Derived Lagrangian Correspondences ~~& Classical Chern-Simons Theory~~.

~~OUTLINE:~~

I
- Recollection of basic concepts from Mike's talk
- Derived geometry & intersections

not at all what I ended up writing →

II
- ~~Quantization: TQFTs — Chern-Simons theory~~
- ~~AKSZ interpretation of classical CS theory. BG~~

→ outline of functor from $\mathrm{Cob}_2 \longrightarrow \mathrm{LagrCorr}$

III
- BG as a stack, differential forms on it
- defn of shifted sympl. str. & Lagrangians

IV
- General AKSZ classical functor for X shifted symplectic.
- ~~Example of classical Chern-Simons theory~~

Last week Mike told us about Weinstein's symplectic "category" — recall

objects: n-dimensional symplectic manifolds

maps (M,N) : Lagrangian submanifolds of the product, $L \subseteq \overline{M} \times N$.

One of the key ideas was that a symplectomorphism $M \xrightarrow{\varphi} N$ yields a map

$\Gamma_\varphi \subseteq \overline{M} \times N$.

In this way, Weinstein's category provides more maps than just symplectos. Of course, category here is in quotes — to compose, we took a fiber product, an

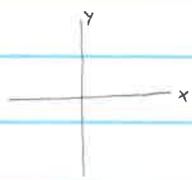
$$\begin{tikzcd}[column sep=small] & & L_1 \times_{M_2} L_2 \arrow[dl] \arrow[dr] & & \\ & L_1 \arrow[dl] \arrow[dr] & & L_2 \arrow[dl] \arrow[dr] & \\ M_1 & & M_2 & & M_3 \end{tikzcd}$$

intersection. This need not exist unless we are given certain transversality-type guarantees on these maps. Today I want to explain one context in which we can remove the quotes: the setting of derived geometry. Once we have this bona fide algebraic structure, which we will call the category of Lagrangian correspondences, we can fit it into the geometric framework of AKSZ theories and Atiyah's notion of a TQFT. While Mike's motivation last week was analytic in nature, mine today will be algebraic/functorial. I will focus on the example of Chern-Simons theory

Let's start with some intuition on derived geometry. There are several motivations for derived geometry, including deformation theory and the study of possibly singular moduli spaces. The caricature to keep in mind is the following: a derived geometric object is characterized by having not a ring of functions on it (whether they be algebraic, holomorphic, or smooth), but instead a cdga^{+} explain this. of functions. By convention we work cohomologically, concentrated in non-positive degrees, so we have "classical" functions in degree zero, satisfying $fg = gf$, but we also

might have functions h_1, h_2 in negative degrees, satisfying $h_1 h_2 = (-1)^{|h_1||h_2|} h_2 h_1$.

In particular, if $|h| = -1$ then $h^2 = -h^2 \Rightarrow h^2 = 0$, so we have nilpotents. Here's a basic example that I like to give — write $R := \mathbb{C}[x,y]/(xy)$, so $\operatorname{Spec} R$ is the scheme given by the union of $\{x=0\} \cup \{y=0\}$. There is



a singularity at $(0,0) \in \mathbb{C}^2$. Instead we might consider the "derived scheme" whose functions are

$$A^* := \overset{-1}{\mathbb{C}[x,y]\cdot z} \longrightarrow \overset{0}{\mathbb{C}[x,y]} \longrightarrow 0$$

$$z \longmapsto xy.$$

In other words, the free graded-commutative polynomial algebra$/\mathbb{C}$ with indeterminates x, y, z, with $|x| = |y| = 0$ and $|z| = -1$, together with the differential δ sending $z \longmapsto x \cdot y$. Notice that the map of cdgas

$$\begin{tikzcd}
A^* \arrow[d] & = & \mathbb{C}[x,y]\cdot z \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}[x,y] \arrow[d] \\
R[0] & = & 0 \arrow[r, "0"] & \mathbb{C}[x,y]/(xy)
\end{tikzcd}$$

is a quasi-isomorphism, i.e. induces isomorphisms on cohomology. In this sence, A^* and R are equivalent models for functions on the same space. There are some benefits to working with A^* over R which I probably don't have time to go into at the moment. But the point is that now the standard constructions that you are used to, such as tangent vectors & differential forms can be straightfwd extended to this setup — you may have a vector field $\partial/\partial z$ of degree -1!

intuition: can always replace nontransverse maps with transverse maps, up to quasi-iso, and really I mean here "homotopy" fiber products, which are invariant under quasi-iso

Lemma Derived geometric objects (dg schemes or derived stacks) are closed under fiber products.

So this is the setup that I'd like to describe Lagrangian correspondences in. Of course, to do this we need a notion of a symplectic structure on derived geometries. Before we get into those details, let me outline where we are headed.

Recall Atiyah's formalization of a topological field theory (TFT): it is the data of a symmetric monoidal functor from a cobordism category under disjoint union to a target symmetric monoidal category $\mathcal{C}^{\otimes}$,

(here let's say oriented cobordisms)

$$F : dCob^{\amalg} \longrightarrow \mathcal{C}^{\otimes}$$

$$\bigcirc \;\; \bigcirc \longmapsto F(\bigcirc)^{\otimes 2}.$$

 $\longmapsto \left(F(\bigcirc) \xrightarrow{F(\;)} F(\bigcirc) \right)$

If we have in mind a topological QUANTUM field theory, then $\mathcal{C}$ might be Hilbert spaces$^{\otimes}$. My goal for today is to describe a CLASSICAL TFT (or family of such) taking values in $\mathcal{C}$ = Weinstein's symplectic category in the derived setting. For X a (shifted) symplectic derived stack,

Maps:

$$\text{Maps}(-, X) : dCob^{\amalg} \longrightarrow \text{LagCorr}^{\times}$$

too ambitious ~~I will outline the example of Chern-Simons theory.~~

Main references:

PTVV - Shifted symplectic structures

Calaque - Lagrangian structures on mapping stacks and semi-classical TFTs

Safronov - Quasi-Hamiltonian reduction via classical Chern-Simons theory.

The first order of business is to explain symplectic and Lagrangian structures in the derived/stacky setting. This is a little more involved than you might otherwise except — the fundamental difficulty is that closed forms are classically defined as cocycles in the de Rham complex. The notion of cocycles is not homotopy-invariant, unfortunately — by changing our model from R to A^*, say, we might enlarge our set of cocycles. The idea instead will be to note that there is a filtration (the Hodge filtration)

$$F^p\Omega^* X := \overset{p-1}{0} \to \overset{p}{\Omega^p X} \to \overset{p+1}{\Omega^{p+1} X} \to \Omega^{p+2} X \to \cdots$$

and $H^0(F^p\Omega^* X[p]) \cong \Omega^p_{cl} X$.

shift left by p

Let's get to formal definitions.

<u>Definition</u>. Let $X = RSpec(A^*)$ be an affine derived scheme, i.e. a derived geometric object whose functions is $A^* \in cdga^{\leq 0}_{/\mathbb{C}}$. Then the <u>de Rham complex of X</u> is

"weight" = monomial degree

e.g.

wt$(z\,dx\,dz) = 2$

$|z\,dx\,dz| = -4$

$$dR(X) = Sym^*_A(\mathbb{L}^*_A[1])$$

← due to Illusie in his thesis

Here $\mathbb{L}^*_A$ is the <u>cotangent complex of A</u>.

Here's how you compute $\mathbb{L}_A$: first you replace A^* with a quasi-isomorphic $\tilde{A}^* \in cdga^{\leq 0}_{/\mathbb{C}}$ such that $\tilde{A}^*$ is a semi-free $\tilde{A}^0$-module, and then take

$$\mathbb{L}^*_A := \Omega^1_{\tilde{A}^*/\mathbb{C}} \quad \text{the module of Kähler differentials of } \tilde{A}^*.$$

(This is, up to quasi-iso., well-defined — we start to see how derived/∞-cat's are convenient ✓)

<u>Example</u>. Consider $R = \mathbb{C}[x,y]/(xy)$ as a $cdga^{\leq 0}_{/\mathbb{C}}$ that is concentrated in degree zero. To compute $\mathbb{L}^*_R$ we replace R with A^* as described above. Now:

$$\mathbb{L}_R \simeq \Omega^1_{A^*/\mathbb{C}} = 0 \to \overset{-1}{\mathbb{C}[x,y]\cdot dz} \xrightarrow{\delta} \overset{0}{\mathbb{C}[x,y]\cdot dx \oplus \mathbb{C}[x,y]\cdot dy} \to 0$$

$$dz \longmapsto d(xy) = y\,dx + x\,dy.$$

No surprises — it's what you would naively expect. Now the de Rham complex $dR(Spec\,R)$ is just the graded polynomial algebra in dx, dy, dz, where $|dx| = |dy| = -1$ and $|dz| = -2$, and $\delta(dz) = y\,dx + x\,dy$. Notice that there is a differential δ of degree $+1$, weight 0, but also $d = d_{dR}$ of degree -1, weight $+1$.

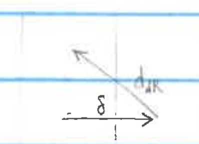
<u>Remark.</u> The previous example shows how to compute the de Rham complex of a derived (affine) scheme. More generally, in working with moduli spaces, etc., it is useful to work with derived stacks. I don't want to go into any defns or details but let me just say that the de Rham complex of a derived stack is computed by Kan extension along $\mathrm{derAffSch} \hookrightarrow \mathrm{derStack}$. I'll gloss over these details when they come up later.

We might visualize the de Rham complex in the example above as something living in the second quadrant:

$dR(\mathrm{Spec}\, R) =$

wt \ deg	−4	−3	−2	−1	0
3	$dz\,dx\,dy$				
2	dz^2	$dzdx$, $dzdy$, $zdxdy$	$dxdy$		
1		zdz	dz, zdx, zdy	dx, dy	
0				z, zx^k, zy^k	x^k, y^k

Differentials:



double check shift signs! 1,2 or −1,−2?

Notice that we have the "expected", say, 2-form $dxdy$ in degree -2. However we also have some "shifted" 2-forms like $dzdx$ or dz^2, which are "(−1)- and (−2)-shifted 2-forms."

<u>Definition</u> A <u>k-shifted p-form</u> on a derived scheme $\mathrm{Spec}\, A^*$ is a degree zero element of $dR(\mathrm{Spec}\, A^*)[k-p](p)$. Here (p) means the p^{th} weight-graded piece (the p^{th} row in the visualization above). In other words it's an element of $dR(\mathrm{Spec}\, A^*)(p)^{k-p}$. (*)

The next step towards defining symplectic structures is to define closed forms. This is a bit painful, not because it's difficult but because of the awkward directions of the differentials. See PTVV §1.2 for details.

(*) Actually, I forgot – it will be useful to work with a space (really only homotopy type) of k-shifted p-forms instead of a complex. Write $|E|$ for the space associated to (the $\tau_{\le 0}$-truncation of E) — this space has pts elements of E^0, paths γ from $x, y \in E^0$ if $\exists \gamma \in E^{-1}$ s.t. $d\gamma = x - y$, and so on

We write

$$\mathcal{A}^p(\mathrm{Spec}\, A^*, k)$$

for this space of k-shifted p-forms. in other words it's $|dR(\mathrm{Spec}\, A^*)[k-p](p)|$.

Okay back to closed forms. A k-shifted closed p-form is a k-shifted p-form

$\omega_1 \xrightarrow{\delta} d\omega_0$, $\omega_0 \xrightarrow{d_{dR}}$

$\omega_0 \in dR(\mathrm{Spec}\, A^*)[k-p](p)^0$ satisfying $\delta\omega = 0$

$\omega_1 \in dR(\mathrm{Spec}\, A^*)[k-p-2](p+1)$ $\quad \delta\omega_1 \pm d\omega_0 = 0$ I don't know the exact signs. Depends on the column.

$\omega_2 \in dR(\mathrm{Spec}\, A^*)[k-p-4](p+2)$ $\quad \delta\omega_2 \pm d\omega_1 = 0$

$\vdots \qquad \vdots$

We write

$$\mathcal{A}^p_{cl}(\mathrm{Spec}\, A^*, k)$$

for the space of k-shifted closed p-forms.

Exercise. Consider the example of the union of the axes above. Check that dx and dy are 0-shifted closed 1-forms. Is dz a (-1)-shifted closed 1-form?

Protip: if it isn't — are there any (-1)-shifted closed 1-forms?

(hint: $(?)dz + z[(?)dx + (?)dy]$.)

In pursuit of symplectic structures we are interested in closed 2-forms. What's great about 2-forms is that they yield maps from tangent to cotangent spaces, by interior multiplication, say. If ω is a k-shifted 2-form on a derived stack X, then it provides a shifted map

$$(*) \qquad \mathbb{T}_X \longrightarrow \mathbb{L}_X[k]$$

from the tangent complex to the cotangent complex.

Definition. A 2-form $\omega \in \mathcal{A}^2(X,k)$ is non-degenerate if the associated map $(*)$ is a quasi-isomorphism, i.e. induces an iso. on cohomologies (i.e. an iso. in the derived category). Write

$$\mathcal{A}^2(X,k)^{nd}$$

for the full subspace of $\mathcal{A}^2(X,k)$ of non-degenerate 2-forms.

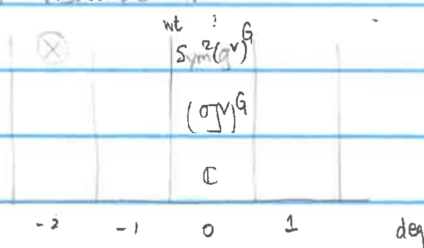
<u>Definition</u>. The space of <u>k-shifted symplectic structures</u> on X is defined to be the homotopy pullback

$$\begin{tikzcd} \mathrm{Symp}(X,k) \arrow[r] \arrow[d] \arrow[dr, phantom, "h\lrcorner"] & \mathcal{A}^2_{cl}(X,k) \arrow[d] \\ \mathcal{A}^2(X,k)^{nd} \arrow[r] & \mathcal{A}^2(X,k). \end{tikzcd}$$

Examples. (most of these examples could be called theorems)

(1) X a smooth scheme. Then $\mathrm{Symp}(X,k)$ is empty for $k \neq 0$ and $\mathrm{Symp}(X,0)$ is equivalent to the set of usual symplectic forms on X.

(2) $X = BG = [*/G]$ for G a reductive algebraic group. Then
$$dR(BG) \simeq (\mathrm{Sym}^* \mathfrak{g}^\vee)^G [0].$$
Let's visualize this



A zero-shifted 2-form would live at the "$\otimes$" sign. So BG has only $(+2)$-shifted 2-forms. The closedness data can be taken to be trivial so each 2-form is closed. It follows that
$$\mathrm{Symp}(BG, 2) \simeq \{\text{non-degenerate } G\text{-invariant bilinear forms on } \mathfrak{g}^\vee\}$$

[PTVV, Calaque] (3) For reasonable derived stacks (Artin, lfp) X, the k-shifted cotangent derived stack,
$$T^*X[k] = d\mathrm{Spec}\ \mathrm{Sym}_{\mathcal{O}_X}(\mathbb{T}_X[-k])$$
has a natural k-shifted symplectic structure.

[PTVV] (4) Let X be a derived (Artin) stack equipped w/ k-shifted symplectic form, and let M be a compact connected oriented topological manifold of dim d. Then the derived mapping stack $\mathrm{Map}(M, X)$ (assuming it is reasonable) has a natural $(k-d)$-shifted symplectic structure.

This last example is a special case of PTVV's much more general Theorem 2.5, for shifted sympl. structures on more general mapping stacks. This is particularly useful for understanding sympl. str's on moduli spaces, which often occur as mapping stacks. eg $\mathrm{Map}(M, BG)$ for M as above is the moduli of flat G-connections on M. PTVV show that these new structures extend those defined classically away from the singular pts of these moduli spaces.

Let's now turn to the notion of Lagrangians.

Definition. Let (X, ω) be a k-shifted symplectic derived Artin stack, and let $f: L \to X$ be a map of derived stacks. The space of isotropic structures on f

$$\mathrm{Isot}(f, \omega)$$

is the space of paths from $f^*\omega$ to 0 in $\mathcal{A}^2_{cl}(L, k)$.

Just as classical Lagrangians are a special class of isotropics, we will need one more condition, of "maximal-ness". In particular, consider the map

$$f^*\mathbb{T}_X \longrightarrow \mathbb{L}_L[k] \quad \text{given by (the underlying form of) } \omega.$$

Write

$$\mathbb{T}_L^{f,\omega} = \mathrm{hofib}\left(f^*\mathbb{T}_X \longrightarrow \mathbb{L}_L[k]\right). = \mathrm{hoeq}\left(f^*\mathbb{T}_X \underset{-\lrcorner\omega}{\overset{0}{\rightrightarrows}} \mathbb{L}_L[k]\right)$$

There is a map $\mathbb{T}_L \xrightarrow{df} f^*\mathbb{T}_X$ and a path from $f^*\omega$ to 0 (an isotropic structure) provides a homotopy between the compositions

$$\mathbb{T}_L \xrightarrow{df} f^*\mathbb{T}_X \underset{-\lrcorner\omega}{\overset{0}{\rightrightarrows}} \mathbb{L}_L[k],$$

whence inducing a map $\mathbb{T}_L \dashrightarrow \mathbb{T}_L^{f,\omega}$.

Definition. We say an isotropic structure $\gamma \in \mathrm{Isot}(f, \omega)$ is Lagrangian if the induced map $\mathbb{T}_L \dashrightarrow \mathbb{T}_L^{f,\omega}$ is a quasi-isomorphism.

Remark. This lines up with our usual intuition: an isotropic is Lagrangian if it has all the directions along which the restriction of ω is zero.

Example. Here is the most basic example, which can be interpreted as showing us how in some sense Lagrangians are more fundamental than symplectic structures. Let f be the map

$$f: X \longrightarrow (pt_k, \omega = 0)$$

for X a derived Artin stack and $(pt_k, \omega=0)$ the point with its unique, trivial k-shifted symplectic structure. An isotropic structure on f is a loop at 0 in the space $\mathcal{A}^2_{cl}(X,k)$. In other words, a point in $\Omega\mathcal{A}^2_{cl}(X,k)$. But

$$\Omega \mathcal{A}^2_{cl}(X,k) \simeq \mathcal{A}^2_{cl}(X,k-1)$$

Now

$$\mathbb{T}^{f,\omega}_X = \mathrm{hofib}\left(f^*\mathbb{T}_{pt} \longrightarrow \mathbb{L}_X[k]\right) \simeq \mathbb{L}_X[k-1]$$

(say by replacing $f^*\mathbb{T}_{pt} \simeq 0$ by the cone of $\mathrm{id}_{\mathbb{L}_X[k]}$ shifted by $[-1]$)

and so a Lagrangian structure on f is an element of $\mathcal{A}^2_{cl}(X,k-1)$ s.t. the induced map

$$\mathbb{T}_X \longrightarrow \mathbb{T}^{f,\omega}_X \simeq \mathbb{L}_X[k-1]$$

is a quasi-isomorphism. We conclude that a Lagrangian structure on

$$f: X \longrightarrow pt_k$$

is precisely a $(k-1)$-shifted symplectic structure on X.

Exercise: Show that the intersection (hofiber prod.) of two Lag's in X_k is $(k-1)$-sympl.

We now turn to the main example of Lagrangians that appear in the Moore–Tachikowa style TFT as detailed by Calaque.

Theorem (Calaque). Let W be a smooth compact oriented manifold of dim $d+1$ with boundary ∂W. Then, if X is a k-shifted symplectic derived Artin Stack, the restriction map

$$\mathrm{Map}(W,X) \longrightarrow \mathrm{Map}(\partial W, X)$$

is canonically a Lagrangian.

We will use this result to study canonical relations as Mike introduced in the smooth setting last time.

Recall that a <u>canonical relation</u> between two symplectic manifolds X and Y was a Lagrangian submanifold of $X \times \overline{Y}$. ($\overline{Y}$ having the opposite sympl. str. of Y)
We will do the same thing, but use different terminology to match the category theory peoples' language.

<u>Definition.</u> Let X and Y be k-shifted sympl. Then a <u>Lagrangian correspondence</u> from X to Y is a map $L \longrightarrow X \times \overline{Y}$ of derived Artin stacks equipped with a Lagrangian structure. We often depict it as

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

Mike described to us several examples in the smooth setting — we are to think of these as generalized maps, generalizing symplectomorphisms. To compose maps we have to intersect Lagrangians.

<u>Theorem</u> Let (X, ω_X), (Y, ω_Y), (Z, ω_Z) be k-shifted symplectic. Then, if

$f_1 : L_1 \to X \times \overline{Y}$ and $f_2 : L_2 : Y \times \overline{Z}$ are Lagrangian correspondences,

then $L_1 \overset{h}{\times}_Y L_2 \longrightarrow X \times \overline{Z}$ is canonically a Lagrangian corr.

<u>Pf sketch.</u> We have a path from $f_1^* \pi_X^* \omega_X$ to $f_1^* \pi_Y^* \omega_Y$ in $\mathcal{A}^2_{cl}(L_1, k)$, a path from $f_2^* \pi_Y^* \omega_Y$ to $f_2^* \pi_Z \omega_Z$, and a path from $\pi_{L_1}^* f_1^* \pi_Y^* \omega_Y$ to $\pi_{L_2}^* f_2^* \pi_Y^* \omega_Y$ (consider diagram to the left). in $\mathcal{A}^2_{cl}(L_1 \overset{h}{\times}_Y L_2, k)$. Putting these together (by pulling back to $L_1 \overset{h}{\times}_Y L_2$) we find a path from $\pi_{L_1}^* f_1 \pi_X^* \omega_X$ to $\pi_{L_2}^* f_2 \pi_Z^* \omega_Z$ in $\mathcal{A}^2_{cl}(L_1 \overset{h}{\times}_Y L_2, k)$, which defines an isotropic str. on the map $L_1 \overset{h}{\times}_Y L_2 \longrightarrow X \times \overline{Z}$. We omit the nondegeneracy.

$$\begin{tikzcd} L_1 \overset{h}{\times}_Y L_2 \arrow[r, "\pi_{L_2}"] \arrow[d, "L_1"'] \arrow[dr, phantom, "h"] & L_2 \arrow[d, "\pi_Y f_2"] \\ L_1 \arrow[r, "\pi_Y f_1"'] & Y \end{tikzcd}$$

Thus in the derived setting, Lagrangian correspondences can always be composed — the transversality / clean conditions are automatic.

<u>Definition.</u> Write $\mathrm{LagCorr}_k$ for the symmetric monoidal category whose objects are k-shifted symplectic derived stacks, and $\mathrm{Hom}_{\mathrm{LagCorr}_k}(X, Y)$ is the weak equivalence classes of Lagrangian maps $L \to X \times \overline{Y}$. The composition of maps is given by homotopy fiber product. The monoidal product is $\times$.

Definition Let $d \geq 0$. Write Cob_d for the category of closed oriented d-manifolds and morphisms diffeomorphism classes of oriented cobordisms. The monoidal product is disjoint union.

Recall that a d-dimensional (oriented) TFT with values in $\mathcal{C}^{\otimes}$ is a symmetric monoidal functor $\mathrm{Cob}_d^{\sqcup} \longrightarrow \mathcal{C}^{\otimes}$.

Theorem (Calaque) Let X be a k-shifted symplectic derived Artin stack. Then the functor $\mathrm{Map}(-, X)$ defines a d-dimensional TFT

$$\mathrm{Map}(-,X) : \mathrm{Cob}_d^{\sqcup} \longrightarrow \mathrm{LagCorr}_{k-d}^{\times}.$$

$$M \longmapsto \mathrm{Map}(M, X) \quad (k-d)\text{-shifted symplectic}$$

$$(M_{in} \xrightarrow{W} M_{out}) \longmapsto \begin{tikzcd} & \mathrm{Map}(W,X) \arrow[dl] \arrow[dr] & \\ \mathrm{Map}(M_{in},X) & & \mathrm{Map}(M_{out},X) \end{tikzcd}$$