A polynomial-time algorithm for ground states of spin trees

ArXiv 1907.04862

Nilin Abrahamsen

December 28, 2019

Local Hamiltonians

- Many-body spin system:

$$n = |\mathsf{V}|, \quad \mathfrak{H}_{\mathsf{v}} \simeq \mathbb{C}^d, \quad \mathfrak{H} = \bigotimes_{\mathsf{v}\in\mathsf{V}} \mathfrak{H}_{\mathsf{v}}, \quad |\mathfrak{H}| = d^n$$

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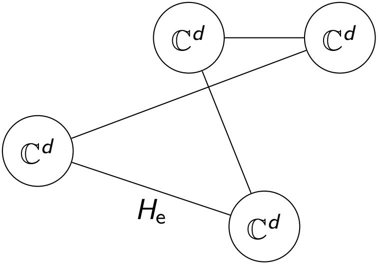

- Local interactions $H_{(\mathsf{v},\mathsf{w})} \in \mathrm{Herm}(\mathfrak{H}_{\mathsf{v}} \otimes \mathfrak{H}_{\mathsf{w}})$.

$$H = \sum_{\mathsf{e} \in \mathsf{E}} H_{\mathsf{e}} \otimes \mathrm{I}_{\mathsf{V} \setminus \mathsf{e}}.$$

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For $\Delta = o(1)$ QMA-hard even in 1D .

Background

matrix product states

[Vidal]

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1D area law

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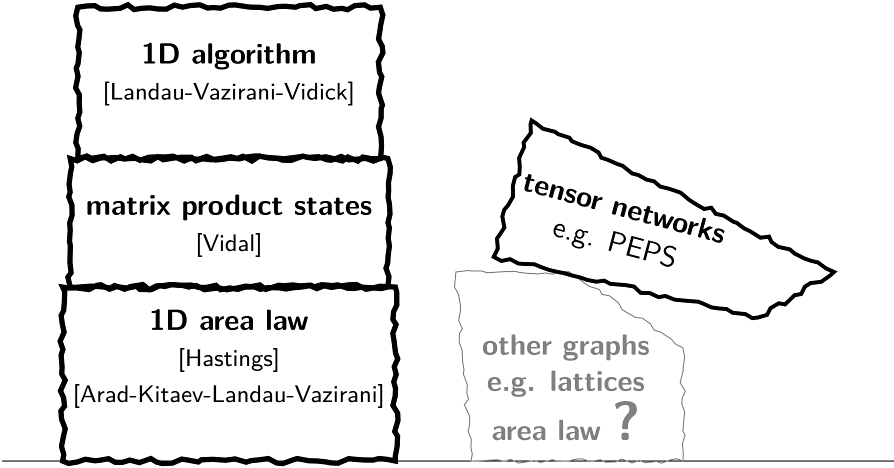
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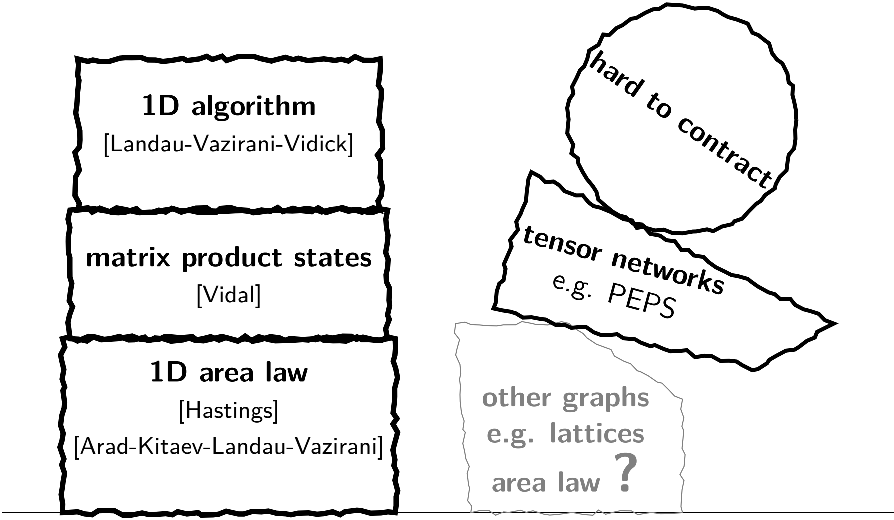
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First area law and efficient algorithm beyond spin chains

Fractal dimension of interaction tree T

Motivation Tree with vertex set $V \subset \mathbb{R}^2$ and $\mathrm{dist}(v_1, v_2) = \Omega(1)$ for vertices $v_1 \neq v_2$.

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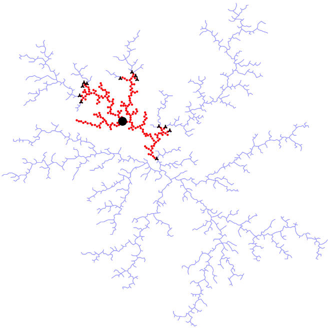

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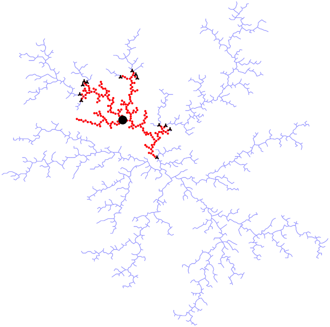

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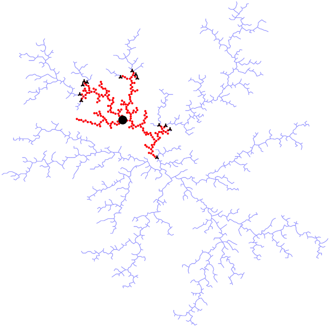

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Empirical observation:

Typically $\boldsymbol{|\mathcal{N}_{v \in T}(r)| = O(r^{\beta})}$ for fixed $\beta < 2$.

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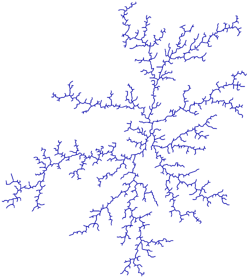

2D DLA

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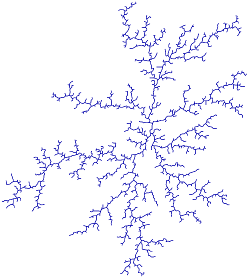

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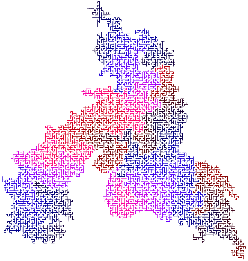

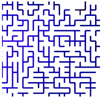

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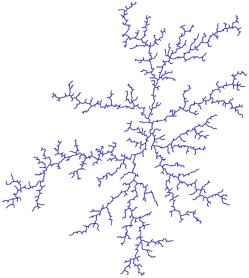

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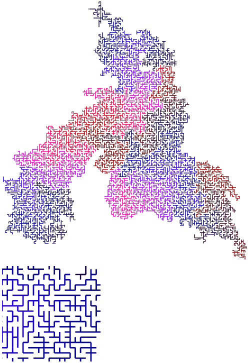

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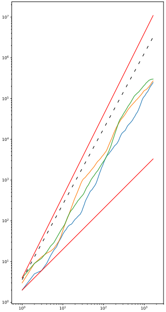

3D lattice UST

Fractal dimension of interaction tree T

We forget ambient space $\mathbb{R}^2$.

Definition

Discrete fractal dimension of T: $\beta = \sup_{\mathsf{v} \in \mathsf{V}, r > C} \dfrac{\log |\mathcal{N}_{\mathsf{v} \in \mathsf{T}}(r)|}{\log r}$.

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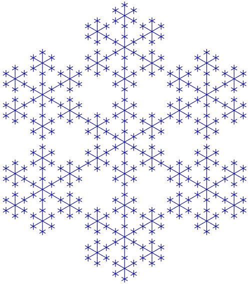
Model of branched polymers [BFJK04]



3D Vicsek tree

$\beta = \log_3 7$

Results

Tree T with discr. fractal dimension $\beta < 2$. Local Hamiltonian H on T with eigenvalues (energies) $E_0 \leq E_1 \leq \cdots$.

Ground states	Degeneracy	Spectral gap
$\mathcal{Z} = \ker(H - E_0)$	$D := \lvert\mathcal{Z}\rvert$	$\Delta = E_D - E_0$

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Theorem (Area law)

Let $\mathrm{T} = L \sqcup R$ *be a partition with* $\lvert\partial L\rvert = O(1)$ *and* $\lvert\psi\rangle \in \mathcal{Z}$. *There exists* $\lvert\phi\rangle \approx \lvert\psi\rangle$ *with*

$$\lvert\phi\rangle = \sum_{i=1}^{r} \lvert\phi_i^L\rangle\lvert\phi_i^R\rangle, \qquad r = D\exp(\tilde{O}(\Delta^{-\frac{\beta}{2-\beta}})).$$

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Theorem (Algorithm)

If $D = n^{O(1)}$*, can compute* $\tilde{\mathcal{Z}} \prec \mathfrak{H}$ *such that w.h.p.* $\tilde{\mathcal{Z}} \approx_\epsilon \mathcal{Z}$ *where* $\epsilon = 1/n^{10}$*. Time:*

$$n^{O(\Delta^{-\frac{\beta+1}{2-\beta}})}.$$

Closeness of subspaces

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- For subspaces: $\mathcal{Y} \succ_\delta \mathcal{Z}$ if

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Viable subspaces

Let $T = \Omega_1 \sqcup \cdots \sqcup \Omega_k$
so $\mathfrak{H} = \mathfrak{H}_{\Omega_1} \otimes \cdots \otimes \mathfrak{H}_{\Omega_k}$

Find local "solutions" $\mathcal{V}_j \prec \mathfrak{H}_{\Omega_j}$ such that $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_k \succ_\delta \mathcal{Z}$.

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Subspace $\mathcal{V} \prec \mathfrak{H}_L$ is $\boldsymbol{\delta}$**-viable** for $\mathcal{Z} \prec \mathfrak{H}_L \otimes \mathfrak{H}_R$ if:

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Definition (σ-PAP)

A σ-PAP for $\mathcal{Z} \prec \mathfrak{H}_L \otimes \mathfrak{H}_R$ is an operator subspace $\mathcal{A} \prec \mathfrak{L}_L$ such that $\mathcal{A} \otimes \mathfrak{L}_R$ contains a σ-approximate projector with target space $\mathcal{Z}$.

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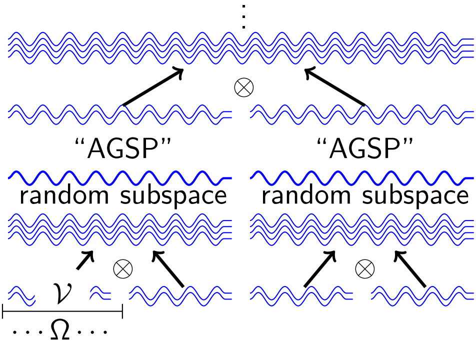
Goal: construct σ-PAP $\mathcal{A}$ with target space $\tilde{\mathcal{Z}} \approx_\delta \mathcal{Z}$ and

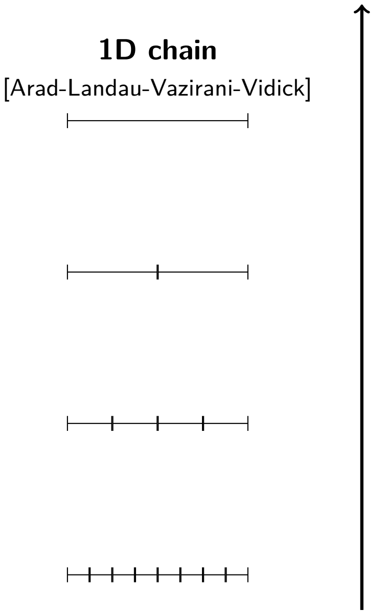
$$\sigma \cdot |\mathcal{A}| \ll 1.$$

1D algorithm

1D algorithm [Arad-Landau-Vazirani-Vidick]:



Bird's eye view of algorithm



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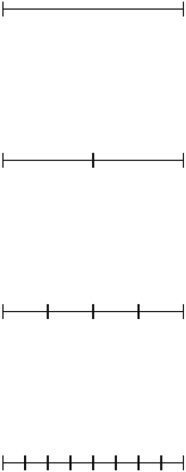
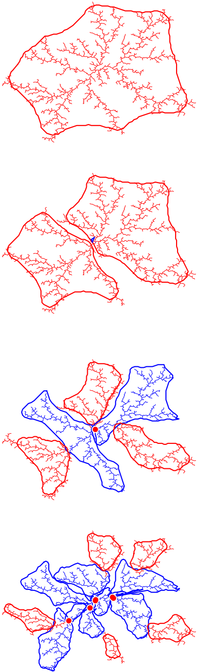
1D chain

[Arad-Landau-Vazirani-Vidick]

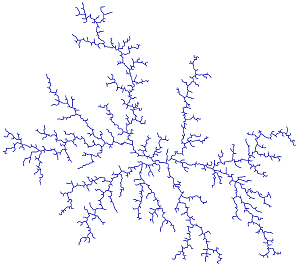


Tree (this work)



META-tree

Given interaction tree



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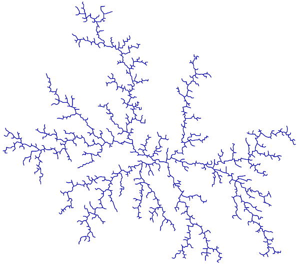
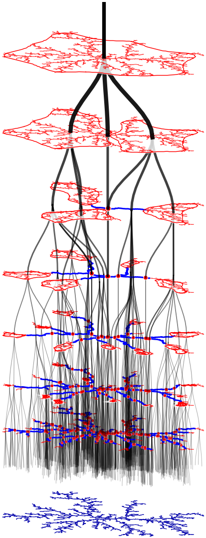


Output state encoded as tensor network on META-tree:



Constructing a σ-PAP

Given subtree $L \subset \mathrm{V}$ with $|\partial L| = O(1)$.

Objective: Construct σ-PAP $\mathcal{A} \prec \mathfrak{H}_L$ with target $\approx \mathcal{Z}$, and

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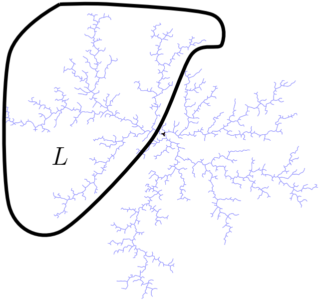

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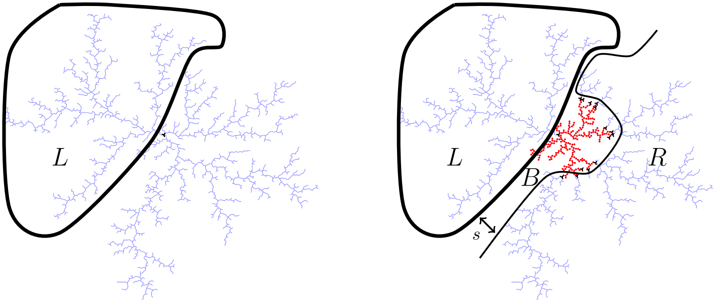

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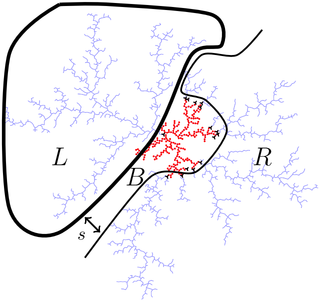

Constructing a σ-PAP

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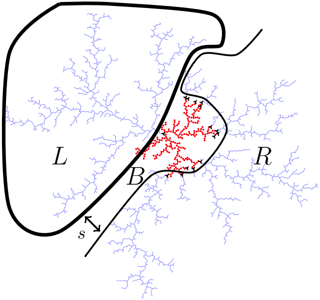

Lemma

We can construct $\mathcal{A} \prec \mathfrak{L}_L$ such that $I, H, \ldots, H^m \in \mathcal{A}_L \otimes \mathfrak{L}_{BR}$, and

$$|\mathcal{A}| = m^{m/s+|B|}$$

*where $s = d(L, R)$. $\mathcal{A}$ is **degree**-m **viable** for H.*

Proof.

pigeonhole principle [Arad-Kitaev-Landau-Vazirani]. $\square$

Why $\beta < 2$?

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$\mathcal{A}$ is σ-PAP for $\mathcal{Z}$ where

$$\sigma = \exp\left[-\Omega\left(m\sqrt{\frac{\Delta}{\|H\|}}\right)\right].$$

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$\mathcal{A}$ contains the degree-m Chebyshev AGSP [Arad-Kitaev-Landau-Vazirani]. $\square$

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Open Questions

- Trees with $\beta \geq 2$. k-regular interaction trees?
- What is β for $3D$ lattice UST? (Simulation: $\beta < 2$)

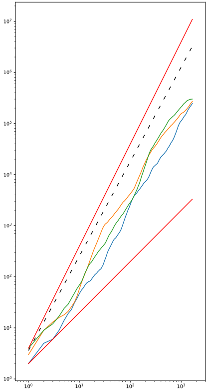

Figure: 3D UST (radius, volume), (log, log)-scale