INDEX THEORY.

Recall L a line bundle,

$$Td(L) = \frac{c_1(L)}{1-e^{-c_1(L)}} = 1 + \frac{1}{2}c_1(L) + \frac{1}{12}c_1(L)^2 - \frac{1}{720}c_1(L)^4$$

$$\zeta(2l) = \frac{(-1)^{l+1}(2\pi)^{2l}}{2(2l)!}B_{2l}.$$

Up to H^4, the last exercise is (so for surfaces)

$$(1+\tfrac{1}{2}x+\tfrac{1}{12}x^2)(1+\tfrac{1}{2}y+\tfrac{1}{12}y^2) = 1+\tfrac{1}{2}(x+y)+\tfrac{1}{12}\left((x+y)^2+xy\right).$$

$E = L_1 \oplus L_2$, $x = c_1(L_1)$, $y = c_1(L_2)$. $c_1(E) = x+y$, $c_2(E) = xy$

$1+\frac{1}{2}c_1(E)+\frac{1}{12}\left(c_1(E)^2+c_2(E)\right) + \cdots$ ← stuff in H^4.

How to compute higher terms uniformly?

$E = L_1 \oplus \cdots \oplus L_r$. ϕ a genus. we introduce a parameter, t.

$$\phi_t(E) = \phi_t(L_1)\cdots\phi_t(L_r)$$
$$= f(tc_1(L_1)) \cdots f(tc_1(L_r)).$$

$$\frac{d}{dt}\log\phi_t(E) = \sum_{i=1}^r \frac{d}{dt}\log f(tc_1(L_i)).$$

$$\frac{d\phi_t}{dt} = \phi_t \cdot \sum_{i=1}^r \frac{d}{dt}\log f(tc_1(L_i)).$$

Case 1. $f(x) = 1-x$

$$\frac{d}{dt}\log(1-tx) = -\frac{x}{1-tx}$$

$$c_t(E) = \sum_{k=0}^r (-1)^k t^k c_k(E).$$

$$\sum_{k=1}^r k(-1)^{k-1}t^{k-1}c_k(E) = \left(\sum_{k=0}^r (-t)^k c_k(E)\right) \cdot \underbrace{\sum_{i=1}^r\sum_{k=1}^\infty t^{k-1}c_1(L_i)^k}_{=\sum_{k=1}^\infty k!\, t^{k-1}ch_k(E)}.$$

"Newton's formula" or something...

Define:

$$ch(E) = \sum_{i=1}^n ch(L_i) = \sum_{i=1}^n e^{c_1(L_i)} = \sum_{k=0}^\infty ch_k(E).$$

$ch_k(E) \in H^{2k}(X;\mathbb{Q})$.

notice: $ch(E\oplus F) = ch(E) + ch(F)$.

$ch(E) = r$, $ch_1(E) = c_1(E)$.

Exercise. Compute ch_2, ch_3 in terms of c_i.

Can now play the same game with the Todd genus.

$$f(tx) = \frac{tx}{1-e^{-tx}}$$

$$\frac{d}{dt}\log f(tx) = \frac{1}{t} + \frac{e^{-tx}}{1-e^{-tx}} = -\frac{1}{t}\sum_{k=1}^\infty \frac{t^k x^{k-1}}{k!}B_k$$

One finds similarly

$$\sum_{k=1}^r k\,t^{k-1}Td_k(E) = \left(\sum_{k=0}^r t^k Td_k(E)\right)\sum_{k=1}^\infty \frac{B_k}{k}t^{k-1}ch_{k-1}(E).$$

Exercise. Check that this is correct.

Before we get to Chern-Weil theory, we will discuss H.R.R.

Let E be a rank r hol'c vector bundle on a complex manifold X.

We have a sheaf of hol'c sections, i.e. locally

$$\frac{\partial}{\partial \bar{z}_j} s^a(z,\bar{z}) = 0.$$

Call the sheaf $\mathcal{O}(E)$.

(Stein cover)

One might try to compute $H^k(X, \mathcal{O}(E))$ via Dolbeault or Čech.

<u>Theorem</u> (Serre duality) $\dim H^k(X, \mathcal{O}(E)) < \infty$ if X is compact.

$H^k = 0$ for $k > n$.

<u>Defn</u>. $\chi(E) = \sum_{k=0} (-1)^k \dim H^k(X, \mathcal{O}(E))$.

<u>Theorem</u>. (Hirzebruch-Riemann-Roch)

$$\chi(E) = \int_X ch(E)\, Td(X) \qquad (Td(X) := Td(TX)).$$

Let's look at $n=1$. Here $ch(E) = r + c_1(E)$ $\quad Td(X) = 1 + c_1(TX) = 1 - c_1(K)$.

In this case the RHS is

$= -\frac{r}{2}\deg(K) + \deg(E)$ $\qquad$ here $\deg(E) = \int c_1(E)$.

Let's take $E = X \times \mathbb{C}$ so $\mathcal{O}(E) = \mathcal{O}$. Of course we know

$$H^0(X, \mathcal{O}) = \mathbb{C} \qquad H^1(X,\mathcal{O}) \cong H^0(X, K)^\vee$$

whence $\deg(K) = 2-2g$.

Another example is $X = \mathbb{CP}^n$, w/ $E = X \times \mathbb{C}$.

$$H^k(X, \mathcal{O}) = \begin{cases} \mathbb{C} & k=0 \\ 0 & k>0 \end{cases} \longrightarrow \chi(\mathbb{CP}^n, \mathcal{O}) = 1.$$

ch will be one so let's compute $Td(\mathbb{CP}^n)$. Let L be the tautological line bundle. What is $c(T\mathbb{CP}^n)$? Notice that

$$0 \to \mathbb{CP}^n \times \mathbb{C} \to (L^\vee)^{n+1} \to T\mathbb{CP}^n \to 0.$$

Whence the latter two have the same total Chern character.

Write $x = c_1(L)$. $c(T\mathbb{CP}^{n*}) = (1 - c_1(L))^{n+1}$. We want to pick out coeff. of x^n of $\mathrm{Td}(\mathbb{CP}^n) = \left(x/e^x - 1 \right)^{n+1}$.

$$= \mathrm{Res}_{x=0}\, x^{-n-1} \left(\frac{x}{e^x - 1} \right)^{n+1} dx = \mathrm{Res}_{y=0} \frac{dy}{y^{n+1}(1+y)} = (-1)^n \qquad (dy = e^x dx).$$

In fact the Todd genus is the unique thing to put in HRR to get the right answer for $\mathbb{CP}^n$.

Suppose we have $L = (g_{\alpha\beta})_{\alpha,\beta \in I}$ cocycle in $\underline{GL(1,\mathbb{C})}$.

$$c_1(L)_{\alpha\beta\gamma} = +\frac{1}{2\pi i}\left(\log g_{\beta\gamma} - \log g_{\alpha\gamma} + \log g_{\alpha\beta} \right).$$

$c_1(L)$ is represented by the closed form

$$+\frac{1}{2\pi i} \sum_{\alpha\beta\gamma} (\text{——}\ {''}\ \text{——})\, \eta_\alpha\, d\eta_\beta\, d\eta_\gamma .$$

Notice the second two terms vanish and we get

$$= +\frac{1}{2\pi i} \sum_{\alpha\beta} \log g_{\alpha\beta}\ d\eta_\alpha\, d\eta_\beta$$

notice this is well-defined in coh. indpt of choice of branch.

<u>Connections on cpx vector bundles</u>. ∇ nabla (harp (?) maybe lute in Aramaic)

$$\nabla : \Gamma(X,E) \longrightarrow \Omega^1(X,E) = \Gamma(X, T^*X \otimes E).$$

with:

- linear
- $\nabla(fs) = df \otimes s + f \nabla s$.

Notice on U_α we can write $s = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix}$, then

$$\nabla s = \begin{bmatrix} ds_1 \\ \vdots \\ ds_r \end{bmatrix} + \underset{\uparrow\ r\times r\text{-matrix of cpx 1-forms}}{[\omega_i^{\,j}]}\,[s_j]$$

Notice that $\nabla_1 - \nabla_0 \in \Omega^1(X, \mathrm{End}\, E)$.

Moreover, if ∇_α is a connection on $E|_{U_\alpha}$ then

$$\nabla = \sum_\alpha \eta_\alpha \nabla_\alpha$$

is a connection on E.

~~Also~~ Finally, check that there is a unique extension

$$\nabla : \Omega^k(X, E) \longrightarrow \Omega^{k+1}(X, E).$$

satisfying the Leibniz rule.

Now ∇ is a differential op. (linear) on $\Omega^\cdot(X, E)$. Consider ∇^2. It turns out that this is just a 0^{th} order diff. op. even though ∇ is 1^{st} order. In particular,

$$\nabla^2 = F \in \Omega^2(X, \mathrm{End}(E)). \qquad \text{"curvature of } \nabla\text{"}$$

How to see this? You can just compute

$$(d+\omega)^2 = d\omega + \omega^2. \qquad \text{(care: these are matrices)}$$

For a line bundle, $F = d\omega$. Let's compute using cocycles. Put on each chart U_α the connection $\nabla_\alpha = d$. Globally we get

$$\nabla = \sum \eta_\alpha \nabla_\alpha$$

$$\nabla|_{U_\alpha} = \sum \eta_\beta \left(g_{\alpha\beta} \circ d \circ g_{\beta\alpha} \right) = d - \sum_\beta \eta_\beta \, d\log g_{\alpha\beta} \qquad (\text{since } g_{\beta\alpha} = g_{\alpha\beta}^{-1}).$$

Now

$$F|_{U_\alpha} = -\sum_\beta d\eta_\beta \, d\log g_{\alpha\beta}.$$

This is global now so extends

$$F = -\sum_{\alpha\beta} \eta_\alpha \, d\eta_\beta \, d\log g_{\alpha\beta}.$$

$$= -\sum \log g_{\alpha\beta} \, d\eta_\alpha \, d\eta_\beta + d\left(\sum_{\alpha,\beta} \eta_\alpha \, d\eta_\beta \log g_{\alpha\beta} \right).$$

The punchline:

$$c_1(L) = -\frac{1}{2\pi i} F \quad \text{in } H^2(X, \mathbb{C}).$$

Now more generally.

E vector bundle, ∇ connection on E, curvature F.

$$\det\left(I - \tfrac{1}{2\pi i}F\right) = c(E) \quad \text{in } \prod H^{2k}(X, \mathbb{C}).$$

To compute, one uses

$$\det(I - A) = \exp \operatorname{Tr} \log (I - A)$$ which makes sense b/c A nilpotent.

Step 1. Prove that this formula yields a closed form.

Step 2 Changing connection changes the form by an exact piece.

Step 3. Check naturality.

Step 4. Normalization.

Step 5. Product formula.

↑ "Chern-Weil theory".

For the Todd genus,

$$Td(E) = \det\left(\frac{-\frac{1}{2\pi i}F}{I - e^{\frac{1}{2\pi i}F}}\right).$$

and the Chern character,

$$ch(E) = \operatorname{Tr}\left(e^{-\frac{1}{2\pi i}F}\right).$$