GDL - Lecture 10 continued

Tensor products, Clebsch-Gordon coefficients, Sherical Harmonics

Tensor products: Recall, G group w/ (ρ_V, V)

and (ρ_W, W) reps.

Tensor product ~~$(\rho_V \oplus \rho_W)$, $V \otimes$~~

$$(\rho_V \otimes \rho_W, V \otimes W)$$

$$V \otimes W = \left\{ \sum_i c_i v_i \otimes w_i \mid v_i \in V, w_i \in W, c_i \in \mathbb{R} \right\}$$

Rep $\rho_V \otimes \rho_W$ defined

$$(\rho_V \otimes \rho_W)(g)\left(\sum c_i v_i \otimes w_i\right) = \sum_i c_i (\rho_V(g) v_i) \otimes (\rho_W(g) w_i)$$

$$\overbrace{\qquad\qquad}^{G \circlearrowright W}$$

$$G \circlearrowright V \left\{ \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \right.$$

(we act simultaneously on the rows and the columns)

Example: $G = SO(2)$ $(\rho_1 \mathbb{R}^2)$

$\rho_1 \otimes \rho_1$, $\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{R}^{2\times 2} = \mathbb{R}^4$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\rho_1 \otimes \rho_1 (Rot\,\theta)} A(Rot\,\theta) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rho_1 (Rot\,\theta)^T$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

— Rest in Mathematica.

Irreps of $SO(2)$:

$$\begin{array}{ccccc} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_k \\ | & | & | & & | \\ 1 & 2 & 2 & & 2 \end{array}$$ – Dimensionality.

$\rho_1(g) \circ Id\, \rho_1(g^T) = c\,\rho_1(g)\,\rho_1(g)^T$

$= c\,\rho_1(g)\,\rho_1(g)^{-1} = c\,Id$

$V_1 \subseteq \rho_1 \otimes \rho_1$

$\parallel$

$\{c\,Id \mid c \in \mathbb{R}\} \cong \rho_0$

$$(\mathrm{Rot}\,\theta)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(\mathrm{Rot}\,\bar{\theta}) =$$

$$= (\mathrm{Rot}\,\theta)(\mathrm{Rot}\,\tfrac{\pi}{2})(\mathrm{Rot}\,-\theta) =$$

$$= \mathrm{Rot}(\theta + \tfrac{\pi}{2} - \theta) = \mathrm{Rot}(\tfrac{\pi}{2}) = \begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}$$

$$V_2 \subseteq \rho_1 \otimes \rho_1$$

$$\|$$

$$\left\{ c\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\} \cong \rho_0$$

$$\begin{pmatrix}c & 0\\ 0 & c\end{pmatrix} + \begin{pmatrix}0 & -s\\ s & 0\end{pmatrix} = \begin{pmatrix}c & -s\\ s & c\end{pmatrix}$$

Going to Mathematica.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \longmapsto \begin{pmatrix} a+d+(a-d)\cos(2\theta) - (b+c)\sin 2\theta & b-c+(a-d)\sin(2\theta) + (b+c)\cos 2\theta \\ -b+c+(a-d)\sin(2\theta) + (b+c)\cos 2\theta & a+d+(-a+d)\cos 2\theta + (b+c)\sin 2\theta \end{pmatrix}$$

$$V_3 = \left\{ \alpha \underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{M_3'} + \beta \underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{M_4} \right\}$$

$$\mathrm{Rot}(\theta)\left(\alpha M_3 + \beta M_4\right)\mathrm{Rot}(\theta)^T =$$

$$= \left(\alpha\cos 2\theta - \beta\sin 2\theta\right)M_3 + \left(\beta\cos 2\theta + \alpha\sin 2\theta\right)M_4$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} = \begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$= \rho_2(\mathrm{Rot}\,\theta)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$V_3 \cong \rho_2, \qquad \rho_1 \otimes \rho_1 \cong \rho_0 \otimes \rho_0 \otimes \rho_2$$

$$\cong \rho_0^2 \otimes \rho_2$$

This is the Clebsch–Gordon decomposition.

<u>Prop.</u> $\rho_k \otimes \rho_\ell \cong \bigotimes \rho_i \qquad |k-\ell| \leq i \leq k+\ell$

Clebsch – Gordan coefficients

- multiplicities (# of times an irrep occurs in a decomposition): ρ_0 had mult. 2 in $\rho_1 \otimes \rho_1$

$C^{0}_{1,1} = 2$

- $\rho_1 \times \rho_1 \longrightarrow \rho_1 \otimes \rho_1 \cong \rho_0 \otimes \rho_0 \otimes \rho_2 \longrightarrow$

$v \quad w \qquad vw^T = cM_1 + sM_2 + \alpha M_3' + \beta M_4$

$\longrightarrow \rho_2$

$\alpha M_3 + \beta M_4$

Clebsch – Gordan Projection

$C^{2}_{1,1}$ 2×4 matrix

$C^{2}_{1,1} : \rho_1 \otimes \rho_1 \longmapsto \rho_2$

$v_1 \quad w \longmapsto \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leftarrow C^{2,1}_{1,1,1,1}$

($C^{1}_{1/2,\,1/2}$ — You'd write this for spin groups.)

3D Rotations Group = SO(3)

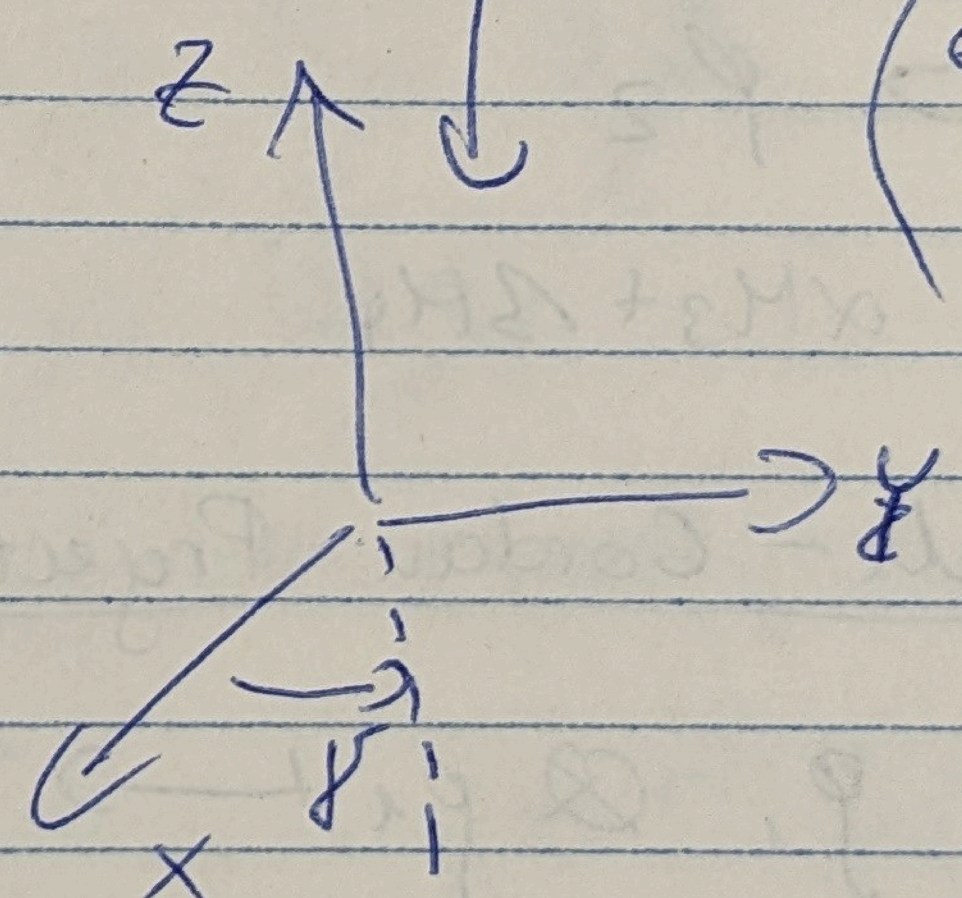
$$SO(3) = \{ g \in GL_3(\mathbb{R}) \mid \underbrace{g^T g = I}_{\text{orthogonal}}, \underbrace{\det(g) = 1}_{\text{orientation preserving}} \}$$

$$= \{ \text{group of rotations in 3D} \}$$

$$= \{ R_z(\alpha) R_x(\beta) R_z(\gamma) \}$$

(extrinsic zxz Euler angles)

There are many options.



$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\underbrace{\{ Rot_{\vec{u}}(\theta) \}}$ axis $\vec{u} \in \mathbb{R}$, $\|\vec{u}\| = 1$

θ angle

$\hookrightarrow$ any sequence of rotations reduces to this

Representations of SO(3)

SO(3) is a compact group $\Longrightarrow$ Peter-Weyl applies

$(\rho, V) \cong \bigoplus_i (\rho_i, V_i)$ irreps

Example: $\rho_0(g) = 1$, $\mathbb{R}$

Example: $SO(3) \circlearrowleft \mathbb{R}^3$

standard rep $\rho_1(g)$ — 3x3 rotation matrix

$\qquad \qquad \qquad \qquad \qquad ||$

D^1_{ij} - matrix coefficient $\qquad D^1(g)$

$\qquad ||$ $\qquad \qquad \qquad \hookrightarrow$ (Darschtellung)

entry $\qquad \qquad \qquad$ Wigner D-matrix

$D^1_{ij} : SO(3) \mapsto \mathbb{R}$

Other Irreps $\quad \rho_\ell, \quad \ell \in \mathbb{N}$

$\dim 2\ell+1$

$$\begin{array}{lccccc} & \rho_0 & \rho_1 & \rho_2 & \rho_3 & \cdots \\ & | & | & | & | & \\ \dim & 1 & 3 & 5 & 7 & \cdots \end{array}$$