$A \in L(\mathbb{R}^n, \mathbb{R}^m)$, $\Omega = \{A \in L(\mathbb{R}^n, \mathbb{R}^m) : A$ is inveritable$\}$

Operator norm

$$\|A\| = \sup\{ \|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| \leq 1\}$$

$$\|Ax\| \leq \sum_{j=1}^{n} \|Ae_j\|$$

ii) $\|Ax\| \leq \|A\| \|x\| \quad \forall x \in \mathbb{R}^n$

iii) If $\|Ax\| \leq \lambda \|x\| \quad \forall x \in \mathbb{R}^n$ then $\|A\| \geq \lambda$

$\lambda < \|A\| \Rightarrow \exists x_n \in \mathbb{R}^n$ with $\|x_0\| \leq 1$

and $\lambda \|x_0\| \leq \lambda \leq \|Ax_0\| \leq \lambda \|x_0\|$

$$Ac_j = \sum_{j=1}^{m} a_{ij} c_i$$

$Mat(A) = [A]$

$Ax \longleftrightarrow mat(A) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \sum a_{1j} c_j \\ \vdots \\ \sum a_{nj} c_j \end{bmatrix}$

$L(\mathbb{R}^n, \mathbb{R}^m) \longleftrightarrow M_n(\mathbb{R})$

$\Omega \longleftrightarrow \det \neq 0$

A $[A]$

A^{-1} $[A]^{-1} = \dfrac{adj(A)}{\det[A]}$

proof: i) $A \in \Omega$ and $\|B - A\| \|A^{-1}\| < 1 \Rightarrow B \in \Omega$

put $\|A^{-1}\| = 1/\alpha$ and $\|B - A\| = \beta \Rightarrow \beta < \alpha$

$0 \neq x \in \mathbb{R}^n$

Now $\alpha \|x\| = \alpha \|A^{-1}(Ax)\| \leq \alpha \|A^{-1}\| \|Ax\| = \|Ax\| \leq \|(A-B)\| \|x\| + \|Bx\| \leq \beta \|x\| + \|Bx\|$

Put $y = Bx$ or $B^{-1} y = x$

$\|B^{-1} y\| \leq (\alpha - \beta)^{-1} \|y\|$

$\Rightarrow (\alpha - \beta) \|x\| \leq \|Bx\|$

Then $x \neq 0 \Rightarrow Bx \neq 0$. Hence B is injective and therefore bijective.

$\Rightarrow B \in \Omega$

ii) inv $\Omega \to \Omega$ given by

$A \mapsto A^{-1}$

Let $A \in \Omega$ If $B \in \Omega$ such that $B \to A$, then

we shall show that

Hilbert norm $= \sqrt{tr(A^t A)}$

Def. $E \subseteq \mathbb{R}^n$ (open), $f : E \to \mathbb{R}^m$ be a function ; $x \in E$

f is diff at x (f has a total derivative at x) if

$\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0 \iff f(x+h) = f(x) + Ah + r(x,h)$$

$$\boxed{f'(x) = A}$$

Ex: Let $f: \underset{\cup}{E} \to R^m$ (with $E \subseteq \mathbb{R}^n$) be a function differentiable at $x \in E$. Suppose

$h: E \to \mathbb{R}^m$, diff. at x. Then show that $f+h$ is also diff at x and cf

is also diff at x $c \in \mathbb{R}$

$$(f+h)'(x) = f'(x) + g'(x)$$
$$(cf)'(x) = cf'(x)$$

Chain rule.

Let $f: E \to \mathbb{R}^m$ be dif at x_0 and $g: G \to \mathbb{R}^l$ be diff at $f(x_0)$, where

G be an open at cortain $f(E)$. Then the composition $F = g \circ f: E \to \mathbb{R}^l$

is diff at x_0 and $F'(x_0) = g'(f(x_0))\, f'(x_0)$

Directional Derivative: Let $\phi: \underset{\cup}{E} \longrightarrow \mathbb{R}$ (with $E \subseteq \mathbb{R}^n$) be a function and $x \in E$. Let $u \in \mathbb{R}^n$

be a unit vector i.e $\|u\| = 1$. If

$\lim\limits_{t \to 0} \dfrac{\phi(x+tu) - \phi(x)}{t}$ exists, then we say ϕ has a directional derivative at x in

direction of u. and we write

$$(D_u \phi)(x) = \lim_{t \to 0} \frac{\phi(x+tu) - \phi(x)}{t} \quad \text{(Primite limit exists)}$$

If $(D_{e_j} \phi)(x)$ exists, then we call it

the partial derivative of ϕ at x w.r.t x_j and denote it by

$$D_j \phi(x) = \frac{\partial \phi}{\partial x_j}(x)$$

$$(D_i f)(x) = \sum_{i=1}^{m} (D_u f_i)(x)\, u_i$$

proposition: If $f: E \to \mathbb{R}^m$ is differentiable at x, then $D_j f_i(x)$ exists for all component

function f_i's at further $f'(x)h = \sum_{i=1}^{m} \sum_{j=1}^{n} D_j f_i(x)\, h_j\, u_i$

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Inverse function Theorem

$E \subseteq \mathbb{R}^n$ $f: E \to \mathbb{R}^n$ is a C'- mapping

Let $a \in E$, $b = f(a)$ and $f'(a)$ is investable

Then i) There are open sets U and V in $\mathbb{R}^n$ with $a \in U$ & $b \in V$ such that f is 1-1 mapping from U onto V

ii) Let $g = f^{-1}$ on V. Then $g: V \to U \subseteq \mathbb{R}^n$ is C'-mapping

Proof: Let $A = f'(a)$ choose $\lambda > 0$, $2\lambda \|A^{-1}\| = 1$

f' is continous at $a \Rightarrow \exists$ on open ball with s.t

$x \in U \Rightarrow \|f'(x) - A\| < \lambda$ — ①

For any $y \in \mathbb{R}^n$, define mapping

$\varphi : E \to \mathbb{R}^n$ by

$\varphi(x) = I + A^{-1}(y - f(x))$ — ②

$\|f'(x) - A\| \|A\| < \lambda \|A^{-1}\| = \frac{1}{2} < 1$

$\Rightarrow \varphi(x) = x \iff y = f(x)$

From $\varphi'(x) = I - A^{-1}f'(x) = A^{-1}(A - f'(x))$

$\Rightarrow \|\varphi'(x)\| \le \|A^{-1}\| \|A - f'(x)\| \le \|A^{-1}\|\lambda = \frac{1}{2}$

By M.V.T for several variable, we have

$\|\varphi(x_1) - \varphi(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$ for $x_1, x_2 \in U$

we see that φ can have at most one fixed point

$\Rightarrow$ for given $y \in \mathbb{R}^n$ there exists at most one $x \in U$ s.t $f(x) = y$

Then f is 1-1 on U put $V = f(U)$

Let $y_0 \in V$ then $\exists x_0 \in U$ s.t $y_0 = f(x_0)$

Choose $r > 0$ so small that B with centre x_0 and radius r, $B = B(x_0, r)$ s.t $\bar{B} \subseteq U$

If $y \in V$ satisfies $\|y - y_0\| < \lambda r$, then we shall show that

$y = f(x)$ for some $x \in \bar{B}$

Let φ be an given by ②

$\|\varphi(x_0) - x_0\| = \|A^{-1}(y - f(x_0))\| \le \|A^{-1}\| \|y - y_0\| < \|A^{-1}\| \lambda r = \frac{r}{2}$

If $x \in \bar{B}$, then $\|\varphi(x) - x_0\| \le \|\varphi(x_0) - \varphi(x)\| + \|\varphi(x_0) - x_0\|$

$\le \frac{1}{2}\|x - x_0\| + \frac{r}{2} = \frac{r}{2} + \frac{r}{2} = r$

$\Rightarrow \varphi(x) \in \bar{B}$ for $x \in \bar{B}$

Then φ is contraction of $\bar{B}$, $\bar{B}$ being a closed subset of $\mathbb{R}^n$ is a complete metric space

$\Rightarrow \varphi$ has a unique real point $\exists! x \in \bar{B}$

s.t $\varphi(x) = x \Rightarrow y = f(x)$

$\Rightarrow y = f(x) \in f(\bar{B}) \subseteq f(U) = V$

$\Rightarrow V$ is open set

Let g be the inverse of f from U onto V then

$$g(f(x)) = x \qquad \forall x \in U$$

Pick $y \in V$ and $y+k \in V$. Then $\exists x \in U$ & $x+h \in U$

with $f(x) = y$ & $f(x+h) = y+k$

for this y, let φ be as in ②

$$\Rightarrow \varphi(x) = x + A^{-1}(y-y) = x$$

$$\& \ \varphi(x+h) = x+h + A^{-1}(y-y-k) = x+h$$

$$\Rightarrow \varphi(x+h) - \varphi(x) = h - A^{-1}k \quad ③$$

from ③

$$\|h - A^{-1}k\| = \|\varphi(x+h) - \varphi(x)\| \le \tfrac{1}{2}\|h\|$$

$$\|h\| - \|A^{-1}k\| \le \|h - A^{-1}k\| \le \tfrac{1}{2}\|h\|$$

$$\|h\| \le 2\|A^{-1}k\| \le 2\|A^{-1}\|\,\|k\| = \tfrac{1}{\lambda}\|k\|$$

If $k \to 0$, then ④ $h \to 0$. Let $T = (f'(x))^{-1}$ (we know $x \in O$ $\Rightarrow f'(x)$ is invertible)

$$\frac{\|g(y+k) - g(y) - Tk\|}{\|k\|} = \frac{\|h - Tk\|}{\|k\|}$$

$$h - Tk = T(k - T^{-1}h)$$

$$= -T(f(x+h) - f(x) - f'(x)h)$$

$$\|h - Tk\| \le \|T\|\,\|f(x+h) - f(x) - f'(x)h\|$$

$$\Rightarrow \frac{\|h - Tk\|}{\|k\|} \le \frac{\|T\|}{\lambda}\,\frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|}$$

$$\Rightarrow g'(y) = T = (f'(x))^{-1} = (f'(g(y)))^{-1}$$

$g' =$ inv of $f' \circ g$. $\qquad g' \in C'(U)$

Corollary: Let $f : \underset{\text{open}}{E \subset \mathbb{R}^n} \longrightarrow \mathbb{R}^n$ be C'-mapping

Suppose $f'(x)$ is invertible for all $x \in E$. Then f is an open mapping (ie $W \subset E$ is open, then $f(W)$ is open)

Corollary: (Global IFT) Suppose f is are in the limit contro. In addiction f is 1-1 then

$$f^{-1} : f(E) \to E \quad \text{is } C'\text{-isomorphic}$$

Implicit function Theorem

Let $E \subseteq \mathbb{R}^{n+m}$

$(x,y) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$

$\hookrightarrow (x_1, x_2, \dots x_n, y_1, \dots y_m)$

$A \in L(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$

$$Mat(A) = \left([A] \,\middle|\, [A] \right] \begin{bmatrix} h \\ 0 \end{bmatrix} \Big)$$

$$A(h,k) = A(h,0) + A(0,k)$$
$$= A_x h + A_y k$$

Let $(a,b) \in E$ such that $f(a,b) = 0$. Then there are openset $U \subseteq E$ in $\mathbb{R}^{n+m}$ and W in $\mathbb{R}^n$ such that $(a,b) \in U$ and $b \in W$.

Suppose $f'(a,b) = A$ and A_x is invertable. For each $y \in W$, $\exists!\, x \in \mathbb{R}^n$ such that $(x,y) \in U$ and $f(x,y) = 0$. We can define a map $g: W \to \mathbb{R}^n$ by $x = g(y)$ and $f(g(y), y) = 0$ for $y \in W$.

Further, g is C'-mapping and $g(b) = A_x^{-1} A_y$

eg: $f(x,y) = x^2 + y^2 - 1$

$f(x,y) = 0$

$$f: \underset{\mathbb{R}^{n+m}}{\overset{\cup}{E}} \longrightarrow \mathbb{R}^n$$

$F: E \longrightarrow \mathbb{R}^{n+m}$

$F(x,y) = (f(x,y), y)$

$f'(a,b)$ is Invertable.