BROWN REPRESENTABILITY AND CW-SPECTRA

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Outline:

(1) Generalized (reduced) cohomology theory, from Ω-spectra
(2) Brown representability
(3) every spectrum is weak equivalent to an omega-spectrum (?)
(4) are these categories equivalent?
(5) properties of CW spectra
(6) smash products, axiomatically, ring spectra?

Note that I will write h instead of $\tilde{h}$, etc. – all cohomology theories will be reduced.

Last time Sean defined the CW-spectra category and its associated homotopy category, the stable homotopy category. The main goal for today is to understand the relation between CW-spectra and generalized cohomology theories. Let us first recall the definitions.

Definition 1. A **(reduced) generalized cohomology theory** is a functor $h^\bullet : \mathrm{CW}_* \to \mathrm{AB}^{\mathbb{Z}}$ from pointed CW-complexes to graded abelian groups together with natural isomorphisms

$$h^\bullet \xrightarrow{\sim} h^{\bullet+1} \circ \Sigma,$$

called suspension isomorphisms, such that

(1) if $f, g : X \to Y$ are homotopic then $f^* = g^* : h^\bullet Y \to h^\bullet X$;
(2) for $\iota : A \to X$ an inclusion and $j : X \to C\iota$ the induced mapping cone, we have an exact sequence

$$h^\bullet(C\iota) \longrightarrow h^\bullet X \longrightarrow h^\bullet A;$$

(3) for a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $\iota_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism

$$h^\bullet(\textstyle\bigvee_\alpha X_\alpha) \longrightarrow \textstyle\prod_\alpha h^\bullet X_\alpha.$$

The examples we've seen so far of such gadgets are ordinary (singular) cohomology and K-theory. A few more examples are cohomotopy and cobordism. Note that in general a generalized cohomology theory need not be multiplicative. Let us now recall the category of CW-spectra.

Definition 2. A **CW-spectrum** E is a sequence of pointed CW-complexes $\{E_n\}_{n\in\mathbb{N}}$ together with structure maps $\varepsilon_i : \Sigma E_n \to E_{n+1}$ such that ΣE_n is mapped isomorphically (as a CW-complex) onto a subcomplex of E_{n+1}. A subspectrum $A \subset E$ is a CW-spectrum for which each A_n is a subcomplex of E_n. A morphism $E \to F$ of

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CW-spectra is an equivalence class of maps $A_n \to F_n$ (commuting with structure maps) of cofinal subspectra $A \subset E$.[1]

The prototypical example is the suspension spectrum $\Sigma^\infty X$ of a CW-complex X, which is given $E_n = \Sigma^n X$ with the obvious structure maps. The sphere spectrum, for instance, is $\mathbb{S} \equiv \Sigma^\infty S^0$. Another important class of CW-spectra is that of Ω-spectra, CW-spectra for which the adjoint structure maps $\hat{\varepsilon}_n : X_n \to \Omega X_{n+1}$ are weak-equivalences.

We have in fact already seen two Ω-spectra, as Sean pointed out. The first is the Eilenberg-Maclane spectrum, given $H\mathbb{Z}_n = K(\mathbb{Z}, n)$. That this yields an Ω-spectrum is clear from the path-fibration over $K(\mathbb{Z}, n)$. The Eilenberg-Maclane spaces have the important property that they represent cohomology, i.e. there is a natural isomorphism

$$H^\bullet X \cong [X, K(n, \mathbb{Z})].$$

Another example we've seen is the K-theory spectrum $BU \times \mathbb{Z}$. Recall that $K^0(X)$ is the group completion of the monoid of complex vector bundles over (a compact space) X and $K^{-1}(X)$ is defined to be $K^0(\Sigma X)$. The fact that rank n vector bundles given by homotopy classes of maps to $BU(n)$ yields

$$K^0(X) \cong [X, BU \times \mathbb{Z}].$$

It follows from loops-suspension adjunction that

$$K^{-1}(X) \cong [X, \Omega(BU \times \mathbb{Z})] \cong [X, \Omega BU] \cong [X, U],$$

where the last isomorphism comes from the homotopy equivalence $\Omega BU \simeq U$. Moving further down, $K^{-2}(X) = K^0(\Sigma^2 X)$, but Bott periodicity tells us that $\Omega U \simeq BU \times \mathbb{Z}$. Hence we obtain a 2-periodic sequence of spaces $\{BU \times \mathbb{Z}, \Omega BU, BU \times \mathbb{Z}, \dots\}$ that obviously form an Ω-spectrum.

It is natural to ask, now, whether every generalized cohomology theory is representable by an Ω-spectrum. This is Brown's representability theorem. Before we prove this, however, let us consider the converse question, which is easier: does every Ω-spectrum yield a generalized cohomology theory?

Proposition 3. *Let E be an Ω-spectrum. Then $E^\bullet \equiv [-, E_\bullet]$ is a generalized cohomology theory.*

Proof. For X a CW-complex, the homotopy classes of maps $[X, E_\bullet]$ forms an abelian group. finish

That $E^\bullet$ is homotopy-invariant is clear: pullback by f is given by precomposition by f so if $f \sim g$ then $f^*\phi \sim g^*\phi$. The suspension isomorphism follows immediately from the fact that E is an Ω-spectrum:

$$\begin{aligned} E^{n+1}(\Sigma X) &= [\Sigma X, E^{n+1}] \\ &\cong [X, \Omega E^{n+1}] \\ &\cong [X, E^n] \\ &= E^n X \end{aligned}$$

The wedge axiom is also straightforward: maps $[\bigvee_\alpha X_\alpha, E_\bullet]$ are in natural bijection with the product $\prod_\alpha [X_\alpha, E_\bullet]$. □

[1] Sean explained what this means, as well as the notion of homotopy but we won't need these precise definition today.