# MICROLOCAL ANALYSIS

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These are notes from Jared Wunsch's "Microlocal analysis" course – errors and inaccuracies are, as usual, mine.

## 1. September 22, 2017

Recall Schwartz space
$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta, |x^\alpha D_x^\beta \phi| \leqslant C_{\alpha\beta}\}$$
for multiindices $\alpha, \beta$, where $D_j = -i\frac{\partial}{\partial x_j}$.

**Exercise 1.** Equivalently, one could instead switch the order of the differentiation and the multiplication by $x$ (with some different constant of course). This follows from the commutation relations.

The Schwartz space can be topologized by the seminorms given by the optimal $C_{\alpha\beta} \equiv \sup |x^\alpha D^\beta \phi|$, and thus becomes a Fréchet space. An alternate notation is: $\forall \beta, k$ we require that $|\langle x\rangle^k D^\beta \phi| \leqslant C_{k\beta} < \infty$ where $\langle x\rangle = \sqrt{1+x^2}$.

Notice that $\mathcal{S} \subset L^1(\mathbb{R}^n)$, because if $\phi \in \mathcal{S}$ then $\sup \langle x\rangle^{n+1}\phi| \equiv C_{n0} < \infty$, i.e.
$$\begin{aligned}\int_{\mathbb{R}^n} |\phi| dx &\leqslant \int \frac{C_{n0}}{\langle x\rangle^{n+1}} dx \\ &= \int \int_{S^{n-1}} 1 d\theta \frac{1}{\langle r\rangle^{n+1}} r^{n-1} dr \\ &= C \int_0^1 + \int_1^\infty \frac{1}{\langle r\rangle^{n+1}} r^{n-1} dr,\end{aligned}$$
which is finite since $(n-1) - (n+1) < -1$.

Set, for $\phi \in \mathcal{S}$,
$$(\mathcal{F}\phi)(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\phi(x) đx$$
where $đx = (2\pi)^{-n/2} dx$. The Riemann-Lebesgue lemma shows that the output is in $C_0(\mathbb{R}^n)$. Note that if $\phi \in \mathcal{S}$ then
$$\begin{aligned}(D_\xi^\alpha \mathcal{F}\phi)(\xi) &= \int \phi(x)(-x)^\alpha e^{-\xi x} đx \\ &= \mathcal{F}\left((-x)^\alpha \phi\right)\end{aligned}$$
where one should justify why we can move the derivative inside. Similarly one checks that
$$\mathcal{F}(D_x^\alpha \phi) = \xi^\alpha \mathcal{F}\phi.$$
Maybe a better notation is to use $M^\alpha$ for multiplication by coordinates; we have shown that for $\phi \in \mathcal{S}$, $\mathcal{F}D^\alpha\phi = M^\alpha \mathcal{F}\phi$ and $\mathcal{F}M^\alpha\phi = (-D)^\alpha \mathcal{F}\phi$.

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**Exercise 2.** Check that the Fourier transform of a Schwartz function is again a Schwartz function. In particular $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a continuous linear map.

Another useful notation: set $Jf(x) = f(-x)$ and set $T = J \circ \mathcal{F} \circ \mathcal{F}$. Then one checks that $T$ commutes with multiplication and differentiation:

$$TD = DT$$
$$TM = MT.$$

**Theorem 3.** *$T : \mathcal{S} \to \mathcal{S}$ is continuous and linear, which implies that $T = \mathrm{id}$.*

**Corollary 4** (Fourier inversion)**.** *$\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is one-to-one and onto and $\phi(x) = \mathcal{F}^{-1}(\mathcal{F}\phi)$ where $\mathcal{F}$ is defined similar to the Fourier transform but with a plus sign.*

*Proof of theorem.* Fix $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\psi(x) \neq 0$ for all $x$, e.g. $e^{-x^2}$. Now for each $\phi \in \mathcal{S}$ and for each $x \in \mathbb{R}^n$,

$$\phi(x) = \phi(a)\frac{\psi(x)}{\psi(a)} + r(x)$$

for $r \in \mathcal{S}$ with $r(a) = 0$. Taylor's theorem allows us to write

$$r(x) = \sum_j (x_j - a_j)\gamma_j(x).$$

with $\gamma_j \in \mathcal{S}$ (near $x = a$ this is just Taylor's theorem, buy away from $x = a$ we can just set $\gamma_j(x) = (x_j - a_j)/|x - a|^2 \cdot r(x)$ and then put these together using a partition of unity). Now we have that for all $\phi \in \mathcal{S}$ and $a \in \mathbb{R}^n$,

$$\phi(x) = \frac{\phi(a)}{\psi(a)}\psi(x) + \sum (x_j - a_j)\gamma_j(x).$$

Hence

$$(T\phi)(x) = \frac{\phi(a)}{\psi(a)}(T\psi)(x) + \sum (x_j - a_j)(T\gamma_j)(x)$$

using one of the above properties of $T$. Evaluating at $a$ we find that

$$(T\phi)(a) = \phi(a)\frac{(T\psi)(x)}{\psi(a)}.$$

Set $g(x) = T\psi(x)/\psi(a)$. Thus for all $a, \phi$, $(T\phi)(x) = \phi(x)g(x)$. Now using that $TD = DT$, it follows that $\phi g' = 0$ identically, whence $g$ is constant. Hence $T$ is multiplication by a constant.

One could show that the constant is 1 by checking for the Gaussian. Instead we will consider $\phi(x) = 1/(1 + x^2)$. In multiple dimensions one uses the product of these for each dimension.

$$(\mathcal{F}\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + x^2} d^{-i\xi x} dx$$

We compute this integral on a contour either an upper semicircle (if $\xi < 0$) or a lower semicircle (if $\xi > 0$). We obtain now: $\sqrt{\pi}/2 \cdot e^{-|\xi|}$. Now we take the Fourier transform again and apply $J$ we get back $1/(1 + x^2)$. $\square$

Note that if $\phi, \psi \in \mathcal{S}$, we write

$$(\phi, \psi) = \int \phi(x)\psi(x) dx$$

for the distributional pairing and

$$\langle \phi, \psi \rangle = \int \phi \bar{\psi} dx$$

for the $L^2$-pairing. It is straightforward, using Fubini, to show that

$$(\phi, \hat{\psi}) = (\hat{\phi}, \psi).$$

Define the tempered distributions

$$\mathcal{S}'(\mathbb{R}^n) = \{u : \mathcal{S} \to \mathbb{C} \mid \text{linear, cts}\}.$$

and define, for $u \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform $\hat{u}$ of $u$,

$$(\hat{u}, \phi) \equiv (u, \hat{\phi}).$$

Now consider for $\phi, \psi \in \mathcal{S}$,

$$\langle \hat{\phi}, \hat{\psi} \rangle = (\hat{\phi}, \bar{\hat{\psi}}) = (\mathcal{F}\phi, \mathcal{F}^{-1}\bar{\psi}) = (\phi, \bar{\psi}) = \langle \phi, \psi \rangle.$$

Hence $\mathcal{F}$ extends by continuity to a unitary map $L^2 \to L^2$ (we are using density of the Schwartz space in $L^2$).

If $p(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ is a polynomial then we consider the constant coefficient differential operator $p(D)$. For each $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\mathcal{F}(p(D)u) = p(\xi)\mathcal{F}(u)$$

(of course multiplication by polynomials preserves tempered distributions). Next time we move to nonconstant coefficient differential operators.

## 2. September 25, 2017

**Definition 5.** We write $A \in \mathrm{Diff}^m(\Omega)$ for $\Omega \subset \mathbb{R}^n$ an open, if there exist $a_\alpha(x) \in C^\infty(\Omega)$ such that

$$(A\phi)(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \phi$$

for $\phi \in D'(\Omega)$ (which recall is the dual to $C_c^\infty(\Omega)$).

**Exercise 6.** $A \in \mathrm{Diff}^m(\Omega)$ if and only if there exists $b_\alpha(x) \in C^\infty(\Omega)$ such that $A\phi(x) = \sum_{|\alpha| \leq m} D^\alpha(b_\alpha(x)\phi(x))$. To see this one uses the commutation rule $xD = Dx + 1/i$.

**Definition 7.** Suppose we have $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \in \mathrm{Diff}^m(\Omega)$. We write

$$\sigma_A(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

which defines $\sigma_A \in C^\infty(\Omega \times \mathbb{R}^n_\xi)$. The map $A \to \sigma_A$ is an isomorphism of graded vector spaces. We call this map the **left total symbol**. It is important to note that this is not a ring homomorphism ($AB \neq BA$ but $\sigma_A \sigma_B = \sigma_B \sigma_A$).

What is the reverse process – how do we get from $\sigma_A(x,\xi) = \sum a_\alpha(x)\xi^\alpha$ back to $A$? Suppose $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then we see that

$$\begin{aligned}(A\phi)(x) &= \sum a_\alpha(x)(D^\alpha\phi)(x)\\ &= \sum a_\alpha(x)\mathcal{F}^{-1}\xi^\alpha\mathcal{F}\phi\\ &= \frac{1}{(2\pi)^n}\sum a_\alpha(x)\int e^{i\xi x}\xi^\alpha\int e^{-i\xi y}\phi(y)dyd\xi\\ &= \frac{1}{(2\pi)^n}\int\int\sum a_\alpha(x)\xi^\alpha e^{i(x-y)\xi}\phi(y)dyd\xi\\ &= \frac{1}{(2\pi)^n}\int\int\sigma_A(x,\xi)e^{i(x-y)\xi}\phi(y)dyd\xi.\end{aligned}$$

In other words, a differential operator can be written as acting by the symbol under iterated integration against a phase. Notice that $\sigma_A$ is very special – it is fiberwise polynomial. A pseudodifferential operator is of this form where we replace the symbol by a more general form than a polynomial.

The main question is: what functions should we use to define what appears in the definition of pseudodifferential operators?

**Definition 8.** Let $\Omega \subset \mathbb{R}^d$ be open. We define $a(x,\theta) \in S^m(\Omega \times \mathbb{R}^N_\theta)$ to mean that $a$ is a smooth function on $\Omega \times \mathbb{R}^N$ (notice the dimensions of $\Omega$ and $\mathbb{R}^N$ need not be the same) if

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \leqslant C_{\alpha\beta}\langle\theta\rangle^{m-|\beta|}.$$

This is known as a **Kohn-Nirenberg symbol**.

Given $a \in S^m(\Omega \times \mathbb{R}^n)$, set

$$(\mathrm{Op}(a)\phi)(x) = (2\pi)^{-n}\int\int a(x,\theta)e^{i(x-y)\theta}\phi(y)dyd\theta.$$

If $\phi \in \mathcal{S}(\mathbb{R}^n)$ this makes sense as an iterated integral. Notice that this is technically $\mathrm{Op}^L$ since we used the left symbol. If we have used the right symbol then instead of $a(x,\theta)$ we would find $a(y,\theta)$ in the integral.

In fact one could put something in much more general than a Kohn-Nirenberg symbol. However, one would like a composition of pseudodifferential operators to again be a pseudodifferential operator. The Kohn-Nirenberg symbol is one of the easier definitions that satisfies this. Shubin, for instance, uses the more general definition

**Definition 9.** We say that $a(x,\theta) \in S^m_{\rho\delta}(\Omega \times \mathbb{R}^N)$ if for all $\alpha, \beta$,

$$|\partial_x^\alpha \partial_\theta^\beta a| \leqslant C_{\alpha\beta}\langle\theta\rangle^{m-\rho|\beta|+\delta|\alpha|}.$$

where $0 \le \delta < \rho \le 1$. These are know as **Hormander symbols**.

It turns out that most of theory still goes through here. Most of what we will say will go through mutatis mutandis for Hormander symbols (e.g. composition), but we will focus on Kohn-Nirenberg symbols for which $\rho = 1, \delta = 0$.

*Remark* 10. A polynomial $\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in S^m$ if $a_\alpha(x) \in C^\infty$ with bounded ?.

One can check that if $a \in S^m, b \in S^{m'}$ then $ab \in S^{mm'}$.

What can we do with these that we couldn't do with just polynomials? Well notice that if $p, q$ are polynomials and $q$ is never zero on $\mathbb{R}^n$ then $p(\xi)/q(\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. For instance $1/(1+\xi^2) \in S^{-2}(\mathbb{R}^n \times \mathbb{R}^n)$.

We can localize these symbols in fact. Localizing to compact sets is not so interesting because then the $m$ can be taken $-\infty$. So let's look at the following.

**Example 11.** Fix $\phi \in C_c^\infty(\mathbb{R})$ which is 1 close to the origin say in a radius of $\varepsilon/2$ and zero outside a radius of $\varepsilon$. Let $\chi$ be a function that is zero for $\xi < 0$ and 1 for $\xi > 1$ Fix $x_0 \in \mathbb{R}^n$ and a direction $\hat{\xi_0} \in S_\xi^{n-1}$. Then set (notice that it is smooth)

$$a(x,\xi) = \langle \xi \rangle^m \phi(|x-x_0|)\chi(|\xi|)\phi\left(\left|\frac{\xi}{|\xi|} - \xi_0\right|\right).$$

We claim that $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. Some arguments follow. This is a localization near $x_0$ and the covector direction $\xi_0$.

**Definition 12.** We define $S^{-\infty}(\Omega \times \mathbb{R}^n) = \cap_{m\in\mathbb{R}} S^m(\Omega \in \mathbb{R}^n)$. These are rather trivial to deal with generally.

An example of such a symbol would be $\phi(|x-x_0|)\phi(|\xi-\xi_0|)$, i.e. having compact support in the $\xi$ direction.

**Theorem 13** (Schwartz kernel theorem)**.** *Suppose we have a linear continuous map $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Then there exists $K_A(x,y) \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that $(A\phi)(x) = \int K_A(x,y)\phi(y)dy$.*

Here one should be careful that the integral is not an integral and the kernel is not really a function. What do we mean by this? We mean that if we took another $\psi \in \mathcal{S}$, then

$$(A\phi, \psi) = (K_A, \psi(x)\phi(y)) = (K_A, \psi \boxtimes \phi).$$

## 3. September 27, 2017

Recall last time we define an operator $\mathrm{Op}^l(a)$ associated to a Kohn-Nirenberg symbol $a$, under the "left choice" that we made. Instead we could make the "right choice,"

$$\mathrm{Op}^r(a) = \sum_{|\alpha|\le m} D^\alpha(a_\alpha \phi)(x).$$

It is easy to see that the resulting operator is the same, at least on fiber polynomials. One annoying thing is that adjoints of left operators will yield right operators and vice versa.

Let's generalize this story. Consider $a \in S^m(\Omega_x \times \Omega_y \times \mathbb{R}^n)$, i.e. we have $\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \alpha| \le C_{\alpha\beta\gamma}\langle\xi\rangle^{m-\gamma}$. Set formally

$$(\tilde{\mathrm{Op}}(a)\phi)(x) = (2\pi)^{-n} \int\int e^{i(x-y)\xi} a(x,y,\xi)\phi(y)dyd\xi.$$

Notice that it is not clear that this iterated integral even makes sense. Recall now the Schwartz kernel theorem, which we stated last time. We won't prove this theorem, even though it's not all that difficult. Instead let's just consider the case of $S^1$. Then say we have $B : C^\infty(S^1) \to \mathcal{D}'(S^1)$. Then

$$B\phi = \int K_B(x,y)\phi(y)dy.$$

We should take

$$\begin{aligned}\hat{K}_B(m,n) &\equiv \int\int K_B(x,y)e^{-imx}e^{-iny}dxdy \\ &= \big(B(e^{-in\bullet}), e^{-im\bullet}\big)\end{aligned}$$

For $\mathbb{R}^n$ one might use instead a basis involving the Hermite polynomials. This theorem is philosophically important: to define operators just write down Schwartz kernels.

We will now define $\tilde{\mathrm{Op}}(a)$ for $a \in S^m(\Omega\times\Omega\times\mathbb{R}^n$, by the Schwartz kernel theorem on $\Omega \times \Omega$, for all $\gamma \in C_c^\infty(\Omega\times\Omega)$, we want

$$(2\pi)^{-n}\int\int\int e^{i(x-y)\xi}a(x,y,\xi)\gamma(x,y)dydxd\xi.$$

In general we will have to think about regularization. However, there is a rich class of $a$ for which this converges already. If $a \in S^m, m < -n$ this does converge and define a distribution, indeed $K_{\tilde{\mathrm{Op}}}(x,y)$ is continuous. To see this one checks that in this case the symbol is in $L^1$ (switch to polar and check the decay rate) and then apply dominated convergence.

Now note that if $a \in S^m, m < -n$, then since for all $k$,

$$\frac{1}{(1+|\xi|^2)^k}(1+D_y^2)^k e^{i(x-y)\xi} = e^{i(x-y)\xi},$$

(where by $D_y^2$ we need the Laplacian) we see that by integrating by parts (there are no boundary terms)

$$\begin{aligned}(\tilde{\mathrm{Op}}(a),\gamma) &= \frac{1}{(2\pi)^n}\int\int\int \frac{1}{(1+|\xi|^2)^k}(1+D_y^2)^k e^{i(x-y)\xi}a(x,y,\xi)\gamma(x,y)dxdyd\xi \\ &= \frac{1}{(2\pi)^n}\int\int\int e^{i(x-y)\xi}\frac{1}{(1+|\xi|^2)^k}(1+D_y^2)^k e^{i(x-y)\xi}a(x,y,\xi)\gamma(x,y)dxdyd\xi\end{aligned}$$

But if $a \in S^m$ the integrand is $O(\langle\xi\rangle^{m-2k})$ i.e. converges if $a \in S^m$ for $m-2k < -n$, in other words $m < -n+2k$. So we see that $a \mapsto \tilde{\mathrm{Op}}(a) \in \mathcal{D}'(\Omega\times\Omega)$ extends to $a \in S^m$ if $m < -n+2k$, in other words for all $m$ by taking $k$ very large. One might worry that the answer might depend on the regularization procedure. However the symbols that are of very negative order work unambiguously and something something about topology shows that its unambiguous in general.

**Lemma 14.** *If* $a \in S^m(\Omega\times\mathbb{R}^m)$ *then* $\tilde{\mathrm{Op}}(a) : C_c^\infty(\Omega) \to C^\infty(\Omega)$.

*Proof.* We know that

$$\begin{aligned}(\tilde{\mathrm{Op}}(a)\phi)(x) &= (2\pi)^{-n}\int\int e^{i(x-y)\xi}a(x,y,\xi)\phi(y)dyd\xi \\ &\equiv (2\pi)^{-n}\int\int e^{i(x-y)\xi}\frac{1}{(1+|\xi|^2)^k}(1+D_y^2)^k(a(x,y,\xi)\phi(y))dyd\xi\end{aligned}$$

but the inside (other than the exponential) is in $L^1$ uniformly in $x, y$ so we obtain something in $C^0(\Omega_x)$. Now what happens if we differentiate with respect to $x$? Applying $D_x^\alpha$ and moving it inside the integral, it is differentiating the whole integrand. What's the worst order behavior we get in $\xi$? Well we have things of order $m-2k+|\alpha|$ in $\xi$ times a function $C^\infty(y)$. This converges if $m-2k+|\alpha| < -n$. So taking $k$ large enough we see that $\tilde{\mathrm{Op}}(a)$ is $C^\infty$. $\square$

Notice that if $A : C_0^\infty \to C^\infty$ (on $\Omega$) then we can define $A^t : \mathcal{E}' \to \mathcal{D}'$ (where $\mathcal{E}'$ is compactly supported distributions, which are dual to smooth functions) by taking

$$(A^t u, \phi) \equiv (u, A\phi).$$

**Exercise 15.** Check that $K_{A^t}(x, y) = K_A(y, x)$. Just apply Fubini.

## 4. September 29, 2017

Recall our definition of the adjoint $A^t$ of $A$ above. We found that for all $A : \mathcal{S} \to \mathcal{S}'$,

$$K_{A^t}(x, y) = K_A(y, x).$$

*Remark* 16. If $A = \tilde{\mathrm{Op}}(a)$ for $a \in S^m(\Omega \times \Omega \times \mathbb{R}^n)$ and

$$K_A = (2\pi)^{-n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi$$

then we check that

$$\begin{aligned} K_{A^t} &= (2\pi)^{-n} \int e^{-i(x-y)\xi} a(y, x, \xi) d\xi \\ &= (2\pi)^{-n} \int e^{i(x-y)\eta} a(y, x, -\eta) d\eta \\ &= \tilde{\mathrm{Op}}(b), \end{aligned}$$

for $b(x, y, \xi) = a(y, x, -\xi)$. Hence for any $A = \tilde{\mathrm{Op}}(a)$ set $A : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$

$$(Au, \phi) = (u, A^t \phi).$$

since $A^t : C_c^\infty(\Omega) = \mathcal{D}(\Omega) \to \mathcal{E}(\Omega) = C^\infty(\Omega)$.

Let's now turn to discussing the singular support.

**Definition 17.** Let $u \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$. We say that $x_0 \notin \operatorname{singsupp} u$ if there exists $\phi \in C_c^\infty(\Omega)$ with $\phi(x_0) \neq 0$ such that $\phi u \in C_c^\infty(\Omega)$.

**Lemma 18.** *For $a$ a symbol of order $m$, take $A = \tilde{\mathrm{Op}}(a)$. Then* $\operatorname{singsupp} K_A \subset \Delta = \{(x, x) \mid x \in \Omega\}$.

**Example 19.** For instance, notice that the Schwartz kernel of the identity operator is the Dirac delta. This is singular precisely along the diagonal. Check for yourself that the identity operator is the (left-)quantization of the function 1.

**Exercise 20.** What is the Schwartz kernel of a differential operator $P = a_\alpha(x) D^\alpha$? Where is it singular?

*Proof of Lemma.* We have that

$$\begin{aligned} D_x^\alpha D_y^\beta K_A(x, y) &= D_x^\alpha D_y^\beta (2\pi)^{-n} \int e^{i(x-y)\xi} a(x, y, \xi) d\xi \\ &= D_x^\alpha D_y^\beta (2\pi)^{-n} \int \frac{1}{|x-y|^{2k}} \Delta_\xi^k (e^{i(x-y)\xi}) a(x, y, \xi) d\xi \\ &= D_x^\alpha D_y^\beta (2\pi)^{-n} \frac{1}{|x-y|^{2k}} \int e^{i(x-y)\xi} \Delta_\xi^k a(x, y, \xi) d\xi \end{aligned}$$

where we have integrated by parts. But now the Laplacian of the symbol is in $S^{m-2k}$. Hence we get integrability if $m - 2k - |\alpha| + |\beta| < -n$. We conclude that $K_A \in C^\infty(\Omega \times \Omega \setminus \Delta)$. □

Say $K_A(x,y) \in C^\infty$ then

$$(A\delta)(x) = \int K_A(x,y)\delta(y)dy = K_A(x,0) \in C^\infty.$$

This is typical of the action of a $C^\infty$ Schwartz kernel. In fact one can check one can "differentiate under the integral sign" (really it's not an integral but a distributional pairing, so it's a bit more subtle).

**Lemma 21.** *If $K_A \in C^\infty(\Omega \times \Omega)$ then $A : \mathcal{E}'(\Omega) \to C^\infty(\Omega)$.*

**Proposition 22.** *If $A = \tilde{\operatorname{Op}}(a)$ for $a \in S^m(\Omega \times \Omega \times \mathbb{R}^n)$ then* $\operatorname{singsupp} Au \subset \operatorname{singsupp} u$ *for all $u \in \mathcal{E}'(\Omega)$.*

This shows, as Nir asked earlier, that translation cannot be a pseudodifferential operator. Some lingo that people like to use: $\Psi$DOs are **pseudolocal**, but not local. Think of microlocal analysis as the study of distributions modulo smooth functions.

*Proof.* Pick $x_0 \notin \operatorname{singsupp} u$. Fix $\chi \in C_c^\infty$ such that $\chi = 1$ on $\operatorname{singsupp} u$ and $\chi = 0$ near $x_0$. Then pick $\psi \in C_c^\infty$ such that $\psi = 1$ at $x_0$ and $\psi\chi = 0$. Near $x_0$, we can write $Au = \psi Au = \psi A\chi u + \psi A(1-\chi)u$. Now

$$K_{\psi A\chi} = \psi(x)K_A(x,y)\chi(y)$$

But by construction of $\psi$ and $\chi$ we find that $K_{\psi A \chi} \in C^\infty$. On the other hand $(1-\chi)u \in C_c^\infty$ whence $A(1-\chi)u \in C^\infty$. □

As things stand, we have the following awkward problem. Pseudodifferential operators take smooth functions with compact support to smooth functions without compact support. If we want to talk about compositions of pseudodifferential operators, we need to fix this.

**Definition 23.** We say that $A : C_c^\infty(\Omega) \to \mathcal{D}'(\Omega)$ has **proper support** if both $\pi_L, \pi_R : \operatorname{supp} K_A \to \Omega$ are proper maps, where $\pi_L, \pi_R$ are the left and right projections on $\Omega \times \Omega$.

Morally you can think that we're requiring that the support of the kernel be in a neighborhood of the diagonal.

**Proposition 24.** *Suppose $A = \tilde{\operatorname{Op}}(a)$ is properly supported. Then $A$ maps $C_c^\infty(\Omega)$ to $C_c^\infty(\Omega)$ and $\mathcal{E}'(\Omega)$ to $\mathcal{E}(\Omega)$. Dualizing, we see that it maps $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ and $C^\infty(\Omega)$ to $C^\infty(\Omega)$.*

*Proof.* Left as an exercise. □

The goal of the next couple lectures is the following.

**Theorem 25** (Left and right reduction)**.** *Suppose $A = \tilde{\operatorname{Op}}(a)$ for $a \in S^m(\Omega \times \Omega \times \mathbb{R}^n)$ is properly supported. Then there exist $b_L, b_R \in S^m(\Omega \times \mathbb{R}^n$ such that $A = \operatorname{Op}^L(b_L) = \operatorname{Op}^R(b_R)$.*

In particular, we see every left quantization can be written as a right quantization and vice versa. This result tells us already that pseudodifferential operators are already closed under adjoints.

**Proposition 26.** *If $A = \tilde{\operatorname{Op}}(a)$ for $a \in S^m$ then $A = A_0 + A_1$ where $A_0$ is a properly supported $\Psi DO$ and $K_{A_1} \in C^\infty$.*

*Proof.* Let $\chi$ be a cutoff function supported on $[0,1]$. Split up the integral expression for $K_A(x,y)$ into two terms by adding $\chi = \chi + (1-\chi)$ into the integral. The two terms will have the claimed properties. □

*Remark* 27. If $A = \mathrm{Op}^L(b)$ then (formally for now)

$$\begin{aligned}\mathrm{Op}^L(b)e^{i\eta x} &= (2\pi)^{-n} \int\int e^{i(x-y)\xi} b(x,\xi) e^{i\eta\xi} dy d\xi \\ &= e^{ix\eta} b(x,\eta).\end{aligned}$$

In other words, we can read off $b(x,y)$ as $b(x,y) = e^{-ix\eta} A e^{ix\eta} \equiv \sigma_A^L(x,y)$. The only question is: is this a symbol? This is the direction to proving the theorem above.

## 5. October 2, 2017

Note: if $A = \tilde{\mathrm{Op}}(a)$ for some $a \in S^m(\Omega \times \Omega \mathbb{R}^N)$, and we set $b(x,\xi) = e^{-ix\xi} A e^{ix\xi} \in S^m$ for $\xi \in \mathbb{R}^n$, then for all $\phi \in \mathcal{S}$,

$$\begin{aligned}(A\phi)(x) &= A\left((2\pi)^{-n/2} \int \hat{\phi}(\xi) e^{i\xi x} d\xi\right) \\ &= (2pi)^{-n/2} \hat{\phi}(\xi) A(e^{i\xi x}) d\xi \\ &= (2\pi)^{-n/2} \hat{\phi}(\xi) b(x,\xi) e^{i\xi x} dx \\ &= (2\pi)^{-n/2} \int\int b(x,\xi) e^{i(x-y)\xi} \phi(y) dy d\xi \\ &= (\mathrm{Op}^L(b)\phi)(x)\end{aligned}$$

where we have moved $A$ into the integral since the integral converges in $\mathcal{D}'$ in the $x$ variable.

Our main goal now is to show that if $a \in S^m(\Omega \times \Omega \times \mathbb{R}^N)$ and $A = \tilde{\mathrm{Op}}(a)$ is properly supported then

$$\sigma_A^L(x,\xi) = e^{-i\xi x} A e^{i\xi x} \in S^m(\Omega \times \mathbb{R}^N).$$

**Definition 28.** Say $a \in C^\infty(\Omega \times \mathbb{R}^N)$ and $a_j(x,\theta) \in S^{m_j}(\Omega \times \mathbb{R}^N)$ such that $m_j$ is a decreasing sequence $m_j \to -\infty$ as $j \to \infty$ for $j \in \mathbb{N}$. We say that

$$a \sim \sum_{j=0}^{\infty} a_j$$

if for each $J$,

$$a - \sum_{j=0}^{J-1} a_j \in S^{m_J}(\Omega \times \mathbb{R}^N).$$

Notice that this implies that $a \in S^{m_0}$. However, there is no guarantee of convergence.

We've all seen something like this before: Taylor's theorem.

**Theorem 29** (Borel). *Given $a_j \in S^{m_j}$ as above, there exists $a \in S^{m_0}$ such that $a \sim \sum_{j=0}^{\infty} a_j$, and any two such $a$ differ by an element of $S^{-\infty}(\Omega \times \mathbb{R}^N)$.*

*Proof.* Fix $\phi \in C^\infty(\mathbb{R}^N)$ such that $\phi(\theta) = 1$ for $|\theta| \geq 1$ and 0 for $|\theta| < 1/2$. Pick some increasing $t_j \to \infty$ and set

$$a(x,\theta) = \sum_{j=0}^{\infty} \phi\left(\frac{\theta}{t_j}\right) a_j(x,\theta).$$

Notice that for any particular $(x,\theta)$ this is a finite sum, due to the construction of $\phi$. So there is no question about the convergence of this series (though it is not uniform). Note that for every $\beta$ for $|\beta| \geqslant 1$ (and say $t_j \geqslant 1$),

$$\begin{aligned} \partial_\theta^\beta \phi(\theta/t_j) &= t_j^{-|\beta|}(\partial_\theta^\beta \phi)(\theta/t_j) \\ &= (\partial_\theta^\beta \phi)(\theta/t_j)|\theta/t_j|^\beta |\theta|^{-\beta} \\ &\leqslant C_\beta \langle\theta\rangle^{-\beta}. \end{aligned}$$

Hence $\phi(\theta/t) \in S^0$ uniformly in $t \geqslant 1$. Similarly for $\beta = 0$, of course. This shows that that

$$|\partial_x^\alpha \partial_\theta^\beta (\phi(\theta/t_j) a_j(x,\theta))| \leqslant C_{\alpha\beta} \langle\theta\rangle^{m_j - |\beta|} 1_{|\theta| > t_j/2}.$$

We claim now that we can choose increasing $t_j \to \infty$ such that

$$|\partial_x^\alpha \partial_\theta^\beta (\phi(\theta/t_j) a_j(x,\theta))| \leqslant 2^{-j} \langle\theta\rangle^{m_{j-1} - |\beta|}.$$

Use that

$$1_{\theta > t_j/2} \langle\theta\rangle^{m_j - |\beta|} = \langle\theta\rangle^{m_{j-1} - |\beta|} \langle\theta\rangle^{m_j - m_{j-1} - 1} 1_{|\theta| > t_j/2} < \frac{1}{2} \langle\theta\rangle^{m_{j-1} - |\beta|}$$

if $t_j$ are sufficiently large. How do we kill the constants $C_{\alpha\beta}$? Then for $J > |\alpha| + |\beta|$, the remainder term satisfies

$$\begin{aligned} |\partial_x^\alpha \partial_\theta^\beta \sum_{j>J} \phi(\theta/t_j) a_j| &\leqslant \sum_{j>J} 2^{-j} \langle\theta\rangle^{m_{j-1} - |\beta|} \\ &\leqslant \langle\theta\rangle^{m_J - |\beta|} 2^{-J}. \end{aligned}$$

Now given $\alpha, \beta$, pick $J > |\alpha| + |\beta|$ and write $a = \sum_{j=0}^{J} \phi(\theta/t_j) a_j + r_J$ where $r_J$ is the remainder as above. Then we find that

$$|\partial_x^\alpha \partial_\theta^\beta a| \leqslant C \langle\theta\rangle^{m_0 - |\beta|} + \langle\theta\rangle^{m_J - |\beta|}$$

i.e. that $a \in S^{m_0}$. By the same argument one finds that $r_J \in S^{m_J}$ for all $J$. Moreover we check that $a \sim \sum a_j$ as

$$a - \sum_{j=0}^{J-1} a_j = \sum_{j=0}^{J-1} a_j(-1 + \phi(\theta/t_j)) + a_{J+1}\phi + r_{J+1}.$$

The sum term is compactly supported whence in $S^{-\infty}$ while the last two terms are in $S^{m_{J+1}}$ and $S^{m_J}$ respectively. This proof can be found in Shubin's book, for instance. □

The following is a tool to check whether something is indeed an asymptotic sum.

**Proposition 30.** *Say $a \in C^\infty(\Omega \times \mathbb{R}^N)$ and $a_j \in S^{m_j}$ for decreasing $m_j \to -\infty$ such that there exists a $\mu$ such that $|\partial_x^\alpha \partial_\theta^\beta a| < C_{\alpha\beta} \langle\theta\rangle^\mu$ for all $\alpha, \beta$. Finally, suppose $|a - \sum_{j=0}^{J-1} a_j| \leqslant C \langle\theta\rangle^{\mu_j}$ for decreasing $\mu_j \to -\infty$. Then $a \sim \sum_{j=0}^{\infty} a_j$.*

We'll prove this next time. Here is a useful calculus fact: say $f \in C^2[-1,1]$ and $A_j = \sup_{[-1,1]} |f^{(j)}(t)|$ for $j = 0, 1, 2$ then

$$|f'(0)| \leqslant 4A_0(A_0 + A_2).$$

## References