18/9/23 $(X,d)$ → order spaces.

$K \subseteq X$, $K$ is compact $\Leftrightarrow$ $K$ is sequential compact

$K$ is compact $\Rightarrow$ $K$ is closed.

$X$ is compact & $K \subset X$ $\Rightarrow$ $K$ is compact / sequentially compact

$X \to Y$ continuous, $K \subseteq X$ sequentially $\Rightarrow$ $f(K)$ is

* $A \subset (X_1, d_1)$ ; $B \subseteq (X_2, d_2)$ seq compact / compact

$\Rightarrow$ $A \times B \subseteq (X_1 \times X_2, d_E)$ so seq compact / compact

$$d_E((x_1,x_2),(y_1,y_2)) = \sqrt{d(x_1,y_1)^2 + d(x_2,y_2)^2}$$

Proof?:

Proposition

$\forall$ n-cube is compact in $\mathbb{R}^n$

Proof: Let $C = [a_1,b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ $a_i < b_i$
$1 \leq i \leq n$

by Induction on $n$,

for $n=1$, $[a,b] \subseteq \mathbb{R}$ is compact

assume $n \geq 2$ and by Induction.

$[a_1,b_1] \times \cdots [a_{n-1}, b_{n-1}] \subseteq \mathbb{R}^{n-1}$ is compact.

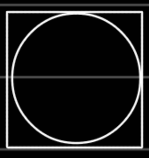
Take

$A = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$ in $X_1 = \mathbb{R}^{n-1}$

$B = [a_n, b_n]$ in $X_2 = \mathbb{R}$

then $A \times B = C$ is compact subset of $\mathbb{R}^n$

Theorem: a subset $K$ of $\mathbb{R}^n$ (metric) is compact $\Leftrightarrow$ (iff) $K$ is closed & bounded. (Heine-Borel thm)



Exercise: $K$ closed & bounded in $\mathbb{R}^n$ $\Rightarrow$ $K$ is compact.

P R1

Let $K$ be subset of metric space $(X,d)$

we say $K$ is totally bounded if

for $\forall\ \varepsilon > 0$, $K$ can be covered by finitely many open-ball

i.e $K$ is said to be totally bounded if $\forall\ \varepsilon > 0$,

$\exists$ a finite set of $\{P_1 \ldots P_n\} \subseteq K$ st

$K \subseteq \bigcup B(P_i, \varepsilon)$ Sometime ?

$\{P_1, P_2 \ldots P_n\}$ is called an $\varepsilon$-net.

Exe totally bounded $\Rightarrow$ bounded.

+ $A \subset (X_1, d_1)$ ; $B \subseteq (X_2, d_2)$ seq compact / compact

$\Rightarrow$ $A \times B \subseteq (X_1 \times X_2, d_E)$ so seq compact / compact

---

compact $\Rightarrow$ sequentially compact

let K be compact on X,

consider $(a_n)$ in K,

$\because$ $(a_n)$ is sequence, it have infinite distinct term / finite any distinct term, let we have infinite subsequence of distinct

by bolzano - weierstrass $\exists$ a convergent subsequence $(a_{n_k})$ with this subsequence

and the subsequence itself is trivially convergent with a lim $a_n$

it it has finitely many term from K, $\exists$ at least one element of K that appear often in the sequence, and denote by $x$.

In both case, we have a convergent subsequence of $(a_n)$

If the subsequence is formed by term in K, $\Rightarrow$ limit of subseq is also K. being K is compact

the subsequence was made up $x$, lim is $x$, which in K.

---

to confirm $A \times B$ is compact we need to check

1) sequential compactness : $\forall (a_n)$ in $A \times B$ is conv.
2) Total boundedness : $\forall \varepsilon > 0$, $\exists$ n it cover $A$, or (B)

Conv: K closed & bounded in $\mathbb{R}^n$ $\Rightarrow$ K is compact.

$\because$ K is bounded $\exists M > 0$ $d(x,y) < M$ $x, y \in K$

a finite collection of open ball of radius M center at each point

K is cover K. $\Rightarrow$ K is 1

Claim : totally bound set is also compact

$\forall \varepsilon > 0$ $\exists$ finite collection balls $\{B(x_i; \varepsilon)\}$ such that

A is contain in $A \subseteq \bigcup_{i=1} B(x_i, \varepsilon)$ n is finite

let $(x_n)$ cauchy

$\forall \varepsilon > 0$, N $d(x_m, x_n) < \varepsilon$

$(x_n)$. $\forall \varepsilon > 0$, $\exists N$, s.t $\forall m, n \geq N$ $d(x_m, x_n) < \varepsilon$

$\because \forall \varepsilon > 0$, $\exists \{B(x_i; \varepsilon/2)\}$ s.t $A \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon/2)$

for $m, n \geq N$ we $d(x_m, x_i) < \varepsilon$

for $i$ $A \subseteq B(x_i, \varepsilon/2)$

$B(x_i, \varepsilon/2)$, $\because$ Cauchy seq, $\exists N_i$ st $m, n \geq N_i$, $d(x_n, x_i) < \varepsilon/2$

$N' = \max\{N_1 \ldots N_n\}$ for $\forall m, n \geq N'$ we have

$d(x_m, x_i) < \varepsilon/2$ $(N' \geq N)$

$x_m \in B(x_i, \varepsilon/2)$ & $x_n \in B(x_i, \varepsilon/2)$ for all $m, n \geq N$

$d(x_m, x_n) \leq d(x_m, x_i) + d(x_i, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\Rightarrow$ $\forall$ Cauchy seq converge, to limit in $A$.

---

-xe totally bounded $\Rightarrow$ bounded

take any $x, y$ in $A$

$\because A \subseteq \bigcup \{B(x_i, \varepsilon)\}$, $\exists i$ s.t

$x \in B(x_i, 1)$

$||^{ly}$ $\exists j$ s.t $y \in B(x_j, 1)$

by triangle inequality

$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y)$

$\because x \in B(x_i, 1)$, $\Rightarrow d(x, x_i) < 1$ $||^{ly}$ $d(y, x_j) < 1$

$d(x, y) < 2 + d(x_i, x_j)$

for any two point $x, y$, $A$ is bounded

sup bound $M = 1 + \max_{ij} d(x_i, x_j)$

① Show that a totally bounded subset is bounded

② Give an example of a bounded set that's not totally bounded.

$X$ = Infinite, $d$ = disc

**Thm:** Generalized Heine–Borel theorem.

A subset $K$ of a complete metric space $(X,d)$ is compact If and only If $K$ is closed & totally bounded.

Idea of Proof: $\Rightarrow$ $K \subseteq \bigcup_{p \in K} B(p,\varepsilon)$ using

$\Leftarrow$ $\varepsilon_n = 1/n$ for any $\varepsilon$, .. only finite

Compact $\Leftrightarrow$ Seq. conv.

fix $x$, take $(a_n)_{n \in \mathbb{N}}$ in $K$.

$\exists$ 1 ball $B(P, \varepsilon_1)$ contains infinitely many terms.

$a_{n_1} \in K_1 = K \cap B(P, \varepsilon_1)$: $K_1$ contains infinite number of $K$.

take $1/\varepsilon_2$,

$a_{n_2} \in K_1 \cap B(P_2, \varepsilon_2)$ $n_2 > n_1$.

Subsequence $(a_{n_l})_{l \in \mathbb{N}}$

this is cauchy sequence.

$d(a_{n_l}, a_{n_{l+k}}) \leq 2/l$

as $l \to \infty$.

$\lim_{l \to \infty} d(a_{n_l}, a_{n_{l+k}}) < \varepsilon$

So Cauchy i.e. convergent.

$\Rightarrow$ it sequential compact

compact set acts as finite set for continuous function.

**Proposition**

Let $(X,d)$ be compact metric space & $f: X \to \mathbb{R}$ be continuous function

then $f$ is bounded and $f$ attains it's bounds

$[\exists M > 0$ s.t. $|f(x)| \leq M \; \forall x \in X]$

$f(x) = 1/x$ $x \in (0,1)$ ; $g(x) = \frac{1}{1+(1-x)^2}$ $x \in (0,1)$

Proof;

Defn: Let $f:(X,d) \to (Y,\rho)$ be function
we say that $f$ is uniformly continous
if $\forall\ \varepsilon>0$, $\exists\ \delta>0$ such that whenever
$x_1, x_2 \in X$ with $d(x_1,x_2)<\delta$ we have
$\rho(f(x_1), f(x_2)) < \varepsilon$

Observation
uniform continuity $\Rightarrow$ continuity

example: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x)=x^2$, $x\in\mathbb{R}$
proff $f$ is continous but not uniformly continous
$g:(0,1) \to \mathbb{R}$ given by $g(x)=1/x$ $x\in(0,1)$
$x_1 = 1/n$ $x_n = 1/(n+1)$ ; $n>>0$

Let $\varepsilon > 1/2$ if $\delta>0$ exists, then choose $n>>0$ st $1/n<\delta$
$|1/n - 1/(n+1)| < 1/n < \delta$
But $|f(x_1)-f(x_2)| = |n-(n+1)| = 1 > 1/2$.

thm: Let $f:(X,d) \to (Y,\rho)$ be continous function
and $(X,d)$ be compact then $f$ uniformly continous.
$\exists\ \varepsilon_0>0$ s.t $\forall \delta = 1/n$
$d(x_1,x_2)<\delta \Rightarrow \rho(f(x_1), f(x_2)) \geq \varepsilon_0$

---

19/9/23

compact $\Leftrightarrow$ sequentially compact

proposition
Let $f:(X,d) \to (Y,\rho)$ be a cont function, suppose
$X$ is compact then $f$ is uniformly cont.

Proof: by contradiction.
Suppose $f$ is not uniformly continous, Then $\exists\ \varepsilon_0>0$ such that
$\forall\ \delta>0$ there are $a,b\in X$ with $d(a,b)<\delta$ but $\rho(f(a),f(b)) \geq \varepsilon_0$

Taking $\delta = 1/n$, there are $a_n, b_n \in X$ with $d(a_n,b_n) < 1/n$
but $\rho(f(a_n), f(b_n)) \geq \varepsilon_0$
Now $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequence in $X$
$\because$ $X$ is compact, it is also sequentially compact
$\exists$ a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$
s.t $\lim_{k\to\infty}(a_{n_k}) = p \in X$ (convergent)

Also $f$ is cont at $p$,
$\therefore$ $f(a_{n_k}) = f(p)$

We have $d(a_k, P) \le d(a_{n_k}, b_{n_k}) - d(b_{n_k}, P)$
$\le 1/n + d(b_{n_k}, P)$

$\therefore$ Subsequence $(b_{n_k})_{k \in \mathbb{N}}$ also converge to $b \in X$

But $\rho(f(b_{n_k}), f(P)) \ge \rho(f(b_{n_k}), f(a_{n_k})) - \rho(f(a_{n_k}), f(P))$
$\ge \varepsilon_0 - \rho(f(a_{n_k}), f(P))$
(as $k \to \infty$ ; $\rho(f(a_{n_k}), f(P \to \varepsilon_0)$)

As $b_{n_k} \to P$; continuity of $f$ implies that
$\rho(f(b_{n_k}), f(P)) \to 0$, as $k \to \infty$

This contradiction $\Rightarrow \Leftarrow$

Hence $f$ must be uniformly continous.

---

Exercise:

1) Let $f\ (X,d) \to (Y,\rho)$ and $g\ (Y,\rho) \to (Z,\sigma)$ continous functions
Then show that:
1) $f$ and $g$ continous $\Rightarrow f \circ g\ (X \to d) \to (Z,\sigma)$ is continous
2) $f$ & $g$ continous both uniformly continous $\Rightarrow$
$g \circ f$ is also uniformly continous.

Proofs: 1) Can be done by contradictions
Inverse Image of $f$ cont is
i.e $(g \circ f)^{-1}$ or $f^{-1} g^{-1}(-1)$)

Defn: Connectedness

A metric space $(X,d)$ is said to be disconnected if there are non-empty disjoint open set $U$ and $V$ such that $X = U \amalg V$

In this case we say that pair $(U,V)$ of nonempty disjoint open set is a disconnected for $X$

If $X$ is not disconnected then $X$ is connected. If $Y \subseteq (X,d)$ then $Y$ is such that connected (or disconnected). If $Y$ is connected or disconnected) as a metric space.

Example: Let $X$ be a st of with atleast two points. Then $X$ with discrete metric is disconnected be.
$X = \{x_1, x_2\}$ let as
then $[x_1]$ is
& $x = P_{11}\} \amalg \{x_1\}$
$\Rightarrow X = U \amalg V$ both $U$ & $V$ is open

Exe:

Show that a subset $I$ of $\mathbb{R}$ is connected iff if $I$ is interval in $\mathbb{R}$

$p \in I$ $\quad$ ( $\Sigma$ that $p < v < q$ $\in I$

i.e $\left(I \cap (-\infty, r)\right) \cup \left(I \cap (r, \infty)\right)$

Thm: If $X$ is connected and $f: X \to Y$ is a contin map then $f(x)$ is connected subset of $Y$.

proof: suppose $f(x)$ is disconnected. then $f(x) = (f(x) \cap U) \amalg (f(x) \cap V)$ where $U$ and $V$ are set $(f(x) \cap U) \neq \phi$ and $f(x) \cap V \neq \phi$

Clearly

$X = f^{-1}(U \cap f(x)) \amalg f^{-1}(V \cap f(x))$

thus $X$ is non empty disjoint union of open set

$\therefore$ $X$ is disconnected.

$\because$ Since $X$ is connected, we must $f(x)$ be connected.

Proof by contrapositive statement)

# path connected

let $S \subseteq (X, d)$

A path $\gamma$ is a continuous function

$\gamma : [0,1] \to S$

If $\gamma(0) = p \in S$ & $\gamma(1) = q \in S$ true

that $\gamma$ is path joining $p$ & $q$

we say $S$ is path connected if $\forall$

2 points in $S$ is connected.

---

$\sin(1/x)$ is connected, if

Connected sets

$(X, d) \to$ disconnected if

$X = U \amalg V$, $U, V$ open ; $U \neq \phi$, $V \neq \phi$

$U \cap V = \phi$ ( $U$ & $V$ are disjoint

example: $I = [0,1]$ is connected

$f: (X, d) \to (Y, \rho)$

Ex: A subset $I$ of $\mathbb{R}$ is connected $\Leftrightarrow$ $I$ is an Interval

Path connected $S \subset (X, d)$

$\gamma : [0,1] \to S$ is continuous ; is called a path.

Considers a continuous function $f : [a,b] \longrightarrow \mathbb{R}$

connected & compact

Intermediate value theorem: suppose $x_1 < x_2$ in $[a,b]$ s.t $f(x_1) < f(x_2)$ (or $f(x_1) > f(x_2)$).

If $c \in \mathbb{R}$ s.t $f(x_1) < c < f(x_2)$ ( or $f(x_1) > c > f(x_2)$)

then $\exists\, y \in (x_1, x_2)$ s.t $f(y) = c$.

Image of connected sets are connected.

Ex: let $S$ be a subset of a metric space $(X,d)$

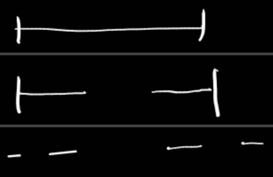
two nonempty subset in $(X,d)$ $H$ & $K$ said separated if $\bar{H} \cap K = \phi$ and $H \cap \bar{K} = \phi$, such that $S$ is disconnected $\Leftrightarrow$ $S$ is disjoint union of (non empty separated subsets of $(X,d)$)

Cantor ternary set

$E_0 =$

$E_1 =$

$E_2 =$



$$C = \bigcap_{n=0}^{\infty} E_n \subseteq E[0,1]$$

$\rightarrow$ a perfect set

measure zero.

---

21/9/23

Connected sets:

$(X,d) \Rightarrow$ is disconnected if

$X = U \amalg V$ ; $U, V$ are open

$U \neq \phi$ & $V = \phi$

$U \cap V = \phi$

If $\exists$ a nonempty exclusive 2 open set, where $\amalg$ is $X$.

If $X$ is not disconnected, then $X$ is connected.

Eg: $[0,1]$ is connected $\rightarrow$ how to prove.

$f : (X,d) \longrightarrow (Y,\rho)$

If $X$ is connected then $X$ is also connected.

Proof: Suppose $X$ is connected

assume "$Y$ is not connected"

$\because$ $Y$ is not connected $\exists$ non empty disjunsive $U, V$ such that $U \amalg V = Y$

$\therefore$ $f^{-1}(U)$ & $f^{-1}(V)$ are open cover which cover

$\Rightarrow$ $X$ is disconnected

But $X$ is connected contradiction

so assumation $Y$ is not connected is wrong.

Note: A subset $I$ of $\mathbb{R}$ is connected if and only if $I$ is an interval

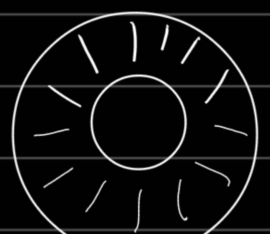
consider $I = [a,b] \in \mathbb{R}$ $a \le b$

$\mathbb{R} = (-\infty, a) \cup [a,b] \cup (b, \infty)$

## Path connected.

$S \subseteq (X, d)$

$\gamma : [0,1]$ to $S$ continous such fn $\gamma$ is called path



Annulus is path connected.

2 circles s.t inter line & exten of outer circle

went in set

But $(\text{Anulus})^c$ is not path connected.

? Any path connected subset is path connected because path connected is rope powerful statement ?

considers a continous function.

$f : [a,b] \to \mathbb{R}$ $[a,b] \Rightarrow$ is connected & compact

#) Intermediate value thm

Suppose $x_1 < x_2$ in $[a,b]$ such that $f(x_1) < f(x_2)$ [or $f(x_2) < f(x_1)$] if $C \in \mathbb{R}$ such that $f(x_1) << f(x_2)$ or $f(x_2) << f(x_1)$] then

$\exists y \in (x_1, x_2)$ s.t $f(y) = C$

This consequence of connectedness as imaged $[x_1, x_2]$ connected should $f(x_1)$ & $f(x_2)$ and this containing set should be connected

$\therefore [f(x_1), f(x_2)]$ so $\exists C \in \mathbb{R}$ $f(x_1), f(x_2)$

which have an in case image in $[x_1, x_2]$

$\therefore$ existence of $Y$ is justified.

## Exercise

Let $S$ be a subset of metric space $(X, d)$, Two subsets $H$ & $K$ are said to be separated if $\bar{H} \cap K = \phi$ & $H \cap \bar{K} = \phi$

$\bar{H} \Rightarrow$ their closure

$S$; $\bar{S} = S \cup D(S)$

Q) Show that $S$ is disconnected $\Leftrightarrow$ $S$ is disjointed union of (nonempty) separated subsets of $(X, d)$

## Cantor ternary set

$E_0 :=$ $0$ ⊢——⊣ $1$ $\quad [0,1]$

$E_1 :=$ $0 \;\; 1/3 \;\; 2/3 \;\; 1$ $\quad [0, 1/3] \amalg [2/3, 1]$

$E_2 :=$ $1/9 \; 2/9 \; 1/3 \;\; 2/3 \; 7/9 \; 8/9$ $\quad [0, 1/9] \amalg [2/9, 1/3] \amalg [2/3, 7/9] \amalg [8/9, 1]$

$E_n$ = disjoint union of $2^n$ closed Interval of length $1/3^n$ each

then cantor ternary set $C = \bigcap_{n=0}^{\infty} E_n \subseteq [0,1]$

## Demorgan's thm.

length of Interval of

$1/3 + 2/3^2 + 2^2/3 + \quad 2^{n-1}/3$

$1/3 \, [1 + 2/3 + 2^2/3 \; - - . (2/3)^n]$

$= 1/3 \times 1/(1-2/3)$

$= 1/3 \times 1/(1/3) = 1$

22/09/23

## Real valued function on Interval

let $f : (a,b) \to \mathbb{R}$ be a function

$a < b$ $(a,b) = \{x \in \mathbb{R} ; a < x < b\}$ then

limit of the function at a point := let $a < x < b$

we want to define $\lim_{t \to x} f(t)$, the limit of $f(t)$ as $t \to x$

We say that $f(t)$ tends to $l \in \mathbb{R}$ as $t$ tend to $x$.

$(f(t) \to l$ as $t \to x)$ If $\forall \varepsilon > 0, \exists \delta > 0$ (St) for $t \in (a,b)$ with $0 < |t-x| < \delta$ we have

$|f(t) - l| < \varepsilon$

# Right hand limit

we say $f(t)$ tends to $l \in \mathbb{R}$ as $t$ tends to $x^+$ and write $\lim f(t) = l$.

If $\forall \varepsilon > 0, \exists \delta > 0$ (St) $t \in (x,b)$ $t \to x^+$ with $0 < t - x < \delta$ we have

$|f(t) - l| < \varepsilon$ $\qquad t > x$ no needed of $|t-x|$

## left hand limit

we say $f(t)$ tends to $l_2 \in \mathbb{R}$ as tends to $x^-$ and write

$\lim f(t) = l_2$ If $\forall \varepsilon > 0, \exists \delta > 0$ (St) $t \in (a,x)$ $t \to x^-$ $0 < |x-t| < \delta$

we have $|f(t) - l_2| < \varepsilon$

Note: If $\lim f(t)$ exists, we say that right hand limit of $f(t)$ as $t \to x^+$ exist and denote it by $\lim_{t \to x^+} f(t) = f(x^+)$

Illy if left hand limit exists, then it is as $\lim_{t \to x^-} f(t) = f(x^-)$

## Observations

- If $\lim f(t)$ exist, then $f(x^+)$ and $f(x^-)$ also exist and furthes $\lim_{t\to x} f(t) = f(x^+) = f(x^-)$
- If $f$ is continous at $x \in (a,b)$ then

$$\lim_{t \to x} f(t) = f(x) = f(x^+) = f(x^-)$$

## Exercise

Suppose $f:(a,b) \to R$ ad $x \in (a,b)$ let $\lim_{t\to x} f(t) = l$, $\lim_{t \to x^+} f(t) = f(x^+) = l_1$, $\lim_{t \to x^-} f(t) = f(x^-) = l_2$. Then show that

① For every seq $(t_n)_{n \in \mathbb{N}}$ in $(a,b)$ with $\lim_{n\to\infty} t_n = x$ with $t_n \neq x$ we show $\lim_{n\to\infty} f(t_n) = l$.

② For every seq $(t_n)_{n\in\mathbb{N}}$ in $(x,b)$ with $\lim t_n = x$. we have $\lim_{n\to\infty} f(t_n) = l_1$ we have $\lim_{n\to\infty} f(t_n) = l_1$

③ for every seq $(t_n)_{n\in\mathbb{N}}$ in $(a,x)$ with $\lim t_n = x$. we have $\lim_{n\to\infty} f(t_n) = l_2$ we have $\lim_{n\to\infty} f(t_n) = l_2$

Further also show that reverse also hold.

For $x \in \overline{\mathbb{R}}$ and $A \in \overline{\mathbb{R}}$. then $\lim_{t\to x} f(t) = A$

mean $\lim_{t \to x} f(t) = l$ (ST)

$t \in (x-\delta, x+\delta) \setminus \{x\}$

$\Rightarrow f(t) \in (l-\varepsilon, l+\varepsilon)$

## Remark

- If $f(x^+)$ and $f(x^-)$ exist and $f(x^+) = f(x^-)$ then

$$\lim_{t \to x} f(t) = f(x^+) = f(x^-)$$

$$\left.\begin{array}{l}\text{Simple discontinuity on} \\ \text{Discontinuity } 1^{st} \text{ kind}\end{array}\right\} \quad \begin{array}{l}\text{either } f(x^+) \neq f(x^-) \\ \text{or } f(t) \neq f(x)\end{array}$$

*** [ Try to follow:= G.F. Simmons Intro to topology & modern Analysis

**Def:** The point $x \in (a,b)$ is a point of discontinuity of $f$ if

① either $f(x^+) = f(x^-)$ or $f(x^+) = f(x^-) \neq f(x)$

② Atleast one of $f(x^+)$ & $f(x^-)$ doesn't exist

Discontinuity of second kind at $x$.

Def: Let $f:(a,b)\to\mathbb{R}$

- we say that $f$ is monotonically strictly increasing, if for $a<x_1<x_2<b$ we have $f(x_1)\le f(x_2)$ $(<)$
- We say that $f$ is monotonically strictly decreasing if for $a<x_1<x_2<b$ where $f(x_1)\ge f(x_2)$ $(>)$
- By a monotonic function, we mean a function which either monotonically increasing, monotonically decreasing

Observation

If $f$ is monotonically increasing, then $(-f)$ is monotonically decreasing

Proposition

Let $f:(a,b)\to\mathbb{R}$ be a monotonically increasing function and $x\in(a,b)$ then

$$\sup\{f(t): t\in(a,x)\} = f(x^-) \le f(x) \le f(x^+) = \inf\{f(t): t\in(x,b)\}$$

Illy, If $f$ is monotonically decreasing function then

$$\inf\{f(t); t\in(a,x)\} = f(x^-) \ge f(x) \ge f(x^+) = \sup\{f(t); t\in(x,b)\}$$

$\therefore$ A monotonic function has discontinuity of 1st kind (ie jumped discontinuity $f(x^+) = f(x^-) \neq f(x)$

For $f(x) = \lfloor x\rfloor$ = floor of $x$.

$n\in\mathbb{Z}$ $\quad f(n^-) = \lim_{x\to n^-} f(x) = (n-1)$

$f(n^+) = \lim_{x\to n^+} f(x) = n$

$f(n) = n$