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Compact set: $(X,d) \rightarrow$ metric space

$K \subseteq X$, K is compact if $\forall$ opencovers, $\mathcal{C} = \{G_\alpha : \alpha \in I\}$ of K, i.e $K \subseteq \bigcup_{\alpha \in I} G_\alpha$, there is a finite subset $\{\alpha_1, \alpha_2, \dots \alpha_n\} \subseteq I$ such that

$K \subseteq G_{\alpha_i}$ for some n.

Proposition: K is compact

$\Rightarrow K$ is closed and K is bounded

Example: $X = l^\infty$, $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$

$$S = \{(a_n) \in l^\infty . \|(a_n)\|_\infty = 1\} \subseteq B(0,2)$$

Ex: Show that the subset

$S = \{x \in X : d(p,x) = 1\}$ is closed.

$$S = \{x \in X : d(p,x) \leq 1\} \setminus \{x \in X \; d(x,p) < 1\}$$
$$= \bar{B}(p,1) \setminus B(p,1)$$
$$= \bar{B}(p,1) \cap (X \setminus B(p,1)) \qquad A \setminus B = A \cap (X \setminus B)$$

$\because \bar{B}(p,1)$ & $X \setminus B(p,1)$ is closed. and Intersection 2 close space is closed

2 .

S's closed.

compactness an intergrane property

Observation: Let Y be a subspace of (X,d) and $K \subseteq Y \subseteq X$ then K is compact of $Y \Leftrightarrow K$ is compact of X

Idea of Proof: $\{G_\alpha : \alpha \in I\}$ is opencover of K in X

then $\{H_\alpha : \alpha \in I\}$; open cover of K $\quad H_\alpha = G_\alpha \cap Y$

Definition:- Let (X,d) be a metric space and $K \subseteq X$. we say that K is sequentially compact subset of X if $\forall$ sequence in K has a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ with $\lim_{k \to \infty}(a_{n_k}) = p \in K$

Propositions: Let K be compact subset of (X,d). then K's sequentially compact

Proof: Suppose K is compact but it is not sequentially compact

Then $\exists$ a sequence $(a_n)_{n \in \mathbb{N}}$ in K such $(a_n)_{n \in \mathbb{N}}$ has no subsequence that converge in K.

Thus for each $p \in K$, p's not an accumulation point of the set $\{a_n . n \in \mathbb{N}\}$. of terms of the sequence $(a_n)_{n \in \mathbb{N}}$

Thus $\exists r_p > 0$ such that the open ball $B(p, r_p)$ contains no point of $\{a_n : n \in \mathbb{N}\}$ exactly possibly p

Now consider the open $\{B(p, r_p) : p \in K\}$ of K. By compactness of K, there are finitely many points, $p_1, p_2 \dots p_n \in K$

s.t $K \subseteq \bigcup_{i=1}^{n} B(p_i, r_{p_i})$

Thus $\exists$ a subsequence (a_{n_k}) of $(a_n)_{n\in\mathbb{N}}$ such that
$a_{n_k} \in B(p_j, r_{p_j})$ for some r;

However that $a_n = p_j$ $\forall k$ $((B(p_j, r_{p_j}) \setminus \{p_j\}) \cap \{a_n : n\in\mathbb{N}\})$

$\Rightarrow$ the subsequence (a_{n_k}) is a constant (and compact) subsequence $=\phi$)

Let $K \subseteq (X,d)$ and $\{G_\alpha : \alpha\in I\}$ be a an open cover of K

A number $\lambda > 0$ is called a lebesgue covering number for the open cover $\mathcal{C}$,

If $\forall$ $a \in K$, $\exists \alpha \in I$ s.t

$B(a, \lambda) \subseteq G_\alpha$

$a \in K \subseteq \bigcup_{\alpha\in I} G_\alpha$

Ex: If K is compact subset of (X,d),
then $\forall$ open cover of K have lebesgue covering number

theorem: (lebesque covering lemma) $\forall$ open cover of a sequentially compact subset of (X,d) has a lebesque covering number

Proof: by contradiction

suppose a open cover $\mathcal{C} = \{G_\alpha : \alpha\in I\}$ of K,

$\lambda > 0$, $\exists a \in K$ s.t $B(a,\lambda) \nsubseteq G_\alpha$ $\forall \alpha\in I$

$\lambda = 1/n$ $a_n \in K$; $B(a_n, 1/n) \nsubseteq G_\alpha$.

(thereby we prove we say the is not sequential subset of (X,d) has a lebesque covering number

yet to be prove

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$(X,d) \rightarrow$ metric space $K \subseteq X$ (subset)

1) K is compact; $\forall$ open cover of K has a finite sub cover
$\Rightarrow$ K is closed.

2) K is sequentially compact, If sequence $(a_n)_{n\in\mathbb{N}}$ in K has sub sequence $(a_{n_k})_{k\in\mathbb{N}}$ that converge in K

Thm: K is subset of a metric (X,d) then follow are equivalent

i) K is compact

ii) K is sequentially compact

Defn: let $\{G_\alpha : \alpha \in I\}$ be an open cover, a subset $K \subseteq (X,d)$, a number $\lambda > 0$ is called covering number for open cover $\{G_\alpha : \alpha \in I\}$ of K, if $\forall a \in K$ $\exists \alpha \in I$ s.t $B(a,\lambda) \subseteq G_\alpha$

{minimum no. of open sets covers space (K)}.

Exercise: If K is compact, then show that every open cover of K, has covering number.

Proof: x be any arbitrary point in K.

there should exist λ open cover from $\{G_\alpha\}$ such that these set cover x. denoted by $\{G_{\alpha_1}, G_{\alpha_2} \dots G_{\alpha_\lambda}\}$.

$$x \in G_{\alpha_1} \cup \dots \cup G_{\alpha_\lambda}$$

let take i, $1 \le i \le \lambda$

we want $B(x,\lambda) \subseteq G_{\alpha_i}$

proposition: Lebesgue's covering number.

$\forall$ open cover of a sequentially compact subset has a covering number

proof: by contradiction

Let K be sequentially compact and $\{G_\alpha : \alpha \in I\}$ be an open cover of K that do not have covering number.

$\therefore$ $\forall$ $\lambda > 0$, $\exists$ $a \in K$ such that

$B(a,\lambda) \nsubseteq G_\alpha$ $\forall \alpha \in I$

let $\lambda = 1/n$

$\exists$ $a_n \in K$ such that $B(a_n, 1/n) \nsubseteq G_\alpha$ $\forall \alpha \in I$

Now since $(a_n)_{n \in \mathbb{N}}$ is a sequence in K and K is sequential compact

There is a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ s.t

$$\lim_{n \to \infty} (a_{n_k}) = P \in K$$

Since $P \in K$, $\exists$ $\alpha_0 \in I$ such that

$B(P,r) \subseteq G_{\alpha_0}$ for some $r > 0$

for this $r > 0$ $\exists$ $k_0 \in \mathbb{N}$ s.t

$$d((a_{n_k}), P) < r/2 \quad \forall k \ge k_0$$

Now pick $k \,(\ge k_0)$ so that $1/n < r/2$

Claim $B(a_{n_k}, 1/n_k) \subseteq B(P,r) \subseteq G_{\alpha_0}$

Suppose $y \in B(a_{n_k}, 1/n_k)$

$d(y, a_{n_k}) < 1/n_k < r/2$

now since

$$d(y,P) \le d(y,a_{n_k}) + d(a_{n_k}, P)$$
$$< r/2 + r/2 = r$$

$\Rightarrow d(y,P) < r$

$\Rightarrow y \in B(P,r)$

$\Rightarrow$ contradiction as $B(a_n, 1/n) \nsubseteq G_\alpha \; \forall \alpha$.

Proof for $2 \Rightarrow 1$

Suppose K is compact sequential compact, we shall show that K's compact

let $\{G_\alpha ; \alpha \in I\}$ be an open cover of K

By lebesgue covering lemma, the given open cover has every number $(\lambda > 0)$

let $a \in K$ then $\exists \alpha_1 \in I$ such that

$B(a_1, \lambda) \subseteq G_{\alpha_1}$

If $K \subseteq G_{\alpha_1}$, then $\{G_{\alpha_1}\}$ is finite subcover

other wise $\exists a_2 \in K \setminus G_{\alpha_1}$ and again $\exists \alpha_2 \in I$

$B(a_2, \lambda) \subseteq G_{\alpha_2}$

If $K \subseteq G_{\alpha_1} \cup G_{\alpha_2}$, we have that

$\{G_{\alpha_1}, G_{\alpha_2}\}$ as a finite subcover

otherwise $\exists a_3 \in K \setminus \{G_{\alpha_1} \cup G_{\alpha_2}\}$

$\vdots$

Continuing this process, if K has no finite subcovers then we obtain sequence $(a_n)_{n \in \mathbb{N}}$ in K and a seq of open set $(G_{\alpha_i})_{\alpha_i \in I}$ s.t

$B(a_n, \lambda) \subseteq G_{\alpha_n}$ and $a_{n+1} \in (K - (\bigcup_{i=1}^{n} G_{\alpha_i}))$

Clearly $d(a_n, a_m) \geq \lambda$ for $m \neq n$

$\therefore$ $(a_n)_{n \in \mathbb{N}}$ has no convergent subseque

$\Rightarrow \Leftarrow$ If K is sequential compact, $\exists$ a convergent subsequence

$\therefore$ If K is sequentially compact, then there must be some finite cover

Exercise

① let $f : (X, d) \to (Y, d)$ be a continuous map, If $K \subseteq X$ is compact then show that $f(K) \subseteq Y$ is compact

② If (X, d) is compact K is closed subset of X, then show that K is compact

③ $d_{sum}((x_1, x_2)(y_1, y_2)) = d(x_1, x_2) + d(y_1, y_2)$

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ans) $f:(X,d)\longrightarrow(Y,\rho)$

let $\mathcal{O}$ be an open cover of $f(K)$ in Y

we want $\exists$ a finite subcover of $\mathcal{O}$.

$\{f^{-1}O : O\in\mathcal{O}\}$ in X. Since f is continuous

we have K in X given $f^{-1}(O)$ ...