CS 7180 – Lecture on convolutions, continued.

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These class notes are a supplement to the slides.

1 Convolutions

1.1 Shifting images exercise

Image $\tilde{I} : \mathbb{Z}^2 \to \mathbb{R}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Operator $(T_{(1,1)}I)(i,j) = I(i-1, j-1)$.

Compute $(T_{(1,1)}I)$.

Answer:

We subtract the shift on the indices because we are using indices of the output, so we are looking backwards.

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1.2 Proof that the above operation is shift equivariant

<u>Claim:</u> $(T_{k,l}I) * K = T_{k,l}(I * K)$.

Proof:

$$
\begin{aligned}
\left[(T_{k,l}I) * K\right](m,n) &= \sum_{i,j\in\mathbb{Z}^2} \left[T_{k,l}I\right](i,j)K(m-i,n-j) && (1)\\
&= \sum_{i,j\in\mathbb{Z}^2} I(i-k,j-l)K(m-i,n-j) && (2)\\
&(x = i-k, y = j-l)\\
&= \sum_{x,y\in\mathbb{Z}^2} I(x,y)K(m-(x+k), j-(y+l)) && (3)\\
&= \sum_{x,y\in\mathbb{Z}^2} I(x,y)K\left((m-k)-x,(n-l)-y\right) && (4)\\
&(\tilde{m} = m-k, \tilde{n} = n-k)\\
&= (I*K)(\tilde{m},\tilde{n}) && (5)\\
&= (I*K)(m-k,n-l) && (6)\\
&= \left[T_{k,l}(I*K)\right](m,n). && (7)
\end{aligned}
$$

1.3 Convolution is linear (optional exercise)

You can put the proof of this into the homework if you'd like:)

Claim: $(a_1 I_1 + a_2 I_2) * K = a_1(I_1 * K) + a_2(I_2 * K)$.

Hint: $(I*K)\overrightarrow{v} = \int_{\overrightarrow{w}} I(\overrightarrow{w})K(\overrightarrow{v} - \overrightarrow{w})d\overrightarrow{w}$.