CS 7180 – Regular representations, subrepresentations, irreducible representations.

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Notation: $\cong$ – isomorphism.

Definition: A representation of a group G is a homomorphism $\rho : G \to GL(V)$ where V a vector space. Data: (V, ρ).

Example: Regular representation of C_4:

$$\rho : C_4 \to GL_4(\mathbb{R}), \tag{1}$$
$$r \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2}$$

Example: Regular representation for a general group G:

$$G \circlearrowleft \mathbb{R}[G] = \{f : G \to \mathbb{R}\}, \tag{3}$$
$$(gf)(h) = f(g^{-1}h), \tag{4}$$
$$\rho_{reg} : G \to GL_{|G|}(\mathbb{R}). \tag{5}$$

Example: Irreducible representation of C_4:

$$\rho_1 : C_4 \to GL_2(\mathbb{R}), \tag{6}$$
$$r \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

Definition: If $\rho : G \to GL(V)$ is a representation then a subrepresentation is a subspace $W \subseteq V$ such that $\rho(g)W \subseteq W, \forall g \in G$. $\rho_W : G \to GL(W)$.

A representation (V, ρ) is irreducible if the only subrepresentations of $W \subseteq V$ are $W = 0$ and $W = V$.

If

$$\rho_1 : G \to GL(V_1), \tag{8}$$
$$\rho_2 : G \to GL(V_2) \tag{9}$$
$$\tag{10}$$

can be combined to give direct product

$$\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2), \tag{11}$$
$$g \to \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{12}$$

Exercise: Show $V_1 \times \{0\}$ is a subrep of $V_1 \oplus V_2 = \begin{pmatrix} \rho_1(g)V_1 \\ \rho_2(g)V_2 \end{pmatrix}$.

Goal: Given a representation ρ find *irreps* (irreducible representations) $\rho_1, ..., \rho_K$ such that $\rho \cong \rho_1 \oplus ... \oplus \rho_K$.

... Robin's mathematica example ...

Theorem (Weyl): If G is a compact or a finite group and ρ is a finite dimensional rep of G then $\rho \cong \oplus \rho_i$, ρ_i are irreps.

We could have infinitely many irreps.

Theorem (Peter-Weyl): If G is compact then

$$L^2(G) = \{f : G \to \mathbb{R} | \int_G |f|^2 d\rho(g) < \inf\} \tag{13}$$

is an infinitely dimensional vector space.

$$(gf)(h) = f(g^{-1}h) \tag{14}$$

is an infinitely dimensional representation of G.

$$L^2(G) \cong \hat{\oplus}_i V_i^{m_i} \tag{15}$$
$$\forall f \in L^2(G) \lim f_i = f, f_i \in \oplus_i V_i^{m_i} \tag{16}$$

Irreps of $SO(2)$:

Trivial: $(\rho_0, V_0 = \mathbb{R})$, $\rho_0(Rot(\theta)) = 1$.

Standard: $(\rho_1, V_1 = \mathbb{R}^2), \rho_1(Rot(\theta)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -sin\theta & \cos\theta \end{pmatrix}$

$(\rho_K, V_K = \mathbb{R}^2)$, $\rho_K(Rot(\theta)) = \begin{pmatrix} \cos K\theta & \sin K\theta \\ -\sin K\theta & \cos K\theta \end{pmatrix}, \forall K \in \mathbb{N}$.

ρ is a rep of $SO(2)$ is well-defined if ...

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$$V_0 \subseteq L^2(SO(2)) \tag{17}$$
$$V_0 = \{f : SO(2) \to \mathbb{R} \; s.t. \; f(g) = c, c \in \mathbb{R}\} \cong \mathbb{R}. \tag{18}$$

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$$V_1 \subseteq L^2(SO(2)) \tag{19}$$
$$Rot(s)(a\cos\theta + b\sin\theta) = ... = a'\cos\theta + b'\sin\theta \tag{20}$$
$$a\cos(\theta - s) + b\sin(\theta - s) \tag{21}$$
$$= a(\cos\theta\cos(-s) - \sin\theta\sin(-s)) + b(\cos\theta\sin(-s) + \sin\theta\cos(-s)) \tag{22}$$
$$= (a\cos(-s) + b\sin(-s))\cos\theta + (-a\sin(-s) + b\cos(-s)) \tag{23}$$
$$= a'\cos\theta + b' \tag{24}$$
$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \cos(-s) & \sin(-s) \\ -\sin(-s) & \cos(-s) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} = \rho_1(Rot(s)) \tag{25}$$

$$V_K = Span(\{\cos K\theta, \sin K\theta\}) \tag{26}$$
$$L^2(SO(2)) = \mathbb{R}_1 \hat{\oplus} \mathbb{R}\cos\theta \hat{\oplus} .. \hat{\oplus} \mathbb{R}\cos]\theta \tag{27}$$