Microlocal methods — Boris 11/01/2016.

"Deformation quantization of symplectic manifolds."

Goal: $(M,\omega) \rightsquigarrow$ sheaf of $\overset{(nc)}{\text{algebras}}$ $(C^\infty_M[[\hbar]], *)$ + more structure

(by gluing over Darboux charts)

Idea: Modules over $C^\infty_M[[\hbar]]$ is an interesting category.

We write $\mathcal{O}_M$ for structure sheaf.

*-product: ~~(locally)~~

$$f*g = fg + \sum_{k=1}^{\infty} (i\hbar)^k P_k(f,g) \qquad \text{formal expression}$$

P_k bidifferential operator (bilinear), i.e. locally:

$$= \sum_{|k|,|l|<N} A_{kl}(x)\, \partial_x^k f \cdot \partial_x^l g,$$

(k, l: multi-index)

with:

(1) $f*(g*h) = (f*g)*h$

(2) $1*f = f*1 = f$

It follows: $P_1(f,g) - P_1(g,f) \equiv \{f,g\}$ "the Poisson bracket".

An isomorphism $* \xrightarrow[G]{\sim} *'$ is

$$G(f) = f + \sum_{k=1}^{\infty} (i\hbar)^k \underbrace{T_k(f)}_{\text{diff. lin. ops.}}$$

and $G(f*g) = G(f) *' G(g)$

Do these exist if we want a fixed $\{-,-\}$?

Yes, c.f. Kontsevich for Poisson case.

Maybe not the best definition... weaker structure:

$M = \bigcup U_\alpha$, $*$ on U_α gluing, i.e.
$$\begin{cases} (1)\ A_\alpha \equiv (\mathcal{O}(U_\alpha)[[\hbar]], *) \\ (2)\ G_{\alpha\beta} : \overset{\alpha}{*} \xleftarrow{\sim} \overset{\beta}{*} \text{ on } U_\alpha \cap U_\beta \text{ with} \\ \qquad G_{\alpha\beta} G_{\beta\gamma} = G_{\alpha\gamma}. \end{cases}$$

(sheaf of algebras)

"Algebroid Stack"

An even more relaxed structure...

$$G_{\alpha\beta}\, G_{\beta\gamma} = \mathrm{Ad}(c_{\alpha\beta\gamma})\, G_{\alpha\gamma} \qquad \text{(cocycle cond. upto inner automorphism)}$$

inner!

$$c_{\alpha\beta\gamma} = 1 + \sum_{k=1}^{\infty} \hbar^k c_{k;\alpha\beta\gamma} \quad \in A_\alpha[[\hbar]](U_{\alpha\beta\gamma}).$$

and. $$\underbrace{G_{\alpha\beta} G_{\beta\gamma}} G_{\gamma\delta} = \left\{ \begin{array}{c} \mathrm{Ad}(c\ldots c\ldots) \\ \| \\ \mathrm{Ad}(c'\ldots c') \end{array} \right\} G_{\alpha\delta},$$

require: $c\ldots c\ldots = c'\ldots c'\ldots$ (in $A_\alpha(U_{\alpha\beta\gamma\delta})$)

(← autom. true upto central element.) (?)

(if $c_{\alpha\beta\gamma} = 1$ then reduces to sheaf)

Weaker than sheaf! Why do this?

(1) arises naturally

(2) in C^∞ case get nothing new, $(*, G_{\alpha\beta}, c_{\alpha\beta\gamma})$ reduces to $*$ on $C^\infty_M[[\hbar]]$.

$c_{\alpha\beta\gamma} = 1 + \hbar c^{(1)}_{\alpha\beta\gamma} + \cdots$

b/c mult. 2-Cech cocycle, $c^{(1)}_{\alpha\beta\gamma}$ additive in $\mathcal{O}_M$.

(3) $(*_\alpha, G_{\alpha\beta}, c_{\alpha\beta\gamma})$ is (glues to) a sheaf of categories on M:

$\alpha \longmapsto A_\alpha$-mod

$\alpha, \beta \longmapsto \mathcal{C}_\alpha \xleftarrow{\sim} \mathcal{C}_\beta$ functor on $U_{\alpha\beta}$

$$\begin{tikzcd} \mathcal{C}_\alpha & & \mathcal{C}_\gamma \arrow[ll, "G_{\alpha\gamma}"'] \arrow[dl] \\ & \mathcal{C}_\beta \arrow[ul] \arrow[u, phantom, "\Uparrow c_{\alpha\beta\gamma}"] & \end{tikzcd}$$

(no higher morphisms, I think).

Now, $U \rightsquigarrow \mathcal{C}(U)$

objects: collections of $M_\alpha \in \mathrm{Obj}(\mathcal{C}_\alpha)$ and $g_{\alpha\beta} : M_\alpha \xleftarrow{\sim} G_{\alpha\beta} M_\beta$ such that...

"twisted objects/modules".

Example 1. $M = T^*X$, $X = \bigcup U_\alpha$, $M = \bigcup_\alpha T^*U_\alpha$ (Darboux chart ?)

$\mathcal{D}^\hbar(U_\alpha) = \mathcal{O}(U_\alpha)[i\hbar\partial_{x_1}, \ldots, i\hbar\partial_{x_n}][\hbar]$ $\hbar$ formal parameter.

Notice: $\xi_k = i\hbar\, \partial/\partial x_k$, $\mathcal{D}^\hbar(U_\alpha) \xrightarrow[\phi_\alpha]{\sim} \mathcal{O}(U_\alpha)[\xi_1, \ldots, \xi_n][\hbar]$ $\quad [\xi_k, x_\ell] = i\hbar\delta_{k\ell}$

i.e. $\sum p_j(x) \cdot (i\hbar\partial_x)^j \longleftrightarrow \sum p_j(x)\, \xi^j$

finite sum if polynomial in ξ

but want smooth in ξ e.g. $\pi^{-1}(U_\alpha) \subset T^*M$.

Now, $(f*g)(x,\xi) = \sum_{j\geq 0} \frac{(i\hbar)^j}{j!} \partial_\xi^j f \cdot \partial_x^j g$, and

$\mathcal{O}(U_\alpha)[\xi_1, \ldots, \xi_n][\hbar] \hookrightarrow \mathcal{O}(U_\alpha \times \mathbb{R}^n)[[\hbar]]$, $*$

We obtain $\phi_\alpha \phi_\beta^{-1} : \mathcal{O}(U_{\alpha\beta} \times \mathbb{R}^n)[[\hbar]] \circlearrowleft$ (does this make sense?)

Cocycle condition holds on the nose! extending ϕ_α to $[[\hbar]]$?

$\hookrightarrow$ what is $G_{\alpha\beta}$?

? Peng asked too, does this help?

Concrete example $\rightarrow$ consider $x + x^2 \equiv g(x)$ coord. change.

$P(x,\xi) = \sum P_n(x)\xi^n \longrightarrow \sum P_n(x+x^2)(i\hbar\partial_x + i\hbar\varphi(x))^n$

$G_{\alpha\beta}$ autom. of $\mathcal{D}^\hbar(\mathbb{R}^1)$ $= \sum Q_n(x) \, (i\hbar\partial_x)^n$

$x \longmapsto x + x^2$

$i\hbar\partial x \longmapsto g \circ i\hbar\partial_x \circ g^{-1}$ $\quad Q(x,\xi) = \sum (i\hbar)^n \,\text{dif. op.}(P(x,\xi))$.

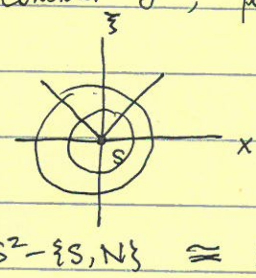
Example 2. Weyl $*$-product: on $\mathbb{R}^{2n}$

$$(f*g)(x,\xi) = \exp\left(\frac{i\hbar}{2}(\partial_\xi\partial_y - \partial_\eta\partial_x)\right) f(x,\xi)g(y,\eta) \Big|_{\substack{x=y \\ \xi=\eta}}$$

is a $*$-product! In fact is $Sp(2n)$-equivariant!

$\hookrightarrow = fg + \frac{i\hbar}{2}\{f,g\} + \cdots$

Example 3. Consider S^2, poles N, S. In Darboux coords, x and y (away from S)



$x = r\cos\varphi \quad y = r\sin\varphi$; $\quad \rho = \frac{r^2}{2} = \frac{x^2+y^2}{2}$

can compute $[\varphi, \rho]_{\text{Weyl}} = i\hbar$ $\quad \rho = \rho(x,y)$, $\varphi = \varphi(x,y)$

"action_angle"

$S^2 - \{S,N\} \cong S^1 \times (0,1)$ have Weyl $*$-prod in φ, ρ
$\qquad\qquad\qquad\qquad\;\; \varphi \quad\;\; \rho$

Now glue with charts near S, near N to get a sheaf of algebras, locally $C^\infty_{S^2}[[\hbar]], *$. How do they glue explicitly?