The Matrix of a Convolution (1D)

In[∘]:=
```
H = 4;
```

In[∘]:=
```
(* Matrix in terms of a basis *)
```

In[∘]:=
```
Em[m_] := Table[If[m == i, 1, 0], {i, 1, H}]
```

In[∘]:=
```
Em[1] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In[∘]:=
```
K = {0, a, b, c, 0}
Ker[a_] := K[[a - 3]]
```

Out[∘]=
```
{0, a, b, c, 0}
```

In[∘]:=
```
Safe[i_, m_] := (i ≥ 1 && i ≤ H) && (m - i ≥ -1 && m - i ≤ 1)
```

In[∘]:=
```
Conv[I_] := Table[
  Sum[If[Safe[i, m], I[[i]] × Ker[m - i], 0], {i, -H, H}],
  {m, 1, H}]
```

In[∘]:=
```
Conv[Em[2]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix}$$

```
Mat = Table[Conv[Em[i]], {i, 1, H}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} b & c & 0 & 0 \\ a & b & c & 0 \\ 0 & a & b & c \\ 0 & 0 & a & b \end{pmatrix}$$

The Matrix of a shift-equivariant, Local Operator

In[∘]:=
```
M = Table[m[i, j], {i, 1, H}, {j, 1, H}];
```

In[∘]:=
```
M // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} m[1,1] & m[1,2] & m[1,3] & m[1,4] \\ m[2,1] & m[2,2] & m[2,3] & m[2,4] \\ m[3,1] & m[3,2] & m[3,3] & m[3,4] \\ m[4,1] & m[4,2] & m[4,3] & m[4,4] \end{pmatrix}$$

In[∘]:=
```
shift[M_] := Append[M[[2 ;;]], 0]
```

```
In[◦]:= shift[Em[2]]

Out[◦]= {1, 0, 0, 0}

In[◦]:= Flatten[Table[(shift[M.Em[i]] - M.shift[Em[i]])[[1 ;; H - 1]], {i, 2, H}]]

Out[◦]= {-m[1, 1] + m[2, 2], -m[2, 1] + m[3, 2], -m[3, 1] + m[4, 2],
  -m[1, 2] + m[2, 3], -m[2, 2] + m[3, 3], -m[3, 2] + m[4, 3],
  -m[1, 3] + m[2, 4], -m[2, 3] + m[3, 4], -m[3, 3] + m[4, 4]}

In[◦]:= sys = # == 0 & /@ Flatten[Table[(shift[M.Em[i]] - M.shift[Em[i]])[[1 ;; H - 1]], {i, 2, H}]]

Out[◦]= {-m[1, 1] + m[2, 2] == 0, -m[2, 1] + m[3, 2] == 0, -m[3, 1] + m[4, 2] == 0,
  -m[1, 2] + m[2, 3] == 0, -m[2, 2] + m[3, 3] == 0, -m[3, 2] + m[4, 3] == 0,
  -m[1, 3] + m[2, 4] == 0, -m[2, 3] + m[3, 4] == 0, -m[3, 3] + m[4, 4] == 0}

In[◦]:= vars = Flatten[M]

Out[◦]= {m[1, 1], m[1, 2], m[1, 3], m[1, 4], m[2, 1], m[2, 2], m[2, 3], m[2, 4],
  m[3, 1], m[3, 2], m[3, 3], m[3, 4], m[4, 1], m[4, 2], m[4, 3], m[4, 4]}

In[◦]:= M /. Solve[sys, vars][[1]]

  Solve: Equations may not give solutions for all "solve" variables.

Out[◦]= {{m[3, 3], m[2, 3], m[1, 3], m[1, 4]}, {m[3, 2], m[3, 3], m[2, 3], m[1, 3]},
  {m[3, 1], m[3, 2], m[3, 3], m[2, 3]}, {m[4, 1], m[3, 1], m[3, 2], m[3, 3]}}

In[◦]:= soln = M /. Solve[sys, vars][[1]];
        soln // MatrixForm

  Solve: Equations may not give solutions for all "solve" variables.

Out[◦]//MatrixForm=
  ( m[3, 3]  m[2, 3]  m[1, 3]  m[1, 4] )
  ( m[3, 2]  m[3, 3]  m[2, 3]  m[1, 3] )
  ( m[3, 1]  m[3, 2]  m[3, 3]  m[2, 3] )
  ( m[4, 1]  m[3, 1]  m[3, 2]  m[3, 3] )

        (* Enforce Locality with N=1*)

In[◦]:= Locality = Flatten[Table[If[Abs[i - j] > 1, m[i, j] → 0, 0 → 0], {i, 1, H}, {j, 1, H}]]

Out[◦]= {0 → 0, 0 → 0, m[1, 3] → 0, m[1, 4] → 0, 0 → 0, 0 → 0, 0 → 0, m[2, 4] → 0,
  m[3, 1] → 0, 0 → 0, 0 → 0, 0 → 0, m[4, 1] → 0, m[4, 2] → 0, 0 → 0, 0 → 0}

In[◦]:= soln /. Locality // MatrixForm

Out[◦]//MatrixForm=
  ( m[3, 3]  m[2, 3]     0        0     )
  ( m[3, 2]  m[3, 3]  m[2, 3]     0     )
  (    0     m[3, 2]  m[3, 3]  m[2, 3]  )
  (    0        0     m[3, 2]  m[3, 3]  )
```