Lecture 7 cont' 2/13

Recall X G-transformable set. Want to learn $\varphi : X \to \mathbb{R}$ invariant $\varphi(x) = \varphi(gx)$

· Use Group-convolutional Neural Network

$$\underbrace{\cdot X \underset{\text{Equ.}}{\xrightarrow{\text{lift}}} \mathbb{R}[G]^{n_1} \underset{\substack{G\text{-conv}\\ \text{Equ.}}}{\longrightarrow} \mathbb{R}[G]^{n_2} \underset{\substack{\text{nonlinear}\\ \text{Equ.}}}{\xrightarrow{\sigma}} \mathbb{R}[G]^{n_e} \to \cdots \quad \cdots \xrightarrow{g\text{-conv}} \mathbb{R}[G]^{n_5} \xrightarrow{\sigma} \mathbb{R}[G]^{n} \to \cdots \to \mathbb{R}[G]^{n_2}}_{\text{Equi.}}$$

$$\underset{\text{inv proj.}}{\xrightarrow{P}} \mathbb{R}^{c}$$

<u>Last time</u>: $(f * k)(g) = \sum_{h \in G} f(h) k(h^{-1} g)$ is equivariant

$f : G \to \mathbb{R}^{c_{in}}$, $f \in \mathbb{R}[G]^{c_{in}}$

$k : G \to \mathbb{R}^{c_{out} \times c_{in}}$

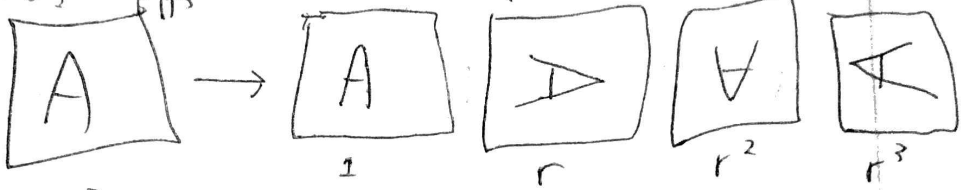
$(gf) * k = g(f * k)$

$f * k : G \to \mathbb{R}^{c_{out}}$

Lift

$I \longrightarrow f_I$ $\quad \mathbb{R}^{H\times W\times C} \quad \mathbb{R}^{H\times W\times C\times 4}$

Ex. Image



$X = \{\mathbb{Z}^2 \longrightarrow \mathbb{R}^3\} \longrightarrow \{C_4 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^3\}$

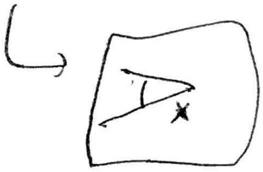
$G = p4 = C_4 \ltimes \mathbb{Z}^2 = \{(r^k, (i,j))\}$

C_4: 90° degree rotation

$\mathbb{Z}^2$: translations

$f_I(g) = g^{-1} I$

$f_I(r, (1,1)) = (0,0,0)$



Lifting is Equivariant

$X \to \mathbb{R}[G]$

$I \to f_I$, $h \in G$

Want $f_{hI} = h f_I$

Proof: $f_{hI}(g) = g^{-1} h I = (h^{-1} g)^{-1} I = f_I(h^{-1} g) = (h f_I)(g)$

$\Rightarrow f_{hI} = h f_I \quad \square$

Pooling

$$f \in \mathbb{R}[G] \longmapsto \rho f \in \mathbb{R},$$

$$\rho f = \frac{1}{|G|} \sum_{g \in G} f(g)$$

<u>Invariant</u>: $h \in G$, $\rho(hf) = \frac{1}{|G|} \sum_{g \in G} (hf)(g)$

$$\frac{1}{|G|} \sum_{g \in G} f(h^{-1} g), \qquad u = h^{-1} g$$

$$\frac{1}{|G|} \sum_{u \in G} f(u)$$ since $g \mapsto h^{-1}g$ bijection w/ inver $g \mapsto hg$

<u>Exercise</u>: $\rho_{max} = \max_{g \in G} f(g)$

Component-wise non-linearities are Equivariant

$\sigma : \mathbb{R} \to \mathbb{R}$ non-linear

$\check{\sigma} : \mathbb{R}[G] \to \mathbb{R}[G]$

$f \in \mathbb{R}[G]$

$g(\check{\sigma}(f)) = \check{\sigma}(gf)$

$$\begin{tikzcd}
\begin{pmatrix} c_1 \\ c_r \\ c_{r^2} \end{pmatrix} \arrow[r, mapsto] \arrow[d, "r"] & \begin{pmatrix} \sigma(c_1) \\ \sigma(c_r) \\ \sigma(c_{r^2}) \end{pmatrix} \arrow[d, "r"] \\
\begin{pmatrix} c_{r^2} \\ c_1 \\ c_r \end{pmatrix} \arrow[r, mapsto, "\check{\sigma}"] & \begin{pmatrix} \sigma(c_{r^2}) \\ \sigma(c_1) \\ \sigma(c_r) \end{pmatrix}
\end{tikzcd}$$

New Groups from Old Groups

Def^n Direct Product of G_1, G_2 groups

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$$

$$(g_1, g_2) \circ (h_1, h_2) = (g_1 \circ_1 h_1, g_2 \circ_2 h_2)$$

$\circ_i$ is composition op in G_i

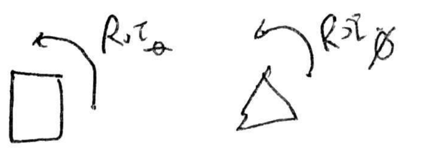
identity: $(1_1, 1_2)$

inverse $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$

"G_1 and G_2 don't interact"

Ex. $(\mathbb{R}, +) \times (\mathbb{R}, +) = (\mathbb{R}^2, +)$

Ex. $SO(2) \times SO(2) = \{(Rot_\theta, Rot_\phi\}$



G_1, G_2 finite, then $|G_1 \times G_2| = |G_1||G_2|$

$G_1 \subset G_1 \times G_2$

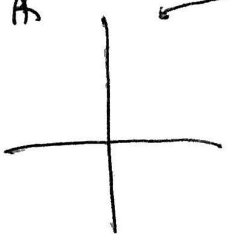
$G_2 \subset G_1 \times G_2$ subgroup

$G_1 \longrightarrow G_1 \times G_2$ As subgroups G_1, G_2

$g_1 \longmapsto (g, 1)$

$\mathbb{R}^2$ commute $(\mathbb{R}, +) \leq (\mathbb{R}^2, +)$

Ex.



$\{(x, 0), x \in \mathbb{R}$

Def^n: Quotient group

$H \trianglelefteq G$ normal subgroup

HW $\Rightarrow$ G/H is group

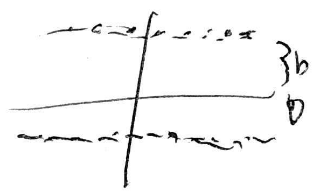
The set $G/H = \{gH \mid g \in G\}$

composition $g_1H \cdot g_2H = (g_1 g_2)H$

Ex. $(\mathbb{R}^2, +)/(\mathbb{R}, +) = \{(x,0) \mid x \in \mathbb{R}\} \cong (\mathbb{R}, +)$

$\parallel$

$\{(a,b)(\mathbb{R},+) = (a,b) + \mathbb{R} \times \{0\} = \{(a+x, b)\}, x \in \mathbb{R}$



cosets are horizontal lines

Ex: D_4

$H \trianglelefteq D_4$ $\quad H = \{1, f\}$ · Normal subgroup

$D_4/H = \{\{1,f\}, \{r, rf\}, \{r, rf\}, \{r^2, r^2f\}, \{r^3, r^3f\}\}$

$\cong C_4 = \{1, r, r^2, r^3\}$

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Defn 1) $f: G \to H$ is a homomorphism if
$$f(g_1 g_2) = f(g_1) \circ f(g_2)$$

2) $f: G \to H$ bijection and homomorphism, group isomorphism

3) $f: G \to G$ is a group isomorphism, then this is an automorphism

Defn N, H groups
$$\varphi: H \to \mathrm{Aut}(N) = \{f: N \to N \text{ automorphism}\}$$
$$h \mapsto \varphi_h$$

The semi-direct product of N, H

$H \ltimes N$: set $H \times N = \{(h,n)\}$

$$h\, n = \varphi(n)\, h$$

Ex. $\underset{\displaystyle N}{D_n \geq C_n} = \{1, r, \dots, r^n\}$ $\quad \underset{\displaystyle H}{D_1} = \{1, f\}$

$$\varphi: D_1 \to \mathrm{Aut}(C_n)$$
$$f \mapsto \varphi_f : C_n \to C_n$$
bijection ✓ $\quad r^i \mapsto r^{n-i}$

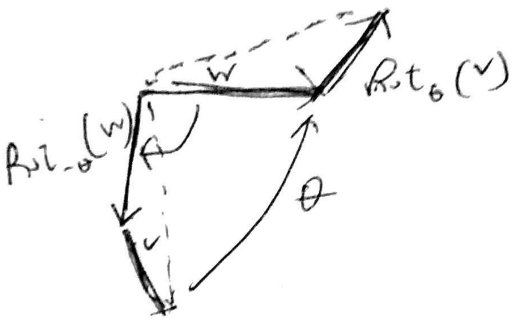
$$\varphi_f(r^i)\,\varphi_f(r^j) = r^{n-i} r^{n-j} = r^{2n-i-j} = r^n r^{n-i-j} = r^{n-i-j}$$
$$= \varphi_f(r^i r^j) \text{ homom}$$

Ex: $SE(2)$ contains $(\mathbb{R}^2, +) \rtimes (SO(2), \ )$

$\qquad\qquad\qquad\qquad \{T_v\} \qquad\qquad \{Rot_\theta\}$

$Rot_\theta \circ T_v \circ Rot_{-\theta}(v)$

$\Downarrow$



$$Rot_\theta T_v = T_{Rot_\theta(v)} Rot_\theta \quad \varphi_{Rot_\theta}(T_v) = T_{Rot_\theta(v)}$$