Topic 05:

Linear Filters & Fourier Analysis

Today

- Filter-based view of image formation
- Linear systems & 1D convolution
- Example 1D filters
- Filtering in 2D
- The Fourier series
- Sampling & aliasing
- Discrete-time filters & the DFT

Covered in detail in the lecture notes

2 ways to think about images

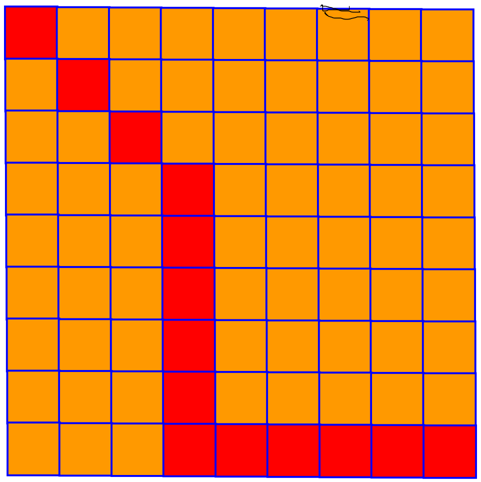
infinitesimal

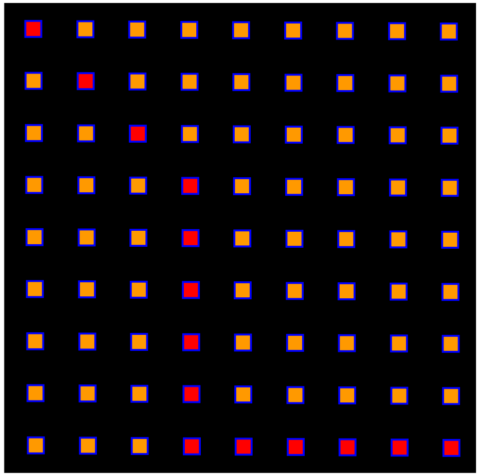
pixel-is-a-square

pixel-is-a-point-sample

$D(m,n)$

discrete array of pixels



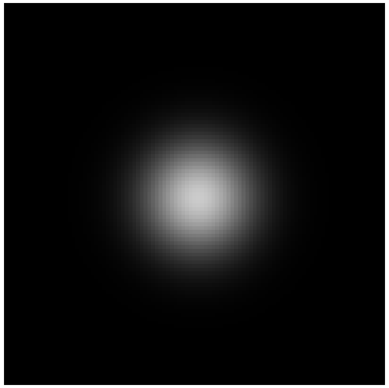


D(x,y)

defined over finite domain $[-\pi,\pi]\times[-\pi,\pi]$

pixel-is-a-point-sample
(alternative visualization)

continuous image



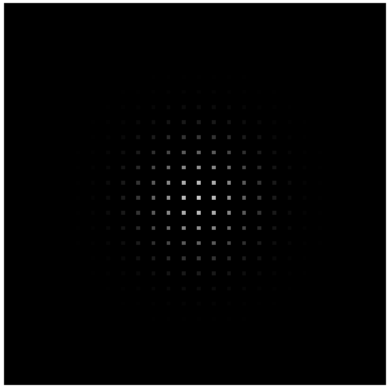


image after sampling

Image formation from a filtering perspective

$I(x,y)$

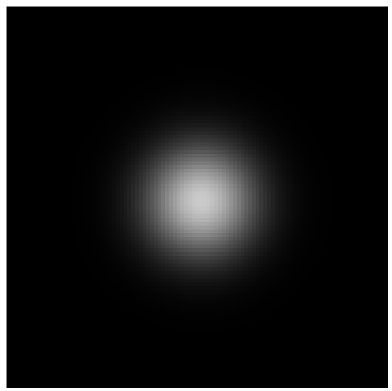
Ideal continuous image

$I(x,y) \quad (x,y) \in [-\pi,\pi] \times [-\pi,\pi]$

Image formation from a filtering perspective

$I(x, y)$

Impulse response
$f(x, y)$



Blur due to lens & finite pixel footprint

$$B(x,y) = (I * f)(x,y)$$

Ideal continuous image

$$I(x,y) \quad (x,y) \in [-\pi,\pi] \times [-\pi,\pi]$$

Image formation from a filtering perspective

$B(x,y)$

Blur due to lens & finite pixel footprint

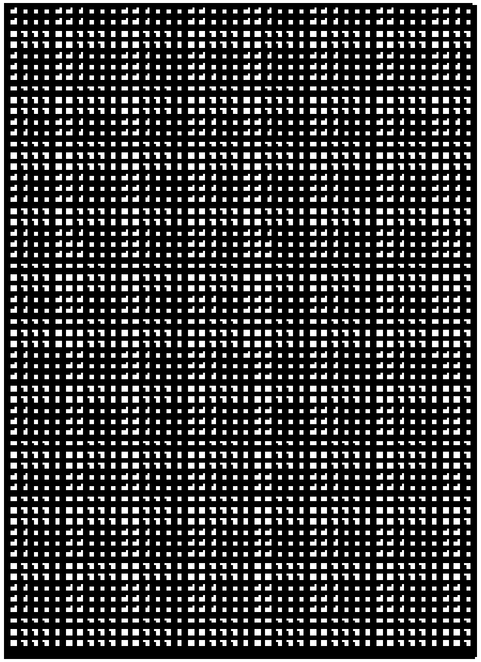
$B(x,y) = (I * f)(x,y)$

$I(x,y)$ $(x,y) \in [-\pi,\pi] \times [-\pi,\pi]$

Image formation from a filtering perspective

$B(x,y)$

Impulse train



Sampled image

$$D(x,y) = B(x,y) \cdot S(x,y)$$

$S(x,y) = 1$ iff x, y integer multiples of pixel pitch $\Delta x, \Delta y$ & zero otherwise

Blur due to lens & finite pixel footprint

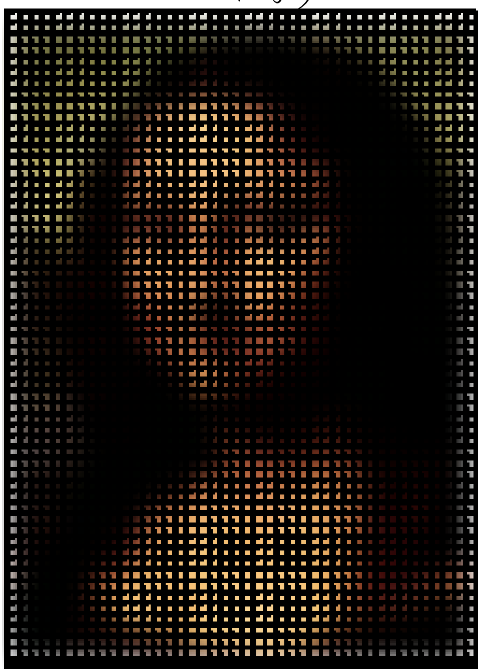
$$B(x,y) = (I * f)(x,y)$$

Ideal sensor irradiance

$$I(x,y) \quad (x,y) \in [-\pi,\pi] \times [-\pi,\pi]$$

Image formation from a filtering perspective

$D(x,y)$



Sampled image

sampling & aliasing

$$D(x,y) = B(x,y) \cdot S(x,y)$$

$S(x,y) = 1$ iff x, y integer multiples of pixel pitch $\Delta x, \Delta y$ & zero otherwise

filtering & convolution

Blur due to lens & finite pixel footprint

$$B(x,y) = (I * f)(x,y)$$

Ideal sensor irradiance

$$I(x,y) \quad (x,y) \in [-\pi, \pi] \times [-\pi, \pi]$$

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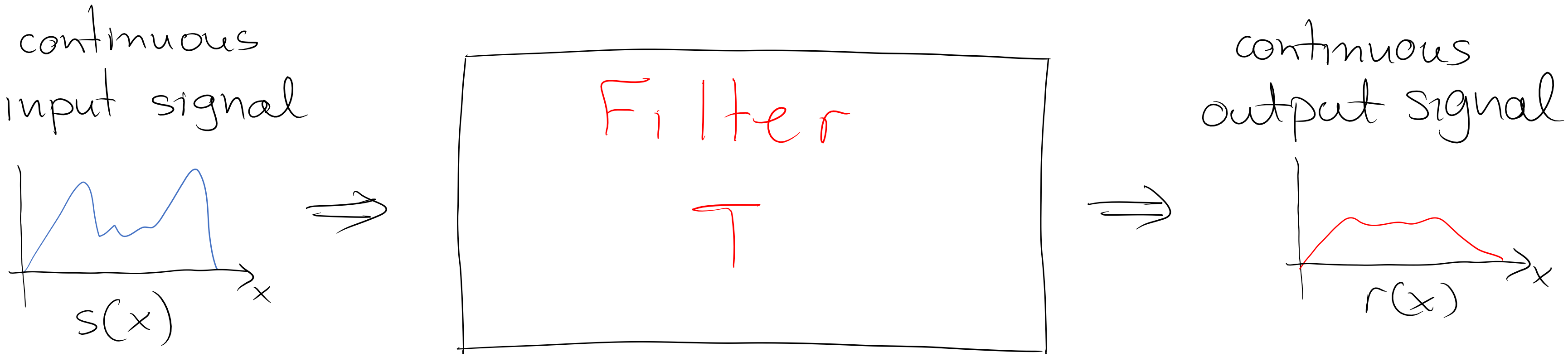
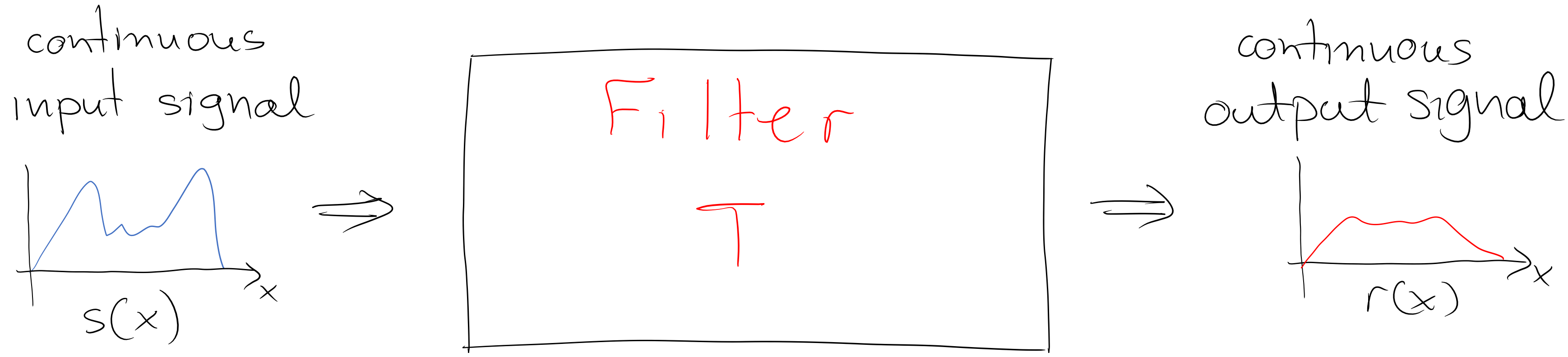
Linear filters



- A filter transforms one signal into another
- Filters are used to describe image formation (lens blur, etc) as well as to implement operations on images (edge detection, denoising, etc)

Linear filters



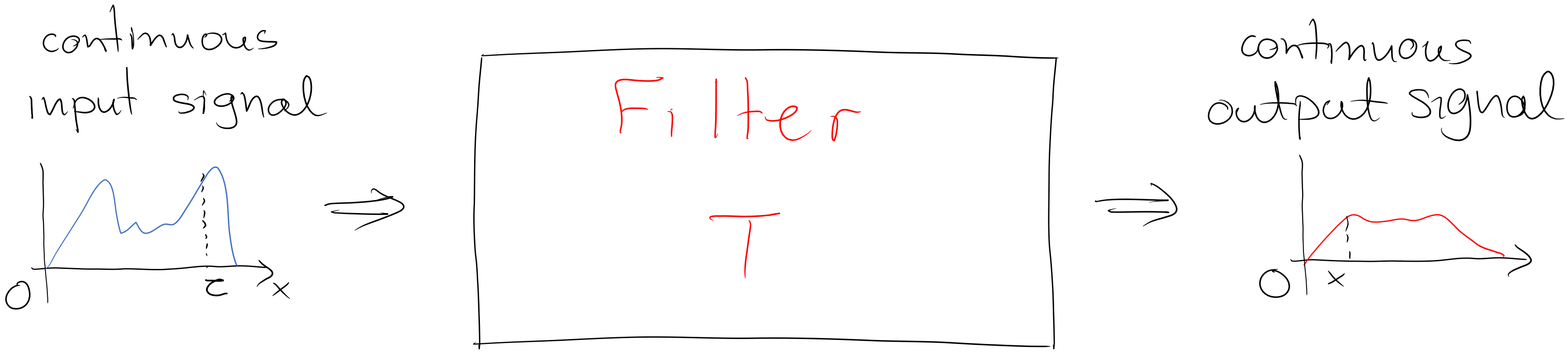
Definition

A transformation T is linear iff it satisfies

$$\boxed{T\left[a_1 s_1(x) + a_2 s_2(x)\right] = a_1 T\left[s_1(x)\right] + a_2 T\left[s_2(x)\right]}$$

for any a_1, a_2 and continuous functions $s_1(x), s_2(x)$

The superposition integral



* Every linear transformation can be expressed as an integral of the form

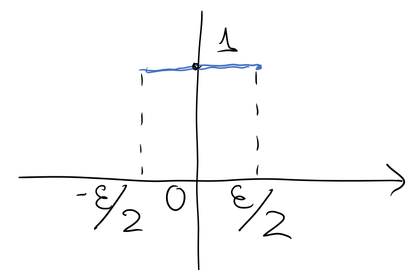
$$r(x) = \int_{-\infty}^{\infty} f(x, \tau)\, s(\tau)\, d\tau$$

describes the contribution of the input signal at τ to the output signal at x

Applying a filter to a delta function

Dirac delta function:

* $$\delta(x) = \lim_{\epsilon \to 0} \frac{box_\epsilon(x)}{\epsilon}$$



$$box_\epsilon(x) = \begin{cases} 1 & |x| < \frac{\epsilon}{2} \\ 0 & |x| > \frac{\epsilon}{2} \end{cases}$$

Properties of $\delta(x)$:

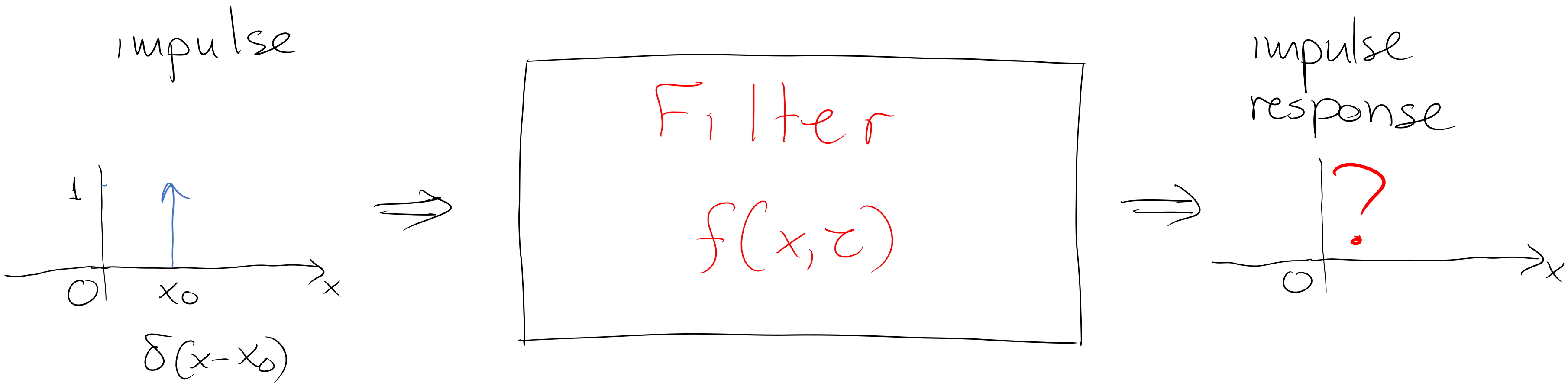
- $\delta(x) = 0 \quad \forall x \neq 0$
- $\int f(x)\,\delta(x)\,dx = f(0)$

Applying a filter to a delta function

Applying the superposition integral to $\delta(x)$:

$$r(x) = \int_{-\infty}^{\infty} f(x, \tau)\, \delta(\tau)\, d\tau = f(x, 0)$$

Applying a filter to a delta function



Applying the superposition integral to $\delta(x)$:

$$r(x) = \int_{-\infty}^{\infty} f(x,\tau)\,\delta(\tau - x_0)\,d\tau = f(x, x_0)$$

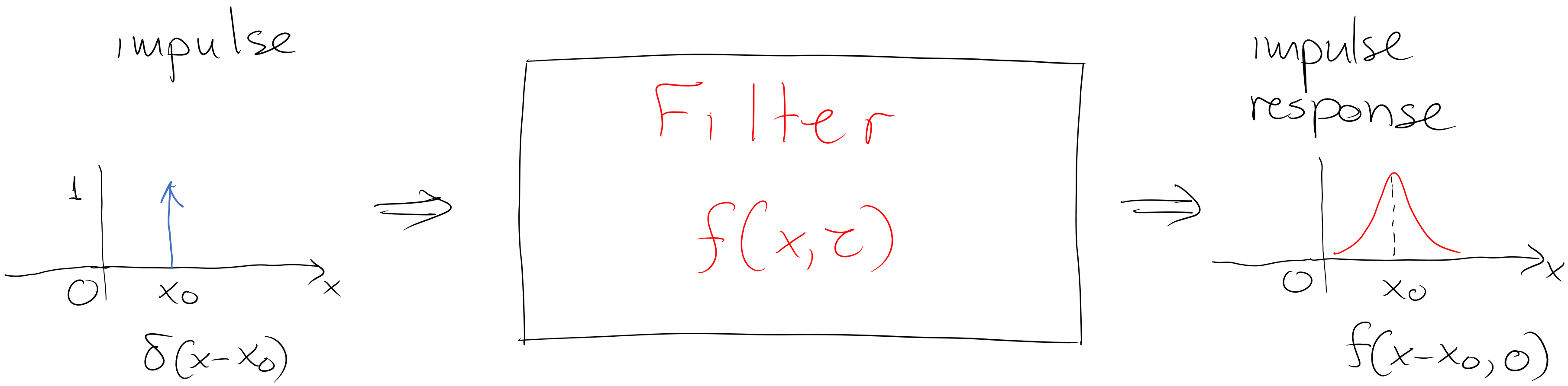
Shift-invariant filtering

A transformation is shift invariant iff shifted impulses produce identical but shifted responses:

$$r(x) = \int_{-\infty}^{\infty} f(x,\tau)\,\delta(\tau - x_0)\,d\tau = f(x, x_0)$$

$$\boxed{= f(x - x_0, 0)}$$

Shift-invariant filtering



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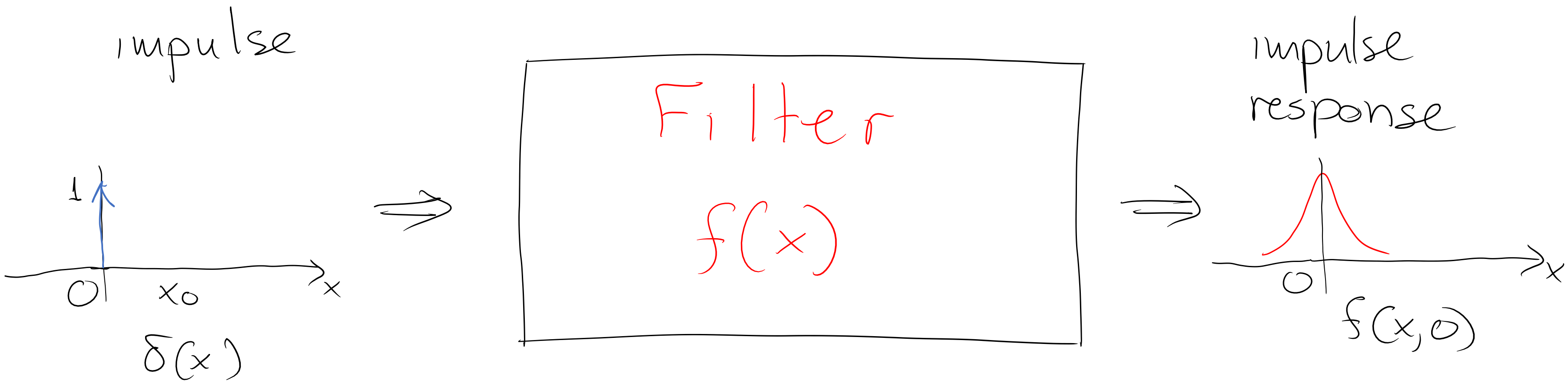
Impulse response of a filter



In this case, the transformation is completely described by $f(x) \equiv f(x,0)$ which is called the impulse response

$$r(x) = \int_{-\infty}^{\infty} f(x,\tau)\,\delta(\tau - x_0)\,d\tau = f(x, x_0)$$

$$\boxed{= f(x - x_0, 0)}$$

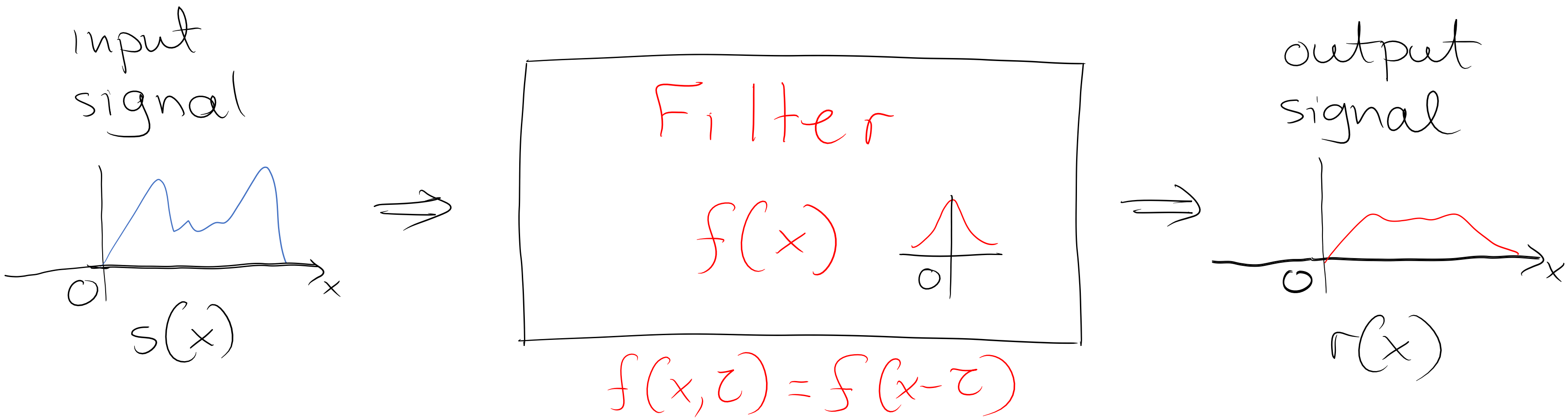
Impulse response of a filter

In this case, the transformation is completely described by $f(x) \equiv f(x,0)$ which is called the impulse response

$$r(x) = \int_{-\infty}^{\infty} f(x,\tau)\,\delta(\tau - x_0)\,d\tau = f(x,x_0)$$

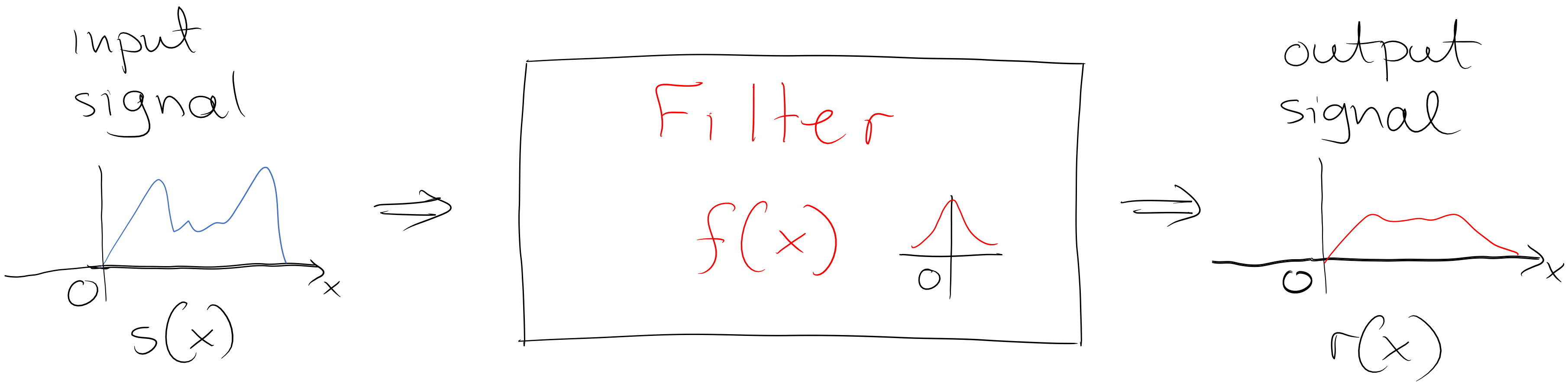
$$\boxed{= f(x-x_0,0)}$$

The convolution operation



$$r(x) = \int_{-\infty}^{\infty} f(x,\tau)\, s(\tau)\, d\tau = \int_{-\infty}^{\infty} f(x-\tau)\cdot s(\tau)\, d\tau$$

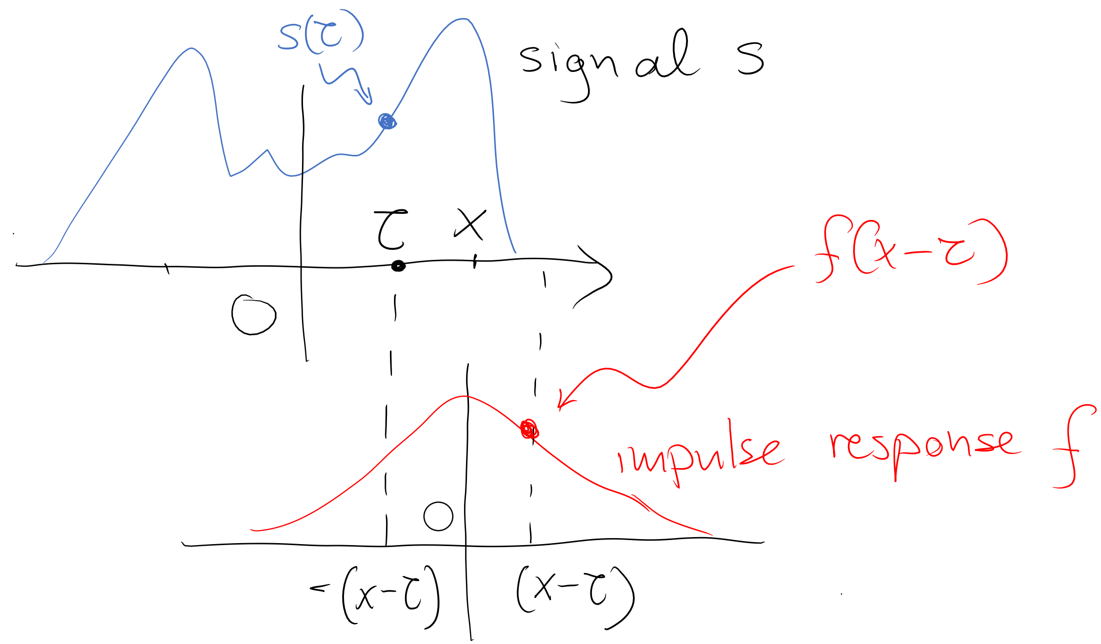
The convolution operation



$$r(x) = \int_{-\infty}^{\infty} f(x-\tau)\cdot s(\tau)\, d\tau$$

$$r \overset{def}{=} f * s$$

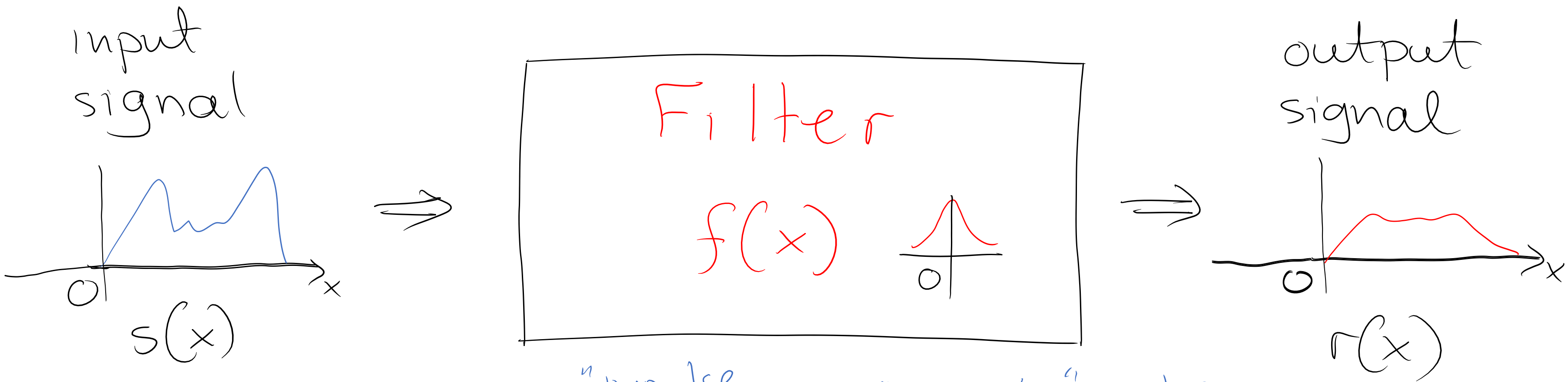
The convolution operation



$$r(x) = \int_{-\infty}^{\infty} f(x-\tau)\cdot s(\tau)\, d\tau$$

$$r \overset{\text{def}}{=} f * s$$

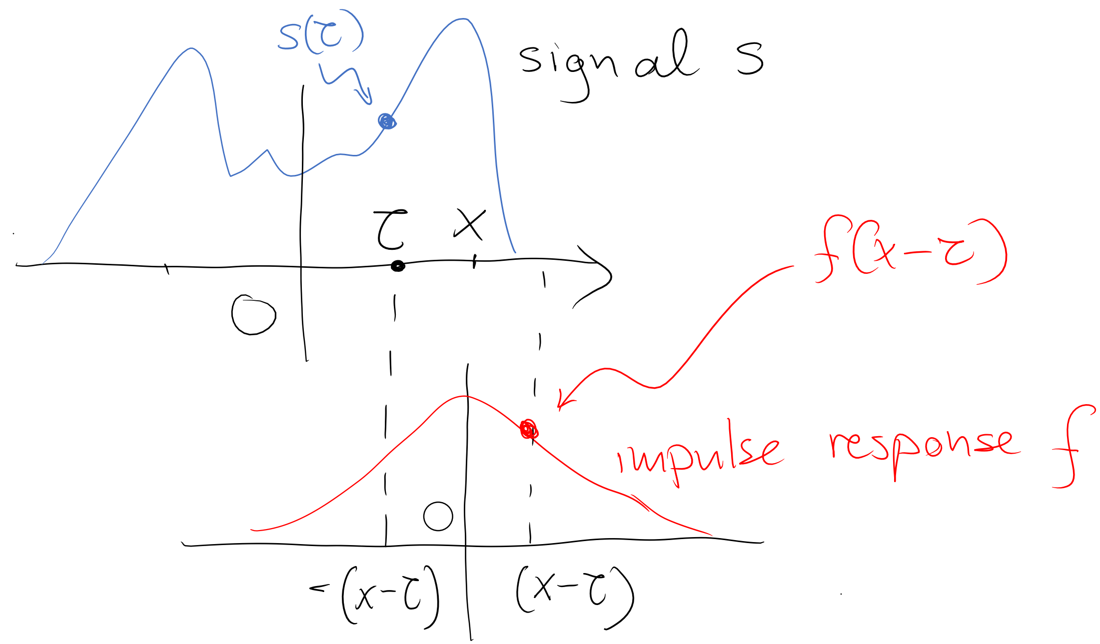
Properties of convolution



"impulse response" "signal" "impulse response" "signal"

- Commutativity: $f * s = s * f$
- Associativity: $f * (g * s) = (f * g) * s$
- Distributivity over addition: $(f+g) * s = f * s + g * s$

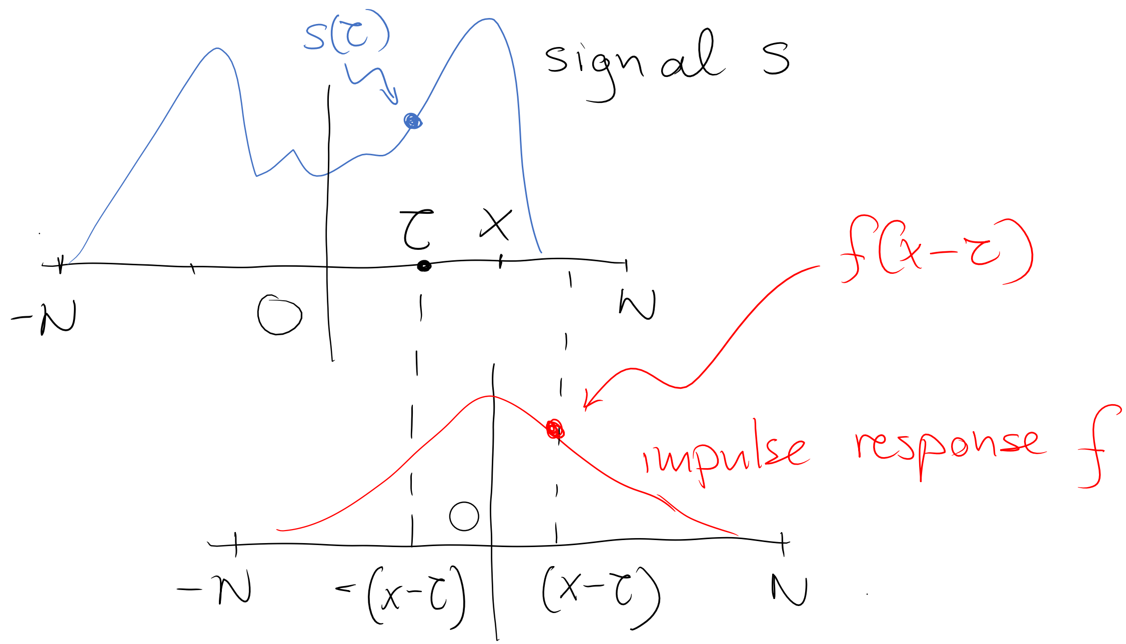
Signals with a bounded domain



$$\boxed{r(x) = \int_{-\infty}^{\infty} f(x-\tau)\cdot s(\tau)\, d\tau}$$

So far we assumed that both f and s are defined over the real line $(-\infty, \infty)$

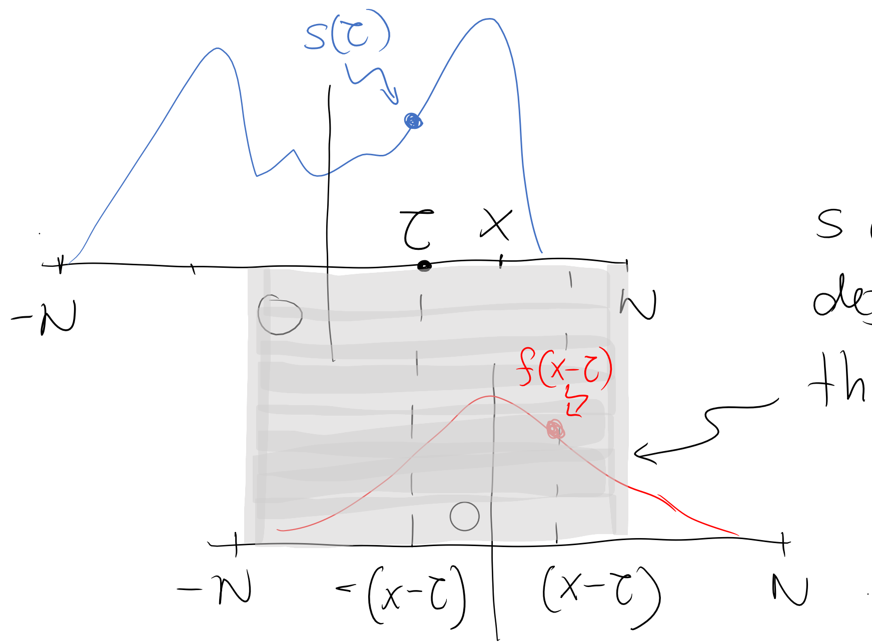
Signals with a bounded domain



$$\boxed{r(x) = \int_{-N}^{N} f(x-\tau)\cdot s(\tau)\, d\tau}$$

If s, f are only defined over $[-N, N]$ we must impose boundary conditions

Signals with a bounded domain

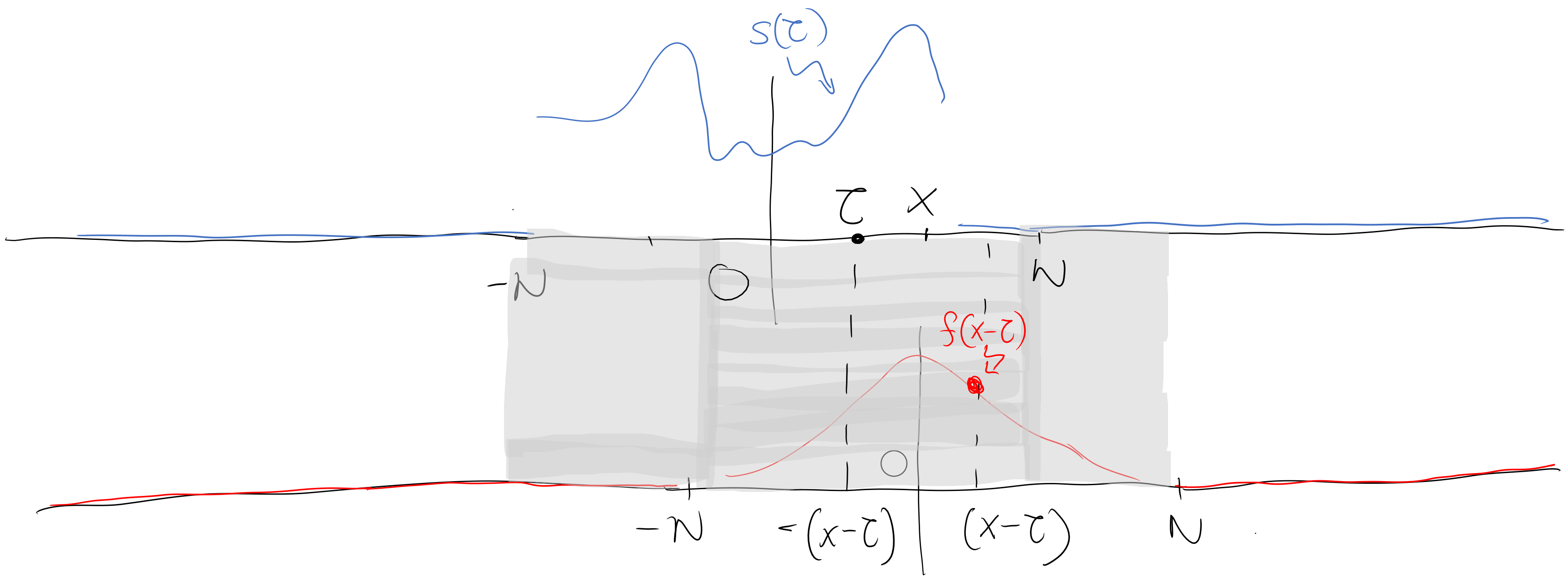


s and f only defined for this interval

$$\boxed{r(x) = \int_{-N}^{N} f(x-\tau)\cdot s(\tau)\,d\tau}$$

If s, f are only defined over $[-N, N]$ we must impose boundary conditions

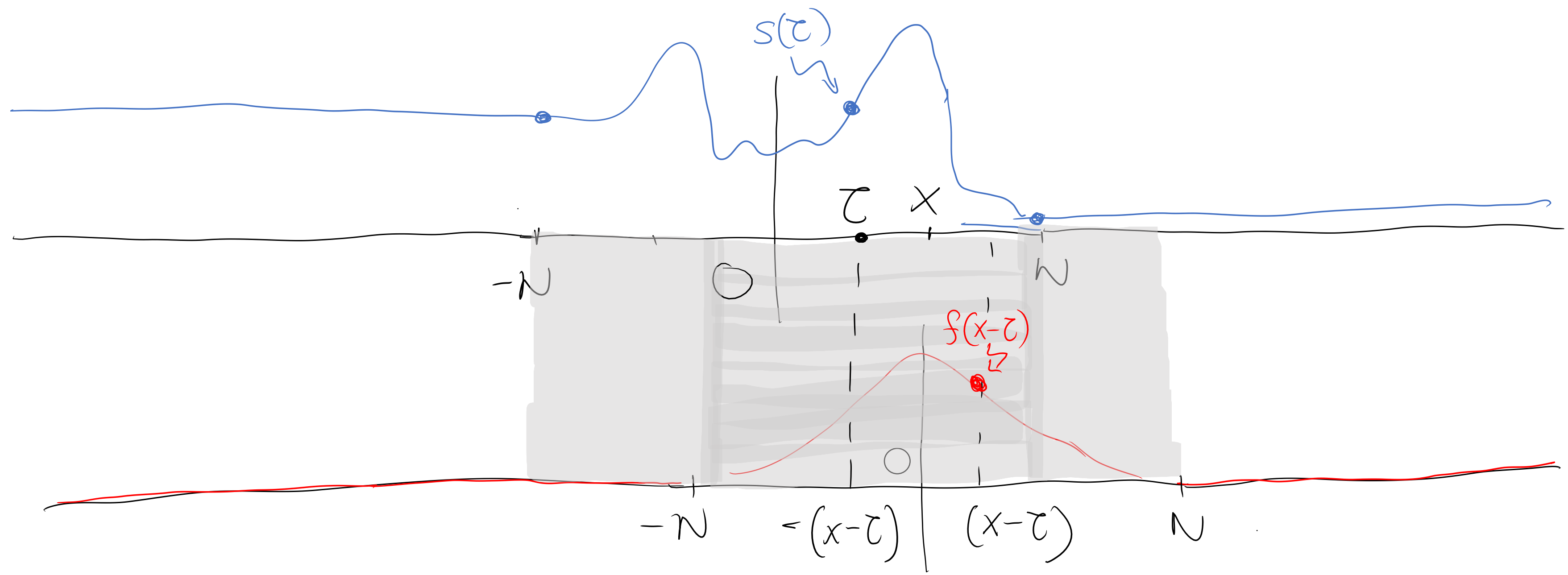
Boundary conditions: zero padding



$$\boxed{r(x) = \int_{-\infty}^{\infty} f(x-\tau) \cdot s(\tau)\, d\tau}$$

Assume both signal and impulse response are zero outside the interval (but shift invariance is lost)

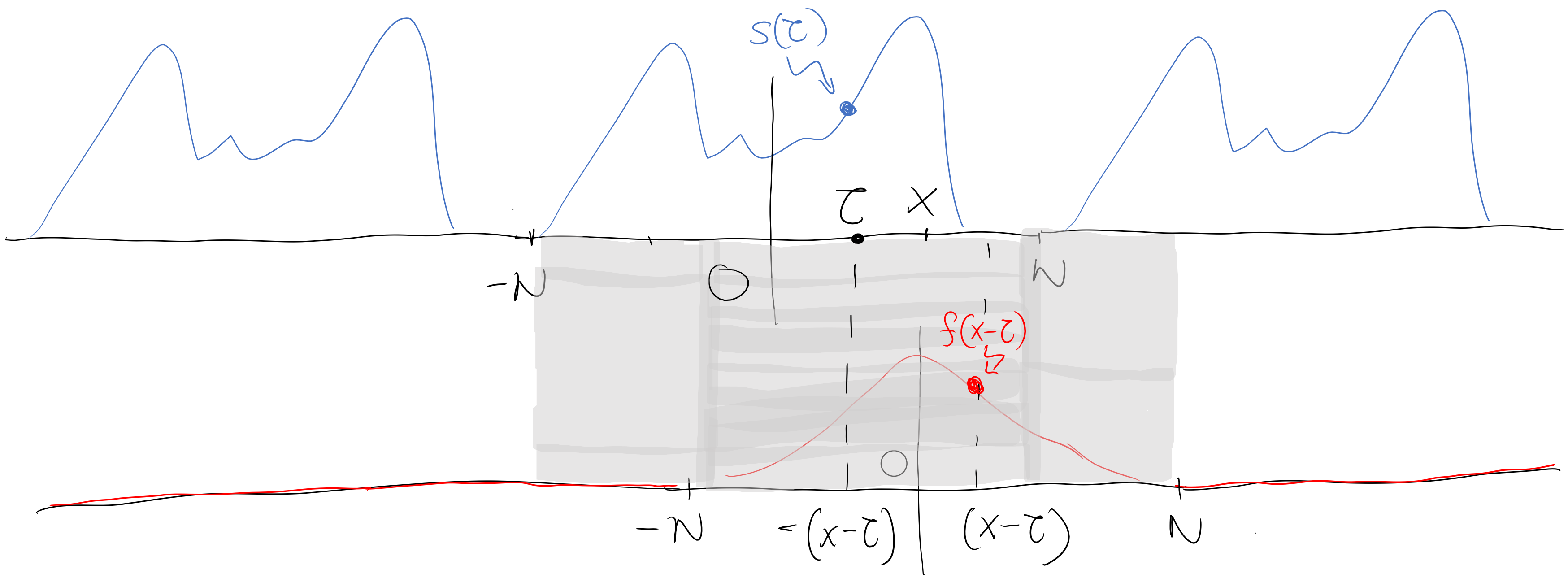
Boundary conditions: endpoint padding



$$\boxed{r(x) = \int_{-\infty}^{\infty} f(x-\tau)\cdot s(\tau)\, d\tau}$$

Pad the signal with the values of s at the two endpoints (shift invariance lost here too)

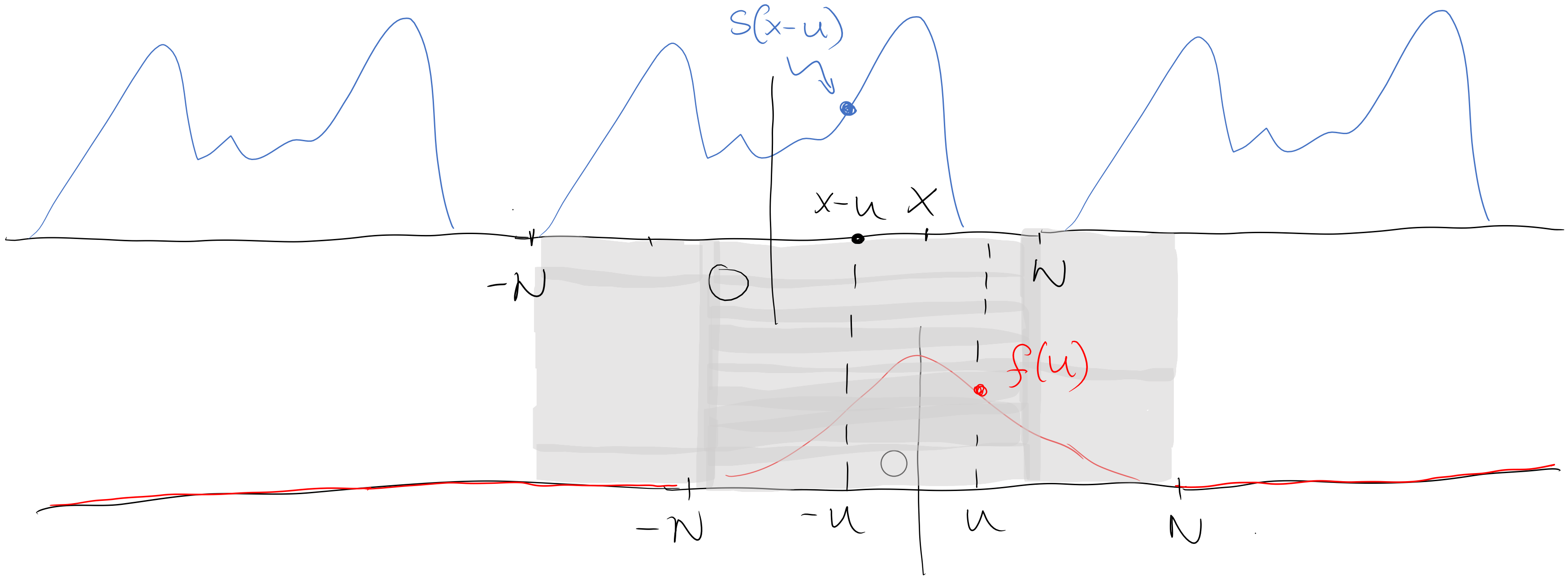
Boundary conditions: periodic signal



$$\boxed{r(x) = \int_{-\infty}^{\infty} f(x-\tau)\cdot s(\tau)\, d\tau}$$

Assume signal is periodic & impulse response is zero outside [−N, N]

Convolution operation: alternative definition



Setting $u = x - \tau$ we get

$$\boxed{r(x) = \int_{-\infty}^{\infty} f(u)\, s(x-u)\, du}$$

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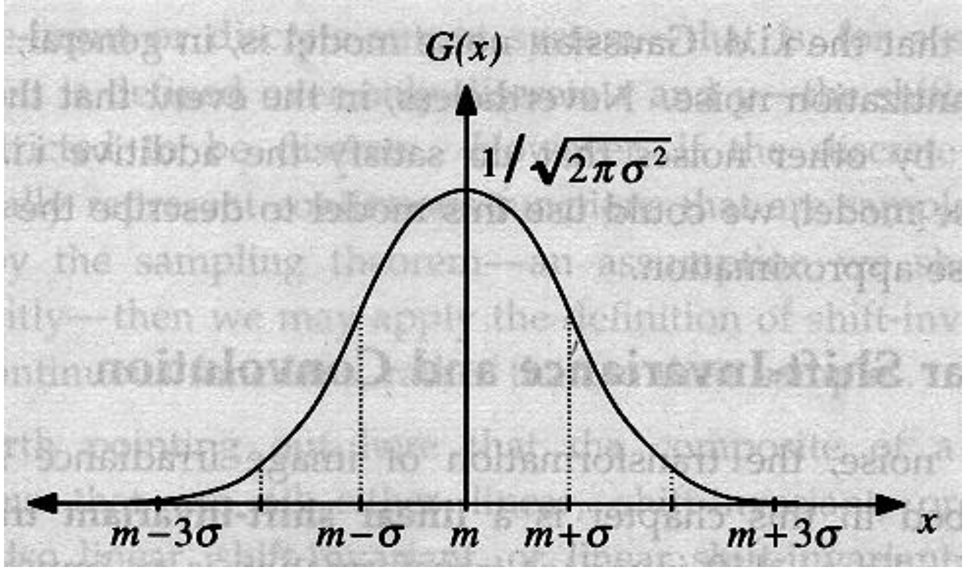
Gaussian in 1D

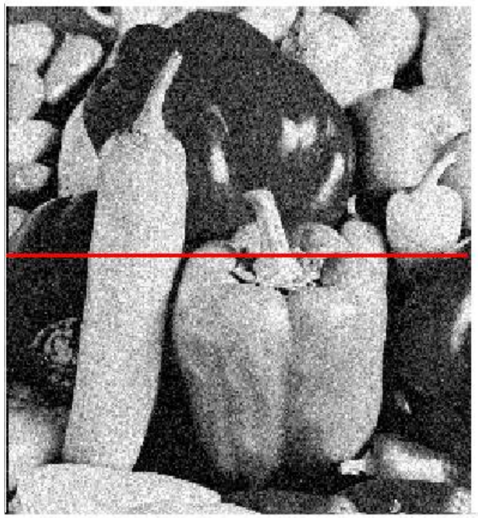


mean

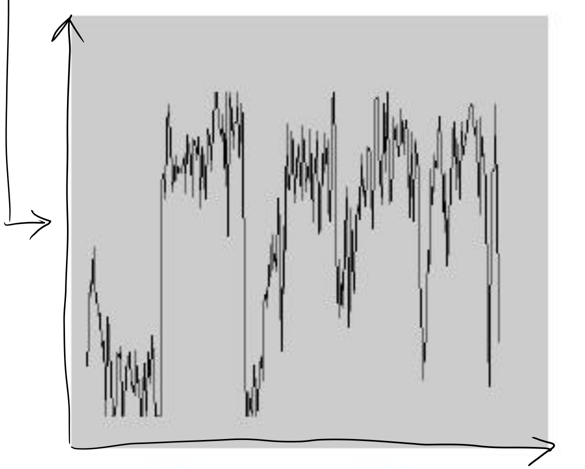
$$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

standard deviation

1D smoothing with Gaussian filters



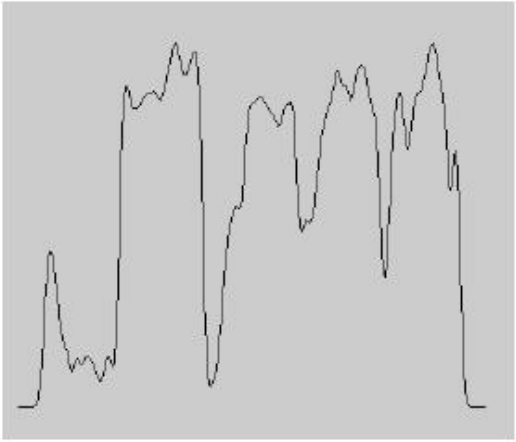
S



x

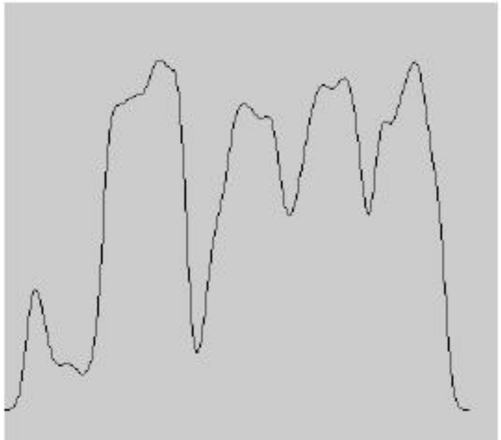
No smoothing

$S * G_2$



$\sigma = 2$

$S * G_4$



$\sigma = 4$

Derivative filters

Gaussian-smoothed signal:

$$r(x) = \int G_\sigma(x-\tau)\, s(\tau)\, d\tau$$

Differentiating the result:

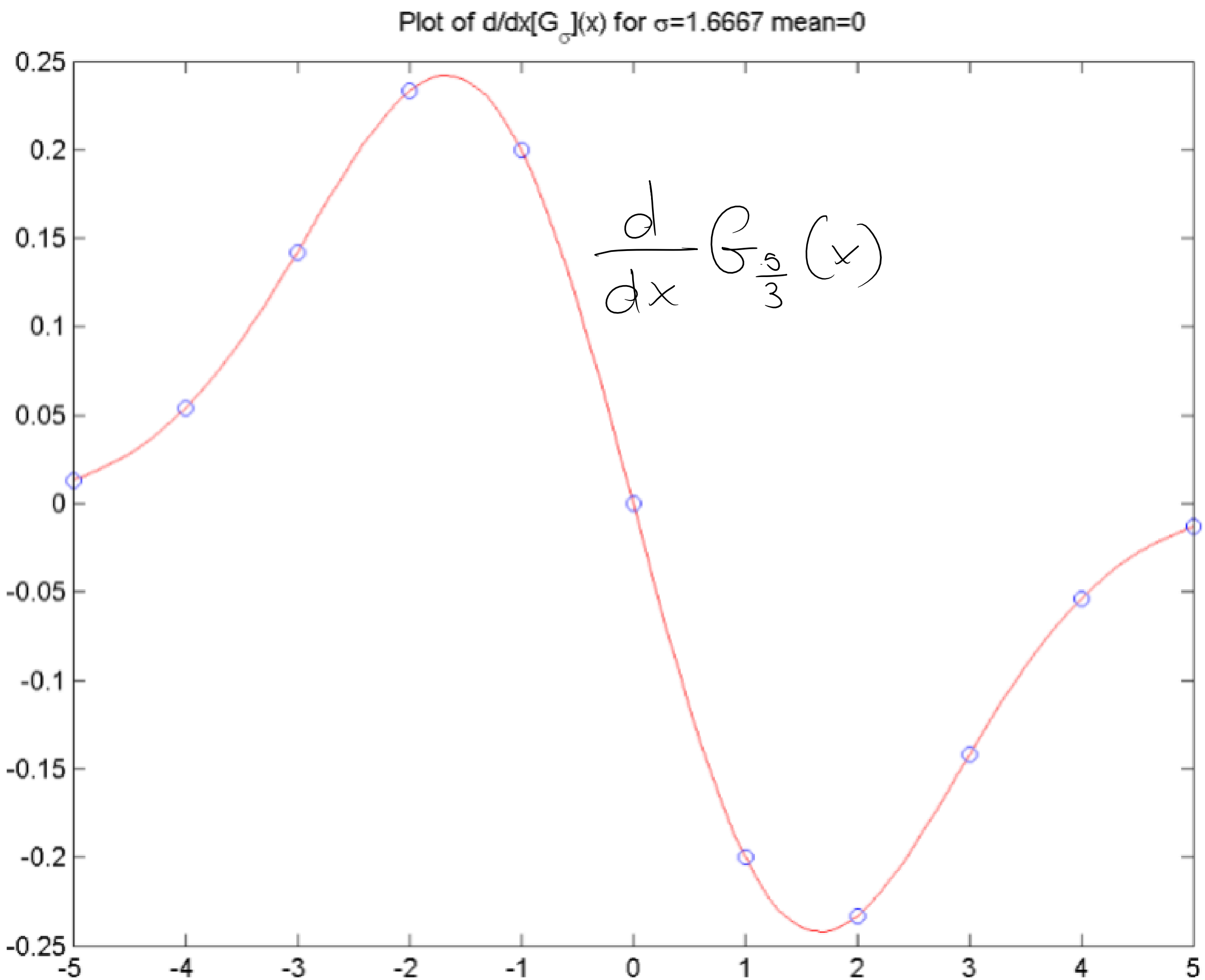
$$\frac{d^n}{dx^n} r(x) = \frac{d^n}{dx^n} \int G_\sigma(x-\tau)\, s(\tau)\, d\tau$$

$$= \int \left[\frac{d^n}{dx^n} G_\sigma(x-\tau) \right] \cdot s(\tau)\, d\tau$$

$$* \quad = \left[\frac{d^n}{dx^n} G_\sigma \right] * s$$

can be computed analytically

1st derivative of a Gaussian

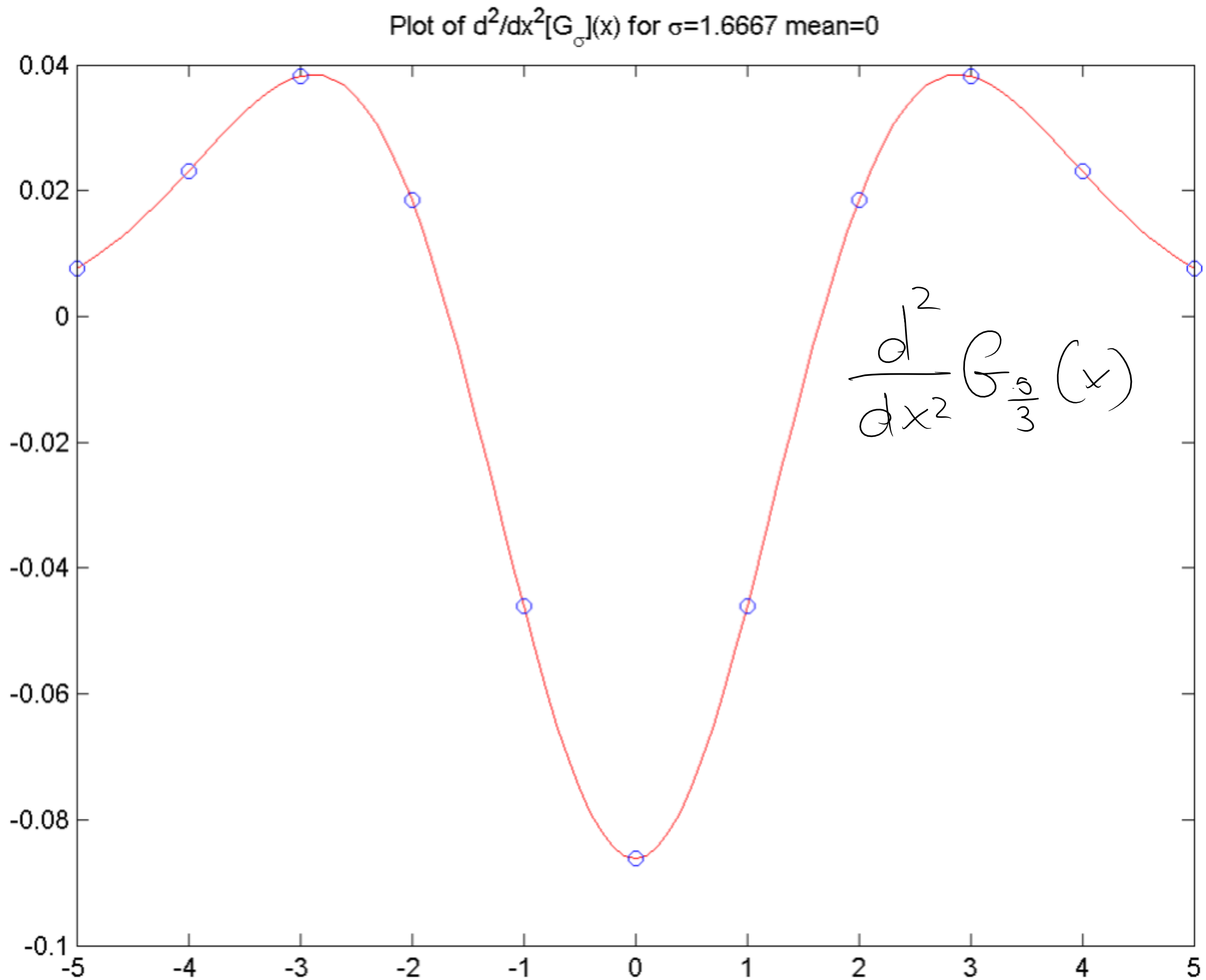


Input image

Convolve each row with a Gaussian derivative filter

Convolve each column with a Gaussian derivative filter

2nd derivative of a Gaussian



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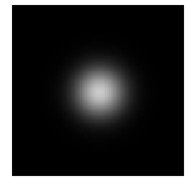
Convolution in 2D

$s(x,y)$

$\Rightarrow$

Filter

$f(x,y)$



$\Rightarrow$

$r(x,y)$

$$\boxed{\begin{aligned} r(x,y) &= \iint f(u,v)\, s(x-u, y-v)\, du\, dv \\ r &\overset{\text{def}}{=} f * s \end{aligned}}$$

Separable 2D filters

A 2D filter is separable iff

$$f(x,y) = f_1(x)\, f_2(y)$$

where f_1, f_2 are 1D filters

2D convolution as cascade of 1D convolutions

$$r(x,y) = \iint f_1(u)\cdot f_2(v)\, s(x-u, y-v)\, du\, dv$$

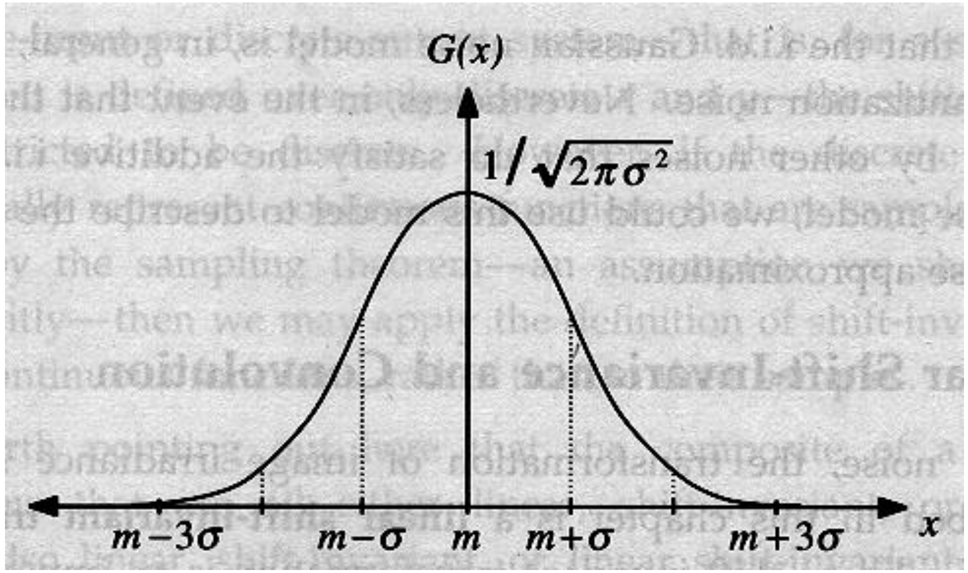
$$* = \underbrace{\int f_2(v) \left[\underbrace{\int f_1(u)\cdot s(x-u, y-v)\, du}_{\text{1D convolution along } x} \right] dv}_{\text{1D convolution along } y}$$

A 2D filter is separable iff

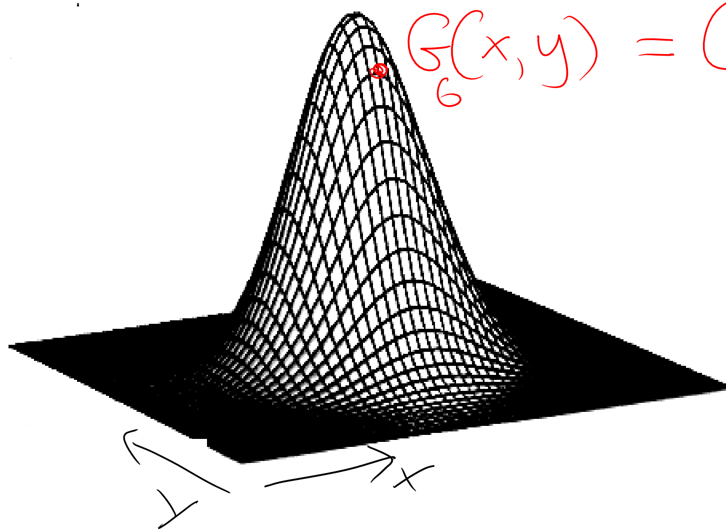
$$f(x,y) = f_1(x)\, f_2(y)$$

where f_1, f_2 are 1D filters

Gaussian in 2D



$G_\sigma(x,y) = G_\sigma(x) \cdot G_\sigma(y)$



Difference-of-Gaussian filters (DOG filters)

$I * G_{\sigma_1}(x,y)$

near 0

original

What does smoothing take away?

$I * G_{\sigma_2}(x,y)$

Difference-Of-Gaussians (DOG) filtering

Difference of two Gaussian-smoothed versions of I:

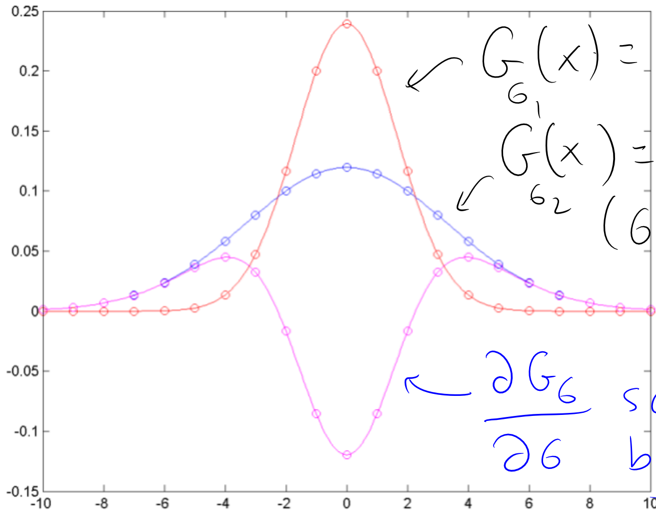
$I * G_{\sigma_1} -$

$I * G_{\sigma_2} =$

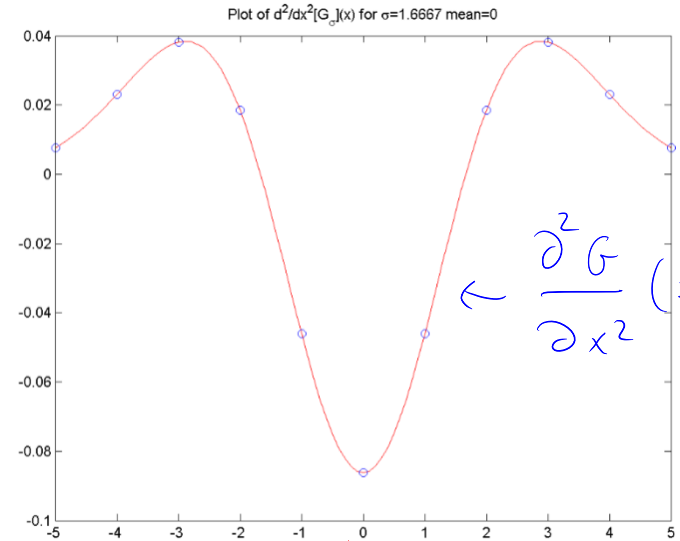
$I * \underbrace{(G_{\sigma_1} - G_{\sigma_2})}_{\text{the DOG filter}}$

Equivalence of DOG & 2nd derivative filter in 1D

① Consider $G_\sigma(x)$ to be a function of σ and calculate its derivative with respect to σ:



② Compare to the 2^{nd} derivative of G with respect to x:



$$\boxed{G_{\sigma_2}(x) - G_{\sigma_1}(x) \approx (\sigma_2 - \sigma_1) \cdot \frac{\partial G_\sigma}{\partial \sigma}(x;\sigma_1)}$$

$\Longrightarrow$

The DOG filter is just a scaled version of the Gaussian 2^{nd} derivative filter

Equivalence of DOG & 2nd derivative filter in 1D

① Consider $G_\sigma(x)$ to be a function of σ and calculate its derivative with respect to σ:

Approximate difference by the derivative at scale σ_1

$$\frac{\partial G(x;\sigma_1)}{\partial \sigma} \approx \frac{G(x;\sigma_2) - G(x;\sigma_1)}{\sigma_2 - \sigma_1}$$

$\Longleftrightarrow$

$$\boxed{G_{\sigma_2}(x) - G_{\sigma_1}(x) \approx (\sigma_2 - \sigma_1) \cdot \frac{\partial G_\sigma}{\partial \sigma}(x;\sigma_1)}$$

② Compare to the 2nd derivative of G with respect to x:

$$\frac{\partial^2 G_\sigma}{\partial x^2} = \left(\frac{x^2}{\sigma^2} - 1\right)\frac{1}{\sigma^2} G_\sigma(x)$$

$$\frac{\partial G_\sigma}{\partial \sigma} = \left(\frac{x^2}{\sigma^2} - 1\right)\frac{1}{\sigma} G_\sigma(x)$$

i.e. $$\boxed{\frac{\partial G_\sigma}{\partial \sigma} = \sigma \frac{\partial^2 G_\sigma}{\partial x^2}}$$

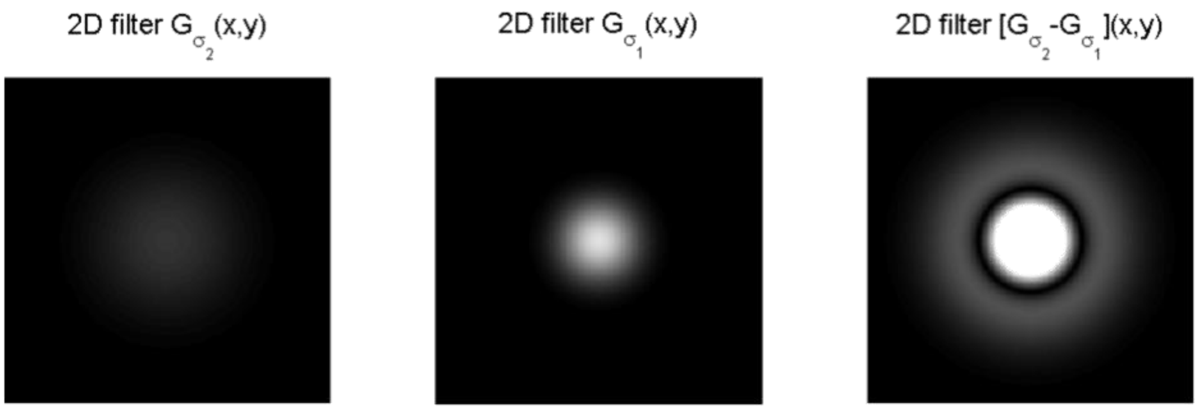
$\Downarrow$

$\Longrightarrow$

$$\boxed{G_{\sigma_2}(x) - G_{\sigma_1}(x) = (\sigma_2 - \sigma_1) \cdot \sigma_1 \cdot \frac{\partial^2 G_\sigma}{\partial x^2}}$$

Difference-Of-Gaussians (DOG) filter in 2D

$$G_{\sigma_2}(x,y) - G_{\sigma_1}(x,y) \simeq \sigma_1(\sigma_1-\sigma_2)\left[\underbrace{\frac{\partial^2}{\partial x^2}G_{\sigma_1} + \frac{\partial^2}{\partial y^2}G_{\sigma_1}}_{\text{Laplacian filter } \nabla^2 G_{\sigma_1}}\right]$$



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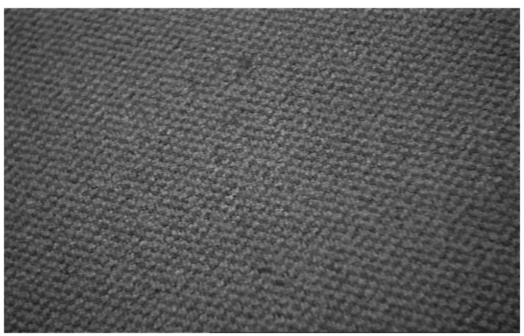
The 2D (Continuous-Time) Fourier Transform

We treat images as infinite-size, continuous periodic functions

Carpet image (black & white)



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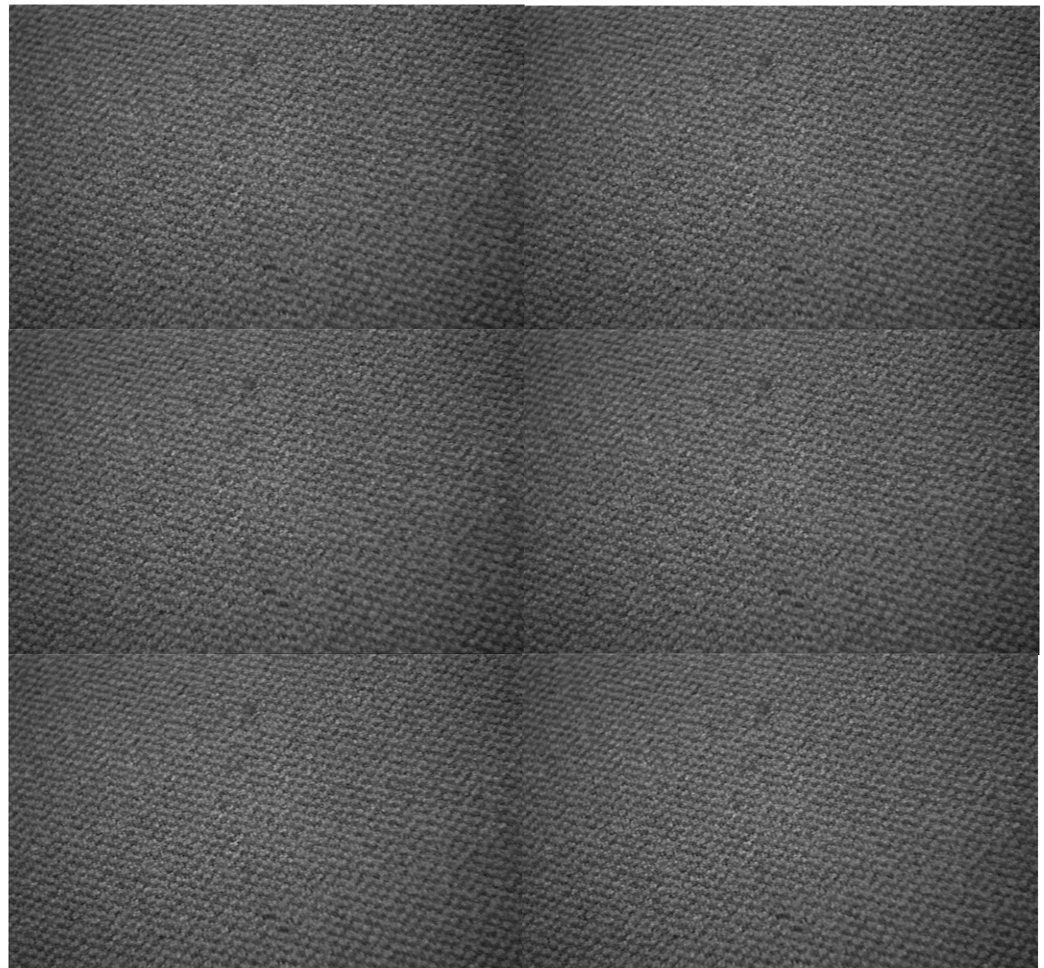


…

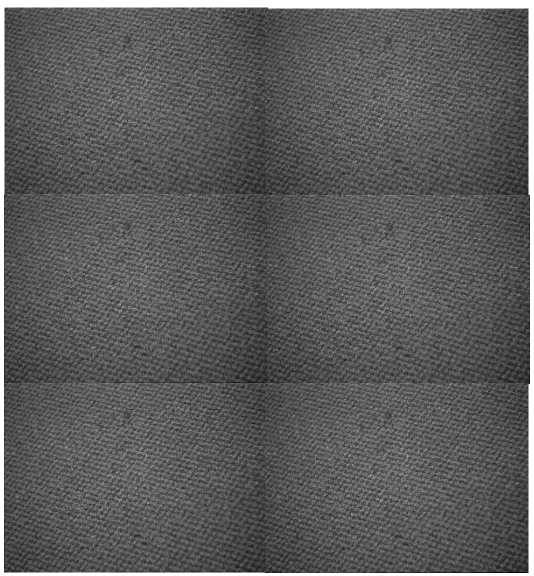
…

The 2D (Continuous-Time) Fourier Transform

We can write such functions as an infinite sum of Fourier Basis images, each corresponding to a different spatial frequency

Carpet image (black & white)

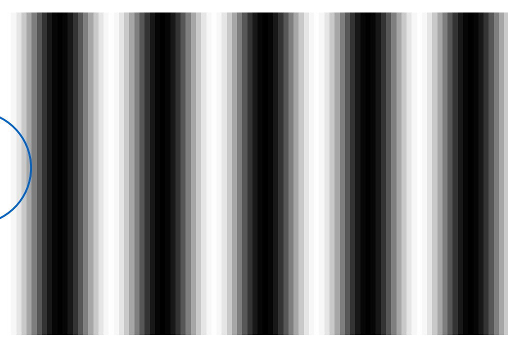
The Fourier Basis Images (real component)



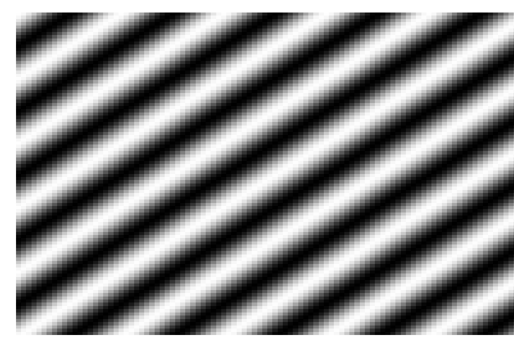
= $F_{5,0}$

+ $F_{10,0}$

+ $F_{0,5}$



+ $F_{5,5}$



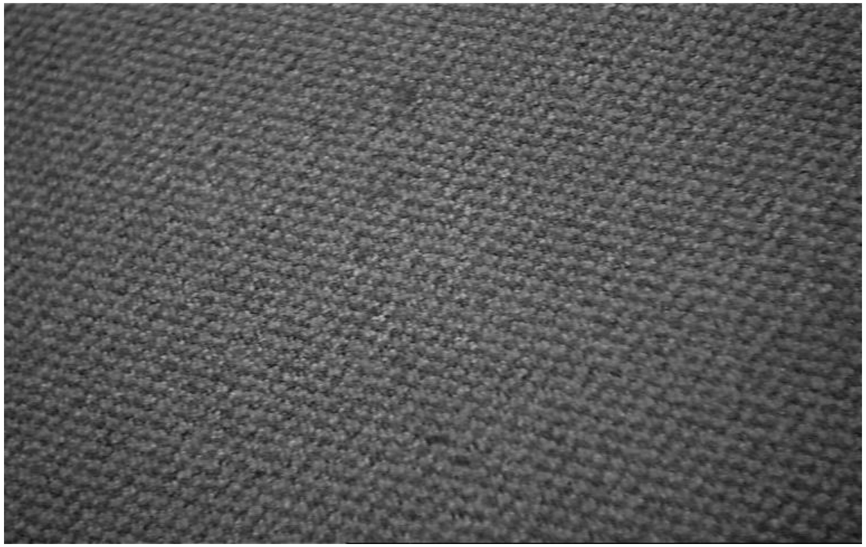
Fourier coefficient

+ ...

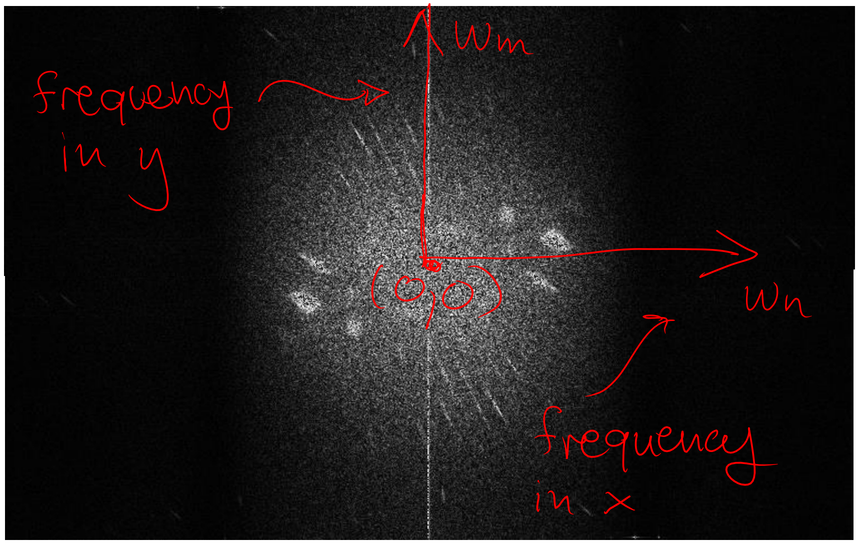
The 2D Discrete Fourier Transform

We can write such functions as an infinite sum of Fourier Basis images, each corresponding to a different spatial frequency

Carpet image (black & white)



Its Fourier Transform...



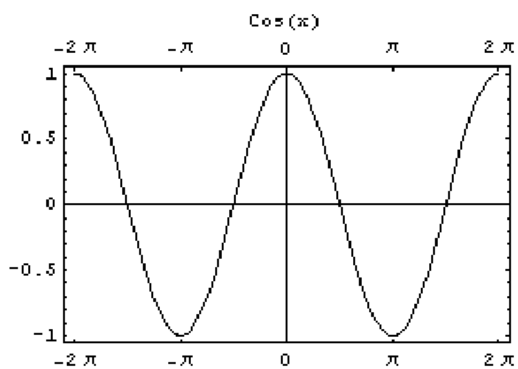
2π-periodic functions & the Fourier series

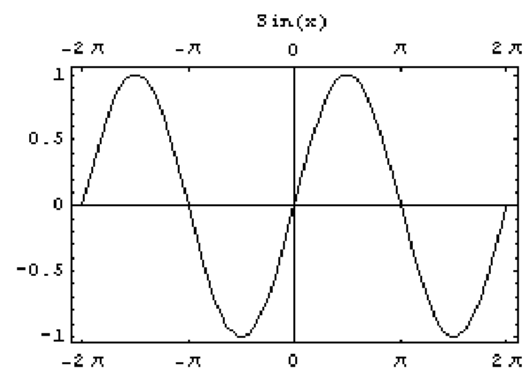
- Consider a bounded, continuous, possibly complex periodic signal with $s(x) = s(x+2\pi)$.
- Then we can express it as

contributions of n-th harmonic

$$s(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + \dots + a_n \cos(nx) + \dots$$
$$+ b_1 \sin(x) + b_2 \sin(2x) + \dots + b_n \sin(nx) + \dots$$

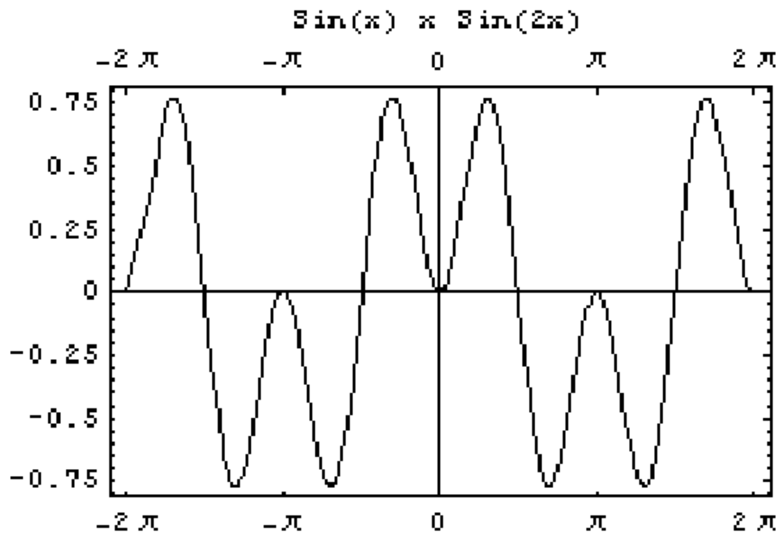
periodic over $[-\pi, \pi]$





Orthogonality of sin(x), cos(x) over (-π, π)

$$\int_{-\pi}^{\pi} \sin(nx)\cdot\sin(mx) = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m > 0 \end{cases}$$



$$\boxed{\begin{aligned} s(x) = \frac{a_0}{2} &+ a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + \cdots \\ &+ b_1\sin(x) + b_2\sin(2x) + \cdots + b_n\sin(nx) + \cdots \end{aligned}}$$

Orthogonality of sin(x), cos(x) over (-π, π)

Multiplying with sin(nx) and integrating

$$* \int_{-\pi}^{\pi} \sin(nx)\, s(x)\, dx = \int_{-\pi}^{\pi} \overbrace{\cancel{\frac{a_0}{2} \cdot \sin(nx)}}^{0} + \sum_{m=1}^{\infty} \overbrace{\cancel{a_m \cos(mx) \cdot \sin(nx)}}^{0} + \sum_{m=1}^{\infty} \overbrace{\cancel{b_m \sin(mx) \sin(nx)}}^{\text{for } m \neq n}\, dx$$

$$\| $$

$$\pi b_n$$

$$\boxed{\begin{aligned} s(x) = \frac{a_0}{2} &+ a_1 \cos(x) + a_2 \cos(2x) + \cdots + a_n \cos(nx) + \cdots \\ &+ b_1 \sin(x) + b_2 \sin(2x) + \cdots + b_n \sin(nx) + \cdots \end{aligned}}$$

The Fourier series coefficients

More generally

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\, s(x)\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\, s(x)\, dx$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} s(x)\, dx \leftarrow \text{average value of signal}$$

$$\boxed{\begin{aligned} s(x) = \frac{a_0}{2} &+ a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + \cdots \\ &+ b_1\sin(x) + b_2\sin(2x) + \cdots + b_n\sin(nx) + \cdots \end{aligned}}$$

Using Euler notation

More generally

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\, s(x)\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\, s(x)\, dx$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} s(x)\, dx$$ ← average value of signal

Euler notation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Fourier Transform of a continuous periodic signal

These relations can now be expressed more compactly

$$\boxed{s(x) = \sum_{n=-\infty}^{\infty} F_n e^{jnx}}$$

Fourier Series (aka Inverse Fourier Transform)

(Continuous-Time) Fourier Transform of s(x)

$$\boxed{F_n = \int_{-\pi}^{\pi} s(x) e^{-jnx} dx}$$

Euler notation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Fourier Transform of a continuous periodic signal

For periodic $s(x)$ with domain $[0,1]$ and period 1:

$$\boxed{s(x) = \sum_{n=-\infty}^{\infty} F_n \, e^{jn2\pi x}}$$

Fourier Series (aka Inverse Fourier Transform)

(Continuous-Time) Fourier Transform of $s(x)$

$$\boxed{F_n = \int_0^1 s(x) \, e^{-jn2\pi x} \, dx}$$

Euler notation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Conventions used here

- Use ω to denote frequency:

eg $\omega_n = \begin{cases} 2\pi n & \text{if } s(x) \in [0,1] \\ n & \text{if } s(x) \in [-\pi,\pi] \end{cases}$

- $F_n\{s\} \stackrel{\text{def}}{=}$ Fourier transform coefficient for freq. ω_n

- $F_n\{s\}$ is complex-valued in general, expressed in the form $\rho e^{i\phi}$

phase

magnitude

$$\rho(\omega_n) = |F_n\{s\}| = |F_n| \qquad \phi(\omega_n) = \arg(F_n\{s\})$$

Fourier Transform properties

Symmetries:

* $\forall x \quad s(x) \in \mathbb{R} \implies F_n\{s(x)\}$ is symmetric

$$F_n = F_{-n}^{*}$$ conjugate

* $\forall x \quad s(x) = s(-x) \implies F_n\{s(x)\}$ is real-valued
* $\forall x \quad s(x) = -s(-x) \implies F_n\{s(x)\}$ is imaginary

Shift property:

* $$F_n\{s(x - x_0)\} = e^{-j\omega_n x_0} F_n\{s(x)\}$$

The Convolution Theorem

$$F\{f*s\} = F\{f\}\cdot F\{s\}$$

the F.T. of the convolution of two functions = the product of the F.T.s of the two functions

Proof_

$$*\ F_n\{f*s\} = \int_{-\pi}^{\pi} (f*s)\, e^{-jw_n x}\, dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\tau)\, s(x-\tau)\, e^{-jw_n x}\, dx\, d\tau$$

$$= \int_{-\pi}^{\pi} f(\tau) \int_{-\pi}^{\pi} s(x-\tau)\, e^{-jw_n x}\, dx\, d\tau$$

from shift property

$$= \int_{-\pi}^{\pi} f(\tau)\, e^{-jw_n \tau} \cdot F_n\{s\}\, d\tau = F_n\{f\}\cdot F_n\{s\}$$

The Convolution Theorem

$$F\{f * s\} = F\{f\} \cdot F\{s\}$$

the F.T. of the convolution of two functions = the product of the F.Ts of the two functions

$$F\{f \cdot s\} = F\{f\} * F\{s\}$$

the F.T. of the product of two functions = the convolution of the F.Ts of the two functions

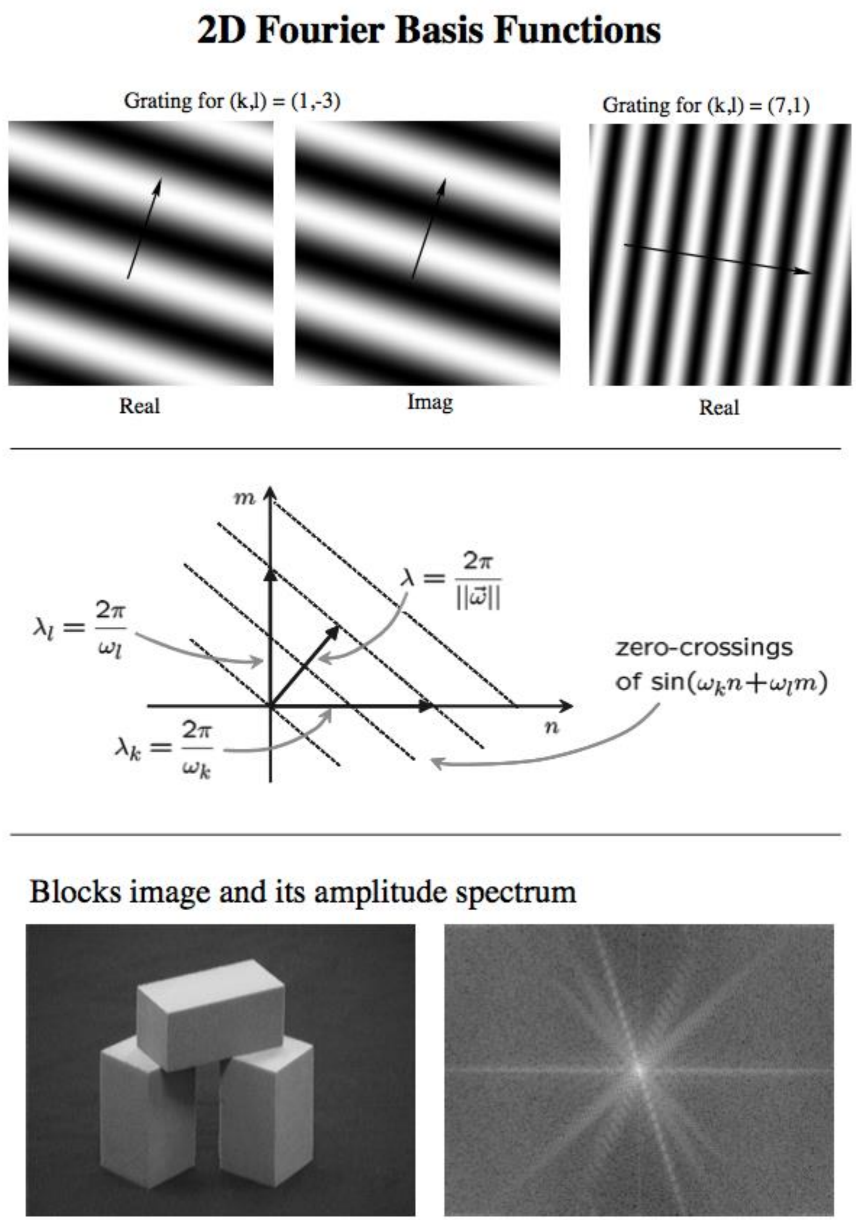
2D Fourier Transform

For periodic $s(x,y)$ with domain $[-\pi,\pi]\times[-\pi,\pi]$ period 1:

$$s(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m}\, e^{j(nx+my)}$$

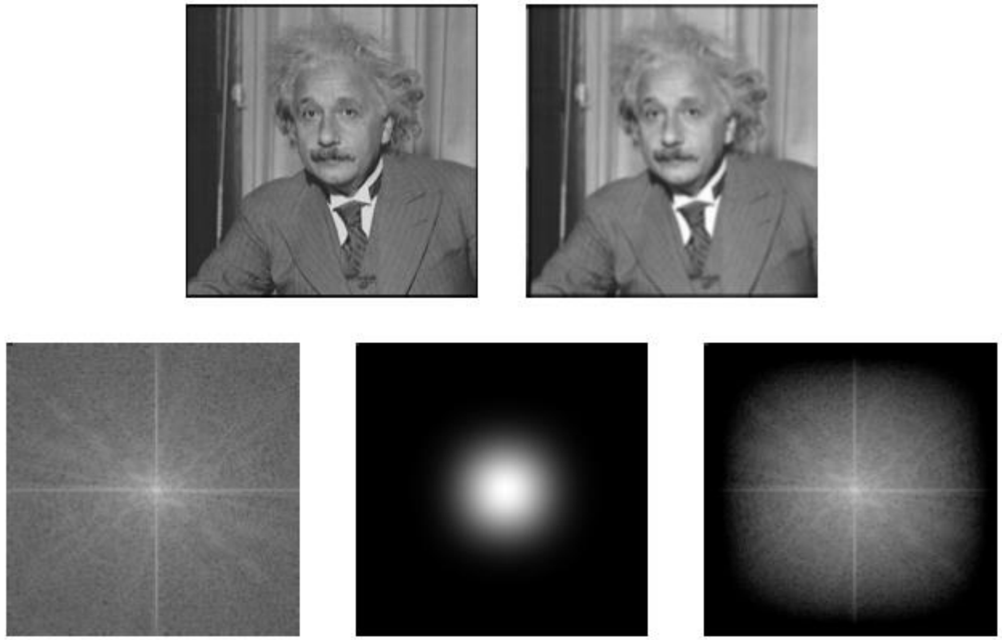
$$F_{n,m} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} s(x,y)\cdot e^{-j(nx+my)}\, dx\, dy$$

Example 2D Fourier basis functions



Examples of 2D filters & their spectra

Common Filters and their Spectra



Top Row: Image of Al and a **low-pass** (blurred) version of it. The low-pass kernel was separable, composed of 5-tap 1D impulse responses $\frac{1}{16}(1, 4, 6, 4, 1)$ in the x and y directions.

Bottom Row: From left to right are the amplitude spectrum of Al, the amplitude spectrum of the impulse response, and the product of the two amplitude spectra, which is the amplitude spectrum of the blurred version of Al. (Brightness in the left and right images is proportional to log amplitude.)

Examples of 2D filters & their spectra

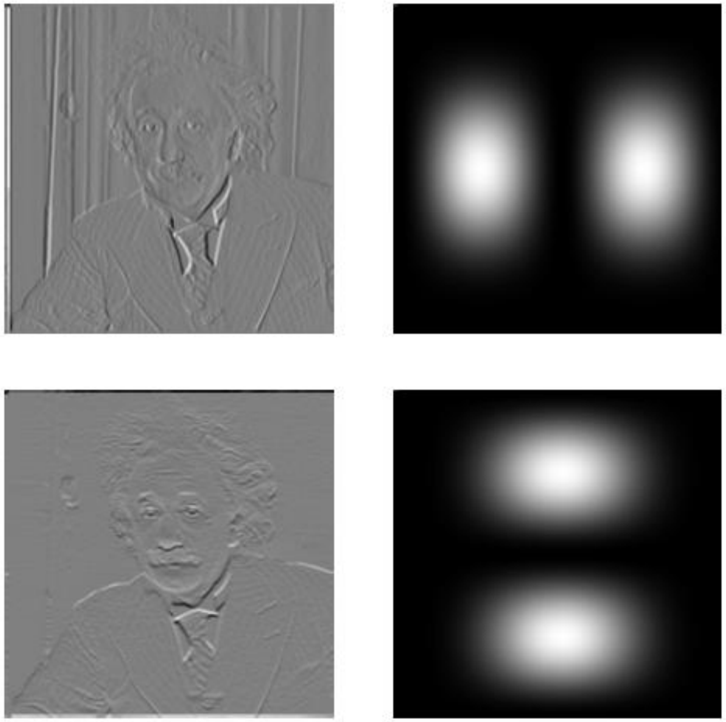
Common Filters and their Spectra (cont)

From left to right is the original Al, a **high-pass** filtered version of Al, and the amplitude spectrum of the filter. This impulse response is defined by $\delta(n) - h(n, m)$ where $h[n, m]$ is the separable blurring kernel used in the previous figure.

From left to right is the original Al, a **band-pass** filtered version of Al, and the amplitude spectrum of the filter. This impulse response is defined by the difference of two low-pass filters.

Examples of 2D filters & their spectra

Common Filters and their Spectra (cont)



Top Row: Convolution of Al with a horizontal derivative filter, along with the filter's Fourier spectrum. The 2D separable filter is composed of a vertical smoothing filter (i.e., $\frac{1}{4}(1, 2, 1)$) and a first-order central difference (i.e., $\frac{1}{2}(-1, 0, 1)$) horizontally.

Bottom Row: Convolution of Al with a vertical derivative filter, and the filter's Fourier spectrum. The filter is composed of a horizontal smoothing filter and a vertical first-order central difference.

Topic 05:

Linear Filters & Fourier Analysis

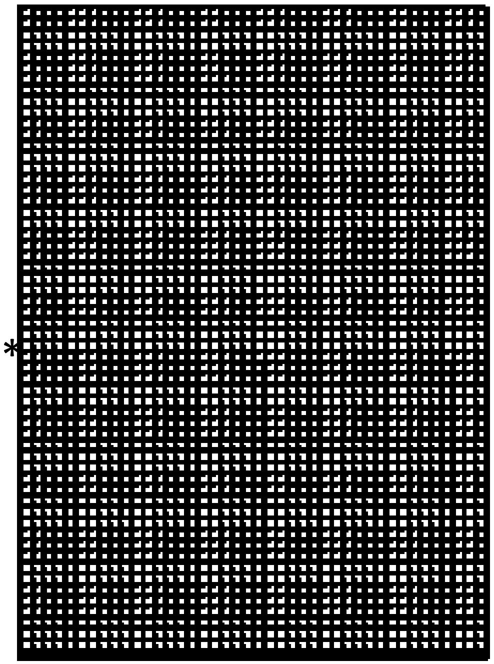
- Filter-based view of image formation
- Linear systems & 1D convolution
- Example 1D filters
- Filtering in 2D
- The Fourier series
- Introduction to sampling & aliasing
- Discrete-time filters & the DFT

Image formation from a filtering perspective

$B(x,y)$

*

Impulse train



Sampled image

$D(x,y) = B(x,y) \cdot S(x,y)$

$S(x,y) = 1$ iff x, y integer multiples of pixel pitch $\Delta x, \Delta y$ & zero otherwise

Blur due to lens & finite pixel footprint

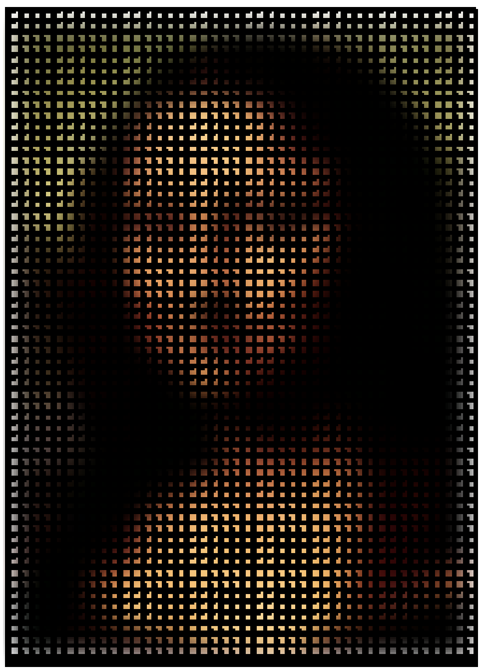
$B(x,y) = (I * f)(x,y)$

Ideal sensor irradiance

$I(x,y) \quad (x,y) \in [-\pi,\pi] \times [-\pi,\pi]$

Image formation from a filtering perspective

$D(x,y)$



Sampled image

sampling & aliasing

$$D(x,y) = B(x,y) \cdot S(x,y)$$

$S(x,y) = 1$ iff x, y integer multiples of pixel pitch $\Delta x, \Delta y$ & zero otherwise

filtering & convolution

Blur due to lens & finite pixel footprint

$$B(x,y) = (I * f)(x,y)$$

Ideal sensor irradiance

$$I(x,y) \quad (x,y) \in [-\pi,\pi] \times [-\pi,\pi]$$

Example: photo with no visible aliasing

Close-up photo of my office carpet (looks pretty good)

Aliased photo

Sampling in spatial vs. Fourier domain

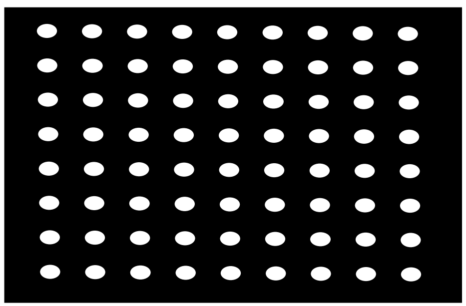
Let us consider the sampling operation in the Fourier Transform domain

$$\sum_{k=-N}^{N}\sum_{\ell=-M}^{M}\delta\left(x-\frac{\pi}{N}k\right)\delta\left(y-\frac{\pi}{M}\ell\right)$$

$\parallel$ $*$

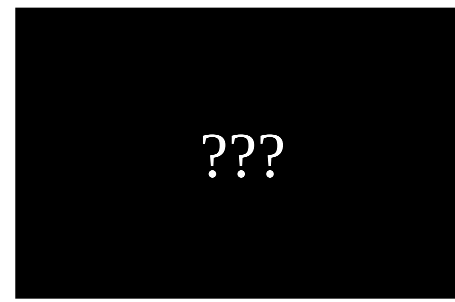
2N samples

2M samples



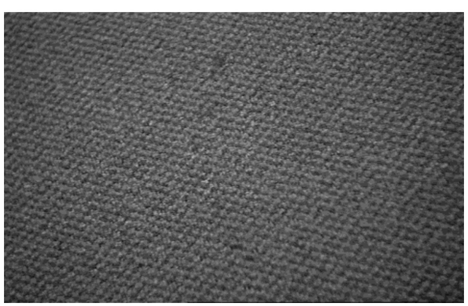
$S(x,y)$

$\mathcal{F}\{S\}$



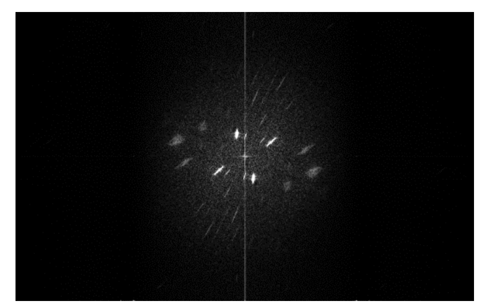
$\times$

$*$



$B(x,y)$

$\mathcal{F}\{B\}$



Sampling in spatial vs. Fourier domain

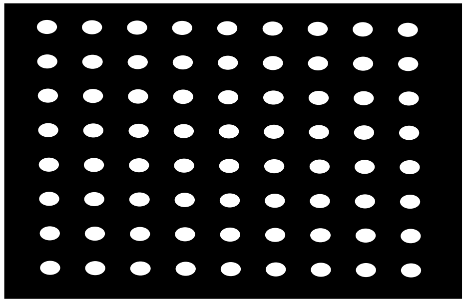
Let us consider the sampling operation in the Fourier Transform domain

$$\sum_{k=-N}^{N}\sum_{l=-M}^{M}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\delta\left(x-\frac{\pi}{N}k\right)\delta\left(y-\frac{\pi}{M}l\right)e^{-j(xn+ym)}\,dx\,dy$$

$* \;||$

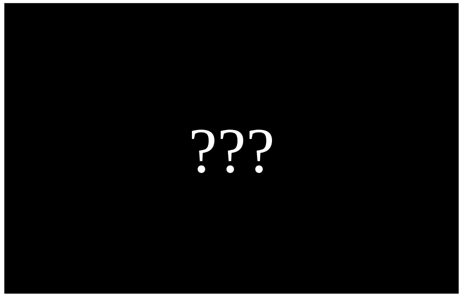
2N samples

2M samples



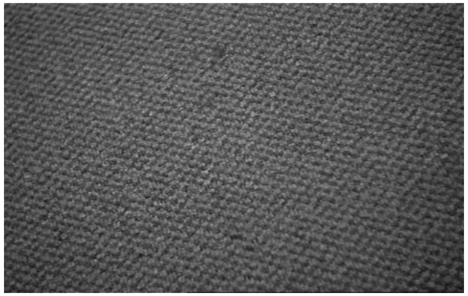
$S(x,y)$

$F\{S\}$



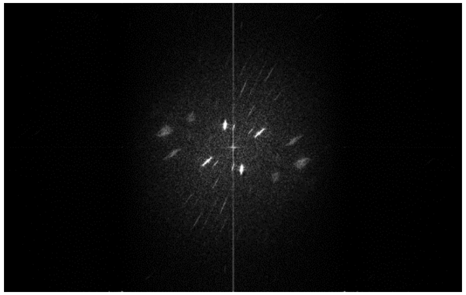
$\times$

$*$



$B(x,y)$

$F\{B\}$



Sampling in spatial vs. Fourier domain

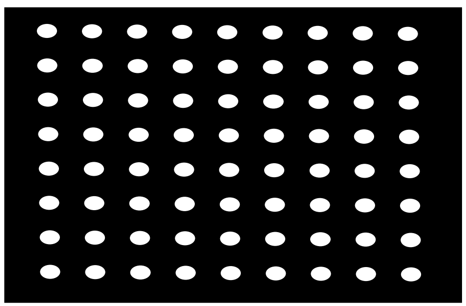
Sample distances in spatial domain inversely proportional to Fourier domain

$$\sum_{k=-N}^{N}\sum_{l=-M}^{M}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\delta\left(x-\frac{\pi}{N}k\right)\delta\left(y-\frac{\pi}{M}l\right)e^{-j(xn+ym)}\,dx\,dy$$

$$\|$$

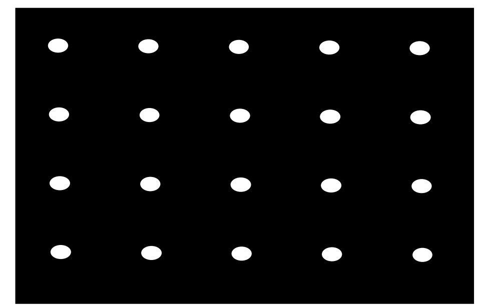
2N samples

2M samples



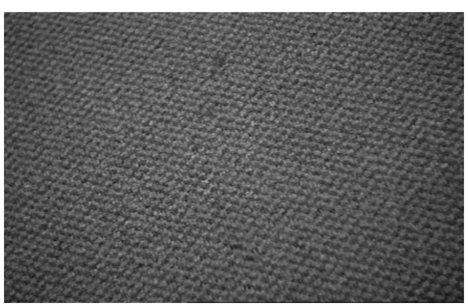
$S(x,y)$ $\frac{\pi}{M}$ $\frac{\pi}{N}$

$\mathcal{F}\{S\}$ $\frac{M}{\pi}$



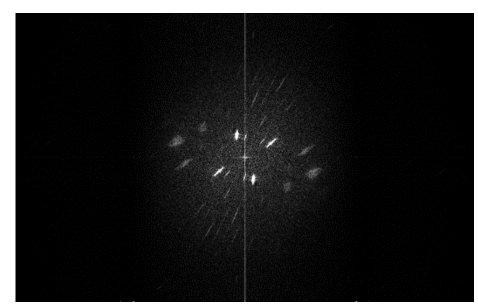
$\frac{N}{\pi}$

$\times$ $*$



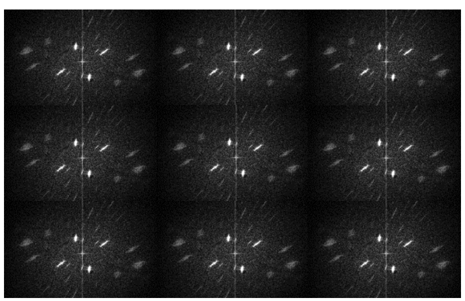
$B(x,y)$

$\mathcal{F}\{B\}$



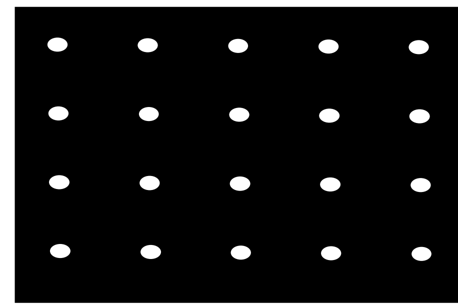
Convolution with an impulse train

a superposition of shifted copies of $F\{B\}$



$\Longleftarrow$

$F\{S\}$

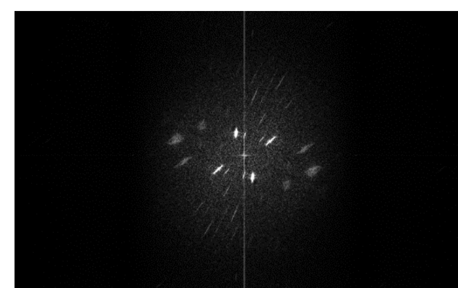


$\frac{M}{n}$

$\frac{N}{n}$

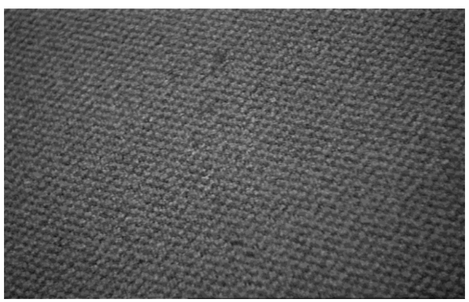
$*$

$F\{B\}$



Sampling in spatial vs. Fourier domain

When region of support of $F\{B\}$ does not overlap with adjacent copies it is possible to reconstruct B from $F^{-1}\{F\{B \cdot S\}\}$



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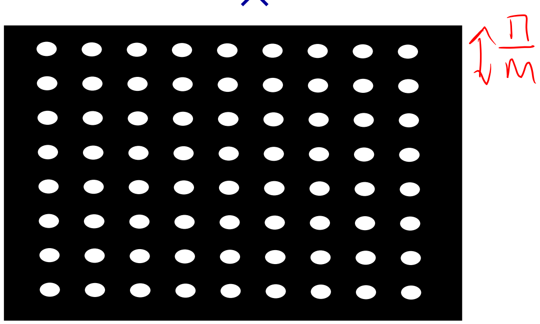

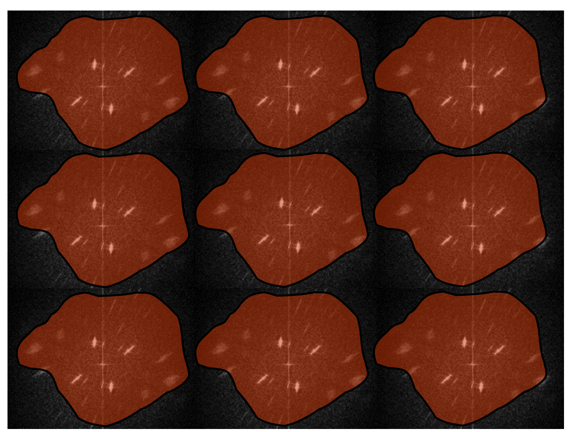

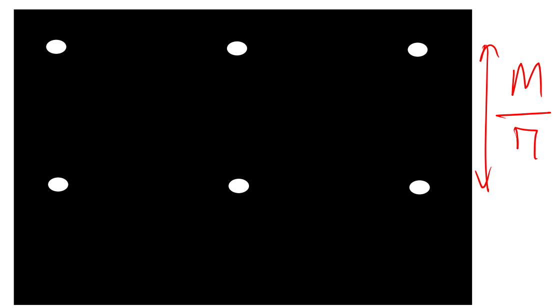

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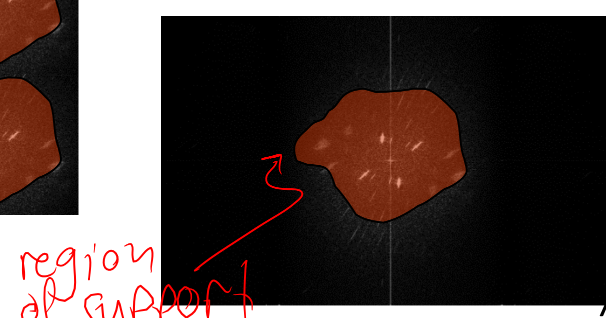
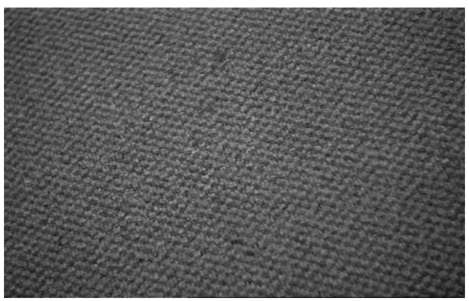


Aliasing

If overlap occurs in $F\{B \cdot S\}$ not possible to reconstruct $F\{B\}$ any longer



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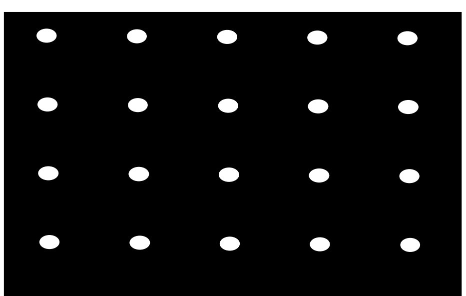
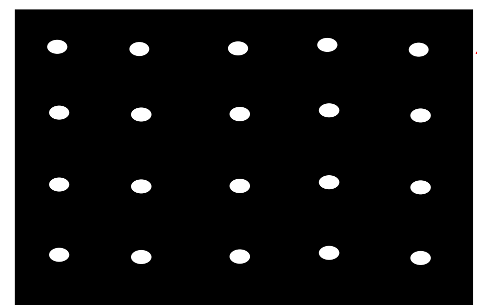
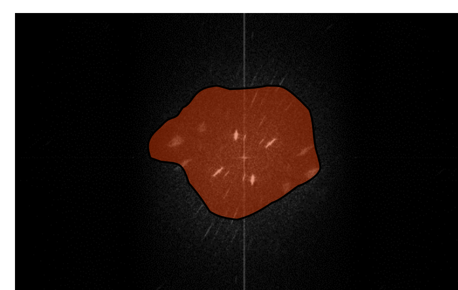


$\frac{2\pi}{M}$



$\frac{M}{2\pi}$

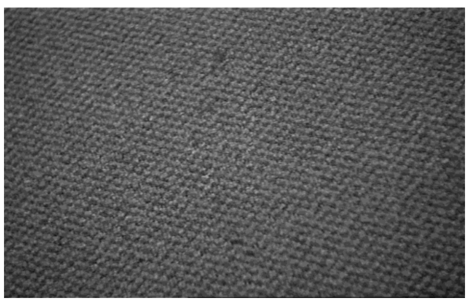
$*$



Nyquist Sampling Theorem

Let $f(x)$ be a band-limited signal such that $F_w\{f\}=0$ for $|w|>w_0$

* Then f uniquely determined by its samples $g(n)=f(n\cdot\Delta x)$ when $\frac{2\pi}{\Delta x}>2w_0$



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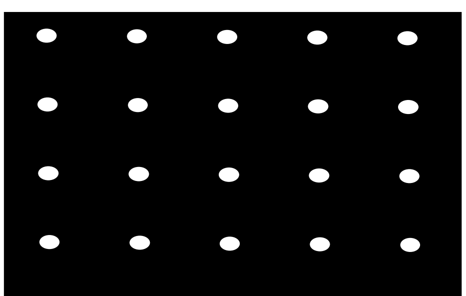
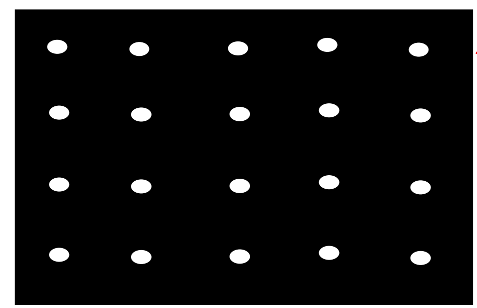
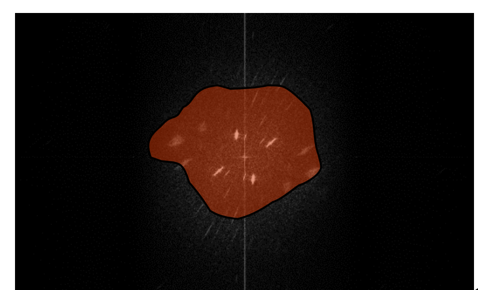


$\frac{2\pi}{M}$



$\frac{M}{2\pi}$

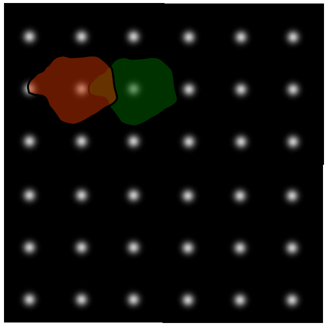
*



The Source of Aliasing…

The end result

- High frequencies are “masked” (aliased) as lower frequencies



Topic 05:

Linear Filters & Fourier Analysis

- Filter-based view of image formation
- Linear systems & 1D convolution
- Example 1D filters
- Filtering in 2D
- The Fourier series
- Sampling & aliasing
- Discrete-time filters & the DFT

Covered in detail in the lecture notes