Q1. i) Let X be an infinite set and $x \in X$. Show that $X \setminus \{x\}$ is also infinite. Further, show that X contains a denumerable subset.

ii) Show that $\mathbb{N} \times \mathbb{N}$ is denumerable. Deduce that if A and B are countable, then $A \times B$ is also countable. Show that a countable union of countable sets is countable. Hence or otherwise, show that $\mathbb{Q}$ = the set of rational number, is countable

iii) Show that $[0,1]$, $(0,1)$ and $\mathcal{P}(\mathbb{N})$ are uncountable.

ans) ① Consider $X \setminus \{x\}$ be finite set

Then $\exists\, n \in \mathbb{N}$ (st)

$|X \setminus \{x\}| = n$

Then $|X| = |X - \{x\}| + |\{x\}|$

$|X| = n + 1$

but it contradictin X being Infinite set

so $X - \{x\}$ should Infinite.

Since X is Infinite set, it should either denarmerable or not.

If denarmerable, and X is subset X itself.

X have subset of denarmerable

If it is not denarmerable.

Let $x_1, x_2 \cdots x_n \cdots$ be elements in X

by cantor digonal methed, we first map ϕ $\mathbb{N} \to X$

$1 \to x_1 \quad 2 \to x_2 \quad \cdots \quad n \to x_n$

Then $\exists\, a$ such that $a \in X \quad a \notin X$

Then $x_i \in X$ in ϕ is denarmerable subset

belongs to X

2) $N \times N$ can be represent (n,m) wher $n,m \in N$

than we enumerate these as

$$
\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & \vdots \\
(3,1) & (3,2) & (3,3) & (3,4) & \\
(4,1) & (4,2) & (4,3) & (4,4) & \\
\vdots & \vdots & & &
\end{array}
$$

where each k^{th} diagonal has k ele

so total number point 1 to k diagonal is

$$d(k) = 1+2+\cdots+k = \frac{k(k+1)}{2}$$

on k^{th} diagonal sum of digit is $k+1$

the point (m,n) would be in $k = m+n-1$

and m^{th} point as we downward. so taken $k-1$ diagonal and then count to m.

so Let $\phi : N \times N \longrightarrow N$

$$\phi(m,n) = d(m+n-2)+m$$

$$= \tfrac{1}{2}(m+n-2)(m+n-1)+m$$

$\Rightarrow$ so their exist bijection b/w $N \times N$ onto. N

b) countable union of countable sets are countable.

Let consider $A_1, \ldots A_n, \ldots$ countable sets.

Then we denote element of these sets as a_{mn}

where: $a_{mn} \Rightarrow$ the m^{th} element of A_n

Then we elements of $\bigcup_{i=1} A_i$ can have bijection

with $\mathbb{N}$ $\left(\bigcup_{i=1} A_i \text{ is } \mathbb{N}\times\mathbb{N} \right.$
and we showed $(\mathbb{N}\times\mathbb{N}$ denumeral 2)

$\therefore$ countable union of countable set are countable.

Let A & B are countable set

$\Rightarrow \exists\ \phi_1 : A \longrightarrow \mathbb{N}$

$\phi_2 : B \longrightarrow \mathbb{N}$

where ϕ_1 & ϕ_2 are Bijection

consider $A\times B$.

then $\exists\ \psi : A\times B \longrightarrow \mathbb{N}\times\mathbb{N}$

$\{(x,y) \longrightarrow (n,m)\ ;\ x\in A, y\in B,\ n,m\in\mathbb{N}\}$

$g : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$ is bij

$g\circ\psi : A\times B \longrightarrow \mathbb{N}$ is bijection.

$\Rightarrow$ $A\times B$ is countable.

II(i) $[0,1]$, $(0,1)$, are countable and represented by

$x_1 = 0.a_{11} a_{12} a_{13} \cdots a_{1m} \cdots$

$x_2 = 0\ a_{21} a_{22} a_{23} \cdots a_{2n} \cdots$

$x_n = 0\ a_{n1} a_{n2} a_{n} \cdots a_{nn} \cdots$

$\vdots \quad \cdot \quad \cdot \quad \cdot$

a_{ij} can be $0-9$

then take $Y = 0.Y_1 Y_2 \cdots Y$

where $y_1 = a_{11} \quad y_2 = a_{22} \quad y_3 = a_{33} \cdots$

then $Y_1 \neq$ any $x_1 \cdots x_n \cdots$ an $Y_1 \in [0,1]$ & $(0,1)$

$\Rightarrow$ $[0,1]$ and $(0,1)$ can't be countable

let $\varphi : N \longrightarrow P(A)$ is a surjection
Since $\varphi(a)$ is subset of N, either n belong to $\varphi(a)$ or it doesn't belong to the set

we let $D := \{ a \in A ; a \notin \varphi(a)$
D is subset of A, if φ is surjection then $D = \varphi(a_0)$ for some $a_0 \in A$.
we must either $a_0 \in D$ or $a_0 \notin D$. If $a_0 \in D$ then $D = \varphi(a_0)$, we must have $a_0 \in D$. contrary to definition of D, Illy if $a_0 \notin D$ then $a_0 \notin \varphi(a_0)$ so that $a_0 \in D$. which also contradiction
Therefore φ can't be surjection
by this φ defined in $\varphi : \mathbb{N} \longrightarrow P(\mathbb{N})$
is not bijection
$\Rightarrow$ $P(\mathbb{N})$ is uncountable.

Q2. Consider the n-dimensional real vector space $\mathbb{R}^n$. For $a=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$, define

$$\|a\|_1=\sum_{i=1}^{n}|a_i|,\quad \|a\|_2=\left[\sum_{i=1}^{n}a_i^2\right]^{1/2}\ \text{and}\ \|a\|_\infty=\max\{|a_i|:1\le i\le n\}.$$

i) For $a=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$, $b=(b_1,b_2,\cdots,b_n)\in\mathbb{R}^n$, prove Lagrange's identity

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2=\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)-\sum_{1\le i<j\le n}(a_ib_j-a_jb_i)^2.$$

Hence, deduce Cauchy-Schwarz inequality $|\langle a,b\rangle|\le\|a\|_2\|b\|_2$.

ii) Show that $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$. Show that these are equivalent norms. Hence the set of open sets in all the associated metric spaces $\mathbb{R}^n$ are the same.

iii) Let $p\in\mathbb{R}$ with $1\le p<\infty$. Set $q\in\mathbb{R}$ with $\frac{1}{p}+\frac{1}{q}=1$ (q is called the conjugate exponent of p.)
For $\alpha,\beta\in\mathbb{R}$ positive, show that $\alpha\beta\le\frac{\alpha^p}{p}+\frac{\beta^q}{q}$.
Hence, prove Hölder's inequality

$$\sum_{i=1}^{n}|a_ib_i|\le\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|b_i|^q\right)^{1/q}.$$

iv) Show that $\|a\|_p=\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}$ for $a=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$ defines a norm on $\mathbb{R}^n$ (called the ℓ^p-norm). Also show that

$$\lim_{p\to\infty}\|a\|_p=\|a\|_\infty \text{ for } a\in\mathbb{R}^n.$$

i) Langrage's theorem

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2=\left(\sum_{i=1}^{n}a_i^2\right)+\left(\sum_{i=1}^{n}b_i^2\right)-2\left(\sum_{1\le i<j\le n}a_ib_j-a_jb_j\right)^2$$

$a=(a_1,a_2,a_3\ldots a_n)\in\mathbb{R}^n$, $b=(b_1,b_2,\ldots b_n)\in\mathbb{R}^n$

$$\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k}b_k^2\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_i^2b_j^2=$$

$$=\sum_{k=1}^{n}a_k^2b_k^2+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}a_i^2b_j^2$$

$$\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}a_i^2b_j^2$$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i b_i a_j b_j$$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i b_j - a_j b_i)^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i b_j a_j b_i\right)$$

$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i^2 b_j^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} - 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i b_j a_j b_i$$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i b_j - a_j b_i)^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n-1}\sum_{i+1}^{n} a_i^2 b_j^2$$

$$-2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i b_j a_j b_i$$

$$\Rightarrow \left(\sum_{k=1}^{n} a_k b_k\right)^2 - \left(\sum_{k=1}^{n} a_k^2 \sum^{n} b_k^2\right) = \sum_{j=1}^{n-1}\sum_{j=i+1}^{n}(a_i b_j - a_j b_i)^2$$

$$|\langle a, b\rangle| \le \|a\|\,\|b\| \qquad \langle a, a\rangle = \|a\|^2$$

$$\left(\sum a_i b_i\right)^2 \le \left(\sum^{n} a_i^2\right)\left(\sum^{n} b_i^2\right)$$

$$\left(\sum a_i b_i\right) \le \left(\sum^{n} a_i^2\right)^{1/2}\left(\sum^{n} b_i^2\right)^{1/2}$$

$$|\langle a, b\rangle| \le \|a\|_2 \|b\|_2$$

$\|\cdot\|_1$

$$\|a\| = \sum_{i=1}^{n} |a_i|$$

$$\|0\|_1 = \sum_{i=1}^{n} |0| = 0$$

$$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

$$\le \|x\|_1 + \|y\|_1$$

$$\|ax\| = \sum_{i=1}^{n} |a x_i| = a\sum_{i=1}^{n} |x_i| = a\|x\|$$

1) Positivity

$$\|x\|^2 = \sum_{i=1}^{n} x_i^2$$

for any x, $x \neq 0$, norm is positive since sum squares always positive

$$\|cx\|^2 = \sum_{i=1}^{n} (cx_i)^2 = \sum_{i=1}^{n} c^2 x_i^2$$
$$= c^2 \sum_{i=1}^{n} x_i^2$$
$$= c^2 \|x\|^2$$

$$\|x+y\|^2 = \sum_{i=1}^{n} (x_i+y_i)^2 = \sum_{i=1}^{n} (x_i)^2 + \sum_{i=1}^{n} (y_i)^2 + 2\sum_{i=1}^{n} x_i y_i$$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2x \cdot y$$

using Cauchy Schwarz inequality

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2x \cdot y \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$
$$= (\|x\| + \|y\|)^2$$

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$$
$$\|x+y\| \leq \|x\| + \|y\|$$

$$\| \ \|_\infty = \sup\{ |a_i| : 1 \leq i \leq n \}$$

assume wlog $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$

If $x_n = 0$

$\forall p \geq 1$ $0 = \|x\|_p \to 0 = x_n$

If $x_n > 0$

$\forall p \geq 1$ $\|x\|_p = x_n \left(\sum_{k=1}^{n} \left(\frac{x_k}{x_n} \right)^p \right)^{1/p} = x_n \left(1 + \sum_{k=1}^{n-1} \left(\frac{x_k}{x_n} \right)^p \right)^{1/p}$

since $0 \leq \frac{x_k}{x_n} \leq 1$

$$x_n = x_n \cdot 1^{1/p} \leq \|x\|_p \leq x_n \cdot (n \cdot 1)^{1/p} = x_n \cdot n^{1/p}$$

$p \to \infty$ $\lim_{p \to \infty} \|x\|_p = x_n$ claimed.

$\therefore$ i.e $\|x\| = \lim_{p \to \infty} \|x\|_p = \max\{|x_i|\}$

1) as $\|x\|_\infty$ is defined as maximum of absolute value of any element

$\Rightarrow \|x\|_\infty < 0$

$\|x\|_\infty = 0 \iff x = 0$

2) $\|ax\|_\infty = \max\{|a x_i| : 1 \le i \le n\}$

$= |a| \max\{|x_i| : 1 \le i \le n\}$

$= |a| \, \|x\|_\infty$

3) $\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty$

$\|x + y\|_\infty = \max(|x_1 + y_1|, |x_2 + y_2| \cdots |x_n + y_n|)$

by triangle inequality

$|x_1 + y_1| \le |x_1| + |y_1|$

$|x_2 + y_2| \le |x_2| + |y_2|$

$\vdots$

$|x_n + y_n| \le |x_n| + |y_n|$

$\Rightarrow \|x + y\|_\infty = \max\{|x_1 + y_1|, |x_2 + y_2| \cdots |x_n + y_n|\}$

$\le \max\{|x_1| + |y_1|, |x_2| + |y_2| \cdots |x_n| + |y_n|\}$

$\le \|x\|_\infty + \|y\|_\infty$

(i) Let $p \in \mathbb{R}$ with $1 \le p < \infty$. Set $q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$ (q is called the conjugate exponent of p.)
For $\alpha, \beta \in \mathbb{R}$ positive, show that $\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$.
Hence, prove Hölder's inequality

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}.$$

1) Young's Inequality

method: 1

$1/p + 1/q = 1 \Rightarrow q = p/p-1$. Let $p = x$

$$f(x) = \frac{u^x}{x} + \frac{v^{x/x-1}}{x/x-1}$$

$$= u^x/x + (x-1)\frac{v^{x/x-1}}{x}$$

$$f'(x) = \frac{x u^x \ln u - u^x}{x^2} + \frac{1}{x^2} v^{\frac{x}{x-1}} + \frac{1}{x^2} + \frac{x}{x-1} + \frac{1}{x^2} \ln(v) v^{x/x-1}$$

h

Q 3) $d(x,y) = \sqrt{|x-y|}$; $d_2(x,y) = \begin{cases} |x-y|+1 & \text{exactly one of the} \\ |x-y| & ; \text{otherwise.} \end{cases}$

i) a) for any x, y

$d(x,y) = \sqrt{|x-y|} \geq 0$

Since $|\ |$ gives no negative values and we are taking + ve.

$d(x,x) = \sqrt{|x-x|} = \sqrt{0} = 0$

b) $d(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|}$ $\quad \because |x-y| = |y-x|$

$= d(y,x)$

c) $d(x,y) \leq d(x,r) + d(r,y)$ for any $r \in X$

$d(x,y) = \sqrt{|x-y|} = (|x-y|)^{1/2}$

$d(x,r) = |x-r|^{1/2}$; $d(r,y) = |r-y|^{1/2}$

$d(x,r) + d(r,y) = \sqrt{|x-r|} + \sqrt{|r-y|}$

$\geq \sqrt{|x-r| + |r-y|}$ $\quad \because \sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$

$\geq \sqrt{|x-z|}$

$\geq d(x,y)$

So $d_1(x,y)$ is metric.

2) $d_2(x,y) \begin{cases} |x-y|+1 & \text{if exactly one positive.} \\ |x-y| \end{cases}$

a) $d(x,y) = \begin{cases} |x-y|+1 \\ |x-y \end{cases}$ is always equal greater than zero

$d(x,x) = |x-y|$ (since both would have same sign or zero.

$= 0$

b) $d(x,y) = 2\text{ case} \begin{cases} |x-y|+1 & 1\text{ pos} \\ |x-y| \end{cases}$

$d(y,x) = 2\text{ case} \begin{cases} |y-x|+1 & 1\text{ pos} \\ |y-x| \end{cases}$

Since $|x-y| = |y-x|$

$d(x,y) = d(y,x)$ for any x, y.

c) $d(x,y) \leq d(x,r)\ \ d(r,x)$

2 case ① exactly one is positive.

② otherwise.

① $d(x,y)$ one of them is positive.

let x be positive (even if is y $\because d(x,y) = d(y,x)$ it doesn't matter)

$d(x,y) \leq d(x,r)\ \ d(r,y)$

here we have 2 case, r is positive / r is not posi.

first can

$|x-y|+1 \le |x-r| + |r-y|$

It can't be metric on $\mathbb{R}$

Q4) (X, d) $A \subseteq X$ $D(A)$ derived set)

$a \in D(A) \iff \exists$ sequence $(a_n)_{n\in\mathbb{N}}$ of distinct points of A

$\lim_{n\to\infty} a_n = a$

Suppose $a \in D(A)$ by definition $\forall\, B(a,r)$ for $r>0$
contains a different point than a.

$n \in \mathbb{N}$, chose $a_n \in B(a, 1/n)$ st $a_n \ne a$

$a_n \in A$

$\Rightarrow |a_n - a| < 1/n$ as $n \to \infty$

$1/n \to 0$

$\lim_{n\to\infty} a_n = a$

Suppose (a_n) sequence in A st $\lim_{n\to\infty} a_n = a$.

we want $a \in D(A)$

Let be $U = B(x, r)$ for arbitrary r. Since $\lim_{n\to\infty} a_n = a$

$\exists\, N \in \mathbb{N}$ st $n \ge N$, a_n is in U

Consider the set $\{a_n \dots a_n\}$ this finite set and a is not in this
finite set because it is distinct from all a_n with $n \ge N$.
So we have a point a_k in A in U. this prove a limit point
$a \in D(A)$

(i) $A \subseteq B \subseteq X$ thus $D(A)$ is

Q7) n-d Euclidean Space $\mathbb{R}^n$ is a complete metric space

a metric (X, d) is complete if for $\forall$ Cauchy sequence (x_n)
$\exists$ $x \in X$ st
$$\lim (x_n, x) = 0$$

$$\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x_1, x_2 \dots x_n) \longrightarrow \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

① $d(x, y) = \sqrt{\sum_{i=1} (x_i - y_i)^2}$

≥ 0 $\because$ Sum of some squares is always non negative

$$d(x, x) = \sqrt{\sum_{i=1} (x_i - x_i)^2} = 0$$

② $d(x, y) = \sqrt{\sum_{i=1} (x_i - y_i)^2}$

$= \sqrt{\sum_{i=1} (y_i - x_i)^2}$ $\because (x-y)^2 = (y-x)^2$

$= d(y, x)$

③ $d(x, r) + d(r, y) \geq d(x, y)$

$d(x, r) + d(r, y) = \sqrt{\sum (x_i - r)^2} + \sqrt{(r-y)^2}$

$\geq \sqrt{\sum (x-r)^2}$

$\geq d(x, y)$

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8888888888888

④

for $\forall$ Cauchy sequence (x_n) in X, $\exists$ $x \in X$ s.t
$$\lim_{n \to \infty} d(x_n, x) = 0$$

Let (x_k) be Cauchy sequence in $\mathbb{R}^n$ with Euclidean metric
i.e $\forall$ $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ st $k, m \geq N$ $d(x_k, x_m) < \varepsilon$.
where $d(x_k, x_m)$ is Euclidean distance b/w x_k and x_m

Claim: Cauchy sequence (x_k) converge to a limit point in $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ limit point (x_k).
because of Euclidean distance in $\mathbb{R}^n$ is the square root of the sum of squared difference of the compone

consider i^{th} component of the vectors x_k & x. Let x_{ki} i^{th} component of x_k and x_i be the i^{th} component of x.

Since $\mathbb{R}$ is complete, every ith component of x_k converge.

now consider vector x with component $(x_1 \dots x_n)$.

by convergence of each component.

$$\lim_{k \to \infty} x_k = x$$

Since (x_k) is an arbitrary cauchy seq..

we have shown that $\forall$ (x_k) cauchy seq converge.

so $\mathbb{R}^n$ - Euclidean is a complete metric space

Q6) i) $[a,b]$; $a<b$ is homeomorphic with $[0,1]$

2 3

let map from $[a,b] \to [0,1]$

$f: x \longrightarrow \frac{x-a}{b-a}$ $\quad x \in [a,b]$, $b-a \neq 0$

claim: f is bijective

to show injectiveness ($f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in [a,b]$

$$\frac{x_1 - a}{b-a} = \frac{x_2 - a}{b-a}$$

$x_1 - a = x_2 - a \Rightarrow x_1 = x_2$ injective

Surjective, let y arbitrary element in $[0,1]$

$$y = \frac{x-a}{b-a}$$

$$(b-a)y = x-a$$

$$(b-a)y + a = x$$

$$g(y) = (b-a)y + a$$

so $\forall y$ in $[0,1]$ there exist preimage in $[a,b]$

Since f is linear function, f is continuous

any $\exists$ $g(y) = (b-a)y + a$

where $g(0) = a$ $\quad g(1) = b$

and the thus $g(y)$ is also bijective. and continuous

these by $[a,b]$ is homeomorphic to $[0,1]$

ii) $(a,b) \to (0,1)$

$$\frac{x-a}{b-a}$$

$(a, \infty) \to (0,1)$

$1 - \frac{a}{x}$ $\quad \frac{x-a}{x}$

$(\infty, b) \to (0,1)$

$$\frac{b}{x}$$

$(-\infty, \infty) \to (0,1)$

$$f(x) = \frac{1}{1+e^{-x+b}}$$

$[a,b]$ and (a,b) are not homeomorphic

consider closed interval $[a,b]$, it's compact set because it satisfies the Heine-Borel theorem.

which state a subset of R^n is compact iff it is closed & bounded.

(a,b) is not compact..

we can construct an open cover of (a,b) that does not have finite subcover eg.

$$\{(a+1/n,\ b-1/n) : n \in \mathbb{N}\}$$

In this for any x in (a,b), $\exists$ n such that

$$a+1/n < x < b-1/n$$

but there no finite subcover of this collection because no matter how many interval you select there always some be points (a,b) that are not cover.

So $[a,b]$ a compact and (a,b) a non compact can't be homeomorphic. because homeomorphism preserve topological property, and compactness is one of them.

Q3) $f:(X,d) \to (Y,\rho)$ be a function. Then show that f is continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for any $A \subseteq X$

Idea of proof: f is continous $\leftrightarrow$ $f(\bar{A}) \subseteq \overline{f(A)}$

If f is continous, we want $f(\bar{A}) \subseteq \overline{f(A)}$

i.e $y \in f(\bar{A}) \Rightarrow y \in \overline{f(A)}$

suppose $y \in f(\bar{A})$, which means

$\exists$ x in $\bar{A}$ st $f(x)=y$

Since $\bar{A}$ is consist of all limit points of A,

so $\forall$ $B(x,r)$ of x, $\exists$ a, such that

$a \in B(x,r)$ $a \neq x$

$\because$ f is continous, any neighborhood V of y

$\exists$ an open neighborhood U

such that $f(U) \subseteq V$

consider the set $f(A)$. If x is limit point of A

$\exists$ sequence (a_n) in A such that

$$\lim_{n\to\infty} a_n = x$$

$\because$ f is continous,

$$\lim_{n\to\infty} f(a_n) = f(x) = y.$$

this shows that y is limit point of $f(A)$

$y \in \overline{f(A)}$

thus we shown that if f continous $f(\bar{A}) \subseteq \overline{f(A)}$

Let $f(\bar{A}) \subseteq \overline{f(A)}$, then f is continuous

Let $f(\bar{A}) \subseteq \overline{f(A)}$

take an arbitrary point x in domain of f.

Let $\varepsilon > 0$ be given. we need $\delta > 0$ such that for all y in domain of f

$d(x, y) < \delta$

then $d(f(x), f(y)) < \varepsilon$

Consider $\bar{A}$ and a point x.

$\therefore$ $x \in \bar{A}$, it is a limit point of A. This means

for every open ball B_x of x. $\exists$ $a_x \in A$ st

$a_x \in B_x$ and $a_x \neq x$

$\because$ $f(\bar{A}) \subseteq \overline{f(A)}$,

$f(x)$ must be limit point of $f(A)$.

$\Rightarrow$ $\forall$ open ball $V_{f(x)}$ of $f(x)$ $\exists$ $y_{f(x)} \in f(A)$

st $y_{f(x)} \in V_{f(x)}$ & $y_{f(x)} \neq f(x)$

we know f maps open ball of x to open ball of $f(x)$ (due continuity)

B_x be an open ball of x st $f(B_x) \subseteq V_{f(x)}$

Consider the set of all such open ball B_x for all limit points of x of A. This forms an open cover of $L(A)$

By compactness, because $\bar{A}$ is compact. $\exists$ finite subcover that there finitely many points $x_1 \cdots x_n$ to $B_{x_1}, B_{x_2} \cdots B_{x_n}$

such that

$\bar{A} \subseteq \bigcup_{i=1}^{n} B_{x_i}$

Let δ be the minimum of the distance from x to each these finitely many points

$x_1 \cdots x_n$

$\delta = \min\{d(x, x_1) \cdots d(x, x_n)\}$

Since each B_{x_i} is an open ball of x_i, and δ is chosen st x is at least δ away from x_i.

we have $x \in B_{x_i}$ for some i

$f(U_{x_i}) \subseteq V_{f(x_i)}$ & $d(x, x_i) < \delta$

$\Rightarrow$ $f(x) \in f(U_{x_i})$

$\Rightarrow$ $d(f(x), f(x_i)) < \varepsilon$

Thus we shown that arbitrary point x in the domain of f and any $\varepsilon > 0$, $\exists$ $\delta > 0$ st $\forall y$ in domain of f

$d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon$

$\Rightarrow$ f is continuous.