NOTES ON SYMPLECTIC GEOMETRY

NILAY KUMAR

Contents

1. Week 1

1.1. The cotangent bundle.

Definition 1. Let X be a smooth n-manifold and $\pi : M = T^*X \to X$ be its cotangent bundle. We define the **canonical one-form** $\theta \in \Omega^1(M)$ as follows. For any $p = (x, \xi) \in M$, set

$$\theta_p(v) = \xi(d_x\pi(v)).$$

The one-form θ is canonical (or tautological) in the sense that its value at a point is simply given by the covector determined by that point. More precisely, we have the following characterization.

Proposition 2. *The canonical one-form θ is the (unique) one-form such that for every $\lambda \in \Omega^1(X)$, $\lambda^*\theta = \lambda$.*

Proof. We compute, for $v \in T_pX$,

$$\begin{aligned}(\lambda^*\theta)_p(v) &= \theta_{\lambda(p)}(d_p\lambda(v))\\ &= \lambda_p(d_p(\pi \circ \lambda)(v))\\ &= \lambda_p(v),\end{aligned}$$

where we have used the fact that λ is a section of π, i.e. $\pi \circ \lambda = \mathrm{id}_X$. Uniqueness is easily checked. □

Date: Fall 2015.

Definition 3. The **canonical symplectic form** $\omega \in \Omega^2(M)$ is now defined to be the exterior derivative

$$\omega = -d\theta,$$

of the canonical one-form. To be symplectic, ω must be closed and nondegenerate. That it is closed is obvious.

Proposition 4. *The form $\omega \in \Omega^2(M)$ is nondegenerate and thus defines a symplectic structure on $M = T^*X$.*[1]

Proof. For ω to be non-degenerate, it must be nondegenerate at each point $p \in M$. Given coordinates $p = (x, \xi) = (x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ in a neighborhood of p, we can compute

$$\begin{aligned}\theta_{(x,\xi)}\left(v^i \frac{\partial}{\partial x^i} + \nu^i \frac{\partial}{\partial \xi^i}\right) &= \xi\left(v^i \frac{\partial}{\partial x^i}\right) \\ &= \xi_i v^i\end{aligned}$$

and hence

$$\theta = \xi_i dx^i.$$

Taking an exterior derivative, we find that

$$\begin{aligned}\omega &= -d\theta \\ &= dx^i \wedge d\xi_i.\end{aligned}$$

Fix $v \in T_pM$ and suppose that $\iota_v \omega_p = 0$, i.e. $\omega_p(v, w) = 0$ for all $w \in T_pM$. In coordinates, this implies that

$$\begin{aligned}\iota_{v^j \frac{\partial}{\partial x^j} + \nu^j \frac{\partial}{\partial \xi^j}}(dx^i \wedge d\xi_i) &= v^i d\xi_i - \nu^i dx^i \\ &= 0,\end{aligned}$$

and hence that $v^i = \nu^i = 0$, i.e. $v = 0$. We conclude that ω_p is nondegenerate at each $p \in M$. $\square$

Remark 5. Note that a 2-form ω on a manifold M is nondegenerate if and only if ω^n is nowhere vanishing. Fix $p \in M$ and consider the vector space (T_pM, ω_p). If ω_p is nondegenerate, we can find a symplectic basis for T_pM, and so ω_p^n evaluated on $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ is nonzero, whence ω_p^n is not zero on V. On the other hand, suppose ω_p is degenerate, i.e. there is a $v \neq 0$ such that $\omega_p(v, w) = 0$ for all $w \in V$. Choosing a basis $v_1, \ldots, v_{2n}$ for V such that $v_1 = v$, we find that $\omega_p(v_1, \ldots, v_{2n}) = 0$ and hence $\omega_p = 0$ on V.

We conclude that every symplectic manifold is orientable.

It is easy to see that ω provides an isomorphism $\iota : T_xX \xrightarrow{\sim} T_x^*X$ between tangent and cotangent spaces at each point $x \in X$: since ω_x is nondegenerate, the linear map $\iota : v \mapsto \omega_x(v, -)$ is injective and hence bijective. In fact, we can say more.

Proposition 6. *The metric ω induces an isomorphism of vector bundles $\iota : TX \xrightarrow{\sim} T^*X = M$.*

[1] Is there a coordinate invariant proof?

Proof. Recall that an isomorphism in the category of smooth vector bundles is a smooth bijection[2] ι such that the diagram

$$\begin{tikzcd} TX \arrow[rr, "\iota"] \arrow[dr, "\pi_1"'] & & T^*X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

commutes and for each $x \in X$, the restriction $\iota_x : T_xX \to T_x^*X$ is linear. The map $\iota : TX \to T^*X$ taking $(x, v) \mapsto (x, \omega(v, -))$ fits into the diagram above and is bijective and fiberwise linear. Moreover, ι is a smooth map, as is seen by its coordinate description computed above. $\square$

Definition 7. A **Hamiltonian** is a smooth function $H : M = T^*X \to \mathbb{R}$. we define the **Hamiltonian vector field** v_H associated to H to be the vector field on M satisfying

$$\iota_{v_H}\omega = dH.$$

The (local) flow $F : (-\varepsilon, \varepsilon) \times M \to M$ determined by v_H is called the **Hamiltonian flow**.[3]

Note that an integral curve $\gamma_{v_H} : (-\varepsilon, \varepsilon) \to M$ of v_H can be thought of as the trajectory of a physical state in phase space. Indeed, Hamilton's equations are given

$$\begin{aligned} \frac{\partial x^i}{\partial t} &= \frac{\partial H}{\partial \xi_i} \\ \frac{\partial \xi_i}{\partial t} &= -\frac{\partial H}{\partial x^i}, \end{aligned}$$

which is precisely the condition that $\gamma'_{v_H}(t) = (v_H)_{\gamma(t)}$. Moreover, H is constant along the Hamiltonian flow, as

$$dH(v_H) = (\iota_{v_H}\omega)(v_H) = \omega(v_H, v_H) = 0,$$

i.e. v_H is perpendicular to the level sets of H. In a physical system, where H is the energy functional on phase space, this phenomenon is the law of conservation of energy.

Proposition 8. *The Hamiltonian flow is a symplectomorphism, i.e.* $F_t^*\omega = \omega$.[4]

Proof. We use the following trick:

$$\int_0^t \frac{d}{dt} F_t^*\omega \, dt = F_t^*\omega - \omega$$

since $F_0 = \mathrm{id}_M$, and hence F_t is a symplectomorphism if and only if the integrand is zero. But

$$\begin{aligned} \frac{d}{dt} F_t^*\omega &= \left.\frac{d}{ds}\right|_{s=0} F_{t+s}^*\omega = F_t^* \left.\frac{d}{ds}\right|_{s=0} F_s^*\omega \\ &= F_t^* \mathcal{L}_{v_H}\omega, \end{aligned}$$

[2]Existence of a smooth inverse is automatic (reference?).
[3]Is this a global flow? Does it depend on X?
[4]Is there a better proof?

and Cartan's magic formula,

$$\mathcal{L}_{v_H}\omega = d\iota_{v_H}\omega + \iota_{v_H}d\omega,$$

tells us that $\mathcal{L}_{v_H}\omega = 0$ since $\iota_{v_H}\omega = dH$ is closed, as is ω. □

Corollary 9 (Liouville's Theorem)**.** *The volume form ω^n on $M = T^*X$ is preserved by the Hamiltonian flow.*

1.2. **Geodesic flow as Hamiltonian flow.** We wish to discuss geodesics and geodesic flow. For this, we need the concept of connections and covariant derivatives.[5]

Definition 10. A **connection** on a vector bundle $E \to X$ is an $\mathbb{R}$-linear map $\nabla : \Gamma(X, E) \to \Gamma(X, E \otimes T^*X)$ such that the Leibniz rule

$$\nabla(f\sigma) = (\nabla\sigma)f + \sigma \otimes df,$$

for all $f \in C^\infty(X)$ and $\sigma \in \Gamma(X, E)$.

Theorem 11. *Given a Riemannian manifold (X, g), there exists a unique connection on $\pi : TX \to X$, known as the* ***Levi-Civita connection****, satisfying*

(i) symmetry:

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

for $X, Y \in \Gamma(X, TX)$;

(ii) compatibility with g:

$$Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,$$

for $X, Y, Z \in \Gamma(X, TX)$.

Definition 12. Let v be a vector field on (X, g); we define the **covariant derivative** of v along a smooth curve $c : I \to X$ to be the vector field

$$\frac{Dv}{dt} = \nabla_{dc/dt} v,$$

where ∇ is the Levi-Civita connection. Explicitly, if we write $v = v^i \partial/\partial x^i$ and $c(t) = (c_1(t), \ldots, c_n(t))$,

$$\frac{Dv}{dt} = \sum_i \frac{dv^i}{dt}\frac{\partial}{\partial x^i} + \sum_{ijk} \frac{dc_i}{dt} v^i \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

Here Γ^k_{ij} are the Christoffel symbols of ∇, determined by

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \sum_{ijk} \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

We say that c is **geodesic** at some $t \in I$ if $D/dt(dc/dt) = 0$ at t, and that c is geodesic if it is geodesic at all $t \in I$. In coordinates, the condition for c to be geodesic is given by a system of second-order differential equations:

$$\frac{d^2c^i}{dt^2} + \sum_{jk} \Gamma^i_{jk} \frac{dc^j}{dt}\frac{dc^k}{dt} = 0,$$

for $i = 1, \ldots, n$.

[5]Reference do Carmo.

For the rest of the section, assume (X, g) is Riemannian and we fix the Hamiltonian $H : M = T^*X \to \mathbb{R}$ as

$$H(x, \xi) = \frac{1}{2}\left|\xi_x\right|_g^2,$$

i.e. consisting of only a kinetic term. Here we are implicitly using the nondegeneracy of g to associate ξ_x with its corresponding vector (or, equivalently, using g^{-1}).

Proposition 13. *The Hamiltonian flow on $M = T^*X$ is dual to the geodesic flow on TX. In other words, the integral curves of the Hamiltonian vector field v_H associated to the Hamiltonian above project to geodesics of g on X.*[6]

Proof. It suffices to show, in coordinates, that Hamilton's equations (i.e. the condition for being on the integral curve) yield the geodesic equations above after the necessary dualization. Note first that in coordinates the Hamiltonian becomes

$$H(x, \xi) = \frac{1}{2} g^{ij} \xi_i \xi_j.$$

For convenience we will denote the components of an integral curve as $x^i(t)$. Hamilton's equations yield

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial}{\partial \xi_i}\left(\frac{1}{2} g^{jk} \xi_j \xi_k\right) \\ &= \frac{1}{2} g^{jk} \delta_{ij} \xi_k + \frac{1}{2} g^{jk} \xi_j \delta_{ik} \\ &= g^{ij} \xi_j \\ \frac{d\xi_i}{dt} &= -\frac{\partial}{\partial x^i}\left(\frac{1}{2} g^{jk} \xi_j \xi_k\right) \\ &= -\frac{1}{2} \frac{\partial g^{jk}}{\partial x^i} \xi_j \xi_k. \end{aligned}$$

Differentiating the first equation with respect to t and using both of Hamilton's equations yields

$$\begin{aligned} \frac{d^2 x^i}{dt^2} &= \frac{\partial g^{ij}}{\partial x^k} \frac{dx^k}{dt} \xi_j + g^{im} \frac{d\xi_m}{dt} \\ &= g^{kl}\left(\frac{\partial}{\partial x^k} g^{ij}\right) \xi_l \xi_j - \frac{1}{2} g^{im}\left(\frac{\partial}{\partial x^m} g^{nr}\right) \xi_n \xi_r. \end{aligned}$$

Next, differentiating the identity $g^{ij} g_{jk} = \delta^i_k$, it easy to see that

$$\frac{\partial}{\partial x^i} g^{kl} = -g^{la} g^{kb} \frac{\partial}{\partial x^i} g_{ab}.$$

[6]Is there a coordinate-free proof? See Paternain's book.

Using this, contracting indices, and using the first Hamilton's equation to dualize ξ's into dx/dt's, we find

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= -g^{ib}\left(\frac{\partial}{\partial x^k}g_{lb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt} + \frac{1}{2}g^{im}\left(\frac{\partial}{\partial x^m}g_{ts}\right)\frac{dx^s}{dt}\frac{dx^t}{dt}\\ &= -\frac{1}{2}g^{ib}\left(\frac{\partial}{\partial x^k}g_{lb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt} - \frac{1}{2}g^{ib}\left(\frac{\partial}{\partial x^l}g_{kb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt}\\ &+ \frac{1}{2}g^{im}\left(\frac{\partial}{\partial x^m}g_{ts}\right)\frac{dx^s}{dt}\frac{dx^t}{dt}\\ &= -\Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt},\end{aligned}$$

as desired. $\square$

2. Week 2

2.1. **Darboux's theorem.**

Theorem 14 (Darboux). *Let (M, ω) be a symplectic $2n$-manifold. Then M is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.*

We prove Darboux's theorem using the following stronger statement.

Theorem 15. *Let M be a $2n$-dimensional manifold and $Q \subset M$ be a compact submanifold. Suppose that $\omega_1, \omega_2 \in \Omega^2(M)$ are closed 2-forms such that at each point q of Q the forms ω_0 and ω_1 are equal and nondegenerate on T_qM. Then there exist neighborhoods N_0 and N_1 of Q and a diffeomorphism $\psi : N_0 \to N_1$ such that $\psi|_Q = \mathrm{id}_Q$ and $\psi^*\omega_1 = \omega_0$.*

Proof. Consider the family of closed two-forms

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0)$$

on M for $t \in [0,1]$. Note that $\omega_t|_Q = \omega_0|_Q$ is nondegenerate and hence there exists an open neighborhood N_0 of Q such that $\omega_t|_{N_0}$ is nondegenerate.[7] Suppose, for now, that there is a one-form $\sigma \in \Omega^1(N_0)$ (possibly shrinking N_0), such that $\sigma|_{T_QM} = 0$ and $d\sigma = \omega_1 - \omega_0$ on N_0. Then

$$\omega_t = \omega_0 + td\sigma$$

and we obtain by nondegeneracy a smooth vector field X_t on N_0 characterized by

$$\iota_{X_t}\omega_t = -\sigma.$$

The condition $\sigma|_{T_QM} = 0$ implies, again by nondegeneracy of ω_t, that $X_t|_Q = 0$. Now consider the initial value problem for the flow ψ_t of X_t,

$$\begin{aligned}\frac{d}{dt}\psi_t &= X_t \circ \psi_t \\ \psi_0 &= \mathrm{id}\,.\end{aligned}$$

This differential equation can be solved uniquely for $t \in [0,1]$ on some open neighborhood of Q contained in N_0, call it again N_0.[8] Note that $\psi_t|_Q = \mathrm{id}_Q$ since $X_t|_Q = 0$. We compute now that

$$\begin{aligned}\frac{d}{dt}\psi_t^*\omega_t &= \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) \\ &= \psi_t^*\left(d\sigma + d\iota_{X_t}\omega_t\right) \\ &= 0.\end{aligned}$$

Hence $\psi_1^*\omega_1 = \psi_0^*\omega_0 = \omega_0$. Thus the desired diffeomorphism is ψ_1 and the desired neighborhoods are N_0 and N_1. The above argument is known as **Moser's trick**, and is extremely useful in symplectic geometry.

It remains to construct a smooth one-form σ satisfying $\sigma|_{T_QM} = 0$ and $d\sigma = \omega_1 - \omega_0$. If Q were a point (or more generally, diffeomorphic to a star-shaped subset of Euclidean space), we could simply use the Poincaré lemma; in general, however the construction is as follows. Fix any Riemannian metric on M and consider the

[7] Why?

[8] Why?

restriction of the exponential map $\exp : TM \to M$ to a neighbhorhood U_ε of the zero section of the normal bundle $TQ^\perp \to M$:

$$U_\varepsilon = \{(q, v) \in TM \mid q \in Q, v \in T_qQ^\perp, |v| < \varepsilon\}.$$

Recall that exp becomes a diffeomorphism for ε sufficiently small, so we choose ε such that $N_0 = \exp(U_\varepsilon)$ is contained in the neighborhood of Q above on which ω_t is nondegenerate. Define now a family of maps $\phi_t : N_0 \to N_0$ for $t \in [0, 1]$ by

$$\phi_t(\exp(q, v)) = \exp(q, tv).$$

Note that ϕ_t is a diffeomorphism onto its image for $t \neq 0$. Moreover, $\phi_t|_Q = \mathrm{id}_Q$, $\phi_0(N_0)$, and $\phi_1 = \mathrm{id}_{N_0}$. If we now write $\tau = \omega_1 - \omega_0$, we find that

$$\begin{aligned}\phi_0^*\tau &= 0\\ \phi_1^*\tau &= \tau,\end{aligned}$$

since $\tau = 0$ on T_QM. Now, for $t \in (0, 1]$, we define a family of vector fields,

$$Y_t = \left(\frac{d}{dt}\phi_t\right) \circ \phi_t^{-1}.$$

Then for any $\delta > 0$,

$$\begin{aligned}\phi_1^*\tau - \phi_\delta^*\tau &= \int_\delta^1 \frac{d}{dt}\phi_t^*\tau dt = \int_\delta \phi_t^*\mathcal{L}_{Y_t}\tau dt\\ &= \int_\delta^1 \phi_t^*(d\iota_{Y_t}\tau)dt\\ &= d\int_\delta^1 \phi_t^*(\iota_{Y_t}\tau)dt\end{aligned}$$

Clearly $\phi_1^*\tau - \phi_\delta^*\tau = \tau - \phi_\delta^*\tau$ approaches τ as $\delta \to 0^+$, so we find that

$$\tau = d\int_0^1 \phi_t^*(\iota_{Y_t}\tau)dt.$$

Defining

$$\sigma = \int_0^1 \phi_t^*(\iota_{Y_t}\tau)dt,$$

we find that $\tau = \omega_1 - \omega_0 = d\sigma$ and $\sigma|_{T_QM} = 0$ because $\phi_t|_Q = \mathrm{id}_Q$ and $\tau = 0$ on Q, forcing the integrand to vanish on T_QM. Hence σ is the one-form required above for Moser's trick, and we are done.[9] $\square$

The proof of Darboux's theorem is now straightforward: we choose a coordinate chart ϕ so that $\phi^*\omega$ is equal to the standard form on a subset of $\mathbb{R}^{2n}$ at a single point, and then apply Moser's theorem with Q equal to the chosen point.

Proof of Darboux's theorem. Let $q \in M$ and fix a symplectic basis $\{u_i, v_i\}$ for the symplectic vector space (T_qM, ω_q). Fix any Riemannian metric on M and pick an open $U \ni 0$ small enough such that exp restricted to $U \subset T_qM$ is a diffeomorphism

[9] Why is σ smooth?

and hence a chart $(x^i, y_i) = \exp : U \subset \mathbb{R}^{2n} \to M$ $(i = 1, \ldots, n)$ such that $x^i(p) = y_i(p) = 0$. Now we can compute, for example,

$$\begin{aligned} \exp^* \omega_p \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k} \right) &= \omega_p \left(\exp_* \frac{\partial}{\partial x^j}, \exp_* \frac{\partial}{\partial y^k} \right) \\ &= \omega_p \left(u_j, v_k \right) = \delta_{jk}, \end{aligned}$$

to check that $\exp^* \omega_p = (\omega_0)_0$ where ω_0 is the standard form on $T_0 U$. Here we have used the fact that $\exp_* = \mathrm{id}$ at $0 \in U$. Applying Theorem 2.1 to U with $Q = 0 \in U$, we obtain a diffeomorphism ψ of (some possibly smaller) U such that $\psi^* \exp^* \omega = \omega_0$ on U. But now $\exp \circ \psi$ provides a symplectomorphism in a neighborhood of q to a neighborhood of $\mathbb{R}^{2n}$ pulling ω back to the standard form ω_0. □

3. Week 3

3.1. Submanifolds of symplectic manifolds.

Definition 16. Let (V,ω) be a symplectic vector space. We define the **symplectic complement** U^ω of a subspace $U \subset V$ as

$$U^\omega = \{v \in V \mid \omega(v,u) = 0 \text{ for all } u \in U\}.$$

Lemma 17. *For any subspace $U \subset V$, $U^{\omega\omega} = U$ and*

$$\dim U + \dim U^\omega = \dim V.$$

Proof. Nondegeneracy of ω yields an isomorphism $\iota_\omega : V \to V^*$ which identifies U^ω with $U^\perp \equiv \{\nu \in V^* \mid \nu(u) = 0 \text{ for all } u \in U\}$. The result now follows from the fact that $\dim U + \dim U^\perp = \dim V$. □

Definition 18. Let (M,ω) be a symplectic manifold. A submanifold $Q \subset M$ is called **symplectic, isotropic, coisotropic, or Lagrangian** if for each $q \in Q$, the linear subspace $T_qQ \equiv V_q$ of (T_qM, ω_q) is
(a) symplectic: $V_q \cap V_q^{\omega_q} = 0$,
(b) isotropic: $V_q \subset V_q^{\omega_q}$,
(c) coisotropic: $V_q^{\omega_q} \subset V_q$,
(d) Lagrangian: $V_q = V_q^{\omega_q}$,
respectively.

Remark 19. Note that $Q \subset M$ is Lagrangian if and only if the restriction of ω to Q is zero and $\dim Q = \dim M/2$.

Example 20. Let X be any manifold, and $(M = T^*X, \omega)$ be its cotangent bundle with the usual symplectic structure. Recall that $\omega = -d\theta$, where $\theta_\xi(v) = \xi(d_x\pi(v))$.[10] In coordinates, if (x^i, ξ^i) are coordinates for M, we can write $\omega = dx^i \wedge d\xi^i$.

It is then easy to see that the fibre $T_x^*X \subset M$ is Lagrangian, as

$$\begin{aligned} 0 &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial \xi^j}, b_k \frac{\partial}{\partial \xi^k} + c_l \frac{\partial}{\partial x^l}\right) \\ &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial \xi^j}, c_l \frac{\partial}{\partial x^l}\right) \\ &= a_i c_i, \end{aligned}$$

forces $c_i = 0$.

Similarly, the zero section $\Gamma_0 \subset M$ is Lagrangian, as

$$\begin{aligned} 0 &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial x^j}, b_k \frac{\partial}{\partial \xi^k} + c_l \frac{\partial}{\partial x^l}\right) \\ &= (dx^i \wedge d\xi^i)\left(a_j \frac{\partial}{\partial x^j}, b_k \frac{\partial}{\partial \xi^k}\right) \\ &= a_i b_i, \end{aligned}$$

forces $b_i = 0$.

[10] Can we do this coordinate-invariantly?

More generally, given a submanifold $Q \subset L$, the annihilator

$$TQ^\perp = \{(q,\nu) \in T^*L \mid q \in Q, \nu|_{T_qQ} = 0\}$$

is Lagrangian.

Example 21. Let (M,ω) be a symplectic manifold. The product $M \times M$ can be given a symplectic structure $\omega' = \alpha\pi_1^*\omega + \beta\pi_2^*\omega$ for $\alpha, \beta \in \mathbb{R}$. Consider in particular the case of $\alpha = 1, \beta = -1$. Then it is clear that $M \times \{m\}$ and $\{m\} \times M$ are symplectic submanifolds. Moreover, the diagonal $\Delta \subset M \times M$ is Lagrangian, as

$$\begin{aligned} 0 &= \omega'\left((u,u),(v,w)\right) \\ &= \omega(u,v) - \omega(u,w) \\ &= \omega(u, v-w) \end{aligned}$$

and hence $v = w$, as desired.

Example 22. Let $S \subset (M,\omega)$ be a codimension 1 submanifold. Then S is coisotropic. Indeed, fix $s \in S$, and note that $T_sS \subset T_sM$ is codimension one. By Lemma 17, $T_sS^{\omega_s}$ is a one-dimensional subspace. Pick any vector $v \in T_sS^{\omega_s}$; v spans the entire symplectic complement, and hence if v is not in $T_sS^{\omega_s}$, $T_sS \cap T_sS^{\omega_s} = 0$ and T_sS is symplectic and thus even-dimensional. This is a contradiction, and hence T_sS must be coisotropic.

Proposition 23. *The graph $\Gamma_\sigma \subset T^*X$ of a one-form is Lagrangian if and only if σ is closed.*

Proof. Note that Γ_σ is defined to be the image of the embedding $\sigma : X \to T^*X$. Then $\dim \Gamma_\sigma = n$, so it remains to show that ω restricts to zero on Γ_σ if and only if σ is closed. Using Proposition 2, we compute

$$d\sigma = d\sigma^*\theta = \sigma^* d\theta = -\sigma^*\omega,$$

which yields the desired statement, as $\sigma^*\omega = 0$ on X if and only if $\omega = 0$ on Γ_σ, by virtue of σ being an embedding. $\square$

With these definitions out of the way, we present a number of theorems characterizing neighbhorhoods of special submanifolds of symplectic manifolds.

Theorem 24 (Symplectic neighborhood theorem)**.** *Let $(M_0,\omega_0),(M_1,\omega_1)$ be symplectic manifolds with compact symplectic submanifolds Q_0, Q_1 respectively. Suppose there is an isomorphism $\Phi : TQ_0^\omega \to TQ_1^\omega$ of symplectic normal bundles covering a symplectomorphism $\phi : (Q_0,\omega_0) \to (Q_1,\omega_1)$. Then ϕ extends to a symplectomorphism $\psi : (N(Q_0),\omega_0) \to (N(Q_1),\omega_1)$ such that $d\psi$ induces the map Φ on TQ_0^ω.*

Proof. Let $\exp_0, \exp_1$ be diffeomorphisms mapping neighborhoods of the zero section in the normal bundle to neighborhoods of Q_0, Q_1 in X, respectively. Then we obtain

$$\phi' = \exp_1 \circ \Phi \circ \exp_0^{-1},$$

a diffeomorphism between these neighborhoods of Q_0 and Q_1. Now $\phi'^*\omega_1$ and ω_0 are two symplectic forms on M_0 whose restrictions to Q_0 agree. Now ϕ' extends to the desired ψ by Theorem 2.1. $\square$

Theorem 25 (Lagrangian neighborhood theorem)**.** *Let (M, ω) be a symplectic manifold and let $L \subset M$ be a compact Lagrangian submanifold. Then there exists a neighborhood $N(\Gamma_0) \subset T^*L$ of the zero section Γ_0, a neighborhood $U \subset M$ of L, and a diffeomorphism $\phi : N(\Gamma_0) \to U$ such that $\phi^*\omega = -d\theta$ and $\phi|_L = \mathrm{id}$, where θ is the canonical one-form on T^*L.*

Proof. □

3.2. **Contact manifolds.** Let X be a differential manifold and $H \subset TX$ be a smooth hyperplane field, i.e. a smooth subbundle of codimension one. Then, locally on some open U, we can write $H = \ker \alpha$, for $\alpha \in \Omega_1(U)$. In fact, if we assume that H is **coorientable**, we can extend U to all of X.[11] We will assume for what follows that H is coorientable.

Definition 26. Let X be a manifold of odd dimension $2n+1$. A **contact structure** on X is a hyperplane field $H = \ker \alpha$ where the top-dimensional form $\alpha \wedge (d\alpha)^n$ is nowhere vanishing. We call α a **contact form**, and the pair (X, H) a **contact manifold**.

Remark 27. Suppose we have $\alpha, \alpha' \in \Omega^1(X)$ such that $H = \ker \alpha = \ker \alpha'$. Then α is a contact form if and only if α' is. This is because the condition that α, α' cut out H requires $\alpha' = f\alpha$ for some nonzero $f : X \to \mathbb{R}$.

Remark 28. In the language of distributions, H can be described as a codimension one distribution that is maximally non-integrable in the following sense. Recall that a distribution on X is said to be integrable if every point p of X is contained in a integral manifold of H, i.e. in a nonempty immersed submanifold $N \subset X$ such that $T_pN = H_p$. The Frobenius theorem tells us that H is integrable if and only if H is involutive, i.e. H is closed under the Lie bracket of local sections. Now, since

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X, Y],$$

we find that H is integrable if and only if $d\alpha = 0$ on H. Thus asking for $d\alpha$ to be nondegenerate on H forces the distribution to be "as non-integrable as possible."

Indeed, we obtain the above definition of a contact structure by noting that $d\alpha$ is nondegenerate on H if and only if $\alpha \wedge (d\alpha)^n$ is nowhere vanishing, as follows. By remark 5, $d\alpha$ is nondegenerate on H if and only if $(d\alpha)^n$ is nowhere vanishing, but this is simply equivalent to asking that $\alpha \wedge (d\alpha)^n$ be nowhere vanishing.

Armed simply with the definition of a contact manifold, one might think that contact geometry is somewhat obscure. We provide the following list of examples as evidence that contact manifolds are actually quite common.

Example 29. Let $X = \mathbb{R}^{2n+1}$ with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n, z)$. The one-form

$$\alpha = dz + x^i dy^i$$

is a contact form, as

$$\alpha \wedge (d\alpha)^n = dz \wedge dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n,$$

which is nowhere vanishing. We define the standard contact structure on $\mathbb{R}^{2n+1}$ to be $H = \ker \alpha$.

For the next few examples the following lemma will be useful.

[11] Why?

Lemma 30. *Let (M, ω) be a symplectic manifold of dimension $2n$. A vector field Y on M satisfying $\mathcal{L}_Y\omega = \omega$ is called a* ***Liouville vector field****. In this case, $\alpha = \iota_Y\omega$ is a contact form on any hypersurface $Q \subset M$ transverse to Y (i.e. at any point p, T_pQ and Y_p span T_pM).*

Proof. Cartan's magic formula in this case tells us that $\omega = d\iota_Y\omega$, and hence

$$\begin{aligned} \alpha \wedge (d\alpha)^{n-1} &= \iota_Y\omega \wedge \omega^{n-1} \\ &= \iota_Y(\omega^n)/n. \end{aligned}$$

Now, since ω^n is a volume form on M, we find that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form when restricted to the tangent bundle of any hypersurface transverse to Y. □

Example 31. Consider $M = \mathbb{R}^4$ with its usual symplectic form $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. The vector field

$$Y = \frac{1}{2}\left(x_1\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial y^1} + x^2\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial y^2}\right)$$

is clearly transverse to the sphere S^3 given by $(x^1)^2 + (y^1)^2 + (x^2)^2 + (y^2)^2 = 1$. It is a straightforward computation to check that Y is Liouville, using the identity

$$(\mathcal{L}_Y\omega)(v, w) = \mathcal{L}_Y(\omega(v, w)) - \omega([Y, v], w) - \omega(v, [Y, w]).$$

We conclude, using the previous lemma, that S^3 is a contact manifold, with a contact structure $\ker \iota_Y\omega$. This example is easily extended to show that S^{2n+1} has a contact structure.

Example 32. Let (M, g) be a Riemannian n-manifold. We define the **unit cotangent bundle**

$$ST^*M = \{(p, \xi) \in T^*M \mid |\xi_p|_g^2 = 1\} \subset T^*M.$$

The unit cotangent bundle is a manifold of dimension $2n-1$ as it can be written as the level set of a Hamiltonian $H(p, \xi) = |\xi_p|_g^2/2$. Moreover, it is a sub-fiber bundle of the cotangent bundle, with fiber S^{n-1}. We claim that the canonical one-form on T^*M is a contact form for ST^*M. Indeed, let Y be a vector field on T^*M given by $\iota_Y\omega = \theta$. Then Y is Liouville: $d(\iota_Y\omega) = d\theta = \omega$. In coordinates, $Y = p^i\partial/\partial p^i$, and hence is transverse to ST^*M. Note that if M is compact, so is SY^*M and in this case ST^*M is an example of a compact contact manifold.

Example 33. Let $(M, H = \ker\alpha)$ be a contact manifold. Then, if $\pi_M : M \times \mathbb{R} \to M$ is the projection onto the second factor, we claim that $(M \times \mathbb{R}, \omega = d(e^t\pi_M^*\alpha))$ is a symplectic manifold. Indeed, if M has dimension $2n-1$, we compute

$$\begin{aligned} \omega^n &= (e^t dt \wedge \pi_M^*\alpha + \pi_M^* d\alpha)^n \\ &= ne^{nt} dt \wedge \pi_M^*\alpha \wedge \pi_M^*(d\alpha)^{n-1} \\ &= ne^{nt} dt \wedge \pi_M^*\left(\alpha \wedge (d\alpha)^{n-1}\right) \\ &\neq 0. \end{aligned}$$

We call $(M \times \mathbb{R}, d(e^t\pi_M^*\alpha))$ the **symplectization** of (M, α). Note that $\partial/\partial t$ is a Liouville vector field for ω[12] and $M \subset M \times \mathbb{R}$ is a hypersurface transverse to $\partial/\partial t$.

[12] why?

4. Week 4

4.1. Symplectic linear group and linear complex structures.

Definition 34. Let (V,ω) be a symplectic vector space. We denote the group of symplectomorphisms from V to itself as $\mathrm{Sp}(V,\omega)$, the **symplectic linear group**. In the case of the standard symplectic structure on $\mathbb{R}^{2n}$ we write the group as $\mathrm{Sp}(2n)$.

Lemma 35. *A real $2n \times 2n$ matrix Ψ is in $\mathrm{Sp}(2n)$ if and only if*

$$\Psi^\top J_0 \Psi = J_0,$$

where

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathrm{Sp}(2n).$$

Proof. Let u_i, v_i be a symplectic basis for V. For $x, y \in V$ write $x = (a,b), y = (c,d)$ for $a,b,c,d \in \mathbb{R}^n$. Then

$$\omega(x,y) = a^i d^i - b^i c^i = -x^\top J_0 y.$$

Clearly $\Psi^*\omega = \omega$ if and only if $\Psi^\top J_0 \Psi = J_0$. □

Definition 36. Let V be a vector space. A **complex structure** on V is an automorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$. We denote the set of all complex structures on V by $\mathcal{J}(V)$. Now suppose (V,ω) is a symplectic vector space. We say that a complex structure J is **compatible** with ω if

$$\omega(Jv, Jw) = \omega(v,w)$$

for all $v, w \in V$, and

$$\omega(v, Jv) > 0$$

for all nonzero $v \in V$. We denote the set of all compatible complex structures on (V,ω) by $\mathcal{J}(V,\omega)$.

Lemma 37. *Let $J \in \mathcal{J}(V,\omega)$ be a compatible complex structure on (V,ω). Then*

$$g_J(v,w) = \omega(v, Jw)$$

defines an inner product on V.

Lemma 38. *Let (V,ω) be a symplectic vector space and J be a complex structure on V. Then the following are equivalent:*

(a) J is compatible with ω;

(b) the bilinear form $g_J : V \times V \to \mathbb{R}$ defined by

$$g_J(v,w) = \omega(v, Jw)$$

is symmetric, positive-definite, and J-invariant.

(c) if we view V as a complex vector space with J as its complex structure, the form $H : V \times V \to \mathbb{C}$ defined by

$$H(v,w) = \omega(v, Jw) + i\omega(v,w)$$

is complex linear in w, complex antilinear in v, satisfies $H(w,v) = \overline{H(v,w)}$, and has a positive-definite real part. Such a form is called a ***Hermitian inner product*** *on (V,J).*

Proof. That (a) implies (b) is clear from Lemma 37. For (b) implies (c), note first that the real part of H is simply g_J and hence is positive-definite. For linearity, we compute

$$\begin{aligned} H(Jv, w) &= \omega(Jv, Jw) + i\omega(Jv, w) \\ &= g_J(Jv, w) - ig_J(w, v) \\ &= g_J(w, Jv) - ig_J(v, w) \\ &= -iH(v, w), \end{aligned}$$

and

$$\begin{aligned} H(v, Jw) &= -\omega(v, w) + i\omega(Jv, Jw) \\ &= -\omega(v, w) + ig_J(Jv, w) \\ &= -\omega(v, w) + i\omega(v, w) \\ &= iH(v, w), \end{aligned}$$

as desired. Finally, note that

$$\begin{aligned} H(w, v) &= \omega(w, Ju) + i\omega(w, v) \\ &= \omega(v, Jw) - i\omega(v, w) \\ &= \overline{H(v, w)}. \end{aligned}$$

For (c) implies (a), $\omega(v, Jv) > 0$ because the real part $\omega(v, Jw)$ is by hypothesis positive-definite. Moreover, $\omega(Jv, Jw) = \operatorname{im} H(Jv, Jw) = \operatorname{im} H(v, w) = \omega(v, w)$. □

The following result shows that all linear complex structures are isomorphic to the standard complex structure.

Proposition 39. *Let V be a $2n$-dimensional real vector space and let $J \in \mathcal{J}(V)$. Then there exists a vector space isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that*

$$J\Phi = \Phi J_0.$$

Proof. Consider the extension $J^{\mathbb{C}}$ of J to the complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \cong V$ given by $J \otimes 1$. Clearly $J^{\mathbb{C}}$ is a complex structure on $V^{\mathbb{C}}$ and thus has eigenvalues $\pm i$. We obtain a direct sum decomposition $V^{\mathbb{C}} \cong E^+ \oplus E^-$ of the $\pm i$ eigenspaces respectively, i.e. $J^{\mathbb{C}}|_{E^\pm} = \pm iI$. Clearly $\dim_{\mathbb{C}} E^\pm = n$. We claim that a basis $w_j = u_j + iv_j$ for E^+ yields a basis u_j, v_j for V. It suffices to show that these vectors are linearly independent. Since w_j is a basis for E^+,

$$\sum_{j=1}^{n} (a_j + ib_j)(u_j \otimes 1 + v_j \otimes i) = 0$$

for $a_j, b_j \in \mathbb{R}$ implies that $a_j = b_j = 0$ for all j. Suppose there exist $\alpha_j, \beta_j \in \mathbb{R}$ such that

$$\sum_{j=1}^{n} \alpha_j u_j + \beta_j v_j = 0.$$

Now since $w_j \in \ker(I - iJ)$, a straightforward computation reveals that $Ju_j = -v_j$ and $Jv_j = u_j$. Applying J to the above equation, we obtain

$$\sum_{j=1}^{n} \beta_j u_j - \alpha_j v_j = 0.$$

Then, taking $a_j = \beta_j, b_j = \alpha_j$, we find that

$$\begin{aligned}\sum_{j=1}^{n}(\beta_j + i\alpha_j)(u_j \otimes 1 + v_j \otimes i) &= \left(\sum_{j=1}^{n} \beta_j u_j - \alpha_j v_j\right) \otimes 1 + \left(\sum_{j=1} \beta_j v_j + \alpha_j u_j\right) \otimes i \\ &= 0.\end{aligned}$$

Linear independence of the w_j now forces $\alpha_j = \beta_j = 0$. Hence u_j, v_j forms a basis for V.

The required $\Phi : \mathbb{R}^{2n} \to V$ can now be written explicitly as

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{j=1}^{n} (x_j u_j - y_j v_j).$$

This map is clearly an isomorphism; moreover, if $x = (r_1, \ldots, r_n, s_1, \ldots, s_n) \in \mathbb{R}^{2n}$ then

$$J\Phi x = -s_1 u_1 - r_1 v_1 - \cdots - s_n u_n - r_n v_n = \Phi J_0 x,$$

as desired. □

Remark 40. Define an action of $\mathrm{GL}(2n, \mathbb{R})$ on the set $\mathcal{J}(V)$ by $g \cdot J = g^{-1}Jg$. By Lemma 39, $\mathrm{GL}(2n, \mathbb{R}) \cdot J_0 = \mathcal{J}(V)$, i.e. the orbit of J_0 is the entire set. Moreover, since $\mathrm{GL}(n, \mathbb{C})$ is naturally embedded (as a Lie subgroup) in $\mathrm{GL}(2n, \mathbb{R})$ as $\{A \in \mathrm{GL}(2n, \mathbb{R}) \mid J_0 A = A J_0\}$, the stabilizer of J_0 is $\mathrm{GL}(n, \mathbb{C})$.[13] We conclude that $\mathcal{J}(V)$ can be given the structure of a smooth manifold such that $\mathcal{J}(V) \cong \mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$.

Proposition 41. *The set $\mathcal{J}(V, \omega)$ of compatible complex structures is naturally identified with the space $\mathcal{P}$ of symmetric positive-definite symplectic matrices. In particular, $\mathcal{J}(V, \omega)$ is contractible.*

Proof. By fixing a symplectic basis for V we may assume that $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$. By the proof of Lemma 35, we note that $J \in \mathrm{Aut}(\mathbb{R}^{2n})$ is a compatible complex structure if and only if the conditions

$$\begin{aligned} J^2 &= -\operatorname{id}_{\mathbb{R}^{2n}}, \\ J_0 &= J^\top J_0 J, \\ 0 &< -v^\top J_0 J v, \end{aligned}$$

hold (for $v \neq 0$). Set $P = J_0 J$. P is symmetric, since

$$(J_0 J)^\top = -J^\top J_0 = J^\top J_0 J^2 = J_0 J,$$

[13] The embedding is given by replacing each entry $a + bi$ with a block of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

as well as positive-definite, and symplectic. Moreover, it is easy to check that if any matrix P has these three properties, then $J = -J_0 P$ is a compatible complex structure. Hence $\mathcal{J}(V, \omega)$ is in bijective correspondence with the space $\mathcal{P}$ of symmetric positive-definite symplectic matrices.[14] □

4.2. **Symplectic vector bundles.**

Definition 42. A **symplectic vector bundle** (E, ω) over X is a real vector bundle $\pi : E \to X$ together with a smooth symplectic bilinear form $\omega \in \Gamma(X, E^* \wedge E^*)$, i.e. a symplectic bilinear form on each E_x that varies smoothly with x. A **complex structure** on $\pi : E \to M$ is a bundle automorphism $J : E \to E$ such that $J^2 = -\operatorname{id}_E$. We say J is **compatible** with ω if the induced complex structure on E_x is compatible with ω_x for all $x \in X$. By the above lemma, we obtain a symmetric, positive-definite bilinear form $g_J \in \Gamma(X, \operatorname{Sym}^2 E^*)$, and we call the triple (E, ω, g_J) a **Hermitian structure** on E.

Theorem 43. *Every symplectic vector bundle (E, ω) over a manifold X admits a Hermitian structure.*

[14] Finish!