Cont → Disc convolutions:

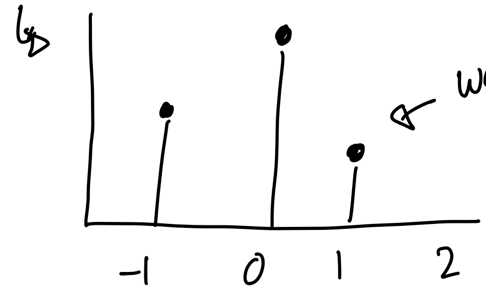
We have a defined a continuous convolution, let us derive a special case, the discrete convolution.

Let input I be a discrete signal given by.

$$I = \sum_{n \in \mathbb{Z}} I_n \delta_n$$

where $I_n \in \mathbb{R}$ and δ_n is a delta distribution with mass 1 at n

I (graphically)



weight at this point is I_n

Then,

$$O(a) = \int_{\mathbb{R}} I(a) K(t-a)\, da \quad \text{for some } K$$

$$= \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} I_n \delta_n(a) K(t-a)\, da$$

$$= \sum_{n \in \mathbb{Z}} I_n \int_{\mathbb{R}} \delta_n(a) K(t-a)\, da$$

some scalar we can pull out

when $n = a$, $\delta_n(a) = 1$

$$= \sum_{n \in \mathbb{Z}} I_n K(t-a)$$

Because $t \in \mathbb{R}$, $O(a) = \sum_{n \in \mathbb{Z}} I_n k(t-a)$ is a discrete to continuous convolution. When we discretize t ($t = m \in \mathbb{Z}$), we have

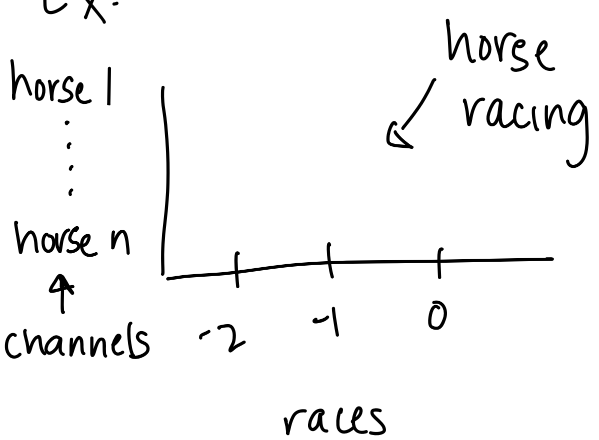
$$O_m = \sum_{n \in \mathbb{Z}} I_n K_{m-n}$$ ← discrete convolution in one dimension.

At this point, I, k, and O are sequences of #s.

Vector Valued Convolution:

When I and k have compact support, then they can be represented by vectors in $\mathbb{R}^n$. In many problems, the input signal is not just a real number at a given position + instead $I_n \in \mathbb{R}^{C_{in}}$ is a "multi-channel input".

Ex:



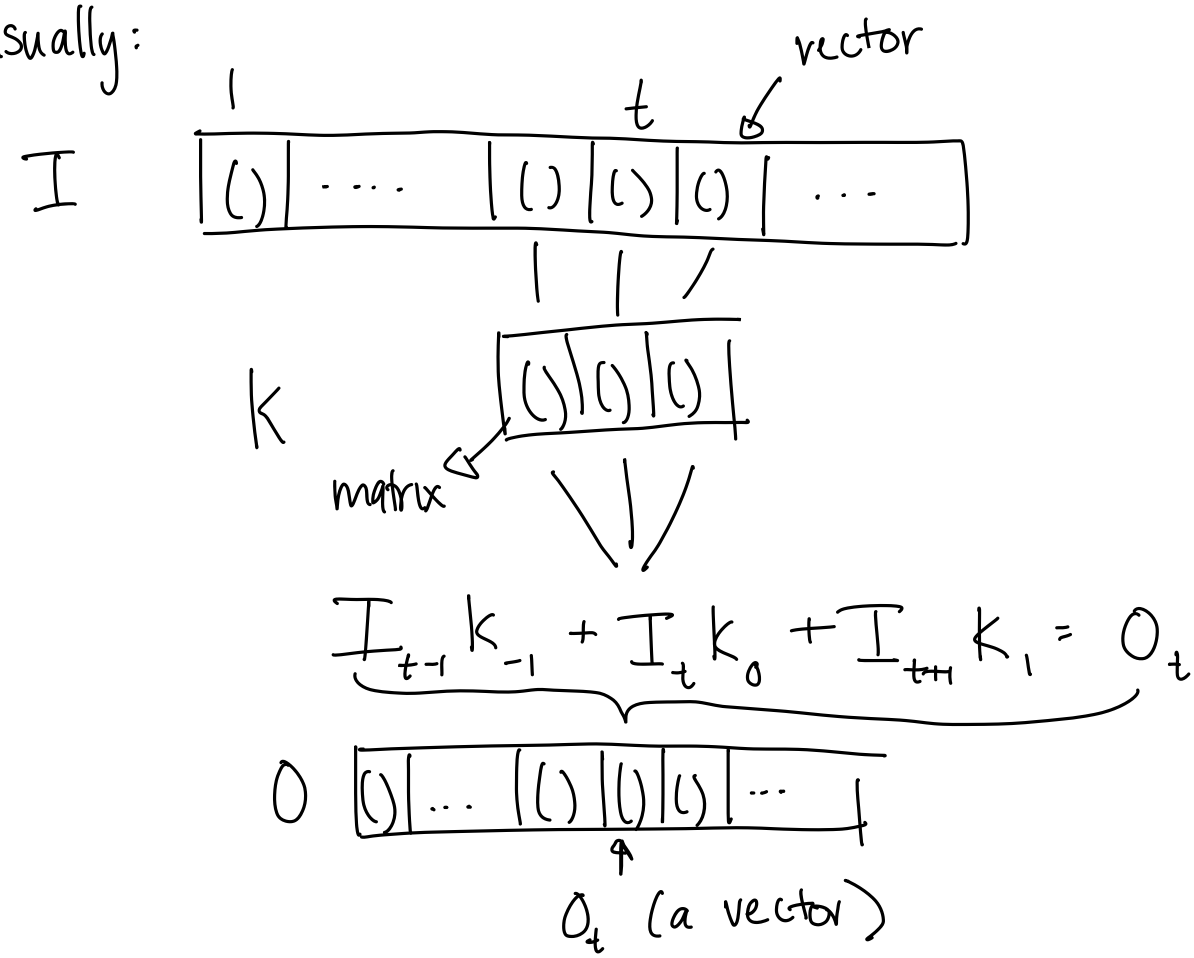
or EEGs

We can also have multiple output channels, ie $O_m \in \mathbb{R}^{C_{out}}$.

If we let K ($K \in \mathbb{R}^{C_{in} \times C_{out}}$) be a matrix, then we can generalize our original disc. conv. to multiple channels.

↳ note: when $O = I * K$, $K \in \mathbb{R}^{C_{in} \times C_{out}}$, when $O = K * I$, $K \in \mathbb{R}^{C_{out} \times C_{in}}$

Visually:



Convolutions over $\mathbb{R}^2$, $\mathbb{Z}^2$, $\mathbb{R}^n$, and $\mathbb{Z}^n$:

So far we've considered sequences (vectors), how do we generalize to spatial data?

Over $\mathbb{R}$:

$$O(t) = \int_{\mathbb{R}} I(a) K(t-a)\, da$$

Over $\mathbb{R}^2$:

$$O(t,s) = \int_{\mathbb{R}} \int_{\mathbb{R}} I(a,b)\, K(t-a, s-b)\, da\, db$$

Let $\vec{w} = (a,b) \in \mathbb{R}^2$, $\vec{v} = (t,s) \in \mathbb{R}^2$, then

$$O(\vec{v}) = \int_{w \in \mathbb{R}^2} I(\vec{w})\, K(\vec{v} - \vec{w})\, d\vec{w}$$

vector notation

Vector notation readily generalizes and over $\mathbb{R}^n$:

$$O(\vec{v}) = \int_{w \in \mathbb{R}^n} I(\vec{w})\, K(\vec{v} - \vec{w})\, d\vec{w}$$ where $\vec{w}, \vec{v} \in \mathbb{R}^n$.

Over $\mathbb{Z}^2$, the discrete conv. is given by

$$O(m,n) = (I * K)(m,n)$$
$$= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} I(i,j)\, K(m-i, n-j)$$

Convolutions are commutative:

Claim: $I * K = K * I$

pf:

$$(I * k)(\vec{v}) = \int_{\vec{w}=-\infty}^{\vec{w}=\infty} I(\vec{w})\, K(\vec{v}-\vec{w})\, d\vec{w}$$

We consider a change of base, let $\vec{u} = \vec{v} - \vec{w}$ then

$$\int_{\vec{w}=-\infty}^{\vec{w}=\infty} I(\vec{w})\, K(\vec{v}-\vec{w})\, d\vec{w} = \int_{\vec{u}=\infty}^{\vec{u}=-\infty} I(\vec{v}-\vec{u})\, k(\vec{u})(-1)^n\, d\vec{u}$$

$$= \cancel{(-1)^n} \int_{\vec{u}=-\infty}^{\vec{u}=\infty} I(\vec{v}-\vec{u})\, k(\vec{u})\cancel{(-1)^n}\, d\vec{u}$$

flipping limits
n times

$$= \int_{\vec{u}=-\infty}^{\vec{u}=\infty} k(\vec{u})\, I(\vec{v}-\vec{u})\, d\vec{u}$$

$$= (k * I)(\vec{v}). //$$

Convolution should work overy any space that has integration and subtraction (a locally compact group).

Flipping the kernel:

Let $\tilde{K}(\vec{u}) = k(-\vec{u})$. What does this look like?

$$K \qquad\qquad\qquad \tilde{K}$$

$$\begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 \end{array} \qquad\qquad \begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}$$

what is $\tilde{K}(1,1)$?

$K(-1,-1)$

$\tilde{K}$ is the same as flipping K through the origin

Most libraries are implemented using flipped kernels. How do we mathematically do that?

$$(K \star I)(\vec{v}) = \int_{\vec{u}=\infty}^{\vec{u}=-\infty} K(-\vec{u})\, I(\vec{v}-\vec{u})\, d\vec{u}$$

Let $\vec{w} = -\vec{u}$

"cross correlation" = conv. using the flipped kernel

$$= \int_{\vec{w}=-\infty}^{\vec{w}=\infty} \tilde{K}(\vec{w})\, I(\vec{v}+\vec{w})\, d\vec{w}$$

flipping the kernel flips this sign + ruins the commutativity

$$K \star I \neq I \star K$$