A polynomial-time algorithm for ground states of spin trees

Nilin Abrahamsen

Local Hamiltonians

- Many-body spin system: n local Hilbert spaces $\mathfrak{H}_{\mathsf{v}} \simeq \mathbb{C}^d$,

$$\mathfrak{H} = \bigotimes_{\mathsf{v} \in \mathsf{V}} \mathfrak{H}_{\mathsf{v}}, \quad |\mathfrak{H}| = d^n.$$

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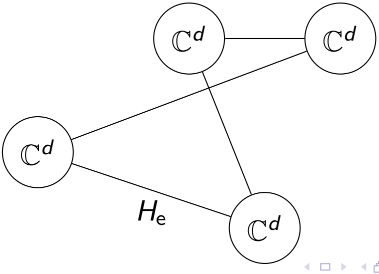
- **Local Hamiltonian:**

$$H = \sum_{\mathsf{e} \in \mathsf{E}} H_{\mathsf{e}} \otimes \mathrm{I}_{\mathsf{V} \setminus \mathsf{e}}.$$

- where $H_{(\mathsf{v},\mathsf{w})} \in \mathrm{Herm}(\mathfrak{H}_{\mathsf{v}} \otimes \mathfrak{H}_{\mathsf{w}})$.



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For $\Delta = o(1)$ **QMA-hard** [Kempe-Kitaev-Regev] **even in 1D** [Aharonov-Gottesman-Irani].

Background

Efficient classical algorithm for spin chains (1D)

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matrix product states

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#P-hard to contract

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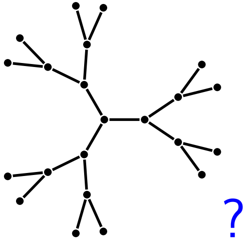
First area law and efficient algorithm beyond spin chains

Fractal dimension of interaction tree T

Motivation. Geometry of physical system (e.g. large branched polymer).



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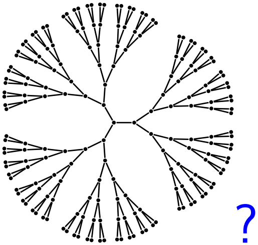

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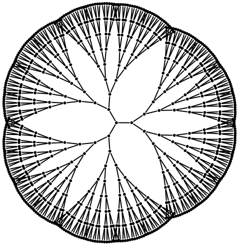 ?

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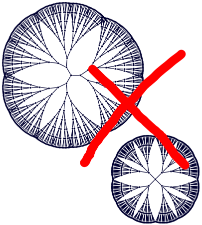

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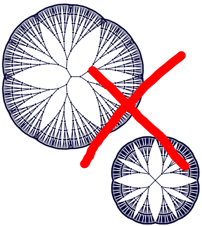

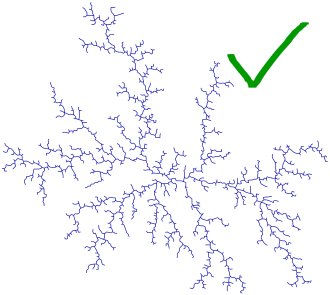

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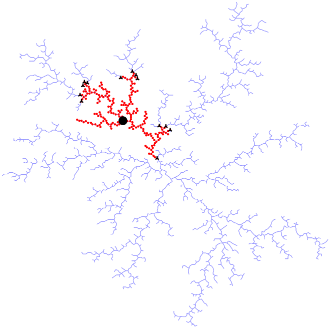
Neighborhood $\mathcal{N}_{\mathrm{v}\in\mathrm{T}}(r)$:



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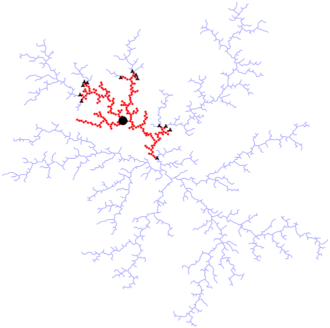

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Condition:

$|\mathcal{N}_{\mathsf{v}\in\mathsf{T}}(r)| = O(r^{\beta})$
for *discrete fractal dimension* $\beta < 2$.

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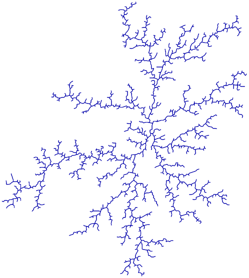

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$\beta \approx \mathbf{1.7}$ [M83]

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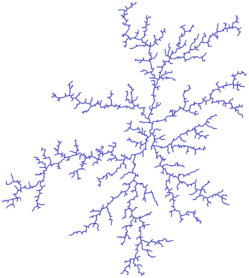

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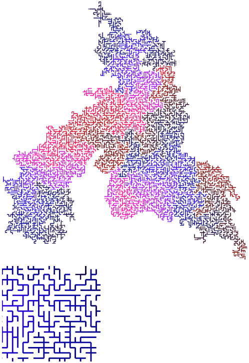

2D lattice UST
$\boldsymbol{\beta = 1.6}$ [BM11].

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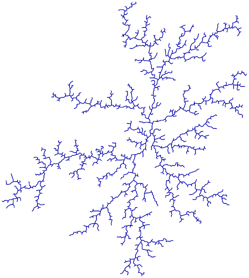

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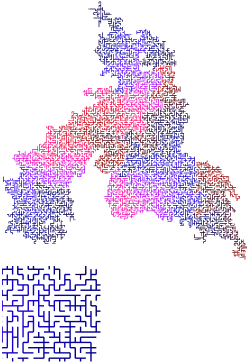

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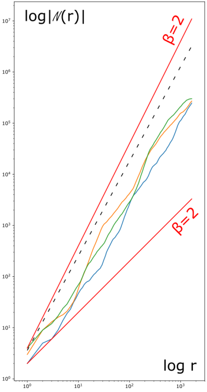

3D lattice UST

Fractal dimension of interaction tree T

Definition

Discrete fractal dimension of T: $\beta = \sup_{\mathsf{v} \in \mathsf{V}, r > C} \frac{\log |\mathcal{N}_{\mathsf{v} \in \mathsf{T}}(r)|}{\log r}$.

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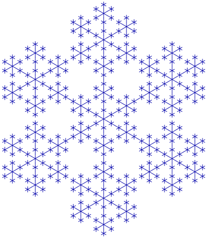
Deterministic example:

Model of branched polymers [Blumen et al. *Macromolecules*]



3D Vicsek tree

$\beta = \log_3 7 = \mathbf{1.77...}$

Results

Local Hamiltonian H on T with eigs $E_0 = \cdots = E_{D-1} < E_D \leq \cdots$.

Ground states		Spectral gap
$\mathcal{Z} = \ker(H - E_0)$	$\lvert\mathcal{Z}\rvert = D$	$\Delta = E_D - E_{D-1}$

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Theorem (Area law)

Let $\mathsf{T} = L \sqcup R$ be a partition with $|\partial L| = O(1)$ and let $|\psi\rangle \in \mathcal{Z}$. There exists $|\phi\rangle \approx |\psi\rangle$ with

$$|\phi\rangle = \textstyle\sum_{i=1}^{r} |\phi_i^L\rangle|\phi_i^R\rangle, \qquad r = D\exp(\tilde{O}(\Delta^{-\frac{\beta}{2-\beta}})).$$

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Theorem (Algorithm)

If $D = n^{O(1)}$, can compute $\tilde{\mathcal{Z}} \prec \mathfrak{H}$ such that w.h.p. $\tilde{\mathcal{Z}} \approx_\epsilon \mathcal{Z}$ where $\epsilon = 1/n^{10}$. Time complexity:

$$n^{O(\Delta^{-\frac{\beta+1}{2-\beta}})}.$$

Closeness of subspaces

For subspaces $\mathcal{Y}, \mathcal{Z} \prec \mathfrak{H}$:

Definition

- $\mathcal{Y} \succ_\delta \mathcal{Z}$ if: For all unit $|z\rangle \in \mathcal{Z}$,

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Viable subspaces

Let $T = \Omega_1 \sqcup \cdots \sqcup \Omega_k$
so $\mathfrak{H} = \mathfrak{H}_{\Omega_1} \otimes \cdots \otimes \mathfrak{H}_{\Omega_k}$

Find local "solutions" $\mathcal{V}_j \prec \mathfrak{H}_{\Omega_j}$ such that $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_k \succ_\delta \mathcal{Z}$.

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Given $\mathcal{Z} \prec \mathfrak{H}_{LR}$ in bipartite $\mathfrak{H}_{LR}$. $\mathcal{V} \prec \mathfrak{H}_L$ is **δ-viable** for $\mathcal{Z}$ if

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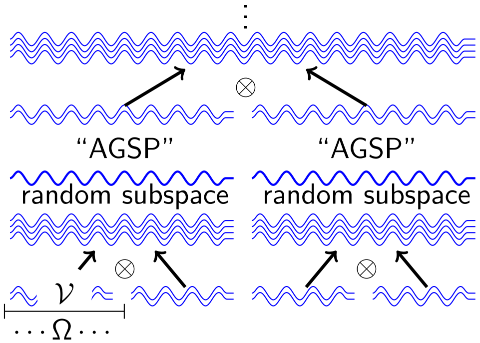
Goal: Given region $L \subset T$, construct σ-PAP $\mathcal{A}$ with

$$|\mathcal{A}| \cdot \sigma \ll 1.$$

1D algorithm

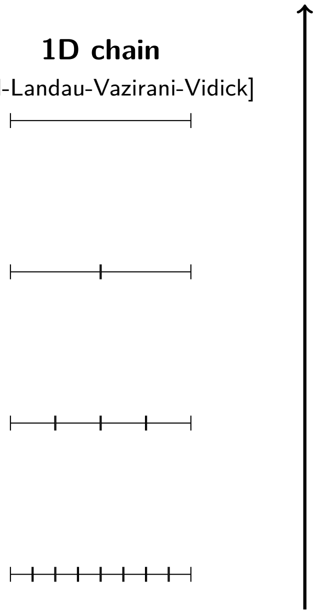
1D algorithm [Arad-Landau-Vazirani-Vidick]:



Bird's eye view of algorithm

1D chain

[Arad-Landau-Vazirani-Vidick]



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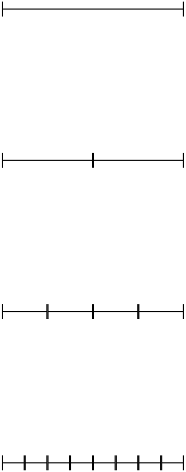
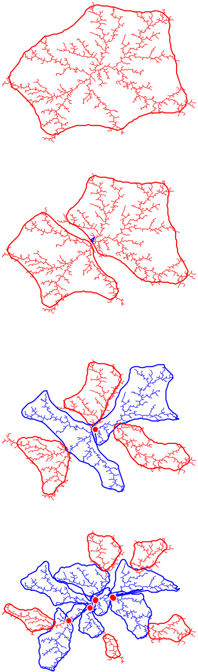
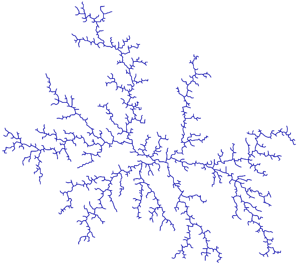


Tree (this work)



META-tree

Given interaction tree



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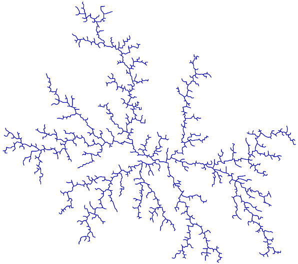
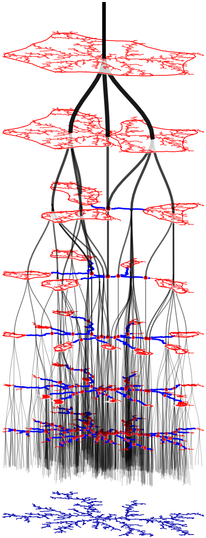


Output state encoded as tensor network on META-tree:

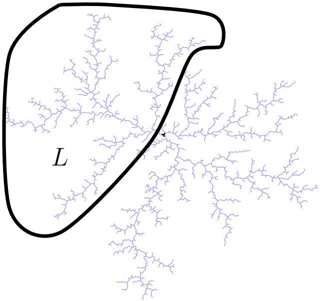


Constructing a σ-PAP

Given subtree $L \subset \mathsf{V}$ with $|\partial L| = O(1)$.

Objective: Construct σ-PAP $\mathcal{A} \prec \mathfrak{H}_L$ with target $\approx \mathcal{Z}$, and

$$\sigma \cdot |\mathcal{A}| \ll 1.$$

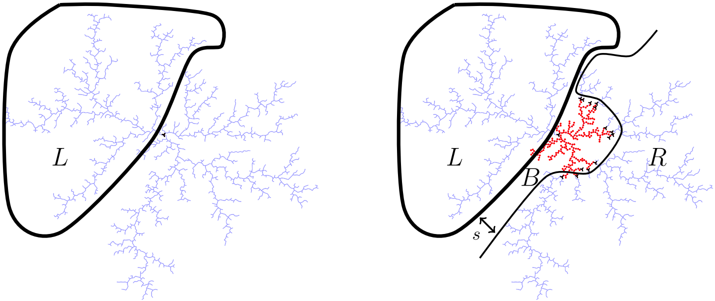


Constructing a σ-PAP

Given subtree $L \subset \mathsf{V}$ with $|\partial L| = O(1)$.

Objective: Construct σ-PAP $\mathcal{A} \prec \mathfrak{H}_L$ with target $\approx \mathcal{Z}$, and

$$\sigma \cdot |\mathcal{A}| \ll 1.$$



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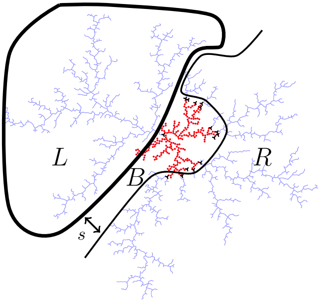

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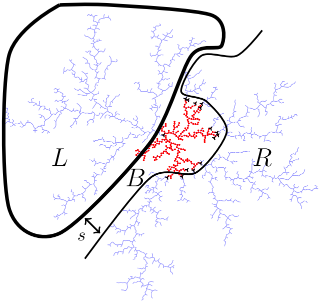

Objective:
Construct σ-PAP $\mathcal{A} \prec \mathfrak{H}_L$.

Lemma

Can construct $\mathcal{A} \prec OP_L$ such that $I, H, \ldots, H^m \in \mathcal{A}_L \otimes OP_{BR}$, and

$$|\mathcal{A}| = m^{m/s+|B|}$$

*$\mathcal{A}$ is **degree**-m **viable** for H.*

Proof.

pigeonhole principle [Arad-Kitaev-Landau-Vazirani]. □

Why $\beta < 2$?

Lemma

$\mathcal{A}$ degree-m viable for H $\quad\Rightarrow\quad$ $\mathcal{A}$ is σ-PAP for $\mathcal{Z}$ where

$$\sigma = \exp\left[-\Omega\left(m\sqrt{\frac{\Delta}{\|H\|}}\right)\right].$$

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$$\mathcal{A} \text{ degree-}m \text{ viable for } H$$
$$\Downarrow$$

$\mathcal{A}$ viable for Chebyshev AGSP [Arad-Kitaev-Landau-Vazirani]. □

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Lemma

$\mathcal{A}$ degree-m viable for $\tilde{H}$ $\quad\Rightarrow\quad$ $\mathcal{A}$ is σ-PAP for $\mathcal{Z}$ where

$$\sigma = \exp\Big[-\Omega\Big(\frac{m}{\sqrt{\cancel{\|H\|}}}\Big)\Big] \xrightarrow{trunc} \exp\Big[-\Omega\Big(\frac{m}{\sqrt{|B|}}\Big)\Big].$$

Proof.

$$\mathcal{A} \text{ degree-}m \text{ viable for } \tilde{H}$$
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Lemma

We can construct $\mathcal{A}$ such that $I, H, \ldots, H^m \in \mathcal{A}_L \otimes OP_{BR}$, and

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$$|B| \ll s^2$$

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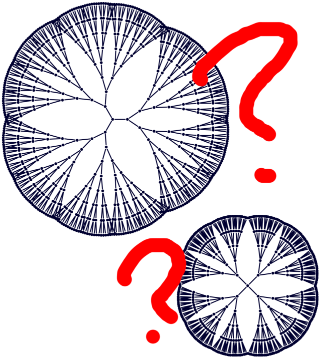
By fractal dimension bound:

$$|B| \ll s^2 \Rightarrow \frac{m}{\sqrt{|B|}} \gg m/s \Rightarrow \sigma|\mathcal{A}| \ll 1.$$

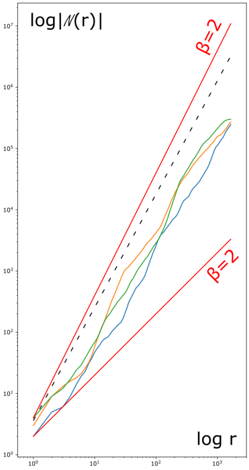
PAP constructed. $\square$

Open Questions

Extension to k-regular trees?



Does 3D lattice UST satisfy $\beta < 2$?



Thanks!

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