CS 7180: GEOMETRIC DEEP LEARNING, HOMEWORK 2

Problem 1. Prove the identity element of a group is unique.

Problem 2. Prove that the inverse of each element in a group is unique.

Definition 3. Let H be a subgroup of G. We say H is normal if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

The key part of the above definition is that H is closed by conjugation of elements of the bigger group G. Closure under conjugation by elements in H itself is true of all subgroups.

Problem 4. Write out the multiplication table of the dihedral group D_4 the set of symmetries of a square. What are the subgroups of D_4? Which are normal? Which are Abelian? Draw the lattice of subgroups (https://en.wikipedia.org/wiki/Lattice_of_subgroups).

Problem 5. Let $[n] = \{1, 2, \ldots, n\}$. Define $\Sigma_n = \{f : [n] \to [n] : f \text{ a bijection}\}$. Prove Σ_n is a group under function composition.

Problem 6. Prove that for an Abelian group all subgroup are normal.

Let G be a group and H a subgroup. A coset of the subgroup G is a subset of the form $gH = \{gh : h \in H\}$. Let $G/H = \{gH : g \in G\}$ be the set of cosets. Note that $|G/H|$ is usually smaller than $|G|$ since $g_1H = g_2H$ whenever $g_2^{-1}g_1 \in H$. That is, the choice of representative may not be unique since $g_1H = g_2H$ for $g_1 \neq g_2$.

Problem 7. Prove that G/H is a group with composition law $g_1H \circ g_2H = g_1g_2H$ if H is normal. (This hinges on the mapping being well-defined. That is, $g_1H \circ g_2H$ should be independent of the choice of g_1, g_2.) Give an example of G and H not normal in G showing the composition law is not well-defined.

Let G be a finite group. Define the group algebra $\mathbb{R}[G]$ to be the space of function defined on the group $\mathbb{R}[G] = \{f : G \to \mathbb{R}\}$. We may also denote it $\mathbb{R}^G$ which is a fitting notation since an element $f \in \mathbb{R}^G$ may be represented by a tuple with one real number for each group element. In particular $\dim_{\mathbb{R}}(\mathbb{R}^G) = |G|$.

The group G acts on $\mathbb{R}[G]$ by $(g_1.f)(g_2) = f(g_1^{-1}g_2)$.

Problem 8. Consider the space of $\mathbb{R}$-linear maps from $\mathbb{R}[D_3] \to \mathbb{R}[D_3]$. Since $|D_3| = 6$, this is the space of 6×6-matrices. Find the subspace of matrices which commute with the group action, i.e. $M : \mathbb{R}[D_3] \to \mathbb{R}[D_3]$ linear such that $g.Mf = Mg.f$ for all $g \in G$.

We proved in class that group convolution

$$(f * k)(g) = \sum_{h \in G} f(h)k(h^{-1}g)$$

is equivariant. To get more practice with this proof, try proving that group cross-correlation is equivariant.

Optional Problem 9. Let $f, k \in \mathbb{R}[G]$. Let $g \in G$. Group cross-correlation is defined

$$(f \star k)(g) = \sum_{h \in G} f(h)k(g^{-1}h)$$

Prove $f \mapsto f \star k$ is G-equivariant.

Another good exercise is showing that all equivariant linear maps $\mathbb{R}[G] \to \mathbb{R}[G]$ can be represented as group convolutions. Recall that the group action on $f \in \mathbb{R}[G]$ and $a \in G$ is defines $af \in \mathbb{R}[G]$ where $(af)(g) = f(a^{-1}g)$.

Optional Problem 10. Let $F \colon \mathbb{R}[G] \to \mathbb{R}[G]$ be linear and G-equivariant. Then there exists $k \in \mathbb{R}[g]$ such that $F(f) = f * k$ for all $f \in \mathbb{R}[g]$.

(Hint: Since F is linear, we can write it in terms of the basis $\{\delta_g\}$ giving a matrix $(F_{gh})_{g,h}$. Then $F(f)(g) = \sum_{h \in G} f(h)F_{gh}$.)