CS 7180: GEOMETRIC DEEP LEARNING, HOMEWORK 3

Problem 1. Consider the standard representation of D_4 on $\mathbb{R}^2$ where

$$\rho(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho(f) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show this representation is well-defined. (That is, show the relations of D_4 are satisfied.)

Problem 2. Let G be a finite group. Then $\mathbb{R}[G]$ is a representation of G where $(g \cdot f)(h) = f(g^{-1}h)$ for $g, h \in G$ and $f \in \mathbb{R}[G]$. Show this is a well-defined representation, i.e. that it is a linear group action.

Problem 3. In class we defined a G-action on G by left-multiplication $g \cdot h = gh$. Show that right-multiplication does not necessarily satisfy the axioms of a group action $a(g, h) = hg$. However, $a'(g, h) = hg^{-1}$ does give a well-defined group action.

Problem 4. Let G act on a set X. Prove $\mathrm{Stab}(x)$ is a subgroup of G.

Problem 5. Let G act on a set X. If x, y are in the same orbit, show that their stabilizers are conjugate subgroups, i.e. there exists $g \in G$ such that $g\mathrm{Stab}(x)g^{-1} = \mathrm{Stab}(y)$.

Problem 6. What are the one-dimensional representations of C_n? What are the one-dimensional representations of D_n?

We say a representation (ρ, V) has a *subrepresentation* W if $W \subset V$ is a subspace and $\rho(g)w \in W$ for all $w \in W$ and $g \in G$. A representation V is called *irreducible* if its only subrepresentations are V itself and $\{0\}$.

Problem 7. Consider the 4-dimensional representation of C_4 given by $\mathbb{R}[C_4]$. Find two 1-dimensional subrepresentations W_1, W_2 and one 2-dimensional subrepresentation W_3 which intersect trivially (i.e. $W_i \cap W_j = \{0\}$) and together span $\mathbb{R}[C_4]$). Observe they are irreducible. Choose bases of W_1, W_2 and W_3 which collectively form a basis of V. Then write $\rho(r)$ with respect to this basis and note that it is block diagonal with blocks of size 1,1,2.

Such a situation is called a *direct sum decompositon* of the representation $V = W_1 \oplus W_2 \oplus W_3$.

Optional Problems

Optional Problem 8. Consider the action of S_3 on S_3 by conjugation $g \cdot h = ghg^{-1}$. Draw a graph illustrating the group action. Draw 6 nodes of the graph labeled by the elements of S_3. Then draw arrows between $g_1 \to g_2$ labeled by h if $h \cdot g_1 = g_2$. This graph makes it easy to see the orbits as connected components and the stabilizers as self-loops.

Optional Problem 9. Let G act on X such that there is only one orbit. (In this case the action is called *transitive* and the set X is called a *homogeneous space*.) Choose and fix $x_0 \in X$. Show there is a bijection $G/\mathrm{Stab}(x_0) \to X$ given by $g\mathrm{Stab}(x_0) \mapsto gx_0$. In particular, show the map is defined (i.e. if $g_1\mathrm{Stab}(x_0) = g_2\mathrm{Stab}(x_0)$ then $g_1x_0 = g_2x_0$.)