For a linear map $\vec{y} = M\vec{x}$, what matrices are equivariant?

A: $M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$, ie: $PMx = MPx$, P = permutation

Problem: this is only for permutations, what about other kinds of symmetries?

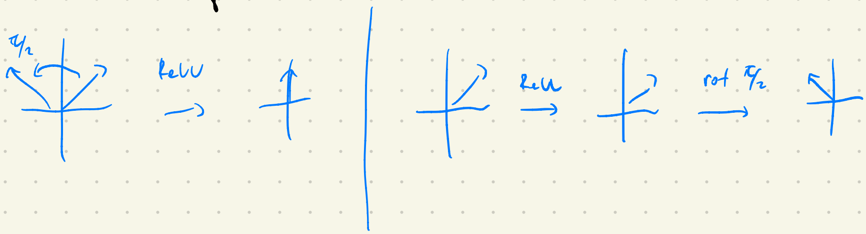
Rotation on different spaces: $R^2 \to R^4$

$$\underbrace{\begin{pmatrix} a & -b \\ b & a \\ -a & b \\ -b & a \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{R_2} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{R_4} \underbrace{\begin{pmatrix} a & -b \\ b & a \\ -a & b \\ -b & a \end{pmatrix}}_{M}$$

commuting past different actions

Note: non-linear operators should also be equivariant

ReLU is NOT equivariant over rotations:



- Symmetries reduce our search space exponentially:
  a $128 \to 128$ CNN has 128^2 weights. CNN has size of much smaller kernel

Intro to Group Theory

Defn A <u>group</u> is:

- A set G
- Binary op $\circ : G \times G \to G$ $\circ$ is closed over G

that obeys:

1. $(a \circ b) \circ c = a \circ (b \circ c)$ associativity
2. $\exists i \in G : \forall g \in G,\ i \circ g = g \circ i = i$ identity
3. $\forall g \in G,\ \exists g^{-1} \in G : g \circ g^{-1} = g^{-1} \circ g = i$ inverses

Defn An <u>Abelian</u> group is a group where

$\forall g, h \in G : g \circ h = h \circ g$

Examples of groups:

1. Cyclic group of order 3:

$G = \{1, r, r^2\},\ r^3 = 1$

$a \circ b$	1	r	r^2
1	1	r	r^2
r	r	r^2	1
r^2	r^2	1	r

→ also abelian

symmetric

2. More generally: cyclic groups of any order:

$C_n = \{1, r^1, r^2, \dots r^{n-1}\}$ $\|C_n\| = n$

$r^i \circ r^j = r^{i+j \bmod n}$

3. Permutation group: $S_n = \Sigma_n = \mathrm{Perm}_n$

$[n] = \{1, 2, \dots n-1\}$

S_n = all bijection from $[n] \to [n]$

$(\sigma \circ \tau)(x) = \sigma(\tau(x))$

$\|S_n\| = n!$

Example: $S_3 = \{ i, (1\,2), (2\,3), (3\,1), (1\,2\,3), (1\,3\,2) \}$

Notation: $\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{matrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\,2\,3)$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\,2)(3)$ ← (3) can be left out

Not abelian: $(1\,2)(2\,3) = (1\,2\,3)$
$(2\,3)(1\,2) = (1\,3\,2)$

composition is left-to-right

4. Orthogonal group $O(n)$

$O(2) = \{$ all 2×2 orthogonal matrices $\}$

$\circ$ = matrix mul.

Defn A subgroup of H of G $(H \leq G)$ are subsets of G closed under $\circ$

$H \subseteq G : \forall h_1, h_2 \in H : h_1 \circ h_2 \in H$

5. Dihedral group Dih_n:

Group of rotations and reflections on an n-gon

Ex: $Dih_3 = \{1, r^2, r^3, s^1, s^2, s^3\}$

r = rotate triangle by $\frac{1}{3}\tau$

s = reflection by axis (f in class)

$$rs\begin{pmatrix}1\\2\\3\end{pmatrix} = r\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix}$$

$$sr\begin{pmatrix}1\\2\\3\end{pmatrix} = s\begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

→ isomorphic to permutation

There are 2 generators : $\langle r, s \mid r^3 = 1,\ s^2 = 1,\ sr = r^2 s \rangle$

- You can obtain all elements in the group by composing one or more of the generators
- All other identities can be reduced to the 3 properties shown above ← presentation

More generally : for any Dih_n, $\langle r, s \mid r^n = 1,\ s^2 = 1,\ sr^i = r^{n-i} s \rangle$