9/10/23 - tutorial

10/10/23 - Recap.

Riemann Integration

$f: [a,b] \longrightarrow \mathbb{R}$

* Partition $P = \{x_0, x_1, x_2 \cdots x_n\}$, $x_0 = a < x_1 < \cdots < x_n = b$

Mesh $\mu(P) = \max\{\Delta x_i : 1 \le i \le n\}$ $\Delta x_i = x_i - x_{i-1}$

Refinement of P $P \subseteq P'$ is called a refinement

$T = \{t_1, t_2, \ldots, t_n\}$ (P, T)

Riemann sum

$$R(f, P, T) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

f is riemann integrable over $[a,b]$ if

$$\left[\lim_{\mu(P)\to 0} R(f,P,T) = L \text{ (exists)}\right]$$

$$\int_a^b f\,dx$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ st for any choice of tag.

for every partition of

Darboux Integrability (only for bounded)

$|f(x)| \le M \ \forall x \in [a,b]$

$m_i(f) = m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\}$

$M_i(f) = M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}$

$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$ $L(P,f) \le U(P,f)$

$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$

Exercise: If P' is refinement of P, then show that

$L(f,P) \le L(f,P')$ & $U(f,P') \le U(f,P)$

2) If P_1 & $P_2 \in Par[a,b]$, then show that

$L(f, P_1) \le L(f, P_1 \cup P_2) \le U(f, P_1 \cup P_2) \le U(f, P_2)$

$$\underline{\int_a^b} f\,dx = \text{lower integrable} = \sup_{P \in Par[a,b]} L(f,P) \ ; \quad \overline{\int_a^b} f\,dx = \text{upper integral} = \inf_{P \in Par[a,b]} U(f,P)$$

we say f is Darboux integrable over $[a,b]$ if $\underline{\int} f\,dx = \overline{\int} f\,dx$.

Proposition.

Let $f: [a,b] \to \mathbb{R}$ be bounded function. Then f is Darboux integrable if and only if

$\forall \varepsilon > 0$, $\exists P_\varepsilon \in Par[a,b]$ s.t

$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

Proof $\Rightarrow$ we have $\underline{\int} f = \overline{\int} f$

Given $\varepsilon > 0$, $\underline{\int} - \varepsilon/2 < L(f, P_1)$ for some $P_1 \in Par[a,b]$

Also $\overline{\int} f + \varepsilon/2 \ge U(f, P_2)$ for some $P_2 \in Par(a,b]$

Let $P_\varepsilon = P_1 \cup P_2$

$$\underline{\int} f - \varepsilon/2 < L(f, P_1) \le L(f, P_\varepsilon) \le U(f, P_\varepsilon) \le U(f, P_2) < \overline{\int} f + \varepsilon/2$$

$\Rightarrow U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

$\Leftarrow$ Given $\varepsilon > 0$, $\exists P_\varepsilon$ s.t

$U(f, P_\varepsilon) < L(f, P_\varepsilon) + \varepsilon$

$\overline{\int} f < U(f, P_\varepsilon) < L(f, P_\varepsilon) + \varepsilon \le \underline{\int} f\,dx + \varepsilon$

$\Rightarrow \overline{\int} f \le (\underline{\int} f \le \overline{\int} f) \rightarrow$ normally.

$\Rightarrow \overline{\int} f = \underline{\int} f$

Let $f: [a, b] \to \mathbb{R}$ be bounded function.

Thm: f is Riemann Integrable $\Leftrightarrow$ f is Darboux Integrable.

Corollary: (Riemann's Integrability Criterion)

Let f be a bounded function on $[a, b]$

Then $f \in \mathcal{R}[a, b] \Leftrightarrow \forall \varepsilon > 0, \exists P \in Par[a, b]$ with $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

Exercise: ① If $f: [a, b] \to \mathbb{R}$ is continous. then show that $f \in \mathcal{R}[a, b]$

② If $f: [a, b] \to \mathbb{R}$ is monotonically increasing then show that $f \in \mathcal{R}[a, b]$

Sol: f is uniformly cont.

Given $\varepsilon > 0$ choose $\delta > 0$ s.t

$|s - t| < \delta \Rightarrow |f(t) - f(s)| < \varepsilon/(b-a+1)$

$P_\varepsilon = P_\varepsilon$ $\mu(P_\varepsilon) = \delta$

$M_i - m_i < \varepsilon/(b-a+1)$

$U(f, P) - L(f, P) = (M_i - m_i)\sum \Delta x_i < \frac{\varepsilon}{(b-a+1)} \sum \Delta x_i < \varepsilon$

$U(f, P) - L(f, P) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\left(\frac{b-a}{n}\right) = \left(\frac{b-a}{n}\right)(f(b) - f(a)) < \varepsilon$

$= \int f\,dx + \varepsilon$

Riemann Lebesgue thm

Let $f: [a, b] \to \mathbb{R}$ be bounded function Then f is Riemann Integral if and only if the set of discontinuities of f zero set. Other words $f \in \mathcal{R}[a, b] \Leftrightarrow f$ is continous a.e. on $[a, b]$

$S \subseteq \mathbb{R}$ is called zero set if $\forall \varepsilon > 0$, there is countable family of Integrable (st) $S \subseteq \bigcup_j I_j$ and $\sum_{j=1}^{\infty} l(I_j) < \varepsilon$

12/10/23 Recap:

Riemann Integration

* $f : [a,b] \to \mathbb{R}$; $f \in \mathcal{R}[a,b]$
* $|f(x)| \le M$; f is Darboux integrable ($\forall \varepsilon > 0, \exists P_\varepsilon$ with $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$)

  f is bounded.
* f Riemann integration $\Leftrightarrow$ f is Darboux integral

Riemann's Integrability criterion

$f \in \mathcal{R}[a,b] \Leftrightarrow$ Madhu

* f continuous $\Rightarrow f \in \mathcal{R}[a,b]$
* f monotonic $\Rightarrow f \in \mathcal{R}[a,b]$

Riemann-Lebesgue theorem

$|f(x)| \le M \quad \forall x \in [a,b]$

$f \in \mathcal{R}[a,b]$ $\Leftrightarrow$ set of points discontinuity of is measure zero set

Cor: $f \in \mathcal{R}[a,b]$ $\varphi : [-M,M] \to \mathbb{R}$ continuous

then $\varphi \circ f \in \mathcal{R}[a,b]$

$\Rightarrow f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$ $\left(M_i(|f|) - m_i(|f|) < M_i(f) - m_i(f) \right)$

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f|\,dx \le M(b-a)$$

Fundamental theorem of calculus

First FT

Let $f \in \mathcal{R}[a,b]$ Define

$$F(x) = \int_a^x f(t)\,dt \quad ; x \in [a,b]$$

Then i) F is continous function over $[a,b]$

ii) If f is continous at $x_0 \in [a,b)$, then

F is differentiable at x_0 and $F'(x_0) = f(x_0)$

$f \in \mathcal{R}[a,b]$; $a<c<b$ $\Leftrightarrow$ $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$

Second F.T

Let $f \in \mathcal{R}[a,b]$. Suppose f has a primitive over $[a,b]$

(Is there is differentiable function $g : [a,b] \to \mathbb{R}$

with $g'(x) = f(x) \ \forall x \in [a,b]$)

$$g(b) - g(a) = \int_a^b f(t)\,dt$$

Proof: i) Let $x, y \in [a,b]$ & $x<y$

Then $F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt$

$$\Rightarrow |F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \le \int_x^y |f(t)|\,dt$$
$$\le M(y-x) \to 0 \text{ as } y \to x$$

then f is continuous (Indeed F is certainly continuous)

Then for $h>0$

$$\frac{F(x_0+h) - F(x)}{h} - f(x_0) = 1/h\left[\int_{x_0}^{x_0+h} f(t)\,dt - \int_{x_0}^{x_0+h} f(x_0)\,dt\right]$$
$$= 1/h \int_{x_0}^{x_0+h} (f(t) - f(x_0))\,dt$$

Now
$$\left|\frac{F(x+h) - F(x_0)}{h} - f(x)\right| \le 1/h \int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,dt$$

$$\le [\sup\{|f(t) - f(x_0)| : t \in [x_0, x_0+h]\}]$$

$\to 0$ as $h \to 0$ because f is continuous at x_0

Similarly we can show that $h>0$

$$\frac{F(x_0-h) - F(x_0)}{(-h)} - f(x) \to 0 \text{ as } h \to 0$$

$$\Rightarrow F'(x_0) = f(x_0)$$

$$j'(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

$$J(x) = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$$

Proof: Let $P = \{x_0, x_1 \ldots x_2 \ldots x_n\}$ of (a,b)

then for $1 \le i \le n$ we have

$$g(x_i) - g(x_{i-1}) = g'(t_i)\Delta x_i \quad ; \ t_i \in (x_{i-1}, x_i)$$
$$= f(t_i)\Delta x_i$$

$$\Rightarrow \sum_{i=1}^n (g(x_i) - g(x_{i-1})) = \sum_{i=1}^n f(t_i)\Delta x_i$$

Since $f \in R[a,b]$

$\lim R(f,P,T) = \int f(t)\,dt$ exists

$g(b) = R(f,P,T)$

$$\Rightarrow \int_a^b f(t)\,dt = g(b) - g(a)$$

$|f(x)| \le M$

$f \in R[a,b] \Rightarrow f$ is Darboux Integrable

* suppose $\lim_{\mu(P)\to 0} R(f,P,T) = L$ exist, Given $\varepsilon > 0$, $\exists (P,T)$ tagged net

$$|R(f,P,T) - L| < \varepsilon/4 \quad \text{— ①}$$

for any tag T of P

$= \sum M_i \Delta x_i$

$R(f,P,T) = \sum f(t_i)\Delta x_i$

$|U(f,P) - R(f,P,T)| < \varepsilon/4$

$|R(f,P,T') - L(f,P)| < \varepsilon/4$

Choose t_i such that $f(t_i)$ is very close to M_i

$f(t_i)$ is very close to m_i

$U(f,P) - L(f,P) = U(f,P) - R(f,P,T) + R(f,P,T)$
$- R(f,P,T') + R(f,P,T') - L(P,f) < \varepsilon$

f is Darboux Integrable

Given $\varepsilon > 0$

$\underline{\int} f = \overline{\int} f = L$

f is Darboux Integrable $\Rightarrow U(f,P_1) - L(f,P_1) < \varepsilon/4$

P_1 is partition with $\{x_0, x_1, \dots x_n\} \to n_1 + 1$

Choose $\delta < \varepsilon/8Mn_1$

Choose $P \in \mathcal{P}$ or $[a,b]$

with $\mu(P) < \delta$

$|R(f,P,T) - L| < \varepsilon$

$P^* = P \cup P_1$

$U(f,P) - U(f,P^*)$

$L(f,P^*) - L(f,P) < \varepsilon/4$

13th/10/23 Recap

$f: [a,b] \longrightarrow \mathbb{R}$ $\quad |f(x)| \le M \quad \forall x \in [a,b]$ (bounded)

Thm: $f \in \mathcal{R}[a,b] \Leftrightarrow f$ is Darboux Integrable

Thm: Riemann-lebesque thm.

$f \in R[a,b] \Leftrightarrow$ "f is continous a.e on $[a,b]$"

$D = \{x \in (a,b);\ f$ is discontinous at $x\}$

is a zero set

$\forall \varepsilon > 0,\ D \subset \bigcup_{J=1}^{\infty} J_j,\ \sum l(J_j) < \varepsilon$

Sketch of proof:

$\Rightarrow f \in \mathcal{R}[a,b]$, Given $\varepsilon > 0$, $\exists \delta > 0$

$\exists L = \int_a^b f\,dx \quad$ s.t $\mu(P) < \delta$

$\Rightarrow |R(f,P,T) - L| < \varepsilon/4 \quad$ —①

$f(P)$

$L(f,P) = \sum m_i(f)\Delta x_i$

$U(f,P) = \sum M_i(f)\Delta x_i$

$L(f,P) \le R(f,P,T) \le U(f,P)$

$R(f,P,T') - L(f,P) < \varepsilon/4$

$U(f,T) - R(f,P,T'') < \varepsilon/4$

$U(f,T) - L(f,P) < U(f,T) - R(f,PT') + R(f,PT')$

$- R(f,P,T') - L(f,P) + R(f,P) < \varepsilon$

$\Leftarrow$

f is Darboux integrable over $[a,b]$

Given $\varepsilon > 0$, $\exists P_1 \in \operatorname{Par}[a,b]$ s.t

$U(f,P_1) - L(f,P_1) < \varepsilon/4$

$P_1 = \{x_0, x_1, \dots, x_{n_1}\}$

Let: $0 < \delta = \varepsilon/8Mn_1$

Take $P \in \operatorname{Par}[a,b]$ with $\mu(P) < \delta$

Let $P^* = P_1 \cup P$

$U(f,P^*) - L(f,P^*) < \varepsilon/4$ — ⓢ

$|R(f,P,T) - L'| < \varepsilon$

$P = \{x_0, x_1 \dots x_n\}$

$U(f,P) - U(f,P^*) < 2M\delta(n-1) < \varepsilon/4$

Illy

$L(f,P^*) - L(f,P) < \varepsilon/4$

$\overbrace{\qquad\qquad\qquad\qquad}^{R(f,P,T)}$

$L(f,P) \le L(f,P^*)$ $\qquad\qquad$ $U(f,P^*) \le U(f,P)$

$L(f,P^*) \le L'' \le U(f,P^*)$

$\Rightarrow \underbrace{L(f,P) \le L' \le U(f,P)}_{R(f,PT)}$

$f:(X,d) \longrightarrow \mathbb{R}$

$x \in X$; $r > 0$

$\Omega_f(x,r) = \Omega_f(x, B_d(x,r)) = \sup\{|f(u) - f(v)| : u, v \in B_d(x,r)\}$

$\omega_f(x) = \inf\{\Omega_f(x,r) ; r > 0\} \to$ oscillation of f at x

Show the above function upper semicontinuous

and also f is continuous at $x \Leftrightarrow \omega_f(x) = 0$

Upper semicontinuous

$g : X \to \mathbb{R}$ upper semicontinuous

$\forall b \in \mathbb{R}$ $\quad g^{-1}(-\infty, b)$ is open

$g^{-1}(a, \infty)$ is op.

$f \in R[a,b]$

$D = \{x \in f ; f \text{ is discontinous at } x\}$

$= \{x \in f . \ w_f(x) > 0\} = \bigcup_{k=1}^{\infty} D_k$

where $D_k = \{x \in [a,b] : w_f(x) \geq 1/2k\}$

Given $\varepsilon > 0$, P_k

$U(f, P_k) - L(f, P_k) < \varepsilon/4k$ $\qquad P_k = \{x_0^k, x_1^k \dots x_n^k\}$

there is a partition,

and if $x \in D_k \cap (x_{k-1}^k, x_j^k)$ $\exists \ r > 0$ (ST)

$(x-r, x+r) \subset (x_{1-1}^k, x_1^k)$

$1/2k < w_f(x) = \Omega_f(x, (x-r, x+r)) = M_1^k(f) - m_1^k(f)$

$1/2k \ \Sigma' \Delta x_1 \leq U(f, P_k) - L(f, P_k) \leq \varepsilon/4k$

$\Sigma \Delta x_i < \varepsilon/2k$

2^k can be covered be finite number. Since there is only finite partitions.

Suppose f is continous almost every
you need to find paration, when
$U(f, P) - L(f, P) < \varepsilon$
to do by yourself.

Let f & g is Riemann Integrable, product is Riemann Integrable,

also composition of $f \circ g$ is also Riemann Integrable.

It is you asked Integrability criterion
dont use Riemann-Lebesgue thm
make def clear Rieman & darboux
&
Imp-

Convergence
when is coled interval should be compact

Sequence and series functions

E set ($E \subseteq \mathbb{R}$ or $E \subseteq (X,d)$)
metric space

suppose for each $n \in \mathbb{N}$

$f_n : E \longrightarrow \mathbb{R}$ is function

Then we say that $(f_n)_{n \in \mathbb{N}}$ is sequence of function over E

suppose $x \in E$ then $(f_n(x))$ is sequence of real n.

suppose $(f_n(x))$ is component $\forall x \in E$

Then we define

$f : E \longrightarrow \mathbb{R}$

$f(x) = \lim_n f(x)$

we call f the limit function of (f_n)