Iffate Jahan Nilima
ID: IT-21063

Number Theory and Abstract Algorithm

① Is 1729 a carmichael number?

→ A carmichael number is a composite number n which satisfies the congruence relation:

$$a^n \equiv a \mod n$$

for all integers a that are relatively prime to n.

To prove that, 1729 is a carmichael number, when we need to show that it satisfies the above condition.

Step 1:

As given, $n = 1729 = 7 \times 13 \times 19$

Let, $P_1 = 7$, $P_2 = 13$ and $P_3 = 19$

Then $P_1 - 1 = 6$, $P_2 - 1 = 12$ and $P_3 - 1 = 18$

Also, $n-1 = 1729 - 1 = 1728$, which is divisible by $P_1 - 1 = 6$

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therefore, $n-1$ is divisible by P_1-1

Step 02:

Similarly, we can show that $n-1$ is also divisible by P_2-1 and P_3-1

Therefore, from the definition of carmichael numbers and the above discussion, we can conclude that 1729 is indeed a carmichael number.

② Primitive root (Generator) of Z_{23}?

→ **Definition:** A primitive root modulo a prime p is an integer r in Z_p such that every non-zero element of Z_p is a power of r.

We want to find a primitive root modulo 23, an element $g \in Z_{23}$ such

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that the powers of generator all non-zero elements of Z_23.

Let,

Z_23 = the set of integers from 1 to 22 under multiplication modulo 23.

since 23 is a prime number;

$|Z_{23}^{*}| = \phi(23) = 22$

So, a primitive root g is an integer such that:

$g^k \not\equiv 1 \mod 23$ for all $k < 22$

and $g^{22} \equiv 1 \mod 23$

We check for $g = 5$:

- Prime factors of 22 = 2, 11

$5^{22/2} = 5^{11} \mod 23 = 22 \neq 1$

$5^{22/11} = 5^{2} \mod 23 = 2 \neq 1$

So, 5 is a primitive root modulo 23

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④ Is $\langle Z-37, + \rangle$, $\langle Z-35, \times \rangle$ are abelian group?

→ $(Z_{37}, +)$:

This is an abelian group under addition mod 37. Always true for Zn with addition.

$(Z_{35}, *)$:

This is not an abelian group. Only the units in Z_{35} from a group under multiplication. But full Z_{35} under multiplication includes 0, non-invertibles.

So, it's not a group.

Iffate Jahan Nilima
ID: IT-24063

(3) Is $\langle Z-11, +, \times \rangle$ a Ring?

→ Yes, $Z_{11} = \{0, 1, 2, \ldots 10\}$ with addition and multiplication modulo 11 is a Ring because:

- $(Z_{11}, +)$ is an abelian group.
- Multiplication is associative and distributes over addition.
- It has a multiplicative identity.

Since 11 is prime, Z_{11} is also a field.

So, $(Z_{11}, +, \times)$ is a Ring.

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Iffate Jahan Nilima
ID: IT-21063

(5) Let's take $P=2$ and $n=3$ that makes the $GF(P^n) = GF(2^3)$, then solve this with polynomial arithmetic approach:

→ Given,

$P=2, n=3$

we want to construct the finite field $GF(2^3)$ which has $2^3 = 8$ elements

Step 1: choose an irreducible polynomial of degree 3 over $GF(2)$

A common choice is

$f(x) = x^3 + x + 1$

Step 2: Define the field elements, every element of $GF(2^3)$ can be express as a polynomial with degree less than 3 and co-efficients in $GF(2)$:

$\{0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1\}$

Step 3: Define addition and multiplication

- Addition is performed by adding corresponding co-efficients modulo 2

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ID: IT-21063

$x^2+x=0$, $x^2+1=x^2+1$

- Multiplication is polynomial multiplication followed by reduction modulo,

$f(x)=x^3+x+1$

$x^3 = x+1 \pmod{f(x)}$

Example calculations:

- $x \cdot x = x^2$ (no reduction needed)
- $x \cdot x^2 = x^3 = x+1$
- $(x+1) \cdot x = x^2+x$

Thus, $GF(2^3)$ is a field with 8 elements and well defined addition and multiplication.