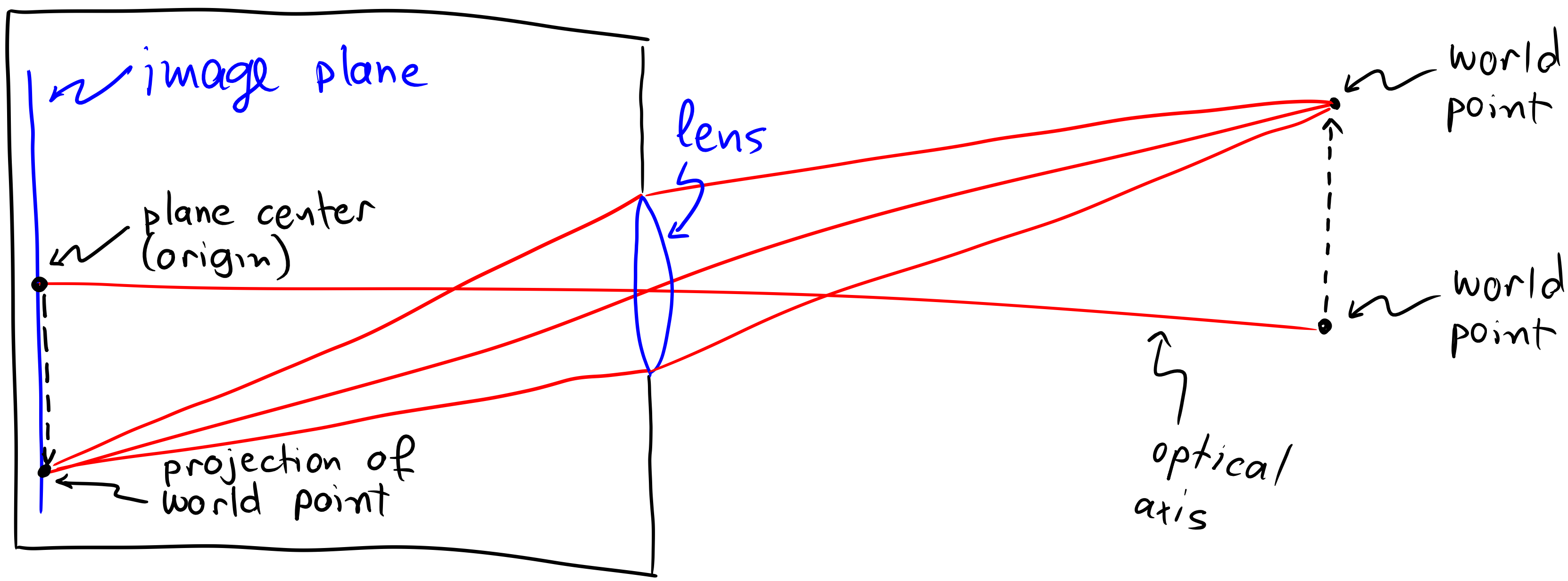
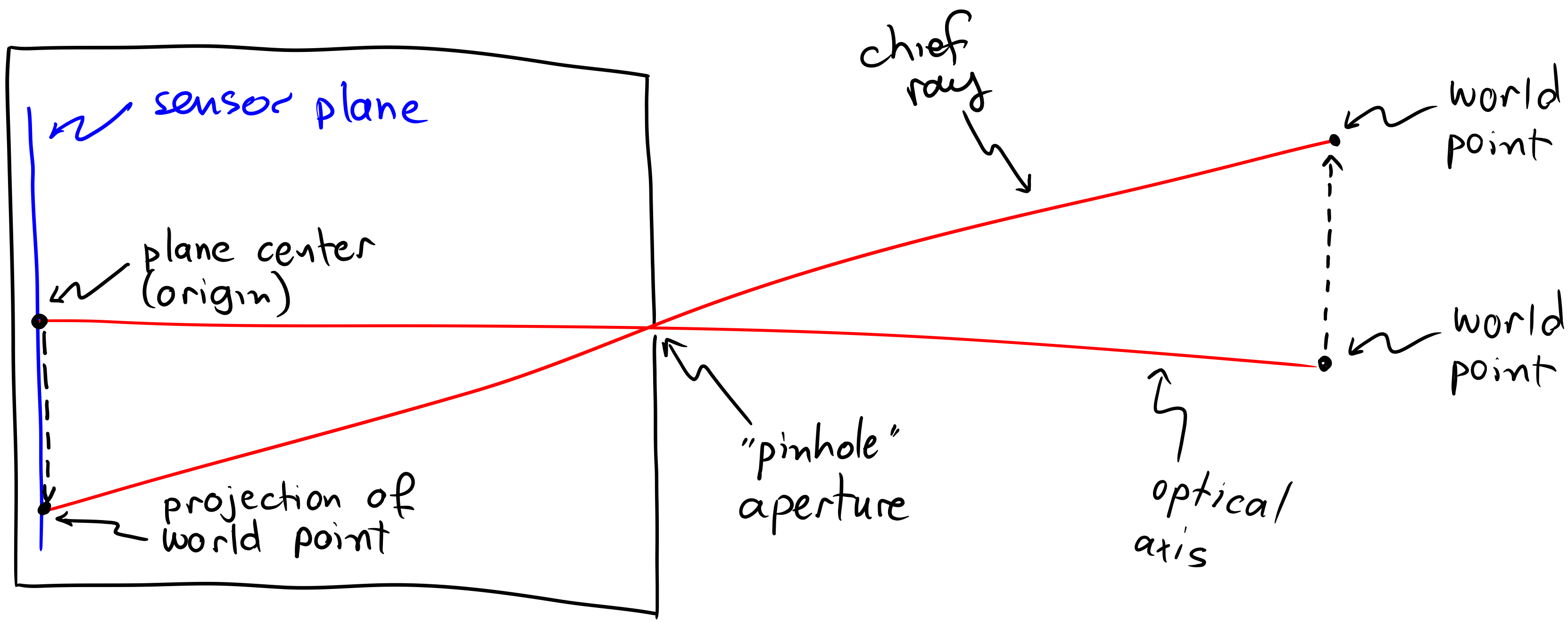
Geometry of Image Projection

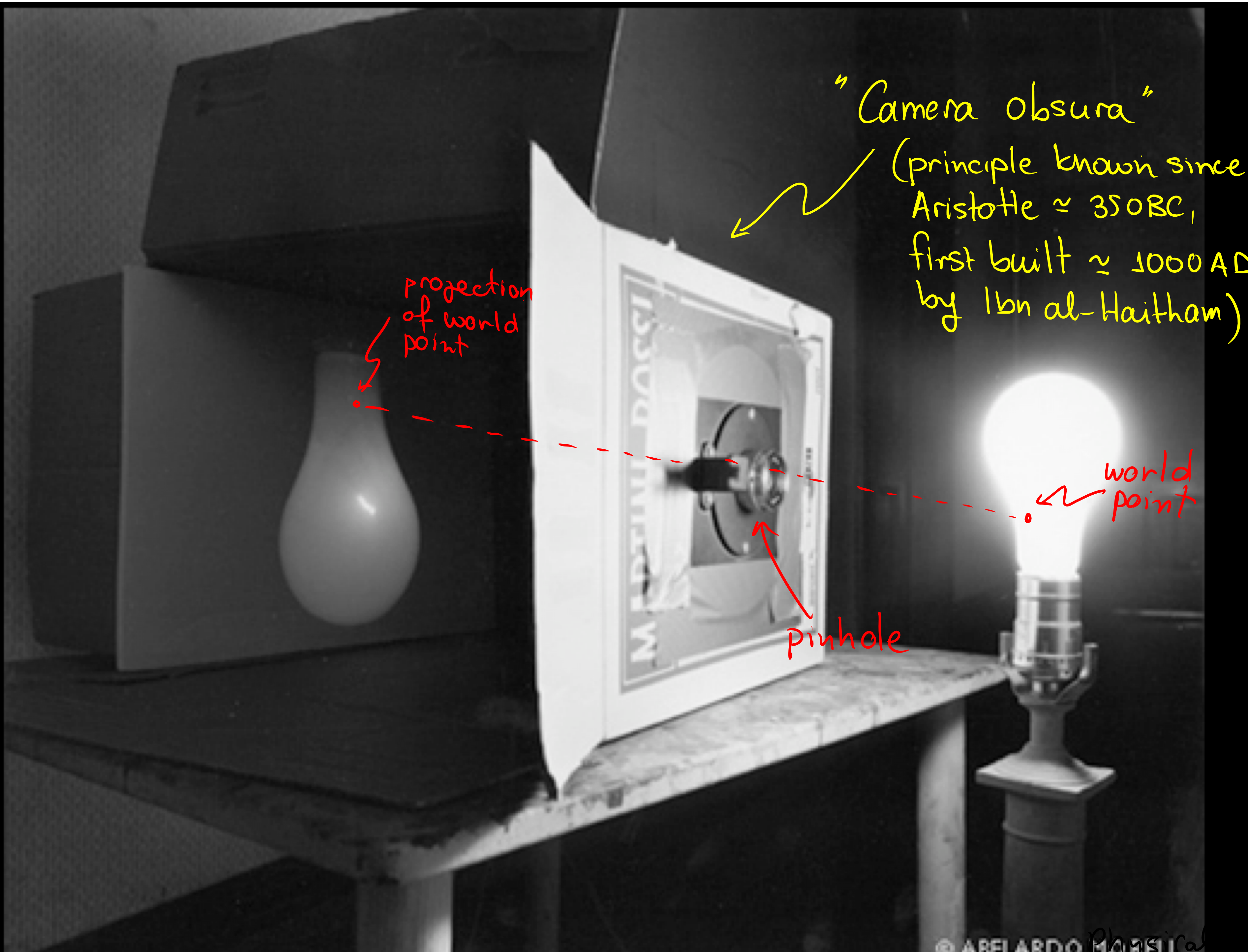
- Perspective projection
- Homogeneous coordinates in 2D
- Homogeneous coordinates in 3D
- Projection matrices
- Geometric transformations
- Parallelism in homogeneous coordinates
- Orthographic projection

Simple Lens-Based Camera



The Pinhole Camera

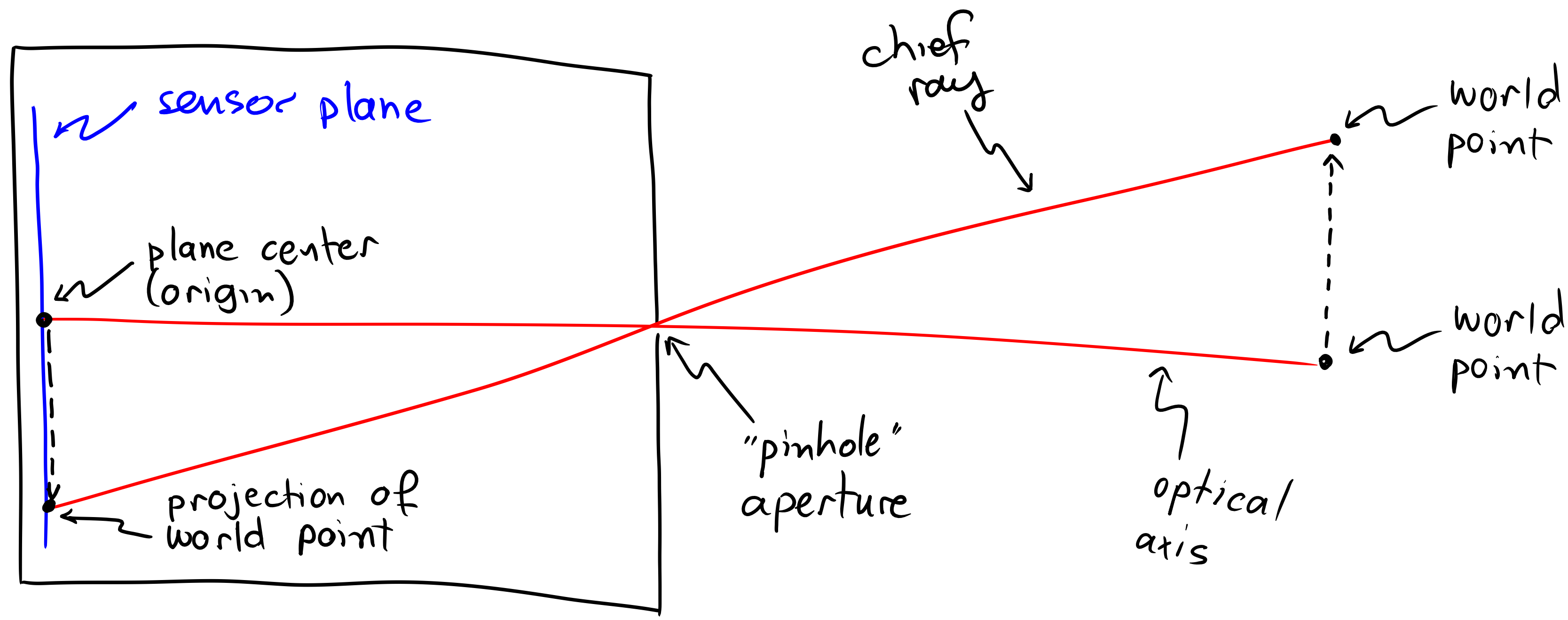




Light Bulb, 1991

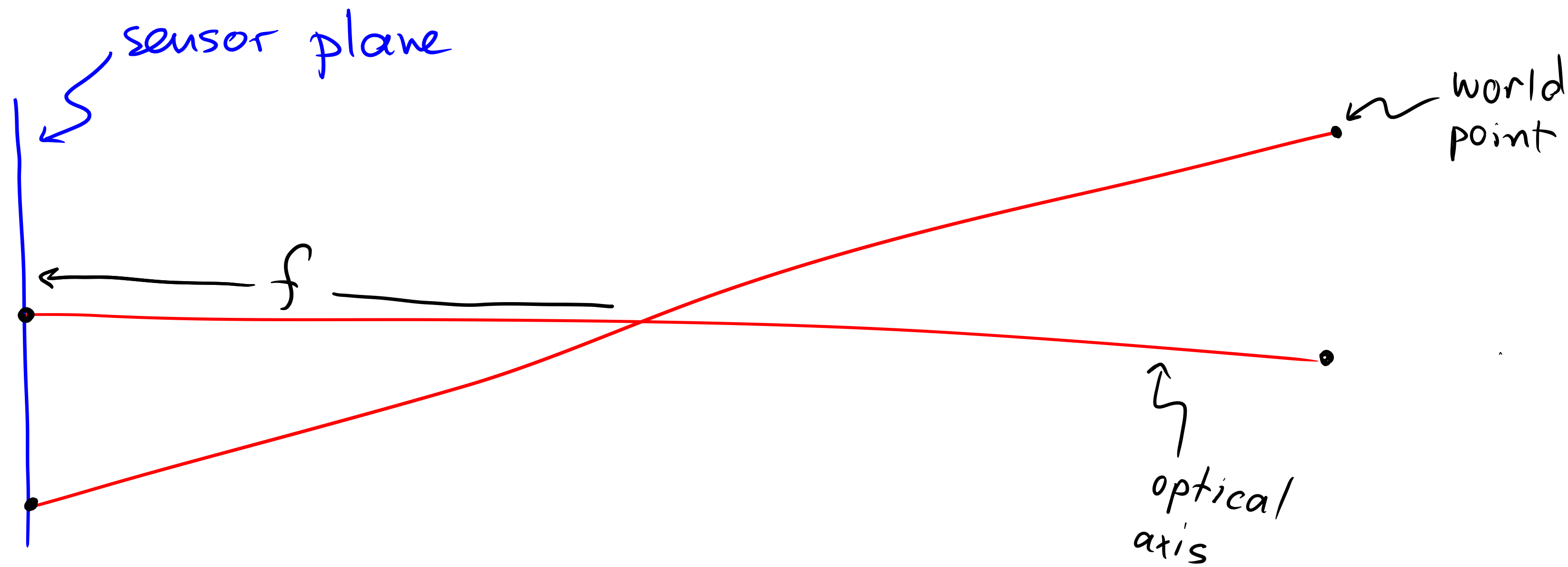
www.lensculture.com/morell.html

The Pinhole Camera



The Pinhole Camera: Basic Geometry in 2D

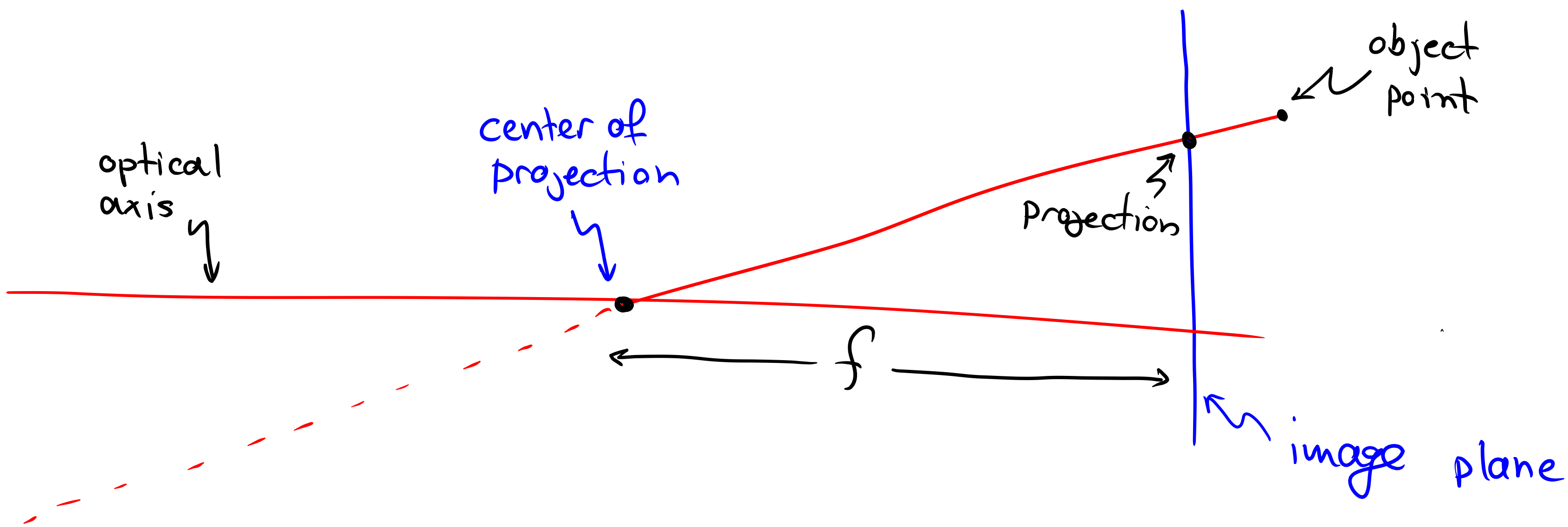
We will first consider the idealized pinhole model (a.k.a. perspective projection)



Convention : Use f for the pinhole-to-image distance

The Pinhole Camera: Basic Geometry in 2D

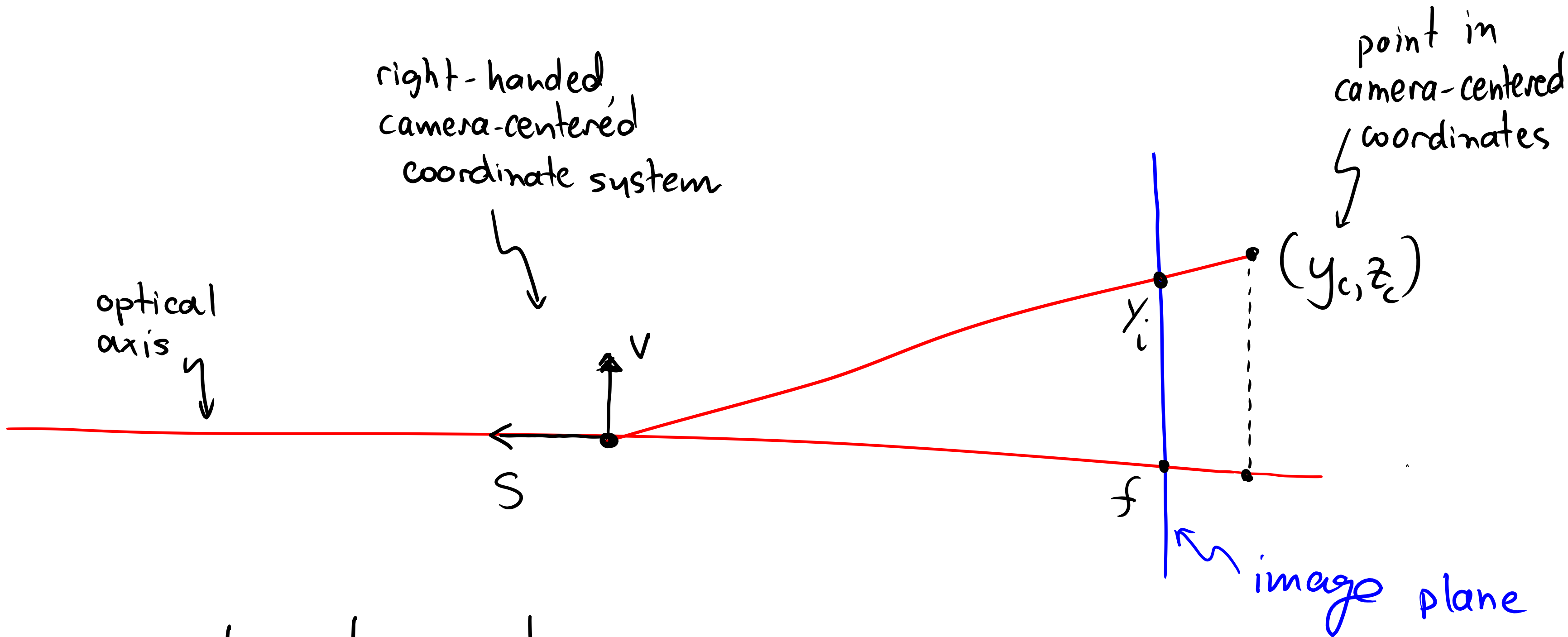
We will only consider the idealized pinhole model here (a.k.a. perspective projection)



Convention : Use f for the pinhole-to-image distance

Simplification : "Undo" image reversal by placing viewing plane in front of pinhole

The Perspective Projection Equation in 2D

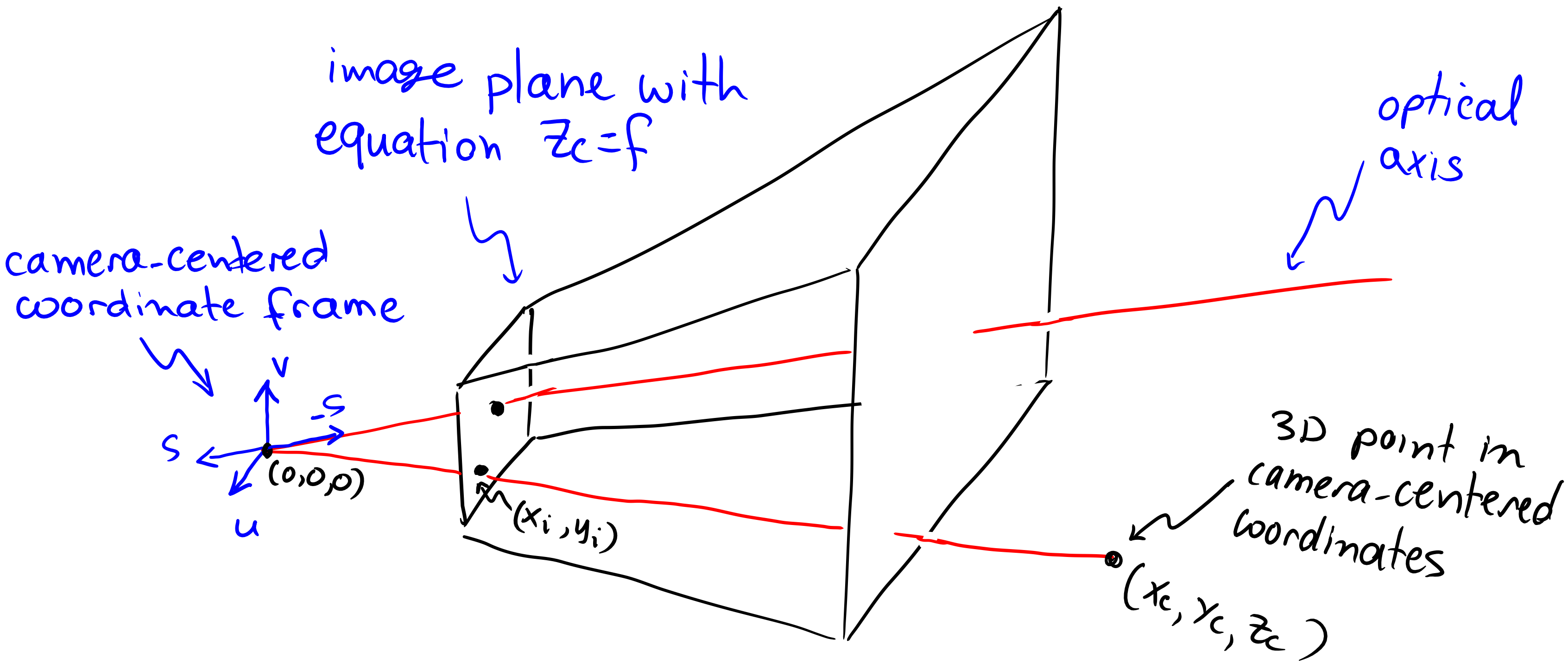


From similar triangles,

$$\frac{y_c}{z_c} = \frac{y_i}{f} \implies \boxed{y_i = \frac{f}{z_c} y_c}$$

The perspective projection equation

The Perspective Projection Equations in 3D



From similar triangles,

$$\frac{Y_c}{Z_c} = \frac{Y_i}{f}$$

analogously, for x_i

$$\frac{X_c}{Z_c} = \frac{X_i}{f}$$

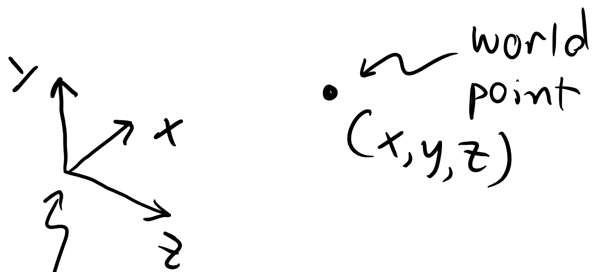
$\Rightarrow$

$$\boxed{X_i = \frac{f}{Z_c} X_c \qquad Y_i = \frac{f}{Z_c} Y_c}$$

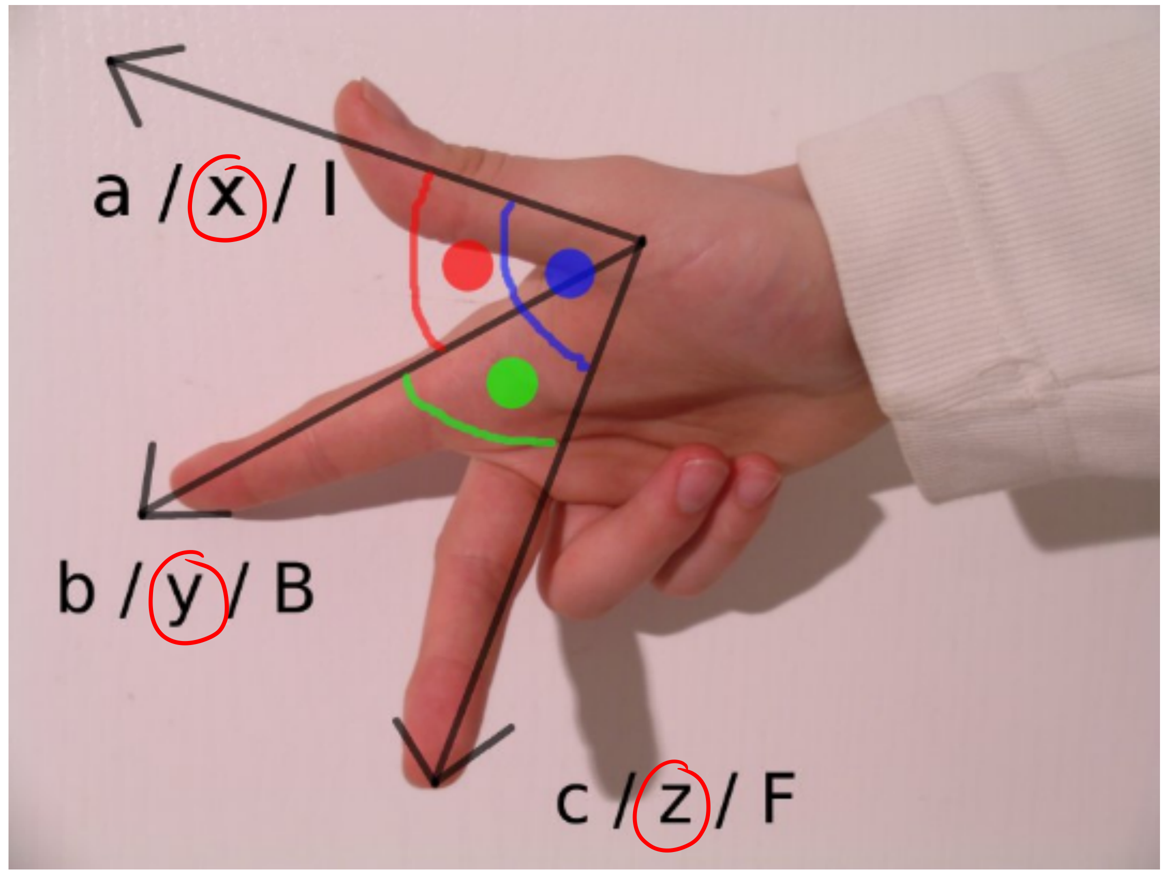
The perspective projection equations

The Camera-Centered Coordinate System

The right-hand rule

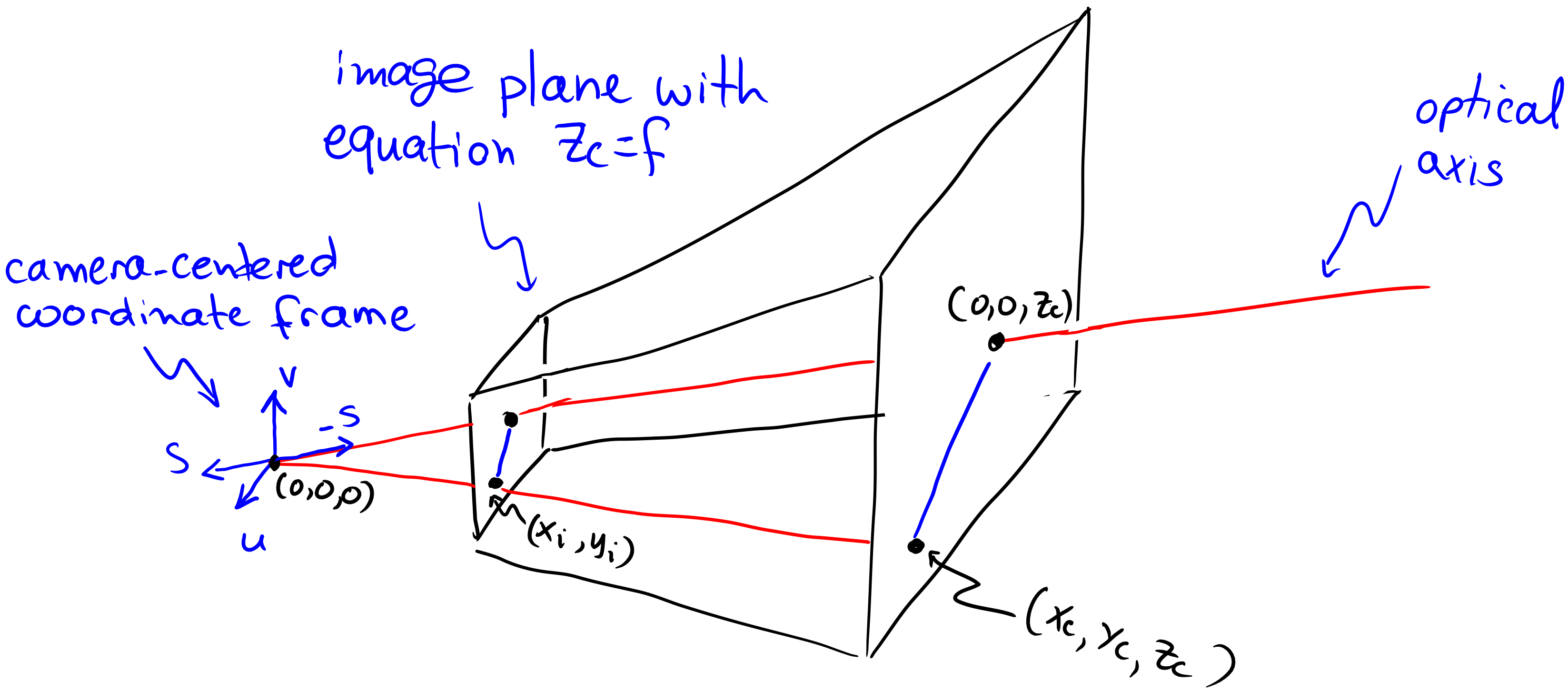


right-handed
coordinate
system



rechte-hand-regel.jpg (wikipedia.com)

The Perspective Projection Equations in 3D



From similar triangles,

$$\frac{y_c}{z_c} = \frac{y_i}{f}$$

analogously, for x_i

$$\frac{x_c}{z_c} = \frac{x_i}{f}$$

$\Rightarrow$

$$\boxed{x_i = \frac{f}{z_c} x_c \qquad y_i = \frac{f}{z_c} y_c}$$

$\Rightarrow$ As objects move farther away (i.e. $|z_c|$ increases) their projection gets smaller and smaller


frankfurt airport tunnel (wikipedia.com)
object-to-camera
distance affects an
object's projection

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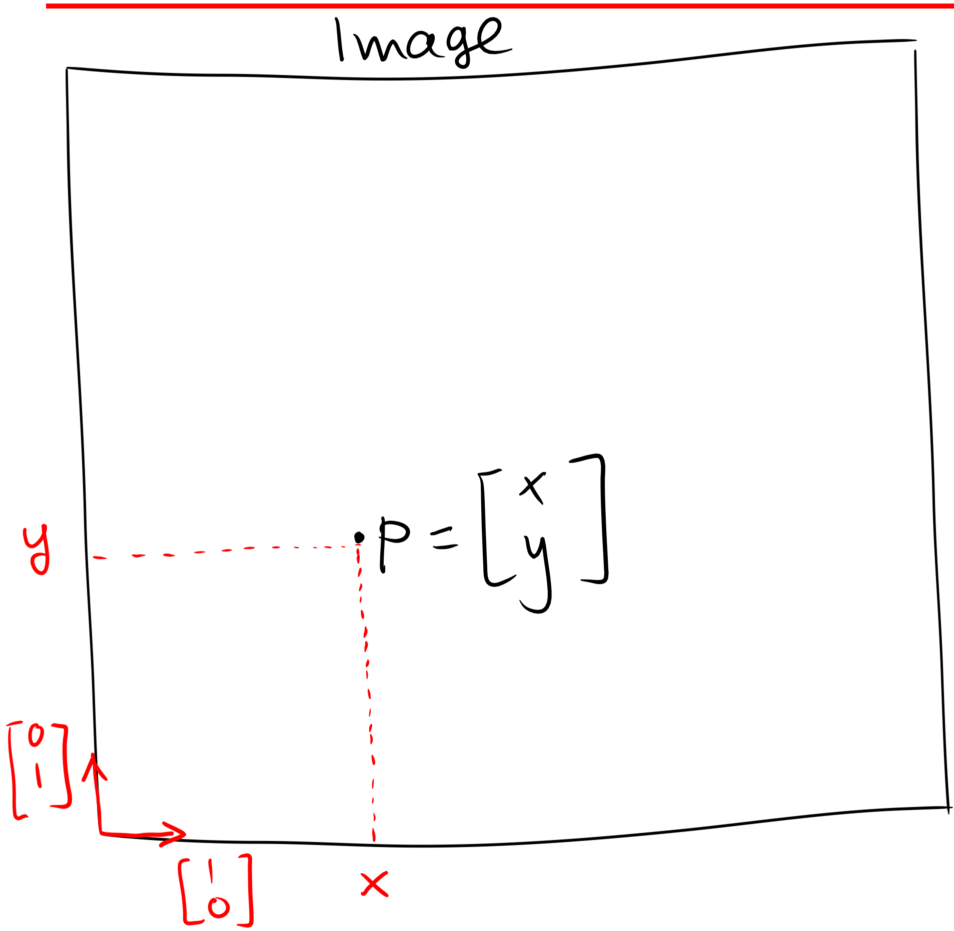
Representing Pixels by Euclidean 2D Coordinates



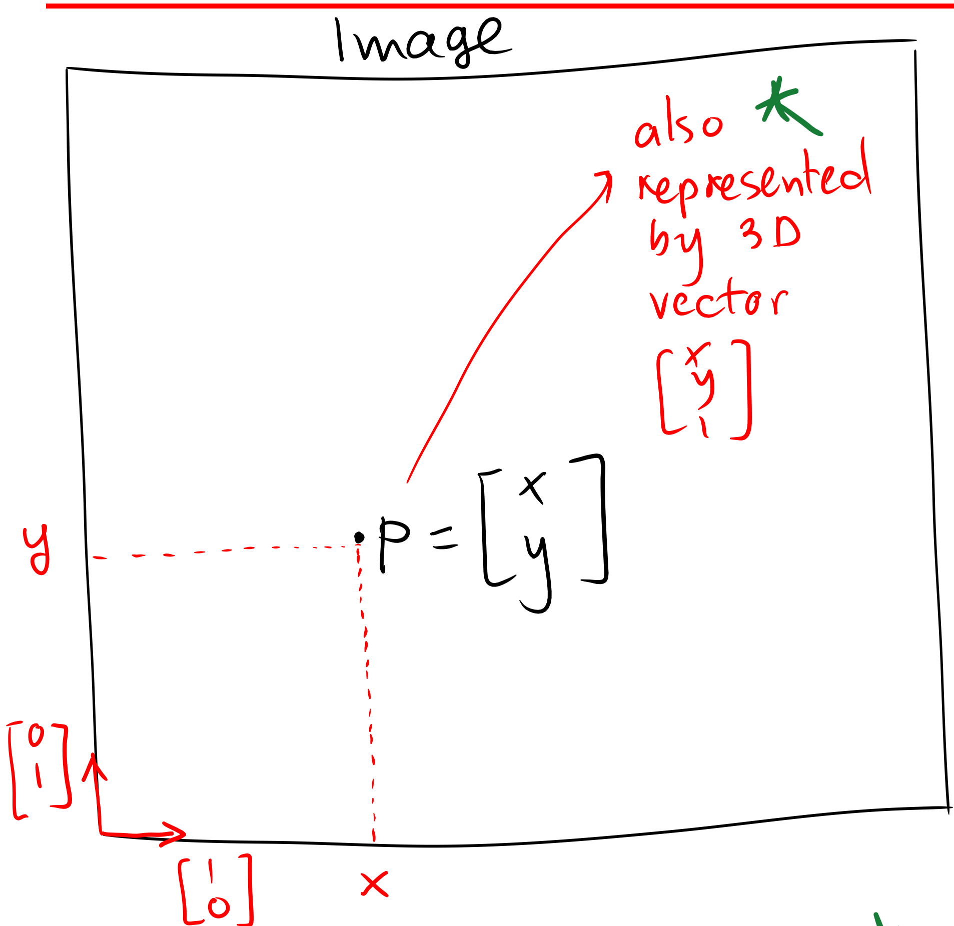
- "Standard" (Euclidean) representation of an image point p:

$$p = x \cdot \begin{bmatrix}1\\0\end{bmatrix} + y \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

basis vectors

Euclidean coordinates

Euclidean Coordinates $\Rightarrow$ Homogeneous Coordinates



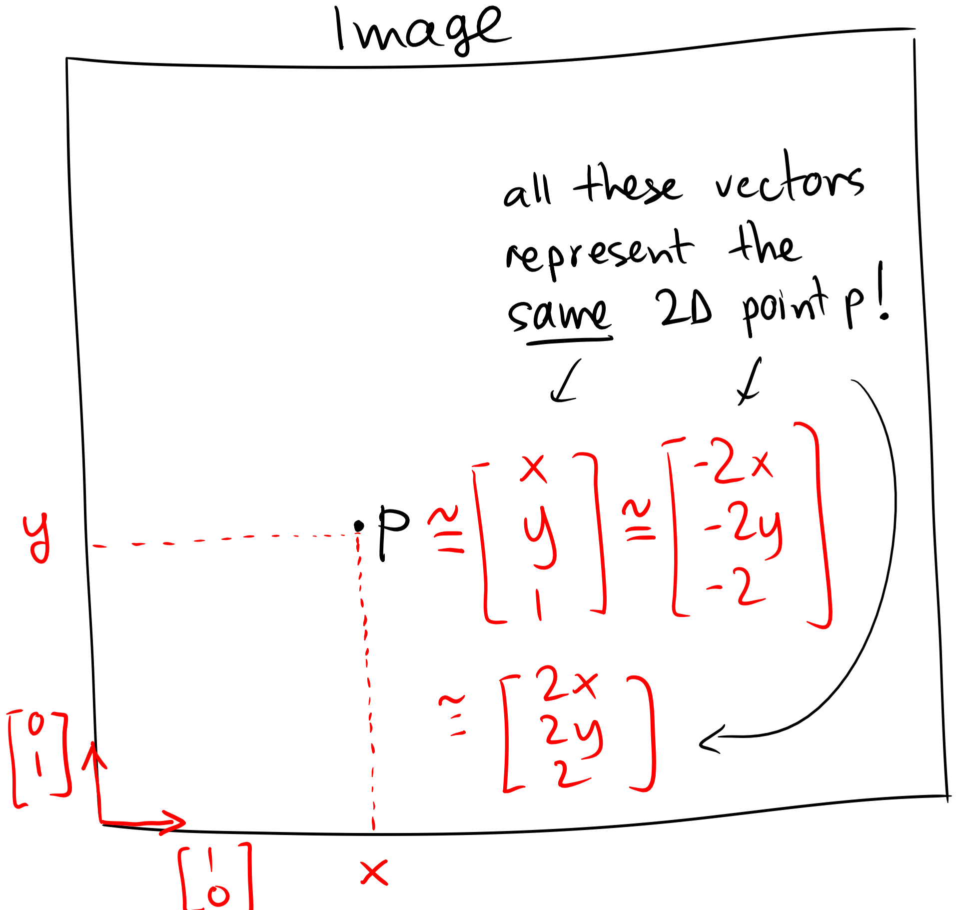
- "Standard" (Euclidean) representation of an image point p:

$$p = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Homogeneous (a.k.a. Projective) representation of p

image coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$ $\longrightarrow$ homogeneous 2D coordinates $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

2D Homogeneous Coordinates: Definition



- For any $\lambda \neq 0$, the numbers $\lambda x, \lambda y, \lambda$ are called the homogeneous coordinates of point p

Definition:

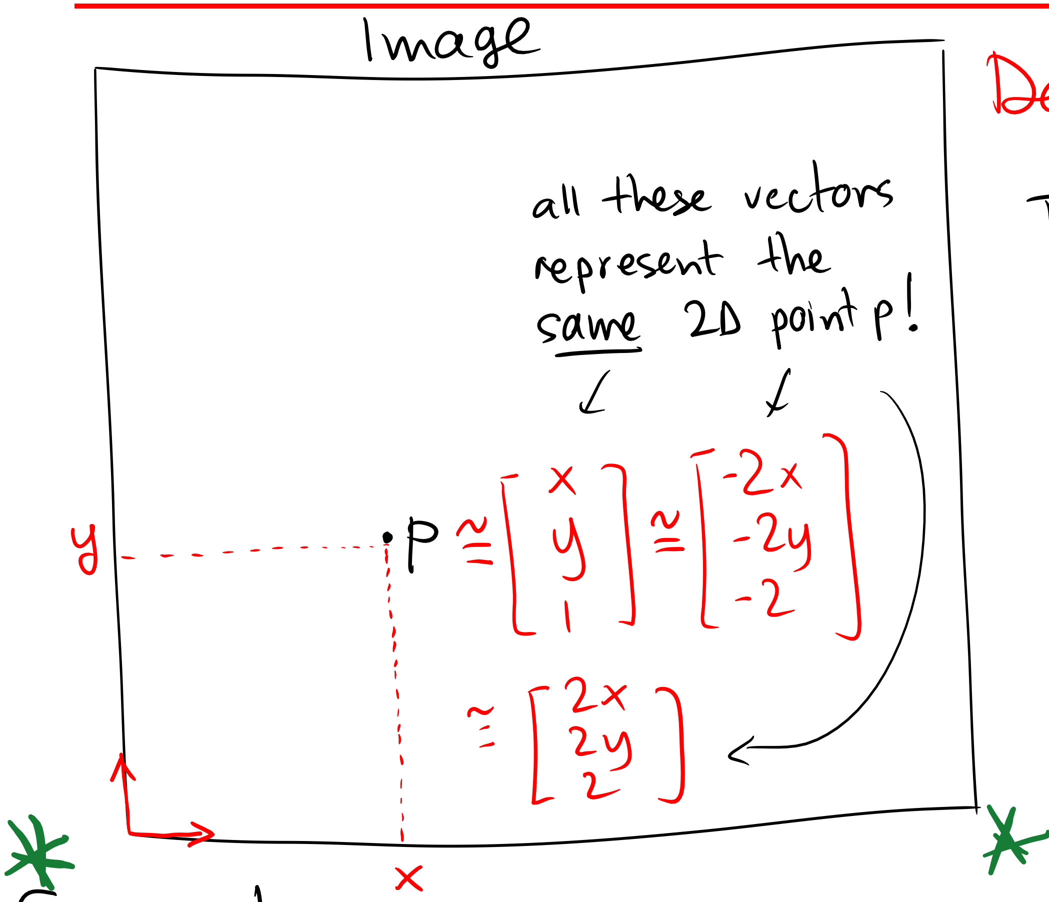
Homogeneous representation of p

p represented by any 3D vector $\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}$ with $\lambda \neq 0$

- Homogeneous (a.k.a. Projective) representation of p

image coordinates	homogeneous 2D coordinates
$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow$	$\begin{bmatrix} \lambda \cdot x \\ \lambda \cdot y \\ \lambda \cdot 1 \end{bmatrix}$ $\lambda \neq 0$

2D Homogeneous Coordinates: Equality



$$p \cong \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cong \begin{bmatrix} -2x \\ -2y \\ -2 \end{bmatrix} \cong \begin{bmatrix} 2x \\ 2y \\ 2 \end{bmatrix}$$

Definition (Homogeneous Equality)

Two vectors of homogeneous coords $v_1 = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and $v_2 = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$ are called <u>equal</u> if they represent the <u>same</u> 2D point:

$v_1 \cong v_2$ — denotes homog. equality

$\Longleftrightarrow$

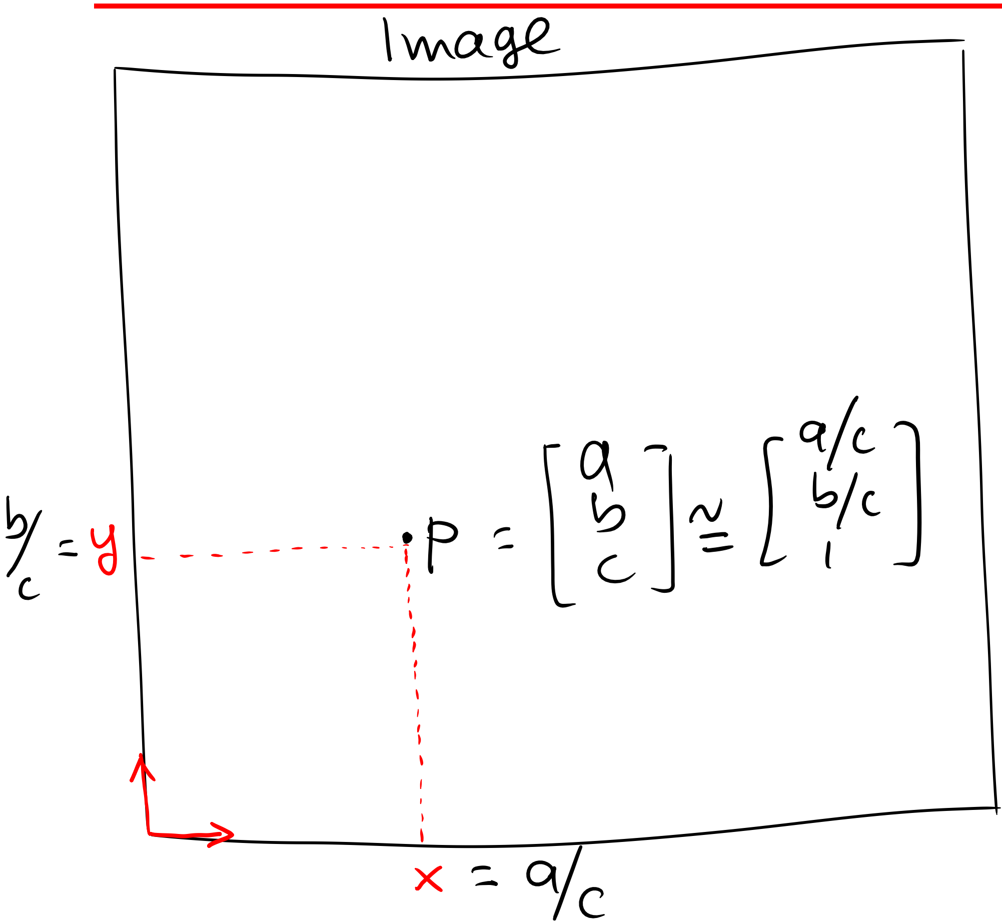
there is a $\lambda \neq 0$ such that

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

Examples:

- Is $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \cong \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$? yes (take $\lambda = 2$)
- Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix}$? yes (take $\lambda = 30$)
- Is $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cong \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$? <u>no</u>!

Homogeneous Coordinates ⇒ Euclidean Coordinates



Converting from homogeneous to Euclidean coordinates:

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a/c \\ b/c \\ 1 \end{bmatrix}$ represent the same 2D point

$\iff$ 2D coordinates are $\begin{bmatrix} a/c \\ b/c \end{bmatrix}$

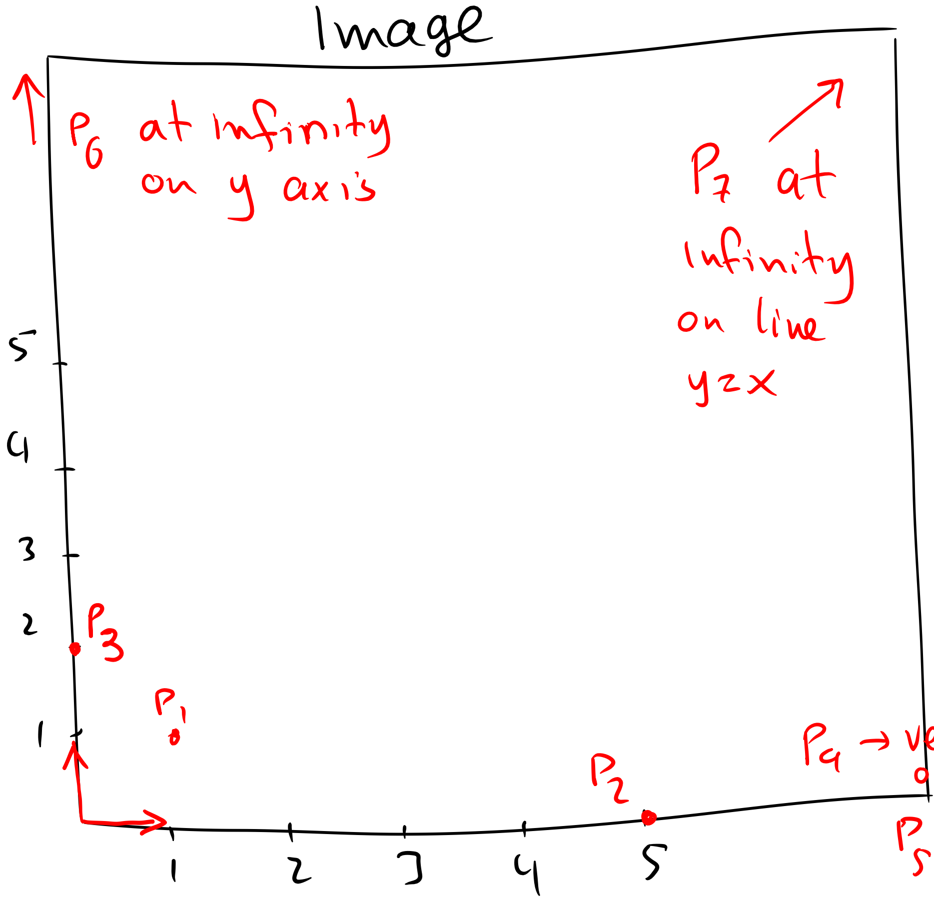
$$v_1 \cong v_2$$

$$\iff$$

there is a $\lambda \neq 0$ such that

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

Homogeneous Coordinates ⇒ Euclidean Coordinates



Converting from homogeneous to Euclidean coordinates:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a/c \\ b/c \\ 1 \end{bmatrix}$$ represent the same 2D point

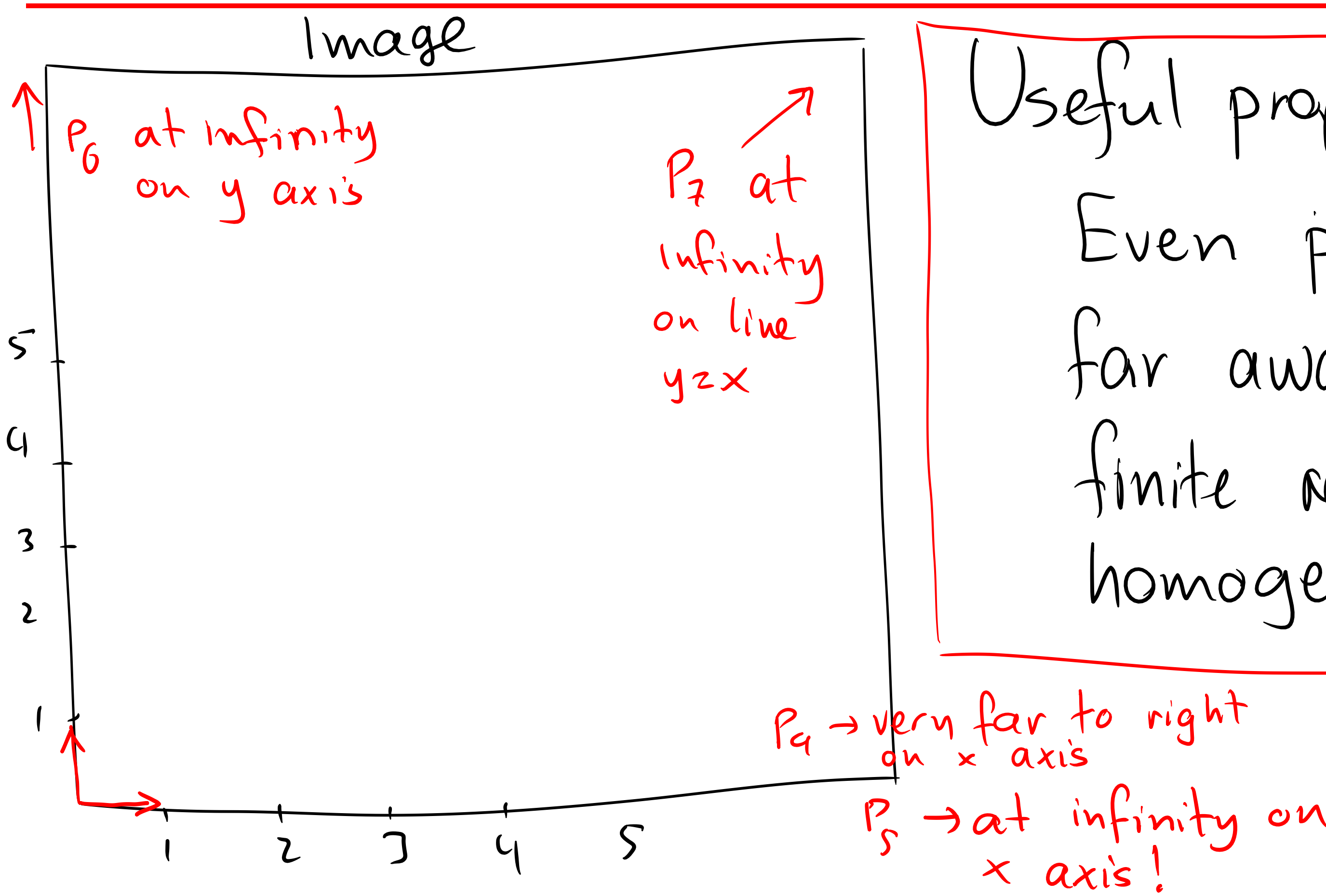
$\Leftrightarrow$ 2D coordinates are $\begin{bmatrix} a/c \\ b/c \end{bmatrix}$

* Practice exercise: Plot positions of the following points

$P_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $P_7 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$P_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $P_2 = \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix}$ $P_3 = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$ $P_4 = \begin{bmatrix} 1 \\ 0 \\ 0.0001 \end{bmatrix}$ $P_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Points at ∞ in Homogeneous Coordinates *



Useful property #1:
Even points infinitely far away have a finite representation in homogeneous coords!

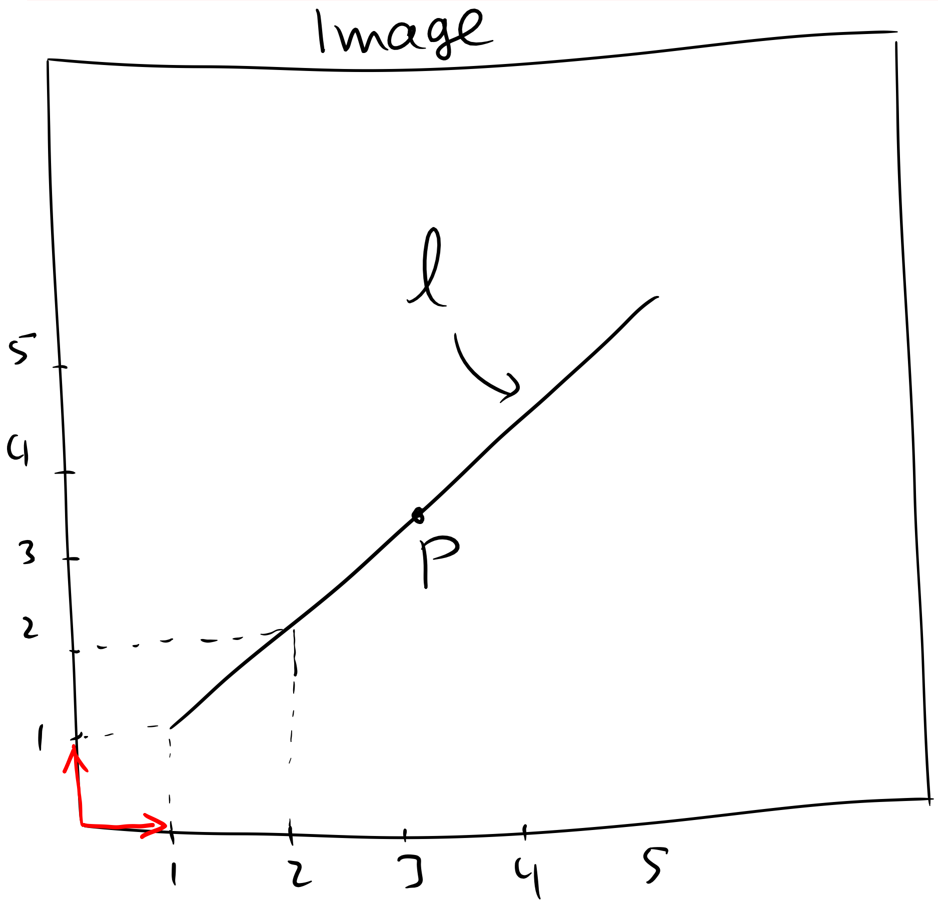
leads to very stable geometric computations

Points at infinity have their last coordinate equal to zero

$P_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad P_7 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$P_4 = \begin{bmatrix} 1 \\ 0 \\ 0.0001 \end{bmatrix} \quad P_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Line Equations in Homogeneous Coordinates



Example: line $y=x$ in homogeneous coords:

$$1 \cdot x - 1 \cdot y + 0 \cdot 1 = 0$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}_{\text{line parameters of } l} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- The equation of a line

$$\underbrace{a x + b y + c}_{\text{line parameters: } a,\, b,\, c} = 0$$

- In homogeneous coordinates

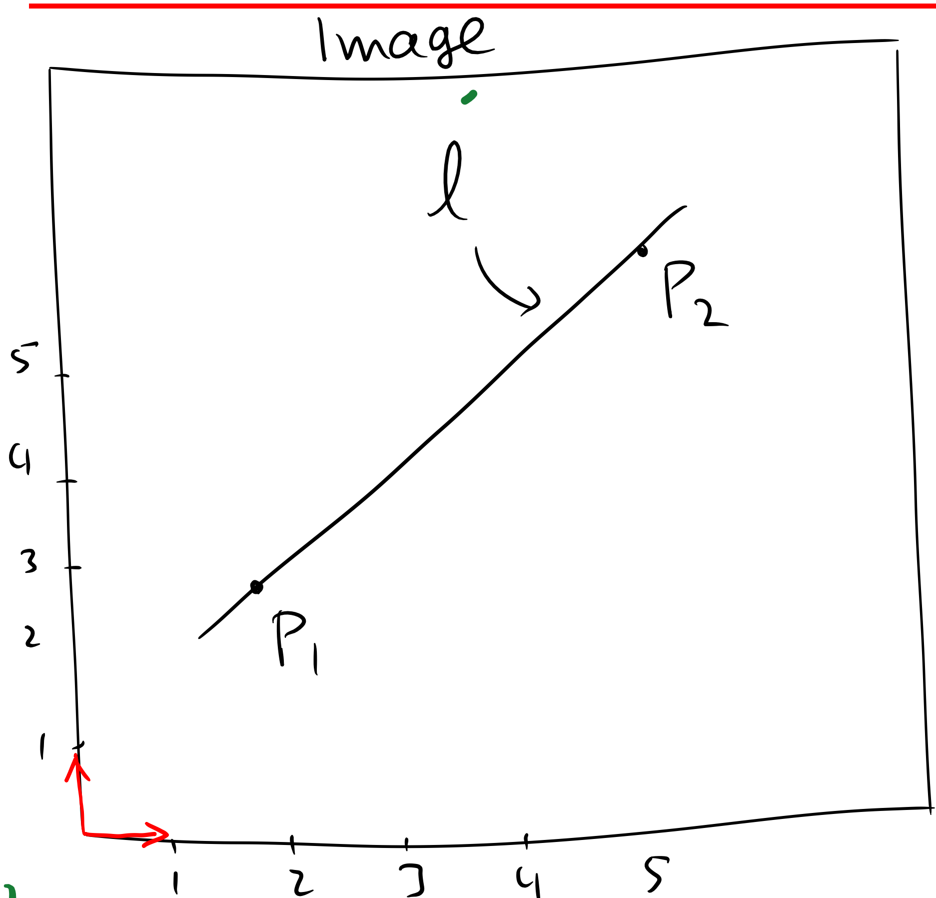
$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or $l^T \cdot p = 0$

l: vector holding line parameters

p: vector holding homogeneous coordinates of a point

The Line Passing Through 2 Points



- Calculating the parameters of a line through two points with homogeneous coordinates p_1, p_2

$$\boxed{\ell = p_1 \times p_2}$$

cross product of two 3D vectors

- ℓ must satisfy $\ell^T \cdot p_1 = 0$, $\ell^T \cdot p_2 = 0$
- taken as 3D vectors, ℓ is perpendicular to both p_1 and p_2

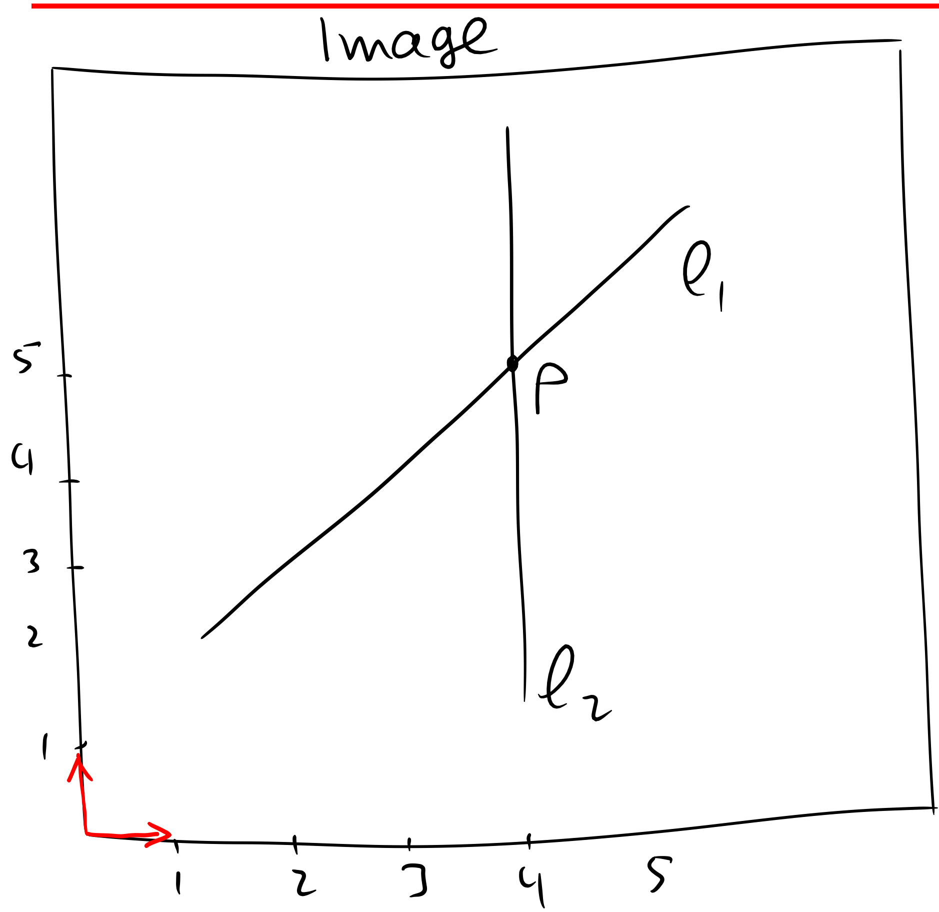
$\Rightarrow$ it is along the cross product, $p_1 \times p_2$

- In homogeneous coordinates

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or $\ell^T \cdot p = 0$

The Point of Intersection of Two Lines



- Calculating the homogeneous coordinates of the intersection of two lines l_1, l_2

$$\boxed{p = l_1 \times l_2}$$

cross product of two 3D vectors

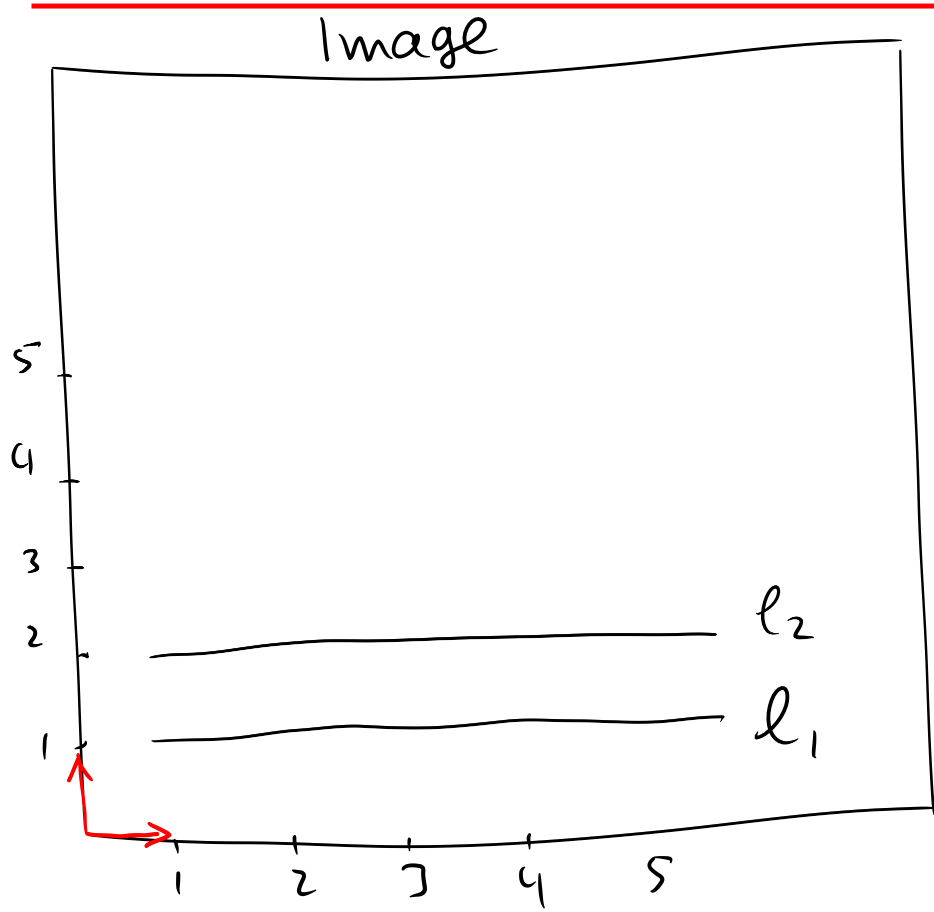
- In homogeneous coordinates

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or $l^T \cdot p = 0$

- p must satisfy $l_1^T p = 0$, $l_2^T p = 0$
- taken as 3D vectors, p is perpendicular to both l_1 and l_2
  $\Rightarrow$ it is along the cross product, $l_1 \times l_2$

Computing the Intersection of Parallel Lines



Calculating the homogeneous coordinates of the intersection of two lines ℓ_1, ℓ_2

$$\boxed{p = \ell_1 \times \ell_2}$$

cross product of two 3D vectors

This calculation works even when ℓ_1, ℓ_2 are parallel!

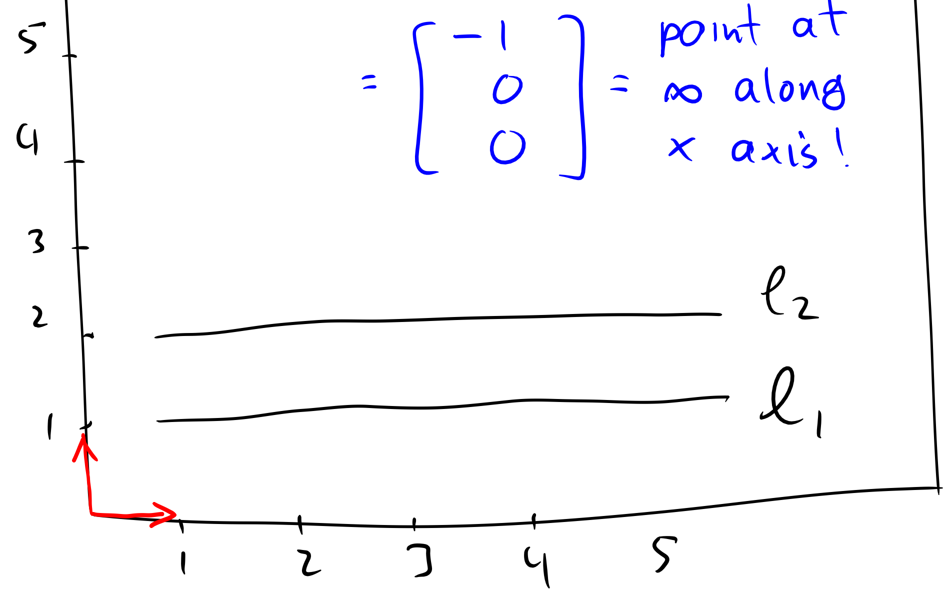
(no floating point exceptions or divide-by-zero errors!)

Computing the Intersection of Parallel Lines

Image

$$\ast\; l_1 \times l_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \text{point at } \infty \text{ along } x \text{ axis!}$$



· Calculating the homogeneous coordinates of the intersection of two lines l_1, l_2

$$p = l_1 \times l_2$$

↑ cross product of two 3D vectors

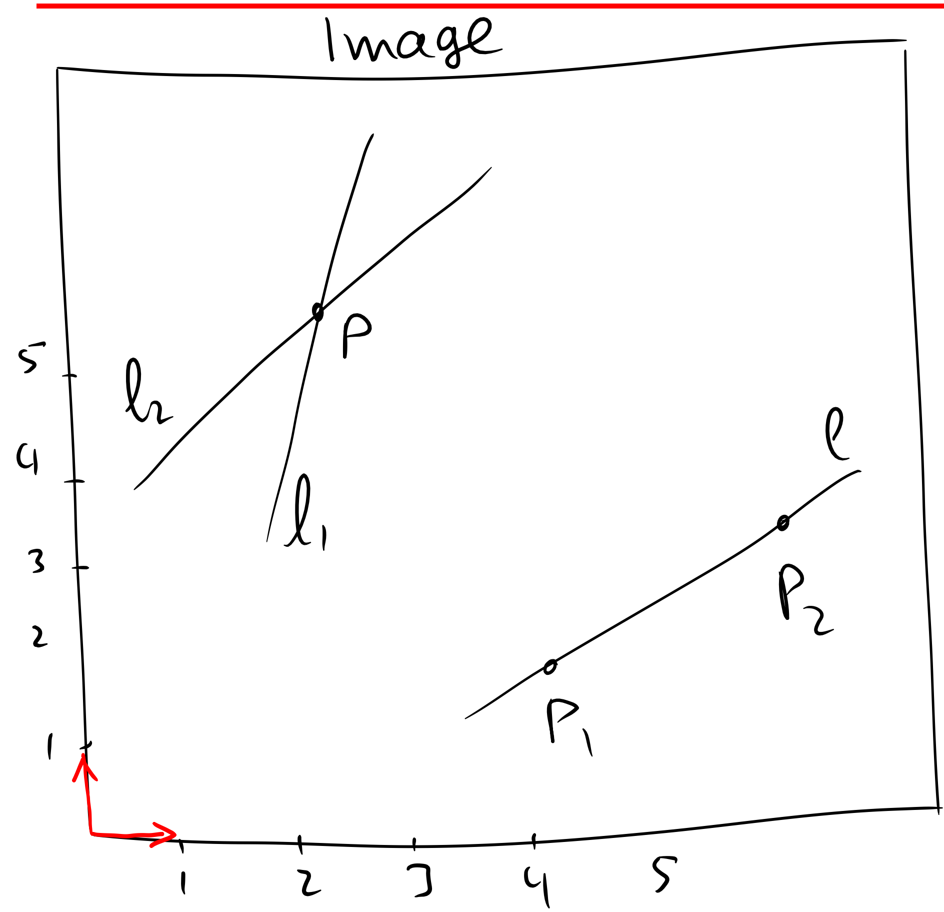
· Line eq. of l_1 is $y=1$. Also written as $0 \cdot x + 1 \cdot y - 1 = 0$. So $l_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

· Similarly $l_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

Aside (calculating cross products): If $l_1 = (a, b, c)$ then

$$l_1 \times l_2 = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} l_2$$

Lines from Points & Points from Lines



Useful property #2

- Very simple way of computing & intersecting lines
- Numerical stability even when result is at ∞

Line through 2 points

$$l = P_1 \times P_2 = \begin{bmatrix} 0 & -P_1^z & P_1^y \\ P_1^z & 0 & -P_1^x \\ -P_1^y & P_1^x & 0 \end{bmatrix} \begin{bmatrix} P_2^x \\ P_2^y \\ P_2^z \end{bmatrix}$$

Intersection of 2 lines

$$P = l_1 \times l_2 = \begin{bmatrix} 0 & -l_1^z & l_1^y \\ l_1^z & 0 & -l_1^x \\ -l_1^y & l_1^x & 0 \end{bmatrix} \begin{bmatrix} l_2^x \\ l_2^y \\ l_2^z \end{bmatrix}$$

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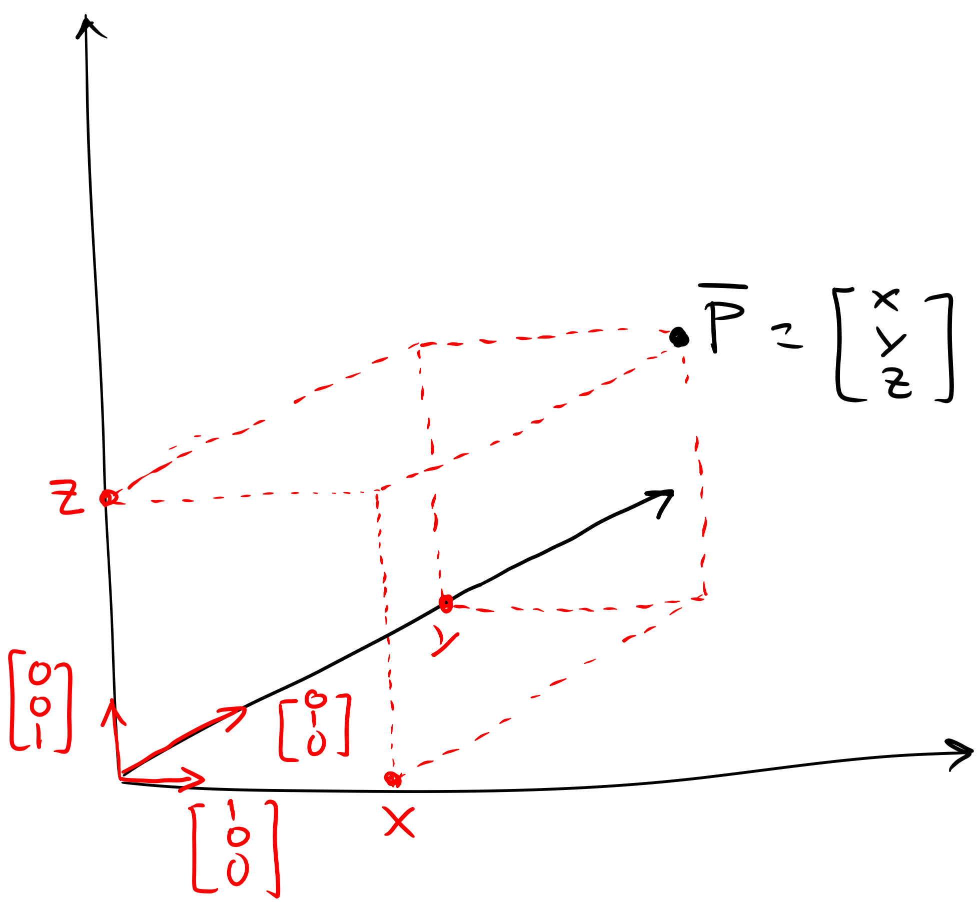
Representing Points by Euclidean 3D Coords



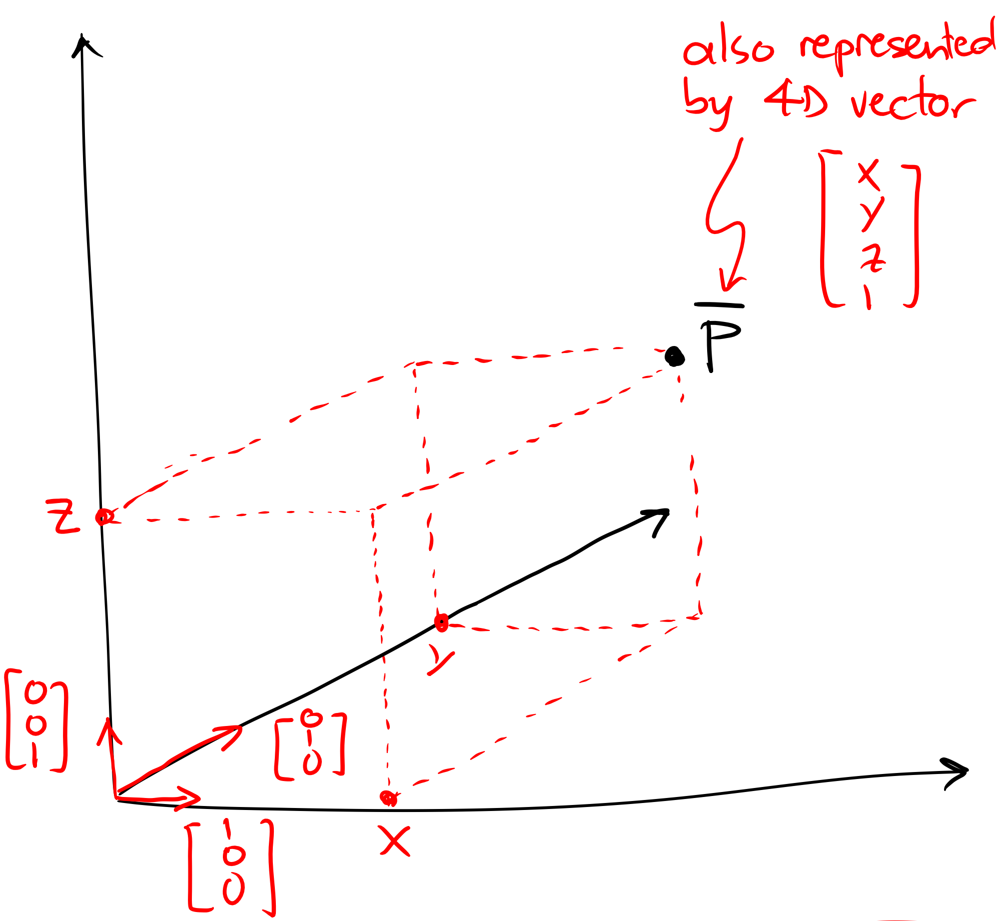
- "Standard" (Euclidean) representation of a point $\bar{p}$:

$$\bar{p} = x \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

basis vectors

Euclidean coordinates

Euclidean Coords ⇒ Homogeneous Coords



- "Standard" (Euclidean) representation of a point $\bar{p}$:

$$\bar{p} = x \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Homogeneous (a.k.a. Projective) representation of $\bar{p}$

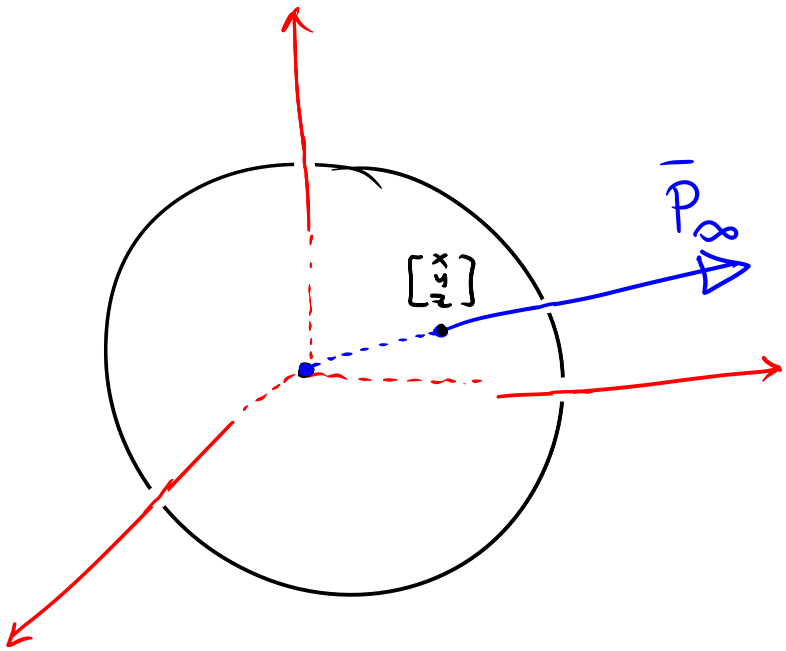
Converting from homogeneous to Euclidean 3D Coords

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \longrightarrow \begin{bmatrix} x/w \\ y/w \\ z/w \end{bmatrix}$$

3D coordinates → homogeneous 3D coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Points at ∞ in Homogeneous Coordinates



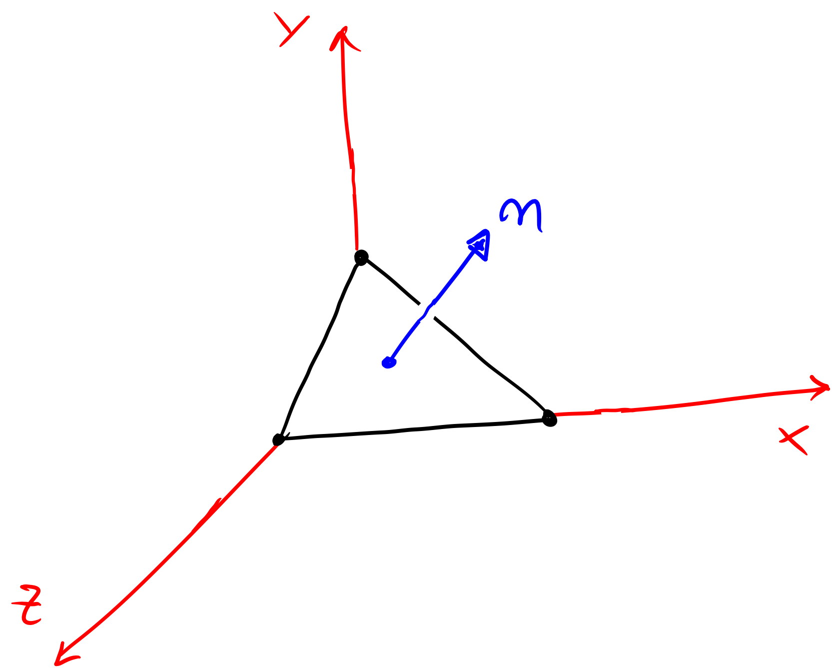
- A point at infinity does not represent a physical location on the plane
- It represents a direction

Points at infinity have their last coordinate equal to zero

$$\bar{P}_\infty = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

i.e. point at ∞ in direction of 3D vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Plane Equation in Homogeneous Coordinates



- The equation of a plane

$$ax+by+cz+d=0$$

plane parameters

- In homogeneous coordinates

$$\begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}=0$$

first 3 coordinates represent direction of the plane's normal, n

or $\bar{l}^T \cdot \bar{p} = 0$

vector holding plane parameters

vector holding homogeneous coordinates of a point

3D Homogeneous Coordinates: Definition

If l_1, l_2 are two planes,
their intersection is the set of points
p for which

$$\left.\begin{aligned} l_1^T p &= 0 \\ l_2^T p &= 0 \end{aligned}\right\} \quad [l_1 \; l_2] p = 0$$

→ p defines the null space of the corresponding linear system.

→ when l_1, l_2 are parallel, all points p will be on a line at infinity (show this!) *

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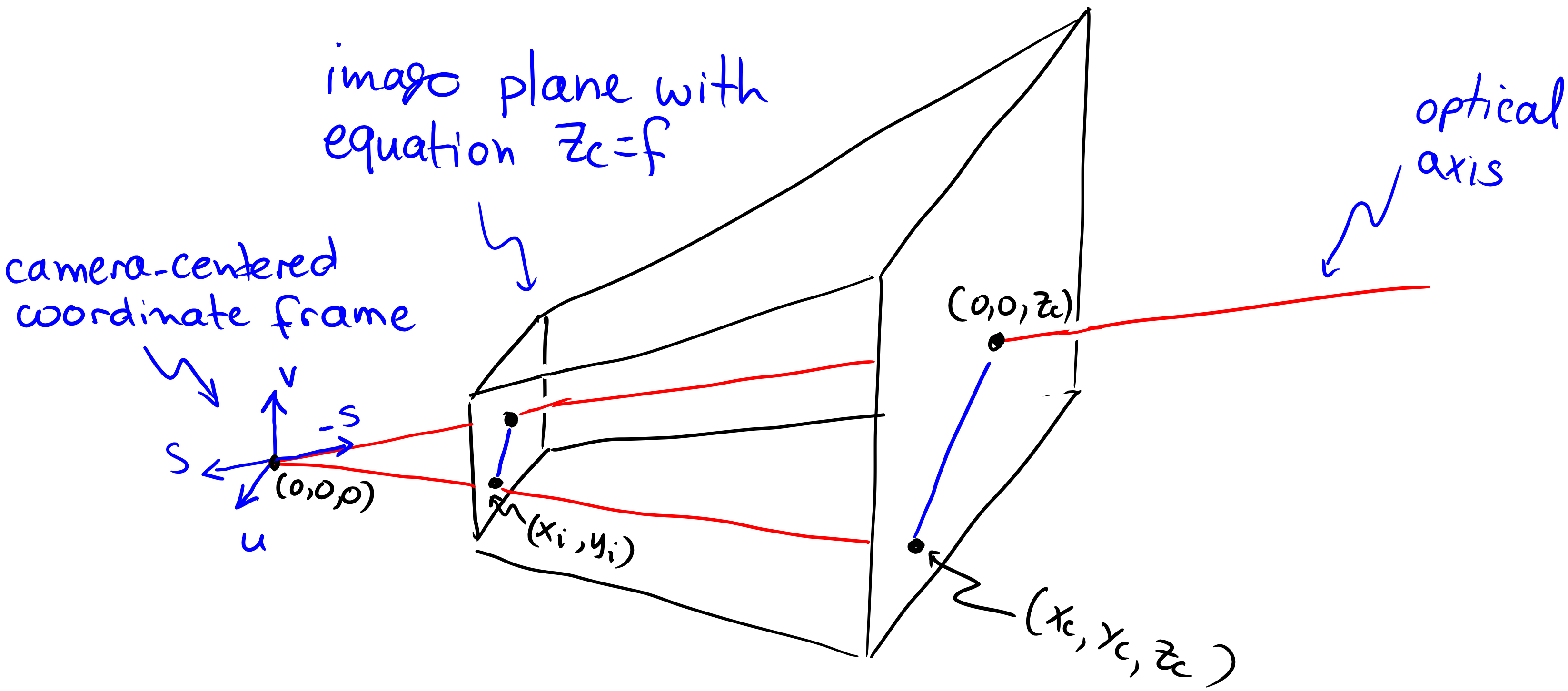
The Perspective Projection Equations in 3D



From similar triangles,

$$\frac{Y_c}{Z_c} = \frac{Y_i}{f}$$

analogously, for x_i

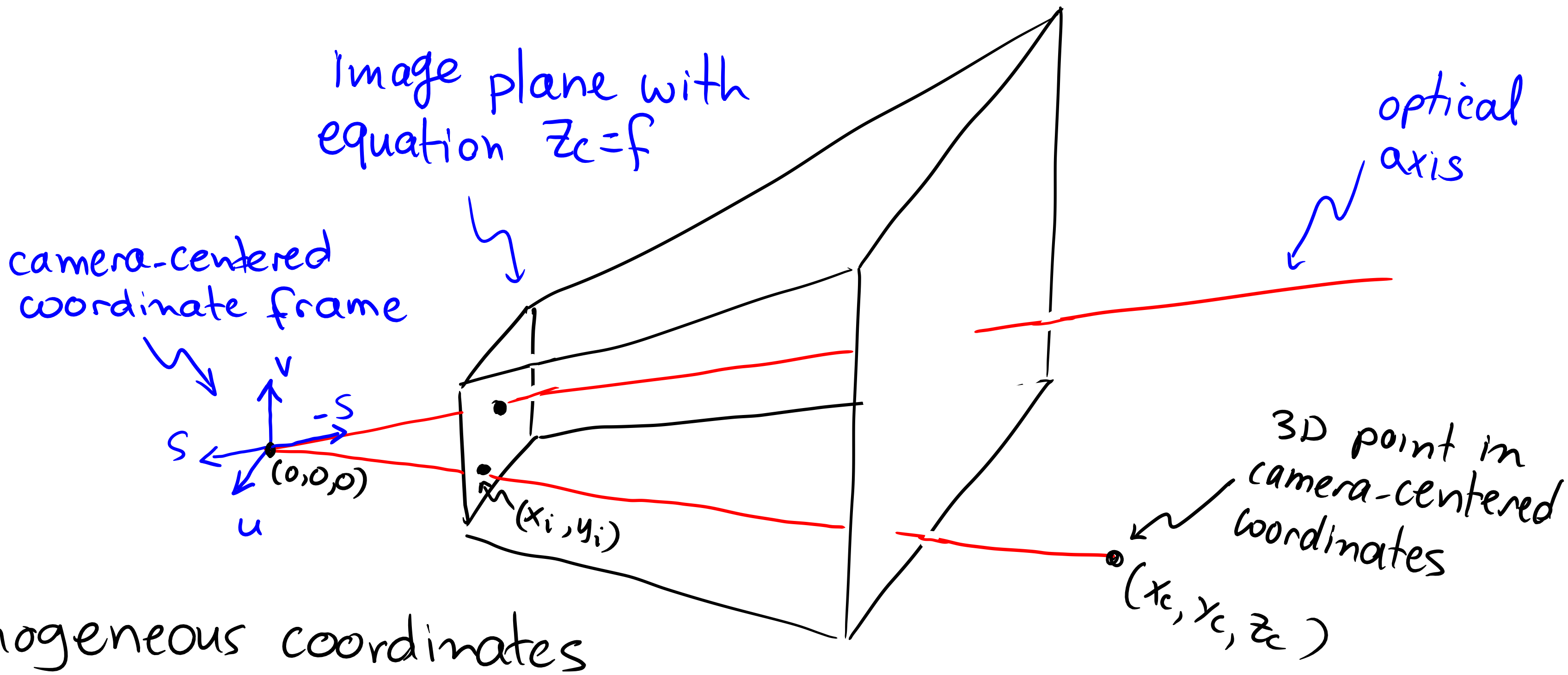
$$\frac{X_c}{Z_c} = \frac{X_i}{f}$$

$\Rightarrow$

$$\boxed{X_i = \frac{f}{Z_c} X_c \qquad Y_i = \frac{f}{Z_c} Y_c}$$

$\Rightarrow$ As objects move farther away (i.e. $|Z_c|$ increases) their projection gets smaller and smaller

Perspective Projection & Homogeneous Coords



In homogeneous coordinates

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{f x_c}{z_c} \\ \frac{f y_c}{z_c} \\ 1 \end{bmatrix} \cong \begin{bmatrix} x_c \\ y_c \\ \frac{z_c}{f} \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_i &= \frac{f}{z_c} x_c \\ y_i &= \frac{f}{z_c} y_c \end{aligned}}$$

Coordinate Systems & Calibration Matrices

- A world-centered coordinate frame (3D)
- A camera-centered coordinate frame (3D)
- An image coordinate frame (2D)

Extrinsic calibration matrix

often the last row of this matrix is dropped making it 3x4

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R_{wc} & e_{wc} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

a 4x4 matrix that maps world-centered coords to camera-centered coords

Intrinsic calibration matrix

often the last column is dropped making it 3x3

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

a 4x3 matrix that maps camera-centered 3D coords to 2D image coords

The Intrinsic Calibration Matrix

The matrix can have a more general form

Example: * a camera with rectangular pixels of size $1/s_x$, $1/s_y$, focal length f piercing point (o_x, o_y) ← intersection of image plane & optical axis

$$M_{c\to i} = \begin{bmatrix} s_x & 0 & o_x/f \\ 0 & s_y & o_y/f \\ 0 & 0 & 1/f \end{bmatrix}$$

Intrinsic calibration matrix $M_{c\to i}$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

a 4x3 matrix that maps camera-centered 3D coords to 2D image coords

Coordinate Systems & Calibration Matrices

- A world-centered coordinate frame (3D)
- A camera-centered coordinate frame (3D)
- An image coordinate frame (2D)

rotation matrix   translation vector

Extrinsic calibration matrix $M_{w \to c}$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R_{wc} & e_{wc} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

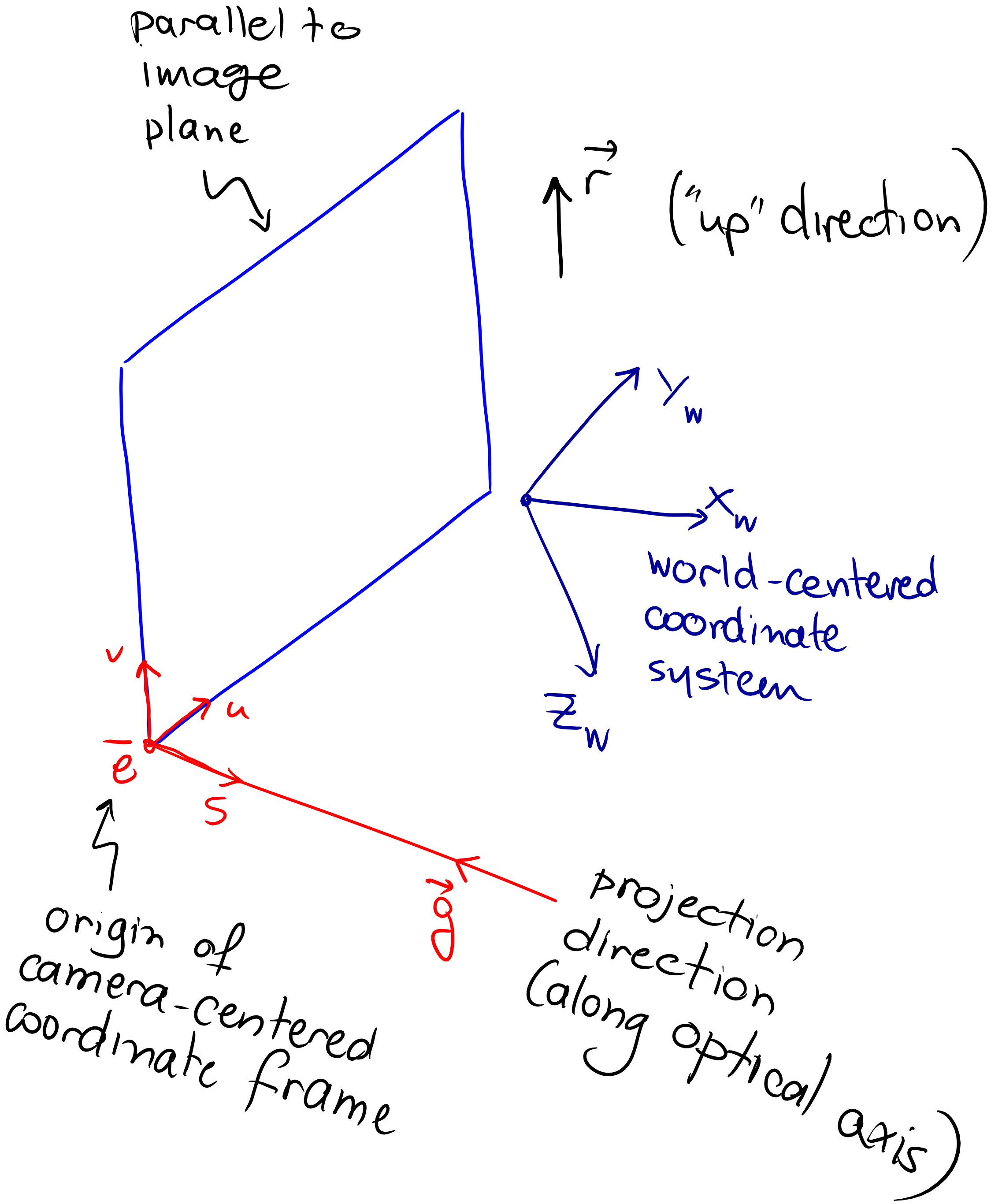
a 4x4 matrix that maps world-centered coords to camera-centered coords

Intrinsic calibration matrix $M_{c \to i}$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

a 4x3 matrix that maps camera-centered 3D coords to 2D image coords

The Camera-to-World Matrix



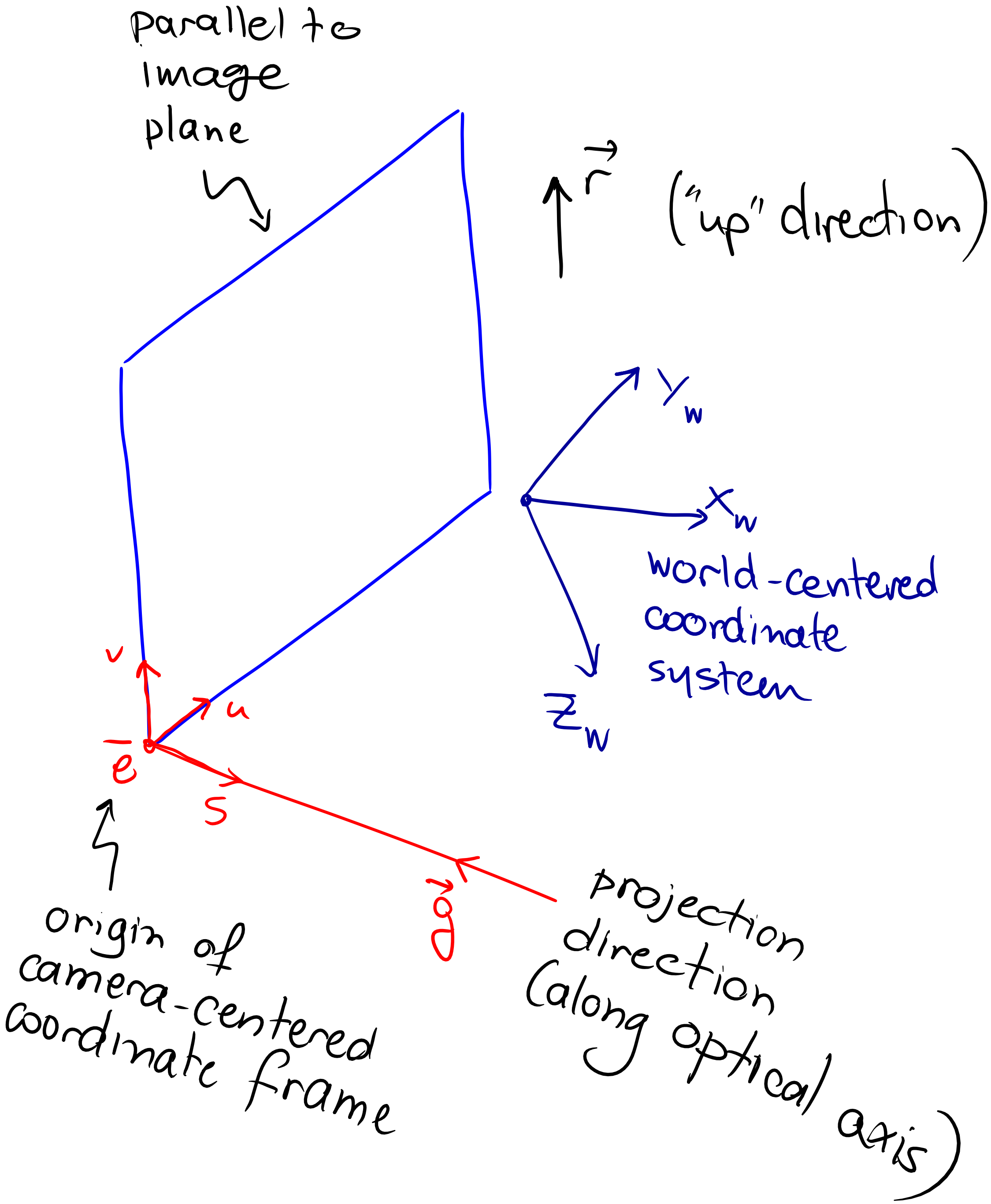
$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} \xleftarrow{M_{w\to c}} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

extrinsic calibration matrix

$$\begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \xrightarrow{M_{c\to w}} \begin{bmatrix} x_c \\ y_c \\ w_c \\ 1 \end{bmatrix}$$

camera-to-world matrix

Computing the Camera-to-World Matrix

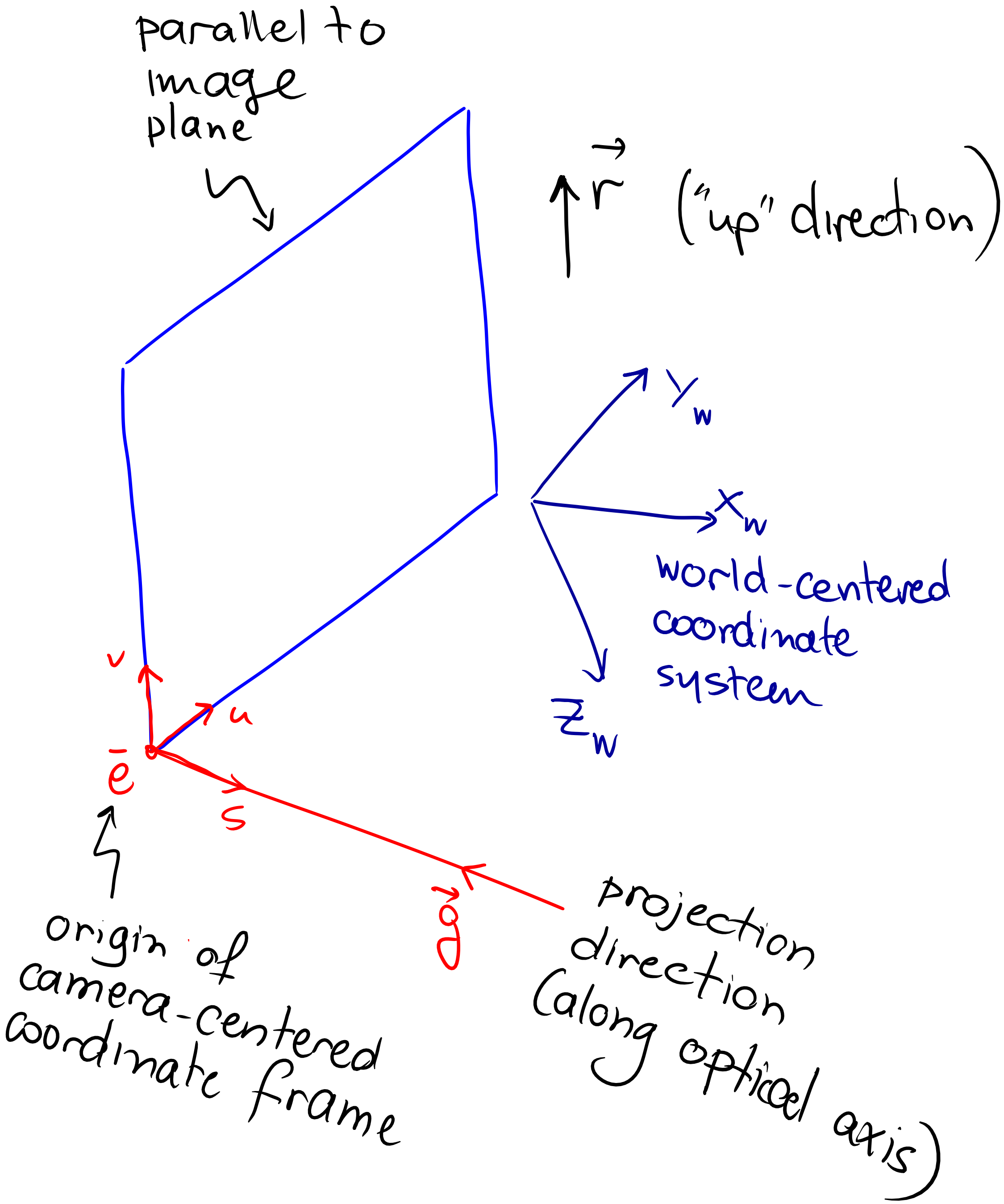


Goal: given ① camera origin $\bar{e}$ and ② a projection direction unit vector $\vec{g}$, compute $M_{c \to w}$

1. Let $\vec{r}$ be the "up" vector in world coords
2. Set $\vec{s} = -\vec{g}$
3. Define 2 perpendicular unit vectors on view plane

$$\boxed{\begin{aligned} \vec{u} &= (\vec{r} \times \vec{s}) / \|\vec{r} \times \vec{s}\| \\ \vec{v} &= (\vec{s} \times \vec{u}) / \|\vec{s} \times \vec{u}\| \end{aligned}}$$

Computing the Camera-to-World Matrix



1. Let $\vec{r}$ be the "up" vector in world coords
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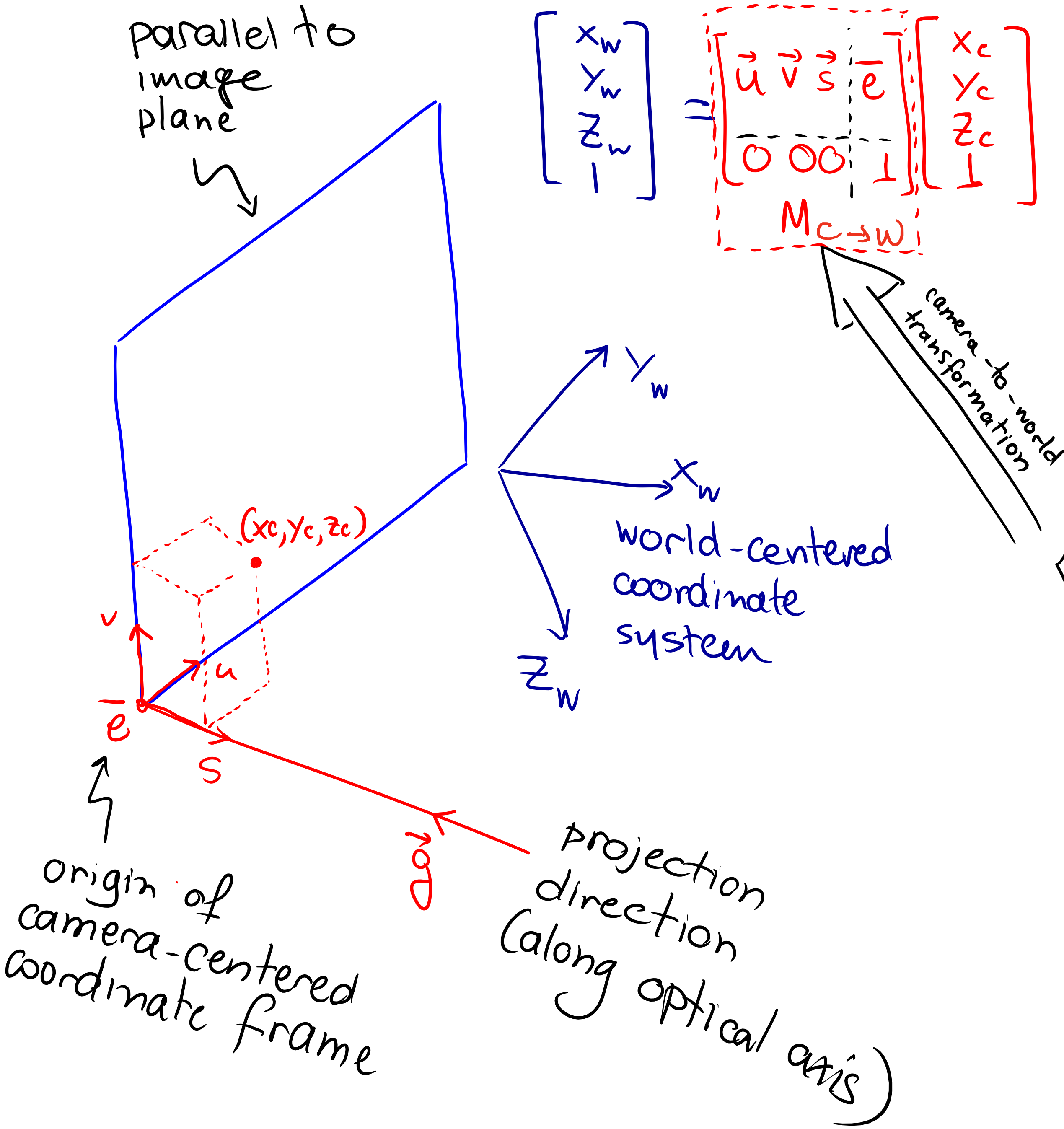
4. Use vectors $\vec{u}, \vec{v}, \vec{s}$ as the right-handed camera coord system

Computing the Camera-to-World Matrix

$$\begin{bmatrix} x_w \\ y_w \\ z_w \\ 1\end{bmatrix} = \underbrace{\begin{bmatrix} \vec{u} & \vec{v} & \vec{s} & \bar{e} \\ 0 & 0 & 0 & 1\end{bmatrix}}_{M_{c\to w}} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1\end{bmatrix}$$



Camera coords $\xrightarrow{\text{camera-to world}}$ world coords

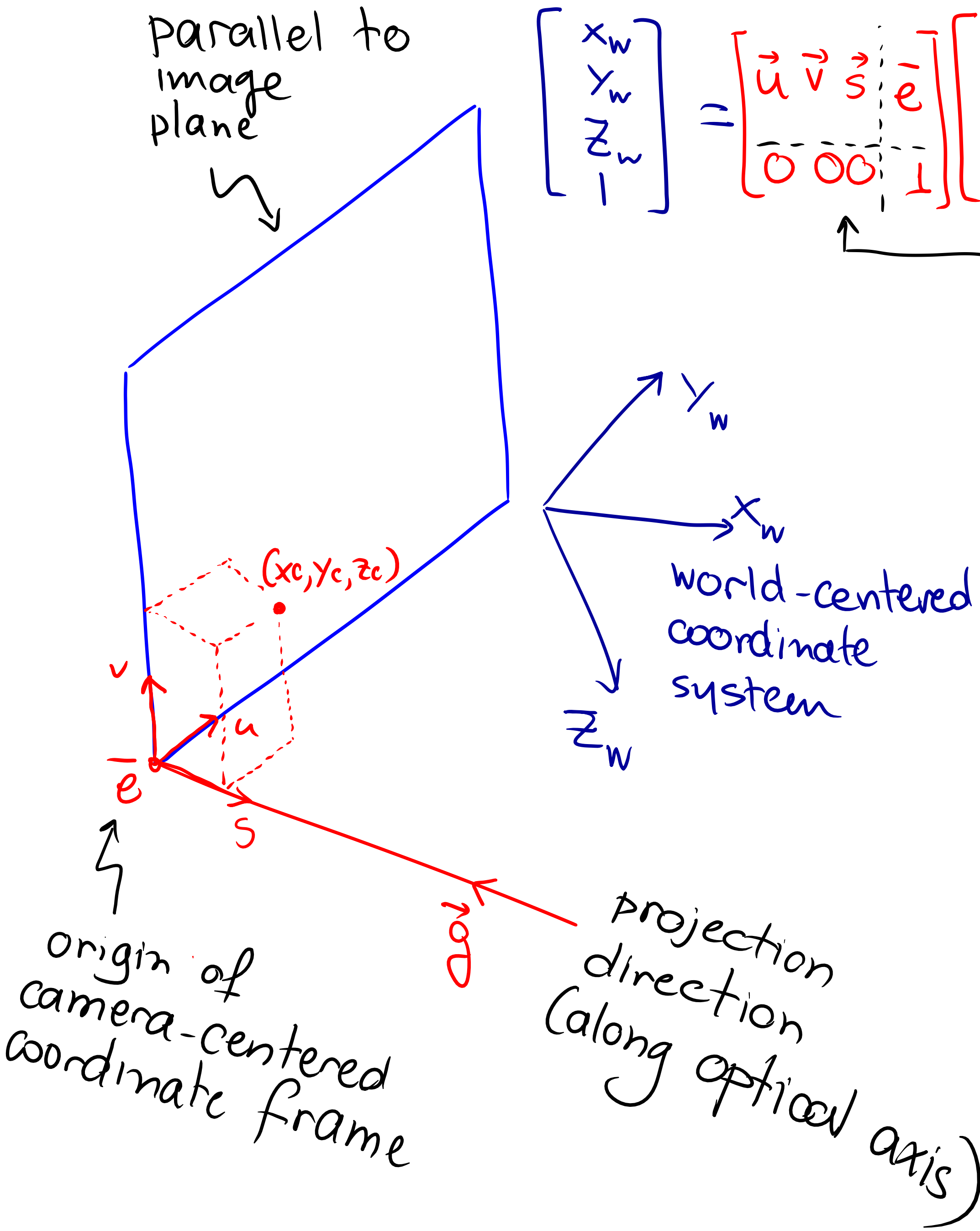
$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1\end{bmatrix}$ becomes $\begin{bmatrix} \bar{e} \\ 1\end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0\end{bmatrix}$ becomes $\begin{bmatrix} \vec{u} \\ 0\end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0\end{bmatrix}$ becomes $\begin{bmatrix} \vec{v} \\ 0\end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0\end{bmatrix}$ becomes $\begin{bmatrix} \vec{s} \\ 0\end{bmatrix}$

$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 0\end{bmatrix}$ becomes $\begin{bmatrix} x_c\vec{u} + y_c\vec{v} + z_c\vec{w} \\ 0\end{bmatrix}$



$$\begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} \vec{u} & \vec{v} & \vec{s} & \bar{e} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

- Extrinsic calibration matrix = inverse of this matrix

- Since $[\vec{u}\ \vec{v}\ \vec{w}]$ orthonormal, (i.e. $\vec{u}, \vec{v}, \vec{g}$ unit-length & perpendicular)

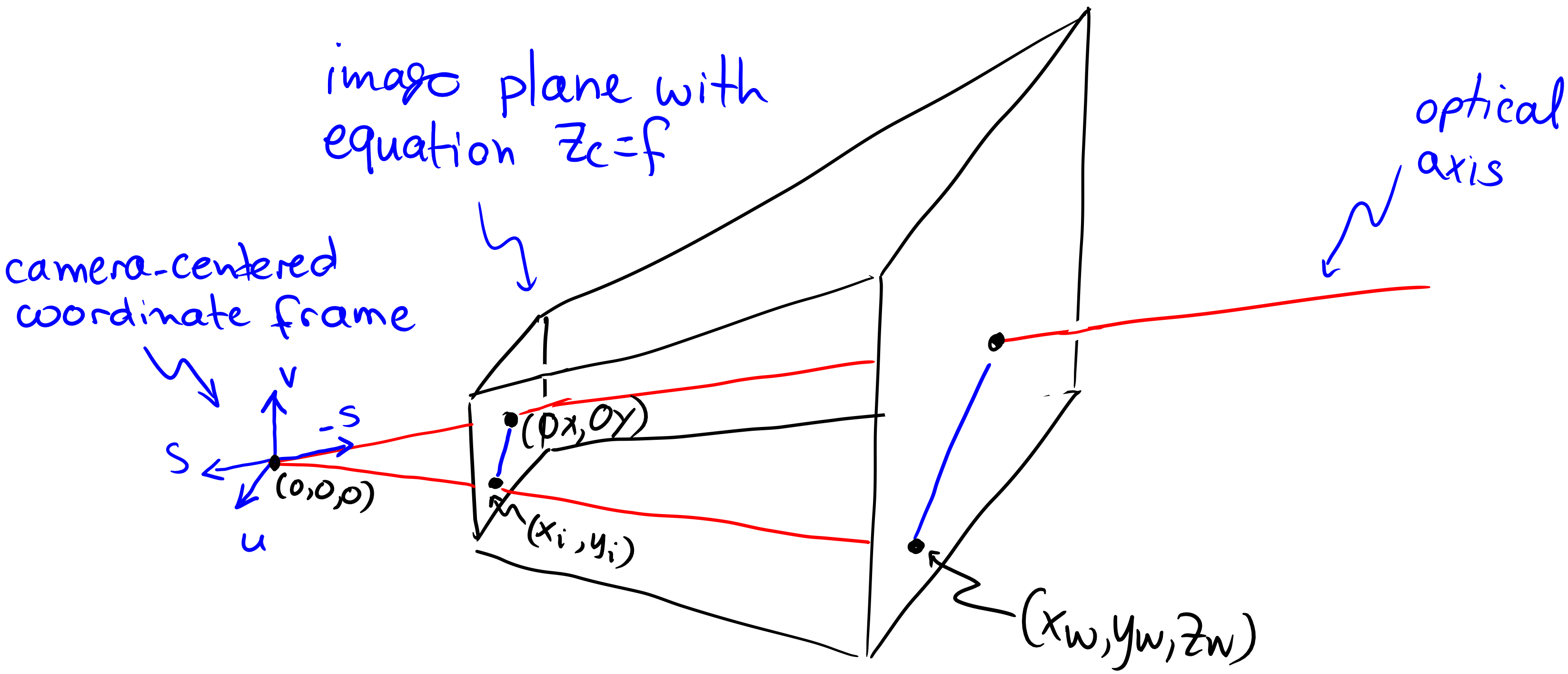
$$[\vec{u}\ \vec{v}\ \vec{s}]^{-1} = [\vec{u}\ \vec{v}\ \vec{s}]^{T}$$

- So M_{wc} is given by

*

$$\boxed{M_{wc} = \left[\begin{array}{c|c} [\vec{u}\ \vec{v}\ \vec{s}]^T & -[\vec{u}\ \vec{v}\ \vec{s}]^T \bar{e} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right]}$$

The Perspective Projection Equations in 3D



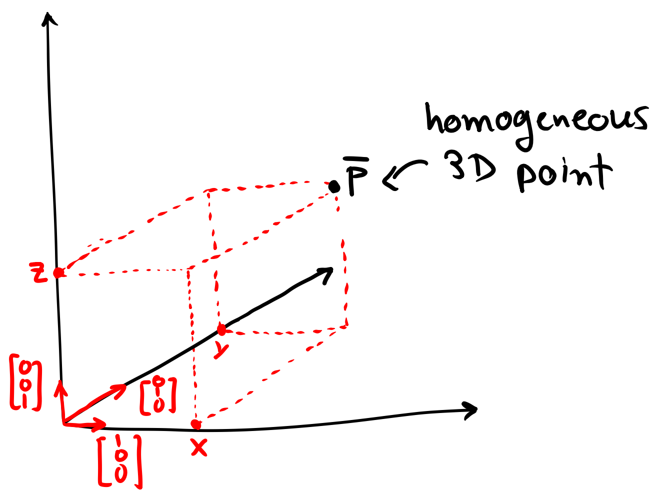
$$\bar{p} = \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \qquad \bar{q} = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

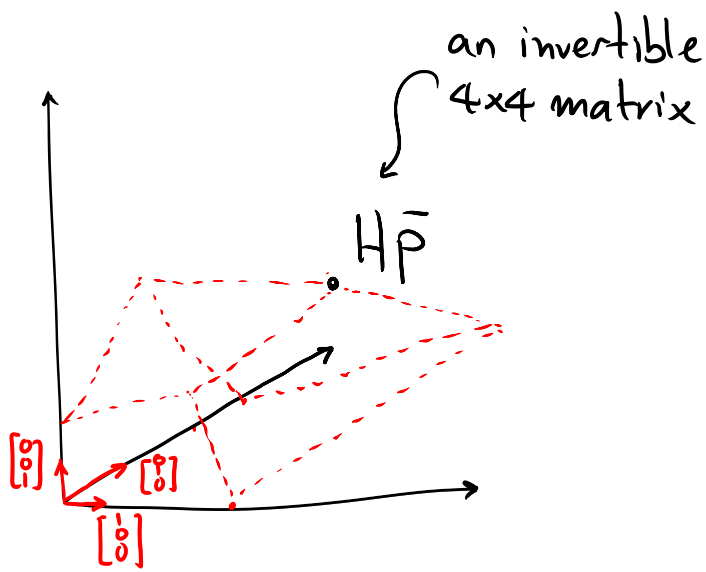
$$\boxed{\bar{q} \cong M_{ci} \cdot M_{wc}\,\bar{p}}$$

Geometry of Image Projection

- Perspective projection
- Homogeneous coordinates in 2D
- Homogeneous coordinates in 3D
- Projection matrices
- Geometric transformations
- Parallelism in homogeneous coordinates
- Orthographic projection

General Linear 3D Transformations





- The matrix H represents a very general set of transformations

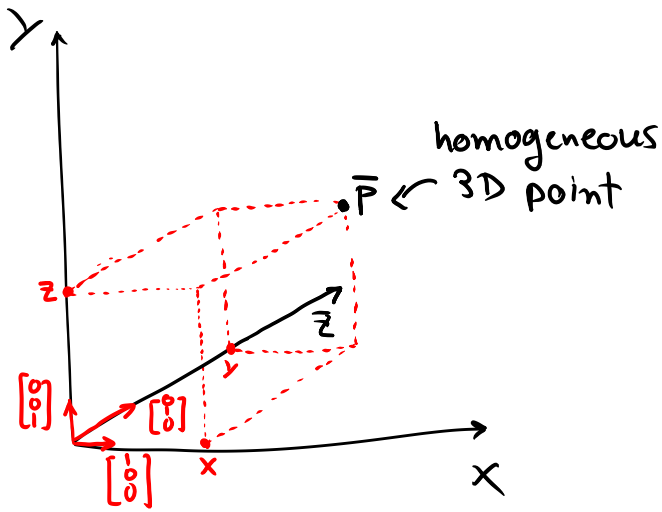
General linear (preserve planes)

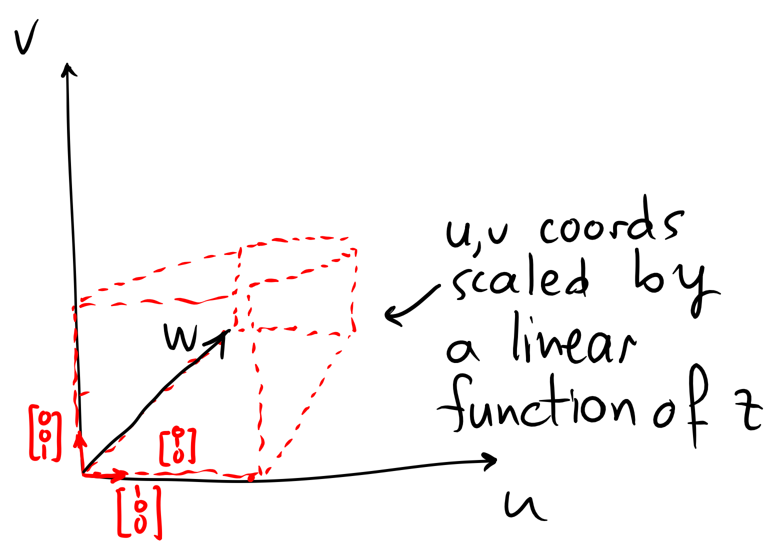
Affine (preserve parallelism)
- Arbitrary shearing
- General scaling

Conformal (preserve angles)
- Uniform scaling

Rigid (preserve lengths)
- Translation
- Rotation

General Linear 3D Transformations





- Example: z-dependent tapering

non-linear in Euclidean coords

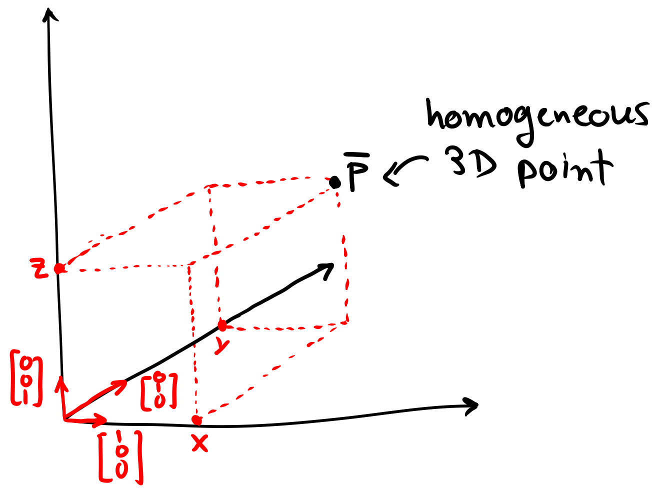
$$(x, y, z) \rightarrow (\alpha(z)x, \alpha(z)y, \alpha(z)z)$$

$$\text{where } \alpha(z) = (\alpha_0 + \alpha_1 z)^{-1}$$

$$\begin{bmatrix} u \\ v \\ w \\ q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\Rightarrow$ linear in homogeneous coords!

Affine Transformations in 3D

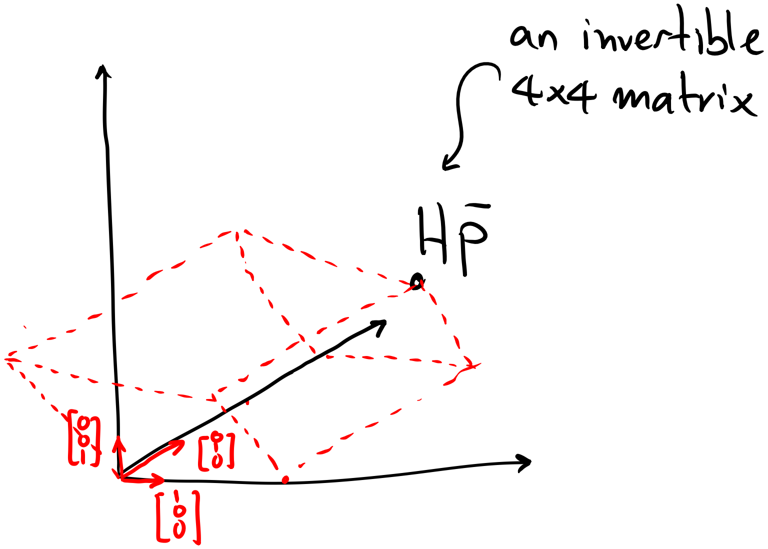


- The matrix H represents a very general set of transformations

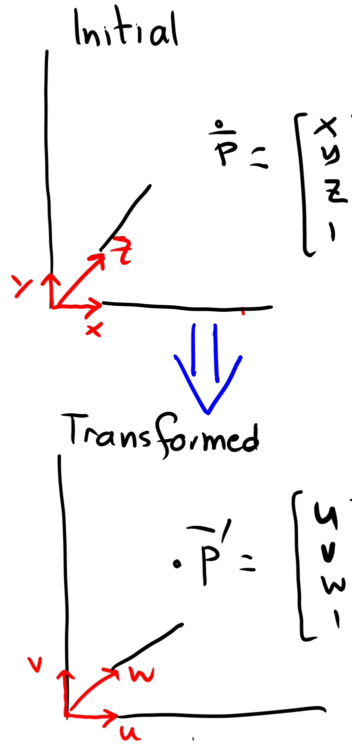
General linear (preserve planes)

Affine (preserve parallelism)

The matrix H now takes a more restricted form!



Affine Transformations: Basic Properties



$$\tilde{p} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \Downarrow \quad \tilde{p}' = \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix}$$

General form of matrix H

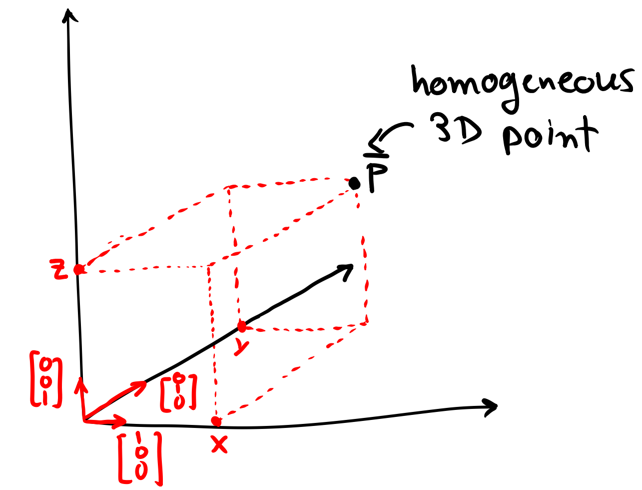
arbitrary 3x3 matrix — 3x1 vector

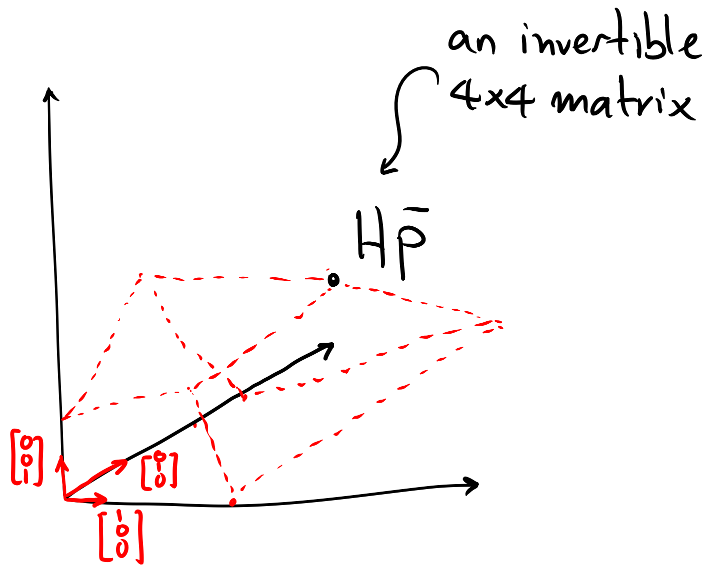
$$H = \left[\begin{array}{c|c} A & \bar{t} \\ \hline 0\,0\,0 & 1 \end{array}\right]$$

3. Affine transforms preserve the value of the last homogeneous coordinate $\Longleftrightarrow$ * points at ∞ before the transform remain at ∞ afterwards

$$\left[\begin{array}{c|c} A & \bar{t} \\ \hline 0\,0\,0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \\ \hline 1 \end{array}\right] = \begin{bmatrix} A\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \bar{t} \\ [0\;0\;0]\begin{bmatrix} x \\ y \\ z \end{bmatrix} + 1 \end{bmatrix} = \begin{bmatrix} A\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \bar{t} \\ 1 \end{bmatrix}$$

From Affine to Rigid Transformations





Affine: $\left[\begin{array}{c:c} A & \bar{t} \\ \hdashline 000 & 1 \end{array}\right]$

General linear (preserve lines)

Affine (preserve parallelism)

- Arbitrary shearing
- General scaling

Conformal (preserve angles)

- Uniform scaling
- Reflection

- Translation
- Rotation

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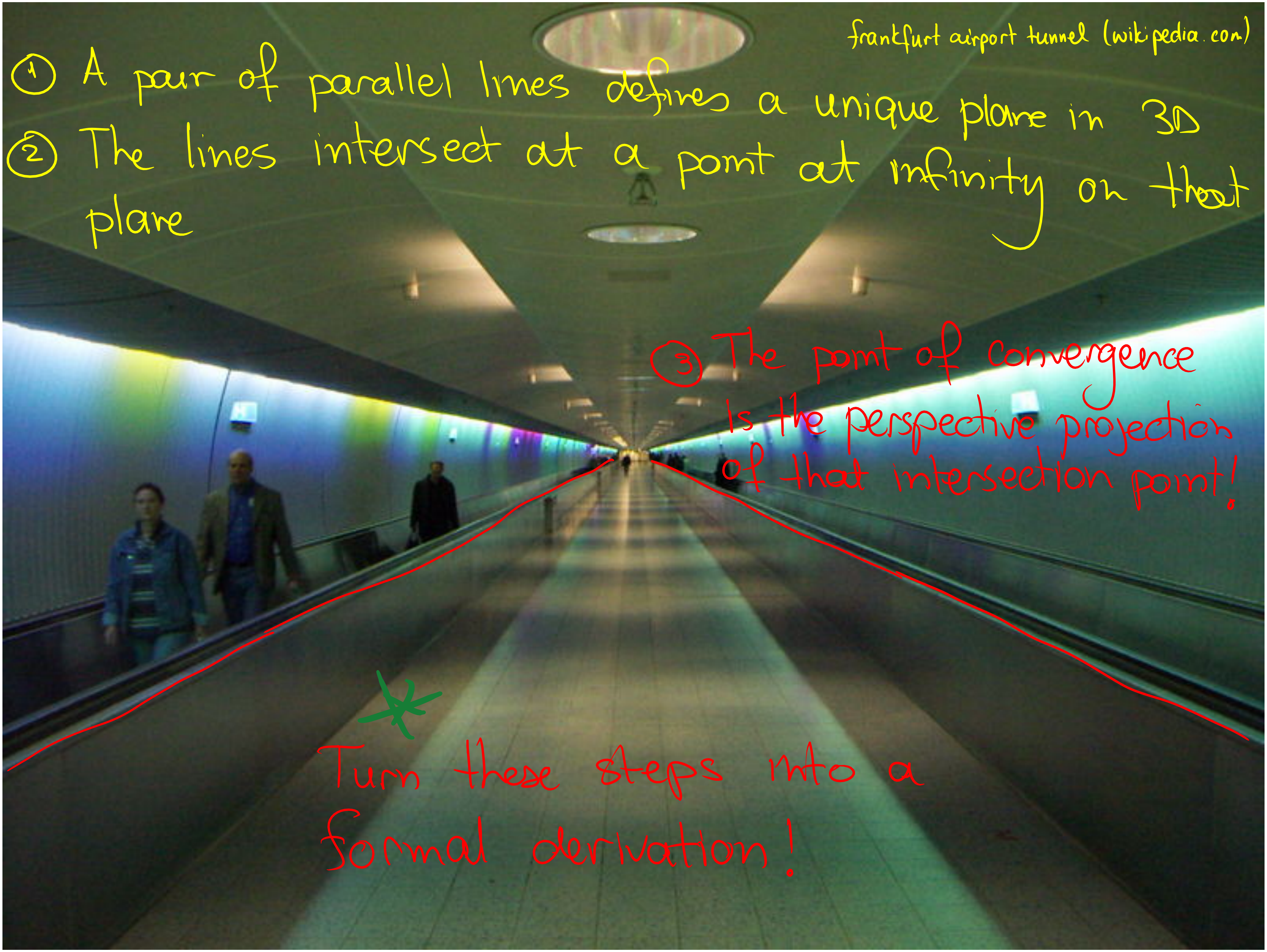


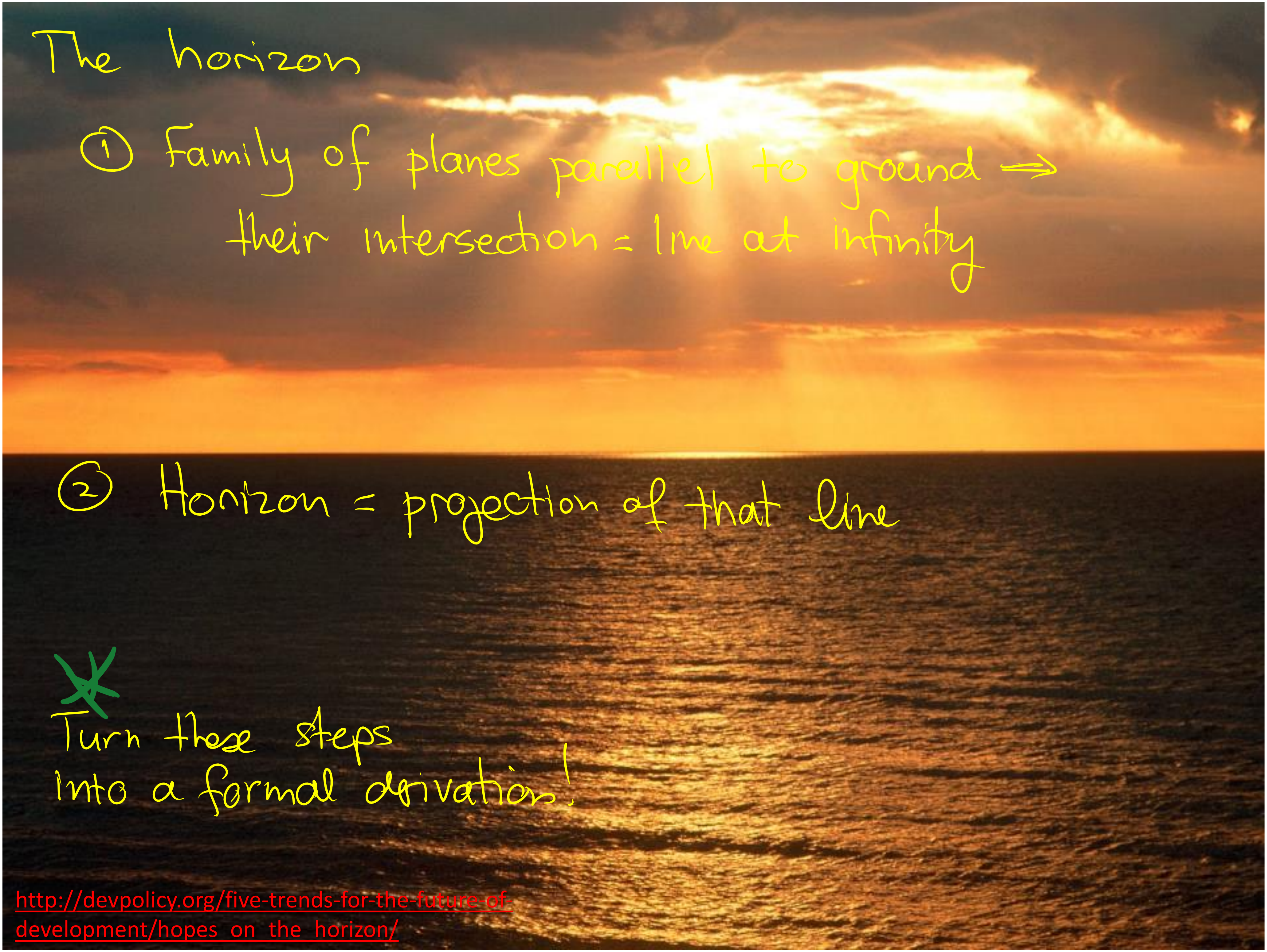
frankfurt airport tunnel (wikipedia.com)

① A pair of parallel lines defines a unique plane in 3D

② The lines intersect at a point at infinity on that plane

③ The point of convergence is the perspective projection of that intersection point!

Turn these steps into a formal derivation!



The horizon

① Family of planes parallel to ground ⇒ their intersection = line at infinity

② Horizon = projection of that line

Turn these steps into a formal derivation!

http://devpolicy.org/five-trends-for-the-future-of-development/hopes_on_the_horizon/

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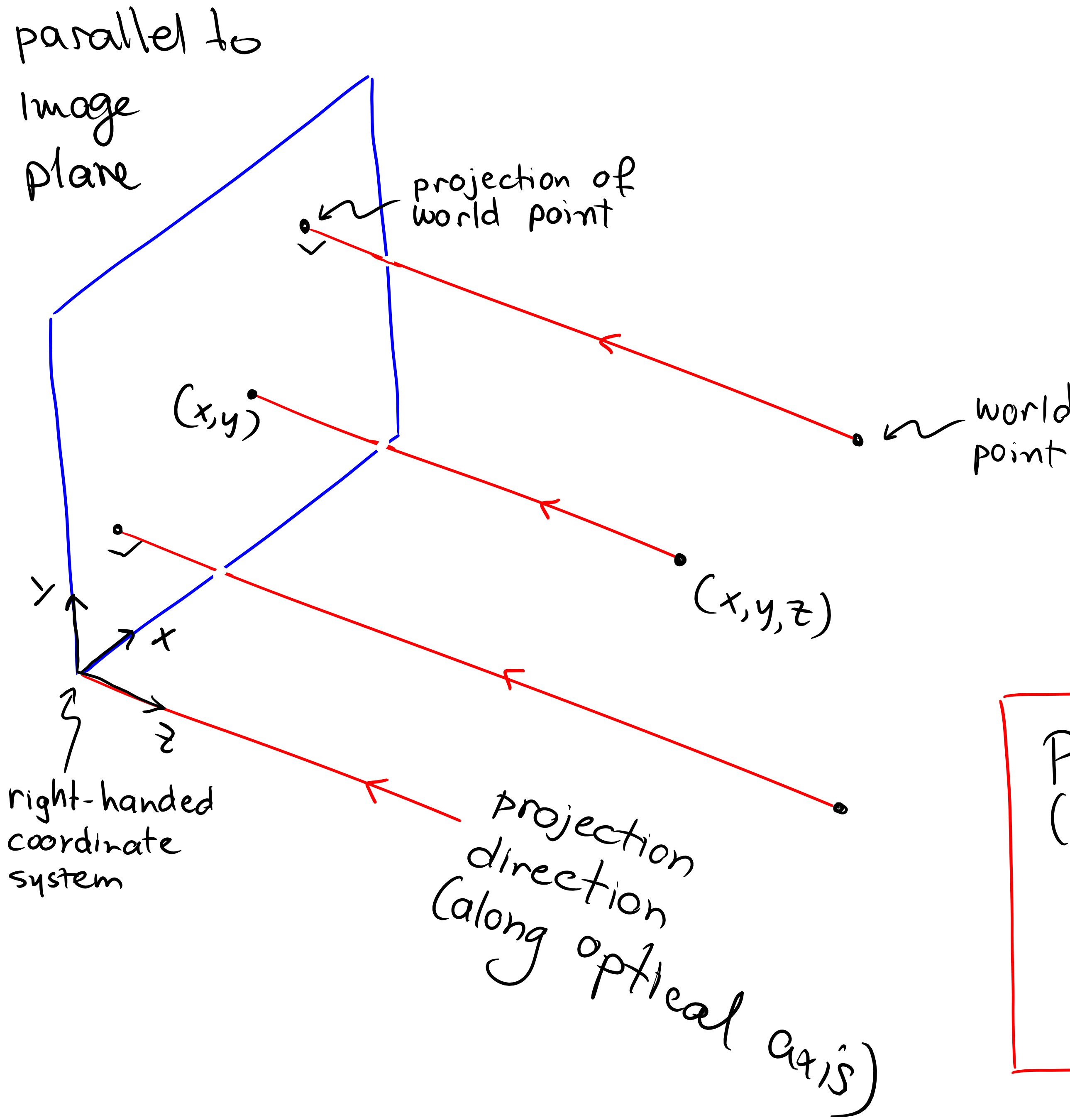
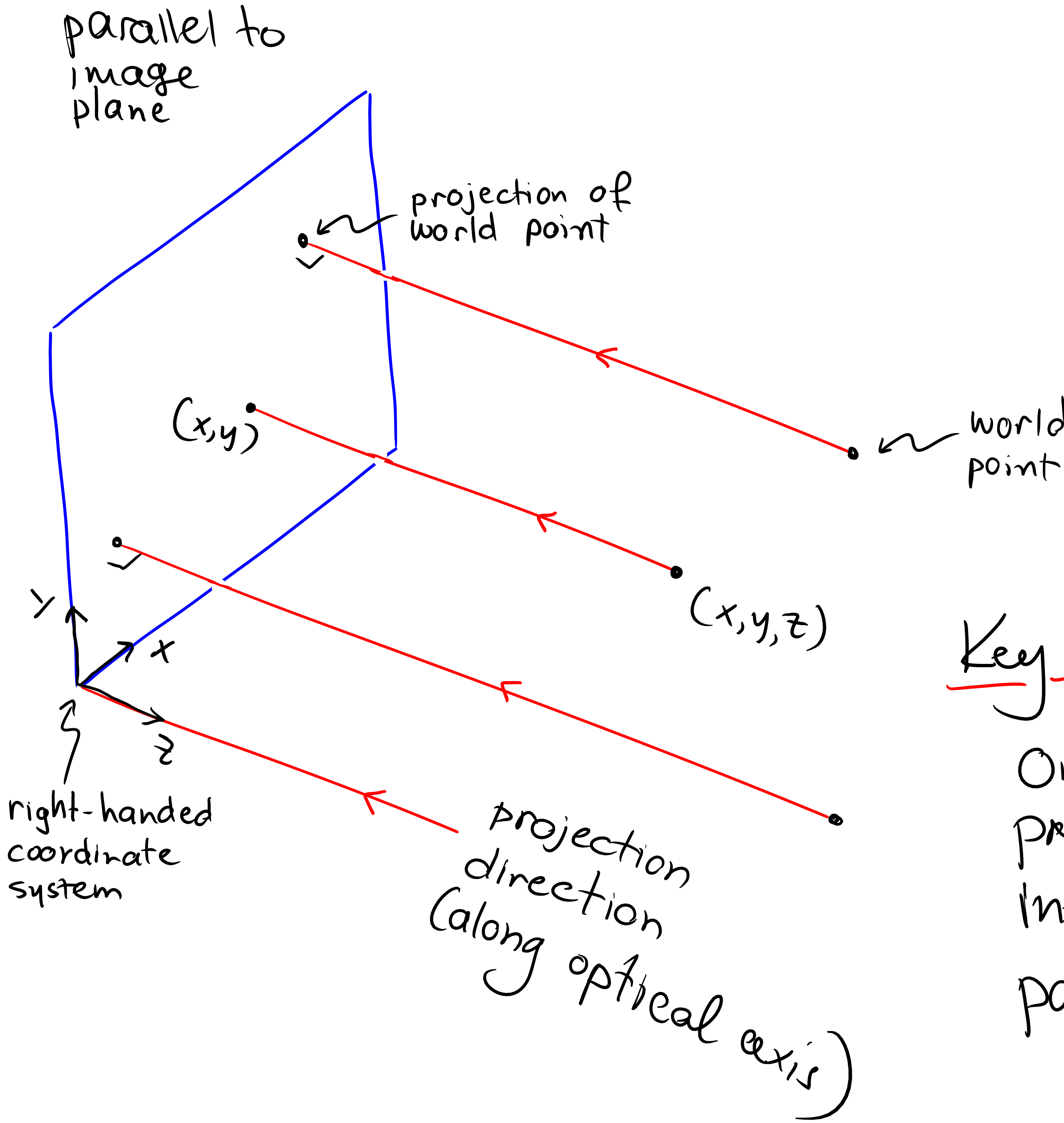
Orthographic Projection



Projection equation (in homogeneous coords)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Orthographic Projection



Key property:

Orthographic projection preserves the plane at infinity (and thus parallelism)