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## Basic concepts of metric space

$(X, d) \rightarrow$ metric space

* $x \in X$, $r > 0$ (real)

$B(x; r) = \{ y \in X : d(y, x) < r \} \rightarrow$ open

$\bar{B}(x; r) = \{ y \in X : d(y, x) \leq r \} \rightarrow$ closed

$S \subseteq X$

* $x \in S$ in Interior point of $S$. $\exists\, r > 0$ st
$B(x, r) \subseteq S$

* $S$ is open if every point of $S$ is an Interior point

* $P \in X$ is an accumulation point of $S$
if $\forall\, r > 0$
$B(P, r) \cap (S \setminus \{P\}) \neq \emptyset$

$D(S)$ = set of accumulation point of $S$

$\bar{S} = S \cup D(S)$

$S = [0, 1) \cup \{2\}$ ; $X = \mathbb{R}$

* $S$ closed if $D(S) \subseteq S$

Exerise : ① $S$ is closed in $(X, d) \Leftrightarrow X \setminus S$ is open in $(X, d)$

② Let $x \in X$ Then show that $X - \{x\}$ is open

$(x_n)_{n \in \mathbb{N}}$ is sequence in $(X, d)$
we say $x_n \rightarrow P \in X$ if
$\forall\, \varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ (St)
$d(x_n, P) < \varepsilon \quad \forall\, n \geq n_0$

* $(x_n)_{n \in \mathbb{N}}$ is a cauchy sequence in $(X, d)$
$\forall\, \varepsilon > 0$, $\exists\, n_0$ st
$d(x_n, x_m) < \varepsilon \quad \forall\, n, m \geq n_0$

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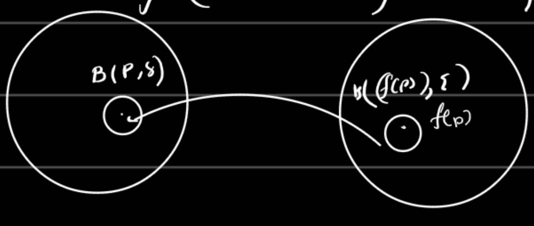
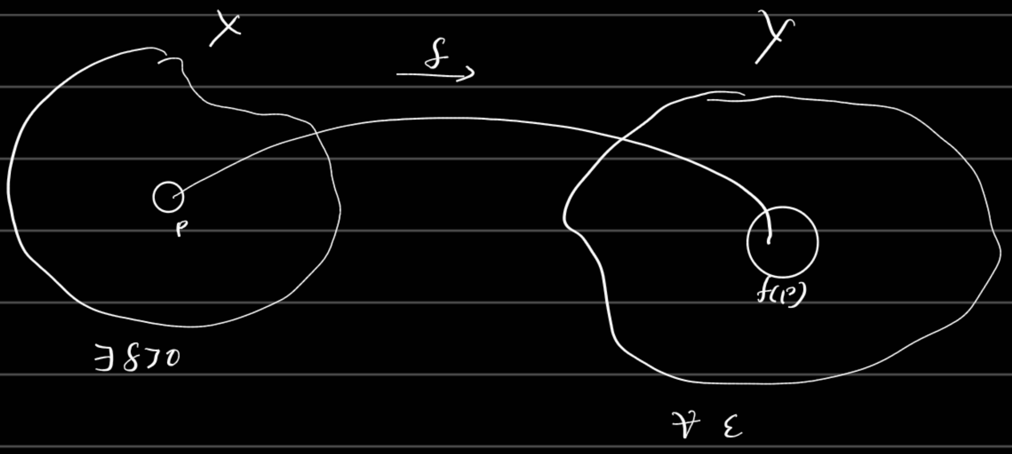
[ε or X]

**Defn :** Suppose $f : (X, d) \longrightarrow (Y, \rho)$

Let $P \in X$ . we say that $f$ in continous @ $P$ if
for $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t
$\forall\, x \in X$ with $d(x, P) < \delta$ , we have
$\rho(f(x), f(P)) < \varepsilon$

$f(B(P, \delta)) \subseteq B_\rho(f(P), \varepsilon)$






## Proposition

Let $f:(X,d) \to (Y,\rho)$ be a function from a metric space $(X,d)$ to a metric space $(Y,\rho)$ & $p \in X$. Then TFAE

i) $f$ is continous @ $p$

ii) for every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ with

$$\lim_{n\to\infty} x_n = p, \quad \text{we have } \lim_{n\to\infty} f(x_n) = f(p) = f\left(\lim_{n\to\infty} x_n\right)$$

Let $(x_n)_{n\in\mathbb{N}}$ be convergent sequence in $X$ with $\lim x_n = p$. we shall show that the seq $(f(x_n))_{n\in\mathbb{N}}$ in $Y$ is convergent with limit $f(p)$

Since $f$ is continous @ $p$, $\forall \varepsilon > 0$, $\exists \delta > 0$

$f(B_d(p,\delta)) \subseteq B_\rho(f(p),\varepsilon)$ $\quad$ $\forall$ open has a preimage which is open. (Superset)

As $x_n \to p$ & $\delta > 0$

$\exists\, n_0 \in \mathbb{N}$ s.t

$x_n \in B_d(p,\delta)\ \forall\, n \geq n_0$

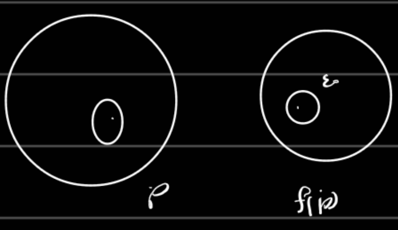
$\Rightarrow f(x_n) \in f(B_d(p,\delta)) \subseteq B_\rho(f(p),\varepsilon)$

$\Rightarrow f(x_n) \to f(p)$

ii $\Rightarrow$ i holds but

$f$ is not continous @ $p$

Thus $\exists\, \varepsilon_0 > 0$ st $\forall \delta > 0$




for some $x \in X$ with $d(x,p) \leq \delta$

but $\rho(f(x), f(p)) \geq \varepsilon_0$

Choose $\delta = 1/n$ $\quad n \in \mathbb{N}$

$\exists\, x_n \in X$ st $d(x_n, p) < 1/n$

but $\rho(f(x), f(p)) \geq \varepsilon$

$f^{-1}(Y \setminus S) = X \setminus f^{-1}(S)$

$f : (X, d) \to (Y, \rho)$ is said to be continous function if $f$ is continous at every point $p \in X$

proposition : Let $f : (X, d) \to (Y, \rho)$ be a function. Then FCAE

i) $f$ is continous
ii) Inverse image of open set is open [If $G$ is open in $\rho$]
iii) Inverse image of closed set is closed.

homeomorphism → con. bijectiv $X\ Y$
→ converse exist
→ $f$ & $f^{-1}$ is continuous.

proposition of openset in $(X, d)$

i) $\phi, X$ are open

2) $G_\alpha$ is open for $\alpha \in B$, then $\bigcup_{\alpha \in B} G_\alpha$ is also open.

iii) $G_1, G_2 \dots G_n$ are open
$G_1 \cap G_2 \dots \cap G_n$ is open.

iv) $\mathcal{J}_d$ = set of open set in $(X, d)$
→ topology.

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Recall

$f : (X, d) \to (Y, d)$ is function
with distance

$p \in X$ we say $f$ is continous @ $p$. If
$\forall \varepsilon > 0, \exists \delta > 0$ (st)
$f(B_d(p, \delta)) \subseteq B_\rho(f(p); \varepsilon)$

→ $f$ is continous if $f$ is continous @ $\forall p \in X$

Prof i) $f$ is continous @ $P$
ii) If $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in $X$ with $\lim_{n \to \infty} x_n = p$, then $(f(x_n))_{n \in \mathbb{N}}$ is convergent in $Y$ and $f(p) = \lim f(x_n)$

Prop FCAE

i) $f$ is continous
ii) $f^{-1}(G)$ is open in $X$ $\forall$ openset $G$ in $Y$
iii) $f^{-1}(F)$ is closed in $X$ for $\forall$ closed set $F$ in $Y$
iv) $\forall$ subset $A \subseteq X$, we have
$f(\bar{A}) \subseteq \overline{f(A)}$

v) If $(x_n)$ is convergent sequence in $X$, then $(f(x_n))$ is convergent in $Y$ and $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$

$\mathcal{T}_d$ = set of open set in $(X,d)$ $\to$ topology.

Properties of opensets

prop: i) $\phi \in \mathcal{T}_d$, $X \in \mathcal{T}_d$

ii) If $G_\alpha \in \mathcal{T}_d$, $\forall \alpha \in \Delta$, then $\bigcup_{\alpha \in \Delta} G_\alpha \in \mathcal{T}_d$.

iii) If $G_1, G_2 \dots G_n \in \mathcal{T}_d$, then $G_1 \cap G_2 \dots \cap G_n \in \mathcal{T}_d$.

Example

1) $X$, set ; $d$ = disc

$\mathcal{T}_d = P(X)$

2) $X$, $\mathcal{T} = \{\phi (X)\}$

$G$ is open in $(X,d)$

$x \in G$, $\exists\, r_x > 0$ (st)

$B(x, r_x) \subseteq G$

$\Leftrightarrow G = \bigcup_{x \in G} B(x, r_x)$

$Y \underset{\text{subset}}{\subseteq} (X,d)$

$Y$ with restriction of $d$ to $Y \times Y$ is metric space $(Y, d|_Y)$

This called a metric subspace.

$X = \mathbb{R}$

$Y = [0,1]$

except 0,1, $\forall y \in Y$ is interior point

$X = \mathbb{R}^2$ ; $d_E$

$Y$ = x axis $\{(a,0) . a \in \mathbb{R}$

Then $Y$ doesn't have any interior pts

Proposition: Suppose $(X,d)$ is a metric space and $Y$ is a (metric) subspace of $(X,d)$ Then we have.

i) $U$ is an open set in $Y$ if and only if $U = G \cap Y$ for some open set $G$ in $X$

2) $F$ is closed set in $Y$ if and only if $F = C \cap Y$ for some set $C$ in $X$

$\mathcal{T}_Y = \{ Y \cap G : G \in \mathcal{T}_d \}$

## Compactness (compact set)

Let $(X,d)$ be a metric space and $K \subseteq X$

1) a collection $\mathcal{C} = \{G_\alpha : \alpha \in \Delta\}$ of open sets in $(X,d)$ is said to be an open cover of $K$ if

$$K \subseteq \bigcup_{\alpha \in \Delta} G_\alpha$$

2) A sub collection $\mathcal{C}' = \{G_\alpha ; \alpha \in \Delta'\}$ for some $\Delta' \subseteq \Delta$ is called a sub cover of $K$ if

$$K \subseteq \bigcup_{\alpha \in \Delta'} G_\alpha$$

~ like finite

③ a subset $K$ of $(X,d)$ is said to be compact if every open cover $K$ has a finite subcover.

### Examples

1) A finite set is compact

$K = \{x_1, x_2, x_n\}$

Let $\mathcal{C} = \{G_\alpha : \alpha \in \Delta\}$ be an open cover of $K$

Then $K \subseteq \bigcup_{\alpha \in \Delta} G_\alpha$ then for each $x_i \in K$

$\exists\ \alpha_i \in \Delta$ s.t $x_i \in G_{\alpha_i}$

Clearly, $K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$

2) $(X,d)$ ; $d = disc$

$K$ is compact $\Leftrightarrow$ $K$ is finite                Heine-Borel thm.

$K = \bigcup_{x \in K} \{x\}$

$K = (0,1)$   $\mathcal{C} = \{(1/n, 1) ; n \geq 2\}$

$(0,1) \supseteq U_{n_1} \cup U_{n_2} \cdots U_{n_k} = \bigcup_{\max\{n_1, \ldots, n_k\}} (1/n_k, 1)$

but $\frac{1}{2n_k}$ not there

Prop   $[0,1]$ is a compact subset of the Euclidean line $\mathbb{R}$

$T = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$

$\{ \frac{1}{n} + 1/m \ \ n, m \in \mathbb{N}\}$

Proposition: Let $(X, d)$ be a metric space and $K$ be a subset

Let $K$ be a subset

If $K$ is compact then

1) $K$ is closed

2) $K$ is bounded (ie $K \subseteq B(p, r)$, $p \in X$

$\{ B(p, n) ; n \in \mathbb{N} \}$