NOTES FOR MATH 520-2

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Contents

1. April 4, 2017

1.1. **Chern-Weil theory.** Recall that the Chern character of a vector bundle with connection (E, ∇) was defined

$$\operatorname{ch}(E, \nabla) = \operatorname{tr}\left(\exp(-\frac{1}{2\pi i}F)\right),$$

where F is curvature of ∇. We need to show two things: that this form is closed and that its cohomology class is independent of the connection ∇. To do the first we will use the following formula; if $a \in \Omega^k(M, \operatorname{End} E)$ then $\operatorname{tr} a \in \Omega^k(M)$ and

$$d\operatorname{tr}(a) = \operatorname{tr}(\nabla^{\operatorname{End} E} a) = \operatorname{tr}[\nabla, a].$$

where we are using the induced connection $\nabla^{\operatorname{End} E} = [\nabla, \cdot]$ on $\operatorname{End} E$. More generally, using the cylicity of the trace,

$$\operatorname{tr}(F^n) = \operatorname{tr}\nabla(F^n) = \sum_{i=1}^{n} \operatorname{tr}(\nabla F \cdot F^{n-1}) = 0,$$

where in the last step we have used the Bianchi identity $\nabla F = 0$. The Bianchi identity is proved straightforwardly:

$$\nabla F = [\nabla, \nabla^2] = 0,$$

via the Jacobi identity. More generally, if α is a closed even degree differential form and ϕ is analytic function, then $\phi(\alpha)$ (defined using the series expansion for ϕ) is a closed even degree form, as

$$d(\alpha^n) = n d\alpha \cdot \alpha^{n-1} = 0.$$

Let us now show that the cohomology class defined by $\operatorname{ch}(E, \nabla)$ is independent of the connection. Take a smooth family of connections ∇^t on E for $t \in [0, 1]$.

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Consider the projection $\pi : X \times [0,1] \to X$ and the pullback bundle $\pi^* E$. We define a connection ∇ on $\pi^* E$ as

$$\nabla s(x,t) = \nabla^t s(x,t) + dt \frac{\partial}{\partial t} s(x,t).$$

Let us compute the curvature of this connection:

$$\nabla^2 = F^t + dt \dot{\nabla}^t,$$

where F^t is the curvature of ∇^t. We already know, by closedness, that

$$d_X \operatorname{tr}(F^n) + dt \frac{\partial}{\partial t} \operatorname{tr}(F^n) = 0$$

As before, it is easy to see that

$$\operatorname{tr}(F^n) = \operatorname{tr}((F^t)^n) + n dt \operatorname{tr}(\dot{\nabla}^t (F^t)^{n-1})$$

Let us extract the coefficient of dt in $(d_X + dt\partial/\partial t) \operatorname{tr}(F^n) = 0$:

$$\frac{\partial}{\partial t} \operatorname{tr}(F^t)^n - nd \operatorname{tr}(\dot{\nabla}^t (F^t)^{n-1}) = 0.$$

Finally, integrating from 0 to 1, we find that the difference

$$\operatorname{tr}(F^1)^n - \operatorname{tr}(F^0)^n = nd \left(\int_0^1 \operatorname{tr}(\dot{\nabla}^t (F^t)^{n-1}) \right)$$

is exact. Applying this to the Chern character, we find that

$$\operatorname{ch}(E, \nabla^1) - \operatorname{ch}(E, \nabla^0) = -\frac{1}{2\pi i} d \int_0^1 \operatorname{tr}(\dot{\nabla}_t e^{-F^t/2\pi}) dt.$$

Often we are interested in families of connections of the form $\nabla^t = \nabla^0 + tA$ (in physics we would call ∇^0 the background field). Applying the above formula, we see that if $\nabla = \nabla^1$,

$$\operatorname{ch}_2(E, \nabla) = \operatorname{ch}(E, \nabla^0) + (-2\pi i)^2 d \operatorname{tr}(\frac{1}{2} A dA + \frac{1}{3} A^3),$$

where the form in the trace is often called the Chern-Simons form. In the special case where ∇ is a gauge transformation of a flat connection, the Chern-Simons form is closed.

It now remains to prove naturality, normalization, and the product formula. Naturality is easy – given a map $f : X \to Y$, a connection on $E \to Y$ pulls back to a connection $f^*\nabla$ on f^*E. Locally, the pullback connection is precisely given by the pullback of the connection one-form. Thus the curvature of the pullback connection is just the pullback of the curvature of ∇. This shows naturality on the nose for the pullback connection, which is enough on cohomology, by the independence proven above.

Normalization we already proved last class, so it remains to prove the product formula,

$$\mathrm{c}(E \oplus F) = \mathrm{c}(E)\mathrm{c}(F),$$

or equivalently (over $\mathbb{Q}$),

$$\operatorname{ch}(E \oplus F) = \operatorname{ch}(E) + \operatorname{ch}(F).$$

As above, we should choose our connection cleverly: the sum of the connections on E and F. The curvature, as an endomorphism, will now be block diagonal, but the trace of powers of block diagonal matrices is the sum of the traces of the powers of those block matrices. One can also prove it directly for $\mathrm{c}(E \oplus F)$ using the determinant.

More generally, if $E \subset G$ is a subbundle (i.e. a map $E \to G$ of bundles that is fiberwise injective) then

$$\mathrm{c}(G) = \mathrm{c}(E)\mathrm{c}(G/E)$$

and

$$\mathrm{ch}(G) = \mathrm{ch}(E) + \mathrm{ch}(G/E).$$

This is proved the same way: realize G/E as $E^{\perp}$ for some Hermitian metric on G. In particular, let P be the endomorphism of G given by the orthogonal projection to the image of E in G. Then by linear algebra (Gram-Schmidt) we have $P^{\perp} = I - P$ which is an orthogonal projection to $E^{\perp}$. Now we introduce the Grassmann connection

$$\nabla^P = P\nabla P + P^{\perp}\nabla P^{\perp},$$

which has the property that it preserves the decomposition $G = E \oplus E^{\perp}$. Explicitly, one can check that

$$\nabla^P s = \frac{1}{2}(2P - I)\nabla(2P - I).$$

Now since the curvature is the sum of the curvatures of the two terms in the Grassmann connection, we obtain the product formula.

1.2. **Superconnections.** Superconnections were introduced by Quillen in the 80's, as a generalization for connections that have much the same properties. Recall that the adjective "super" indicates a $\mathbb{Z}/2\mathbb{Z}$-grading. In particular, a superbundle is a bundle $E = E^+ \oplus E^-$.

Remark 1. Topological K-theory of a locally compact space X is generated by triples $(E^0, E^1, f : E^0 \to E^1)$ where f – the clutching function – is invertible outside a compact set. This is perhaps the starting point for superconnections.

Notice that the bundle $\mathrm{End}\, E$ inherits a natural superbundle structure from E where the even endomorphisms are those preserving the parity on E. Consider the algebra $\Omega(X, \mathrm{End}\, E)$ as a superalgebra with respect to the total degree, i.e.

$$\Omega^{\pm}(X, \mathrm{End}\, E) = \sum_i \Omega^{2i}(X, \mathrm{End}^{\pm} E) \oplus \sum_i \Omega^{2i+1}(X, \mathrm{End}^{\mp} E).$$

Notice that $\Omega(X, E)$ then is a supermodule over $\Omega(M, \mathrm{End}\, E)$.

Definition 2. A superconnection $\mathbb{A}$ on a superbundle $E \to X$ is an odd operator on $\Omega(X, E)$ satisfying the Leibniz rule

$$\mathbb{A}(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^k \alpha \wedge \mathbb{A}\omega,$$

where $\alpha \in \Omega^k(X, E)$ and $\omega \in \Omega(X, E)$.

Once you define these guys, you find that they're everywhere! Especially in algebraic geometry.

Consider $s \in \Gamma(X, E)$. Then

$$\mathbb{A}(\alpha s) = d\alpha \cdot s + (-1)^{|\alpha|}\alpha \wedge \mathbb{A}s.$$

Decomposing $\mathbb{A}s$ into differential forms of different degrees, $\sum \mathbb{A}_{[k]}s$, we see that if $k \neq 1$, $\mathbb{A}_{[k]}$ is an k-form valued endomorphism, and if $k = 1$ $\mathbb{A}_{[1]}$ is a block diagonal connection on E. We conclude that

$$\mathbb{A} = \mathbb{A}_{[1]} + \omega$$

where $\omega \in \Omega^-(X, \operatorname{End} E)$.

The curvature of a superconnection is $\mathbb{A}^2 \in \Omega^+(X, \operatorname{End} E)$, as one can compute. Notice however that the curvature is now inhomogeneous in form degree. We can now define the Chern character of a superconnection as

$$\operatorname{ch}(\mathbb{A}) = \operatorname{str}(e^{-\mathbb{A}^2}),$$

where $\operatorname{str} : \Omega(X, \operatorname{End} E) \to \Omega(X)$ is the supertrace (signed according to parity). We will now go on to show that this is in fact independent of the superconnection.

Of course, the point is that now we have something that looks much like the supertrace of a heat kernel.

2. April 11, 2017

Recall that when we had a connection on E and P was projection onto a subbundle (i.e. $P^2 = P$) then we had the Grassmann connection $\tilde{\nabla} = P\nabla P + P^\perp \nabla P^\perp = \nabla + (2P - I)\nabla P$. Notice that the covariant derivative of P with respect to this connection is zero. It is easy to see that $P(\nabla P) = (\nabla P)P$ whence the curvature of the Grassmann connection is given

$$\tilde{\nabla}^2 = P\nabla^2 P + P^\perp \nabla^2 P^\perp + (\nabla P)^2.$$

Let's return now to the case of our rescaled superconnection (where D is self-adjoint)

$$\mathbb{A} = t^{1/2} D + \nabla + t^{-1/2}\mathbb{A}_{[2]} + \cdots .$$

Let us assume now that $\ker D$ is a subbundle of E, which is naturally a superbundle, $\ker D = \ker D_+ \oplus \ker D_-$. The idea is that $\exp(-\mathbb{A}_t^2)$ has a "factor" that looks like $\exp(-tD^2)$ whence, intuitively, $\exp(-\mathbb{A}_t^2)$ converges as $t \to \infty$ to an element of $\Omega^\bullet(M, \operatorname{End} \ker D)$.

Let P be the projection onto $\ker D$ and consider the Grassmann connection ∇^P and consider

$$\operatorname{str}(Pe^{-(\nabla^P)^2}) = \operatorname{str}|_{\ker D} e^{-(\nabla^P)^2}.$$

It is a theorem of Berline-Vergne (1987) that

$$\operatorname{str}|_{\ker D} e^{-(\nabla^P)^2} = \lim_{t\to\infty} \operatorname{str}(e^{-t\mathbb{A}_t^2}).$$

and furthermore that as $\to \infty$ the error is $O(t^{-1/2})$. For the proof, see BGV section 9.1.

Notice that if D is invertible then $\ker D = 0$ whence the supertrace exponentially decays to zero as $t \to \infty$. It turns out that we obtain an asymptotic expansion in powers of $t^{-1/2}$ as $t \to 0$. In particular $f(t) = O(e^{-\varepsilon t})$ for $\varepsilon > 0$ as $t \to \infty$ and asymptotically $\sim \sum_{k \le N} a_k t^{-k/2}$. We define the Mellin transform of a function

$$(Mf)(s) = \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^{s-1} dt.$$

Recall that $1/\Gamma(s)$ is an entire function:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

where $\Re s > 0$, but can be extended to the left meromorphically using the identity $\Gamma(s+1) = s\Gamma(s)$. Next notice that

$$M \sum a_i e^{-\lambda_i t} = \sum a_i \lambda_i^{-s}.$$

Let us check whether we can apply this transform to our case of f. We split the integral into two pieces from $0 \to 1$ and $1 \to \infty$. The second term ends up being a product of two entire functions. For the first, we will need the asymptotic expansion $f(t) = \sum_{k=N}^{M} a_k t^{k/2} + O(t^{(M+1)/2})$ as $t \to 0$. The big O term if $(M-1)/2 + \mathrm{Re}\ s - 1 > -1$ this defines a holomorphic function of s when $\mathrm{Re}\ s < -(1+M)/2$. The sum term..?

Define $\mathrm{Lim}_{t\to 0} f(t)$ as the coefficient of t^0 in the asymptotic expansion of $f(t)$ as $t \to 0$. What we showed in the paragraph above is that $Mf(0) = \mathrm{Lim}_{t\to 0} f(t)$. We can use this machinery to renormalize products of eigenvalues of an operator.

In the case where we have a superconnection on a zero-dimensional manifold, say a point, $\mathbb{A} = D$ and $\mathbb{A}_t^2 = tD^2$. Then $\mathrm{str}(e^{-tD^2}) = \sum_{D_- D_+} e^{-\lambda t} - \sum_{D_+ D_-} e^{-\lambda t}$, where the sums are over the eigenvalues over the denoted operators. If we now take the Mellin transform,

$$\begin{aligned} M\,\mathrm{str}(e^{-tD^2})(0) &= \lim_{t\to 0} \mathrm{str}(e^{-tD^2}) \\ &= \mathrm{rk}\, E^+ - \mathrm{rk}\, E^- = 0. \end{aligned}$$

One can similarly compute the derivative of the Mellin transform at 0 and we get zero as well. All of this is a finite-dimensional model of what we will eventually be discussing

We now introduce Dirac operators, and to do this we will need the language of Clifford algebras. Given a vector space V and a bilinear form $(-,-)$ the Clifford algebra $C(V)$ is the algebra generated by $\{c(v) \mid v \in V\}$ subject to the relations

(a) $c(v+w) = c(v) + c(w)$;
(b) $c(tv) = tc(v)$;
(c) $c(v)c(w) + c(w)c(v) = -2(v,w)$.

Notice that if the quadratic form is zero, then we simply obtain the exterior algebra, so we might think of the Clifford algebra as a quantization of the exterior algebra. (In particular there is some connection with the deformation quantization of the ring of functions over a supermanifold, with repsect to a Poisson structure given by the quadratic form).

Theorem 3. *The Clifford algebra* $C(V)$ *exists and has finite dimension* $\dim C(V) = 2^{\dim V}$.

In fact the Clifford algebra is a superalgebra $C^+(V) \oplus C^-(V)$ (where these subspaces are of the same dimension unless $\dim V = 0$). Notice that $c(v) \in C^-(V)$ whence we can rewrite the last axiom of Clifford algebras as a graded commutator $[c(v), c(w)] = -2(v,w)$.

Exercise: check that the span of $\{c_i c_j \mid i < j\}$ forms the Lie algebra $\mathrm{Lie}\, so(V) \subset C^+(V)$.