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Construction of Real numbers system

$\mathbb{N}$ = Set of natural numbers

$\phi \longleftrightarrow 0$ $\qquad$ $0, 1, 2, \cdots n$

$\{\phi\} = \{0\} \longleftrightarrow 1$ $\qquad$ $n^+ = n+1 = n \cup \{n\}$

$\{0,1\} \longleftrightarrow 2$ $\quad \cdots$

$\mathbb{Z} = \mathbb{N} \times \mathbb{N}$

$(n,m) \sim (n',m')$

$n+m' = n'+m'$

This $\sim$ is equivalence relation

reflexive $\forall (n,m) \sim (n,m)$ $\quad (n+m = n+m)$

symmetry $(n,m) \sim (n',m)$

$\Rightarrow (n',m) \sim (n,m)$ $\quad (n+m' = n'+m)$

$n'+m = n+m'$

transitive $(n,m) \sim (n',m')$

$(n',m) \sim (n'',m')$

$\Rightarrow (n,m) \sim (n'',m')$

$[(n,m)] + [(r,s)] = [(n+r), (m+s)]$

$[(n+m)] \cdot [(r,s)] = [(nr+ms) \quad (mr+ns)]$

Ex: $(\mathbb{Z}, +, \cdot)$ is commutative ring

Proof: $[(1,1)] = 0 = [(n,n)]$; additive identity)

$[(2,1)] = [(n+1, n)]$; multiplicative identity.

$[(n+1, 1)] \longleftrightarrow n$

$[(1, n+1)] \longleftrightarrow -n$ $\quad$ Inverse

$\mathbb{Z} = \{0, \pm 1, \pm 2, \qquad \}$

$\underset{n}{[(n+1,1)]} \cdot \underset{m}{[(m+1,1)]} \longrightarrow [(nm+m+n+2, m+n+2)]$

$\rightarrow \underset{nm}{[(nm+1, 1)]}$

$\mathbb{Q}$ = field of rational numbers

$\mathbb{Z} \times \mathbb{Z}^*$, $\approx$

$(p,q) \approx (r,s)$ if $ps = qr$

Ex: i) $\approx$ is equivalence.

ii) $\approx (p,q) = p/q$ is the equivalence of (p,q)

$$\mathbb{Q} = \frac{\mathbb{Z}\times\mathbb{Z}^*}{\approx} = \text{Set of equivalence class of } \approx$$

Define

$$p/q + r/s = \frac{ps+qr}{qs} \quad ; \quad p/q \cdot r/s = pr/qs$$

iii) $(\mathbb{Q}, +, \cdot)$ is field

$(0, p) = 0/r$ additive

$(p, p) = p/p = 1$ multiplicative

$p = p_1 d$, $q = q_1 d$; $d = \gcd(p,q)$

$$p/q = \frac{p_1}{q_1} = \frac{-p_1}{q_1}$$

Standard form

$p/q > r/s$ if $ps > qr$

otherwise $p/q > 0$ if $pq > 0$

Ex: Show that $\mathbb{Q}$ is an ordered field

Ex: Show that $\mathbb{Q}$ is not Dedekind complete

$A = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 0\}$

Let $p \in \mathbb{Q}$. (There exist rational numbers whose square is 2?)

$$q = p - \left(\frac{p^2-2}{p+2}\right) = \frac{2(p+1)}{p+1} > 0$$

$$q^2 - 2 = \frac{4(p+1)^2}{(p+2)^2} - 2 = \frac{2(p^2-2)}{(p+2)^2}$$

If $p^2 < 2$, then $p < q$ and $q^2 < 2$

If $p^2 > 2$, then $q < p$ and $q^2 > 2$

Dedekind Construction of Real numbers

We consider the set of real numbers $\mathbb{R}$ as a subset of the power set $\mathcal{P}(\mathbb{Q})$ of $\mathbb{Q}$

Constructing of so called cuts

Def: A subset $\alpha \subseteq \mathbb{Q}$ is called Dedekind cut if

i) $\phi \neq \alpha \neq \mathbb{Q}$

ii) If $p \in \alpha$ & $q < p$, then $q \in \alpha$.

iii) If $p \in \alpha$, then $\exists\, r \in \alpha$ s.t $p < r$

Ex: Show that $\mathbb{Q}$ is an ordered field

Ex: Show that $\mathbb{Q}$ is not Dedekind complete

Proof: $\mathbb{Q}$ is ordered field.

$P \subset F$ subset such

$x \in F$, exactly one of the following

$x \in P$, $-x \in P$ $x = 0$

$x, y \in P$, $xy \in P$ $x+y \in P$

$x \leq y \Leftrightarrow y - x \in P \vee x = y$.

let $a, b \in S$ $S \subset \mathbb{Q}$

$a+b \in S$, $a \cdot b \in S$,

So $\mathbb{Q}$ is ordered field

Dedekind complete (LUB) $\forall$ non empty bounded above subset of K has supremum in itself (K). Then K is dedekind complete

complete field:- (cauchy) If $\forall$ cauchy sequence of element in K converges to a limit that is also in K. i.e for $\forall$ (a_n) $\lim_{n \to \infty}(a_n) \in K$.

Example $\mathbb{R}(x)$ complete but not dedekind complete

$\mathbb{A}$ (field of algebraic numbers) is dedekind complete

but not complete or constructible numbers

$\mathbb{Q}$ is not dedekind: for $\forall$ non empty bounded able subset of K isn't have suporm in K.

$$A = \{x : x > 0, x^2 \leq 2 \quad x \in \mathbb{Q}\}$$

let $x = 1$, $x^2 = 1$

$\Rightarrow x \in A$

A is nonempty.

Consider $x = 3$

$x > 3$ but $x^2 = 9 \nleq 2$

So $x \neq 3$

A is bounded above.

Since A is nonempty bounded above set,

let $U = \sup(A)$

let $x = \sqrt{2}$, $x^2 = 2$

claim $\sup(A) = \sqrt{2}$

If b, $\sup(A)$

$\Rightarrow b < \sqrt{2}$,

by dense $b < c < \sqrt{2}$

squaring $b^2 < c^2 < \sqrt{2}$

$c \in A$, $b < c$.

so b is not $\sup(A)$.

So $\sup(A) = \sqrt{2}$

but $\sqrt{2}$ doesn't belong to $\mathbb{Q}$. So

$\mathbb{Q}$ is not dedekind complete

$\{ r \in \mathbb{Q} : r < 1 \}$

$\mathbb{R} = \{ \alpha \subseteq \mathbb{Q} : \alpha \text{ is a cut} \}$

Define an order on $\mathbb{R}$ by

$\alpha \leq \beta$ if $\alpha \subseteq \beta$

Ex: 1) Show that $\leq$ is total order on $\mathbb{R}$

2) $\forall$ nonempty bdd ve subset of $\mathbb{R}$ has the supremum in $\mathbb{R}$

$\mathcal{A} \subseteq \mathbb{R}$

$\exists \gamma \in \mathbb{R}$

$\alpha \in \mathcal{A} \Rightarrow \alpha \subseteq \gamma$

$\bigcup_{a \in \mathcal{A}} \alpha = \beta$

try to explain $\mathbb{R}$ order & sth..

Addition

$\alpha, \beta \in \mathbb{R}$

$\alpha + \beta = \{ p + q : p \in \alpha, q \in \beta \}$ is cut

$p \in \mathbb{Q}$, $p^* = \{ r \in \mathbb{Q} : r < p \}$ $\rightarrow$ rational cut

0^*, 1^*

$\alpha \in \mathbb{R}$

Let γ be given by

$\gamma = \{ p \in \mathbb{Q} : -p - r \notin \alpha$ for some $r \in \mathbb{Q}$ with $r > 0 \}$

$-(q - r)$

Check: $\alpha + \gamma = \gamma + \alpha = 0^*$ $\qquad$ $\gamma = -\alpha$

Exercise $(\mathbb{R}, +, 0^*)$ is an Abelian group

$\alpha > 0^*$, $\beta > 0^*$

$\alpha\beta = \{ r \in \mathbb{Q} : r \leq pq$ for some positive $p \in \alpha$ and positive $q \in \beta \}$ is a cut

$\alpha 1^* = 1^* \alpha = \alpha$

$S = \{ r \in \mathbb{Q} : rp < 1 - s \;\; \forall\, 0 < p \in \alpha \quad s \in \mathbb{Q}$ with $s > 0 \}$

$(\mathbb{R}^+, \cdot, 1^*)$

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Dedekind construction of Real number

$\mathbb{Q} \to P(\mathbb{Q}) =$ the set of all subsets of $\mathbb{Q}$

power set of $\mathbb{Q}$

$\mathbb{R} = \{ \alpha \in P(\mathbb{Q}) : \alpha$ is a Dedekind cut $\}$

$\alpha \subseteq \mathbb{Q}$ $\quad \phi \neq \alpha \neq \mathbb{Q}$

$\alpha \leq \beta$ if $\Rightarrow \alpha \subseteq \beta$

$A = \{ \alpha_j \; j \in \mathcal{J} \} \subseteq \mathbb{R}$

$\alpha_j \leq \Rightarrow \bigcup_{j \in J} \alpha_j$

$\alpha + \beta = \{ p + q \; ; \; p \in \alpha \; q \in \beta \}$

$p^* = \{ r \in \mathbb{Q} ; r < p \}$

0^* $\to$ additive identity

$\alpha + 0^* = 0^* + \alpha = \alpha$

do what

$\alpha \in \mathbb{R}$

$\gamma = \{ p \in \mathbb{Q} : \exists r \in \mathbb{Q} ; r > 0$ with $-p - r \notin \alpha \}$

$-(\mathbb{Q} \setminus \alpha)$

$\alpha > 0^*$, $\beta > 0^*$

$\alpha\beta = \{ r \in \mathbb{Q} : r \leq pq$ for $p \in \alpha, q \in \beta$ both $p > 0, q > 0 \}$

$\alpha \, 1^* = 1^* \alpha = \alpha$

$p \in \alpha ; \; \exists p' \in \alpha$ with $p < p'$

$p = p'(p/p')$

$\alpha > 0$ $\quad \exists s \in \mathbb{Q} ; s > 0$

$\delta = \{ p \in \mathbb{Q} : pr < 1 - s \; , \forall r \in \alpha, r > 0 \}$

$\alpha \delta = \delta \alpha = 1^*$

$$\alpha\beta \begin{cases} -((-\alpha)\beta) & \alpha < 0^* \quad \beta > 0^* \\ -((\alpha)(-\beta)) & \alpha > 0^* \quad \beta < 0^* \\ ((-\alpha)(-\beta)) & \alpha < 0^* \quad \beta < 0^* \\ (\alpha)(\beta) & \alpha > 0^* \quad \beta > 0^* \end{cases}$$

$\alpha \, 0^* = 0^* \alpha = 0^*$

$\mathbb{Q} \longrightarrow \mathbb{R}$

$p \longmapsto p^*$

$(p+q^*) = p^* + q^* \; ; \; (pq^*) = p^* q^*$

$p_1 < p_2 \Rightarrow p_1^* < p_2^*$

$$\begin{array}{ccc} & \mathbb{Q} & \xrightarrow{j} \\ \nearrow & & \\ \mathbb{R} & \xrightarrow[f]{} & \mathbb{K} \end{array}$$

doubt

$f(r) = j(r)$ if $r \in \mathbb{Q}$

$\alpha = \{ p \in \mathbb{Q} : p^* < \alpha \}$

$f(\alpha) = \sup \{ j(r) : r^* < \alpha \}$

$$x - \frac{\lfloor 2^n x \rfloor}{2^n} = \frac{2^n x - \lfloor 2^n x \rfloor}{2^n} \leq \frac{1}{2^n}$$

$$j\left(\frac{\lfloor 2^n x \rfloor}{2^n} \right) \longrightarrow x \qquad n \to \infty$$

$$f(x) = \lim_{n \to \infty} j\left(\frac{\lfloor 2^n x \rfloor}{2^n} \right)$$

$\phi \leq \alpha \leq \mathbb{Q}$

$-\infty < \quad < \infty$

$\bar{\mathbb{R}} = \mathbb{R} \cup \{ -\infty, \infty \}$

$\infty + \infty = a + \infty = \infty \quad ; \quad a - \infty = (-\infty) + a = -\infty$

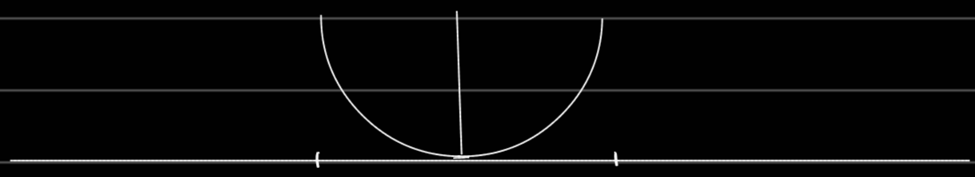
$a \infty = \infty a = \begin{array}{l} \to \infty \quad a > 0 \\ \to -\infty \quad a < 0 \end{array}$

$\infty(-\infty) = (-\infty) a = \begin{array}{l} \to \infty \quad a < 0 \\ \to -\infty \quad a > 0 \end{array}$

$\infty \infty = (-\infty)(-\infty) = \infty$

$(-\infty)(\infty) = (\infty)(-\infty) = (-\infty)$

$\left. \begin{array}{l} \infty - \infty \\ 0 \infty \end{array} \right]$ not defined.



Set theory

- If $A \to B$ is bijection, then we say that A & B are equivalent

$A \sim B$

- X is if X is equivalent to proper subset

$X \to X$ Injective but not surjective

X is finite, if it not infinite

example : $\mathbb{N}$'s infinite

we if X is hh

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$A \sim B$ if $\exists f: A \to B$. f is bijection
↳ equipotent

ex: $a < b$ are real numbers

$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$

$(a,b) = \{x \in \mathbb{R} \quad a < x < b\}$

$(-\infty, b) = \{x \in \mathbb{R} : \quad x < b\}$

$(-\infty, \infty) = \mathbb{R}$

$[0,1] \sim [a\, b]$

$t \longmapsto (1-t)a + tb$

$(0,1) \sim \mathbb{R}$?

X is equipotent to a proper subset of X

$X \sim A \; ; \quad A \subsetneq X$

Def: X is Infinite [If $f: X \to$ Injective but not bijective]

Ex: $\mathbb{N}$ is Infinite

$X \subset Y$

If X is infinite $\to$ Y is Infinite

If Y is finite $\to$ X is finite

X is Infinite $\to$ $X \setminus \{x\}$ is infinite

* X is denumerable If $X \sim \mathbb{N}$

countable If either finite or denumerable

uncountable if it is not countable

Ex: $\mathbb{N} \times \mathbb{N} \longleftarrow \mathbb{N}$

$(m,n) \longleftarrow P = 2^{m-1}(2n-1)$

$(m,n) \longrightarrow 2^{m-1}(2n-1)$

Ex: Countable union of countable sets are countable

abcdefghijklmnopqrstuvwxyz

$[0,1]$ is uncountable

$x_1 \to 0.\, x_{11} \; x_{12} \; x_{13} \; \ldots\ldots$

$x_2 \to 0 \; x_{21} \; x_{22} \; x_{31} \; \ldots$

$x_3 \to 0.\, x_{31} \; x_{32} \; x_{33} \; \ldots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

we take $Y = 0.\, y_1 y_2 y_3 \ldots$

$$y_i = \begin{cases} 1 & \text{if } x_{ii} \neq 1 \\ 2 & \text{if } x_{ii} = 1 \end{cases}$$

then Y would be different from $x_1, x_2 \ldots$ [complete different from the list]

This is called cantor's digonal trick.

Q have bijection with $\mathbb{N}$. ?

Proof construct countable seq with [illegible]

Thm: Schroder - Benstein

If $f: x \to y$ and $g: Y \to X$ are Injective map, then $X \sim Y$
The have same cardinal.

$[1,0] \subseteq \mathbb{R}$

$\tan: (-\pi/2, \pi/2) \to \mathbb{R}$ $\overset{tant}{\rightsquigarrow}$ $(0,1) \subseteq [0,1]$

$[0,1] \setminus \{0, 1, 1/2, 1/3 \ldots\ldots$
$(0,1) \setminus \{1, 1/2, 1/3, 1/4 \ldots\ldots$

Norms :- a function $f: \mathbb{R}/\mathbb{C} \to \mathbb{R}^+ \cup \{0\}$

Vector space

behave like distance from origin.

absolute value on $\mathbb{R}$

$|x| = \max\{x, -x\}$

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R} = \{(a_1, a_2 \cdots a_n) \; a_i \in \mathbb{R}\}$

$\mathbb{R}^n$ is vector space over $\mathbb{R}$

$(a_1, a_2, a_3 \ldots a_n) + (b_1, b_2, b_3, \ldots b_n) = (a_1+b_1, a_2+b_2, \ldots a_n+b_n)$

$\alpha(a_1, \alpha_2, \ldots \alpha_n) = (\alpha a_1, \alpha a_2, \alpha a_3 \ldots \alpha a_n)$

$e_1 = (1, 0, 0 \ldots 0)$

$e_2 \; (0, 1, 0 0 \ldots)$

$e_n = (0, 0 0 \ldots, 1)$

$\{e_1, e_2 \ldots e_n$ is basis of $\mathbb{R}^n$

$a = (a_1, a_2, a_3 \ldots a_n)$

$b = (b_1, b_2, b_3 \ldots b_n) \quad \in \mathbb{R}^n$

$\langle a, b \rangle$ = inner product of a & b $\quad |a \cdot b|$

Eulidean inner product in $\mathbb{R}^n$

$= a_1 b_1 + a_2 b_2 \cdots \quad a_n b_n = \sum_{i=1}^{n} a_i b_i$

$(\mathbb{R}^n, \langle , \rangle)$ is n dimensional Euclidean space

Euclidean norm on $\mathbb{R}^n$

$\|a\| = +\sqrt{\langle a, a \rangle} = \sqrt{\sum_i a_i^2}$

properties

1) $\|a\| \geq 0$ and $\|a\| = 0 \iff a = 0$
2) $\|\alpha a\| = |\alpha| \|a\|$
3) $\|a+b\| \leq \|a\| + \|b\|$ (Triangle inequality)

$0 \leq \|a + tb\|^2 = \langle a+b, a+b \rangle$

$= \sum (a+b)^2$

$= \sum a_i^2 + t^2 \sum b_i^2 + 2t \sum ab$

$= A + t^2 C + 2t B$

Then discriminant $\leq 0 \Rightarrow (2B)^2 - 4AC \leq 0$

$B^2 \leq AC$

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \sqrt{\left(\sum a_i^2\right)} \sqrt{\sum^n b_i^2}$$

$$|\langle a, b\rangle| \le \|a\| \, \|b\|$$ → cauchy – Schwarz inequality

$$\Rightarrow |a \cdot b| \le \|a\| \|b\|$$

Proof: $\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i$

$\|a\| = \left(\sum_{i=1}^{n} a_i\right)$

$\|b\| = \left(\sum_{i=1}^{n} b_i\right)$

we know

$$\left(\sum a_i b_i\right)^2 < \left(\sum^n a_i^2\right)\left(\sum^n b_i^2\right)$$ → by lagrange identity

$$\left(\sum a_i b_i\right) < \left(\sum a_i^2\right)^{1/2}\left(\sum^n b_i^2\right)^{1/2}$$

$$|\langle a, b\rangle| \le \|a\|_2 \|b\|_2$$

lagrange

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right) + \left(\sum_{i=1}^{n} b_i^2\right) - 2\left(\sum_{1 \le i < j \le n}^{n} a_i b_j - a_j b_i\right)^2$$

$a = (a_1, a_2, a_3 \dots a_n) \in \mathbb{R}^n$, $b = (b_1, b_2, \dots b_n) \in \mathbb{R}^n$

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k}^{n} b_k^2\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 =$$

$$= \sum_{k}^{n} a_k^2 b_k^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} a_j^2 b_i^2$$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i b_i a_j b_j$$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left(a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i b_j a_j b_i\right)$$

$$= \sum^{n-1}\sum_{j=i+1}^{n} a_i^2 b_j^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} - 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i b_j a_j b_i$$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n-1}\sum_{i+1}^{n} a_j^2 b_i^2 - 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i b_j a_j b_i$$

$$\rightarrow \left(\sum_{k=1}^{n} a_k b_k\right)^2 - \left(\sum_{k=1}^{n} a_k^2 \sum^{n} b_k^2\right) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2$$

By this.

$$\|a+b\|^2 = \langle a+b, a+b\rangle$$

$$= \langle a, a\rangle + 2\langle a, b\rangle + \langle b, b\rangle$$

$$= \|a\|^2 + 2\langle a, b\rangle + \langle b, b\rangle$$

$$= \|a\|^2 + 2\langle a, b\rangle + \|b^2\|^2$$

$$\le \|a\|^2 + 2\|a\| \|b\| + \|b\|$$

$$\le \|a\| + \|b\|$$

$$\Rightarrow \|a+b\| \le \|a\| + \|b\|$$ → triangle Inequality

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$\| \; \| : V \to \mathbb{R}$

V is a vector space over $\mathbb{R}$

$1 \le p < \infty$

$$\|a\|_p = \left[\sum^n |a_i|^p\right]^{1/p}$$

other norms

$\|a\|_2 = \left(\sum_i a_i^2\right)^{1/2}$

$\|a\|_1 = \sum_i |a_i|$

$\|a\|_\infty = \sup\{|a_i| : 1 \le i \le n\}$

$d_e(a, b) = \|a-b\|_2$

$d_1(a, b) = \|a-b\|_1$

$d_{\sup}(a, b) = \|a-b\|_\infty$