GDL 2/16 Lecture 8

Group Actions and representations

Def: X set, G group, G acts on X if

$$a: G \times X \to X$$
$$g, x \to gx \text{ or } g \cdot x \text{ or } g.x \text{ or } a(g,x)$$

Such that 1) $1 \cdot x = x$

2) $g_1 g_2 x = a(g_1, a(g_2, x))$

$$= a(g_1 \circ g_2, x)$$

Write $G \circlearrowright X$: "G acts on X", called a group action

Ex: $D_4 \circlearrowright \mathbb{R}^2$ $\quad D_4 = \langle r, f \mid f^2 = 1, r^4 = 1, rf = fr^3 \rangle$

$r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ → define r action

$f \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$ → define f action

$$r^2 \begin{pmatrix} x \\ y \end{pmatrix} = r\left(r\begin{pmatrix} x \\ y \end{pmatrix}\right) = r\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$r^3 \begin{pmatrix} x \\ y \end{pmatrix} = r\left(r^2\begin{pmatrix} x \\ y \end{pmatrix}\right) = r\begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

Make sure to check that all relations are satisfied

$r^4 = 1$ so $r^4 x = 1 \cdot x = x$

$$r^4 \begin{pmatrix} x \\ y \end{pmatrix} = r\left(r^3\begin{pmatrix} x \\ y \end{pmatrix}\right) = r\begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \checkmark$$

Ex 2: $D_4 \curvearrowright D_4$ Group acts on itself

what does $a: G \times G \to G$ mean in this context?

Needs to satisfy axioms of group action.

1) $1 \circ g = g$ and 2) $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

Turns out $\circ$ satisfies this:

$1 \circ g = a(1, g) = g$ from identity of group axiom.

$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ from Associativity of $\circ$

Ex 3: $D_4 \curvearrowright \{1\}$

Trivially $g \cdot 1 = g$ so group action axioms are satisfied.

Ex 4: $S_4 \curvearrowright \{1, 2, 3, 4\}$

$a(\sigma, k) = \sigma(k)$, $\sigma \in S_4$

Ex 5: $S_4 \curvearrowright S_4$

$a(g, h) = g \cdot h = g h g^{-1}$

Does this satisfy our group axioms?

Proof for Ex 5:

1) $a(1,h)=h \qquad 1\cdot h = 1\cdot h\cdot 1^{-1} = 1\cdot h\cdot 1 = h$ ✓

2) $g_1\cdot g_2\cdot h = (g_1\circ g_2)(\;) = g_1(g_2(h))$

$g_1(g_2(h)) = g_1(g_2\, h\, g_2^{-1}) = g_1\, g_2\, h\, g_2^{-1} g_1^{-1}$

$= (g_1\circ g_2)(h)(g_2^{-1}\circ g_1^{-1})$ $\qquad (g_1\circ g_2)^{-1}(g_1\circ g_2) = 1$

$= (g_1\circ g_2)(h)(g_1\circ g_2)^{-1}$ $\qquad \underbrace{g_2^{-1}\circ \underbrace{g_1^{-1}\circ g_1}_{x}\circ g_2}_{x} = 1$

so $(g_1\circ g_2)^{-1} = (g_2^{-1}\circ g_1^{-1})$
by uniqueness

Ex 6: $\underset{\{n\times n \text{ inv. matrices}\}}{GL_n} \curvearrowright \underset{\{n\times n \text{ matrices}\}}{M_n}$

$gM \mapsto gMg^{-1}$

Ex 7: $G \curvearrowright \mathbb{R}[G] = \{f: G\to \mathbb{R}\}$

$g\cdot f = f\circ g^{-1} \qquad h, g\in G,\ f\in \mathbb{R}[G]$

$(g\cdot f)(h) = f(g^{-1}h)$ $\boxed{\text{Hw 3 check this is well-defined}}$

Ex 8: $GL_n \curvearrowright \mathbb{R}^n$

Def: If $G \curvearrowright X$ then the orbit of $x \in X$ is

$$O_x = \text{Orbit}_g(x) = Gx$$
$$= \{g \cdot x \mid g \in G\} \subseteq X$$

Def: The stabilizer of x

$$\text{Stab}_G(x) = G_x = \{g \in G \mid gx = x\} \subseteq G$$

subscript vs. Gx above

Prove $\text{Stab}(x)$ is a subgroup for Hw 3

Thm: If G is finite then

$$|G| = |\text{stab}(x)||\text{Orbit}(x)|$$

Intuitively, a group element must either

1) $gx = x$ so $g \in \text{stab}(x)$

2) $gx = y$ so $y \in \text{orbit}(x)$

In class exercise: Find orbits and stabilizers

1) $SO(2) \curvearrowright \mathbb{R}^2$ $\quad a = \mathrm{rot}_\theta \begin{pmatrix} x \\ y \end{pmatrix}$

2) $SO(3) \curvearrowright S^2 = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \}$

3) $GL_2(\mathbb{C}) \curvearrowright M_2(\mathbb{C})$, $g \cdot M \mapsto g M g^{-1}$

4) $Aff_2 = (\mathbb{R}^2, +) \rtimes GL_2(\mathbb{R}) \curvearrowright \mathbb{R}^2$

$(v, g) \cdot w = gw + v$

5) $S_3 \curvearrowright S_3$ by conjugation → $\boxed{\text{Optional HW}}$

$\sigma \tau = \sigma \tau \sigma^{-1}$

Opt. HW: Prove stabilizers of pairs in the same orbit are conjugate.

Def: A representation of G is a linear group action on a vector space V.

$$v \mapsto gv \text{ is linear } \forall g \in G$$

Since g acts linearly, it is represented by a matrix $\rho(g)$.

From group action axioms:

Identity of V

1) $\rho(1)v = v \quad \forall v \in V$ so $\rho(1) = I_V$

2) $g^{-1}gv = 1v = v$ so $\rho(g^{-1})\rho(g) = I_V$

and $\rho(g)\rho(g^{-1}) = I_V$

so $\rho(g)$ is an invertible matrix and $\rho(g^{-1}) = \rho(g)^{-1}$

Think of ρ:

invertible matrices of V

$$\rho : G \to GL(V)$$

$$\rho(1) = I_V, \quad \rho(g_1 g_2) = \rho(g_1)\rho(g_2)$$

This means ρ is a homomorphism.

Linear Group Action G on V $\leftrightarrow$ $\rho : G \to GL(V)$ homomorphism

these are identical

Ex) $C_4 = \{1, r, r^2, r^3 \mid r^4 = 1\}$

$C_4 \curvearrowright \mathbb{R}^4$ Linear so

$\rho: C_4 \to GL_4(\mathbb{R})$

$$r \to \rho(r) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

need to show $\rho(r)^4 = I_{4\times4}$

Ex 2: $\rho_2: C_4 \to GL(\mathbb{R}^2)$

$$r \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2(r)^4 = I_2$$

Ex3: $\rho_1: C_4 \to GL_1(\mathbb{R})$

$$r \to (x), \quad x \neq 0$$

$\boxed{\text{Hw: what are 1 dim reps of } C_n, D_n\text{?}}$

$(x)^4 = 1$ so two options

$x = 1 \to$ trivial rep.

$x = -1 \to$ sign rep. or parity rep.