NOTES ON INTEGRABILITY OF COMPLEX STRUCTURES

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Recall that every complex manifold is by definition a smooth manifold. This raises the following question: when can a smooth manifold be given the structure of a complex manifold, i.e. a complex structure?

We begin by reviewing some definitions.

Definition 1. An almost complex structure J on a $2n$-dimensional smooth manifold M is a (real) vector bundle endomorphism $J : TM \to TM$ satisfying $J^2 = -\operatorname{id}$.

Exercise 2. Show that if M has an almost complex structure then it must be even-dimensional.

Suppose (M, J) is an almost complex manifold of dimension $2n$. Define the complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ to be the tensor product of the real tangent bundle with the trivial complex line bundle. Notice that $T_{\mathbb{C}}M$ has fibers that are $4n$ dimensional. Complexifying the almost complex structure J, we obtain $J_{\mathbb{C}} = J \otimes \mathbb{C}$. Notice that $J_{\mathbb{C}}^2 = -\operatorname{id}$ and hence $J_{\mathbb{C}}$ has eigenvalues $\pm i$. Thus we can decompose

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

the $\pm i$-eigenbundles of $J_{\mathbb{C}}$.[1] These are the holomorphic and antiholomorphic tangent bundles of (M, J).

Proposition 3. *There is a natural almost complex structure J on any complex manifold X.*

Proof. Let $\{U_\alpha, \phi_\alpha\}$ be an atlas for X. Each U_α is biholomorphic to an open subset of $\mathbb{C}^n$, and hence has real coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ comprising the coordinates $z^i = x^i + iy^i$. Consider, in this chart, the endomorphism J of TU_α sending $\partial/\partial x^i \mapsto \partial/\partial y^i$ and $\partial/\partial y^i \mapsto -\partial/\partial x^i$ at every point of U_α. This is an almost complex structure on U_α – it remains to patch these structures together on X.

To do this it suffices to show that if $f : U \to V$ is a holomorphic map between subsets of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, then $(f_*)_{\mathbb{C}}(T^{1,0}U) \subset T^{1,0}V$. In other words, the complexified pushforward of a holomorphic map preserves the direct sum decomposition with respect to the almost complex structure defined above. Notice that $T^{1,0}U$ is trivialized by $\{\partial/\partial z^i\}_{i=1,\ldots,n}$, as $J\partial/\partial z^i = i\partial/\partial z^i$ (here $z^i = x^i + iy^i$, where x^i, y^i are coordinates on U), and similarly $T^{0,1}U$ is trivialized by $\{\partial/\partial \bar{z}^i\}_{i=1,\ldots,n}$. Fixing these vector fields as a basis for $T_{\mathbb{C}}U$, and similarly for the target $T_{\mathbb{C}}V$, one computes

$$(f_*)_{\mathbb{C}} = \begin{pmatrix} \frac{\partial f^i}{\partial z_j} & \frac{\partial f^i}{\partial \bar{z}^j} \\ \frac{\partial \bar{f}^i}{\partial z_j} & \frac{\partial \bar{f}^i}{\partial \bar{z}^j} \end{pmatrix} = \begin{pmatrix} f_* & 0 \\ 0 & \overline{f_*}, \end{pmatrix}$$

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[1] It suffices to check this fiberwise, as $J_{\mathbb{C}}$ is a smooth map of bundles.

where we have used the fact that f is holomorphic. This shows that $(f_*)_{\mathbb{C}}$ respects the almost complex structure. $\square$

Definition 4. Let (M, J) be an almost complex structure. We say that J is integrable if

$$[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M.$$

The following (highly nontrivial) theorem justifies our interest in "integrability" of almost complex structures.

Theorem 5 (Newlander-Nirenberg)**.** *Every integrable almost complex structure is induced by a unique complex structure.*

By definition, an almost complex structure comes from a complex structure if locally $(M, J) \cong (\mathbb{C}^n, i)$. The Newlander-Nirenberg theorem relaxes this condition to an integrability condition reminiscent of the theorem of Fröbenius. In fact, if J is real analytic, then the proof follows from a version of Fröbenius.

Exercise 6. Prove uniqueness.

Example 7. Even familiar manifolds need not admit almost complex structures. For instance, the only spheres with almost complex structures are S^2 and S^6, by topological arguments involving characteristic classes. Moreover, one can show that there exists an almost complex structure on S^6 that is not integrable. The existence of an integrable almost complex structure on S^6 is an open problem.

Let us introduce an equivalent condition for integrability.

Definition 8. Let (M, J) be an almost complex manifold. The Nijenhuis tensor $N_J \in \Gamma(M, T^{(1,2)}M)$ is defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

Lemma 9. *The Nijenhuis tensor satisfies the following properties:*

(i) it is tensorial ($C^\infty(M)$-linear) and antisymmetric;
(ii) $N_J(X, JX) = 0$;
(iii) J is integrable if and only if $N_J = 0$.

Proof. We prove (i). Antisymmetry is clear; hence to show that $N_J(fX, gY) = fgN_J(X, Y)$ it suffices to show that $N_J(fX, Y) = fN_J(X, Y)$. Notice first that

$$[fX, Y] = f[X, Y] - Y(f)X,$$

from which we obtain

$$\begin{aligned} N_J(fX, Y) &= [JfX, JY] - J[JfX, Y] - J[fX, JY] - [fX, Y] \\ &= fN_J(X, Y) - (JY)(f) \cdot JX + Y(f)J^2X + (JY)(f) \cdot JX + Y(f)X \\ &= fN_J(X, Y), \end{aligned}$$

as desired. Statement (ii) is clear from the antisymmetry of the Lie bracket.

To prove (iii) we first claim that any section X of $T^{1,0}M$ can be written as $X = A - iJA$ for A a real valued vector field. This follows immediately from the condition that $JX = iX$. Now if $X = A - iJA, Y = B - iJB$ are sections of $T^{1,0}M$, the Lie bracket

$$[A - iJA, B - iJB] = [A, B] - [JA, JB] - i([JA, B] + [A, JB])$$

is of the form $C - iJC$ if and only if

$$J([A, B] - [JA, JB]) = [JA, B] + [A, JB].$$

This is equivalent to $N_J(A, B) = 0$. As A, B were arbitrary, integrability is equivalent to the vanishing of the Nijenhuis tensor. □

Proposition 10. *A two-dimensional smooth manifold admits the structure of a Riemann surface (a one-dimensional complex manifold) if and only it is oriented.*

Lemma 11. *Let (M, J) be an almost complex manifold with $\dim_{\mathbb{R}} M = 2$. Then $N_J = 0$.*

Proof. As this is a local statement, we fix $p \in M$ and compute $N_J(X, Y)(p)$ for two real vector fields X and Y. The key point is that $\{X_p, JX_p\}$ (for X nonvanishing at p) furnishes a basis for the two-dimensional real vector space T_pM: if $X = aJX$ for some $a \in \mathbb{R}$ then $JX = -aX = -a^2 JX$ whence $X = -a^2 X$, a contradiction. Hence

$$N_J(X, Y)(p) = N_J(X, \alpha X + \beta JX) = 0.$$

□

Proof of Proposition 10. A Riemann surface has an orientation induced by the complex structure. The tangent space at any point p has a canonically ordered $\mathbb{R}$-basis $\{1, i\}$, and any holomorphic transition function f preserves this orientation as the pushforward

$$f_* = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

has positive determinant (via the Cauchy-Riemann equations).

Conversely, by Lemma 11 and Theorem 5 the integrability of almost complex structures on two-dimensional smooth manifolds is automatic. Hence it suffices to show that there always exists an almost complex structure on an oriented smooth manifold M.

Fix a metric g on M (whose existence is guaranteed by partitions of unity). Locally, on a coordinate patch U, we can fix an oriented orthonormal frame $\{e_1, e_2\}$ and define the almost complex structure to rotate by $+\pi/2$:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Does this local definition globalize? Suppose we have another coordinate chart V with $U \cap V \neq \varnothing$ and an oriented orthonormal frame $\{f_1, f_2\}$, with J defined on V as above. If ϕ is the transition function from U to V then the almost complex structure agrees on the overlap if $J_U = \phi_*^{-1} J_V \phi_*$, i.e. the matrix above is stable under conjugation by ϕ_*. As M is oriented and Riemannian, $\phi_* \in \mathrm{SO}(2)$, which commutes with rotations.

□