EXERCISES FOR 520-2

1. March 28, 2017

Exercise 1. Let $E \to X$ be a smooth complex vector bundle and $\mathscr{E}$ be its sheaf of smooth sections. Show that if $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ is a good cover for X then $\check{H}^k(\mathscr{U}, \mathscr{E}) = 0$ for $k > 0$. [Hint: fix a partition of unity η for $\mathscr{U}$ and define a contracting homotopy $s : \check{C}^k(\mathscr{U}, \mathscr{E}) \to \check{C}^{k-1}(\mathscr{U}, \mathscr{E})$ by

$$(s\omega)_{\alpha_0 \cdots \alpha_{k-1}} = \sum_\alpha \eta_\alpha \omega_{\alpha_0 \cdots \alpha_{k-1} \alpha}. \tag{1}$$

where $\omega \in \check{C}^k(\mathscr{U}, \mathscr{E})$.]

Exercise 2. Construct explicitly the long exact sequence for Čech cohomology.

Exercise 3. Check that for a good cover $\mathscr{U}$ of X, the group $\check{\mathrm{H}}^1(\mathscr{U}, \underline{\mathrm{GL}(1,\mathbb{C})})$ is isomorphic to the group $\operatorname{Pic} X$ of isomorphism classes of complex line bundles on X.

Exercise 4. Recall that the Todd genus of a complex line bundle $L \to X$ is the (inhomogeneous) cohomology class $\mathrm{Td}(L) = f(c_1(L))$ where $c_1(L)$ is the first Chern class of L and $f(x) = x/(1 - e^{-x})$. Compute the Todd genus of a rank 2 complex vector bundle $E \to X$ in terms of $c_1(E)$ and $c_2(E)$.

2. March 30, 2017

Exercise 5. Recall that the Chern character of $E = L_1 \oplus \cdots \oplus L_r$ is defined

$$\mathrm{ch}(E) = \sum_{i=1}^r \mathrm{ch}(L_i) = \sum_{i=1}^r \exp c_1(L_i) = \sum_{k=0}^\infty \mathrm{ch}_k(E)$$

where $\mathrm{ch}_k(E) \in \mathrm{H}^{2k}(X; \mathbb{Q})$. Check that $\mathrm{ch}_0(E) = r$ and $\mathrm{ch}_1(E) = c_1(E)$, then compute ch_2 and ch_3 in terms of c_i.

Exercise 6. Let X be a compact complex manifold and $E \to X$ be a holomorphic vector bundle whose sheaf of holomorphic sections we write $\mathscr{E}$. The Hirzebruch-Riemann-Roch theorem is the equality

$$\sum_{k=0}^\infty (-1)^k \dim \mathrm{H}^k(X, \mathscr{E}) =: \chi(E) = \int_X \mathrm{ch}(E)\, \mathrm{Td}(X)$$

where $\mathrm{Td}(X) := \mathrm{Td}(TX)$. We will prove this theorem later in the course. First recover the Riemann-Roch theorem from this equality, then prove the theorem for $X = \mathbb{C}P^n$ and $E = \mathbb{C}P^n \times \mathbb{C}$ the trivial bundle.