16/10/23.

Sequence and series of function.

$E \subseteq \mathbb{R}$ or $E \subseteq (X,d)$

For each $n \in \mathbb{N}$, let $f_n : E \to \mathbb{R}$ be a function then $(f_n)_{n\in\mathbb{N}}$ is called a sequence of function out suppose of function on E

Suppose $x \in E$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ of real numbers is convegent.

Then we define function

$f : E \to \mathbb{R}$ by

$f(x) = \lim_{n\to\infty} f_n(x)$

we say f is the (pointwise) limit function of the sequence $(f_n)_{n\in\mathbb{N}}$

Equivalently, we say that f_n convergent of pointwise on F

($f_n \to f$ pointwise on E)

Example: let $f_n(x) = x^n$ $n \in \mathbb{N}$ $E = [0,1]$

Ans) we have

$$\lim_{n\to\infty} x^n = \begin{cases} 1 & x = 1 \\ 0 & 0 \le x < 1 \end{cases}$$

we define $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

$f_n \to f$ pointwise on $E = [0,1]$

All f_n are on continuous on E but f is discontinuous

2) let $f_n(x) = x/n$ $n \in \mathbb{N}$ & $E = \mathbb{R}$

$\lim_{n\to\infty} f = 0$

Certainly, we have f_n converges to f pointwise of $\mathbb{R}$

(3) let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ and $g_n(x) = \frac{\sin nx}{n}$ $x \in \mathbb{R}$

Then se that $f_n \to f$ and $g_n \to g$, when $f \equiv 0$ $g \equiv 0$

also $f'_n(x) = \sqrt{n}\cos nx$ and $g'_n(x) = \cos nx$.

In fact $f'_n(0) \to +\infty$ $n\to\infty$ & $g'_n(0) = 1$ $\forall n$

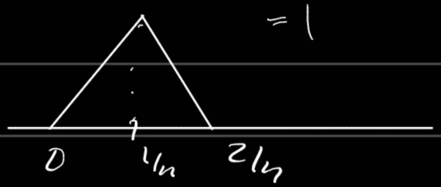
Since $f' \equiv 0$ and $g' \equiv 0$, we see that
$f_n' \not\to f$ and $g_n' \to g$

a) let $f_n(x) = n x(1-x^2)^n$ $\quad x \in E = [0,1]$ $\quad n = 1, 2, \ldots$
Then $f_n(0) = 0$ $\quad f_n(1)$ $\forall n$ $\quad$ If $x \in [0,1)$
$f_n(0) = 0 = f_n(1)$ $\forall n$ $\quad$ if $x \in (0,1)$ then
$f_n(x) \to 0$ as $n \to \infty$ therefore $f_n \to f = 0$

we see that $f_n \in R[0,1]$
also $\int_0^1 f_n dx = \int_0^1 x(1-x^2)^n dx = \frac{x}{2n+2} = 0$

$$f_n(x) = \begin{cases} n^2 x & ; 0 \le x \le 1/n \\ 2n - n^2 x & ; 1/n < x \le 2/n \\ 0 & ; 2/n < x \le 1 \end{cases}$$

Continuous limit function



$f_n \to 0$.
$$\lim_{n\to\infty} \int_0^1 f_n dx = 1 \ne 0 = \int_0^1 0\, dx$$

Series of functions: If $(f_n)_{n\in\mathbb{N}}$ is a sequence of function on E then we can form a series $\sum_{n=1}^{\infty} f_n$ of function on E.
Suppose $x \in E$, the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent $\forall x \in E$
then we define a function $g(x) = \sum_{n=1}^{\infty} f_n(x)$
and g a the sum of the series $\sum_{n=1} f_n$

6 let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ $\quad ; n = 0, 1, 2, \ldots$ $\quad$ for $x \in \mathbb{R}$

Then the series $\sum_{n=0}^{\infty} f_n(x)$ is pointwise convergent and its sum is
$$g = \begin{cases} 0 & ; x = 0 \\ 1+x^2 & ; x \ne 0 \end{cases}$$

Clearly $\sum f_n$ is a series of continuous function but the sum g is not continuous.

7) $f_n(x) = x^2/x^2+n^2$

uniform convergence: Suppose $(f_n)_{n\in\mathbb{N}}$ is sequence of function f_n convergence to a function on E. then we say f_n convergent to a function f on E uniformly if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$
If $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t

Q Once of the example 's not conformity.

$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in E$
and $\forall n \ge n_0$

$f_n(x_0) \to f(x_0)$ and $n \to \infty$

$\forall \varepsilon > 0, \exists n_0(\varepsilon, x_0) \in \mathbb{N}$

$|f_n(x) - f(x)| < \varepsilon$

$\forall n \geq n(\varepsilon, n)$