FACTORIZATION HOMOLOGY

JOHN FRANCIS

Contents

These are notes from John Francis' "Factorization homology" course taught during the Fall quarter of the 2017 year at Northwestern. Errors and inaccuracies are, as usual, due to the notetaker(s).

1. What is factorization homology? [09/20/17]

1.1. **Introduction.** What is factorization homology? Well, if it were an animal, I could describe it in two ways: distribution and phylogeny. More specifially, we will

Date: Fall 2017.

first see how factorization homology is distributed over the face of the planet. Then we will describe how it evolved from single-celled organisms, i.e. how you might come up with it yourself.

For the moment you can think of factorization homology as a sort of

generalized (co)sheaf homology.

Notice that this phrase can be hyphenated in two different ways. In one sense it is a generalization of the ideas of sheaf cohomology, and in the other it is a homology theory for generalized sheaves (or sheaf-like objects). In particular, factorization homology is a machine that takes two inputs: a geometry M and an algebraic object A. The output is

$$\int_M A,$$

the factorization homology of with coefficients in A.

1.2. **Examples.** Let's look at the first description: what are some examples of factorization homology that appear naturally in mathematics?

(1) **Homology.** Here M is a topological space and A is an abelian group. In this case the output is a chain complex

$$\int_M A \simeq \mathrm{H}_\bullet(M, A),$$

quasiisomorphic to singular homology with coefficients in A.

John: What is a theorem you can't prove without ordinary homology?

(2) **Hochschild homology.** Here M is a one-dimensional manifold – let's take in particular $M = S^1$ – and A will be an associative algebra. In this case

$$\int_{S^1} A \simeq \mathrm{HH}_\bullet A,$$

the Hochschild homology of A. You might be less familiar with this algebraic object than ordinary homology. It's importance comes from how it underlies trace methods in algebra (e.g. characteristic 0 representations of finite groups). Hochschild homology is a recipient of "the universal trace" and hence an important part of associative algebra. Note that $\mathrm{HH}_0 A = A/[A, A]$.

(3) **Conformal field theory.** This is in some sense the real starting point for the ideas we will develop in this class. Here M is a smooth complete etc. algebraic curve over $\mathbb{C}$ and A is a vertex algebra. In this case the output $\int_M A$ was constructed by Beilinson and Drinfeld, and is known as chiral homology of M with coefficients in A. It is a chain complex, with $\mathrm{H}_0(\int_M A)$ being the space of conformal blocks of the conformal field theory.

(4) **Algebraic curves over $\mathbb{F}_q$.** Here M is an algebraic curve over $\mathbb{F}_q$ and G is a connected algebraic group over $\mathbb{F}_q$. In this case $\int_M G$ is known as the Beilinson-Drinfeld Grassmannian and is a stack. One interesting property that it has is that

$$\mathrm{H}_\bullet\left(\int_M G, \bar{\mathbb{Q}}_\ell\right) \simeq \mathrm{H}_\bullet(\mathrm{Bun}_G(M), \bar{\mathbb{Q}}_\ell),$$

where here we are taking ℓ-adic cohomology. Although the Beilinson-Drinfeld Grassmannian is more complicated than the stack of principal G-bundles, it is more easily manipulated. We note that the equivalence above is a form of nonabelian Poincaré duality.

In particular, one might be interested in computing

$$\chi(\mathrm{Bun}_G(M)) = \sum_{[P]} \frac{1}{|\operatorname{Aut}(P)|},$$

which makes sense over a finite field. The computation of this quantity is known as Weil's conjecture on Tamagawa numbers.

(5) **Topology of mapping spaces.** Now M is an n-manifold without boundary and A will be an n-fold loop space, $A = \Omega^n Z = \mathrm{Maps}((D^n, \partial D^n), (Z, *))$. The output is a space weakly homotopy equivalent to $\mathrm{Maps}_c(M, Z)$ if $\pi_i Z = 0$ for $i < n$. This is also known as nonabelian Poincaré duality. Again the left hand side is more complicated but more easily manipulated.

(6) **n-disk algebra (perturbative TQFT).** Here M is an n-manifold and A is an n-disk algebra (or an E_n-algebra) in chain complexes. The output is a chain complex and has some sort of interpretation in physics. One thinks of A as the algebra of observables on $\mathbb{R}^n$, and $\int_M A$ is the global observables (in some derived sense). In a rough cartoon of physics, one assigns to opens sets of observables, and a way to copmute expectation values. Factorization homology puts together local observables to global observables:

$$\mathrm{Obs}(M) \simeq \int_M A,$$

at least if we are working in perturbative QFT.

(7) **TQFT.** Here M is an n-manifold (maybe with a framing) and A is an (∞, n)-category (enriched in $\mathcal{V}$). The output is a space (if enriched, an object of $\mathcal{V}$), which is designed to remove the assumptions from the examples above.

That's all the examples for now. Next class we'll go over how one might have come up with factorization homology. It is worth noting that in this class we will focus on learning factorization homology as a **tool** instead of aiming to reach some fancy theorem. Hopefully this will teach you how to apply it in contexts you might be interested in.

Pax: What is the physical interpretation of the first and second chiral homologies? **John:** One might be interested in things like Wilson lines, where these higher homology groups come into play.

2. How to come up with factorization homology yourself [09/22/17]

2.1. **Kan extensions.** Consider the following thought experiment. Suppose you want to study objects in some context $\mathcal{M}$. Unfortunately objects here are pretty hard in general. Inside $\mathcal{M}$, however, we have some objects $\mathcal{D} \subset \mathcal{M}$ that are particulary simple, and moreover we know that everything else in $\mathcal{M}$ is "built out of" objects in $\mathcal{D}$.

Let's consider the example where $\mathcal{M}$ is a nice category of (homotopy types of) topological spaces. Let $\mathcal{D}$ consist of the point, i.e. all contractible spaces. Now to study $\mathcal{M}$ we might map functors out of it into some category $\mathcal{V}$. Let's start with $\mathcal{D}$ instead. Consider $\mathrm{Fun}(*, \mathcal{V})$. Of course this is canonically just $\mathcal{V}$. How do we extend this to studying $\mathcal{M}$? We have an obvious restriction map

$$\mathrm{Fun}(\mathcal{M}, \mathcal{V}) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}(*, \mathcal{V}).$$

John: If you don't know what a left adjoint is you should learn it because I won't tell you. No, I'm not joking (laughs).

We want to look for a left adjoint to this functor ev_* There are two different things we could do. We could ignore the homotopy-ness of everything, and take the naive categorical left-adjoint. If, say $\mathcal{V}$ is the category of chain complexes, this naive left-adjoint produces a stupid answer…depending on what our precise definitions are. Let's suppose that by $\mathcal{M}$ we meant the homotopy category of spaces (here objects are spaces and maps are sets of homotopy classes of maps). Then we are extending

$$\begin{tikzcd} * \arrow[r, "A"] \arrow[d] & \mathcal{V} \\ \mathsf{hoSpaces} \arrow[ur, dashed] & \end{tikzcd}$$

A naive left adjoint would take the functor A to the functor sending a space X to the stupid answer $A^{\oplus \pi_0 X}$ (on morphisms take summands to summands corresponding to where the connected components are sent). Similarly if we take $\mathcal{M}$ to be just spaces and all continuous maps, X would be sent to $A^{\oplus X}$. Here by X we mean the underlying set of elements of X.

Why is this a left adjoint?

There is a more sophisticated notion of a derived or homotopy left adjoint. Suppose now that by $\mathcal{M}$ we mean the topological category of spaces, where the mapping sets are spaces equipped with the compact-open topology. Now we take a *homotopy* Kan extension. This fancy left adjoint will now send a space X to the the chain complex $C_*(X, A)$ (up to equivalence). Hence we see that we can recover homology from this paradigm of extending a simpler invariant to the whole category.

How do we choose what $\mathcal{D}$ and $\mathcal{M}$ are? Suppose we want to study $\mathcal{F}(M)$ for $M \in \mathcal{M}$. For concreteness, let's say we're studying manifolds. The most basic question to ask: is there a local-to-global principle for $\mathcal{F}$? The simplest case is for $\mathcal{F}$ to be a sheaf, i.e.

$$\mathcal{F}(M) \xrightarrow{\sim} \lim_{U \in \mathcal{U}} \mathcal{F}(U).$$

If so you don't need factorization homology, and you can just leave.

For instance, consider $\mathcal{F} = C_*(\mathrm{Maps}(\cdot, Z))$ taking spaces to chain complexes. Is this a sheaf? Well if we forget about C_*, we get a sheaf, as a map into Z is the same as giving maps on subsets of the domain that agree on overlaps. What does taking chains do? Well notice that

$$\begin{aligned} C_*(\mathrm{Maps}(U \coprod V, Z)) &= C_*(\mathrm{Maps}(U,Z) \times \mathrm{Maps}(V,Z)) \\ &= C_*(\mathrm{Map}(U,Z)) \otimes C_*(\mathrm{Maps}(V,Z)). \end{aligned}$$

This is not a sheaf because in this case tensor products and direct sums are never the same for these chain complexes! So what can we do? We need to change what we consider $\mathcal{D}$ to be from open coverings to something else. Why?

Idea: to study $\mathcal{F}$ maybe there are more general arrangements of $\mathcal{D} \subset \mathcal{M}$ such that local-to-global principles still apply, without $\mathcal{F}$ being a sheaf.

2.2. **Manifolds.** The following problem will guide us for the next few weeks.

> Let M be a manifold and let Z be a space. Calculate the homology of the mapping space $\mathrm{H}_\bullet \mathrm{Maps}(M, Z)$.

To begin, let us specify which categories we will be working with.

Definition 1. Let Mfld_n be the (ordinary) category of smooth n-manifolds, with $\mathrm{Hom}(M,N) = \mathrm{Emb}(M,N)$ the set of smooth embeddings of M into N. Similarly, let $\mathcal{M}\mathsf{fld}_n$ be the *topological* category of smooth n-manifolds, with $\mathrm{Hom}(M,N) = \mathrm{Emb}(M,N)$ the *space* of smooth embeddings of M into N, equipped with the compact open smooth topology.

The compact open smooth topology takes a bit of work to define, so we'll leave that as background reading. A good reference is Hirsch's book on differential topology [Hir94]. Roughly, convergence in this topology is pointwise in the map as well as all its derivatives. To get a feel for what this entails, consider a knot. Locally tighten the knot until the knot turns (locally) into a line. These knots would would converge in the usual compact-open topology to another knot, but in the smooth topology, they do not converge as the tightening procedure creates sharp kinks. In particular $\pi_0\mathrm{Emb}(S^1, \mathbb{R}^3)$ is very different from $\pi_0\mathrm{Emb}^{\mathrm{top}}(S^1, \mathbb{R}^3)$.

Definition 2. We define the category Disk_n to be the full subcategory of Mfld_n where the objects are finite disjoint unions of standard Euclidean spaces $\coprod_I \mathbb{R}^n$. Similarly the category $\mathcal{D}\mathsf{isk}_n$ is the full *topological* subcategory of $\mathcal{M}\mathbf{fld}_n$ where the objects are finite disjoint unions of Euclidean space.

Observe that $\mathrm{Hom}_{\mathcal{D}\mathsf{isk}_n}(\mathbb{R}^n, \mathbb{R}^n) = \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma 3. *The map* $\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n) \to GL_n(\mathbb{R}) \simeq O_n\mathbb{R}$ *given by differentiating at the origin is a homotopy equivalence.*

Proof sketch. There is an obvious map $GL_n\mathbb{R} \to \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$. One of the composites is thus clearly the identity. It remains to show that the other composition is homotopic to the identity. The homotopy is given by shrinking the embedding down to zero. □

This fact should fill you with hope. The objects which are building blocks of manifolds have automorphism spaces that are, up to homotopy, just finite-dimensional manifolds. Actually it will be useful to think of the n-disks as some sort of algebra.

Definition 4. An n-disk algebra in $\mathcal{V}$ is a symmetric monoidal functor $A : \mathcal{D}\mathsf{isk}_n \to \mathcal{V}$.

As we stated before, our first goal in this class is to understand the homology $\mathrm{H}_* \operatorname{Maps}(M, Z)$ using n-disk algebras and factorization homology.

Question from someone: what's the relation with E_n-algebras? John: It turns out that E_n-algebras are equivalent to n-disk algebras with framing.

Question from Tochi: what if you work with manifolds with boundary? John: well if you require boundaries to map to boundaries you can make the same definitions. You then have to work with Euclidean spaces and half-spaces. You'll end up with two types of algebras instead of just n-disk algebras.

3. Framings [09/25/17]

3.1. **Framed embeddings, naively.**

Definition 5. A **framing** of an n-manifold M is an isomorphism of vector bundles $TM \cong M \times \mathbb{R}^n$.

Of course, not all manifolds have framings. For instance, one can check that all (compact oriented) two-manifolds except for $S^1 \times S^1$ do not admit framings. You might use the Poincaré-Hopf theorem, which expresses the Euler characteristic as a sum of the index of the zeroes of a vector field v on M that has isolated zeroes. Hence if M is framed, the Euler characteristic of M must be zero.

Here is an example of a theorem that John does not know how to prove without the use of homology.

Theorem 6 (Whitney or Wu)**.** *Every orientable three-manifold admits a framing.*

Pax: isn't there a later proof of this via geometric methods by Kirby? John: well ok I don't know how to prove it without homology...

Notice that any Lie group has a framing, as one takes a basis for the Lie algebra and pushes it forward by the group action. On the other hand, manifolds of dimension four generally do not have framings (at least in John's experience).

We can ask the following question: what is a framed open embedding? There are a few options. The naive (strict) option is as follows. Suppose that we have an open embedding $M \hookrightarrow N$ of framed manifolds. The pullback of TN is TM, we have two different trivializations of TM. We might ask that the induced map of trivial bundles $M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ be the identity. In other words, we ask the two framings to be the same.

Okay fine, but lets think about what we want the answer to be. Embeddings are very flexible you can stretch them and twist them. But strict framed embeddings are very rigid the way we've defined them above. For instance, they are automatically isometries (giving the fibers the usual Euclidean metric). But of course there aren't very many isometric embeddings into a compact manifold. Thus the strict definition of a framed embedding is not what we want to work with.

3.2. **Framed embeddings, homotopically.**

Let's consider a more lax definition. Thinking homotopy theoretically, recall that the tangent bundle is classified by a map $TM : M \to \mathrm{Gr}_n\,\mathbb{R}^\infty$. This map is of course defined only up to homotopy. That's fine, just choose a representative. Over the infinite Grassmannian we have the infinite Stiefel manifold $V_k(\mathbb{R}^\infty) \to \mathrm{Gr}_n\,\mathbb{R}^\infty$. Choosing a lift

$$\begin{tikzcd} & V_n\mathbb{R}^\infty \arrow[d] \\ M \arrow[ur, dashed, "\phi_M"] \arrow[r, "TM"'] & \mathrm{Gr}_n\,\mathbb{R}^\infty \end{tikzcd}$$

is precisely the data of a framing. Suppose now that we have an embedding $M \hookrightarrow N$ where M, N are framed by ϕ_M and ϕ_N respectively. The lax definition of a framed embedding is now going to be extra data: an embedding together with a homotopy between the framings ϕ_M and $\phi_N|_M$. Why?

Definition 7. The space of framed embeddings $\mathrm{Emb}^{fr}(M,N)$ is the homotopy pullback

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(M,N) \arrow[r] \arrow[d] & \mathrm{Emb}(M,N) \arrow[d] \\
\mathrm{Maps}_{V_n\mathbb{R}^\infty}(M,N) \arrow[r] & \mathrm{Maps}_{\mathrm{Gr}_n\mathbb{R}^\infty}(M,N)
\end{tikzcd}$$

In particular a framed embedding is an embedding $M \hookrightarrow N$ and a homotopy in $\mathrm{Map}_{\mathrm{Gr}_n\mathbb{R}^\infty}(M,N)$ between the images along each map.

Exercise 8. Check that $V_n\mathbb{R}^\infty \simeq *$.

With all this talk of homotopy pullbacks (which we'll talk about in more detail next time) it looks like we've made things more complicated, whereas we introduced framings to make things simpler. Let's calculate $\mathrm{Emb}^{fr}(\mathbb{R}^n,\mathbb{R}^n)$ as an example. By definition, this sits in the following diagram

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(\mathbb{R}^n,\mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n) \arrow[d] \\
\mathrm{Maps}_{V_n\mathbb{R}^\infty}(\mathbb{R}^n,\mathbb{R}^n) \arrow[r] & \mathrm{Maps}_{\mathrm{Gr}_n\mathbb{R}^\infty}(\mathbb{R}^n,\mathbb{R}^n).
\end{tikzcd}$$

Notice that the bottom left object is homotopy equivalent to $\mathrm{Maps}_*(\mathbb{R}^n,\mathbb{R}^n) \simeq *$. The bottom right space is homotopy equivalent to the loop space $\Omega\,\mathrm{Gr}_n\mathbb{R}^\infty \simeq \Omega BO(n) \simeq O(n)$. From last time, $\mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n) \simeq \mathrm{Diff}(\mathbb{R}^n) \simeq GL(n) \simeq O(n)$ (this is **homework 1**). Now the vertical map on the right is a homotopy equivalence. This implies (by some machinery) that the vertical map on the left is an equivalence. We conclude that

$$\mathrm{Emb}^{fr}(\mathbb{R}^n,\mathbb{R}^n) \simeq *.$$

The rest of **homework 1** is to show that $\mathrm{Emb}(\mathbb{R}^n,N)$ is homotopy equivalent to the frame bundle of TN. Applying this to the diagram above where we replace the second copy of $\mathbb{R}^n$ with N, we obtain

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(\mathbb{R}^n,N) \arrow[r] \arrow[d] & \mathrm{Emb}(\mathbb{R}^n,N) \arrow[d] \\
\mathrm{Maps}_{V_n\mathbb{R}^\infty}(\mathbb{R}^n,N) \arrow[r] & \mathrm{Maps}_{\mathrm{Gr}_n\mathbb{R}^\infty}(\mathbb{R}^n,N).
\end{tikzcd}$$

Now the same argument will show that the vertical map on the right is an equivalence, and that the map of the left is an equivalence. It follows now that

$$\mathrm{Emb}^{fr}(\mathbb{R}^n,N) \simeq N.$$

Hence we see that by adding framings we are replacing the role of the orthogonal group by that of a point. Indeed, this will allow for an easier transition between algebra and topology.

Definition 9. We define the category $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}$ to be the topological category consisting of finite disjoint unions of open unit disks $\coprod_I D$ under rectilinear embeddings. In other words, embeddings which can be written as a composition of translations and dilations. Here we use the usual topology indcued from the smooth compact-open topology.

One advantage of rectilinear embeddings is that they are easy to analyze. For instance, the space of embeddings from a single disk to a single disk is contractible: take an embedding, translate it to the origin, and the expand it outwards. In this way $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(D, D) = \mathrm{Emb}^{\mathrm{rect}}(D, D)$ deformation retracts onto the identity map. More generally, one checks that there is a homotopy equivalence

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(\coprod D^n, D^n) \xrightarrow{\sim} \mathrm{Conf}_k(D^n)$$

Next time we will prove the following.

Proposition 10. *There is a homotopy equivalence $\mathscr{D}\mathsf{isk}_n^{rect} \simeq \mathscr{D}\mathsf{isk}_n^{fr}$.*

4. Homotopy pullbacks and framing [09/27/17]

Let's define more precisely some of the terms we used last time.

4.1. Homotopy pullbacks.

Definition 11. Suppose we have a map $f : X \to B$ together with a point $* \in B$. The homotopy fiber of $X \to B$ over $* \in B$ is the fiber product

$$\mathrm{hofiber}(f : X \to B) := \{*\} \times_B \mathrm{Maps}([0,1], B) \times_B X.$$

In particular it is the space of triples $(*, \phi, x)$ where $\phi(0) = *$ and $\phi(1) = f(x)$.

$$\begin{tikzcd} \mathrm{hofiber}(f) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & B \end{tikzcd}$$

Lemma 12. *The formation of homotopy fibers is homotopy invariant. More precisely, given an weak equivalence of spaces $X \to X'$ over B a pointed space via maps f and g,*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & X' \arrow[dl] \\ & B & \end{tikzcd}$$

then the homotopy fiber of f is weakly equivalent to the homotopy fiber of g.

Why are these fibrations?

Proof. Simply apply the (naturality of the) long exact sequence on homotopy groups for a Serre fibration to the map of fibrations

$$\begin{tikzcd} \mathrm{hofiber}(f) \arrow[rr] \arrow[d] & & \mathrm{hofiber}(g) \arrow[d] \\ \mathrm{Maps}([0,1], B) \times_B X \arrow[rr] \arrow[dr] & & \mathrm{Maps}([0,1], B) \times_B X' \arrow[dl] \\ & B & \end{tikzcd}$$

We conclude that $\pi_* \,\mathrm{hofiber}(f) \cong \pi_* \,\mathrm{hofiber}(g)$. ☐

Homework 2: Show, more generally, that homotopy pullbacks are homotopy invariant.

Recall last time we were discussing $\mathrm{Maps}_B(M, N)$ for some space B: maps "over" B. This object is defined to be the homotopy

$$\begin{tikzcd} \mathrm{Maps}_B(M,N) \arrow[r] \arrow[d] & \mathrm{Maps}(M,N) \arrow[d] \\ * \arrow[r] & \mathrm{Maps}(M,B) \end{tikzcd}$$

In our case the map on the bottom is (a choice of) the map classifying the tangent bundle of M. Returning to last lecture, notice that by homotopy invariance we can argue that $\mathrm{Maps}_{V_n\mathbb{R}^\infty}(M,N) \simeq \mathrm{Maps}(M,N)$ since $V_n\mathbb{R}^\infty \simeq *$. Hopefully this background fills in some of the gaps we left open during last lecture.

But here we are using homotopy invariance in the base?

4.2. **Framed vs rectilinear n-disks.** Let us now return to our assertion from last time.

What is a homotopy equivalence of topological categories?

Proposition 13. *There is a functor $\mathscr{D}\mathsf{isk}_n^{rect} \to \mathscr{D}\mathsf{isk}_n^{fr}$ which is a homotopy equivalence.*

Proof. Using the computations from last lecture we see that

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}}(\mathbb{R}^n, \mathbb{R}^n) \simeq * \simeq \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(D^n, D^n).$$

What this functor does on objects is clear. On morphisms, the framing is determined by the dilation factor present in the rectilinear embeddings. More generally, consider

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}\left(\coprod_I D^n, \coprod_J D^n\right) = \coprod_{\pi: I \to J} \prod_J \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}\left(\coprod_{\pi^{-1}(j)} D^n, D^n\right).$$

So it suffices to show that

$$\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}}\left(\coprod_I \mathbb{R}^n, \mathbb{R}^n\right) \simeq \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}\left(\coprod_I D^n, D^n\right).$$

Recall that $\mathrm{ev}_0 : \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}(\coprod D^n, D^n) \to \mathrm{Conf}_I(D^n)$ is a homotopy equivalence, which we mentioned ast time. Returning to our homotopy pullback square

$$\begin{tikzcd}
\mathrm{Emb}^{rect}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Emb}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[d] \\
* \simeq \mathrm{Maps}_{EO(n)}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[r] & \mathrm{Maps}_{BO(n)}(\coprod \mathbb{R}^n, \mathbb{R}^n)
\end{tikzcd}$$

notice that

$$\begin{tikzcd}
\mathrm{Fr}(TM) \simeq \mathrm{Emb}(\mathbb{R}^n, M) \arrow[d, "\mathrm{ev}_0"] & \mathrm{Emb}((\mathbb{R}^n, 0), (M, x)) \simeq O(n) \arrow[l] \arrow[d] \\
M & \{x\} \arrow[l]
\end{tikzcd}$$

Likewise

$$\begin{tikzcd}
\mathrm{Emb}(\coprod \mathbb{R}^n, M) \arrow[d, "\mathrm{ev}_0"] & \prod_I O(n) \arrow[l] \arrow[d] \\
\mathrm{Conf}_I(M) & \{x_1, \ldots, x_I\} \arrow[l]
\end{tikzcd}$$

Hence $\mathrm{Maps}_{BO(n)}(\coprod \mathbb{R}^n, \mathbb{R}^n) \simeq \prod_I \mathrm{Maps}_{BO(n)}(\mathbb{R}^n, \mathbb{R}^n) \simeq \prod_I O(n)$.

Up to homotopy, we now obtain

$$\begin{tikzcd}
\mathrm{Emb}^{fr}(\coprod \mathbb{R}^n, \mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Conf}_I(\mathbb{R}^n) \times \prod_I O(n) \arrow[d] \\
* \arrow[r] & \prod_I O(n)
\end{tikzcd}$$

so we conclude that $\mathrm{Emb}^{fr}(\coprod \mathbb{R}^n, \mathbb{R}^n) \simeq \mathrm{Conf}_I(\mathbb{R}^n)$ which concludes the proof of the proposition. $\square$

Example 14. Consider the case $n = 1$. What do the framed and rectilinear embeddings look like in this case? Well $\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}}(\coprod_I \mathbb{R}^1, \mathbb{R}^1) \simeq \mathrm{Conf}_I(\mathbb{R}^1)$ is discrete up to homotopy, and thus identified noncanonically with the symmetric group on I letters.

Recall a definition from the first day.

Definition 15. An $\mathcal{E}_n$ algebra in $\mathcal{V}$ is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathcal{V}^{\otimes}$.

Next time we will see that $\mathcal{E}_1$-algebras are, in a suitable sense, equivalent to associative algebras.

5. Examples of n-disk algebras [09/29/2017]

Notice that we have a functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}} \to \mathscr{D}\mathsf{isk}_n$. In particular, the former category has *less* structure than the latter. Why is this?

Let's recall the following way of thinking about a commutative algebra.

Definition 16. A commutative algebra in $\mathcal{V}^{\otimes}$ (a symmetric monoidal category) is a symmetric monoidal functor

$$(\mathsf{Fin}, \coprod) \xrightarrow{A} (\mathcal{V}, \otimes),$$

where Fin is the category of finite sets.

This probably looks a little unfamiliar, so let's unpack it. Observe that the underlying object is $A = A(*)$. The unit morphism is $A(\varnothing) = 1_{\mathcal{V}} \to A(*)$. Here $1_{\mathcal{V}}$ is the symmetric monoidal unit in $\mathcal{V}$. The multiplicative structure comes from the map from the two-point set to the one-point set, and the commutativity follows from the fact that this map is Σ_2-invariant and that A is a *symmetric* monoidal functor so that $A^{\otimes 2} \to A$ is Σ_2-invariant as well.

Definition 17. For $\mathcal{V}$ a symmetric monoidal topological category, an **n-disk algebra** is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n \to \mathcal{V}$. Similarly a **framed n-disk algebra** is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{fr}} \to \mathcal{V}$ and a **$\mathcal{E}_n$-algebra** is a symmetric monoidal functor $\mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathcal{V}$.

Today we will discuss examples of n-disk algebras for $\mathcal{V}$ being chain complexes and toplogical spaces.

(1) There are the trivial n-disk algebras. For instance, consider $A = \mathbb{Z}$, which sends

$$\textstyle\coprod_I \mathbb{R}^n \longrightarrow \mathbb{Z}^{\otimes I} \cong \mathbb{Z}$$

and any embedding

$$\textstyle\coprod_I \mathbb{R}^n \hookrightarrow \coprod_J \mathbb{R}^n \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}.$$

We can all agree that this is pretty trivial. More generally, we might take $A = \mathbb{Z} \oplus B$, which sends $\coprod_I \mathbb{R}^n$ to $(\mathbb{Z} \oplus B)^{\otimes I}$ and sends $\coprod_I \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ to a map $(\mathbb{Z} \oplus B)^{\otimes I} \to \mathbb{Z} \otimes B$. What is this map? Let's start by looking at the case where $|I| = 2$. In that case take the map

$$\mathbb{Z} \oplus \mathbb{Z} \otimes B \oplus B \otimes \mathbb{Z} \oplus B \otimes B \xrightarrow{\mathrm{id}_{\mathbb{Z}} \oplus \mathrm{id}_B \oplus \mathrm{id}_B \oplus 0} \mathbb{Z} \oplus B.$$

You can generalize this for larger I – just take the product on the B factors to be zero. This map looks weird. Fix it.

(2) Now let $A : (\mathsf{Fin}, \coprod) \to (\mathsf{Ch}, \otimes)$ be a commutative dg algebra. There is a natural symmetric monoidal functor $\pi_0 : (\mathscr{D}\mathsf{isk}_n, \coprod) \to (\mathsf{Fin}, \coprod)$ which sends $\coprod_I \mathbb{R}^n \mapsto \pi_0(\coprod_I \mathbb{R}^n) = I$. The composition of these maps gives us an n-disk algebra. The idea here is that in an n-disk algebra there is not just one way of multiplying things. Indeed, there are $\mathrm{Emb}(\coprod_2 \mathbb{R}^n, \mathbb{R}^n)$ multplications. What we have just done is used the π_0 functor to reduce these various multiplications into the unique multiplication coming from the unique map from the two-point set to the one-point set.

(3) The next example is that of an n-fold loop space of a pointed space $(Z, *)$. We will construct a functor $\mathscr{D}\mathsf{isk}_n \to \mathsf{Top}$ and then postcompose with C_* to obtain a chain complex. This first functor is $\Omega^n Z : \mathscr{D}\mathsf{isk}_n \to \mathsf{Top}$, which we will now define. Recall that for M a space and Z a pointed space, we say that a map $M \to Z$ is **compactly supported** if there exists $K \subset M$ with K compact and such that $g|_{M \setminus K} = * \in Z$. Then we define

$$\Omega^n Z := \mathrm{Maps}_c(-, Z) : (\mathscr{D}\mathsf{isk}_n, \coprod) \to (\mathsf{Top}, \times).$$

If you haven't thought much about compactly supported maps then there is something you have to check. Observe that if

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d] & Z \\ V \arrow[ur, dashed] & \end{tikzcd}$$

the map $U \hookrightarrow V$ is an open embedding then the map $\bar{g}$, given by sending a point v to $g(v)$ for $v \in U$ and $*$ otherwise, is continuous (**homework 3**). Hence Maps_c is covariant via this extension by zero procedure. Moreover it is symmetric monoidal as it sends disjoint unions to products.

Why is this called the n-fold loop space? Well notice that

$$\begin{aligned} \Omega^n Z &= \mathrm{Maps}((D^n, \partial D^n), (Z, *)) \\ &\simeq \mathrm{Maps}_c(\mathbb{R}^n, Z) \end{aligned}$$

where we identify $\mathbb{R}^n$ with the interior of the closed disk D^n. In total, we get

$$\mathscr{D}\mathsf{isk}_n \xrightarrow{\mathrm{Maps}_c(-,Z)} \mathsf{Top} \xrightarrow{C_*} \mathsf{Ch}$$

whose composite we write $C_*\Omega^n Z$.

What happens if we don't use compactly supported and take values in cochains? What is this n-disk algebra in terms of things we know?

(4) At the opposite end of the spectrum from trivial algebras are free algebras. The **free $\mathcal{E}_n$ algebra** on $V \in (\mathsf{Ch}, \otimes)$, which we'll notate as

$$\mathcal{F}_{\mathcal{E}}(V) : \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathsf{Ch},$$

sends

$$\mathbb{R}^n \mapsto \bigoplus_{k \geq 0} C_* \left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n) \right) \otimes_{\Sigma_k} V^{\otimes k}.$$

Here the Σ_k denotes the diagonal quotient. We will define what it does on morphisms in a moment.

This has the universal property that given any map of chain complexes $V \to A$ for A an $\mathcal{E}_n$-algebra (by this we mean a map of chain complexes $V \to A(\mathbb{R}^n)$), there exists a unique map of $\mathcal{E}_n$-algebras such that the diagram

$$\begin{tikzcd} V \arrow[r, "\mu"] \arrow[d] & A \\ \mathcal{F}_{\mathcal{E}_n}(V) \arrow[ur, dashed] & \end{tikzcd}$$

commutes. The vertical map $V \to \mathcal{F}_{\mathcal{E}_n}$ is given by the inclusion into the $k = 1$ summand which is just V.

What is this dashed map? For each k we need a map

$$C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} V^{\otimes k} \to A(\mathbb{R}^n).$$

To do this we use the map

$$C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} V^{\otimes k} \xrightarrow{\mu^{\otimes k}} C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} A^{\otimes k}$$

and then use the multiplication for A. Let's explain this. Notice that $A : \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}} \to \mathsf{Ch}$ and we have $\mathrm{Emb}^{rect}(\coprod_k D^n, D^n) \to \mathrm{Maps}_{\mathsf{Ch}}(A^{\otimes k}(D^n), A(D^n))$ which by Dold-Kan (recall that everything is enriched in Top) corresponds to a map

$$C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_k D^n, D^n)\right) \to \underline{\mathrm{Hom}}_{\mathsf{Ch}}(A^{\otimes k}(D^n), A(D^n))$$

that is Σ_k-equivariant. Because of the equivariance it factors to the quotient, which gives us the multiplication map. Now apply (equivariant) tensor-hom adjunction to obtain this multiplication map. Okay, but we haven't yet shown that the free thing is actually an $\mathcal{E}_n$-algebra, but we're out of time.

(5) The last example we were gonna talk about is pretty awesome. Too bad we're out of time.

Notice that if Z was an Eilenberg-MacLane space, there is overlap between examples 2 and 3.

6. Examples, continued [10/02/2017]

Last lecture we ran out of time in the proof of the following result.

Proposition 18. *The functor $\mathcal{F}_{\mathcal{E}_n}(V)$ sending*

$$D^n \mapsto \bigoplus_{k\geqslant 0} C_*\left(\mathrm{Emb}^{rect}(\coprod_k D^n, D^n)\right) \otimes_{\Sigma_k} V^{\otimes k}$$

is the free $\mathcal{E}_n$-algebra on $V \in \mathsf{Ch}$.

Proof. Last time we showed that this functor satisfied the correct universal property though we hadn't yet specified what it did on morphisms. To define what it does to morphisms we need to construct a map

$$\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n) \to \mathrm{Maps}_{\mathsf{Ch}}(\mathcal{F}_{\mathcal{E}_n}(V)^{\otimes I}, \mathcal{F}_{\mathcal{E}_n}(V)).$$

By the Dold-Kan correspondence this is equivalent to specifying a map

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n)) \to \underline{Hom}(\mathcal{F}_{\mathcal{E}_n}(V)^{\otimes I}, \mathcal{F}_{\mathcal{E}_n}(V))$$

which by the tensor-(internal)hom adjunction, is equivalent to the data of a map

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n)) \otimes \mathcal{F}_{\mathcal{E}_n} \to \mathcal{F}_{\mathcal{E}_n}(V),$$

in other words, a map

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_I D^n, D^n)) \otimes \left(\bigoplus_{k\geqslant 0} C_*(\mathrm{Emb}(\coprod_k D^n, D^n)) \otimes_{\Sigma_k} V^{\otimes k}\right)^{\otimes I} \to \mathcal{F}_{\mathcal{E}_n}(V).$$

Let's maybe just look at the left hand side in the case where $I \cong \{0,1\}$:

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n) \otimes \bigoplus_{k_0,k_1} \left(C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_0} D^n, D^n)) \otimes_{\Sigma_{k_0}} V^{\otimes k_0} \otimes C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_1} D^n, D^n)) \otimes_{\Sigma_{k_1}} V^{\otimes k_1}\right)$$

$$= \bigoplus_{k_0,k_1\geq 0} C_*\left(\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n) \times \mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_0} D^n, D^n) \times \mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_1} D^n, D^n)\right) \otimes_{\Sigma_{k_0}\times\Sigma_{k_1}} V^{\otimes(k_0+k_1)}$$

But from this last expression it is easy to see now that we have a map from what's in the parentheses to $\mathrm{Emb}^{\mathrm{rect}}(\coprod_{k_0+k_1} D^n, D^n) \otimes_{\Sigma_{k_0+k_1}} V^{\otimes(k_0+k_1)}$ by composing the embeddings (up to keeping track of the symmetric group). □

Let's talk about the example that we didn't have time to discuss at the end of last class. This is the class of $\mathcal{E}_n$ enveloping algebras of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. For simplicity we'll work over $\mathbb{R}$.

John: this works for Lie algebras valued in spectra too, up to some changes.

We define a functor $\mathscr{D}\mathsf{isk}_n \to \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Ch}_{\mathbb{R}})$ which sends $U \mapsto \Omega^*_c(U, \mathfrak{g})$, i.e. a Euclidean space to its space of compactly supported de Rham forms. Notice that this construction sends disjoint unions to direct sums. We now postcompose with the Chevalley complex C^{Lie}_* (or if you like $C^{\mathsf{Lie}}_*(\mathfrak{g}) \simeq \mathbb{R} \otimes^{\mathbb{L}}_{\mathcal{U}\mathfrak{g}} \mathbb{R}$). We will write this composite functor as $C^{\mathsf{Lie}}_*(\Omega^*_c(\bullet, \mathfrak{g}))$, and it sends disjoint unions to tensor products.

$$\mathscr{D}\mathsf{isk}_n \longrightarrow \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Ch}_{\mathbb{R}}) \longrightarrow \mathsf{Ch}_{\mathbb{R}}$$

We will use the fact that

$$C_*^{\mathsf{Lie}}(\mathfrak{g} \oplus \mathfrak{g}') \simeq C_*^{\mathsf{Lie}}(\mathfrak{g}) \oplus C_*^{\mathsf{Lie}}(\mathfrak{g}').$$

We claim that for $n = 1$,

$$C_*^{\mathsf{Lie}}(\Omega_c^*(\mathbb{R}^1, \mathfrak{g})) \simeq \mathcal{U}\mathfrak{g}.$$

In particular this functor which maps Lie algebras to $\mathcal{E}_n$-algebras is left-adjoint to the forgetful functor $\mathsf{Alg}_{\mathcal{E}_n} \to \mathsf{Alg}_{\mathsf{Lie}}$.

Here's a small aside. Where is this Lie algebra structure coming from? Well notice that we have a map

$$C_*(\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n)) \otimes A^{\otimes 2} \to A.$$

But notice that the left-hand side is homotopic to $C_*(S^{n-1})$ (do this exercise!). At the level of homology, this gives a map $\mathrm{H}_* S^{n-1} \otimes_{\mathbb{R}} (\mathrm{H}_* A)^{\otimes 2} \to \mathrm{H}_* A$. There are two generators for the homology of S^{n-1} and so we a degree 0 map

$$\mathrm{H}_* A \otimes \mathrm{H}_* A \to \mathrm{H}_* A,$$

which is the associative algebra structure. However, we have another map coming from the fundamental class of S^{n-1},

$$(\mathrm{H}_* A \otimes \mathrm{H}_* A)[n-1] \to \mathrm{H}_* A,$$

is a Lie algebra structure on $\mathrm{H}_* A[1-n]$. (Everything here should be valued in Ch) A reference for this forgetful functor is a paper by F. Cohen.

Since we're almost out of time, let me give you a hint of what we'll be doing next. In factorization homology we are given some functor $A : \mathscr{D}\mathsf{isk}_n \to \mathsf{Ch}$ (or into Top). Factorization homology is an extension

$$\int_M A = \mathrm{hocolim}\ (\mathscr{D}\mathsf{isk}_{n/M} \xrightarrow{A} \mathsf{Ch})$$

an extension that fits into

$$\begin{tikzcd} \mathscr{D}\mathsf{isk}_{n/M} \arrow[r, "A"] \arrow[d] & \mathsf{Ch} \\ \mathsf{Mfld}_n \arrow[ur, dashed] & \end{tikzcd}$$

We need to define not only the homotopy colimit but also what we mean by $\mathscr{D}\mathsf{isk}_{n/M}$. What do we want it to be? Its mapping spaces should fit into the homotopy pullback diagram

$$\begin{tikzcd} \mathrm{Maps}_{\mathscr{D}\mathsf{isk}_{n/M}}(U, V) \arrow[r] \arrow[d] & \mathrm{Emb}(U, V) \arrow[d, "V \hookrightarrow M"] \\ * \arrow[r, "U \hookrightarrow M"] & \mathrm{Emb}(U, M) \end{tikzcd}$$

As usual, if we require this to be a pullback instead of a homotopy pullback this space will be too small. In fact, it will be empty. Okay, you say – so let's just define a category of n-disks with these mapping spaces. The problem that you will run into here is that the composition will be associative only up to homotopy due to the composition of the paths in $\mathrm{Emb}(U, M)$ required by the adjective "homotopy". So we'll have to dip our toes into the theory of infinity-categories, which neatly deals with both this issues and homotopy colimits.

7. Homotopy colimits [10/04/2017]

We are interested in proving the following result.

Theorem 19. *The homotopy colimit is homotopy invariant. More precisely, given two functors $F, G : \mathcal{C} \to \mathsf{Top}$ and any natural transformation $\alpha : F \implies G$ such that for all $c \in \mathcal{C}$, $\alpha(c) : F(c) \to G(c)$ is a homotopy equivalence, then*

$$\operatorname{hocolim}_{\mathcal{C}} F \simeq \operatorname{hocolim}_{\mathcal{C}} G$$

is a homotopy equivalence.

Notice that we can replace homotopy equivalence everywhere with weak homotopy equivalence. Actually we will sketch the proof. The details will be left as **homework 4.** There is a problem in the usual theory of colimits: they are not homotopy invariant. Consider the following simple example. We have a map of spans

$$\begin{tikzcd} D^n \arrow[d] & S^{n-1} \arrow[l] \arrow[r] \arrow[d] & D^n \arrow[d] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd}$$

where the vertical arrows are homotopy equivalences. But the colimits of the top and bottom rows are S^n and $*$ respectively, which are of course not homotopy equivalent.

We have two basic tools that we will use to fix this: Mayer-Vietoris and Seifert-van Kampen.

Lemma 20. *Mapping cones are homotopy invariant. More precisely, if we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

where the vertical arrows are homotopy equivalences then there is an induced homotopy equivalence on cones, $\operatorname{cone} f \simeq \operatorname{cone} f'$.

Proof. Recall that the cone is written as the colimit

$$\operatorname{cone} f = * \sqcup_{X \times \{0\}} X \times [0,1] \sqcup_{X \times \{1\}} Y.$$

Notice that we have maps $\operatorname{cyl} f \to \operatorname{cyl} f'$ inducing an H_*-isomorphism by Mayer-Vietoris applied to the obvious cover. It remains to argue about the fundamental group. Applying the Seifert-van Kampen (for fundamental groupoids) to this cover shows that the fundamental groupoids are equivalent. We conclude that $\operatorname{cyl} f \simeq \operatorname{cyl} f'$. □

John: the most important thing you should take away from a point-set topology course is that a closed embedding of compact Hausdorff spaces is a cofibration.

Likewise for the homotopy pushout. Given $Y \leftarrow X \to Z$ the homotopy pushout is $Y \sqcup_{X \times 0} X \times [0,1] \sqcup_{X \times 1} Z$. This is homotopy invariant as well, which is proved in an identical fashion.

Recall that Δ is the category of finite nonempty ordered sets with nondecreasing functions between them.

Definition 21. A **simplicial space** is a functor $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Top}$. The **geometric realization** $|X_\bullet|$ is the colimit

$$|X_\bullet| \longleftarrow \coprod_{n\geq 0} X_n \times \Delta^n \leftleftarrows \coprod_{[m]\to[l]} X_l \times \Delta^m$$

The basic principle is that the "generators" are given by coproducts and the "relations" are given by reflexive coequalizers. For homotopy colimits the generators will still be coproducts, but the relations will be handled by the geometric realization.

Definition 22. For $X_\bullet$ a simplicial space, the n**th latching object** $L_nX_\bullet$ is

$$L_nX_\bullet = \operatorname*{colim}_{(\Delta^{\mathrm{op}}_{<n})_{/[n]}} X_m \subset X_n \tag{1}$$

The index category is the category of maps $[n] \to [m]$ for $m < n$.

Think of this as all the degenerate simplices induced from everything below n.

We say that X is **Reedy cofibrant** if the map $L_nX\bullet \to X_n$ is a cofibration for all n.

Lemma 23. *If we have a map of simplicial spaces $X_\bullet \to Y_\bullet$ such that both X and Y are Reedy cofibrant with the induced maps $X_n \simeq Y_n$ homotopy equivalences then $|X_\bullet| \simeq |Y_\bullet|$.*

Proof outline. We proceed by induction on skeleta. In particular we have the geometric realization of the n-skeleton

$$|\operatorname{sk}_n X_\bullet| \longleftarrow \coprod_{k\leqslant n} X_k \times \Delta^k \leftleftarrows \coprod_{[m]\to[l];m,l\leqslant n} X_l \times \Delta^m$$

These skeleta sit inside the total geometric realization as closed embeddings whence $|X_\bullet| = \lim |\operatorname{sk}_n X_\bullet|$. So we will prove $|X_\bullet| \simeq |Y_\bullet|$ by proving that $|\operatorname{sk}_n X_\bullet| \simeq |\operatorname{sk}_n Y_\bullet$. The base case just says that $\operatorname{sk}_0 X_\bullet X_0 \simeq Y_0 = \operatorname{sk}_0 Y_0$. For the inductive step check that there is a pushout

$$\begin{tikzcd}
L_nX \times \Delta^n \coprod_{L_nX\times\partial\Delta^n} X_n \times \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} X_\bullet| \arrow[d] \\
X_n \times \Delta^n \arrow[r] & \operatorname{sk}_n X_\bullet|
\end{tikzcd}$$

Likewise for Y. By the inductive hypothesis we know that the map from the top right of the diagram for X to the top right of the diagram for Y is a homotopy equivalence. By assumption the same is true for the bottom left corner. Similarly one has to prove that the top left is a homotopy equivalence. It is then important that the top and left arrows are cofibrations to conclude that the n-skeleton of X is homotopy equivalent to the n-skeleton of Y. $\square$

Write out the details of this proof as homework 4.

Let's now turn to homotopy colimits. Given $\mathcal{C}$ a category we have a simplicial object $N\mathcal{C}_* : \Delta^{\mathrm{op}} \to \mathsf{Set}$, the nerve of $\mathcal{C}$. Observe that the ordinary colimit always receives a surjective map from the coproduct of the functor applied to all the objects in the indexing category. In particular the colimit will always be this coproduct quotiented by a relation coming from morphisms in $\mathcal{C}$. For homotopy colimits we will get a map $F : N\mathcal{C} \to \mathsf{Top}$ sending $[p] \mapsto \sqcup_{N\mathcal{C}_p} F$ and $\operatorname{hocolim}_{\mathcal{C}} F = |F_\bullet|$.

8. Homotopy colimits, continued [10/06/2017]

8.1. **Homotopy colimits.** Recall that if we have an ordinary functor $F : \mathcal{C} \to \mathsf{Top}$ then the colimit $\operatorname{colim}_{\mathcal{C}} F$ can be expressed as a coequalizer: a quotient of the coproduct of $F(c)$ for all $c \in \mathcal{C}$ by the maps in $\mathcal{C}$ (every colimit is a reflexive coequalizer of coproducts).

Definition 24. Given $F : \mathcal{C} \to \mathsf{Top}$ we write $F_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Top}$ for the functor sending $[n] \mapsto \coprod_{N\mathcal{C}_n} F(c_0)$ (where c_0 is the first object in the simplex). The simplicial structure maps are given by copmosition and identities as usual. Then we define the **Bousfield-Kan** homotopy colimit

$$\operatorname{hocolim}_{\mathcal{C}} F := |F_\bullet|$$

Notice that every homotopy colimit is a geometric realization of coproducts.

Theorem 25 (Homotopy invariance of hocolim)**.** *Suppose we have two functors $F, G : \mathcal{C} \to \mathsf{Top}$ such that $F(c)$ and $G(c)$ are cofibrant (i.e. CW complexes) for all $c \in \mathcal{C}$, and there is a natural transformation α such that $\alpha(c)$ is a homotopy equivalence. Then* $\operatorname{hocolim}_{\mathcal{C}} F \simeq \operatorname{hocolim}_{\mathcal{C}} G$*.*

Proof. This is homework 4 (from last time). Recall that the lemma from last time tells us that given a map of Reedy cofibrant simplicial spaces $X_\bullet \to Y_\bullet$ inducing equivalences on n-simplices for every n, the geometric realizations are equivalent. Hence we need only check Reedy cofibrancy for $F_\bullet$ and $G_\bullet$.

In this case the nth latching object of $F_\bullet$ is

$$L_n F_\bullet = \coprod F(c_0)$$

where the coproduct is taken over all degenerate n-simplices of $N\mathcal{C}$. But by the CW complex assumption above the maps $L_n F_\bullet \hookrightarrow F_\bullet$ is a cofibration, as desired. □

8.2. **Factorization homology—a predefinition.**

Definition 26. We define $\mathsf{Disk}_{n/M}$ to be the category of n-disks embedding in M with morphisms given by inclusion (it is equivalent to the subposet of opens on M such that the image is diffeomorphic to an n-disk).

We can make the following predefinition (easier to make, harder to work with). Given $A : \mathsf{Disk}_n \to \mathsf{Top}$ we define the factorization homology

$$\int_M A := \operatorname{hocolim}_{\mathsf{Disk}_{n/M}} A.$$

Really we should be working with the topological version $\mathscr{D}\mathsf{isk}_{n/M}$ but it will end up being homotopy equivalent.

We want factorization homology $\int_M A$ to be M, where we replace $\mathbb{R}^n$ with $A(\mathbb{R}^n)$.

Example 27 (Desiderata)**.**

(1) if $A = *$? Then we would like $\int_M * \simeq *$.
(2) if $A(\coprod_I \mathbb{R}^n) = \coprod_I \mathbb{R}^n$ then $\int_M \mathrm{id} \simeq M$.
(3) be able to compute $\int_M A$ for A belonging to the examples we discussed earlier. For instance, commutative algebras, n-fold loop spaces, free n-disk algebras, trivial n-disk algebras, and enveloping algebra of a Lie algebra.
(4) if A lands in Ch sending $\coprod_I \mathbb{R}^n \mapsto A^{\oplus I}$ then $\int_M A \simeq C_*(M, A)$ (and likewise for spectra).

We need tools for computing homotopy colimits. For instance, it is useful to introduce the topological version,

$$\mathsf{Disk}_{n/M} \to \mathscr{D}\mathsf{isk}_{n/M},$$

and it turns out that homotopy colimits over these two categories are equivalent. To make statements like this, we need crieria for when two homotopy colimits are equivalent when they're indexed by different categories.

Since we don't have time left today to introduce ∞-categories, let's go over some properties of hocolim.

Theorem 28 (Quillen's theorem A)**.** *Let $g : \mathcal{C} \to \mathcal{D}$ is a functor. If F is some functor from $\mathcal{D}$ to some target (such as topological spaces). Then*

$$\operatorname{hocolim}_{\mathcal{C}} F \simeq \operatorname{hocolim}_{\mathcal{D}} F$$

if and only if g is ***final****. In other words, for $d \in \mathcal{D}$, define $\mathcal{C}^{d/} := \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{d/}$, and say that g is final if $B(\mathcal{C}^{d/}) \simeq *$ where $B\mathcal{C} := \operatorname{hocolim} *$.*

There is another key property of homotopy colimits involving hypercovers. Suppose we have a functor $\mathcal{C} \to \mathrm{Opens}(X) \hookrightarrow \mathsf{Top}$. When is

$$\operatorname{hocolim}_{\mathcal{C}} F \simeq X?$$

Define, for $x \in X$, $\mathcal{C}_x$ to be the full subcategory of objects c such that $x \in F(c)$. If $B\mathcal{C}_x \simeq *$ for each $x \in X$ then $\operatorname{hocolim}_{\mathcal{C}} F \simeq X$.

Exercise 29. $\operatorname{hocolim}_* F = F(*)$.

9. ∞-categories [10/09/2017]

9.1. **Topological enrichment.** Suppose we have a category $\mathcal{T}$ with products as well as a functor $\Delta \to \mathcal{T}$ from the ordinal category. Then $\mathrm{Maps}_{\mathcal{T}}(s,t)$ is a simplicial set with

$$\mathrm{Maps}_{\mathcal{T}}(s,t)_p = \mathrm{Hom}_{\mathcal{T}}(s \times [p], t),$$

where by $[p]$ we denote the image of the functor. If we now apply geometric realization, we obtain mapping spaces.

Consider for example $\mathcal{T} = \mathsf{Top}$ (as a non-enriched, ordinary category). There is a functor $\Delta \to \mathsf{Top}$ which sends $[p]$ to the geometric p-simplex. Then we get a simplicial set $\mathrm{Maps}_{\mathsf{Top}}(X,Y)_\bullet$, and notice that

$$\mathrm{Maps}_{\mathsf{Top}}(X \times \Delta^p, Y) \cong \mathrm{Maps}_{\mathsf{Top}}(\Delta^p, \mathrm{Maps}_{\mathsf{Top}}(X,Y))$$

where we equip the set $\mathrm{Maps}_{\mathsf{Top}}(X,Y)$ with the compact-open topology, as usual. Hence the simplicial set $\mathrm{Maps}_{\mathsf{Top}}(X,Y)_\bullet$ is isomorphic to the singular simplicial set $\mathrm{Sing}\,\mathrm{Maps}_{\mathsf{Top}}(X,Y)$. If we apply the geometric realization, since $|\mathrm{Sing}\,A| \simeq A$, we see that we obtain the usual topological enrichment (at least up to homotopy) on the category Top.

This trick allows us to enrich various categories in Top. As we have seen above the category of topological spaces is an immediate example, and it is not hard to do similarly for the category of simplicial sets. Another two familiar examples are those of chain complexes and natural transformations of functors. For a slightly unfamiliar example one could use the functor $\Delta \to \mathsf{CAlg}_{\mathbb{R}}^{\mathrm{op}}$ of de Rham forms on simplices, which sends $[p] \mapsto \Omega^*(\Delta^p)$, to give the (opposite) category of commutative $\mathbb{R}$-algebras a topological enrichment.

This leads us to the following general idea, which highlights the importance of topological enrichment.

> **Principle:** Everywhere where there is a notion of homotopy, there exists an enrichment in Top such that this is an actual homotopy.

9.2. **Complete Segal spaces and quasicategories.** Now, whatever ∞-categories are, they should have two properties:

1. The collection of ∞-categories up to some notion of equivalence should be equal to the collection of topological categories modulo homotopy equivalence (see below for the formal definition).
2. Colimits, limits, functor categories, over/undercategories in ∞-categories are homotopy colimits, homotopy limits, etc. in the corresponding topological category.

Definition 30. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between topological categories. We say that F is a **homotopy equivalence** if for each $c, c' \in \mathcal{C}$, $\mathrm{Maps}_{\mathcal{C}}(c,c') \simeq \mathrm{Maps}_{\mathcal{D}}(Fc, Fc')$ and every object $d \in \mathcal{D}$ is homotopy equivalent to some $F(c)$ for $c \in \mathcal{C}$. In other words, there exists a map $d \to Fc$ such that $\mathrm{Maps}_{\mathcal{D}}(e,d) \to \mathrm{Maps}(e, Fc)$ is a homotopy equivalence for all $e \in \mathcal{D}$.

So that's roughly the philosophy of ∞-categories. They are a nice ground to work on when dealing with homotopy invariance. When it comes to actually defining ∞-categories, there is a conceptual option and a more economical option: complete Segal spaces and quasicategories, respectively. Let me tell you briefly about complete Segal spaces.

When we are given $\mathcal{C}$ a category, there is a set of objects and a set of morphisms. However, the only way we ever use categories is up to equivalence, and these underlying sets have no invariance properties with respect to equivalence of categories (for instance the sets of objects or corresponding sets of morphisms need not have the same number of elements). This leads us to the question: how can we think of a category in a way that better reflects the homotopy theory (i.e. equivalences) of categories.

It turns out that we can construct *spaces* of objects and morphisms of $\mathcal{C}$ in the following way. Consider the underlying groupoid $\mathcal{C}^0 \subset \mathcal{C}$ where we have thrown out all the noninvertible maps. Taking the nerve (classifying space) $N\mathcal{C}^0$ gives us a simplicial set. The associated space is of course the geometric realization $|N\mathcal{C}^0|$. For morphisms, consider the category $\mathrm{Fun}^{\mathrm{iso}}([1], \mathcal{C})$ of functors $[1] \to \mathcal{C}$ with natural transformations through isomorphisms. This category is a also a groupoid, so we obtain a space $|N\,\mathrm{Fun}^{\mathrm{iso}}([1], \mathcal{C})|$.

Observe now that if we have two equivalent categories $\mathcal{C} \simeq \mathcal{C}'$ then the spaces of objects and morphisms that we have defined above will be homotopy equivalent. Generalizing these constructions for higher $[p]$ we obtain a fully faithful functor

$$C_\bullet : \mathsf{Cat} \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Top})$$

sending a category $\mathcal{C}$ to the simplicial space that sends $[p] \mapsto N\,\mathrm{Fun}^{\mathrm{iso}}([p], \mathcal{C})$. It moreover has the property that the diagram

$$\begin{tikzcd}
\mathcal{C}_\bullet[2] \arrow[r] \arrow[d] & \mathcal{C}_\bullet\{1<2\} \arrow[d] \\
\mathcal{C}_\bullet\{0<1\} \arrow[r] & \mathcal{C}_\bullet\{1\}
\end{tikzcd}$$

is actually a homotopy pullback square (this turns out to more or less characterizes the image of $C_\bullet$). In particular, one should suspect (correctly) that colimits and limits will be mapped to homotopy colimits and homotopy limits. Unfortunately, going down this path to ∞-categories quickly turns into messing around with bisimplicial sets, which starts to get a bit complicated.

This leads us to the more economical option of quasicategories. For quasicategories there is only one simplicial index involved and there is the important advantage that there are thousands of pages of reference material.

Definition 31. A **quasicategory** $\mathcal{C}$ is a simplicial set such that every inner horn (for $n \geq 2$) has a filler.

Let's explain these terms. Write $\Delta[n]$ for the geometric n-simplex (the functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$ given $\Delta[n] = \mathrm{Hom}_\Delta(-, [n])$). There are a number of maps $\Delta[n-1] \to \Delta[n]$ induced by maps $[n-1] \to [n]$ that skipping some i. Then the ith horn of the geometric n-simplex is defined to be

$$\Lambda_i[n] = \bigcup \Delta[n-1]$$

where the union is over all faces $\Delta[n-1] \hookrightarrow \Delta[n]$ except for the ith. For instance there are three horns of the 2-simplex. An inner horn is a horn where the missing face is neither the 0th face or the nth face (so the 2-simplex has only one inner horn, $\Lambda_1[2]$). Now we can define what it means for a simplicial set $\mathcal{C}$ to have inner horn fillings. It means that for every map $\Lambda_i[n] \to \mathcal{C}$ there is a lift, or "filling",

$$\begin{tikzcd} \Lambda_i[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

of the map to the simplex making the diagram commute.

Example 32. Spaces and categories are two natural sources of quasicategories.

(1) Consider $\mathcal{C} = \mathrm{Sing}(X)$. By the adjunction between geometric realization and Sing the data of a map $\Lambda_i[n] \to \mathrm{Sing}\, X$ is precisely the data of a map $|\Lambda_i[n]| \to X$. Now one can choose (there is no unique choice) say a retraction $|\Delta[n]| \to |\Lambda_i[n]|$. Composing with the map to X and again applying adjointness, we obtain a map $\Delta[n] \to \mathrm{Sing}\, X$ making the relevant diagram commute. We conclude that the singular simplicial set of a space is a quasicategory.

(2) Given a category C consider the nerve $\mathcal{C} = NC$. By the Yoneda lemma a map $\Lambda_1[2] \to NC$ is precisely the data of a composable pair of morphisms in C. In particular there is a unique way of filling the horn into the simplex by using the composition of these two maps. A similar argument holds for higher-dimensional horns. We conclude that the nerve of any category is a quasicategory.

What we will do next is give the definitions of colimits, limits, functor categories, and over/undercategories in quasicategories. We will also discuss a variant of the nerve functor, $N : \mathsf{TopCat} \to \mathsf{QuasiCat}$, which in good cases will send hocolim $\mapsto$ colim.

Grisha: I understand your philosophy that complete Segal spaces are more compelling than quasicategories. Is there some concrete statement that backs this claim up? **John**: I don't know if this is getting at your question but here's an example. There is an inclusion $\mathsf{Cat}_\infty \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Top})$ so it's easy to give an internal definition of the ∞-category of ∞-categories. However the corresponding construction is not so easy with quasicategories.

10. Colimits in ∞-categories [10/11/2017]

Recall the definition of a colimit of a functor $F : \mathcal{C} \to \mathcal{D}$ in the theory of ordinary categories. We define the **right cone** $\mathcal{C}^{\triangleright}$ of $\mathcal{C}$ as follows. It has objects the objects of $\mathcal{C}$ together with an object we denote $*$. For morphisms we take

$$\mathrm{Hom}_{\mathcal{C}^{\triangleright}}(x, y) = \begin{cases} * & y = * \\ \varnothing & x = * \\ \mathrm{Hom}_{\mathcal{C}}(x, y) & \text{otherwise} \end{cases}$$

Next we define the **undercategory** $\mathcal{D}^{F/}$ as the fiber

$$\begin{array}{ccc} \mathcal{D}^{F/} & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D}) \\ \downarrow & & \downarrow \\ \{F\} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \end{array}$$

It has as objects pairs $d \in \mathcal{D}$ with a natural transformation $F \implies \underline{d}$, where $\underline{d}$ is the constant functor. With these definitions we can now define colimits.

What are the morphisms in the undercategory?

Definition 33. An object $d \in \mathcal{D}$ is a **colimit** of the functor $F : \mathcal{C} \to \mathcal{D}$ if there exists a functor $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ with $\overline{F}(*) \cong d$ and such that the natural restriction $\mathcal{D}^{\overline{F}/} \to \mathcal{D}^{F/}$ is an equivalence.

If you're a bit confused, like Nilay is, about why this is a colimit, observe that the category $\mathcal{C}^{\triangleright}$ has a final object. For any $\mathcal{C}'$ which has a final object, if we have a functor $G : \mathcal{C}' \to \mathcal{D}$ then there is an equivalence $\mathcal{D}^{G/} \cong \mathcal{D}^{G'/}$, where $G' : \mathcal{C} \to \mathcal{D}$ is the restriction of G.

We'd like to make a similar definition for quasicategories. We will need to be able to define equivalence, right cones, and undercategories in that context.

Definition 34. For $\mathcal{C}$ a quasicategory and any objects x and y (i.e. $x, y \in \mathcal{C}[0]$) we define the **mapping space** $\mathrm{Maps}_{\mathcal{C}}(x, y)$ as the fiber

$$\begin{array}{ccc} \mathrm{Maps}_{\mathcal{C}}(x, y) & \longrightarrow & \mathrm{Hom}_{\mathsf{sSet}}(\Delta[1], \mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_1} \\ \{x, y\} & \longrightarrow & \mathrm{Hom}_{\mathsf{sSet}}(\Delta[0], \mathcal{C}) \times \mathrm{Maps}(\Delta[0], \mathcal{C}) \end{array}$$

Here $\mathrm{Maps}(X, Y)$ for X and Y simplicial sets is a simplicial set whose set of n-simplices is the set $\mathrm{Hom}_{\mathsf{sSet}}(X \times \Delta[n], Y)$.

Recall that Kan complexes—simplicial sets that satisfy the horn filling condition for all horns (not just inner horns)—are the combinatorial analog of spaces.

Lemma 35. *As defined above,* $\mathrm{Maps}_{\mathcal{C}}(x, y)$ *is a Kan complex.*

Definition 36. If $F : \mathcal{C} \to \mathcal{D}$ is a functor between quasicategories (i.e. a map of simplicial sets), then F is a **categorical equivalence** if

1. (1) the induced map $F : h\mathcal{C} \to h\mathcal{D}$ is an equivalence of categories, where $h\mathcal{C}$ is the category with objects that of $\mathcal{C}$ and morphisms the set $\pi_0 \mathrm{Maps}(X, Y)_n$.
2. (2) for any $x, y \in \mathcal{C}$ the induced map $F : \mathrm{Maps}_{\mathcal{C}}(x, y) \to \mathrm{Maps}_{\mathcal{D}}(Fx, Fy)$ is a homotopy equivalence (equivalently a homotopy equivalence after geometric realization).

Now we need the notion of the right cone of a simplicial set. Well for a hint of what this definition should be, let's look at the nerve of the right cone construction above:

$$N(\mathcal{C}^{\triangleright})_k = \mathrm{Fun}([k], \mathcal{C}^{\triangleright}) = * \sqcup \coprod_{i=0}^{k} \mathrm{Fun}([i], \mathcal{C})$$
$$= * \sqcup \coprod_{i=0}^{k} N(\mathcal{C})_i.$$

This leads us to the following definition.

Definition 37. For S a simplicial set, define the **right cone** on S to be

$$S_k^{\triangleright} = * \sqcup \coprod_{i \leqslant k} S_i.$$

One checks that this naturally forms a simplicial set.

Definition 38. For $F : \mathcal{C} \to \mathcal{D}$ a functor of quasicategories, we define the **under-category** $\mathcal{D}^{F/}$ to be the fiber

$$\begin{array}{ccc} \mathcal{D}^{F/} & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D}) \\ \big\downarrow & & \big\downarrow \\ \{F\} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \end{array}$$

Definition 39. We say that $d \in \mathcal{D}$ is a **colimit** of F if there exists a functor $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ with $F(*) = d$ and $\mathcal{D}^{F/} \simeq \mathcal{D}^{\overline{F}/}$ is a categorical equivalence.

Theorem 40.

(1) *Colimits in a quasicategory are invariant upto categorical equivalence. In other words, if we have an equivalence $\mathcal{C} \simeq \mathcal{C}'$ and $\mathcal{D} \simeq \mathcal{D}'$ with a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F'} \\ \mathcal{D} & \longrightarrow & \mathcal{D}' \end{array}$$

What does this mean?

then $\mathrm{colim}_{\mathcal{C}} F = \mathrm{colim}_{\mathcal{C}'} F'$.

(2) *The simplicial set* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is a quasicategory if* $\mathcal{C}, \mathcal{D}$ *are quasicategories, and is invariant up to categorical equivalence. In other words, there is a categorical equivalence of quasicategories* $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}', \mathcal{D}')$.

The proof is a straightforward exercise in model categorical language and we might work through this in the future. First let's explain why these results are so great, and why they motivate working with quasicategories.

Consider $\mathscr{D}\mathsf{isk}_\infty^{\mathrm{rect}} = \varinjlim \mathscr{D}\mathsf{isk}_n^{\mathrm{rect}}$ the sequential limit. One finds that $\mathscr{D}\mathsf{isk}_\infty^{\mathrm{rect}} \simeq \mathsf{Fin}$. For motivation for why this might be true, recall that $\mathrm{Emb}^{\mathrm{rect}}(\coprod_2 D^n, D^n) \simeq S^{n-1}$ and as n grows large we obtain S^∞, which is contractible. A basic question one might ask is whether there is a factorization

$$\begin{tikzcd} \mathscr{D}\mathsf{isk}_\infty^{\mathrm{rect}} \arrow[r] \arrow[d] & \mathsf{Top} \\ \mathsf{Fin} \arrow[ur, dashed] & \end{tikzcd}$$

It turns out that there does not exist such a factorization in general:

$$\mathrm{Fun}(\mathsf{Fin}, \mathsf{Top}) \not\simeq \mathrm{Fun}(\mathscr{D}\mathsf{isk}_\infty^{\mathrm{rect}}, \mathsf{Top})$$

This is stemming from the fact that infinite loop spaces are not equivalent to topological groups.

Expand on what this has to do with ∞-categories. Also, isn't this the opposite of homotopy invariance?

11. Homotopy invariance I [10/13/2017]

Recall there was a homework problem to show that there is a continuous assignment $\mathrm{Maps}_c(U, Z) \to \mathrm{Maps}_c(V, Z)$. I should have specified that we are to give $\mathrm{Maps}_c(U, Z)$ the subspace topology as inherited from $\mathrm{Maps}_*(U^+, Z)$. If we view it as a subspace of $\mathrm{Maps}(U, Z)$ this is statement is not true. Of course, this not seem to prevent you from proving it... you know what they say—where there's a will, there's a way. Anyway, for the next homework revise that solution. In addition, do the following for **homework.**

Exercise 41. Prove that there is a homeomorphism $|\Delta[n]| \cong \Delta^n$. Moreover, show that the geometric realization $|X|$ of a simplicial set X has the structure of a CW complex with an n-cell for each nondegenerate n-simplex.

I want to give you a good taste of proofs in quasicategory theory, without having to prove absolutely everything. The following (the homotopy invariance of colimits) should be a good pedagogical example with which we can "get in and get out" of the theory of quasicategories. The main reference will of course be Jacob Lurie's Higher Topos Theory [Lur09].

Proposition 42 (HTT proposition 1.2.9.3)**.** *Let $p : \mathcal{C} \to \mathcal{D}$ be a map of quasicategories and $j : K \to \mathcal{C}$ be any map. Then if p is an equivalence, so is the induced map*

$$\mathcal{C}^{j/} \xrightarrow{\sim} \mathcal{D}^{p\circ j/}.$$

Matt: how does this relate to the notion of pointwise homotopy invariance? John: this result is a bit harder than that one.

Let's outline the proof:

(1) $\mathcal{C}^{j/}$ is a quasicategory
(2) $\mathcal{C}^{j/} \to \mathcal{C}$ is a left fibration
(3) Given two left fibrations $\mathcal{C}', \mathcal{C}''$ over $\mathcal{C}$, and a compatible map g between them, then g is an equivalence if it is an equivalence on fibers. In other words it is an equivalence if for all $x \in \mathcal{C}$ the map $\mathcal{C}' \times_{\mathcal{C}} \{x\} \to \mathcal{C}'' \times_{\mathcal{C}} \{x\}$ is an equivalence of Kan complexes.

We'll begin by showing (2). We first need a definition

Definition 43. For X, Y simplicial sets, the **join** $X \star Y$ is the simplicial set given on totally ordered sets as

$$(X \star Y)(J) = \coprod_{J = I \coprod I'} X(I) \times Y(I')$$

where in the coproduct, every element of I is less than every element of I'. Equivalently,

$$(X \star Y)([n]) = X_n \sqcup \left(\coprod_{i+j=n-1} X_i \times Y_j \right) \sqcup Y_n$$

In the case where C, D are categories, one can check that the usual join $C \star D$, which has

$$\mathrm{Hom}_{C\star D}(x, y) = \begin{cases} \mathrm{Hom}(x, y) & x, y \in C \text{ or } x, y \in D \\ * & x \in C, y \in D \\ \varnothing & \text{otherwise} \end{cases}$$

has the property that its nerve is the quasicategorical join of the corresponding nerves of C and D. A similar statement is true for spaces. Another thing to check is that $X^{\triangleright} = X \star \Delta[0]$. **Homework 6:** Check that $\Delta[n] \star \Delta[m] \cong \Delta[n+m+1]$

Definition 44. A class of morphisms $S \subset C$ (for C an ordinary category) is **weakly saturated** if it is

(1) closed under pushouts,
(2) closed under (transfinite) composition,
(3) and closed under retracts.

This notion is important because any map that has a lifting property with respect to some class of morphisms S then it will also have the lifting property with respect to the weakly saturated closure.

Definition 45. We say that $A \to B$ is **left/right/inner anodyne** if it belongs to the smallest weakly saturated class containing (for $n \geq 1$), $\{\Lambda_i[n] \hookrightarrow \Delta[n], i < n\}$ (left), $\{\Lambda_i[n] \hookrightarrow \Delta[n], i > n\}$ (right), $\{\Lambda_i[n] \hookrightarrow \Delta[n], 0 < i < n\}$ (inner).

Notice that "anodyne" is an english word meaning inoffensive, bland, or unproblematic.

Lemma 46 (HTT proposition 2.1.2.3)**.** *Given inclusions of simplicial sets $f : A_0 \subset A, g : B_0 \subset B$ such that f is right anodyne or g is left anodyne, then*

$$A_0 \star B \coprod_{A_0 \star B_0} A \star B_0 \hookrightarrow A \star B$$

is inner anodyne.

Proof. The two cases are dual so we will just do the case where f is right anodyne. Define class $= \{f \mid \hookrightarrow$ is inner anodyne$\}$, where by the arrow we mean the desired inclusion. We claim that this class is weakly saturated. It suffices to show that it contains $\Lambda_i[n] \subset \Delta[n]$ for $0 < i \leqslant n$. Reduce $g : \partial\Delta[m] \subset \Delta[m]$, where the domain is gen. the weakly saturated class of inclusions. We now have

$$\Lambda_i[n] \star \Delta[m] \coprod_{\Lambda_i[n] \star \partial\Delta[m]} \Delta[n] \star \partial\Delta[m] \hookrightarrow \Delta[n+m+1].$$

More homework: $\Lambda_i[n] \star \partial\Delta[m] \cong \Lambda_i[n+m+1]$, which concludes the proof. □

Definition 47. We say that $X \to Y$ is a **inner/left/right fibration** if it has the right lifting property with respect to inner/left/right anodyne maps.

Proposition 48 (HTT proposition 2.1.2.1)**.** *Given $A \subset B \xrightarrow{p} X \xrightarrow{q} S$, with $r = q \circ p$ and $r_0 : A \subset B$, with q an inner fibration. Then $X^{p/} \to X^{p_0/} \times_{S^{r_0/}} S^{r/}$ is a left fibration.*

This is all to show that $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration (and that the domain is a quasicategory).

Sam: what does a left fibration geometrically realize to? A quasifibration? John: Yeah. [Correction next lecture: I meant to say no. There is a paper of Quillen in the Annals, titled something like: "the geometric realization of a Kan fibration is a Serre fibration." You can guess what the main theorem is. It turns out that for $X \to Y$ a left (or right) fibration it is not necessarily true that $|X| \to |Y|$ is a quasifibration. As an example, consider the left fibration $\mathcal{C}^{x/} \to \mathcal{C}$. Consider the fiber

$$\begin{tikzcd} \mathrm{Maps}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}^{x/} \arrow[d] \\ \{y\} \arrow[r] & \mathcal{C} \end{tikzcd}$$

This cannot possibly yield a quasifibration. Consider a y' with a map $y \to y'$. We get a similar fiber $\mathrm{Maps}_{\mathcal{C}}(x,y)$. But there is no reason for these mapping spaces to be homotopy equivalent (and similarly after taking geometric realization).

Let's recall Quillen's theorme B. Given $\mathcal{C} \to \mathcal{D}$ and the fiber diagram

$$\begin{tikzcd} \mathcal{C}^{d/} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \{d\} \arrow[r] & \mathcal{D} \end{tikzcd}$$

one might ask when taking classifying spaces yields again a homotopy pullback. This is true if for all $d \to c$ in $\mathcal{D}$, $B\mathcal{C}^{d/} \simeq B\mathcal{C}^{c/}$.]

12. Homotopy invariance II [10/16/2017]

Recall that last time we were proving:

Corollary 49 (HTT 2.1.2.2). *For all $K \xrightarrow{p} \mathcal{C}$ the associated undercategory $\mathcal{C}^{p/}$ is a quasicategory.*

Proof. We claim that $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration. Left implies inner, so composing with the map to the point implies that $\mathcal{C}^{p/} \to \mathcal{C} \to \Delta[0]$ is an inner fibration whence $\mathcal{C}^{p/}$ is a quasicategory.

Hence it remains to show that $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration. We show this below. $\square$

Recall from last time we had shown (if you include the homework) the following.

Lemma 50 (HTT 2.1.2.3). *Given $f : A_0 \hookrightarrow A, g : B_0 \hookrightarrow B$ with either f right anodyne or g left anodyne then*

$$A_0 \star B \coprod_{A_0 \star B_0} A \star B_0 \hookrightarrow A \star B$$

is inner anodyne.

This implies

Proposition 51 (HTT 2.1). *Give $A \subset B \xrightarrow{p} X \xrightarrow{q} S$ with the inclusion denoted r_0 and the composition $q \circ p =: r$ where q is an inner fibration, then*

$$X^{p/} \to X^{p_0/} \times_{S^{r_0/}} S^{r/}$$

is a left fibration.

Proof. Recall that the data of a map $J \to \mathcal{C}^{p/}$ for $p : K \to \mathcal{C}$ is precisely the data of a map $K \star J \to \mathcal{C}$ such that $K \star \varnothing \to \mathcal{C}$ is p. To check that the given map is a left fibration we look at a diagram

$$\begin{tikzcd} \Lambda_k[n] \arrow[r] \arrow[d] & X^{p/} \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & X^{p_0/} \times_{S^{r_0/}} S^{r/} \end{tikzcd}$$

Let's apply our adjunction to obtain compatible maps

$$B \star \Lambda_k[n] \longrightarrow X$$

$$A \star \Delta[n] \longrightarrow X$$

$$B \star \Delta[n] \longrightarrow S$$

$$A \star \Delta[n] \longrightarrow S$$

Putting these together,

$$\begin{tikzcd} B \star \Lambda_k[n] \cup_{A\star\Lambda_k[n]} A \star \Delta[n] \arrow[r] \arrow[d, hook] & X \arrow[d] \\ B \star \Delta[n] \arrow[r] & S \end{tikzcd}$$

and applying the lemma above, the vertical map on the left is inner anodyne, whence because $X \to S$ is an inner fibration, there exists a left. $\square$

This now concludes the proof that the undercategory is a quasicategory. To see this, we apply the proposition to the case where $X = \mathcal{C}$, $A = \varnothing$, and $B = *$. Hence $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration.

Sean: is there a time when it matters that these were left fibrations and not just inner? John: absolutely. Think of inner as a technical condition but left/right as a homotopy invariant property. In particular, every functor is equivalent to an inner fibration. This is not at all true for left fibrations. In particular, $\mathsf{LFib}_{\mathcal{D}} \simeq \mathrm{Fun}(\mathcal{D}, \mathsf{Spaces})$—they're like "fiber bundles with connection on $\mathcal{D}$."

Proposition 52 (HTT 1.2.5.1). *For $\mathcal{C}$ a simplicial set the following are equivalent:*

(1) $\mathcal{C}$ is a quasicategory and $\mathsf{h}\mathcal{C}$ is a groupoid;
(2) $\mathcal{C} \to *$ is a left fibration;
(3) $\mathcal{C} \to *$ is a right fibration;
(4) $\mathcal{C} \to *$ is a Kan fibration, i.e. $\mathcal{C}$ is a Kan complex.

*If any of these are try we call $\mathcal{C}$ an **∞-groupoid** or **space**.*

Proposition 53 (HTT 1.2.4.3). *A morphism $\phi : \Delta[1] \to \mathcal{C}$ in a quasicategory $\mathcal{C}$ is an equivalence if and only if for any extension f_0*

$$\begin{tikzcd} \Delta[1] \arrow[r, "\phi"] \arrow[d, hook] & \mathcal{C} \\ \Lambda_0[n] \arrow[ur] \arrow[d, hook] & \\ \Delta[n] \arrow[uur, dashed] & \end{tikzcd}$$

there is a lift to a map f.

Proof. By adjunction

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{C}/\Delta[n-2] \arrow[d] \\ \Delta[0<1] \arrow[r] \arrow[ur, dashed] & \mathcal{C}/\partial\Delta[n-2] \end{tikzcd}$$

That proves one direction. For the other direction take a map $\phi : x \to y$. We have a filler $\Lambda_0[2] \hookrightarrow \Delta[2] \xrightarrow{\psi} \mathcal{C}$ call it σ. This 2-simplex σ gives a homotopy $\mathrm{id}_x \simeq \psi \circ \phi$. Show that $\phi \circ \psi \simeq \mathrm{id}_y$ for **homework.** $\square$

This implies the equivalence of (1) $\iff$ (2) and dually (1) $\iff$ (2). But then (1) $\iff$ (2) + (3) = (4).

13. Homotopy invariance III [10/18/2017]

Recall last time we wanted to prove Proposition 1.2.5.1.

Proof. To show that (1) $\implies$ (2) notice that every $\Delta[1] \to \mathcal{C}$ is a homotopy equivalence. Choose any f_0

$$\begin{tikzcd} \Lambda[n] \arrow[r, "f_0"] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

and now by the previous proposition there exists an extension

$$\begin{tikzcd} \Delta[0<1] \arrow[d] \arrow[dr] & \\ \Lambda_0[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

To see that (2) $\implies$ (1) draw the same picture and apply the proposition, which implies that ϕ is an equivalence.

Notice that by taking opposites (1) $\iff$ (2) implies (1) $\iff$ (3), since taking opposites takes left fibrations to right fibrations. Hence (1) $\iff$ (2) + (3). But being a left and right fibration is the same as being a Kan fibration, which completes the proof. □

Corollary 54. *If $\mathcal{C}$ is a quasicategory there exists a maximal sub-Kan complex $\mathcal{C}^0$ whose morphisms consist of the homotopy equivalences in $\mathcal{C}$.*

Figure out the HTT number.
Fix the notation to tilde.

Proof. We can define $\mathcal{C}^0$ as the subsimplicial set with 1-simplices the homotopy equivalences. $\mathcal{C}^0$ is a quasicategory, and $h\mathcal{C}^0$ is a groupoid if and only if $\mathcal{C}^0$ is a Kan complex. □

In particular we have $\mathsf{Kan} \subset \mathsf{QCat} \subset \mathsf{sSet}$ and the construction in the corollary is the right adjoint to the inclusion $\mathsf{Kan} \hookrightarrow \mathsf{QCat}$.

Recall that our purpose was to show that colimits in quasicategories are invariant with respect to categorical equivalence. In particular, given $J \xrightarrow{j} \mathcal{C} \xrightarrow{p} \mathcal{D}$ where p is a categorical equivalence, we want to show that $\mathcal{C}^{j/} \simeq \mathcal{D}^{p\circ j/}$. We first needed the undercategories to be quasicategories. This we showed last time. Next we show that the two horizontal arrows

$$\begin{tikzcd} \mathcal{C}^{j/} \arrow[r] \arrow[dr] & \mathcal{D}^{p\circ j} \times_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D}^{p\circ j/} \\ & \mathcal{C} & \end{tikzcd}$$

are equivalences. To see that the first is an equivalence we observe first that the vertical maps in the triangle are left fibrations, whence it is enough to show that it produces an equivalence on fibers. Thus we need the following. Given

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "p"] \arrow[d, "g"] & \mathcal{C}'' \arrow[dl] \\ \mathcal{D} & \end{tikzcd}$$

where g and h are left fibrations, we wish to show that p is an equivalence if and only if $\mathcal{C}'_d \to \mathcal{C}''_d$ is an equivalence of Kan complexes for all $d \in \mathcal{D}$. To prove this it's easiest to prove a slightly more general result. Then we need

Lemma 55 (HTT 2.5.4.1)**.** *Given $J \to \mathcal{C}$ $\mathcal{D}$ with the map from $\mathcal{C}$ to $\mathcal{D}$ an equivalence, then $\mathcal{C}^{j/} \times_{\mathcal{C}} \{x\} \to \mathcal{D}^{p\circ j/} \times_{\mathcal{D}} \{px\}$ is an equivalence of Kan complexes for all $x \in \mathcal{C}$.*

The following picture is good to keep in mind:

$$\begin{tikzcd} & \text{Kan fibration} \arrow[dl] \arrow[dr] & \\ \text{left fibration} \arrow[d] & & \text{right fibration} \arrow[d] \\ \text{coCartesian fibration} \arrow[dr] & & \text{Cartesian fibration} \arrow[dl] \\ & \text{inner fibration} & \end{tikzcd}$$

Recall that an inner fibration is more of a technical condition rather than having some homotopy invariant meaning. Each of these fibrations (except for inner fibrations) are classified by functors to a representing object.

Example 56. Consider a functor $F : [1] \to \mathsf{Cat}$. Call $F(0) = \mathcal{C}, F(1) = \mathcal{D}$. From this we can construct a category $\mathcal{M} = \mathrm{cyl}(F) := \mathcal{C} \times [1] \coprod_{\mathcal{C}\times\{1\}} \mathcal{D}$ which sits over $[1]$, $\mathcal{M} \to [1]$. This cylinder construction is a map $\mathrm{Fun}([1], \mathsf{Cat}) \to \mathsf{Cat}_{/[1]}$. You should think of this as the most basic example of an "unstraightening construction". The categories you obtain are the coCartesian fibrations over [1].

Definition 57. We say that a **correspondence** between two categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $\mathcal{M} \to [1]$ with $\mathcal{M}_0 \cong \mathcal{C}$ and $\mathcal{M}_0 \cong \mathcal{D}$.

This construction gives us correspondences but not all correspondences arise this way. If we consider a span $\mathcal{E}$ from $\mathcal{C}$ to $\mathcal{D}$ we can produce a correspondence. In particular take the parameterized join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D} = \mathcal{C} \coprod_{\mathcal{E}\times\{0\}} \mathcal{E} \times [1] \coprod_{\mathcal{E}\times\{1\}} \mathcal{D}$ over [1].

If our fibrations in our diagram above are over $\mathcal{C}$ (except for the inner fibration, say), then we get a diagram before unstraightening

$$\begin{tikzcd} & \mathrm{Fun}(\mathcal{C}, \mathsf{Gpd}^0_\infty) \arrow[dl] \arrow[dr] & \\ \mathrm{Fun}(\mathcal{C}, \mathsf{Gpd}_\infty) \arrow[d] & & \mathrm{Fun}(\mathcal{C}, \mathsf{Gpd}^{\mathrm{op}}_\infty \arrow[d] \\ \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty) & & \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}^{\mathrm{op}}_\infty \end{tikzcd}$$

Note: unstraightening is also known as the "Grothendieck construction." In particular given $F : \mathcal{C} \to \mathsf{Cat}_\infty$ then the fiber is just $F(x)$ over x:

$$\begin{tikzcd} F(x) \arrow[r] \arrow[d] & \mathcal{C}_F \arrow[d] \\ \{x\} \arrow[r] & \mathcal{C} \end{tikzcd}$$

Corollary 58. *Given a map $\mathcal{C}' \to \mathcal{C}''$ of left fibrations over $\mathcal{C}$, if $\mathcal{C}' = \mathcal{C}_F$ and $\mathcal{C}'' = \mathcal{C}_G$ (unstraightening) with the map of fibrations being induced by a natural transformation α sending $F \implies G : \mathcal{C} \to \mathsf{Gpd}_\infty$. Then the map is an equivalence if and only if α is an equivalence if and only if $F(x) \xrightarrow{\alpha} G(x)$ is an equivalence i.e. $\mathcal{C}'_x \to \mathcal{C}''_x$ is an equivalence.*

We would have to prove a lot of this stuff to prove our fact.

Principle: any construction from category theory that only uses universal properties goes through for ∞-categories.

The reason this unstraightening stuff is important because it's generally easier to think about functors out of things instead of fibrations. On the other hand, fibrations (unstraightened things) are generally easier to construct.

References

[Hir94] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.