)**.** Consider again the phase space $(T^*\mathbb{R}^3, \omega)$ of a 1-particle system in $\mathbb{R}^3$. Rotation provides an action of $\mathrm{SO}(3,\mathbb{R})$ on $\mathbb{R}^3$ by matrix multiplication $q \mapsto A_{ij} q^j$ for $A \in \mathrm{SO}(3,\mathbb{R})$. An easy computation reveals that this lifts (in the sense of Lemma 93) to an action ψ of $\mathrm{SO}(3,\mathbb{R})$ on $T^*\mathbb{R}^3$ as $\psi_A : (q,p) \mapsto (A_{ij}q^j, A_{ij}p_j)$. This action is exact by Lemma 93 and hence Hamiltonian by Proposition 92. Recall that the Lie algebra $\mathfrak{so}(3,\mathbb{R})$ is the space of 3×3 antisymmetric matrices, with the commutator as the Lie bracket. Computing the infinitesimal action, we find

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \psi_{\exp(t\xi)}(q,p) &= \left.\frac{d}{dt}\right|_{t=0} \left(q + t\xi_{ij}q^j + \cdots, p + t\xi_{ij}p_j + \cdots \right) \\ &= \sum_{i,j} \xi_{ij} q^j \frac{\partial}{\partial q^i} + \sum_{i,j} \xi_{ij} p_j \frac{\partial}{\partial p^i} \end{aligned}$$

for $\xi \in \mathfrak{so}(3,\mathbb{R})$. Therefore the associated Lie algebra action of $\mathfrak{so}(3,\mathbb{R})$ by vector fields is given by

$$\xi \mapsto \xi_{T^*\mathbb{R}^3}|_{(q,p)} = \sum_{i,j} \xi_{ij} q^j \frac{\partial}{\partial q^i} + \sum_{i,j} \xi_{ij} p_j \frac{\partial}{\partial p^i}$$

Now we find that

$$(\iota_{\xi_{T^*\mathbb{R}^3}}\omega)_{(q,p)} = \left(\sum_k dq^k \wedge dp^k\right)\xi_{T^*\mathbb{R}^3} = \sum_{i,j}\xi_{ij}(q^j dp^i - p_j dq^i)$$
$$= \sum_{i,j}\xi_{ij}(q^j dp^i + p_i dq^j) = d\left(\sum_{i,j} p_i\xi_{ij}q^j\right),$$

by antisymmetry of ξ_{ij}. Hence we obtain a comoment map $\kappa : \mathfrak{so}(3,\mathbb{R}) \to C^\infty(T^*\mathbb{R}^3)$ for the action given by

$$\kappa(\xi)(q,p) = \sum_{i,j} p_i\xi_{ij}q^j.$$

The moment map $\mu : T^*\mathbb{R}^3 \to \mathfrak{so}(3,\mathbb{R})^*$ is therefore determined by

$$\sum_{i,j} p_i\xi_{ij}q^j = \langle \mu(q,p), \xi\rangle.$$

Identifying $\mathfrak{so}(3,\mathbb{R})^*$ with $\mathfrak{so}(3,\mathbb{R})$ via the standard Euclidean metric on $\mathbb{R}^9$ which yields an isomorphism $(\mathbb{R}^9)^* \cong \mathbb{R}^9$, we conclude that

$$\mu(q,p) = \begin{pmatrix} 0 & q_2p_1 - q_1p_2 & q_3p_1 - q_1p_3 \\ q_1p_2 - q_2p_1 & 0 & q_3p_2 - q_2p_3 \\ q_1p_3 - q_3p_1 & q_2p_3 - q_3p_2 & 0 \end{pmatrix}.$$

Moreover, using the Lie algebra isomorphism $\mathfrak{so}(3,\mathbb{R}) \cong (\mathbb{R}^3, \times)$, we can write

$$\mu(q,p) = q \times p,$$

the usual notation for angular momentum.

Example 105 (Lifted actions)**.**

We now turn to the equivariance properties of moment maps.

Definition 106. Let $\mathfrak{g}$ be a real Lie algebra. Recall that $\mathfrak{g}$ can be viewed as a $\mathfrak{g}$-module under the **adjoint action** $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ where

$$\operatorname{ad}_\xi \eta = [\xi, \eta],$$

for $\xi, \eta \in \mathfrak{g}$. Dually, $\mathfrak{g}$ acts on $\mathfrak{g}^*$ through the **coadjoint action** $\operatorname{ad}^* : \mathfrak{g} \to \operatorname{End}\mathfrak{g}^*$ where

$$\langle \operatorname{ad}^*_\xi \sigma, \eta\rangle = \langle \sigma, -\operatorname{ad}_\xi \eta\rangle = \langle \sigma, [\eta, \xi]\rangle,$$

for $\xi, \eta \in \mathfrak{g}$ and $\sigma \in \mathfrak{g}^*$. Note that these actions can be seen as derivatives of actions by G on $\mathfrak{g}$ and $\mathfrak{g}^*$, denoted by Ad and Ad^*, respectively, confusingly also called adjoint and coadjoint actions.

Definition 107. Suppose (M, ω) has a weakly Hamiltonian G-action with moment map $\mu : M \to \mathfrak{g}^*$. We say that μ is **$\mathfrak{g}$-equivariant** if

$$d\mu_p(\xi_M) = -\operatorname{ad}^*_\xi(\mu(p)).$$

for all $\xi \in \mathfrak{g}$ and $p \in M$. Here ad^* is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

Remark 108. The term $\mathfrak{g}$-equivariant comes from a more global G-equivariance: there are actions of G on both the domain (the given action) and target (the coadjoint action of G on $\mathfrak{g}^*$) of μ. Hence we say that μ is G-equivariant if

$$\psi_g^* \mu = \operatorname{Ad}_{g^{-1}}^* \mu$$

for all $g \in G$. It can be shown that these two notions of equivariance are equivalent if G is connected. However, we will concern ourselves only with $\mathfrak{g}$-equivariance.

Our interest in Hamiltonian actions (or $\mathfrak{g}$-equivariance) stems from the following.

Proposition 109. *Suppose (M, ω) has a weakly Hamiltonian G-action with comoment map κ and moment map μ. Then the action is Hamiltonian if and only if μ is $\mathfrak{g}$-equivariant.*

Example 110 (Coadjoint actions)**.**

Show that μ is a moment map for a Hamiltonian action if and only if μ is equivariant.

Is there anything that Morse theory has to say?

Maybe make the whole next section on coadjoint orbits?