BOTT PERIODICITY

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What assumptions do we need to make on the spaces? All vector bundles throughout will be complex.

Recall from last time the definition of K-theory.

Definition 1. Let $\mathrm{Vect} : \mathrm{HT\textsc{op}}^{\mathrm{op}} \to \mathrm{CR\textsc{ig}}$ denote the functor taking any space X to the commutative semiring of vector bundles over X with the operations of direct sum and tensor product. Notice that $f : X \to Y$ is taken to $f^* : \mathrm{Vect}\, Y \to \mathrm{Vect}\, X$. Then, by the usual Grothendieck construction, there exists a universal functor

$$K : \mathrm{HT\textsc{op}}^{\mathrm{op}} \to \mathrm{CR\textsc{ing}}$$

such that if R is a commutative ring then the diagram

$$\begin{tikzcd} \mathrm{Vect}(X) \arrow[r] \arrow[d] & R \\ K(X) \arrow[ur, dashed] & \end{tikzcd}$$

commutes. It follows that for a map $f : X \to Y$ the diagram

$$\begin{tikzcd} \mathrm{Vect}\, Y \arrow[r, "f^*"] \arrow[d] & \mathrm{Vect}\, X \arrow[d] \\ K(Y) \arrow[r, "Kf"] & K(X) \end{tikzcd}$$

commutes.

Remark 2. By construction, every element in $K(X)$ is of the form $[E] - [F]$ for $E, F \in \mathrm{Vect}(X)$. There exists $G \in \mathrm{Vect}\, X$ such that $F \oplus G$ is trivial of rank n: $F \oplus G \cong \underline{n}$. Then $[E] - [F] = [E] + [G] - ([F] + [G]) = [E \oplus G] - [\underline{n}]$. We conclude that every element of $K(X)$ is of the form $[H] - [\underline{n}]$.

Moreover, suppose $E, F \in \mathrm{Vect}\, X$ are such that $[E] = [F]$. Then, by definition of the Grothendieck construction, there exists $G \in \mathrm{Vect}\, X$ such that $E \oplus G \cong F \oplus G$. If $G' \in \mathrm{Vect}\, X$ is such that $G \oplus G' \cong [\underline{n}]$ then adding G', we find that $E \oplus \underline{n} \cong F \oplus \underline{n}$. We conclude that $[E] = [F]$ if and only if E and F are *stably equivalent*, i.e. there exists a suitable trivial bundle such that they become equivalent after adding the trivial bundle.

Definition 3. If $X \in \mathrm{HT\textsc{op}}_+^{\mathrm{op}}$, the inclusion of the basepoint into X yields a map $K(X) \to K(+)$. We define the reduced K-theory $\tilde{K}(X)$ to be the kernel

$$0 \longrightarrow \tilde{K}(X) \longrightarrow K(X) \longrightarrow K(+) \longrightarrow 0.$$

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The constant map $X \to +$ yields a natural splitting of this exact sequence

$$K(X) \cong \tilde{K}(X) \oplus K(+) \cong \tilde{K}(X) \oplus \mathbb{Z},$$

hence inducing a functor $\tilde{K} : \mathrm{HT\scriptstyle{OP}}_+^{\mathrm{op}} \to \mathrm{CR\scriptstyle{ING}}$. A straightforward computation shows that $[E] = [F]$ in $\tilde{K}(X)$ if and only if $E \oplus \underline{n} \cong F \oplus \underline{m}$ for some $m, n \in \mathbb{N}$.

Let $A \subset X$. As in cohomology, we wish to obtain a long exact sequence on K-theory. Consider the cofiber sequence associated to $i : A \to X$:

$$A \xrightarrow{\ i\ } X \xrightarrow{\ j\ } Ci \xrightarrow{\ \pi\ } \Sigma A \xrightarrow{-\Sigma i} \Sigma X \xrightarrow{-\Sigma\pi} \Sigma Ci \xrightarrow{\Sigma^2 i} \Sigma^2 X \longrightarrow \cdots$$

where $(-\Sigma i)(x \wedge t) = f(x) \wedge (1-t)$. Applying K-theory, we claim:

Proposition 4. *For $i : A \hookrightarrow X$ there is a natural long exact sequence*

$$\cdots \longrightarrow K^{-1}(X) \longrightarrow K^{-1}(A) \longrightarrow K^0(X, A) \longrightarrow K^0(X) \longrightarrow K^0(A),$$

where $K^{-n}(X) \equiv K(\Sigma^n X)$ and $K^{-n}(X, A) \equiv K(\Sigma^n Ci)$ for all $n \geqslant 0$, and the maps are induced from the cofiber sequence above.

Proof. Work. □

Bott periodicity is a fundamental result stating that K-theory is 2-periodic.

Theorem 5 (Bott periodicity)**.** *$K^{-n}(X) \cong K^{-n-2}(X)$ for all $n \geqslant 0$.*

In particular, defining

$$K^{2n}(X) \equiv K^0(X)$$
$$K^{2n+1}(X) \equiv K^{-1}(X)$$

for $n \geqslant 0$, the data of the long exact sequence for a pair (X, A) reduces to the exact sequence

$$\begin{tikzcd}
K^0(X,A) \arrow[r] & K^0(X) \arrow[r] & K^0(A) \arrow[d] \\
K^1(A) \arrow[u] & K^1(X) \arrow[l] & K^1(X,A) \arrow[l]
\end{tikzcd}$$

To prove Bott periodicity, we will use a series of results. We will work with reduced K-theory for convenience.

Lemma 6. *Write $BU = \operatorname{colim}_n BU(n)$. Then there is a natural isomorphism of functors $\tilde{K}$ and $[-, BU]$. That is, there is a natural bijection*

$$[X, BU] \cong \tilde{K}(X).$$

Proof. Recall that $[X, BU(n)] \cong \operatorname{Vect}^n X$, whence

$$\begin{aligned}
[X, BU] &= [X, \operatorname*{colim}_n BU(n)] \\
&= \operatorname*{colim}_n [X, BU(n)] \\
&= \operatorname*{colim}_n \operatorname{Vect}^n X.
\end{aligned}$$

It is straightforward to see that the map $\operatorname{Vect}^n X \to \operatorname{Vect}^{n+1} X$ is given by adding a trivial bundle. Hence elements $E, F \in \operatorname{colim}_n \operatorname{Vect}^n X$ – say, vector bundles of rank r and s respectively – are equal in the colimit if and only if there exist $l, k \in \mathbb{N}$ such that $E \oplus \underline{l} \cong F \oplus \underline{k}$. Thus there is a natural isomorphism $\operatorname{colim}_n \operatorname{Vect}^n X \cong \tilde{K}(X)$, completing the proof. why does colim commute with maps? □

Assume, for the moment, the following proposition.

Proposition 7. *There is a homotopy equivalence $BU \simeq \Omega SU$.*

Then we can prove Bott periodicity as follows.

Proof of Theorem 5. It suffices to show that there is a natural isomorphism $\tilde{K}(\Sigma^2 X) \cong \tilde{K}(X)$. Using Lemma 6 and the above proposition, we have natural isomorphisms

$$\begin{aligned}
\tilde{K}(\Sigma^2 X) &\cong [\Sigma^2 X, BU] \\
&\cong [X, \Omega^2 BU] \\
&\cong [X, \Omega SU] \\
&\cong [X, BU] \\
&\cong \tilde{K}(X).
\end{aligned}$$

Why is $\Omega BU \simeq SU$? □

All the hard work lies lies in the proof of the homotopy equivalence $BU \simeq \Omega SU$.