NOTES FOR MATH 520-1

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Abstract.

1. September 28, 2018

1.1. **Introduction.** References:

- Gunning, Rossi
- Hörmander
- Donaldson, Kronheimer
- Kobayashi, Nomizu

The first has a great chapter covering the basics of sheaf cohomology and the second has a lot of material on holomorphic vector bundles.

Our first goal in the course is the following result.

Theorem 1. *Let X be a compact complex manifold and $\mathcal{S}$ be a coherent sheaf on X. Then the sheaf cohomology groups are finite-dimensional, and moreover $H^p(X, \mathcal{S}) = 0$ if $p > \dim X$.*

Our second goal, which is wildly ambitious, is to prove what is in a sense a relative version of the above.

Theorem 2. *Let $\pi : X \to Y$ be a proper map of complex manifolds. If $\mathcal{S}$ is a coherent sheaf on X then the higher direct images $R^*\pi_*\mathcal{S}$ are coherent sheaves.*

Notice that for $Y = *$ we obtain the result above.

The third goal, in the context of the second, is to prove a generalization of the Hirzebruch-Riemann-Roch computes the Euler characteristic of a coherent sheaf in terms of a certain characteristic class. The Grothendieck-Riemann-Roch theorem is a relative generalization. We probably won't have time for this.

1.2. **Review of complex geometry.** Consider the coordinates $(z^1, \dots, z^n)$ on $\mathbb{C}^n$. We will write $z^j = x^j + iy^j$. We define

$$\bar{\partial}_j = \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

One checks that $\bar{\partial}_j z^k = 0$. Similarly we have the complex conjugate

$$\partial_j = \frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)$$

with the property that $\partial_j z^k = \delta_j^k$.

Definition 3. Let $U \subset \mathbb{C}^n$ be an open subset, and let $f : U \to \mathbb{C}$ be a differentiable function. Then f is holomorphic if $\bar{\partial}_j f = 0$, for $1 \le j \le n$.

In particular holomorphic functions are smooth. The set of holomorphic functions on U forms a $\mathbb{C}$-algebra denoted $\mathcal{O}(U)$. Now suppose we have $g : U \subset \mathbb{C}^l \to \mathbb{C}^m$ and $h : V \subset \mathbb{C}^m \to \mathbb{C}^n$ holomorphic functions such that $g(U) \subset V$.

Exercise 4. The composite $h \circ g : U \to \mathbb{C}^m$ is holomorphic.

Consider the case $m = n = 1$, where $g \in \mathcal{O}(U)$ is nowhere zero and $h(w) = 1/w$. We conclude that $1/g \in \mathcal{O}(U)$.

Let M_r be the set of complex-valued $r \times r$ matrices. Observe that $M_r(\mathcal{O}(U))$ is isomorphic to the holomorphic functions from U to M_r. The map

$$\det : M_r(\mathcal{O}(U)) \to \mathcal{O}(U)$$

is a polynomial map, whence holomorphic.

Corollary 5. $f \in M_r(\mathcal{O}(U))$ *has a holomorphic inverse if it is everywhere invertible on* U*.*

Proof. Use Cramer's rule. □

Recall that a real manifold is a topological space together with an atlas: a covering by Euclidean spaces and smooth transition functions between them.

Definition 6. A complex manifold X is the following data. We have a holomorphic atlas $\{U, \phi)\}$, i.e. the $U \subset X$ form an open cover of X and $\phi : U\mathbb{C}^n$ homeomorphisms, such that given U_1, U_2 with nontrivial intersection, the map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is holomorphic.

By definition of complex manifold the notion of a holomorphic functions is well-defined: it is simply a function which when transported to a function on $\mathbb{C}^n$ by the holomorphic atlas, is holomorphic.

Now let X be a complex manifold. It is of course a real manifold, whence we have a de Rham complex of smooth differential forms,

$$(A^*(X, \mathbb{C}), d).$$

Locally we can use coordinates $dz^1, \ldots, dz^n, d\bar{z}^1, \ldots, d\bar{z}^n$ to decompose

$$A^1(X, \mathbb{C}) \cong A^{1,0}(X) \oplus A^{0,1}(X).$$

It is then straightforward linear algebra to check that

$$A^k(X, \mathbb{C}) \cong \oplus_{p+q=k} A^{p,q}(X)$$

and that these notions globalize (as already done by our notation). Now one introduces locally operators

$$\partial = \sum dz^j \partial_j$$
$$\bar{\partial} = \sum d\bar{z}^j \bar{\partial}_j$$

and that these are invariant under change of coordinates. We thus define

$$d = \partial + \bar{\partial}.$$

which satisfies

$$\partial^2 = \bar{\partial}^2 = 0 \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Definition 7. The qth Dolbeault cohomology of a complex manifold X is defined

$$H^{p,q} X = H^q(X, (A^{p,*}, \bar{\partial})).$$

Eventually we will show that $H^{p,q}X \cong H^q(X, \Omega^p)$, i.e. Dolbeault cohomology computes sheaf cohomology.

2. October 1, 2018

2.1. **Chern classes.** For more details on characteristic classes see Milnor and Stasheff.

We will be discussing the map

$$H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{C})$$

where the first is singular cohomology and the latter is identified with de Rham cohomology. This map is an injection for projective spaces and more generally, Grassmannians.

Let's start by defining the first Chern class. Eventually we will define the kth Chern class of a complex vector bundle $E \to X$ for X a nice enough topological space. The technical condition on X necessary is paracompactness. The kth Chern class is

$$c_k(E) \in H^{2k}(X,\mathbb{Z}).$$

If X is an oriented surface (Riemann surface if its differentiable) then $c_1(E)$ is the degree of the vector bundle E. Say the rank of E is r. Then we may form a line bundle $\Lambda^r E$, which is constructed by applying the map $\det : GL_r \to GL_1$ to the cocycle defining E. Then

$$c_1(E) = c_1(\Lambda^r E) = \deg(E) \cdot [X],$$

where $[X]$ is the fundamental class of X.

Let's begin by defining the first Chern class for a line bundle $L \to X$. Cover X by open sets $\{U_\alpha\}_{\alpha\in I}$. By paracompactness we may assume that this is locally finite – if you like you may as well assume X is compact and the cover is finite, since that's what we'll mostly be thinking about. We may assume that each U_α is contractible (this is a good cover) as are all the finite intersections, if we're on a manifold. Now over each U_α choose a nowhere vanishing section (since L is trivial over contractibles), which yields transition functions on overlaps

$$g_{\alpha\beta} : U_{\alpha\beta} \to GL_1(\mathbb{C}) = \mathbb{C}^\times.$$

satisfying

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \qquad g_{\alpha\alpha} = 1,$$

a normalized cocycle. Now since each $U_{\alpha\beta}$ is contractible, we define

$$h_{\alpha\beta} = \log g_{\alpha\beta}$$

where we take any branch of log. Sheaf theoretically, we have a short exact sequence

$$2\pi i\mathbb{Z} = \mathbb{Z}(1) \to C^0 \xrightarrow{\exp} GL_1(\mathbb{C})$$

and we're looking at

$$0 = \check{H}^1(\mathcal{U}, C^0) \to \check{H}^1(\mathcal{U}, GL_1(\mathbb{C})) \to \check{H}^2(\mathcal{U}, Z(1)) \to \check{H}^2(\mathcal{U}, C^0) = 0$$

where the middle map is sending $[g] \mapsto 2\pi i c_1(L)$. We have

$$(\delta h)_{\alpha\beta\gamma} = h_{\alpha\beta} - h_{\alpha\gamma} + h_{\beta\gamma} \in \mathbb{Z}(1)$$

and $\delta^2 = 0$. Hence we obtain an explicit cocycle representative

$$\delta h \in \check{Z}^2(\mathcal{U}, \mathbb{Z}(1))$$

Of course, one should check that this is well-defined (we have chosen a cover as well as nonvanishing sections over the corresponding opens).

Instead of constructing explicitly the higher Chern classes, we will give an axiomatic definition, as it is significantly simpler. For $E \to X$ a vector bundle of rank r the Chern classes sit in a power series

$$c_t(E) = 1 + tc_1(E) + t^2 c_1(E) + \cdots + t^r c_r(E) \in \prod_k H^{2k}(X, \mathbb{Z}).$$

Notice that $c_k(E) = 0$ if $k > \operatorname{rk} E = r$. Given a short exact sequence

$$0 \to E \to F \to F/E \to 0$$

we require that

$$c_t(F) = c_t(E)c_t(F/E).$$

Moreover we require that $c_1(E) = c_1(\Lambda^r E)$, where the right hand side we have already defined in the paragraph above. Finally we require functoriality: given a map $f : X \to Y$ we obtain that $c_t(E) = c_t(f^*E)$.

The idea behind the construction of the Chern classes is the splitting principle. Given $E \to X$ we have the corresponding projectivization $\mathbb{P}E \to X$ whose fibers are $\mathbb{C}P^{r-1}$. Using the Leray spectral sequence one can show that

$$H^*(\mathbb{P}E, Z) \cong H^*(X, \mathbb{Z}) \otimes H^*(\mathbb{C}P^{r-1}, \mathbb{Z}).$$

Actually we don't even need this – all we need is that the pullback

$$H^*(X, \mathbb{Z}) \hookrightarrow H^*(\mathbb{P}E, \mathbb{Z})$$

is an injection. There is a tautological line bundle over $\mathbb{P}E$, call it $\mathcal{O}(-1)$, and we get a short exact sequence

$$0 \to \mathcal{O}(-1) \to \pi^*E \to \pi^*E/\mathcal{O}(-1) \to 0$$

which completely determines the Chern classes by induction on rank of r.

The key case to bear in mind is the sum of line bundles $L_1 \oplus \cdots \oplus L_r$. One computes that

$$c_k(L_1 \oplus \cdots \oplus L_r) = \sum_{1 \le i_1 < \cdots < i_k \le r} c_1(L_i) \smile \cdots \smile c_1(L_{i_k}).$$

2.2. **Chern-Weil theory.** Let's shift gears now and assume that our space X is a smooth manifold and that E is a smooth vector bundle. In this case we can give a formula for the Chern classes in de Rham cohomology. Before doing this let me remind you of Weil's proof of the de Rham theorem (see for instance Bott and Tu). The point is that we choose a smooth partition of unity for a good open cover on X (recall that this means each intersection is empty or diffeomorphic to a starlike region). Good open covers can be constructed by taking (a locally finite subcover of) a cover by small enough geodesic balls for some arbitrary Riemannian metric. Anyway, choose a smooth partition of unity $\phi_\alpha \in C_c^\infty(U_\alpha)$, i.e. $\sum_\alpha \phi_\alpha = 1$. Then we obtain a map as follows. Consider $[c_{\alpha_0,\ldots,\alpha_k}] \in \check{C}^k(\mathcal{U}, \mathbb{Z})$ and map it to the differential form

$$\sum c_{\alpha_0,\ldots,\alpha_k} \phi_{\alpha_0} d\phi_{\alpha_1} \wedge \cdots \wedge d\phi_{\alpha_n}.$$

After tensoring with $\mathbb{R}$ (or $\mathbb{C}$) this yields a quasi-isomorphism

$$\check{C}^*(\mathcal{U}, \mathbb{R}) \to A^*(X, \mathbb{R}).$$

This map arises in the study of a certain double complex.

We now quickly review connections on vector bundles. A connection ∇ on E is a map

$$\nabla : \Gamma(X, E) \to A^1(X, E),$$

that is often written

$$\nabla_\xi s = \iota(\xi)\nabla s$$

for ξ a vector field on X, that satisfies:

- $\mathbb{C}$-linearity,
- $\nabla(fs) = df \otimes s + f\nabla s$.

In other words,

$$\nabla_\xi(fs) = \mathcal{L}_\xi fs + f\nabla_\xi s.$$

Of course, a general vector bundle need not have a natural connection, whereas local systems, for instance, do. Notice that the difference of two connections is

$$\nabla - \nabla' \in A^1(X, \mathrm{End}(E))$$

while the connection itself is not tensorial. Why do connections exist in the first place? Well choose a connection ∇^α on each open set: for instance if the cover is fine enough one can take the zero connection. Now combine then via a partition of unity $\nabla = \sum \phi_\alpha \nabla^\alpha$.

Remarkably, though ∇ is a first differential operator we can construct a tensor, i.e. a zeroth order differential operator, out of it as an invariant. This is the curvature

$$F \in A^2(X, \mathrm{End}(E)),$$

defined by

$$F(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}$$

for ξ, η vector fields. Then one checks that

$$F(\xi, \eta)(fs) = fF(\xi, \eta)s.$$

Now the punchline is that the Chern classes are constructed by choosing a connection and taking polynomials in F. Indeed, locally, after choosing a frame of E we obtain a matrix of 2-forms F^a_b. We consider

$$\det\left(\mathrm{id} - \frac{t}{2\pi i}F^a_b\right) \in \prod_{k=0}^{[\dim X/2]} \Omega^{2k}(X)$$

Since determinant is conjugation invariant what we obtain is a global differential form. We claim that this is in fact a closed form, and in cohomology is independent of the connection chosen, and equal to the Chern class defined earlier.

In the next lecture we'll refine this story by considering connections on holomorphic vector bundles.

3. October 3, 2018

3.1. **Computations with connections.** Take a cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of our manifold. Fix a frame $e^\alpha_a \in \Gamma(U_\alpha, E)$ for $1 \le a \le r$ over each open. These frames are related on overlaps by

$$e^\alpha_a = \sum_{b=1}^r (g_{\alpha\beta})^b_a e^\beta_b,$$

where $g_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, GL_r(\mathbb{C}))$. One immediately finds that

$$g_{\alpha\beta} = g^{-1}_{\beta\alpha} \qquad g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$$

where the latter takes place on $U_{\alpha\beta\gamma}$. In other words we have a normalized 1-cocycle. Now since any section can be written locally in terms of the frame $s = \sum_a s^a e^\alpha_a$ for $s^a \in C^\infty(U_\alpha)$, connections are determined by how they act on the frame:

$$\nabla s = \sum_a ds^a \otimes e^\alpha_a + \sum_a s^a \nabla e^\alpha_a.$$

Write

$$\nabla e^\alpha_a = \sum_{b=1}^r A^{\alpha b}_a e^\alpha_b$$

where $A^\alpha \in A^1(U_\alpha, M_r(\mathbb{C}) = \mathfrak{gl}_\mathfrak{r}(\mathbb{C}))$ are the connection one-forms of ∇. How do these connection one-forms transform? A straightforward local computation shows that

$$A^\alpha = dg_{\alpha\beta} g^{-1}_{\alpha\beta} + g_{\alpha\beta} A^\beta g^{-1}_{\alpha\beta}.$$

Notice that if $r = 1$ then $GL_1(\mathbb{C}) = \mathbb{C}^\times$ is abelian whence this formula simplifies to

$$A^\alpha = A^\beta + dg_{\alpha\beta} g^{-1}_{\alpha\beta}.$$

Consider, on U_α, the trivial connection $\nabla^\alpha e^\alpha_a = 0$. Now suppose that we have a partition of unity subordinate to U_α. We are going to use that convex combinations of connections are again connections. This is because the difference between two connections forms a vector space. We now define $\nabla = \sum_\alpha \phi_\alpha \nabla^\alpha$.

Exercise 8. Check that ∇ as defined is a connection.

On U_α we now have that the connection one-form is

$$A^\alpha = \sum \phi_\beta dg_{\alpha\beta} g^{-1}_{\alpha\beta} \in A^1(U_\alpha, \mathfrak{gl}_r(\mathbb{C})).$$

One might continue then and compute the curvature.

Exercise 9. For $r = 1$, compute the curvature and make sure that the result is a global two-form. In particular check that

$$F^\alpha = g_{\alpha\beta} F^\beta g^{-1}_{\alpha\beta}.$$

In general the formula is

$$F^\alpha = dA^\alpha + A^\alpha \wedge A^\alpha.$$

3.2. **Hermitian and holomorphic vector bundles.** Recall that a Hermitian vector bundle is a vector bundle equipped with a continuously/smoothly/etc. varying (nondegenerate) positive sesquilinear form. Recall that a positive sesquilinear form satisfies for $\lambda \in \mathbb{C}$,

$$(\lambda s, t) = \lambda(s,t), \qquad (s,t) = \overline{(t,s)}$$

with $(s,s) > 0$ if $s \neq 0$. On a vector bundle we now insert sections instead of vectors, $s, t \in \Gamma(X, E)$ and $\lambda \in C^\infty(X, \mathbb{C})$. Hermitian vector bundles exist because locally one might take the (complex) Euclidean inner product and then globalize via partitions of unity:

$$(s,t) = \sum \phi_\alpha (s,t)_\alpha$$

where

$$(e_a^\alpha, e_b^\alpha) = \delta_{ab}.$$

On a Hermitian vector bundle there is a notion of a compatible connection. One way to say it is the following. Notice that $h = (-,-) \in \Gamma(X, E^* \otimes \bar{E}^*)$ so a connection on E yields a connection on $E^* \otimes \bar{E}^*$. Then the compatibility condition is simply that $\nabla h = 0$, where ∇ is the induced connection. Another, more practical approach, is to require that

$$d(s,t) = (\nabla s, t) + (s, \nabla t).$$

What does this say about the connection one-form? Choose e_a^α an orthonormal frame, using say Gram-Schmidt. Now we have $(e_a^\alpha, e_b^\alpha) = \delta_{ab}$ whence the compatibilty condition becomes

$$\begin{aligned} 0 &= (A_a^{\alpha c} e_c^\alpha, e_b^\alpha) + (e_a^\alpha, A_b^{\alpha c} e_c^\alpha) \\ &= A_{ab}^\alpha + A_{ba}^\alpha. \end{aligned}$$

In other words the connection one-form matrix needs to be skew-Hermitian. Of course this is coming from the Lie algebra of the unitary group, i.e.

$$g_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, U(r))$$

where $U(r)$ is the unitary group. Similarly the curvature will satisfy

$$F_{ab}^\alpha + \overline{F_{ba}^\alpha} = 0.$$

Now a holomorphic vector bundle is a bundle whose cocycles $(g_{\alpha\beta})_a^b$ are holomorphic. Now we can define the holomorphic sections as those whose coefficients locally are holomorphic — this is well-defined because the transition cocycles are holomorphic. Of course the local frames themselves are thus holomorphic. We denote the sheaf of holomorphic sections by $\mathcal{O}(U, E)$.

If we have a connection on a holomorphic bundle, we say that ∇ is holomorphic if

$$\nabla s \in A^{1,0}(U, E)$$

for $s \in \mathcal{O}(U, E)$. For instance the de Rham differential is a holomorphic connection on the trivial line bundle. From the definition we immediately conclude that the curvature of a holomorphic connection lies in $A^{2,0}(X, \operatorname{End} E)) \oplus A^{1,1}(X, \operatorname{End} E)$, by say the formula given above. Our first main theorem about holomorphic connections is that this is an if and only if! We'll explain more next time.

Finally, consider a holomorphic vector bundle with a Hermitian structure. In this case we can ask for a connection to simultaneously be Hermitian (compatible with the Hermitian metric) and holomorphic. In this case we will find that the curvature is a $(1,1)$-form. Remarkably, it turns out that there's a uniquely defined such connection! One constructs it locally and then by uniqueness globalize.

Exercise 10. Figure out this formula for ∇e_a^α for e_a^α a holomorphic frame. Hint: consider the inner product $(\nabla e_a^\alpha, e_b^\alpha)$.