NOTES ON SYMPLECTIC GEOMETRY

NILAY KUMAR

Contents

1. Week 1

1.1. The cotangent bundle.

Definition 1. Let X be a smooth n-manifold and $\pi : M = T^*X \to X$ be its cotangent bundle. We define the **canonical one-form** $\theta \in \Omega^1(M)$ as follows. For any $p = (x, \xi) \in M$, set

$$\theta_p(v) = \xi(d_x\pi(v)).$$

The one-form θ is canonical (or tautological) in the sense that its value at a point is simply given by the covector determined by that point. More precisely, we have the following characterization.

Proposition 2. *The canonical one-form θ is the (unique) one-form such that for every $\lambda \in \Omega^1(X)$, $\lambda^*\theta = \lambda$.*

Proof. We compute, for $v \in T_pX$,

$$\begin{aligned}(\lambda^*\theta)_p(v) &= \theta_{\lambda(p)}(d_p\lambda(v)) \\ &= \lambda_p(d_p(\pi \circ \lambda)(v)) \\ &= \lambda_p(v),\end{aligned}$$

where we have used the fact that λ is a section of π, i.e. $\pi \circ \lambda = \mathrm{id}_X$. Uniqueness is easily checked. □

Definition 3. The **canonical symplectic form** $\omega \in \Omega^2(M)$ is now defined to be the exterior derivative

$$\omega = -d\theta,$$

of the canonical one-form. To be symplectic, ω must be closed and nondegenerate. That it is closed is obvious.

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Proposition 4. *The form $\omega \in \Omega^2(M)$ is nondegenerate and thus defines a symplectic structure on $M = T^*X$.*[1]

Proof. For ω to be non-degenerate, it must be nondegenerate at each point $p \in M$. Given coordinates $p = (x, \xi) = (x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ in a neighborhood of p, we can compute

$$\begin{aligned} \theta_{(x,\xi)}\left(v^i \frac{\partial}{\partial x^i} + \nu^i \frac{\partial}{\partial \xi^i}\right) &= \xi\left(v^i \frac{\partial}{\partial x^i}\right) \\ &= \xi_i v^i \end{aligned}$$

and hence

$$\theta = \xi_i dx^i.$$

Taking an exterior derivative, we find that

$$\begin{aligned} \omega &= -d\theta \\ &= dx^i \wedge d\xi_i. \end{aligned}$$

Fix $v \in T_pM$ and suppose that $\iota_v \omega_p = 0$, i.e. $\omega_p(v, w) = 0$ for all $w \in T_pM$. In coordinates, this implies that

$$\begin{aligned} \iota_{v^j \frac{\partial}{\partial x^j} + \nu^j \frac{\partial}{\partial \xi^j}}(dx^i \wedge d\xi_i) &= v^i d\xi_i - \nu^i dx^i \\ &= 0, \end{aligned}$$

and hence that $v^i = \nu^i = 0$, i.e. $v = 0$. We conclude that ω_p is nondegenerate at each $p \in M$. □

It is easy to see that ω provides an isomorphism $\iota : T_xX \xrightarrow{\sim} T_x^*X$ between tangent and cotangent spaces at each point $x \in X$: since ω_x is nondegenerate, the linear map $\iota : v \mapsto \omega_x(v, -)$ is injective and hence bijective. In fact, we can say more.

Proposition 5. *The metric ω induces an isomorphism of vector bundles $\iota : TX \xrightarrow{\sim} T^*X = M$.*

Proof. Recall that an isomorphism in the category of smooth vector bundles is a smooth bijection[2] ι such that the diagram

$$\begin{tikzcd} TX \arrow[rr, "\iota"] \arrow[dr, "\pi_1"'] & & T^*X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

commutes and for each $x \in X$, the restriction $\iota_x : T_xX \to T_x^*X$ is linear. The map $\iota : TX \to T^*X$ taking $(x, v) \mapsto (x, \omega(v, -))$ fits into the diagram above and is bijective and fiberwise linear. Moreover, ι is a smooth map, as is seen by its coordinate description computed above. □

Definition 6. A **Hamiltonian** is a smooth function $H : M = T^*X \to \mathbb{R}$. we define the **Hamiltonian vector field** v_H associated to H to be the vector field on M satisfying

$$\iota_{v_H} \omega = dH.$$

[1] Is there a coordinate invariant proof?

[2] Existence of a smooth inverse is automatic (reference?).

The (local) flow $F : (-\varepsilon, \varepsilon) \times M \to M$ determined by v_H is called the **Hamiltonian flow**.[3]

Note that an integral curve $\gamma_{v_H} : (-\varepsilon, \varepsilon) \to M$ of v_H can be thought of as the trajectory of a physical state in phase space. Indeed, Hamilton's equations are given

$$\begin{aligned} \frac{\partial x^i}{\partial t} &= \frac{\partial H}{\partial \xi_i} \\ \frac{\partial \xi_i}{\partial t} &= -\frac{\partial H}{\partial x^i}, \end{aligned}$$

which is precisely the condition that $\gamma'_{v_H}(t) = (v_H)_{\gamma(t)}$. Moreover, H is constant along the Hamiltonian flow, as

$$dH(v_H) = (\iota_{v_H}\omega)(v_H) = \omega(v_H, v_H) = 0,$$

i.e. v_H is perpendicular to the level sets of H. In a physical system, where H is the energy functional on phase space, this phenomenon is the law of conservation of energy.

Proposition 7. *The Hamiltonian flow is a symplectomorphism, i.e.* $F_t^*\omega = \omega$.[4]

Proof. We use the following trick:

$$\int_0^t \frac{d}{dt} F_t^*\omega \; dt = F_t^*\omega - \omega$$

since $F_0 = \mathrm{id}_M$, and hence F_t is a symplectomorphism if and only if the integrand is zero. But

$$\begin{aligned} \frac{d}{dt} F_t^*\omega &= \frac{d}{ds}\bigg|_{s=0} F_{t+s}^*\omega = F_t^* \frac{d}{ds}\bigg|_{s=0} F_s^*\omega \\ &= F_t^* \mathcal{L}_{v_H}\omega, \end{aligned}$$

and Cartan's magic formula,

$$\mathcal{L}_{v_H}\omega = d\iota_{v_H}\omega + \iota_{v_H} d\omega,$$

tells us that $\mathcal{L}_{v_H}\omega = 0$ since $\iota_{v_H}\omega = dH$ is closed, as is ω. $\square$

Corollary 8 (Liouville's Theorem)**.** *The volume form ω^n on $M = T^*X$ is preserved by the Hamiltonian flow.*

1.2. **Geodesic flow as Hamiltonian flow.** We wish to discuss geodesics and geodesic flow. For this, we need the concept of connections and covariant derivatives.[5]

Definition 9. A **connection** on a vector bundle $E \to X$ is an $\mathbb{R}$-linear map $\nabla : \Gamma(X, E) \to \Gamma(X, E \otimes T^*X)$ such that the Leibniz rule

$$\nabla(f\sigma) = (\nabla\sigma)f + \sigma \otimes df,$$

for all $f \in C^\infty(X)$ and $\sigma \in \Gamma(X, E)$.

Theorem 10. *Given a Riemannian manifold (X, g), there exists a unique connection on $\pi : TX \to X$, known as the **Levi-Civita connection**, satisfying*

[3] Is this a global flow? Does it depend on X?
[4] Is there a better proof?
[5] Reference do Carmo.

(i) symmetry:
$$\nabla_X Y - \nabla_Y X - [X,Y] = 0,$$
for $X, Y \in \Gamma(X, TX)$;

(ii) compatibility with g:
$$Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,$$
for $X, Y, Z \in \Gamma(X, TX)$.

Definition 11. Let v be a vector field on (X,g); we define the **covariant derivative** of v along a smooth curve $c : I \to X$ to be the vector field
$$\frac{Dv}{dt} = \nabla_{dc/dt} v,$$
where ∇ is the Levi-Civita connection. Explicitly, if we write $v = v^i \partial/\partial x^i$ and $c(t) = (c_1(t), \ldots, c_n(t))$,
$$\frac{Dv}{dt} = \sum_i \frac{dv^i}{dt}\frac{\partial}{\partial x^i} + \sum_{ijk} \frac{dc_i}{dt} v^i \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$
Here Γ^k_{ij} are the Christoffel symbols of ∇, determined by
$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \sum_{ijk} \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$
We say that c is **geodesic** at some $t \in I$ if $D/dt(dc/dt) = 0$ at t, and that c is geodesic if it is geodesic at all $t \in I$. In coordinates, the condition for c to be geodesic is given by a system of second-order differential equations:
$$\frac{d^2 c^i}{dt^2} + \sum_{jk} \Gamma^i_{jk} \frac{dc^j}{dt}\frac{dc^k}{dt} = 0,$$
for $i = 1, \ldots, n$.

For the rest of the section, assume (X,g) is Riemannian and we fix the Hamiltonian $H : M = T^*X \to \mathbb{R}$ as
$$H(x,\xi) = \frac{1}{2}\left|\xi_x\right|^2_g,$$
i.e. consisting of only a kinetic term. Here we are implicitly using the nondegeneracy of g to associate ξ_x with its corresponding vector (or, equivalently, using g^{-1}).

Proposition 12. *The Hamiltonian flow on $M = T^*X$ is dual to the geodesic flow on TX. In other words, the integral curves of the Hamiltonian vector field v_H associated to the Hamiltonian above project to geodesics of g on X.*[6]

Proof. It suffices to show, in coordinates, that Hamilton's equations (i.e. the condition for being on the integral curve) yield the geodesic equations above after the necessary dualization. Note first that in coordinates the Hamiltonian becomes
$$H(x,\xi) = \frac{1}{2} g^{ij} \xi_i \xi_j.$$

[6] Is there a coordinate-free proof? See Paternain's book.

For convenience we will denote the components of an integral curve as $x^i(t)$. Hamilton's equations yield

$$\begin{aligned}\frac{dx^i}{dt} &= \frac{\partial}{\partial \xi_i}\left(\frac{1}{2}g^{jk}\xi_j\xi_k\right)\\ &= \frac{1}{2}g^{jk}\delta_{ij}\xi_k + \frac{1}{2}g^{jk}\xi_j\delta_{ik}\\ &= g^{ij}\xi_j\\ \frac{d\xi_i}{dt} &= -\frac{\partial}{\partial x^i}\left(\frac{1}{2}g^{jk}\xi_j\xi_k\right)\\ &= -\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i}\xi_j\xi_k.\end{aligned}$$

Differentiating the first equation with respect to t and using both of Hamilton's equations yields

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= \frac{\partial g^{ij}}{\partial x^k}\frac{dx^k}{dt}\xi_j + g^{im}\frac{d\xi_m}{dt}\\ &= g^{kl}\left(\frac{\partial}{\partial x^k}g^{ij}\right)\xi_l\xi_j - \frac{1}{2}g^{im}\left(\frac{\partial}{\partial x^m}g^{nr}\right)\xi_n\xi_r.\end{aligned}$$

Next, differentiating the identity $g^{ij}g_{jk} = \delta^i_k$, it easy to see that

$$\frac{\partial}{\partial x^i}g^{kl} = -g^{la}g^{kb}\frac{\partial}{\partial x^i}g_{ab}.$$

Using this, contracting indices, and using the first Hamilton's equation to dualize ξ's into dx/dt's, we find

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= -g^{ib}\left(\frac{\partial}{\partial x^k}g_{lb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt} + \frac{1}{2}g^{im}\left(\frac{\partial}{\partial x^m}g_{ts}\right)\frac{dx^s}{dt}\frac{dx^t}{dt}\\ &= -\frac{1}{2}g^{ib}\left(\frac{\partial}{\partial x^k}g_{lb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt} - \frac{1}{2}g^{ib}\left(\frac{\partial}{\partial x^l}g_{kb}\right)\frac{dx^k}{dt}\frac{dx^l}{dt}\\ &\quad + \frac{1}{2}g^{im}\left(\frac{\partial}{\partial x^m}g_{ts}\right)\frac{dx^s}{dt}\frac{dx^t}{dt}\\ &= -\Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt},\end{aligned}$$

as desired. $\square$

2. Week 2

2.1. **Darboux's theorem.**

Theorem 13 (Darboux). *Let (M, ω) be a symplectic $2n$-manifold. Then M is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.*

We prove Darboux's theorem using the following stronger statement.

Theorem 14. *Let M be a $2n$-dimensional manifold and $Q \subset M$ be a compact submanifold. Suppose that $\omega_1, \omega_2 \in \Omega^2(M)$ are closed 2-forms such that at each point q of Q the forms ω_0 and ω_1 are equal and nondegenerate on T_qM. Then there exist neighborhoods N_0 and N_1 of Q and a diffeomorphism $\psi : N_0 \to N_1$ such that $\psi|_Q = \mathrm{id}_Q$ and $\psi^*\omega_1 = \omega_0$.*

Proof. Consider the family of closed two-forms

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0)$$

on M for $t \in [0, 1]$. Note that $\omega_t|_Q = \omega_0|_Q$ is nondegenerate and hence there exists an open neighborhood N_0 of Q such that $\omega_t|_{N_0}$ is nondegenerate.[7] Suppose, for now, that there is a one-form $\sigma \in \Omega^1(N_0)$ (possibly shrinking N_0), such that $\sigma|_{T_QM} = 0$ and $d\sigma = \omega_1 - \omega_0$ on N_0. Then

$$\omega_t = \omega_0 + td\sigma$$

and we obtain by nondegeneracy a smooth vector field X_t on N_0 characterized by

$$\iota_{X_t}\omega_t = -\sigma.$$

The condition $\sigma|_{T_QM} = 0$ implies, again by nondegeneracy of ω_t, that $X_t|_Q = 0$. Now consider the initial value problem for the flow ψ_t of X_t,

$$\begin{aligned} \frac{d}{dt}\psi_t &= X_t \circ \psi_t \\ \psi_0 &= \mathrm{id}\,. \end{aligned}$$

This differential equation can be solved uniquely for $t \in [0, 1]$ on some open neighborhood of Q contained in N_0, call it again N_0.[8] Note that $\psi_t|_Q = \mathrm{id}_Q$ since $X_t|_Q = 0$. We compute now that

$$\begin{aligned} \frac{d}{dt}\psi_t^*\omega_t &= \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) \\ &= \psi_t^*\left(d\sigma + d\iota_{X_t}\omega_t\right) \\ &= 0. \end{aligned}$$

Hence $\psi_1^*\omega_1 = \psi_0^*\omega_0 = \omega_0$. Thus the desired diffeomorphism is ψ_1 and the desired neighborhoods are N_0 and N_1. The above argument is known as **Moser's trick**, and is extremely useful in symplectic geometry.

It remains to construct a smooth one-form σ satisfying $\sigma|_{T_QM} = 0$ and $d\sigma = \omega_1 - \omega_0$. If Q were a point (or more generally, diffeomorphic to a star-shaped subset of Euclidean space), we could simply use the Poincaré lemma; in general, however the construction is as follows. Fix any Riemannian metric on M and consider the

[7] Why?

[8] Why?

restriction of the exponential map $\exp : TM \to M$ to a neighbhorhood U_ε of the zero section of the normal bundle $TQ^\perp \to M$:

$$U_\varepsilon = \{(q, v) \in TM \mid q \in Q, v \in T_q Q^\perp, |v| < \varepsilon\}.$$

Recall that exp becomes a diffeomorphism for ε sufficiently small, so we choose ε such that $N_0 = \exp(U_\varepsilon)$ is contained in the neighborhood of Q above on which ω_t is nondegenerate. Define now a family of maps $\phi_t : N_0 \to N_0$ for $t \in [0, 1]$ by

$$\phi_t(\exp(q, v)) = \exp(q, tv).$$

Note that ϕ_t is a diffeomorphism onto its image for $t \neq 0$. Moreover, $\phi_t|_Q = \mathrm{id}_Q$, $\phi_0(N_0)$, and $\phi_1 = \mathrm{id}_{N_0}$. If we now write $\tau = \omega_1 - \omega_0$, we find that

$$\begin{aligned} \phi_0^* \tau &= 0 \\ \phi_1^* \tau &= \tau, \end{aligned}$$

since $\tau = 0$ on $T_Q M$. Now, for $t \in (0, 1]$, we define a family of vector fields,

$$Y_t = \left(\frac{d}{dt} \phi_t \right) \circ \phi_t^{-1}.$$

Then for any $\delta > 0$,

$$\begin{aligned} \phi_1^* \tau - \phi_\delta^* \tau &= \int_\delta^1 \frac{d}{dt} \phi_t^* \tau dt = \int_\delta \phi_t^* \mathcal{L}_{Y_t} \tau dt \\ &= \int_\delta^1 \phi_t^* (d\iota_{Y_t} \tau) dt \\ &= d \int_\delta^1 \phi_t^* (\iota_{Y_t} \tau) dt \end{aligned}$$

Clearly $\phi_1^* \tau - \phi_\delta^* \tau = \tau - \phi_\delta^* \tau$ approaches τ as $\delta \to 0^+$, so we find that

$$\tau = d \int_0^1 \phi_t^* (\iota_{Y_t} \tau) dt.$$

Defining

$$\sigma = \int_0^1 \phi_t^* (\iota_{Y_t} \tau) dt,$$

we find that $\tau = \omega_1 - \omega_0 = d\sigma$ and $\sigma|_{T_Q M} = 0$ because $\phi_t|_Q = \mathrm{id}_Q$ and $\tau = 0$ on Q, forcing the integrand to vanish on $T_Q M$. Hence σ is the one-form required above for Moser's trick, and we are done.[9] □

The proof of Darboux's theorem is now straightforward: we choose a coordinate chart ϕ so that $\phi^* \omega$ is equal to the standard form on a subset of $\mathbb{R}^{2n}$ at a single point, and then apply Moser's theorem with Q equal to the chosen point.

Proof of Darboux's theorem. Let $q \in M$ and fix a symplectic basis $\{u_i, v_i\}$ for the symplectic vector space $(T_q M, \omega_q)$. Fix any Riemannian metric on M and pick an open $U \ni 0$ small enough such that exp restricted to $U \subset T_q M$ is a diffeomorphism

[9] Why is σ smooth?

and hence a chart $(x^i, y_i) = \exp : U \subset \mathbb{R}^{2n} \to M$ $(i = 1, \ldots, n)$ such that $x^i(p) = y_i(p) = 0$. Now we can compute, for example,

$$\begin{aligned} \exp^* \omega_p \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k} \right) &= \omega_p \left(\exp_* \frac{\partial}{\partial x^j}, \exp_* \frac{\partial}{\partial y^k} \right) \\ &= \omega_p \left(u_j, v_k \right) = \delta_{jk}, \end{aligned}$$

to check that $\exp^* \omega_p = (\omega_0)_0$ where ω_0 is the standard form on $T_0 U$. Here we have used the fact that $\exp_* = \mathrm{id}$ at $0 \in U$. Applying Theorem 2.1 to U with $Q = 0 \in U$, we obtain a diffeomorphism ψ of (some possibly smaller) U such that $\psi^* \exp^* \omega = \omega_0$ on U. But now $\exp \circ \psi$ provides a symplectomorphism in a neighborhood of q to a neighborhood of $\mathbb{R}^{2n}$ pulling ω back to the standard form ω_0. $\square$