# Introduction

*Overview of the course.*

## 0.1 What is homological algebra?

Homological algebra is a place to get answers. All kinds of subtle and interesting answers! Homological algebra is a collection of tools and techniques which are used in any field with algebra in its name: Algebra, algebraic topology, algebraic geometry, algebraic number theory, etc.

With homological algebra, we can reduce difficult questions about complex objects to basic linear algebra problems. Albeit an infinite sequence of problems, but basic problems nonetheless! One might compare this with the way an analytic function can be understood entirely by its Taylor coefficients: Derivatives are easy, and you can understand something subtle (an arbitrary function) by doing an infinite sequence of easy computations.

In practice, this means that the sheer magnitude of objects in homological algebra can be overwhelming to the novice. In this course we'll give an organized introduction and overview of the main ideas. We'll work through them in some of the classicand most usefulapplications, and we'll introduce enough special topics to pique the interest of students from a variety of backgrounds.

## 0.2 Outline

Our text will be Weibel's An Introduction to Homological Algebra, and most of the course will follow this text. We'll cover the basic concepts of homological algebra with most of our attention focused on central applications. At relevant points in the course, we'll foray into related topics which are of interest to the students.

Where there are expositional choices to be made, we tend toward topological and categorical descriptions as a conceptual framework for the techniques of homological algebra. Background in these areas may be helpful, but is not required. The only true prerequisite for the course is familiarity with abelian groups and quotients thereof. More general familiarity with modules over commutative rings, homomorphisms, and tensor products will also be useful.

A potential syllabus is given below, although the pacing and selection of additional topics will be revised depending on the audience.

## 0.3 The concepts

- Chain complexes and homology

- Derived functors and derived categories
- Spectral sequences

## 0.4 The applications

- Homology and cohomology of spaces and of finite groups
- Ext and Tor
- The Serre spectral sequence and spectral sequences arising from exact couples

## 0.5 The additional topics

- Homological dimension
- Lie algebra co/homology
- Hochschild co/homology
- Sheaf cohomology
- Model categories and derived categories

## 0.6 Course status

This Ximera course is based on a course taught to graduate students at the University of Georgia in Spring 2012. Each "activity" here represents one day's lecture, and the exercises are those assigned in class.

At present, not all lectures have been transcribed to this Ximera course, and the transcription progress is not linear. Below is the list of activities, together with their completion status.

(a) (not done) Chain complexes

(b) (not done) LES in homology

(c) (not done) Chain homotopy

(d) (not done) Mapping Cones and Cylinders

(e) (not done) Resolutions

(f) (not done) Derived functors

(g) (not done) Exact functors

(h) (not done) Categories, functors, and natural transformations

(i) (not done) Adjunctions

(j) (not done) Adjunctions and the Yoneda lemma

(k) (not done) Adjunctions and exactness

(l) (not done) Balancing Tor and Ext

(m) (not done) Universal coefficient theorem

(n) (not done) Homological dimension

(o) (not done) Local rings, Koszul complexes

(p) (not done) Gorenstein rings

(q) (not done) Group cohomology

(r) (not done) Local cohomology

**Exercise 1** *What is homological algebra useful for? Choose the best answer.*

(a) *Getting concrete answers to subtle questions*

(b) *Understanding higher-order structure*

(c) *Proving theorems*

(d) *All of these are good answers* ✓