INDEX THEORY. 05/09/2017.

$$D^{+} = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \qquad D = D^*, \text{ self-adjoint w/ } M \text{ compact.}$$

$\ker D = \ker D^+ \oplus \ker D^-$

By elliptic regularity.

$$D^+ : \Gamma_{H^1}(M, E^+) \to \Gamma_{L^2}(M, E^-).$$

$$\operatorname{coker}(D^+) \cong \ker(D^-) \quad \text{since} \quad \underset{\text{(elliptic reg.)}}{D^- = (D^+)^*}.$$

Last time $E - M$ (vertical Clifford module.), with $M \to B$. $\pi^* T^*B \otimes E$ Clifford module on M

$$g(\nabla^g_X Y, Z) = g(\nabla^\oplus_X Y, Z) + \omega(X)(Y, Z). \qquad (\omega(X)(Y,Z) = -\omega(X)(Z,Y)).$$

There's some formula for ω ... cf. last time hopefully.

Last time, $g_{M/B}(S(Y)X, Z) = g_{M/B}(S(T)Z, X)$, the 2^{nd} fund. form was symm.

Dirac operator D^u on $\pi^* T^*B \otimes E$,

$A = \lim_{u \to 0} D^u$ superconnection.

$A_{[0]} = D^{M/B, E}$ vertical Dirac ops on E restricted to fibers of π.

$A_{[2]} = \frac{1}{2} C^{M/B}(\Omega)$ ← curvature of vert. bundle.

$A_{[1]} = \nabla^{M/B} + \frac{1}{2} \operatorname{Tr} S.$ $\quad (\operatorname{Tr}(S) = g^{M/B}(Se_i, e_i) = (Se_i, e^i) =: k$

$k \in \Gamma(M, \pi^* T^*B) \cong \Gamma(M, T^*_H M).)$

Connection on the bundle w/ fiber at $b \in B$,

$\Gamma(M_b, E|_{M_b} \otimes |\Lambda|^{1/2})$, $\quad |\Lambda^{top} T^*(M/B)|$ vertical densities

↳ preHilbert space

$(s_1, s_2) \int_{M/B} h(s_1 \otimes \overline{s_2})$. Turns out the canonical connection is

hermitian metric on E $\quad$ on $E \otimes \overline{E} \otimes (|\Lambda|^{1/2})^2$ $\quad$ precisely $A_{[1]}$.

volume density along fibers of π associated to $g^{M|B}$.

$\mathbb{A}_s = s^{1/2}\mathbb{A}_{[0]} + \mathbb{A}_{[1]} + s^{-1/2}\mathbb{A}_{[2]}$.

<u>Lemma</u>. $\nabla^{M|B} |d\,\mathrm{vol}| = |d\,\mathrm{vol}|\,\mathrm{Tr}(S)$.

Consider now $\underbrace{\mathbb{A}_s^2}_{\mathbb{F}_s :=} : \Gamma(B, \pi_* E) \to \Omega^*(B, \pi_* E)$. It is in fact a diff. op along the fibers and in the horizontal directions it is a tensor – it commutes w/ mult. by forms on B

$$(\mathbb{F}_s)_{[0]} = s\,(D^{M|B,E})^2 + s^{1/2}\,\nabla^{\pi_* E} D^{M|B,E} + \dots$$

To make sense of $e^{-\mathbb{A}_s^2}$ need to construct heat kernel fore an op. whose coefficients in $\Lambda^\bullet T_b^* B$ a graded algebra. (since these are nilpotent, the perturbation is relatively minor).

$$e^{A+B} = \sum_{n=0}^{\infty} \int_{0 \le t_1 \le \dots \le t_n \le 1} e^{t_1 A} B e^{(t_2 - t_1)A} B \cdots B e^{(1-t_n)A}\, dt_1 \cdots dt_n$$

Here $A = -sD^2$, $B = -\mathbb{A}_s^2 + sD^2$. Can make sense of $e^{-\mathbb{A}_s^2}$.

We are interested in $\mathrm{Str}^{\pi_* E}(e^{-\mathbb{A}_s^2})_b \in \Lambda^{ev} T_b^* B$

(kernel of $e^{-\mathbb{A}_s^2}$)$(x,x) \in \Lambda^\bullet T_b^* B \otimes \mathrm{End}\, E_x \otimes |\Lambda|_x$.

<u>Claim</u>: local index theorem for generalized Dirac operators, still goes through in this infinite-dimensional setting.

Generalized Dirac operator on $S^B \otimes \pi_* E$ (over an open subset of B) [spinor]

→ Local index theorem for $D^{B,\pi_* E, \mathbb{A}}$ on $S_B \otimes \pi_* E$

just $D^{M, \pi^* S_B \otimes E}$, identified.

Local index theorem for D^B, taking the pointwise supertrace at time t of $\mathbb{A}_{t^{-1}}^2$.

$$(2\pi)^{-\dim B/2} \det{}^{1/2}\left(\frac{R^B/2}{\sinh R^B/2}\right) \mathrm{Str}^{\pi_* E}(e^{-\mathbb{A}^2}) = (2\pi)^{-\frac{\dim(M)}{2}} \pi_*\left(\det{}^{1/2}\left(\frac{R^M/2}{\sinh R^M/2}\right) \cdot \mathrm{Str}^{E/S}(e^{-F^{E/S}})\right).$$

integration along fibers!

This is the local family index theorem.

(assuming fibers are even-dimensional).

$$\mathrm{Str}^{\pi_* E}\left(e^{-\mathbb{A}^2}\right) = (2\pi)^{-\frac{\dim(M/B)}{2}} \pi_* \left(\underbrace{\frac{\det^{1/2}\left(\frac{R_M/2}{\sinh R_M/2}\right)}{\det^{1/2}\left(\frac{R_B/2}{\sinh R_B/2}\right)}}_{\Omega^{ev}(M)} \underbrace{\mathrm{Str}^{E/S}\left(e^{-F^{E/S}}\right)}_{\Omega^{ev}(M)} \right)$$

↪ even degree + closed, is a rational class in de Rham.

Atiyah + Singer: index bundle interpret g as a Chern character.

The Chern character of a vertical bundle $[E_+]-[E_-]$ associated to a $\mathbb{Z}/2$-graded v.b. $E^+ \oplus E^-$ has 0-degree component $\mathrm{rk}\, E_+ - \mathrm{rk}\, E_-$.

In our case, $\left(\mathrm{Str}\, e^{-\mathbb{A}^2}\right)_{[0]} = \mathrm{Str}\left(e^{-(D^{M/B,E})^2}\right) = \mathrm{ind}\left(D_b^{M/B,E_b}\right) \in \mathbb{Z}$.

so the index is locally constant. (smooth homotopy invariance of the Dirac operator).

Meanwhile the 2-form piece is harder to interpret — it shows up in the determinant line bundle;

$$\mathrm{Str}\left(e^{-\mathbb{A}^2}\right)_{[2]} = c_1\left(\det\left(D^{M/B}\right)\right)$$

$$\left(\Lambda^{top} \ker D^+\right)^{-1} \otimes \Lambda^{top} \ker D^-.$$

Dirac operator D^{M_b}.

Consider $b \mapsto \ker(D^b)$; ~~if this~~ is "$\in$" $K(B)$ virtual bundle.

Moduli space of elliptic curves, $\{\mathrm{Im}\,\tau > 0\}/SL(2,\mathbb{Z})$. $=: \mathcal{M}$.

i.e. $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$.

Can complete $\overline{\mathcal{M}} = \mathcal{M} \cup \{i\infty\}$ to a projective variety.

(Deligne–Mumford)

Need to construct an ample line bundle.

Consider $\begin{array}{ccc} C_\tau & \hookrightarrow & \mathcal{C} \\ & & \downarrow \\ & \tau \in & \mathcal{M} \end{array}$ universal curve. Have

$\pi_* \mathcal{O}$ trivial line bundle $\mathcal{O}$

$(R^1 \pi_* \mathcal{O})_\tau = H^1(C_\tau, \mathcal{O}) = H^0(C_\tau, T^*)^\vee$

← abelian diff of first kind

$E = A^{0,*}(C_\tau)$ C^∞-sections of $\mathbb{C} \oplus \Lambda^{0,1} T^* C_\tau$

Dirac operator along the fibers is $(\bar\partial + \bar\partial^*)_{\mathcal{C}/M}$ and $D^{\mathcal{M}}$ is $(\bar\partial + \bar\partial^*)_{\mathcal{C}}$.