Nov. 6

$(X, d) \rightarrow$ compact metric space

$C(X) = \{ f: X \rightarrow \mathbb{R} \mid f \text{ continous} \} \rightarrow$ function space

Supnorm $\quad \|f\|_\infty = \sup \{ |f(x)| : x \in X \}$

$d_\infty(f, g) = \|f - g\|_\infty$

prop: $C(X)$ is complete metric space (under sup-metric)

Aim: characterize compact subset of $C(X)$

Define: Equicontinous family $\quad \mathcal{E} \subseteq C(X)$ is an equicontinous family,

If $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t for $x, y \in X$ with $d(x,y) < \delta$ we have

$|f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{E}$

Theorem: If $\mathcal{E}$ is compact subset of $C(X)$, then $\mathcal{E}$ is closed, $\mathcal{E}$ is bounded and $\mathcal{E}$ is an equicontinous family.

Arzela-Ascoli theorem: $\mathcal{E} \subseteq C(X)$ is compact $\Leftrightarrow$ $\mathcal{E}$ is uniform closed, uniform bounded. and $\mathcal{E}$ is equicontinuss

pointwise bounded: $\forall x \in X$. $\exists M_x \in \mathbb{N}$ s.t

$|f(x)| \leq M_x \quad \forall f \in \mathcal{E}$

Uniformly bounded: $\exists M \in \mathbb{N}$ s.t $\quad |f(x)| \leq \|f\| \leq M \quad \forall f \in \mathcal{E}$ & $\forall x \in X$

Ex: If $\mathcal{E}$ is equicontinous and pointwise bounded, then show that $\mathcal{E}$ is uniformly bounded. (Prove this result for $X = [a, b]$)

Lemma: Let $E$ be a countable set and $f_n: E \rightarrow \mathbb{R}$ be a function for $n \in \mathbb{N}$. If $(f_n)_{n \in \mathbb{N}}$ is pointwise bounded on $E$, then $\exists$ a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ s.t $(f_{n_k})_{k \in \mathbb{N}}$ converges pointwise on $E$.

pf: If $E$ is finite, proof is easy. Assume that $E$ denumerable and

$E = \{ x_1, x_2 \ldots \}$

Let $\quad S^0 = (f_1, f_2, f_3 \ldots)$

Since $(f_n(x_1))$ is a bounded sequence of reals, there is subsequence

say $(f'_k)_{k \in \mathbb{N}}$ of $S^0$ such that $(f'_k(x_1))$ is convergent

put $\quad S' = (f'_1, f'_2, f'_3, \ldots)$

again $(f'_k(x_2))_{k \in \mathbb{N}}$ is bounded sequence of reals, there is a subsequence of $(f^2_k)_{k \in \mathbb{N}}$ of $S'$

s.t $(f'_k(x_2))_{k \in \mathbb{N}}$ is convergent.

Put $\quad S^2 = (f_1^2, f_2^2, f_3^2, \ldots)$

Since $S^2$ is subsequence of $S'$, we see that $(f^2_k(x))$ is convergent

for $x = x_1$ and $x_2$

continuing in this manner, if the subsequence

$S^{-1} = (f^{-1}, f^{-2}, f^{-3} \ldots)$

has been constructed such that $(f^{n-1}_k(x))_{k \in \mathbb{N}}$

Then since $(f_k^{n-1}(x_n))_{k\in\mathbb{N}}$ is a bounded sequence of reals,

there is a subsequence, say $(f_k^n)$ of $(f_k^{m-1})$ such that $(f_k^n(x))_{k\in\mathbb{N}}$ is convergent for $x = x_1, x_2 \dots x_m$.

Let $S^m = (f_1^m, f_2^m, f_3^m \dots)$

$S_* = (f_1^1, f_2^2, f_3^3 \dots f_m^m)$ be the diagonal subsequence

Let $x_m \in E$ since deleting the first $m-1$ term of $S_*$,

we see that $S_*$ in a subsequence of $S^m$

$\Rightarrow (f_k^k(x_m))_{k\in\mathbb{N}}$ is convergent

Then $S_*$ is a subsequence that has pointwise limit at $\forall x \in E$

**Theorem:** Let $\mathcal{E} \subseteq [a,b]$. Suppose $\mathcal{E}$ is uniformly closed, $\mathcal{E}$ is uniformly bound & $\mathcal{E}$ is equicontinous. Then $\mathcal{E}$ is compact subset of $C[a,b]$

*proof:* We shall prove $\mathcal{E}$ is sequential compact

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{E}$.

Let $E = [a,b] \cap \mathbb{R}$. Then $E$ is countable subset of $[a,b]$ and and $\mathcal{E}$ is uniformly bounded, the sequence $(f_n|_E)_{n\in\mathbb{N}}$ is a pointwise bounded sequence on $E$. Then by last lemma there is a subsequence $(f_{n_k}|_E)_{k\in\mathbb{N}}$ of $(f_n|E)_{n\in\mathbb{N}}$ such that $(f_{n_k}(x))$ is convergent for $\forall x \in E$

Put $g_i = f_n$

Claim: $g_i$ is uniformly convergent

Suppose $\forall \varepsilon > 0$, is given. Chose $\delta > 0$ s.t for $x, y \in [a,b]$ with $|x-y| < \delta$, we have $|f_n(x) - f_n(y)| < \varepsilon/3$ — ①

($\because$ $\mathcal{E}$ is equicontinous and $f_n \in \mathcal{E}$ $\forall n \in \mathbb{N}$)

Choose partition $P = \{P_0, P_1, P_2 \dots P_m\}$ of $[a,b]$ with

$a = P_0 < P_1 < < P_2 \dots < P_n = b$ and $\Delta P_i = P_i - P_{i-1} < \delta$

For each $x \in E$, consider open interval $(x-\delta, x+\delta)$ in $\mathbb{R}$

we have $\{(x-\delta, x+\delta) : x \in E\}$ is an open cover of $[a,b]$

($\because$ $E$ is dense in $[a,b]$). But $[a,b]$ is compact,

so there finitely many $x_1, x_2 \dots x_s \in E$ s.t

$[a,b] \subseteq \bigcup_{i=1}^{s} (x_i - \delta, x_i + \delta)$

Since $(g_i(x))_{i\in\mathbb{N}}$ converge for every $x \in E$, we see that for $\varepsilon > 0$

$\exists n_0 \in \mathbb{N}$ s.t

$|g_i(x) - g_j(x)| < \varepsilon/3$ $\forall i, j \geq n_0$ & $x \in \{x_1, x_2 \dots x_s\}$

let $x \in [a,b]$. Then $a \cup [a,b] \subseteq \bigcup_{i}^{} (x_i - \delta, x_i + \delta)$

$\exists$ $k$ s.t $x \in (x_k - \delta, x_k + \delta)$ for some $k$ with $1 \le k \le \delta$

$\Rightarrow$ $|x - x_k| < \delta$

For $i, j \ge n_0$, we have

$$|g_i(x) - g_j(x)| = |g_i(x) - g(x_k) + g_i(x_k) - g_{x_k} + g_j(x_k) - g_j(x)|$$

$$\le |g_i(x) - g(x_k)| + |g_i(x_k) - g_j(x_k)| + |g_j(x_k) - g_j(x)|$$

$$\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$\Rightarrow$ By co sequence

Theorem : Let $\mathcal{E} \subseteq C[a,b]$ suppose $\mathcal{E}$ is uniform closed, $\mathcal{E}$ is uniformly bound and $\mathcal{E}$ is equicontinuous

this $(g_i)_{i \in \mathbb{N}}$ is uniformly convergent. As $\mathcal{E}$ is uniformly closed, the uniform limit $\lim_{j \to \infty} g_j \in \mathcal{E}$. Hence $\mathcal{E}$ is sequential compact

Weierstrass Approximation theorem.

let $f \in C[a,b]$. then there is sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ s.t $P_n \to f$ uniformly on $[a,b]$

Ex: $f \in C[a,b]$ and $\int_0^1 f(x) x^n dx = 0$ $n = 0, 1, 2$

then that $f = 0$

$P_n \to f(x)$

$\Rightarrow P_n f \to f^2$

$f^2 dx = \lim \int f P_n dx = 0$

$\int_0^1 f(x) x^n dx = 0$ $n = 0$

Nov 7

## Weierstrass approximation theorem

Let $f \in C[a,b]$. Then there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials such that $P_n \to f$ uniformly on $[a,b]$.

Proof: If $a=b$ then $C[a,b] \simeq \mathbb{R}$ and there is nothing to prove.

Assume $a<b$. We see that $[0,1] \simeq [a,b]$ are homeomorphic with $\varphi: [0,1] \longrightarrow [a,b]$ given by $\varphi(t) = (1-t)a + tb$ ; $t \in [0,1]$

Further $\varphi^{-1}(x) = \dfrac{x-a}{b-a}$ $\quad x \in [a,b]$

This show that enough to prove this theorem for $C[0,1]$ i.e $a=0$ $b=1$

Case : 1

Let $f \in C[0,1]$ and $f(0) = f(1) = 0$

Then extend $f$ to continous function on $\mathbb{R}$ by putting $f(t) = 0$ $\forall t \in \mathbb{R}-[0,1]$

Clearly, $f$ is an uniformly continous function on $\mathbb{R}$

Let $M = \sup\limits_{x\in[0,1]} |f(x)|$

Consider a polynomial

$$Q_n(x) = C_n(1-x^2)^n \quad \text{on } [-1,1]$$

where $C_n$ is constant so that

$$\int_{-1}^{1} Q_n(x)\,dx = 1 \quad — ①$$

Observe that $(1-x^2)^n - (1-nx^2)$ is increasing on $[0,1]$.

Then $(1-x^2)^n \geq 1-nx^2$ $\forall x \in [0,1]$

We now have

$$\int_{-1}^{1}(1-x^2)^n dx \geq 2\int_0^{1/\sqrt{n}}(1-x^2)^n \geq \int_0^{1/\sqrt{n}}(1-nx^2)\,dx = 2\left[x - \frac{nx^3}{3}\right]_0^{1/\sqrt{n}}$$

$$= 2\left[\frac{1}{\sqrt{n}} - \frac{1}{3\sqrt{n}}\right] = \frac{4}{3}\frac{1}{\sqrt{n}} > \frac{C_n}{\sqrt{n}}$$

$\Rightarrow C_n < \sqrt{n}$ $\quad \forall n$

If $\delta > 0$ & for $x \in [-1,1]$ with $\delta \leq |x| \leq 1$, we have

$$Q_n(x) \leq \sqrt{n}(1-\delta^2)^n \longrightarrow 0 \quad n \to \infty$$

$1-x^2 \leq 1-\delta^2$

$\dfrac{n^\alpha}{(1+\beta)^n} \to 0$ when $n \to \alpha$.

Then $Q_n(x) \longrightarrow 0$ uniformly on $[-1,-\delta] \cup [\delta,1]$

Put $$P_n(x) = \int_{-1}^{1} f(x+t)\,Q_n(t)\,dt \quad — ②$$

Let put $s = x+t$, then $ds = dt$

$f \in C[-1,1]$

$$P_n(x) = \int_{x-1}^{x+1} f(s)\,Q_n(s-t)\,ds = \int_0^1 f(s)\,Q_n(s-t)\,ds$$

$Q_n(x-s) = q_0(s) + q_1(s)x + \cdots + q_{2n}(s)x^{2n}$ ..

$\int f(s)\,Q_n(x-s) = \int_0^1 q_0(s)\,f(s) + \int_0^1 q_1(\ldots \quad \int q_{2n}(s)x^{2n} f(s)$

Since $Q_n(x-s)$ is a polynomial in $x$ and $s$, we see that $P_n(x) = \int_{-1}^{1} f(s) Q_n(x-s) ds$

is a polynomial in $x$

claim : $P_n \to f$ uniformly on $[0,1]$

Let $\varepsilon > 0$ be given. Since $f$ is uniformly continous on $\mathbb{R}$,

$\exists \delta > 0$ such that

If $x, y \in \mathbb{R}$ with $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon/2$ — (3)

Now for any $x \in [0,1]$ we have

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t) Q_n(t) dt - f(x) \int_{-1}^{1} Q_n(t) dt \right|$$

$$= \left| \int_{-1}^{1} (f(x+t) - f(x)) Q_n(t) dt \right|$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t) dt$$

$$= \int_{-1}^{-\delta} |f(x+t) - f(x)| Q_n(t) dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)| Q_n(t) dt + \int_{\delta}^{1} |f(x+t) - f(x)| Q_n(t) dt$$

$$\le 4M\sqrt{n}\,(1-\delta^2)^n + \varepsilon/2$$

$$\le \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{for } n >> 0$$

So $(P_n = f * Q_n)$, where $Q_n$ is sequence of function

In which $n \to \infty$, $P_n \to f$

$P_n(x) = \int f(t) Q_n(x-t) dt$

Case: 2 $f \in C[0,1]$

put $g(x) = (f(x) - f(0)) - x(f(1) - f(0))$ $g \in C[0,1]$

and $g(0) = g(1) = 0$ from case (1) $P_n^* \to g$ uniformly on $[0,1]$

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Let $(X,d)$ be compact metric space

and $C(X)$ be the space of continous function on $X$ with sup-metric

Let $\mathcal{A} \subseteq C(X)$ be a sub algebra $(f, g \in \mathcal{A} \Rightarrow f+g,\ fg,\ cf \in \mathcal{A}\ \forall c \in \mathbb{R})$

1. $\mathcal{A}$ vanish at no point of $X$ (for every $x \in X$, $\exists g_x \in \mathcal{A}$ s.t. $g_x(x) \neq 0$)

2. $\mathcal{A}$ seperates points of $X$ (If $x_1 \neq x_2$ are point in $X$. $\exists h \in \mathcal{A}$ s.t. $h(x_1) \neq h(x_2)$)

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Stone - weierstrass theorem

$\bar{\mathcal{A}} = C(X)$

1) $\mathcal{A}$ is also satisfy

2) $f \in \bar{\mathcal{A}} \Rightarrow |f| \in \bar{\mathcal{A}}$

3) $f, g \in \bar{\mathcal{A}} \Rightarrow \max(f,g)$ $\min(f,g) \in \bar{\mathcal{A}}$

$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}$

$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$