EXERCISES FOR 520-2

1. March 28, 2017

Exercise 1. Let $E \to X$ be a smooth complex vector bundle and $\mathscr{E}$ be its sheaf of smooth sections. Show that if $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ is a good cover for X then $\check{H}^k(\mathscr{U}, \mathscr{E}) = 0$ for $k > 0$. [Hint: fix a partition of unity η for $\mathscr{U}$ and define a nullhomotopy $s : \check{C}^k(\mathscr{U}, \mathscr{E}) \to \check{C}^{k-1}(\mathscr{U}, \mathscr{E})$ by

$$(s\omega)_{\alpha_0 \cdots \alpha_{k-1}} = \sum_\alpha \eta_\alpha \omega_{\alpha_0 \cdots \alpha_{k-1} \alpha}. \tag{1}$$

where $\omega \in \check{C}^k(\mathscr{U}, \mathscr{E})$.]

Exercise 2. Construct explicitly the long exact sequence for Čech cohomology.

Exercise 3. Check that for a good cover $\mathscr{U}$ of X, the group $\check{\mathrm{H}}^1(\mathscr{U}, \underline{\mathrm{GL}(1, \mathbb{C})})$ is isomorphic to the group of isomorphism classes of complex line bundles on X.

Exercise 4. Recall that the Todd genus of a complex line bundle $L \to X$ is the (inhomogeneous) cohomology class $\mathrm{Td}(L) = f(c_1(L))$ where $c_1(L)$ is the first Chern class of L and $f(x) = x/(1 - e^{-x})$. Compute the Todd genus of a rank 2 complex vector bundle $E \to X$ in terms of $c_1(E)$ and $c_2(E)$.