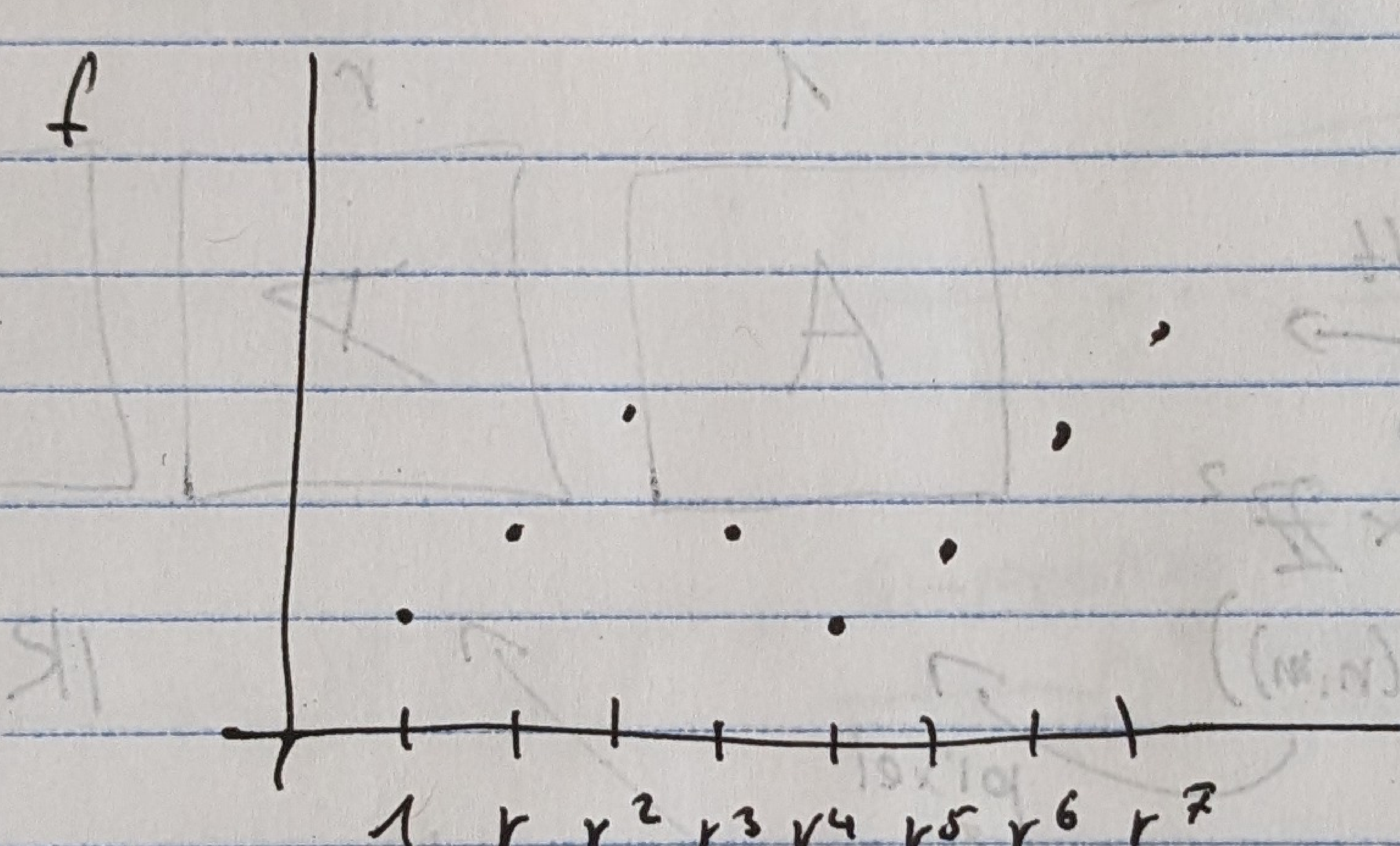
Lecture 7: Graph-valued Signals and Group Convolutional Neural Networks

Data over a group. Let G be a finite group.

$$f : G \mapsto \mathbb{R}$$

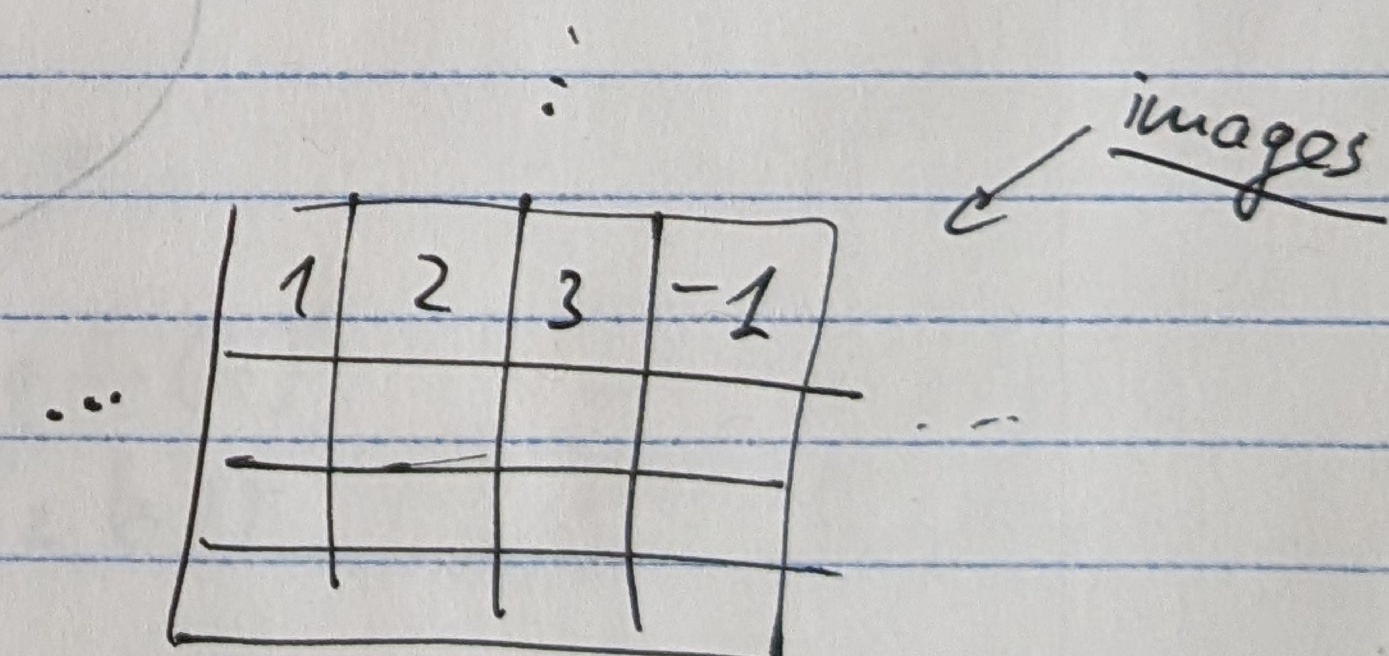
Ex.: $G = C_8 = \{1, r, r^2, \dots, r^7\}$



Ex.: $G = (\mathbb{Z}^2, +)$

$$\begin{array}{c} (\mathbb{Z}^2, +) \\ \shortparallel \\ (n,m) \end{array}$$

$f(n,m) \in \mathbb{R}$



Definition: $\mathbb{R}[G] = \{f : G \mapsto \mathbb{R}\}$ is a group algebra.

$\left(\mathbb{R}[G] = \mathbb{R}^G = \{(c_1, c_r, \dots, c_{r_7})\} - \text{example} \right)$

$\mathbb{R}[G]$ is a vector-space.

$$\{\delta_g : g \in G\}$$

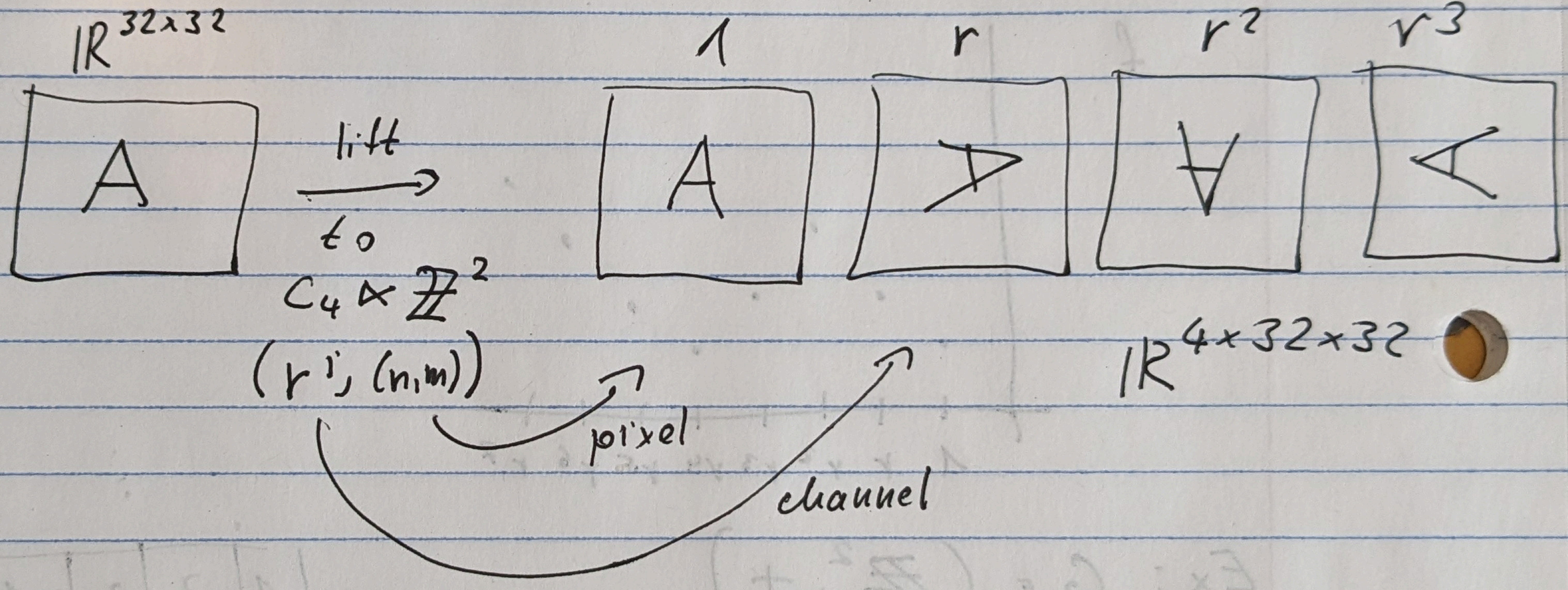
Lifting signal which lives in a G-tractable space,
for example images $f: \mathbb{Z}^2 \mapsto \mathbb{R}$.
But we want to consider this as a signal over G.

Ex.: $G = C_4 \ltimes \mathbb{Z}^2 = \langle T_{(n,m)}, Rot(\frac{\pi}{2}) \rangle$

lift of x to $\mathbb{R}[G]$ is

$$f_x(g) = g^{-1}x \quad , \quad f_x : G \mapsto \mathbb{R}$$

Ex.:



Idea of a Group-convolutional network

Invariant task $x \mapsto c$. Model with invariant network.
$gx \mapsto c$

signals

$$\overbrace{X \xrightarrow{\text{lift}} \mathbb{R}[G]^{n_1} \xrightarrow{\text{Equiv. Group. Conv.}} \mathbb{R}[G]^{n_1} \xrightarrow{\text{Equiv. non-linearity}} \mathbb{R}[G]^{n_1}}^{\times \ell}$$

$$X \mapsto f_x \mapsto h_1 \mapsto a_1$$

$\cdots$

$$\mathbb{R}[G]^{n_\ell} \xrightarrow{G\text{-pooling}} \mathbb{R}^{n_{\ell+1}}$$

$$a_\ell \longrightarrow y$$

Proof: i) $f_1(f_2(gx)) = f_1(g f_2(x))$ — f_2 equiv.
$= g f_1(f_2(x))$ — f_1 equiv.
$\Rightarrow f_1 \circ f_2$ equiv.

ii) $f_1(f_2(gx)) = f_1(g f_2(x)) = f_1(f_2(x))$

$\Rightarrow f_1 \circ f_2$ invariant. $\square$

So NNs (with projection) are invariant
Group convolutional?

Definition: Group convolution

Original conv.: $f, k : \mathbb{R} \mapsto \mathbb{R}$

$$(f * k)(x) = \int_{a \in \mathbb{R}} f(a)\, k(x-a)$$

Conv2d: $f, k : \mathbb{Z}^2 \mapsto \mathbb{R}$

$$(f * k)(n,m) = \sum_{i,j \in \mathbb{Z}^2} f(i,j)\, k(n-i, m-j)$$

Group convolution: $f, k : G \mapsto \mathbb{R}$, $f * k : G \mapsto \mathbb{R}$

$$(f * k)(g) = \sum_{h \in G} f(h)\, k(h^{-1}g)$$

Ex: $G = (\mathbb{R}, +)$ $\quad \int_{h \in \mathbb{R}} f(h)\, k(g-h)$

Ex: $G = (\mathbb{Z}^2, +)$ $\quad \sum_{\underbrace{i,j}_{h} \in \mathbb{Z}^2} f(i,j)\, \overbrace{k(n-i, m-j)}^{h^{-1}g \,=\, -(i,j) + (n,m)}$ $\quad (n,m) = g$

Property: $(f * k)$ is equivariant.

Proof: $a, g, h \in G$.

WTS $(af) * k = a(f * k)$

WTS $((af) * k)g = (a(f * k))(g) \quad \forall g \in G$

$$\| $$

$$\sum_{h \in G} (af)(h)\, k(h^{-1}g)$$

$$\| $$

$$\sum_{g \in G} f(a^{-1}h)\, k(h^{-1}g)$$

$$(f*k)(a^{-1}g) = \sum_{n \in G} f(n)\,k(n^{-1}a^{-1}g) =$$

$n = a^{-1}h$

bijection

$$= \sum_{h \in G} f(a^{-1}h)\,k((a^{-1}h)^{-1}a^{-1}g) =$$

$$= \sum_{h \in G} f(a^{-1}h)\,k(h^{-1}\cancel{a a^{-1}}g)$$

$$= \sum_{h \in G} f(a^{-1}h)\,k(h^{-1}g)$$

<u>Property:</u> $F: \mathbb{R}[G] \longmapsto \mathbb{R}[G]$ linear,

G-equivariant then F is a G-conv,
i.e. $\exists k: G \to \mathbb{R}$ s.t.
$f \in \mathbb{R}[G]$: $F(f) = f*k$

<u>Proof:</u> Optional exercise.