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Let's compute some commutators. Recall

$$[a,bc] = [a,b]c \pm b[a,c]$$ Assuming Q is nondegenerate

$$[a,[b,c]] = [[a,b],c] \pm [b,[a,c]].$$

Now if e_i is an ONB and c_i is Clifford mult. by e_i,

$$[c_i c_j, c_k] = c_i[c_j,c_k] - [c_i,c_k]c_j = -2\delta_{jk} c_i + 2\delta_{ik} c_j$$

Consider the basis $E_{ij} - E_{ji}$ of $so(n)$; we see

$$(E_{ij} - E_{ji}) e_k = \delta_{jk} e_i - \delta_{ik} e_j.$$

Hence get a homomorphism

$$so(n) \longrightarrow \text{Lie algebra associated to } \underbrace{C_+}_{\text{even part}}(\mathbb{R}^n)$$

$$E_{ij} - E_{ji} \longmapsto -\frac{1}{2} c_i c_j.$$

natural thing to do: exponentiate. On the left,

$$\exp(t(E_{ij} - E_{ji})) = I \quad \text{for } t \in 2\pi\mathbb{Z}.$$

On the right, we get the Spin group, Spin(n),

$$\exp(t(-\tfrac{1}{2}c_i c_j)) = \exp(-\tfrac{1}{2}t\, c_i c_j)$$

$$(c_i c_j)^2 = c_i c_j c_i c_j = -c_i c_i c_j c_j = -1.$$

$$= \cos(-\tfrac{1}{2}t)\, 1 + \sin(-\tfrac{1}{2}t)\, c_i c_j. \qquad = \begin{cases} +1 & t \in 4\pi\mathbb{Z} \\ -1 & t \in 2\pi(2\mathbb{Z}+1). \end{cases}$$

The center of the group obtained, the Spin(n) gp, is precisely $\{\pm 1\}$.

direction of map? topology?

$$\{\pm 1\} \longrightarrow Spin(n) \longrightarrow SO(n)$$

Recall $\pi_1(SO(n)) = C_2$ if $n \geq 3$, so $Spin(n)$ is simply-connected if $n \geq 3$.

Also:

$$spin_c(n) = \text{real span of } c_i c_j \text{ and } \sqrt{-1}$$

The exponential of this subalgebra $spin(n) \oplus i\mathbb{R}$

$$Spin_c(n) \subset C_+(V) = Spin(n) \times_{\{\pm 1\}} U(1)$$

Spinor representations.

Take n even and an orientation on $\mathbb{R}^n$.

$$C(V) = C^+(V) \oplus C^-(V).$$

Then

$$C(V \oplus W) = C(V) \otimes C(W) \qquad \text{(tensor product of superalgebras).}$$

Here the product is

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = \pm a_1 a_2 \otimes b_1 b_2 .$$

-1 if a_2 and b_1 are both odd.

If we decompose

$$\mathbb{R}^n = \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2 , \quad C(\mathbb{R}^n) = C(\mathbb{R}^2) \oplus \cdots \oplus C(\mathbb{R}^2).$$

What does $C(\mathbb{R}^2)$ look like?

$$c_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad c_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \qquad \text{satisfying } c_i^* = c_i$$

even odd

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad c_1 c_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

Notice that

$$\mathrm{Str}(1) = \mathrm{Str}(c_1) = \mathrm{Str}(c_2) = 0$$

$$\mathrm{Str}(c_1 c_2) = -2i .$$

We know $C(\mathbb{R}^2) \cong \mathrm{End}(\mathbb{C} \oplus \Pi\mathbb{C})$

so $C(\mathbb{R}^2)^{\otimes n/2} \cong \mathrm{End}((\mathbb{C} \oplus \Pi \mathbb{C})^{n/2})$.

This defines the spinors, $(n > 0)$

$$S = (\mathbb{C} \oplus \Pi\mathbb{C})^{\otimes n/2} \cong \underbrace{\mathbb{C}^{2^{n/2-1}}}_{S_+} \oplus \underbrace{\Pi\mathbb{C}^{2^{n/2-1}}}_{S_-}$$

$\dim C(V) = 2^n$

$\dim S = 2^{n/2}$

Clearly $\mathrm{Spin}(n) < C_+(\mathbb{R}^n)^\times \circlearrowleft S_\pm$

We can now define, using this action

$\mathrm{Str} : C(\mathbb{R}^n) \longrightarrow \mathbb{C}$ $\quad$ n even, uses orientation

$$= \mathrm{Tr}|_{S_+} - \mathrm{Tr}|_{S_-}$$

$$\underset{i_1 < \cdots < i_k}{\mathrm{Str}(c_{i_1} \cdots c_{i_k})} = \begin{cases} 0 & k < n \\ (2i)^{n/2} & k = n \end{cases} \qquad \text{i.e. } \mathrm{Str}(c_1 \cdots c_n) = (-2i)^{n/2}.$$

Let n be even.

$c_1 \cdots c_n \cdot c_{i_1} \cdots c_{i_k} = \, ? \, *(c_{i_1} \cdots c_{i_k})$

get a sign $(-1)^{kn + \binom{k}{2} + k}$. Similarly,

$(c_1 \cdots c_n)^2 = (-1)^{\binom{n}{2} + n}$.

If $4 | n$, $(c_1 \cdots c_n)^2 = 1$

$n \equiv 2 \, (4)$ $\quad = -1$.

(Generalized) Dirac operators.

generalized Laplacians...

Riemannian manifold X

Hermitian vector bundle E.

Δ is a generalized Laplacian

(i) Δ is formally self-adjoint

(i.e. on comp. supp. C^∞-sections)

(ii) in local coordinates,

$$\Delta = -\sum g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} \, \mathrm{id} + \text{first order}.$$

$$\Leftrightarrow [[\Delta, f], f] = -2|df|^2.$$

(notice metric determined by Δ)

A generalized Dirac operator D is a first order differential operator on E

(i) formally self-adjoint

(ii) D^2 is a generalized Laplacian.

i.e. $\underset{\text{correct?}}{4}[D, f]^2$

If we write $D = c^i \partial_i$ + zeroth order

$$D^2 = \tfrac{1}{2}(c^i c^j + c^j c^i) \partial_i \partial_j + \text{lower order}$$

c^i is a representation of the Clifford algebra, $C(T_x^* X)$ on E_x

$(c^i)^* = -c^i$. $\leftarrow$ self-adjoint.

Let us review de Rham cpx and Laplace–Beltrami.

How to define an inner product on $\Omega^k(M,E)$? Declare, at a point,

$$(e_{i_1}\wedge\cdots\wedge e_{i_k}\otimes s_1;\ e_{j_1}\wedge\cdots\wedge e_{j_k}\otimes s_2) = \delta_{i_1j_1}\cdots\delta_{i_kj_k}(s_1,s_2).$$

Now for $\alpha,\beta\in\Omega_c^*(X,E)$,

$$(\alpha,\beta)=\int_X(\alpha,\beta)_x\,|e_1\wedge\cdots\wedge e_n|.$$

We then introduce $d^*:\Omega_c^{k+1}\to\Omega_c^k$ defined by:

$$(\alpha,d^*\beta)=(d\alpha,\beta) \qquad \alpha\in\Omega_c^k,\ \beta\in\Omega_c^{k+1}$$

In particular, for $k=0$, RHS $=\beta(\text{grad}\,\alpha)$. It turns out

$$d^*=-*d*\cdot(-1)^{kn}.$$

Indeed,

$$(d\alpha,\beta)=\int d\alpha\wedge*\beta = (-1)^{k+1}\int\alpha\wedge d(*\beta)$$

since $d\alpha\wedge*\beta+(-1)^k\alpha\wedge d(*\beta)=d(\alpha\wedge*\beta)$ so used Stokes. Now,

$$=(-1)^{k+1}\int(\alpha,*^{-1}d*\beta)$$

Recall $*^2=(-1)^{k(n-k)}$, so

$$=(-1)^{k+1+k(n-k)}\int(\alpha,*d*\beta)$$

In coordinates

$$d=\sum_i\varepsilon^i\partial_i$$

$$\Rightarrow d^*=\sum_i(+\partial_i)^*(\varepsilon_i)^*$$

but $(\partial_i)^*=-\partial_i$ + zeroth order via integration by part, so

$$d^*=-\sum\varepsilon^{*i}\partial_i + \text{zero}^{th}\text{ order}.$$

Now let $D=d+d^*$. Then

$$\Delta=D^2=d^2+d^*d+dd^*+(d^*)^2=\underbrace{d^*d+dd^*}_{\text{Laplace–Beltrami}}.$$

<u>Exercise:</u> check D^2 is a generalized Laplacian, hence D is gen. Dirac.

(also notice that $d^*d+dd^*=[d,d^*]$).

(should get $g^{ij}\partial_i\partial_j+\cdots$ where $g^{ij}=(dx^i,dx^j)$).

Notice that D^2 commutes w/ d,d^*.

Notice

$[d+d^*, f] = \varepsilon(df) - \varepsilon^*(df) =: c(df)$.

We have a covariant derivative on T^*X (Levi-Civita)

- compatible w/ metric
- torsion-free.

Extend ∇^{T^*X} to $\nabla^{\Lambda^\bullet T^*X}$. We now obtain

$$\nabla : \Gamma(X, \Lambda^k T^*X) \longrightarrow \Gamma(X, T^*X \otimes \Lambda^k T^*X).$$

So if $\alpha \in \Gamma_c(X, \Lambda^k T^*X)$, $\beta \in \Omega^1_c(X, \Lambda^k T^*X)$, can take

$$(\nabla\alpha, \beta) = (\alpha, \nabla^*\beta).$$

Locally,

$$\nabla = \sum_i dx^i \otimes \nabla_{\partial/\partial x^i}$$

$$\nabla^* = -\sum (dx^i)^* \otimes \frac{\partial}{\partial x^i} + \text{zero}^{\text{th}} \text{ order}$$

whence $\nabla^*\nabla$ is generalized Laplacian — Bochner's Laplacian.

<u>Theorem</u> (Bochner). $\Delta - \nabla^*\nabla$ is a zeroth-order operator

Next time we'll do this via normal coords.