Brief Review of QM (No Spin): $\hbar = c = k_B = G = 1$

$\mathcal{H} = \mathcal{L}^2(\mathbb{R}^4)$, $\hat{H}$ unitary, $\hat{H}|\psi\rangle = i\partial_t|\psi\rangle$. $\hat{H}$ is not time-dependent

Generic Example: $\hat{H} = \frac{-1}{2m}\nabla^2 + V(\hat{X})$, $\hat{X}\phi(y) = y\phi(y)$.

If we let $\hat{P}_i = -i\partial_i$ we can write this as $\frac{1}{2m}\hat{P}^2 + V(\hat{X})$.

Trick: To solve this, take $\psi(x,y,z)$, $|\psi(x,y,z,t)\rangle = e^{-it\hat{H}}|\psi(x,y,z)\rangle$

To compute $e^{it\hat{H}}$, we must compute this in a basis of $\mathcal{L}^2(\mathbb{R}^3)$

There are two common choices:

Let $|\vec{x}\rangle = \delta(y-x)$. Then $\int d^3x\, |x\rangle\langle x| = 1$. We also have a wave-plane basis $|\vec{p}\rangle = e^{ip\cdot y}$. $\int \frac{d^3p}{(2\pi)^3}|p\rangle\langle p| = 1$.

Then we note $\langle x|p\rangle = e^{ip\cdot x}$. These are Eigenbasis of $\hat{X}$ and $\hat{P}$.

We compute ~~...~~ $|\vec{x}, t\rangle = \delta(\vec{x}-\vec{y})\delta(t-t_0)$.

$$\langle \vec{x}, t|\vec{y}, t_0\rangle \overset{\text{Mixing Picture}}{=} \langle \vec{x}|e^{-i(t_1-t_0)\hat{H}}|\vec{y}\rangle = \int d^3x \langle \vec{x}|e^{i(\frac{t_1-t_0}{2})\hat{H}}|\vec{x}_1\rangle\langle \vec{x}_1|e^{i(\frac{t_1-t_0}{2})\hat{H}}|\vec{y}\rangle$$

$$= \cdots \int d^3x_1 \cdots d^3x_{n-1} \langle \vec{x}|e^{-i\frac{\Delta t}{n}\hat{H}}|\vec{x}_{n-1}\rangle\langle \vec{x}_{n-1}|e^{-i\frac{\Delta t}{n}\hat{H}} \cdots |\vec{y}\rangle$$

$$= \int d^3x_1 \cdots d^3x_{n-1} \langle \vec{x}|e^{-i\frac{\Delta t}{n}(\hat{P}^2 + V(\hat{X}))} \cdots$$

$$= \int d^3x_1 \cdots d^3x_{n-1} \langle \vec{x}|e^{-i\frac{\Delta t}{n}\hat{P}^2} e^{\frac{i\Delta t}{n}V(\hat{X})} \cdots$$

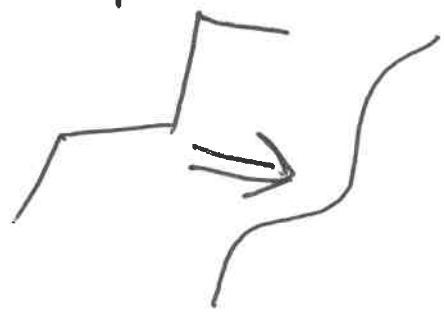
$$= \int d^3x_1 \cdots d^3x_{n-1} \langle \vec{x}|e^{-i\frac{\Delta t}{n}\hat{P}^2} e^{-i\frac{\Delta t}{n}V(x_{n-1})} \cdots$$

$$= \int_{d^3p_1 \cdots d^3p_{n-1}} d^3x_1 \cdots d^3x_{n-1} \langle \vec{x}|e^{\frac{i\Delta t}{n}\hat{P}^2}|p_{n-1}\rangle\langle p_{n-1}|x_{n-1}\rangle\langle x_{n-1}|e^{\frac{i\Delta t}{n}\hat{P}^2} \cdots \prod e^{\frac{i\Delta t}{n}V(x_i)}$$

$$= \int \prod dx_i \prod dp_i \prod \langle x_i | p_i \rangle \langle p_i | x_{i-1} \rangle \prod e^{-i\frac{\Delta t}{n} V(x_i)} \prod e^{-\frac{i\Delta t}{n} p_i^2/2m}$$

$$= \int \prod dx_i\, dp_i \prod e^{ip_i(x_i - x_{i-1})} \cdots = \int \prod dx_i \prod dp_i\, e^{i\left(\sum p_i \Delta x_i - \left(\frac{\Delta t}{n}\right) V(x_i) - \frac{p^2}{2m}\frac{\Delta t}{n}\right)}$$

$$= \int_{\substack{\text{along} \\ Map(\mathbb{R}^4,\mathbb{R}^3)\times Map(\mathbb{R},\mathbb{R}^3)}} Dx\,Dp\, e^{i\int dt\,(p\dot{x} - \frac{p^2}{2m} - V(x))}.$$



How Do We Evalute things like this:

Gaussian:

$$\int dx\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi} \rightarrow \int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \Rightarrow \int dx\, e^{-\frac{1}{2}ax^2 + Jx}$$

$$= \int dx\, e^{-\frac{1}{2}a(x - J/a)^2 + J^2/2a} = e^{J^2/2a}\sqrt{\frac{2\pi}{a}}.$$

$$\int_V dx\, e^{-\frac{1}{2}x^t A x + Jx} = \sqrt{\frac{(2\pi)^{\dim(V)}}{\det(A)}} \exp\left(\tfrac{1}{2} J A^{-1} J\right)$$

Reduce to Diagonal and use Fubini.

In Continuum limit: $\int_H dv\, e^{-\frac{i}{2}(x,Ax) + i(J,x)} = N e^{-\frac{i}{2} J A^{-1} J}$

A^{-1} can be encoded in a basis. Find $A(D_x(y)) = |x\rangle \Rightarrow D(y-x)$

We have $A^{-1}\psi(x) = \int dy\, D(x-y)\psi(y) \Rightarrow e^{-\frac{i}{2}\int dx\,dy\, J(x) D(x-y) J(y)}$

$$= e^{-iV(x)} \int \mathcal{D}x \int \mathcal{D}p \, e^{-i\, p^2/2m - p\dot{x}} = \int \mathcal{D}x \, e^{-iV(x)} e^{i\int dt\, \dot{x}(t)\, m\, \dot{x}(t)/2} = \int \mathcal{D}x \, e^{i\int dt\, p^2/2m - iV(x)}$$

$$= \int \mathcal{D}x \, e^{i\int dt\, \mathcal{L}(x,\dot{x})} = \int \mathcal{D}x \, e^{iS}$$

How to Compute Path integrals in classical Regime:

Restoring units: $S \mapsto \frac{1}{\hbar} S \qquad \hbar = 6.626 \times 10^{-34}\ J\cdot s$

We want to know $\int \mathcal{D}x \, e^{i\lambda S}$ for $\lambda \gg 0$

Add $\theta \in i\mathbb{R}$, $0 < \theta \ll 1$

$$\int_{x_0}^{x_1} \mathcal{D}x \, e^{(i\lambda + \theta i i \lambda) S} = \int \mathcal{D}x \, e^{i\lambda - \theta\lambda S} \cong \sum_{\substack{S \text{ local} \\ \text{maximum} \\ (x_0 - x_1)}} e^{i\lambda - \theta\lambda S}$$

$\theta \to 0$

$$= \sum_{\substack{\delta S = 0 \\ (x_0 - x_1)}} e^{i\lambda S}. \quad \text{So } \delta S = 0 \text{ gives classical equations of motion.}$$

Now $\langle x_2, t_2 | \hat{X}(t_1) | x_0, t_0 \rangle = \int \mathcal{D}x \, x e^{iS}$, $\langle x_2, t_2 | \hat{P}(t_1) | x_0, t_0 \rangle = \int \mathcal{D}x \, p(t_1) e^{iS}$

We can also show $\langle x_2, t_f | \hat{X}_n(t_n) \cdots \hat{X}_1(t_1) | x_i, t_i \rangle$ $\quad t_i < t_1 < \cdots < t_n < t_f$

$= \int \mathcal{D}x \prod X_i(t_i) e^{iS}$. The Right Doesn't Depend on the left, so if we define $T\{\mathcal{O}(t_n) \cdots \mathcal{O}(t_1)\}$

$= \mathcal{O}(t_{\sigma(n)}) \cdots \mathcal{O}(t_{\sigma(1)})$ where $t_{\sigma(i)} < t_{\sigma(i+1)}$, then we see:

$$\langle x_f, t_f | T\{\hat{X}_n(t_n) \cdots \hat{X}_1(t_1)\} | x_i, t_i \rangle = \int \mathcal{D}x \prod x_i(t_i) e^{iS}.$$

How Do we Compute these?

Let $S' = S + \int dt\, (J_k x^i) + \int dt\, J'_i p^i$

Let $Z[J, J'] = \int \mathcal{D}x \, e^{iS'}$. $\left(\frac{1}{i}\right)\frac{\delta}{\delta J_k} Z[J, J'] = \frac{1}{\epsilon} Z[J + \epsilon\delta(t_1), J'] - Z[J, J']$

$= \int \mathcal{D}x \, e^{iS' + \epsilon_k x^k} - \int \mathcal{D}x \, e^{iS'} = \int \mathcal{D}x \, e^{iS'} \int \delta(t_1) \epsilon_k x^k) = \frac{i\epsilon}{i} \int \mathcal{D}x \, e^{iS'} x^k(t_1)$.

So we can understand all observables we know by computing: $Z[J, J']$.

We Can Perform Computations of this type outside of a position basis by being extending.

Guess for first Quantum Field Theory:

$E^2 = p^2c^2 + m^2c^4 = p^2 + m^2$. $(+,-,-,-)$. $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$

Let $E = p_0$, let $p^\mu = \eta^{\mu\nu} p_\nu$, then $p^\mu p_\mu = m^2$.

Note: $E = \sqrt{p^2c^2 + m^2c^4} = c^2 m \sqrt{\frac{p^2}{m^2c^2} + 1} = mc^2 + \frac{1c^2 m}{2m^2c^2} p^2 + O(c^{-2}) \Rightarrow$

$E_0 = E - E_{\text{restmass}} = \frac{1}{2m} p^2$. This Reproduces the Free Schrodinger Equation. We want to try to use the relativistic version

$E\psi = \sqrt{p^2 + m^2}\,\psi$ is a guess, but it is hard to make Sense of $\sqrt{-\nabla^2 + m^2}$ even when $m = 0$,

So instead we will guess something like $E^2\psi = p^2\psi + m^2\psi$, or, $\eta^{\mu\nu}\partial_\mu\partial_\nu\psi = m^2\psi$.

Explain Lorentz-invariance: We want a theory with $SO(3,1)$-invariance.

$\Box\psi - m^2\psi = 0$ is the equation of motion obtained from minimizing $\cancel{\mathcal{L} = \partial_\mu\bar\psi\partial_\nu\psi}$

$S = \int d^4x\, \mathcal{L}_{(\psi,\partial_\mu\psi)} = (-\partial_\mu\bar\psi\,\partial_\nu\psi\,\eta^{\mu\nu} - m^2\bar\psi\psi)$ w/ a new field $\bar\psi$, which we think of as the imaginary part.

Do this explicitly: $\delta_\phi S = 0 \Leftrightarrow \frac{\delta\mathcal{L}}{\delta\psi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi} = 0$.

$S = \int \mathcal{L}\, d^4x = \int +\bar\psi\,\partial_\mu\partial_\nu\psi\,\eta^{\mu\nu} - m^2\bar\psi\psi \Rightarrow \delta_{\bar\psi} S = -\int dx\, \epsilon(x)(\Box\psi - m^2\psi) \Rightarrow \Box\psi - m^2\psi = 0$

$\Rightarrow \Box\bar\psi - m^2\bar\psi = 0$. We will just assume ψ is complex from now on.

We will form our first attempt at a quantum field theory:

$Z[J] = \int D\phi\, \exp(i\int \mathcal{L}_{KG} + J(x)\phi)$ We hope to use this to compute Everything about the Theory.

$Map(\mathbb{R}^4, \mathbb{C})$

~~Lets Compute this: We will compute~~ $\cancel{Z = \int D\phi \exp(i\int d^4x\, \phi(\Box+m^2)\phi)}$

$\mathcal{L}_{KG} = \frac{-1}{2}\psi(\Box + m^2)\psi$, where now ψ is complex valued. $\psi = (\phi, \phi^*)$

$i(\phi^*(\Box+m^2)\phi + \phi(\Box+m^2)\phi^*) + \phi(\Box+m^2)\phi - \phi^*(\Box+m^2)\phi^*)$

~~Complex~~ Imaginary part of $\delta S = 0 \Rightarrow (\Box+m^2)\phi = (\Box+m^2)\phi^* = 0 \Rightarrow \mathrm{Re}(\delta S) = 0$, So $\mathcal{L}_{KG}$ is a good choice.

We will compute $Z[J] = \int D\psi\, \exp(i\int \frac{-1}{2}\psi\hat{A}\psi + J\psi)$.

We first do the trivial case: $\int dx\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$ $\quad \int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}$

$\int dx_1 \cdots dx_n\, e^{-\frac{1}{2}(x^t A x) + J^t x} = \sqrt{\frac{(2\pi)^n}{\det(A)}}\exp(\frac{1}{2}JA^{-1}J)$

Reduce to $A = $ Diagonal

Proof by induction & Fubini.

$\int dx\, e^{-\frac{1}{2}ax^2 + Jx} = \int dx\, e^{-\frac{a}{2}(x-\frac{J}{a})^2 + J^2/2a}$

$= e^{J^2/2a}\int dx\, e^{-\frac{a}{2}(x-J/a)^2}$

$\sqrt{\frac{2\pi}{a}}e^{J^2/2a}$

$$\int_{\mathcal{H}} D\varphi\, e^{-\frac{1}{2}\int \varphi A\varphi + \int J\varphi} = \frac{\sqrt{(2\pi)^{\dim(\mathcal{H})}}}{(\det(A))^{1/2}} \exp\left(\frac{1}{2} J A^{-1} J\right) = C \exp\left(\frac{1}{2}\int d^4y\, d^4x\, J \underbrace{A^{-1}}_{D(x-y)} J\right)$$

$A\, D_y(x) = \delta_y(x)$. Let $A^{-1} = \int d^4_y\, D(x-y)$; $D(x-y) = D_y(x)$

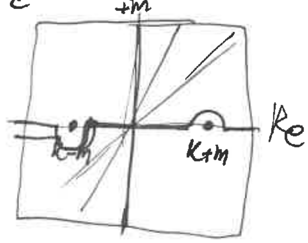
So $$\int_{Map(\mathbb{R}^4,\mathbb{C})} D\phi\, e^{i\int \frac{1}{2}L_{KG} + J\phi} = C \exp\left(\frac{i}{2}\int d^4y\, d^4x\, J(x) D(x-y) J(y)\right).$$

We have Reduced Quadratic Lagrangians (N.D) are done now! Next time we will explore how to evaluate and renormalize this expression.

~~$D(x-y)$~~ Lets calculate $\Delta D(x-y)$. First we solve a related problem:

$(t e^{i\theta}, x \cdots)$ which changes $(\Box + m^2) D = \delta \Rightarrow (+\Delta^2 \mp m^2) D_E = \delta$ Fourier transform

$\Rightarrow (+p^2 \mp m^2)\tilde{D}_E = 1 \Rightarrow D_E = \int d^4x \frac{e^{ipx}}{p^2 - m^2}$. We rotate



$D_E = \int_C dt \int d^3x \frac{e^{ipx}}{p^2 - m^2}$. Smarter people have evaluated this.