THE SYMPLECTIC "CATEGORY"

MLG

1

The goal of this talk will be to describe what Weinstein has called the symplectic category. It has symplectic manifolds as points and morphisms between them are Lagrangian submanifolds of the product, called canonical relations.

The importance of this construction is that composition in this category is a geometric model for composition of quantized objects, and can make calculations involving the latter easier. To explain where this is coming from recall the Schrodinger picture of quantization. Classically states are points in T^*X and the hamiltonian $H(x,\xi)$ generates a hamiltonian flow. The evolution of a point is its trajectory under the flow. On the other hand, a quantized state is an element of $L^2(X)$ and this evolves by the Schrodinger equation $\left(ih\partial_t - \hat{H}\right) = 0$. It is natural to view this quantized evolution as corresponding to the Hamiltonian flow of H, which is a symplectomorphism. More generally, we can quantized other types of 'symplectic maps' which aren't diffeomorphisms. This leads us to canonical relations.

The construction of this category answers the following question: If $A : M_1 \to M_2$ and $B : M_2 \to M_3$ are operators of this kind associated to canonical relations $\Gamma_1 \subset M_1 \times M_2$, $\Gamma_2 \subset M_2 \times M_3$ Is BA associated to $\Gamma_2 \circ \Gamma_1$?

2. The Symplectic Category

2.1. **Canonical relations.** Given two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) of dimensions $2n_i$ we denote by M_1^- the symplectic manifold $(M_1, -\omega_1)$. A canonical relation $\Gamma : M_1 \to M_2$ is a Lagrangian submanifold of $M_1^- \times M_2$, that is of $M_1 \times M_2$ with the symplectic form $-\omega_1 \oplus \omega_2$.

The symplectic category consists of symplectic manifolds as its objects, and canonical relations as morphisms. We will mostly be occupied with understanding how to compose two canonical relations, but first we note an important special case:

Lemma. *A canonical relation is the graph of a symplectomorphism if and only if the maps down to each factor are diffeomorphisms.*

Proof. Suppose $\Phi : M_1 \to M_2$ is a canonical transformation (symplecto.) Consider its graph $\Gamma_\Phi = \{(x, \Phi(x)) \mid x \in M_1\} \subset M_1 \times M_2$. We claim that Γ_Φ is a canonical relation. (Called a canonical graph). To see this we note that $2n_1 = 2n_2$ and so Γ_Φ is half the dimension of $M_1 \times M_2$. The tangent space to the graph at $(x, \Phi(x))$ is the graph of $d\Phi_x$ and since Φ is canonical,

$$-\omega_1 \oplus \omega_2((v_1, d\Phi_x v_2), (v_2, d\Phi_x v_2)) = -\omega_1(v_1, v_2) + \omega_2(d\Phi_x v_1, d\Phi_x v_2) = 0$$

Conversely if the two projections are diffeos, the map $\Phi = \pi_2 \circ \pi_1^{-1} : M_1 \to M_2$ is a diffeomorphism and Γ is its graph. The assertion that Γ be Lagrangian is equivalent to Φ being canonical, as the previous line shows. $\square$

Given two canonical graphs $\Gamma_\Phi : M_1 \to M_2$ and $\Gamma_\chi : M_2 \to M_3$, we set $\Gamma_\chi \circ \Gamma_\Phi = \{(x,z) \in M_1 \times M_3 \mid \exists y \in M_2; (x,y) \in \Gamma_1 \,\&\, (y,z) \in \Gamma_2\}$. It is easy to see that this set is the graph of $\chi \circ \Phi$. This motivates us to try this definition of composition in general.

2.2. **Examples of CR's.**

(1) Lagrangian submanifolds. We can view $\Lambda \subset (M, \omega)$ as a canonical relation $\Lambda : \mathrm{pt} \to M$.

(2) The canonical relation of a smooth map. Suppose $f : X \to Y$ is a smooth map. There is an associated canonical relation $\Gamma_f : T^*X \to T^*Y$

$$\Gamma_f = \{(x, \xi, y, \eta) \mid y = f(x) \,\&\, \xi = df_x^* \eta\}$$

Another way to describe Γ_f is the following. Consider the graph of f, $G = \{(x, f(x)) \mid x \in X\} \subset X \times Y$. The conormal bundle N^*G is a Lagrangian submanifold of $T^*X \times T^*Y$ with respect to the standard symplectic form. If we swap the sign of the first component it becomes Γ_f, which is Lagrangian with respect to the flipped symplectic structure.

(3) The graph of the Geodesic flow.

$$C_t = \{(x, \xi, y, \eta) \mid G^t(x, \xi) = (y, \eta)\} \subset T^*X \times T^*X$$

This is a canonical graph for small time. (The flipped Lagrangian is the conormal bundle to distance spheres when they are submanifolds) At larger times this is no longer a symplectomorphism (cut locus)

(4) The graph of the canonical map $\chi : (x, \xi) \to (x, -\xi)$ on $T^*\mathbb{R}^n$. This is a horizontal Lagrangian; it is the graph of the differential of a phase function $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. In this case it is $-x \cdot \xi$. It corresponds to the semi-classical Fourier transform.

(5) The graph of the identity map. These are associated to semi-classical Pseudodifferential operators.

2.3. **Linear Composition.** We first describe the linear case where symplectic manifolds are replaced by symplectic vector spaces. Here composition is always defined. Suppose V_i are three symplectic vector spaces of dimensions $2n_i$ and $\Gamma_1 \subset V_1^- \oplus V_2$, $\Gamma_2 \subset V_2^- \oplus V_3$ are Lagrangian subspaces. Set $\Gamma_2 * \Gamma_1 = \{(x, y, y, z) \mid (x, y) \in \Gamma_1; (y, z) \in \Gamma_2\} = \Gamma_1 \times \Gamma_2 \cap (V_1 \times \Delta_{V_2} \times V_3)$. We have maps $\tau : \Gamma_1 \times \Gamma_2 \to V_2$ given by $\tau(x, y, y', z) = y - y'$ and $\alpha : \Gamma_2 * \Gamma_1 \to \Gamma_2 \circ \Gamma_1$ given by $\alpha(x, y, y, z) = (x, z)$. Then

$$\dim \Gamma_2 * \Gamma_1 + \dim \mathrm{Im}\tau = n_1 + 2n_2 + n_3$$

$$\dim \ker \alpha + \dim \Gamma_2 \circ \Gamma_1 = \dim \Gamma_2 * \Gamma_1$$

In order for $\Gamma_2 \circ \Gamma_1$ to be the correct dimension we need $\dim \ker \alpha + \dim \mathrm{Im}\tau = 2n_2$. This follows from the fact that $\{y \in V_2 \mid (0, y, y, 0) \in \Gamma_2 * \Gamma_1\} \cong \ker \alpha$ is symplectic orthogonal to $\mathrm{Im}\tau$. No trick just write it out. This shows that the composition is the correct dimension to be Lagrangian, and it is easy to check that it is isotropic. Now we consider general composition.

2.4. **Transverse Composition.** Suppose that $\Gamma_1 \subset M_1^- \times M_2$, $\Gamma_2 \subset M_2^- \times M_3$ are two canonical relations. We would like to define the composed canonical relation set theoretically as

$$\Gamma_2 \circ \Gamma_1 = \{(x,z) \in M_1 \times M_3 \mid \exists y \in M_2 : (x,y) \in \Gamma_1 \,\&\, (y,z) \in \Gamma_2\}$$

The problem is that we need conditions to ensure that this is a Lagrangian submanifold of $M_1^- \times M_3$. The most ideal case is transversality. We have the projection $\pi : M_1 \times M_2 \times M_2 \times M_3 \to M_1 \times M_3$ and two submanifolds of $M_1 \times M_2 \times M_2 \times M_3$, $\Gamma_1 \times \Gamma_2$ and $M_1 \times \Delta_{M_2} \times M_3$ where Δ_{M_2} is the diagonal in $M_2 \times M_2$. Clearly we have

$$\Gamma_2 \circ \Gamma_1 = \pi\left(\Gamma_1 \times \Gamma_2 \cap M_1 \times \Delta_{M_2} \times M_3\right)$$

To shorten notation, we define $\Gamma_2 * \Gamma_1$ to be the above intersection.

Lemma. *Suppose $\Gamma_1 \times \Gamma_2$ and $M_1 \times \Delta_{M_2} \times M_3$ intersect transversally. Then $\Gamma_2 * \Gamma_1$ is a submanifold of $M_1 \times M_2 \times M_2 \times M_3$ and $\pi|_{\Gamma_2 * \Gamma_1}$ is a Lagrangian immersion.*

Proof. We first show that the projection is an immersion. Fix $m = (m_1, m_2, m_2, m_3) \in \Gamma_2 * \Gamma_1$ and suppose that (v_1, v_2, v_2', v_3) is an element of $T_m(\Gamma_2 * \Gamma_1)$ lying in the kernel of $d\pi$. Then we must have $v_1 = v_3 = 0$ and $v_2 = v_2'$ and $(0, v_2) \in T_{(m_1,m_2)}\Gamma_1$ and $(v_2, 0) \in T_{(m_2,m_3)}\Gamma_2$. Our goal is to show that $v_2 = 0$ by demonstrating it is symplectic orthogonal to everything in $T_{m_2}M_2$.

Let $v' \in T_{m_2}M_2$ be arbitrary. By transversality we can write

$$(0, v_2, v', 0) = (x, y, z, w) + (\cdot, v'', v'', \cdot)$$

Where the first vector is tangent to $\Gamma_1 \times \Gamma_2$. Thus we have $v_2 = y + v''$ and $v' = z + v''$, or $v' = z + v_2 - y$ But now note that

$$\omega_2(v_2, v') = \omega_2(v_2, z + v_2 - y) = \omega_2(v_2, z) - \omega_2(v_2, y)$$

But we have $(0, v_2), (x, y) \in T_{(m_1,m_2)}\Gamma_1$ and $(v_2, 0), (z, w) \in T_{(m_2,m_3)}\Gamma_2$ Since these are Lagrangian we have $-\omega_1 \oplus \omega_2((0, v_2), (x, y)) = \omega_2(v_2, y) = 0$ and similarly $\omega_2(v_2, z) = 0$. The previous then shows that $v_2 \in (T_{m_2}M_2)^\perp$

Since the intersection is transverse the codimensions add up which implies that $\dim \Gamma_2 * \Gamma_1 = n_1 + n_3$, which is the correct dimension for the immersed image to be Lagrangian. Therefore it suffices to check that $d\pi^*(\omega_1^- \oplus \omega_3) = 0$. For this we simply notice that

$$d\pi^*(-\omega_1 \oplus \omega_3)((v_1, v_2, v_2, v_3), (v_1', v_2', v_2', v_3')) = -\omega_1(v_1, v_1') + \omega_2(v_2, v_2') - \omega_2(v_2, v_2') + \omega_3(v_3, v_3') = 0$$

□

If we want to ensure that the the composition be a submanifold of $M_1^- \times M_3$, then we need to further assume that $\pi|_{\Gamma_2 * \Gamma_1}$ is proper, injective map.

2.5. **Pushforward and pullback of Lagrangian submanifolds.** Given a smooth map $f : X \to Y$ we have the canonical relation $\Gamma_f : T^*X \to T^*Y$. If $\Lambda : \mathrm{pt} \to T^*X$ we can consider the composition $\Gamma_f \circ \Lambda : \mathrm{pt} \to T^*Y$. This is the pushforward Lagrangian, written $df_*\Lambda$. It is defined as long as the projection down $\pi : \Gamma_f \to T^*X$ is transverse to Λ.

The image of this projection is a subbundle of T^*X if f has constant rank, the fiber over x being the image of $(T^*_{f(x)}Y)$ under df_x^*. Here are two special cases: If f is an immersion, then df_x^* is surjective and hence the image of the projection is all of T^*X and is therefore transverse to anything. If f is a submersion, then the image of the pullback $df_x^*(T^*_{f(x)}Y)$ is the annihilator of the tangent space to the fiber, the horizontal bundle $H(X) \subset T^*X$. In order for pushforward to be defined, the horizontal bundle must be transverse to Λ.

On the other hand we have $\Gamma_f^\dagger = \{(y,\eta,x,\xi) \mid y = f(x) \,\&\, \xi = df_x^*\eta\} : T^*Y \to T^*X$ and if $\Lambda : \mathrm{pt} \to T^*Y$ is a Lagrangian then $\Gamma_f^\dagger \circ \Lambda$ is called the pullback Lagrangian, $df^*\Lambda$. Since η is unconstrained, the issue of transversality is only related to the first component. The composition is defined as long as f is transverse to the map $\pi|_\Lambda : \Lambda \to Y$.

2.6. **Application: The symbolic wave trace.** We consider the Lagragian associated to the Unitary wave group on a Riemannian manifold (X,g). This is the Lagrangian

$$\Lambda = \{(t,\tau,x,\xi,y,\eta) \in T^*\mathbb{R} \times T^*X \times T^*X \mid \tau = |\xi|_x \; G^t(x,\xi) = (y,\eta)\}$$

Consider the map $id \times \Delta : \mathbb{R} \times X \to \mathbb{R} \times X \times X$. The transversality of the projection down with $id \times \Delta$ holds and the pull-back Lagrangian is

$$(id \times \Delta)^*(\Lambda) = \{(t,\tau,x,\eta-\rho) \mid \tau = |\eta|_x \; ; \; G^t(x,\eta) = (x,\rho)\}$$

Consider the map $\pi : Y \times X \to Y$ given by projection onto the first component. The pushforward of a Lagrangian $\Lambda \subset T^*(Y \times X)$ is defined as long as the normal bundle to the fibers intersects the Λ transversally. In this case the tangent space to the fiber is the set of tangent vectors of the form $(0,v)$ so the normal bundle consists of all covectors of the form $(\eta,0)$ (the X 0 section). If the intersection with Λ is transverse then the pushforward

$$\pi_*(\Lambda) = \{(y,\eta) \in Y \mid \exists (y,\eta,x,0) \in \Lambda\}$$

When $Y = \mathbb{R}$ we can use this to compute the Lagrangian of the symbolic trace of the wave group

$$\pi_*(id \times \Delta)^*\Lambda_U = \{(t,\tau) \mid \exists (x,\eta) \in T^*X \, ; \, \tau = |\eta|_x \, ; G^t(x,\eta) = (x,\eta)\}$$

which proves that the singular set consists of $t \in \mathbb{R}$ such that $t = 0$ or $|t|$ is the length of a closed geodesic.