Fact: The field of $\mathbb{R}$ of real number is complete, ordered field

Field: A set $\mathbb{K}$ with 2 binary operation : (Addition & multiplication)

* $(\mathbb{K},+,0)$ $\rightarrow$ abelian Group with zero as additive Identity
* $(\mathbb{K}^*,\cdot,1)$ $\rightarrow$ abelian Group with 1 as multiplicative Identity

$$[\mathbb{K}^* = \mathbb{K} - \{0\}]$$

* Multiplication distributes over addition

$$x\cdot(y+z) = x\cdot y + yz \qquad \forall\, x,y,z \in \mathbb{K}$$

Partial order

$\leq$, is relation on a set, X is partial order if

i) $\forall a \in X$, $a \leq a$ (reflexive)

ii) $a \leq b \Rightarrow b \leq a$ $\quad a,b \in X$ (symmetric)

iii) $a \leq b,\ b \leq c \Rightarrow a \leq c$ $\quad a,b,c \in X$ (transitive

Strict order

A relation $<$ on X is strict If

1) $a \nless a$ (non-reflexive)

2) transitive

Ex: A partial order $\leq$ on X induces a strict $<$ on X

Partial order :- reflexive, antisymetric, trash

Strict :- anti reflexive, antisymmetric, transitive

To induce a strict order $<$ from a partial order

we have $a < b \iff (a \leq b \wedge \neg(b \leq a))$

i.e for any element a,b in set

$a < b$ If and only if $a \leq b$ and not $(b \leq a)$

So partial order

Total order

A relation is total order if

* It is partial order
* $\forall a,b$ either $a \leq b$ or $b \leq a$

Ex: $\leq$ is partial order on X $\iff$ $<$ satisfies law trichotomy.

Law of trichotomy: $\forall a,b \in X$, $a<b$, $a=b$, $b<a$

Ordered Field

A field $\mathbb{K}$ is strictly ordered if there exist strict order $<$ on $\mathbb{K}$ st:

* $\forall\, a, b \in \mathbb{K}$, $a<b$, $a=b$, $b<a$
* $a<b \Rightarrow a+c < b+c \quad \forall c \in \mathbb{K}$

  $a<b \Rightarrow ac < bc \quad c>0$

Ex: * $\mathbb{Q} \Rightarrow$ rationals

* $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \quad p \in \mathbb{N}$ (Prime)

Ex: assignment Q 1

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ans) Let $P \subseteq \mathbb{K}$ st i and ii holds Then the relation defined by $a<b$ if $b-a \in P$ is strict order on $\mathbb{K}$ making it an ordered field

Order in $\mathbb{Q}$

$p \in \mathbb{Z}, q \in \mathbb{N}$

then $x = p/q > 0$ if $pq > 0$

$p/q < r/s$ if $(rq - ps)\, qs > 0$

Ex: If $x \in (\mathbb{K}, <)$ $x \neq 0$ then $x^2 > 0$

ans) consider $P = \{x \mid x \in \mathbb{K}, x < 0\}$ (ie +ve numbers in $\mathbb{K}$)

conside $x \in \mathbb{K}$, $x \neq 0$

then $x \in P$, $-x \in P$

Case 1 $x \in P$ $(x>0)$

$x \times x = x^2 \Rightarrow x^2 \in P$ (Product of Postive belong in

$x^2 > 0$

Case 2 $-x \in P$ $(x<0)$

$-x \times -x = x^2 \Rightarrow x^2 \in P$

$x^2 > 0$

Proposition

An ordered field $(\mathbb{K}, <)$ contains the field of rational numbers as ordered subfield

Proof: consider +ve map $\phi : \mathbb{Z} \to \mathbb{K}$ given by

$$\phi(n) = \begin{cases} 1+1+\dots+1 \ (n\text{ times}) & ; n>0 \\ 0 & ; n=0 \\ (-1)+(-1)\dots+(-1) & ; n<0 \end{cases}$$

Then ϕ in Injective (one-to-one)

Also ϕ is ring homomorphism

i.e. $\phi(n+m) = \phi(n) + \phi(m)$ (closed)

$\phi(n \cdot m) = \phi(n) \cdot \phi(m)$ (closed)

$\phi(0_{\mathbb{Z}}) = 0_{\mathbb{K}}$ (zero map to zero)

$\phi(n) + \phi(-n) = \phi(0_{\mathbb{Z}}) = 0_{\mathbb{K}}$ (additive inverse map)

$\phi(1_{\mathbb{Z}}) = 1_{\mathbb{K}}$ (multiplicative identity)

Now map ϕ induces a map $f : \mathbb{Q} \to \mathbb{K}$

Given by $f(p/q) = \phi(p)\,\phi(q)^{-1}$ ($\forall$) $p/q \in \mathbb{Q}$

Then f induces an isomorphisms b/w $\mathbb{Q}$ and subfield $f(\mathbb{Q}) \subseteq \mathbb{K}$

Def Let $\mathbb{K}_1$ & $\mathbb{K}_2$ be field Then a map $f : \mathbb{K}_1 \to \mathbb{K}_2$ called field homomorphism if

* $f(x+y) = f(x) + f(y)$
* $f(x \cdot y) = f(x) \cdot f(y)$ $\forall x, y \in \mathbb{K}_1$
* $f(1) = 1$

* If f is bijection in addition, then we say f is an isomorphism In this $\mathbb{K}_1$ & $\mathbb{K}_2$ are isomorphic field

* If $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$ $f : \mathbb{K} \to \mathbb{K}$ is isomorphism then we say f is automorphism of $\mathrm{Aut}(\mathbb{K})$

Proposition

Let $f : \mathbb{Q}$ to $\mathbb{Q}$ to be an field automorphism, Then

$f(1) = 1$ Identity Isomorphism

$f(2) = f(1) + f(1) = 2$

$f(n) = f(1) + f(1) \cdots f(1) = n$

$f(n + (-n)) = \underbrace{f(n)}_{n\text{ times}} + f(-n)$ $\qquad 1 \to 1$

$= f(0) = 0$ $\qquad n \to n$

$\Rightarrow f(n) = -f(-n)$ $\qquad \frac{1}{n} \to 1/n$

Let $m/n \in \mathbb{Q}$ $n \in \mathbb{N}$ $m \in \mathbb{Z}$

$f(1/n + \cdots 1/n) = f(1/n) \cdots f(1/n) = n f(1/n)$

$f(1/n) = 1/n$

$f(m/n) = f(m) \cdot f(1/n) = m \times 1/n = m/n$

Automorphism $(\mathbb{Q}) = \{1_{\mathbb{Q}}\}$ $\qquad \mathbb{Q}[\sqrt{2}]$

Complete ordered field

an ordered field $(\mathbb{K}, <)$ with strict order, Let $\leq$ be total associated with $\mathbb{K}$.

consider S, an nonempty subset of $\mathbb{K}$, then

* U is called upper bound of S i.e $a \leq U \quad \forall a \in S$
* U^* be least upper bound S if
  i) $a \leq U^* \quad \forall a \in S$
  ii) for $U \in \mathbb{K}$, $a \leq U \quad \forall a \in S$
  then $U^* \leq U$
* Illy Greatest lower bound. i) $a \geq V^* \quad \forall a \in S$
  2) $V \in \mathbb{K}$, $V \leq a \quad \forall a \in S$
  then $V \leq V^*$

Dedekind complete

for ord $(\mathbb{K}, <)$, if $\forall$ nonempty bounded above subset of $\mathbb{K}$ has supremum in $\mathbb{K}$.

Ex: $\mathbb{Q}$ is not dedekind complete

$S = \{x \in \mathbb{Q},\ 0 < x \ \&\ x^2 > 0\}$

sup(S) doesn't exist.

Ex: $(\mathbb{K}, <)$, $S \neq \phi$ $\quad S = \{-a \in \mathbb{K};\ a \in S\}$

i) show that if U is lub of S then $-a$ lub of $-S$ Vice versa

ii) If sup(S) exist then Inf(-S) exist vice versa

Inf(S) $\rightarrow$ sup(-S)

iii) $(\mathbb{K}, <)$ is complete $\Leftrightarrow$ $\forall$ S bounded below of $\mathbb{K}$ has Inf in $\mathbb{K}$

$(\mathbb{K}, <)$ is complete.

Let take S bounded below subset

then look into $S' = (-a : a \in S)$

Since S is bounded below, S' is bounded above

any be completeness, S' have sup in $\mathbb{K}$, let U be $\sup(S')$

then S has Inf and it is $-U$.

Since $\mathbb{K}$ is field, $-U \in \mathbb{K}$ If $U \in \mathbb{K}$.

SO $\text{Inf}(S) \in \mathbb{K}$.

$\Longrightarrow$ $\forall S$, bounded below subset has Inf in $\mathbb{K}$.

Let A be any bounded above subset

Consider $A' = (-a : a \in A)$

Since A bounded below $\Rightarrow$ A' is bounded below

we have given $\forall$ bounded below nonempty subset has Inf in $\mathbb{K}$

Let $\text{Inf}(A') = l \in \mathbb{K}$

then $\text{Sup}(A) = -l \Rightarrow$ an $-l \in \mathbb{K}$ ($\because l \in \mathbb{K}$, $\mathbb{K}$ is field)

$\Rightarrow$ $\forall$ nonempty bounded above subset has Sup in $\mathbb{K}$

$\Rightarrow$ complete ordered field.

1) If sup(s) exist then Inf(-s) exist viceversa

Inf(s) $\to$ sup(-s)

ans) 1) Sup(S) exist $\Rightarrow$ Inf (-S) exist & sup(S) = - Inf(S)

let S be non empty set bounded above and

$U = \sup(S)$.

consider $-S = (-a : a \in S)$, set of negative number of S.

Claim : - S is bounded below.

we know $\forall a \in S$,

$a \leq U$ ①

① × -1 $\Rightarrow$ $-a \geq -U$

Since a is arbitrary, -a depend set -S.

so $\forall$ $-a \in S$ -a bounded below. i.e $-U$ is lower bound.

$\Rightarrow$ $\inf(-S) = -U$

so $\sup(S) = U$ & $\inf(-S) = -U$

$\Rightarrow \sup(u) = -\inf(u)$

Thm: $\mathbb{R}$ is complete ordered field.

Thm: If $(\mathbb{K}, <)$ & $(\mathbb{K}', <')$ are complete ordered, then there exist isomorphism $f: \mathbb{K} \to \mathbb{K}'$ st f is preserv order.

Properties of $\mathbb{R}$

1) If $x, y \in \mathbb{R}$ $\quad y > x > 0$
   then $nx > y$ $\quad n \in \mathbb{N}$ [Archimedian property]

2) If $x, y \in \mathbb{R}$ then
   $x < p < y$ where $p \in \mathbb{Q}$ [Denseness of $\mathbb{Q}$ in $\mathbb{R}$]

Proof:

$x > y$ & $y \neq 0$, we ned show

$$ny > x$$

take $S = \{ny : n \in \mathbb{N}, y \in \mathbb{R}\}$

Let $ny < x$. this Implies S have bounded

let α be $\sup(S)$.

$$\alpha - y < \alpha$$

So $\exists$ $\alpha - x < my$ for some m

$\Rightarrow \alpha < (m+1)y$

$\Rightarrow \alpha$ is is sup.

So $ny > \alpha$ for any α.

$\rightarrow ny >$

Denc. let $x < y$,

$y - x > 0$

$n(y - x) > 1$

$ny > nx + 1$

$\lfloor nx \rfloor \leq nx < \lfloor nx \rfloor + 1 \leq nx + 1 < ny$

$nx < \lfloor nx + 1 \rfloor < ny \Rightarrow x < \frac{\lfloor nx \rfloor + 1}{n} < y$

Absolute valued Real numbers

$x \in \mathbb{R}$ then:

$$|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

properties of absolute value

- $f : \mathbb{R} \to \mathbb{R}^+_{\geq 0}$
- * $|x| \geq 0$ and $|x| = 0 \iff x = 0$
- * $|xy| = |x|\,|y|$
- * $|x+y| \leq |x| + |y|$ $\quad \forall x, y \in \mathbb{R}$

$|\cdot| : (\mathbb{K}, <) \to \mathbb{R}_0$ called absolute value on $(\mathbb{K}, <)$

Sequence & Convergence

$\phi : \mathbb{N} \to \mathbb{R}$

$\phi \equiv (a_n)_{n \in \mathbb{N}}$

limit

$l \in \mathbb{R}$, $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$

st $|a_n - l| < \varepsilon$ $\forall n \geq n_0$

17/8/23 Recap.

Complete ordered field

$(\mathbb{K}, <)$ is an ordered field

* Dedekind complete :- $\forall$ nonempty bdd above subset has sup in $\mathbb{K}$.
  * Existence :- $\exists$ a complete ordered field

    Ex: $\mathbb{R} \Rightarrow$ field of real numbers
  * Uniqueness := $f : (\mathbb{K}, <) \to (\mathbb{K}', <')$ is Isomorphism, which also order preserving

Sequence of $\mathbb{R}$

$\phi : \mathbb{N} \to \mathbb{R}$

If $\phi(n) = a_n \in \mathbb{R}$, then we denote by

$\phi_n \equiv (a_n)_{n \in \mathbb{N}}$

Convergence :-

$\lim_{n \to \infty} a_n = l$ if

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ st $|a_n - l| < \varepsilon$, $\forall n \geq n_0$

proposition: Every convergent seq is bdd

(a_n is bdd if $\exists M > 0$ st $|a_n| \leq M$ $\forall n$)

proof: convergent seq : $\forall \varepsilon > 0$, $\exists n \in \mathbb{R}$ s.t $\forall N \geq n$,

$|a_N - l| < \varepsilon$

then $\forall$ term after a_n is in the interval $[a-\varepsilon, a+\varepsilon]$

let $U = \max\{a_1, a_2, \ldots a_n, [a-\varepsilon, a+\varepsilon]\}$

then U is upperbound. Illy $\exists$ lowerbound $\Rightarrow a_n$ is bounded

Cauchy Sequence

A Seq. (a_n) of $\mathbb{R}$ is s.t cauchy seq if $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t $|a_n - a_m| < \varepsilon$ s.t $n, m \geq n_0$

proposition: Every convergent seq. is cauchy seq.

Proof: convergent sequence (a_n): $\forall \varepsilon > 0, \exists n \in \mathbb{R}$ st $\forall N \geq n$
$|a_N - l| < \varepsilon$

$\Rightarrow$ let for $\forall N \geq n_1$ $|a_N - l| < \varepsilon_1$ & $\forall M \geq n_2$
$|a_m - l| < \varepsilon_2$ $\quad |a| - |b| \geq$

$n = \max\{n_1, n_2\}$ then $|a_N - l| + |a_m - l| < \varepsilon$

$|a_N - l - a_m + l| \leq |a_N - l| + |a_m - l| < \varepsilon_1 + \varepsilon_1$

$|a_N - a_m| < \varepsilon$ // $\quad \frac{\varepsilon}{2} = \varepsilon_1 = \varepsilon_2$

$\Rightarrow$ $\forall$ convergent sequence is cauchy

Sub Sequence

Seq $:= \phi : \mathbb{N} \to \mathbb{R}$

then $\phi : A \in \mathbb{N} \to \mathbb{R}$ is subsequence

proposition :- i) $\forall$ subsequence of convergent seq is convergent and has the same limit

ii) $\forall$ Real sequence has either a monotonically increasing sub seq or strictly decreasing subsequence

- (a_n) is montonically $\uparrow$ if $a_n \leq a_{n+1}$ $\forall n$
- (a_n) $n \in \mathbb{N}$ strictly $\downarrow$ if $a_n > a_{n+1}$

ii) $\forall$ bdd montonic seq is convergent

Proof: $X = (x_n)$ is bounded.

let U be an Supremum of X and L infimum of
then $\forall x_n$ $\quad L \leq x_n \leq U$
since X is monotonic sequence either
$\forall x_n \geq x_{n+1}$ or $x_n \leq x_{n+1}$

let X be a increasing sequence
Claim: U is limit point of x_n
to prove U is the limit, we need show $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ st $\forall N \geq n_0$

$|x_n - U| < \varepsilon$

for any monotonically sequence, $\exists N$ s.t
$U - \varepsilon < x_N \leq U$ $\quad$ ($\because$ x_n is bounded.

now consider $\forall \varepsilon > 0$, using above
$U - \varepsilon < x_N \leq U \Rightarrow U - \varepsilon < x_n < U + \varepsilon$
$\Rightarrow |x_n - U| < \varepsilon$

so U is limit point

Ily for monotonic decreasing sq
$L + \varepsilon > x_N \geq L$
$L + \varepsilon > x_N > L - \varepsilon$
$|x_n - L| < \varepsilon$
L is limit point

Ex: i) Cauchy sq is bdd. defn d $|a_n - a_m| < \varepsilon$

ii) Cauchy seq has convergent subsequence. then seq is also convergent

(a_n) $(a_{n_k})_{k\in N}$ $l \in R$ $\forall \varepsilon > 0 ; \exists k_0$

$|a_n - a_m| < \varepsilon/2$ $\forall n \geq n_0$ st $|a_{n_k} - l| < \varepsilon/2$ $\forall k > k_0$

Let $n_1 = \max\{n_0, k_0\}$

then $n \geq n_1$ then

$|a_n - l| = |a_n - a_{n_k} + a_{n_k} - l|$

$\leq |a_n - a_{n_k}| + |a_{n_k} - l|$?

thm: $\forall$ cauchy seq. of real is convergent

proof: easy?

In other words, $\mathbb{R}$ is cauchy complete

i.e $\mathbb{R}$ is dedekind complete and cauchy complete

but Cauchy $\Leftarrow$ Dedekind ?

doubt

$f.b : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ the trival fn $\left\{ \begin{matrix} P/q = 0 \\ P/q \neq \gamma \end{matrix} \right\}$

is complete as given constand seq with limit. but $\mathbb{Q}$ not dedekind complete ; since x^2- with ywt ?

Thm: $(\mathbb{K}, <)$ ord. field, Thn following are equivalent

* $\mathbb{K}$ is dedekind complete
* $\mathbb{K}$ is Archimedian and $\mathbb{K}$ is cauchy complete

In simple terms Dedekind $:\equiv$ Archimedian + cauchy.
(stronger).

proposition:

$Aut(\mathbb{R}) = \{1_{\mathbb{R}}\}$ Automorphism on $\mathbb{R}$ is Identity ? doubt

If $f : \mathbb{R} \to \mathbb{R}$ is field isomorphism

step1) $f|_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}$ is field automorphism

$f|_{\mathbb{Q}} = 1_{\mathbb{Q}}$ (Identity map)

step2) f is strictly increasing :

$x > 0$; $\exists y > 0$ s.t $x = y^2$ (i.e y is +ve root of x)

$f(x) = f(y \times y) = f(y)^2 > 0$

Now claim; If $\alpha \in \mathbb{R} - \mathbb{Q}$ then

$f(x) = x$

Suppose a claim is not then

either $f(\alpha) < \alpha$ or $f(\alpha) > \alpha$

$f(\alpha) < r < \alpha$ (By denses $r \in \mathbb{Q}$)

$r < \alpha \Rightarrow f(r) < f(\alpha) \Rightarrow r < f(\alpha)$ [$\because r \in \mathbb{Q}$]

$\mathbb{C} := \mathbb{R} \times \mathbb{R}$

$(a,b) + (c,d) = (a+c, b+d)$

$(a,b) \cdot (c,d) = (ac-bd, ad+bc)$

$\Rightarrow \mathbb{C} = \mathbb{R} \times \mathbb{R}$ is field

where $i = (0,1)$; $i^2 = (-1,0)$

$a \mapsto a,0$

IIly $(a,b) \to a+ib$

* $(0,0)$ is additive identity

* $\mathbb{C}$ is not ordered ($i^2 = -1$)

$|z| = |a+ib| = \sqrt{a^2+b^2}$

$\Rightarrow \mathbb{C}$ is not dedekind complete

$\Rightarrow \mathbb{C}$ is cauchy complete

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$\text{Aut}(\mathbb{Q}) = \{1_\mathbb{Q}\}$

$\text{Aut}(\mathbb{R}) = \{1_\mathbb{R}\}$

$\text{Aut}(\mathbb{C}) = \{1_\mathbb{C}, \text{conjugate}\}$ doubt ?

$\text{Aut}(\mathbb{Q}[\sqrt{i}]) = \{\text{conjugate}\}$

proposition

let $x \in \mathbb{R}$ and $x > 0$. If $n > 1$ then $\exists$ a unique $y > 0$ $y \in \mathbb{R}$

s.t $y^n = x$

[This unique y is called the positive n^{th} root x and we write $\sqrt[n]{x}$ or $x^{1/n}$]

Proof. Consider the subset

$E = \{t \in \mathbb{R},\ t > 0,\ t^n < x\}$

Let $t_0 = \dfrac{x}{x+1} < 1$ also $x/x+1 < x$

$\Rightarrow t_0^n < t_0 < x$

so E is non empty

$t_1 \geq x+1 > 1$

$\Rightarrow t_1^n > t_1 > x$

$t_1 \notin E$

But $1+x$ is an upper bound of E

Then by complete of $\mathbb{R}$

$\exists\, y = \sup E \in \mathbb{R}$

Claim $y^n = x$

then $y^n = x$ $y^n > x$ $y^n < x$

$b^n - a^n < (b-a)nb^{n-1}$

$y^n < x < (y+1)^n$

$0 < x - y^n < (y+1)^n - y^n$

Claim $y^n = x$

then $y^n = x$ $\quad y^n > x$ $\quad y^n < x$

$b^n - a^n < (b-a)nb^{n-1}$

$y^n < x < (y+1)^n$

$0 < x - y^n < (y+1)^n - y^n$

$0 < \frac{x-y^n}{n(y+1)^{n-1}} < \frac{(y+1)^n - y^n}{n(y+1)^{n-1}} < 1$

Choose h such that

$0 < h < \frac{x-y^n}{n(y+1)^{n-1}} < \frac{(y+1)^n - y^n}{n(y+1)^{n-1}} < 1$

then

$(y+h)^n - y^n < hn(y+h)^{n-1} < x - y^n$

?

$(y+h)^n < x$

but y is sup contr

Assume $y^n > x$

$y^n - x < y^n$

$0 < k < \frac{y^n - x}{ny^{n-1}} \le \frac{y^n - x}{y^{n-1}} < y$

Choose $k = \frac{y^n - x}{ny^{n-1}}$ $\quad (0 < k < y$

$y^n - (y-k)^n < kny^{n-1} = y^n - x$

$-(y-k)^n < -x$

$\Rightarrow (y-k)^n > x$

$\Rightarrow (y-k) \notin E$

But contrd

Only $y^n = x$

Rational function field over $\mathbb{R}$

$\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x) + q(x)r(x)}{q(x)s(x)}$

$\frac{p(x)}{q(x)} \cdot \frac{r(x)}{s(x)} = \frac{p(x)}{q(x)} \frac{r(x)}{s(x)}$

$\frac{p(x)}{q(x)} = \frac{a_0 + a_1x_1 \dots a_nx^n}{b_0 + b_1x_1 \dots b_mx^m} > 0$

if $a_nb_m > 0$

$r \in \mathbb{R}$; $x - r > 0$

$\frac{p(x)}{q(x)} > \frac{r(x)}{s(x)} \Rightarrow \frac{p(x)}{q(x)} - \frac{r(x)}{s(x)} > 0$

for $\forall$ $\varepsilon > 0$, $1/x < \varepsilon$ true for $\{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$

Construction of $\mathbb{Z}$

on $\mathbb{N} \times \mathbb{N}$ define $(n,m) \sim (n', m')$

iff $n + m' = n' + m$

then from this equivalence class

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$$

$$[(n,m)] + [(a,b)] = [(n.a, m.b)]$$

$$[(n,m) . (a,b)] = [(na + mb, nb + ma)]$$

$[(2,2)] \sim [(1,1)] \Rightarrow$ additive Identity

$[(0,1)] \Rightarrow -1$ (in $\mathbb{Z}$)

Rational function field over $\mathbb{R}$

doubt

$$\frac{P(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{P(x)s(x) + q(x)r(x)}{q(x)s(x)}$$

$$\frac{P(x)}{q(x)} \cdot \frac{r(x)}{s(x)} = \frac{P(x)}{q(x)} \frac{r(x)}{s(x)}$$

$$\frac{P(x)}{q(x)} = \frac{a_0 + a_1 x_1 \ldots a_n x^n}{b_0 + b_1 x_1 \ldots b_m x^m} > 0$$

if $a_n b_m > 0$

$r \in \mathbb{R}$; $x - r > 0$

$$\frac{P(x)}{q(x)} > \frac{r(x)}{s(x)} \Rightarrow \frac{P(x)}{q(x)} - \frac{r(x)}{s(x)} > 0$$

for $\forall$ $\varepsilon > 0$, $1/x < \varepsilon$ true for $\{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$

The field of rational function is non archimedean and nondedekind.

because if taking two elements,

$$r(x) = \frac{ax^n + \ldots}{bx^m}$$

$r(x) \geq 0$ iff $a/b > 0$

$h(x) = f(x) - g(x)$

if $h(x) \geq 0$

then $f(x) \geq g(x)$

If $h(x) \leq 0$

then $g(x) \geq f(x)$

to be archimedean $\qquad f(x) = x \quad g(x) = 1$