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Generalized Dirac operator D, and we consider the kernel of the operator e^{-tD^2}, which is the soln to the heat eqn

$$(\partial_t + D_x^2)\, k_t(x,y) = 0 \qquad \lim_{t\to 0} k_t(x,y) = \delta(x,y).$$

Fixing a point $y \in X$.

$k_t(\exp_y x, y)$ is a soln to the heat eqn on a ball in T_yX near 0. (I'm conflating Ezra's x and x!) So we have a heat eqn for each $y \in X$. $\hookrightarrow \begin{cases} (\partial_t + D_x^2)\, k_t(x) \\ \lim_{t\to 0} k_t(x) = \delta(x). \end{cases}$

Strictly speaking, $k_t(x) \in \mathrm{Hom}(E_y, E_{\exp_y x})$

$\cong \mathrm{End}(E_y)$ by parallel transport

$\cong C(T_yX) \otimes \mathrm{End}_{C(T_yX)} E$

$\cong \Lambda^{\bullet} T_y^*X \otimes \mathrm{End}_{C(T_yX)} E$

<u>Theorem.</u> $k_t(x)_{[i]} \in \Lambda^i T_yX \otimes \mathrm{End}_{C(T_yX)} E$

has an asymp. expansion $\sim (4\pi t)^{-\frac{m}{2}} \sum_{j=i}^{\infty} t^{j/2} a_j(x)_{[i]}\, e^{-|x|^2/4t}$.

Furthermore, there is an explicit formula

$$\sum_{i=0}^{m} a_i(x)_{[i]} = \cdots$$

where $\sum_{i=0}^{m} a_i(0)_{[i]} = \det^{1/2}\left(\frac{R/2}{\sinh R/2}\right) e^{-tF^{E/S}}$.

We rescale $\begin{cases} t \longmapsto ut & \partial_t \longmapsto u^{-1}\partial_t \\ x \longmapsto u^{1/2}x & \partial_x \longmapsto u^{-1/2}\partial_x \end{cases}$ "making the heat eqn homogeneous"

$\varepsilon^* \longmapsto u^{1/2}\varepsilon^* \qquad \varepsilon \longmapsto u^{-1/2}\varepsilon$

We can write our connection

$$\nabla_\mu = \partial_{x^\mu} + \tfrac{1}{2} c(\omega_\mu) + \omega_\mu^{E/S}$$

Levi-Civita, $\omega_\mu \in \underline{so}(T_yX) \cong \Lambda^2 T_y^*X$.

The Laplacian is

$$-g^{\mu\nu}\nabla_\mu\nabla_\nu + g^{\mu\nu}\Gamma_{\mu\nu}^{\lambda}\nabla_\lambda.$$

We want to compute the leading term in the rescaling parameter (in our normal coordinates). It looks roughly, like

.... ?

superconnections $\longleftrightarrow$ gen. Dirac op's

Q. What if instead of a Dirac op. associated to a connection, we use a superconnection?

Given

V - graded bundle w/ superconnection

S - spinor bundle

we have D defined by

$$\begin{tikzcd}
\Gamma(X, S\otimes V) \arrow[rr, "D"] \arrow[dr, "\nabla^S\otimes 1 + 1\otimes \mathbb{A}^V"'] & & \Gamma(X, S\otimes V) \\
& \Omega^\bullet(X, S\otimes V) \arrow[ur, "c"'] &
\end{tikzcd}$$

(Indeed for any clifford module E, locally: $E \cong S\otimes V$, $\mathbb{A}^V$ defined locally, but $\cancel{(\mathbb{A}^V)^2}$ is global
curvature $F^{E/S}$

Here $F^{E/S} \in \Omega^\bullet(X, \mathrm{End}_{C(X)} E)$ of even degree.

Given a superconnection $\mathbb{A}$, we can rescale

$$(\mathbb{A}_u)_{[i]} = u^{1-i/2}\mathbb{A}_{[i]} \qquad \text{(can't rescale } i=1 \text{ b/c)}$$

$$F_u = \sum_{i=0}^m u^{2-i/2} F_{[i]} = u F_{[0]} + u^{1/2} F_{[1]} + F_{[2]} + \cdots$$

Now we define D_u as above now using $\mathbb{A}_u$.

Recall the simple case: $\mathbb{A} = \mathbb{D} + \nabla^V \rightsquigarrow \mathbb{A}_u = u^{1/2}\mathbb{D} + \nabla^V$. $D_u = D + u^{1/2}\mathbb{D}$.

For Dirac operators associated to supercon's admit a Lichnerowicz formula

$$D_u^2 = \nabla^*\nabla + c(F_u^{E/S})$$

Consider $k_{t,u}(x,y)$ the heat kernel of D_u

<u>Theorem</u>. $k_{t,t^{-1}}(x,y)$ satisfies the same theorem as before, and

$$\sum k_{t,t^{-1}}(x,x)_{[i]} = (4\pi t)^{-\frac{m}{2}} \det{}^{1/2}\left(\frac{tR/2}{\sinh tR/2}\right) e^{-F^{E/S}}.$$

We will now think of a fiber bundle $M \xrightarrow{\pi} B$ that is Riemannian, i.e. M, B are Riemannian, and this in particular yields a connection on π, $H_x M$ is the orthogonal complement of $V_x M = \ker \pi_*$.
horizontal ($H_x M$), vertical ($V_x M$)

We require that $d_x\pi : H_x M \to T_x B$ is an isometry for all B.

Equivalently:
1. connection on M/B
2. metric on $T(M/B)$
3. metric on TB.

We will study Dirac op's on M (w.r.t. conn) as generalized Dirac op's (w.r.t. sup. conn.) on B.

Ex. Laplace–Beltrami

$\Omega^\bullet(M) = \Omega^\bullet(B, \mathcal{E})$ $\qquad \mathcal{E}$ = diff. forms on the fibers of π

$\Gamma(B, \pi_*(\Lambda^\bullet V_\pi^\vee)) = \Gamma(M, \Lambda^\bullet V_\pi^\vee)$. $\qquad \pi_*(\Lambda^\bullet V_\pi^\vee)$

Check that $d+d^*$ gives a $\not{D}$ gen. Dirac op. on B. In this case, Bismut's theorem is precisely this statement (and that it's associated to a superconnection on $\mathcal{E}$. Turns out that $A_{[0]} = d_\pi + d_\pi^*$. and

$A_{[1]}$ = connection on E induced by L.C. and that on π.

$A_{[2]} = c$(curvature of the connection on M/B.)

$= (\varepsilon - \varepsilon^*)(\dots)$

vertical Clifford multiplication.

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Recall the setup from last time $M \xrightarrow{\pi} B$. + data. We obtain

$$TM \cong \underset{\substack{\downarrow\wr \\ T_H M}}{\pi^* TB} \oplus T(M|B).$$

and a connection on TM, $\nabla^{\oplus} = \pi^*\nabla^B \oplus \nabla^{M|B}$ ← this kinda complicated ↪ understand!

If $\nabla^{M/B}$ is the connection $T(M|B)$ and g_B is a Riemannian on B, we obtain a metric

$$g = g_M = \pi^* g_B \oplus g_{M|B}.$$

We also have a connection ∇^g from Levi-Civita construction. If we rescale

$$g_u = u^{-1}\pi^* g_B. \oplus g_{M/B}.$$

Invariants of Riemannian fiber bundle.

- Second fundamental form $S \in \Gamma(M, \pi^* T^*B \otimes \cancel{\mathrm{Sym}}\otimes T^*(M/B) T^*(M/B))$.

$$S(Z)(X,Y) = g_{M|B}(\nabla_Z^{M/B} X - P[Z,X]Y)$$

↑ projection to vertical tangent bundle. $\{f_\alpha\}$ ONF of $T(M/B)$.

exercice. { (1) tensor

(2) symmetric (?) maybe. check this!

$\sum S(Z)(f_\alpha, f_\alpha)$

We write $\mathrm{Tr}(S)$ = contraction of S with $(g^{M|B})^{-1} \in \Gamma(M, \pi^* T^*B)$.

Curvature of $\nabla^{M/B}$ is a tensor (exercise)

$$\Omega(X,Y) = -P[X,Y] \qquad \in \Gamma(M, \pi^*\Lambda^2 T^*B \otimes T(M/B))$$

(Locally) Choose a frame of TB and lift it to $T_H M$, $\{e_i\}$, a horiz. frame.

Let $\{e^i\}$ be the dual frame of $(T_H M)^* \cong T(M/B)^\perp \subset T^*M$.

$\{e_i, f_\alpha\}$ frame of TM ; $\{e^i, f^\alpha\}$ frame of T^*M.

→ Take a Clifford module E for the vertical cotangent bundle $T^*(M/B)$ (take the dual metric of $g_{M/B}$). Now consider $\pi^*\Lambda^* T^*B \otimes E$. Consider the metric induced on T^*M by the metric g_u on TM. On $\pi^*(T^*B)$ this equals $u\pi^* g_B$. On T^*M we have $u\pi^* g_B \oplus g_{M/B}$.

Clifford action : ξ vertical $\Gamma(M, T^*(M/B))$

$$c_u(\xi) = \pm 1 \otimes c^{M/B}(\xi).$$

$\pi^* \eta$ horizontal (η 1-form on B)

$$c_u(\eta) = \varepsilon(\eta) - u\varepsilon^*(\eta)$$

Bismut's idea ("astuce") French

Take the Dirac operator D^u for the metric g_u and the Clifford action c_u on $\pi^* \Lambda^* T^* B \otimes E$. Here we use a connection on E over M s.t. restricted to the fibers it is a Clifford connection.

i.e. $\quad D^u = \sum_i c_u(e^i) \nabla_{e_i}^{\pi^* \Lambda^* T^* B \otimes E} + \sum c(f^\alpha) \nabla_{e_i}^{\pi^* \Lambda^* T^* B \otimes E}$ ← not quite!

What is the connection on $\nabla^{\pi^* \Lambda^* T^* B \otimes E}$?

Connections on $\mathbb{E} = \pi^* \Lambda^* T^* B \otimes E$.

$$\nabla^{\mathbb{E}, \oplus} = \pi^* \nabla^{g_B} \otimes 1 + 1 \otimes \nabla^E$$

$$\nabla_X^{\mathbb{E}, \oplus}(c_u(\alpha) s) = c_u(\alpha) \nabla_X^{\mathbb{E}, \oplus} s + c_u(\nabla_X^{\oplus} \alpha) s$$

want Levi-Civita connection here instead

$$\nabla^g = \nabla^{\oplus} + \omega \qquad \omega \in \Omega^1(M, \mathrm{End}\, TM)$$

where:

$$(\omega(X) Y, Z) = S(Y)(X, Z) - S(Z)(X, Y) + \tfrac{1}{2}(\Omega(X, Z), Y) - \tfrac{1}{2}(\Omega(X, Y), Z) + \tfrac{1}{2}(\Omega(Y, Z), X).$$

This is proved by the usual Koszul formula.

Now it turns out that

$$\nabla_X^{\mathbb{E}} = \nabla_X^{\mathbb{E}, \oplus} - \tfrac{1}{2} c(\omega^*(X)).$$

is a Clifford connection on $\mathbb{E}$ for the Levi-Civita connection.

Then we can do similarly for $\nabla_X^{\mathbb{E}, u} = \nabla_X^{\mathbb{E}, \oplus} - \frac{1}{2} c_u(\omega^*(X))$. and

Conclusion. $\nabla^{\mathbb{E}, u}$ is a Clifford connection for the Clifford action c_u of (T^*M, g_u) on $\mathbb{E}$.

Now take the associated Dirac operator

$$D^u = \sum_i c_u(e^i)\left[\nabla^{E,0}_{e_i} - \tfrac{1}{2} c_u(\omega^*(e_i))\right] + \sum_\alpha c(f^\alpha)\left[\nabla^E_{f_\alpha} - \tfrac{1}{2} c_u(\omega^*(f_\alpha))\right]$$

$D^0 = \lim_{u\to 0} D^u$ is a superconnection $\mathbb{A}$ with $\mathbb{A}_{[0]} = D^{M/B,E}$

and $\mathbb{A}_{[2]} = \frac{1}{2}\, c^E(\Omega)$

and $\mathbb{A}_{[1]}$ is the connection w/

$\nabla_X = \nabla_X^{E,0}$ (X horizontal).

$\mathbb{A}$ is a superconnection on the bundle $\pi_* E$.

Suppose we have a Clifford module S with Clifford connection ∇^S on (B, g_B).

~~Then the generalized Dirac operator on $\pi^* S \otimes E$~~

<u>Theorem</u>. Generalized Dirac operator on $S \otimes \pi_* E$ associated to the superconnection $\mathbb{A} = \lim_{u\to 0} D^u$ is the Dirac operator on $\pi^* S \otimes E$.