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(X,g) oriented Riemannian. $\dim X = m = 2n$.

The Levi-Civita connection on any tensor bundle is induced in the obvious way.

$$\nabla_\mu (dx^\nu) = -\Gamma_{\mu}{}^{\nu}{}_{\lambda}\, dx^\lambda \qquad \nabla_\mu \partial_\nu = \Gamma_{\mu\nu}^{\lambda}\partial_\lambda \qquad (\nabla \curvearrowright dx^\mu(\partial_\nu) = \delta^\mu_\nu)$$

Torsion-free: $\nabla_X Y - \nabla_Y X = [X,Y]$

If θ^a is an ON frame, $\nabla_\mu \theta^a = \omega_\mu{}^a{}_b\,\theta^b$. $\quad (g = \sum \theta^a \otimes \theta^a)$.

We obtain:

$$(\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = X(g(X,Y)).$$

We have a frame bundle $SO(X)$ a $SO(m)$-principal bundle w/ fiber

$SO(X)_x = \{$orientation preserving isometries b/w $\mathbb{R}^m$ and $T_xX\}$

Then if E is an $SO(m)$-module then get an associated bundle

$$SO(X) \times_{SO(m)} E.$$

Now suppose $E = E_+ \oplus E_-$ Hermitian superbundle. Say E is a $C(X) = C(T^*X)$-module.

i.e. E_x is a $C(T^*X)$-module (recall $C(T^*X)$ is a bundle of superalgebras). In particular $c(\alpha): \Gamma(E_\pm) \longrightarrow \Gamma(E_\mp)$

Is X complex mfld here?

Theorem. $\mathrm{End}(E) = C(X) \otimes \mathrm{End}_{C(X)}(E)$ $\qquad \mathrm{End}_{C(X)}(E) = \{A : E \to E \mid c(\alpha)A \mp Ac(\alpha) = 0\}$.

locally $C(X)$ is simple — endomorphisms of the spinor bundle S_x.

Proof. We have

$$E_x \cong S_x \otimes \mathrm{Hom}_{C(T_x^*X)}(S_x, E_x). \qquad \left(\text{this comes from taking invariants } (\mathrm{End}\, S_x \otimes E_x)^{C(T_x^*X)}\right).$$

where $C(T_x^*X)$ acts only in the first factor.

Spin$_c$ is related to Atiyah's Thom iso.

(Spin$_c$-manifolds: $\exists$ global bundle $S \to X$ such that $\mathrm{End}(S) \cong C(X)$, equivalently, there is a Spin$_c$-principal bundle $P \to X$ such that $P \times_{\mathrm{Spin}_c} SO(m) \cong SO(X)$.)

The result now follows by applying $\mathrm{End}(-)$.

Last time, Laplace-Beltrami:

$$c(\alpha) = \varepsilon(\alpha) - \varepsilon^*(\alpha) \qquad \text{for } E = \Lambda^\bullet T^*X.$$

$$\hat{c}(\alpha) = \varepsilon(\alpha) + \varepsilon^*(\alpha) \qquad \text{can check } [c(\alpha), \hat{c}(\alpha)] = 0.$$

Thus: $\Lambda^\bullet T^*X$ spinors for $C(X,g) \otimes C(X,-g) = C(T^*X \oplus T^*X, g \oplus (-g))$.

Connection on Clifford module E:

i) preserves $E = E_+ \oplus E_-$ "Clifford connections"

ii) Hermitian inner product

iii) $\nabla_X(c(\alpha)s) = c(\nabla_X\alpha)s + c(\alpha)\nabla_X s.$ $\alpha \in \Omega^1(X).$ i.e. $[\nabla_X, c(\alpha)] = c(\nabla_X\alpha).$

Say we have a local frame $\{s^i\}$ of E, then (and ON frame θ^a for T^*X)

$$\nabla_\mu s = \sum_{1 \le a_1 < \cdots \le m} c^{a_1} \cdots c^{a_k}\, \omega_{\mu a_1 \cdots a_k} s.$$

locally-defined section of $\mathrm{End}_{C(X)}(E)$.

If we write (iii) in coordinates,

$$\nabla_\mu(c^a s) = \omega_\mu{}^a{}_b c^b s + c^a \nabla_\mu s$$

one finds that in the case of a Clifford connection,

$$\nabla_\mu s = \tfrac{1}{2}\omega_{\mu ab} c^a c^b + \omega_\mu. \qquad (\text{here } \textstyle\sum_{a<b})$$

In the Laplace-Beltrami case: $-\omega_{\mu a}{}^b \varepsilon^a \varepsilon^*_b = \omega_{\mu ab}\varepsilon^a \varepsilon^{*b} = \frac{1}{4}\omega_{\mu ab}(c^a + \hat{c}^a)(-c^b + \hat{c}^b).$

$$= \sum_{a,b} -\tfrac{1}{4}\omega_{\mu ab} c^a c^b + \tfrac{1}{4}\omega_{\mu ab}\hat{c}^a\hat{c}^b.$$

Now we obtain a Dirac operator $D = c^\mu \nabla_\mu$.

<u>Theorem</u>. D is formally self-adjoint,

$$\int (Ds, s')\, dVol = \int (s, Ds')\, dVol.$$

~~~ 5 min break ~~~

Goal: Lichnerowicz formula. (or a generalization thereof).

Consider $D^2 = c^\mu \nabla_\mu c^\nu \nabla_\nu = c^\mu c^\nu \nabla_\mu \nabla_\nu - \underbrace{c^\mu c^\lambda \Gamma_\mu{}^\nu{}_\lambda \nabla_\nu}_{g^{\mu\lambda}\Gamma^\nu_{\mu\lambda}\nabla_\nu}$

The first term,

$$c^\mu c^\nu \nabla_\mu \nabla_\nu = \tfrac{1}{4}\underbrace{(c^\mu c^\nu + c^\nu c^\mu)}_{-2g^{\mu\nu}}(\nabla_\mu\nabla_\nu + \nabla_\nu\nabla_\mu) + \tfrac{1}{4}(c^\mu c^\nu - c^\nu c^\mu)\underbrace{(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)}_{\text{curvature of the Clifford connection.}}.$$

Curvature of the Clifford connection: $F_{\mu\nu} = \frac{1}{2}\sum_{a,b} R_{\mu\nu ab} c^a c^b + F^{E/S}_{\mu\nu}$

End up w/ three terms in D^2 (with some work)

$$-\tfrac{1}{4}\sum_{abcd} R_{abcd} c^a c^b c^c c^d + \tfrac{1}{2}\sum_{\mu\nu} F^{E/S}_{\mu\nu} c^\mu c^\nu + \nabla^*\nabla.$$
~~~

Looking at the first term, and doing some VOODOO:

<u>Theorem</u> (Lichnerowicz). $D^2 = \nabla^*\nabla + \frac{R}{4} + c(F^{E/S})$.

In the Spin_c case, $F^{E/S}$ is an imaginary closed 2-form.

If we have a spin-mfld then $D^2 = \nabla^*\nabla + R/4$. If moreover, ~~M~~ X is cpt,

$$|Ds|^2 = (s, D^2 s) = \underbrace{(s, \nabla^*\nabla s)}_{= |\nabla s|^2} + \frac{1}{4}(s, Rs) \geq \frac{1}{4}(s, Rs) \qquad \text{(imitation of Kodaira)}$$

Now we see that if X is a compact spin-manifold of positive scalar curvature ($R>0$). Then D is invertible, i.e. $\ker(D) = 0$.

<u>Remark</u>. Atiyah-Singer thm says:

$$\mathrm{ind}(D) = \dim(\ker D_+) - \dim(\ker D_-) = \int \hat{A}(X)$$

where this is the $\hat{A}$-genus: for line bundles

$$\hat{A}(L) = \frac{c_1(L)/2}{\sinh c_1(L)/2}$$

Can generalize the above to Clifford superconnections,

$\mathbb{A}$ superconnection on E $: \Omega^\cdot(X,E) \to \Omega^\cdot(X,E)$

satisfying $[\mathbb{A}, c(\alpha)] = c(\nabla\alpha)$, and some compatibility w/ metric.

Indeed, one finds $\mathbb{A}_{[1]}$ is a Clifford connection and $\mathbb{A}_{[k]}$ for $k \neq 1$ $\in \Omega^k(X, \mathrm{End}_{C(X)} E)$.

There is a natural isomorphism of $\mathbb{Z}_2$-graded vector spaces

$$\Omega^\cdot(X) \cong \Gamma(X, C(X)) \qquad \text{"some sort of PBW"}$$

$$\Omega^\cdot(X, \mathrm{End}_{C(X)} E) \cong \Gamma(X, C(X) \otimes \mathrm{End}_{C(X)} E) \cong \Gamma(X, \mathrm{End}\, E).$$

Now we can

$$\Gamma(X,E) \xrightarrow{\mathbb{A}} \Omega^\cdot(X,E) \xrightarrow{c} \Gamma(X, C(X) \otimes E) \to \Gamma(X,E).$$

<u>Exercise</u>. (Weitzenböck identity).

Let Δ be a generalized Laplacian on some Hermitian bundle E, i.e.

$$\Delta = -g^{\mu\nu}\partial_\mu\partial_\nu + 1^{\text{st}} \text{ order}.$$

"Elliptic regularity"

↳ Solve the heat equation — given an initial value $s \in \Gamma(X, E)$, find $s_t \in \Gamma(X \times [0,\infty), \pi_1^* E)$ such that

i) $\frac{\partial s_t}{\partial t} + \Delta s_t = 0$

ii) $s_0 = s$.

Furthermore, there is a section

$$k_t \in \Gamma(X \times X \times (0,\infty), \pi_1^* E \otimes \pi_2^* E^*)$$

such that

$$s_t(x) = \int_X k_t(x,y)\, s(y)\, dVol.$$

$$\frac{\partial k_t}{\partial t} + \Delta_x k_t(x,y) = 0$$

$$\lim_{t\to 0} k_t(x,-)\, dVol = \delta_x \otimes \mathrm{Id}_{E_x}.$$

On $\mathbb{R}^m$, m even or odd

$$k_t(x,y) = (4\pi t)^{-m/2} \exp(-|x-y|^2/4t).$$

Can check: $\int k_t(x,y)\, dy = 1$ and $\frac{\partial k_t}{\partial t} - \sum \partial_\mu^2 k_t = 0.$

Let $L \subset \mathbb{R}^m$ be a lattice in $\mathbb{R}^m$.

$$\theta_t(x,y) = \sum_{z \in L} k_t(x, y+z)$$

Can show θ and its derivatives converge on the torus. (EXERCISE).

Moreover:

$$k_t \sim \sum_{l=0}^{\infty} t^{-\frac{m}{2}+l} \exp(-d(x,y)^2/4t)\, \underbrace{k_l(x,y)}_{\in\, \Gamma(\text{tub. nbhd } X \overset{\Delta}{\subset} X \times X,\ \pi_1^* E \otimes \pi_2^* E^*)}$$

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$k_t(x,y) \in \mathrm{Hom}(E_y, E_x)$. $\lim_{t\to 0} k_t(x,y) = \delta(x,y)\, \mathrm{Id}_{E_x}$.

Fix y, and express x in exponential coordinates around y.

$X \in T_yX$, $x = \exp_y X$. $\exp_y sX \to$ geodesic rays

The connection on E gives a parallel translation map along the geodesic $\exp_y(sx)$ which identifies the fibers of E at $\exp_y(sX)$ with the fiber E_y at $\exp_y 0 = y$.

If $x = \exp_y X$ is close enough to y, we get a C^∞-identification

$$\mathrm{Hom}(E_y, E_x) = \pi_1^* E \otimes \pi_2^* E^* \quad \text{with} \quad \mathrm{End}\, E_y = \pi_2^* \mathrm{End}\, E \quad (= \pi_2^* C(X) \otimes \mathrm{End}_{C(X)}(E)).$$

We may think of $k_t(x,y)$ as a t-dependent family of sections of the bundle $\pi_2^*(\Lambda^* T^* X \otimes \mathrm{End}_{C(X)} E)$.

If $x=y$, $k_t(x,x)$ is a t-dependent differential form with values in the bundle of algebras $\mathrm{End}_{C(X)} E$. Later we will see that

$$k_t(x,x) \sim (4\pi t)^{-m/2} \sum_{i=0}^{\infty} t^i k_i(x) \qquad \text{as } t \to 0.$$

Conjectured by: McKean-Singer

Proved by: Patodi (1976?)

__Theorem.__ $k_i \in \Omega^{\leq 2i}(X, \mathrm{End}_{C(X)} E)$. In particular,

$$\mathrm{Str}\; k_i(x,x) = 0 \quad \text{if} \quad i < m/2.$$

w.r.t. local factorization $E_x \cong S_x \otimes \mathrm{Hom}_{C(T^*x)}(S_x, E_x)$.

Moreover, there is a formula for the highest-order term

$$\sum_{i=0}^{m/2} (k_i)_{[2i]} = (4\pi)^{-n/2} \det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right) e^{-F^{E/S}}.$$

whence

coming from Str on spinors (c.f. 2 lectures ago).

$$\mathrm{Str}\; k_{m/2}(x,x) = (-2i)^{m/2} (4\pi)^{-m/2} \det{}^{1/2}(\cdots)\, \mathrm{Str}_{E/S}(e^{-F^{E/S}}).$$

"Local index theory"

Let's see how this implies the global index theorem.

Recall that we have a Dirac operator D and since X is compact, $\ker D = \ker D^2$. Whence

$$\ker D = \{\text{harmonic sections of } E\}.$$

D and D^2 have eigenvalue expansions — for D^2, we have $\lambda_i \geq 0$ with $\lim_{i\to\infty} \lambda_i = \infty$ and the eigenspace of D^2 w/ value λ_i, $\mathcal{H}_{\lambda_i} \subset L^2(X,E)$, is finite-dim'l subspace of $\Gamma(X,E)$. (elliptic regularity).

By formal self-adjointness, $\mathcal{H}_{\lambda_i}$ are orthogonal, and span a dense subspace of $L^2(X,E)$. Moreover, each $\mathcal{H}_\lambda$ decomposes, for $\lambda \neq 0$, into eigenspaces $\mathcal{H}_{\pm\sqrt{\lambda}, D}$ for D.

i.e. $K_t s$ solves heat eq. w/ limit $K_t s = s$ ($t \to$)

Suppose we consider the heat kernel with initial values in $\mathcal{H}_\lambda^r$, we see that the integral operator K_t is multiplication by $e^{-\lambda t}$. Pick any $t > 0$, K_t is a formally self-adjoint operator:

$$\frac{d}{dt}\left[(K_t s, s') - (s, K_t s') \right] = 0$$

and this fn $= 0$ at $t = 0$.

K_t has a continuous kernel (in fact C^∞) on a compact space with finite measure. Indeed, K_t is a compact self-adjoint operator.

Recall: compact operators

- closure of finite-rank operators, OR
- take bounded subsets of Hilbert space to precompact subsets.

Exercise. Show K_t has compact.

All the eigenvalues of K_t are strictly positive, so we define $\mathcal{H}_\lambda$ to be the eigenspace of K_t with eigenvalue $e^{-t\lambda}$. Now elliptic regularity is clear because k_t is smooth.

Next notice that operators with continuous kernel are trace-class.

Defn. A is trace-class if $\mathrm{Tr}(A^*A)^{1/2} < \infty$.

↑ sum of (positive) eigenvalues w/ multiplicities.

Our setting is easy because $(K_t)^* = K_t$, $K_t^* K_t = K_{2t}$, $(K_{2t})^{1/2} = K_t$.

Recall the Hilbert-Schmidt inner product

$$L^2(X \times X) \longrightarrow \text{(bded) operators on } L^2(X), \text{ compact}.$$

Exercise. $$\|a(x,y)\|^2_{L^2(X\times X)} = \mathrm{Tr}(A^*A).$$

In short, can establish that K_t is trace-class.

$$\mathrm{Str}(K_t) = \mathrm{Tr}_{\Gamma(X,E^+)} K_t - \mathrm{Tr}_{\Gamma(X,E^-)} K_t.$$

Key idea: $\mathrm{Str}\, K_t$ is independent of t.

$$\frac{d}{dt}\left(\mathrm{Str}\, e^{-tD^2}\right) = -\mathrm{Str}\left(D^2 e^{-tD^2}\right) = -\frac{1}{2}\mathrm{Str}\left([D, De^{-tD^2}]\right) = 0.$$

More precisely, (since D is unbd'd have to be careful), restrict to eigenspaces:

$$= -\frac{1}{2}\sum_\lambda \underset{\substack{\uparrow \\ \ker(D^2-\lambda)}}{\mathrm{Str}_{\mathcal{H}_\lambda}}\left([D_x, D_x e^{-tD_\lambda^2}]\right) = 0$$

Another justification:

$-\frac{1}{2}\mathrm{Str}\left([De^{-tD^2/2}, De^{-tD^2/2}]\right) = 0$ since these are Hilbert-Schmidt.

b/c $De^{-tD^2/2} = (De^{-tD^2/4})e^{-tD^2/4}$ ← (why?)

$|\mu e^{-t\mu^2/4}|$ is bounded.

~~McKean*-Singer's proof~~.

Sending $t \to \infty$,

$$\mathrm{Str}(e^{-tD^2}) = \lim_{t\to\infty} \mathrm{Str}(e^{-tD^2}) = \mathrm{Str}(\text{projection to kernel of } D^2)$$
$$= \dim\ker D_+ - \dim\ker D_-.$$
$$= \mathrm{ind}(D).$$

McKean-Singer's argument: for $\lambda > 0$.

$\mathcal{H}_{\lambda,+} \xrightarrow{D_+} \mathcal{H}_{\lambda,-}$ but D_+ has inverse $\lambda^{-}D_-$

whence we are left only with the $\lambda = 0$ terms.

Notice that if $t \to 0$ the trace blows up but the supertrace is constant!

Now:

$$\mathrm{Str}(K_t) = \int_X \mathrm{str}_x(k_t(x,x))$$

McKean-Singer conjectured: $\lim_{t\to 0} \mathrm{str}_x(k_t(x,x)) = \underbrace{\left[(-2i)^{m/2}(4\pi)^{-m/2}\det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right)\mathrm{Str}_{E/S}\, e^{-F^{E/S}}\right]_{\text{top coef.}}}_{d\mathrm{Vol}}$

So this is how to get the global from the local.

$\Delta = D^2$; write this in local coords $\exp_y(x)$. for $x \in T_yX$. Use the trivializations of the bundles T^*X and E by parallel transport along geodesics, $\exp_y(tx)$.

$$\Delta = D^2 = \nabla^*\nabla + \frac{R}{4} + F^{E/S}.$$

Since we're working locally, $E_y = S_y \otimes V$, whence $F^{E/S}$ is just a curvature on V.

Locally

$$= -g^{\mu\nu}(\nabla_\mu \nabla_\nu - \Gamma_{\mu\nu}^{\lambda} \nabla_\lambda) + \frac{R}{4} + F^V$$

Curvature form

$$F = d\omega + \omega \wedge \omega = d\omega + \tfrac{1}{2}[\omega,\omega],$$

"E-swedish"

$$\mathring{E} = \sum x^\mu \frac{\partial}{\partial x^\mu}$$

$$\rightsquigarrow \omega_\mu = -\tfrac{1}{2} F_{\mu\nu} x^\nu + O(x^2)$$

$$\iota(\mathring{E})\omega = 0$$

More generally there are universal formulae for the Taylor coefficients of ω in terms of iterated covariant derivatives of curvature.

May well assume $dx^\mu(0)$ are orthormal. Define θ^μ to be the parallel transport of this orthonormal frame to $\exp_y(x)$. Let e_μ be the dual (orthonormal) frame of $T_{\exp_y x}X$. At 0, $\theta^\mu(0) = \delta^\mu_\nu dx^\nu$, $\theta^\mu = \theta^\mu_\nu dx^\nu$, $g_{\mu\nu} = \delta_{\kappa\lambda}\theta^\kappa_\mu \theta^\lambda_\nu$.

Since the torsion is zero,

$$d\theta^\mu + \omega^\mu \wedge \theta^\nu = 0$$

and we also have

$$d\omega + \omega \wedge \omega = R.$$

<u>Claim</u>. $\iota(\mathring{E})\theta^\mu = x^\mu$

Since the tangent vector to the geodesic is covariant constant,

$$\nabla_{\mathring{E}} \mathring{E} = \mathring{E}.$$

Moreover $\nabla_{\mathring{E}} \partial_\mu - \nabla_{\partial_\mu} \mathring{E} = -\partial_\mu$, since $[\partial_\mu, \mathring{E}] = \partial_\mu$. Now

$$\mathcal{L}_{\mathring{E}} |\mathring{E}|^2 = 2|\mathring{E}|^2 \quad \rightsquigarrow |\mathring{E}|^2 = |x|^2$$

Now since $\iota(\mathring{E})\theta = x$, $(L(\mathring{E}) - 1)x = 0$. Consider

$$(L(\mathring{E}) - 1) L(\mathring{E})\theta = (L(\mathring{E}) - 1)(\iota(\mathring{E}) d\theta + \cancel{dx})$$

$$= (L(\mathring{E}) - 1) \cdot -\iota(\mathring{E})(\omega \wedge \theta)$$

$$= (L(\mathring{E} - 1) \cdot \omega x$$

Working this out carefully, one finds that

$$\theta_\mu^\gamma = \delta_\mu^\gamma - \frac{1}{6} R^\nu_{\ \alpha\beta\mu} x^\alpha x^\beta + O(|x|^3)$$

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3} R_{\alpha\mu\beta\nu} x^\alpha x^\beta + O(|x|^3).$$

Conclude everything is given by coefficients in terms of R.

$$\omega_\mu{}^\lambda{}_\nu = -\frac{1}{2} R_\mu{}^\lambda{}_{k\lambda} x^k + O(x^2)$$

$$\Gamma^\lambda_{\mu\nu} = -\frac{1}{2} R_\mu{}^\lambda{}_{k\nu} x^k + O(x^2).$$

So just for functions (usual Laplacians)

$$\Delta = -g^{\mu\nu}\partial_\mu\partial_\nu + g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\partial_\nu$$

$$= \underbrace{-\delta^{\mu\nu}\partial_\mu\partial_\nu}_{\Delta_0} + O(x^2)\partial^2 + O(x)\partial + \cancel{O(x)}.$$

$$e^{-t\Delta_0}\delta_0 = \underbrace{(4\pi t)^{-m/2} e^{-|x|^2/4t}}_{k_t^0(x)}.$$ known classically for flat space

Then

$$e^{-t\Delta}\delta_0 = e^{-t(\Delta-\Delta_0)-t\Delta_0}\delta_0 = e^{-t(\Delta-\Delta_0)-t\Delta_0} e^{+t\Delta_0} k_t^0(x)$$

We will use the Baker-Campbell-Hausdorff formula.

$$\log(e^X e^Y) = \sum c_{k_1 \cdots} \, ad(Y)^{k_1} ad(X) \cdots ad(X) ad(Y)^{k_n} X + Y.$$

i.e. $X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots$

Let's take $X = -t(\Delta-\Delta_0) - t\Delta_0$, $Y = t\Delta_0$. We in fact obtain a "convergent" power series.

$$-t(\Delta-\Delta_0) + \sum_{\substack{k_1,\dots,k_n\geq 0\\ k_1+\cdots+k_n>0}} (-t)^{k_1+\cdots+k_n+n} ad(\Delta-\Delta_0)\, ad(\Delta_0)^{k_1}\cdots ad(\Delta_0)^{k_n}(\Delta-\Delta_0)$$

If we filter (power series in t and $x^1 \dots x^m$) $k_t^0(x)$, by giving x deg 1 and t deg 2.
It turns out that of each degree there are only finitely many terms
convergence in this filtration is what we mean.

Exercise. Instead of doing this, do it using Hadamard's and Minakshisundaram/Pleijel technique: connect k_i to k_{i+1} by an ODE $\nabla_{\dot{E}} k_{i+1}$ = explicit exp. in $k_i(x)$. Then k_0 can be written down explicitly: $\det^{1/2}(d_x \exp_y)$.

Modify filtration in the following way. We are studying the heat equation on $\Omega^*(X, \mathrm{End}_{C(x)} E)$ (holding y fixed).

X: nbhd of y.

Key Trick: rescale in the following way:

$$x \longmapsto u^{1/2} x \qquad \frac{\partial}{\partial x} \longmapsto u^{-1/2} \frac{\partial}{\partial x}$$

$$t \longmapsto ut \qquad \frac{\partial}{\partial t} \longmapsto u^{-1} \frac{\partial}{\partial t}$$

Recall $c^\mu = \varepsilon^\mu - \varepsilon^{*\mu}$ — we rescale

$$\varepsilon^\mu \longmapsto u^{1/2} \varepsilon^\mu$$

$$\varepsilon^{*\mu} \longmapsto u^{-1/2} \varepsilon^{*\mu}.$$