INDEX THEORY. March 28^{th}, 2017.

· Bott–Tu & Gunning–Rossi

Weil's outlook on de Rham theorem. We have a resolution of the constant sheaf

$$0 \to \mathbb{C} \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots$$

since locally, (for $k>0$) closed forms are exact. This comes from the Poincaré Lemma on star-shaped subsets (recall $tU \subset U$ if $0 \le t < 1$) which gives $h: \Omega^i(U) \to \Omega^{i-1}(U)$ for $i>0$ $\quad dh+hd = 1$

$$\varepsilon: \Omega^0(U) \to \mathbb{C} \quad \text{evaluation at } 0.$$

Now we take a good cover $\{U_\alpha\}_{\alpha\in I}$ $= \mathcal{U}$ that is

- locally finite
- $U_{\alpha_0\ldots\alpha_k} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ "completeness"
  - ↳ either empty or diffeomorphic to a ball.

Partition of unity subordinate to U_α:

$$\eta_\alpha \in C_c^\infty(U_\alpha) \qquad \sum \eta_\alpha = 1 \qquad \text{(don't require positive)}$$

Recall that the Čech k-cochains for a sheaf A

$$\check{C}^k(\mathcal{U}, A) = \prod_{\alpha_0\ldots\alpha_k} A(U_{\alpha_0\ldots\alpha_k}) \quad , \quad \delta: \check{C}^k \to \check{C}^{k+1} \quad \delta^2 = 0.$$

$$(\delta c)_{\alpha_0\ldots\alpha_{k+1}} = \sum_{i=0}^{k+1} (-1)^i c_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{k+1}}\big|_{U_{\alpha_0\ldots\alpha_{k+1}}}$$

$$\check{Z}^k(\mathcal{U}, A) = \ker(\delta: \check{C}^k \to \check{C}^{k+1})$$

$$\check{B}^k(\mathcal{U}, A) = \mathrm{im}(\delta: \check{C}^{k-1} \to \check{C}^k)$$

$$\check{H}^k(\mathcal{U}, A) = \check{Z}^k / \check{B}^k$$

If $H^i(U_{\alpha_0\ldots\alpha_k}, A) = 0$ $i>0$ for all $\alpha_0,\ldots,\alpha_k$. then

$$\check{H}^k(\mathcal{U}, A) \cong H^k(X, A).$$

} Serre / Grothendieck

Example. If A is a constant sheaf and $\mathcal{U}$ is a good cover.

Example. (Fine sheaves). Let $E \to X$ be a smooth v.b., $\mathcal{E}$ be the sheaf of smooth sections. Given a good cover, $\check{H}^k(\mathcal{U}, \mathcal{E}) = 0$ if $i>0$.

Take $\omega \in \check{C}^k(\mathcal{U}, \mathcal{E})$. Define $s: \check{C}^k(\mathcal{U}, \mathcal{E}) \to \check{C}^{k-1}(\mathcal{U}, \mathcal{E})$

$$(s\omega)_{\alpha_0\ldots\alpha_{k-1}} = \sum \eta_\alpha \, \omega_{\alpha_0,\ldots,\alpha_{k-1}\alpha}$$

"contracting htpy"

We claim that s is a ~~nullhomotopy~~ away from $k=0$ that is.

<u>Exercise.</u> $(\delta s + s\delta)\omega = \omega$ for $k>0$

~~...? $s\delta\omega = \omega$ $k=0$~~

Weil's idea is to construct a double complex

$C^{k,l} = \check{C}^k(\mathcal{U}, \Omega^l)$ $k,l \geq 0$.

Define $\mathrm{Tot}(C)^k = \bigoplus_{l=0}^{k} C^{k-l,l}$ with $\delta + (-1)^k d : C^{kl} \to C^{k+1,l} \oplus C^{k,l+1}$.

He constructs a quasi-iso

$\mathrm{Tot}(C)^k \longrightarrow C^{0,k}$

We will use $\tilde{s} = \sum_{m=0}^{\infty} s(-ds)^m : \mathrm{Tot}(C)^k \to \mathrm{Tot}(C)^{k-1}$ which gives a homotopy to the de Rham complex. In particular, $\tilde{\eta} = \sum_{m=0}^{\infty} \eta(-ds)^m$ projects from $\mathrm{Tot}(C)$ to $C^{0,\bullet}$. Let $c \in \check{C}^k(\mathcal{U}, \mathbb{C})$, consider it as $c \in \check{C}^k(\mathcal{U}, \Omega^0)$. Apply $\tilde{\eta}$ — we get $c \mapsto \sum c_{\alpha_0 \cdots \alpha_k} \eta_{\alpha_0} d\eta_{\alpha_1} \cdots d\eta_{\alpha_k}$. Should check that the Čech δ goes to the de Rham d.

$$\delta c \mapsto \sum_{\alpha_0 \cdots \alpha_{k+1}} (\delta c)_{\alpha_0 \cdots \alpha_{k+1}} \eta_{\alpha_0} d\eta_{\alpha_1} \cdots d\eta_{\alpha_{k+1}}$$
$$= \sum_{i=0}^{k+1} (-1)^i \sum_{\alpha_0 \cdots \alpha_{k+1}} c_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{k+1}} \eta_{\alpha_0} d\eta_1 \cdots d\eta_{\alpha_{k+1}}$$

and use $\sum d\eta_\alpha = 0$.

Of course, now one should show that $\mathrm{Tot}(C)$ also projects down to the Čech side.

$\rightsquigarrow$ characteristic classes.

$\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z} \subset \mathbb{C}$. Have $0 \to \mathbb{Z}(1) \to \mathbb{C} \to GL(1,\mathbb{C}) \to 0$ and then an exact sequence of sheaves

nowhere vanishing fns

$$0 \to \mathbb{Z}(1) \to C^\infty \xrightarrow{\exp} \underline{GL(1,\mathbb{C})} \to 0$$

Notice that $\check{Z}^1(\mathcal{U}, \underline{GL(1,\mathbb{C})})$ is the data $(g_{\alpha\beta})_{\alpha,\beta \in I}$

$$\begin{cases} g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \\ \cancel{g_{\alpha\alpha} = 1} \end{cases}$$

~~not true but can find something cohomologous~~

$\Rightarrow$ implies $g_{\alpha\alpha} = 1$, $g_{\alpha\beta} g_{\beta\alpha} = 1$.

Say we have a SES of sheaves, $0 \to A \to B \to C \to 0$. Then get LES

$$\check{H}^k(\mathcal{U},A) \to \check{H}^k(\mathcal{U},B) \to \check{H}^k(\mathcal{U},C)$$

$$\xrightarrow{\partial} \check{H}^{k+1}(\mathcal{U},A) \to \cdots$$

Exercise

Let's check explicitly what this boundary map looks like

$c \in \check{Z}^k(\mathcal{U},C)$, lift c to $\tilde{c} \in \check{C}^k(\mathcal{U},B)$. Define $\partial c = \delta\tilde{c}$ and check that this makes sense (first need exactness at $\check{H}^\cdot(\mathcal{U},B)$.)

From the exponential sequence get

$$\underset{0}{\underset{\parallel}{\check{H}^1(\mathcal{U},C^\infty)}} \to \check{H}^1(\mathcal{U},\underline{GL(1,\mathbb{C})}) \xrightarrow[c_1]{\sim} \check{H}^2(\mathcal{U},\mathbb{Z}(1)) \to \underset{0}{\underset{\parallel}{\check{H}^2(\mathcal{U},C^\infty)}}$$

$\longrightarrow$ by above $\leftarrow$

This isomorphism is the first Chern class. Recall $\{g_{\alpha\beta}\}$ is a line bundle

$L = (\coprod_\alpha U_\alpha \times \mathbb{C})/\sim$ equiv. relation coming from $g_{\alpha\beta}$.

We fix some branch of log and have

$c_{\alpha\beta\gamma} = \log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\beta\gamma}$.

$c_1(L) =$ 2-cocycle given by mult. by $2\pi i$.

Let E be a complex vector bundle.

$$c(E) = \sum_{i=0}^\infty c_i(E) \in \prod_{i=0}^\infty H^{2i}(X,\mathbb{Z})$$

$c_i \in H^{2i}(X;\mathbb{Z})$.

<u>Axiomatically</u>:

1. naturality: given $f: X \to Y$, f^*E is over X now, then $c(f^*E) = f^*(c(E))$.
2. if E is a line bundle then $c(L) = 1 + c_1(L)$.
3. $E \oplus F$ the Whitney sum (pullback along diagonal of $E \times F$) $c(E\oplus F) = c(E)c(F)$.

<u>Splitting principle</u>. $c_k(E) = \sum_{1 \le i_1 < \cdots < i_k \le r} c_1(L_{i_1}) \cdots c_1(L_{i_k})$.

Proof. We construct a fiber bundle F over X such that

i) $H^\bullet(F;\mathbb{Z})$ is a free module over $H^\bullet(X;\mathbb{Z})$.

ii) the pullback of E to F decomposes into a direct sum.

How to do this? Consider $\mathbb{P}(E)$ which is locally $\cong U\times \mathbb{C}P^{r-1}$. We have a taut. line bundle $L\to\mathbb{P}(E)$. If $\pi:\mathbb{P}(E)\to X$,

$$0\to \underset{\text{subbundle}}{L}\to \pi^*E\to \underset{\text{rank } r-1}{\pi^*E/L} \qquad x=c_1(L)\in H^2(\mathbb{P}(E),\mathbb{Z}),$$

and by Leray–Hirsh, $H^\bullet(\mathbb{P}(E),\mathbb{Z})$ is a free module spanned by $\langle 1,x,\dots,x^{r-1}\rangle$

A genus, more generally is

$$\phi(E)\in \prod_{i=0}^{\infty} H^{2i}(X,\mathbb{Q})$$

satisfying 1) naturality

2) $$\phi(E\oplus F)=\phi(E)\phi(F)$$

$$\Rightarrow \underset{\text{line bundle}}{\phi(L)}=f(c_1(L)) \qquad f(x) \text{ some power series.}$$

We've seen c_1 already, where $f(x)=1+x$. The next is the Todd genus

$$Td(E)\ ,\quad f(x)=x/1-e^{-x}.$$

Exercise. Read about the Bernoulli numbers,

$$B_0=1,\quad B_1=-1/2,\quad B_{2l+1}=0 \text{ for } l>0.$$

Exercise. If E is rank 2, compute $Td(E)$ in terms of $c_1(E)$, $c_2(E)$.