1) (a) field :- A set K with 2 binary operation $+, \cdot$ where $(K,+,0)$ and $(K^*,\cdot,1)$ is an abelian group $K^* = K - \{0\}$.
where 'addition distributes over multiplication i.e $a(b+a) = ab + ba$
is called a field $(K,+,\cdot)$.
A field $(K,+,\cdot)$ is to with total order i.e for all a, b
$a<b$, $b>a$, $a=b$ one of them is true. Then this field is orded field.

(2) field of rational functions

let p, q be polynomial function on $\mathbb{Q}$, $g \neq 0$
then $R(x) = \frac{f(x)}{g(x)}$ is rational on $\mathbb{Q}$

$f(x) = f_n x^n + \cdots f_0$

$g(x) = g_m x^n + \cdots g_0$

$f > 0 \Leftrightarrow f_n / g_m > 0$

we define order as $r_1 > r_2$ if $r_1 - r_2$ is grater $>0, =0$ or <0
for the co-efficient
choose $f(x) = x$ and $g(x) = 1$
for any $n \in \mathbb{N}$ $f(x) > n \cdot g(n)$
becase $(f - n \cdot g)(x) = x - n$ and coffient is 1

consider set A of rational form $r(x) = 1/x$, $x \neq 0$
A is subset of field and bounded above.
but A has no lub in the field of rational.

1) for any positive real number x, $\phi(x) > 0$

if x positive, we know ϕ strictly increasing

$\therefore$ $\forall x$, $\exists y$ st. $x = y^2$

$\phi(x) = \phi(y y) = \phi(y)^2 \geq 0$

Since $\phi(0) = 0$ and ϕ is bijection

$\Rightarrow \phi(x) > 0$

2) for any $x, y \in \mathbb{R}$, $x > y \Rightarrow \phi(x) > \phi(y)$

$x > y \Rightarrow x - y > 0$

by ① $\phi(x-y) > 0$

$\phi(x) - \phi(y) > 0 \Rightarrow \phi(x) > \phi(y)$

3) ϕ is identity on $\mathbb{Z}^+$

$\phi(n) = \phi(1+1+\cdots+1) = n\phi(1) = n$

Since $\phi(1) = 1$

4) ϕ is on rational numbers

$\phi(q) = \phi(\pm m/n) = \pm \frac{\phi(m)}{\phi(n)} = \frac{\pm m}{n} = q$ //

using (3rd)

5) ϕ is identity on Real numbers

Now claim; if $\alpha \in \mathbb{R} - \mathbb{Q}$ then

$f(\alpha) = \alpha$

Suppose a claim is not then

either $f(\alpha) < \alpha$ or $f(\alpha) > \alpha$

$f(\alpha) < r < \alpha$ (By denses $r \in \mathbb{Q}$)

$r < \alpha \Rightarrow f(r) < f(\alpha) \Rightarrow r < f(\alpha)$ $[\because r \in \mathbb{Q}]$

contradiction contradiction.

Illy $f(\alpha) > \alpha$ contradicts

so $f(\alpha) = \alpha$.

$\phi : \mathbb{R} \Rightarrow \mathbb{R}$ is identity homomorphism cauchy sequence: * cauchy sequence of $\mathbb{R}$ is bonded

let (a_n) be cauchy sequence

let $0 < \varepsilon_0$, $\exists N_0$ st for $m, n \geq N$

$|a_m - a_n| < \varepsilon$.

$|a_m| - |a_n| \leq |a_m - a_n| < \varepsilon$ by triangle inequality

$|a_m| - |a_N| < \varepsilon$ taking $n = N$

we have $|a_m| < \varepsilon + |a_N|$ for $m \geq N$

Thus $|a_m| \leq \max\{|a_0|, |a_1| \ldots |a_{N-1}|, |a_N|, |a_N| + \varepsilon\}$.

$\Rightarrow a_m$ is bounded above.

it is sandwiched b/w $\pm \max\{|a_1|, |a_2| \ldots |a_N|, \varepsilon + |a_N|\}$

② cauchy sequence is convergent iff it has convergent sequence

$(a_n)_n$ be a cauchy sequence

(a_n) is bounded has a convergent subsequence $(a_{p_n})_n$.

$a = \lim_n a_{p_n}$.

let $\varepsilon > 0$, $\exists n_p$ such that

for $n > n_p$ $|a_{p_n} - a| < \varepsilon/2$ due to converging subsequence

$\because (a_n)$ is cauchy sequence, we have

$n, m < n_0$ $|a_n - a_m| < \varepsilon/2$

let let $N = \max\{n_p, n_0\}$

so for $n > N$

$|a_n - a| \leq |a_n - a_{p_n}| + |a_{p_n} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\Rightarrow \lim a_n = a$.

Define a compact subset of metric space. Show that compact subset is closed. $T = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is compact subset of real line $\mathbb{R}$?

Compact subset: A subset K of metric space X is said to be compact for $\forall$ open cover, $\exists$ finite subcover of K from this open cover.

for $\forall$ $a \in A$ let

$$V_a = B\left(a, \frac{d(a,x)}{2}\right).$$

$$V_x = B\left(x, \frac{d(a,x)}{2}\right)$$

then V_x & V_a are disjoint open ball of a & x

since V_a, $a \in A$ together cover of A.

by compactness of A we get finitely many $V_{a_1} \ldots V_{a_n}$ that cover A.

but the open neighborhood of $V_{a_1} \ldots V_{a_n}$ do not intersect A.

so A is closed.

$x \notin A$, $a \in A$ $\exists$ $r_a > 0$ st

$B(a, r_a) \cap B(x, r_a) = \phi$

so $B(a, r_a)$ $a \in A$ is open cover.

finite subcover $B(a_i, r_{a_i})_{i=1}^{n}$

then $x \in \bigcap_{i=1}^{n} B(x, r_a) \subseteq A^c$ and this is open.

$\Rightarrow$ A^c is open

$T = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is compact subset