Q1) (X,d) a metric space, $K \subset X$

① K is compact $\Rightarrow$ K is sequentially compact

② K is sequentially compact $\Rightarrow$ K is compact.

③ K is compact & F is closed subset of X $\Rightarrow$ $K \cap F$ is compact

$\lim_{t \to x} f(t) = l \iff \forall (t_n)_{n \in \mathbb{N}}$ in $(a,b) \setminus \{x\}$

if $t_n \to x$, then $f(t_n) \to l$ as $n \to \infty$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ st

if $t \in (a,b)$ with $0 < |x-t| < \delta \Rightarrow |f(t) - l| < \varepsilon$

negations of this

$\exists \varepsilon_0 > 0$, $\forall \delta > 0$

$t \in (a,b)$ $0 < |x-t| < \delta \Rightarrow |f(t) - l| \geq \varepsilon_0$

take $\delta = 1/n$.

Dirichlet function

$K \subseteq \bigcup_{\alpha \in I} G_\alpha$.

If $p \in K$, then $p \in G_\alpha$ for some $\alpha \in I$. Since G_α is open, $\exists \delta_p > 0$ st $B(p, \delta_p) \subseteq G_\alpha$.

let $p \in X$ for some $\exists x$

$f(B(p, \delta p)) \supseteq B(f(p), \varepsilon)$?

Cancelle $\{B(p, \varepsilon_2) : p \in X\}$

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25/9/23 Recap

let $f:(a,b)\to\mathbb{R}$; $x\in(a,b)$: $l\in\mathbb{R}$

Then $\lim_{t\to x} f(t)=l$

$\Leftrightarrow$ $\forall (t_n)_{n\in\mathbb{N}}$ in $(a,b)-\{x\}$, if $t_n\to x$ then $f(t_n)\to l$

as $n\to\infty$.

Def of $\lim_{t\to x} f(t)=l$:

$\forall \varepsilon>0$, $\exists \delta>0$ (st) $t\in(a,b)$ with $0<|x-t|<\delta \Rightarrow |f(t)-l|<\varepsilon$

negation of this

$\exists \varepsilon>0$, $\forall\delta>0$ (st) $t\in(a,b)$ with $0<|x-t|<\delta\Rightarrow |f(t)-l|\geq\varepsilon$

how construct a seq funct this

take $\delta=1/n$ $n\in\mathbb{N}$

$\exists t_n\in(a,b)-\{x\}$

with $|x-t_n|<1/n$ but $|f(t_n)-l|\geq\varepsilon_0$

Dirichlet fn

$f(x)=\begin{cases}1 ; & x\in\mathbb{Q}\\ 0 ; & x\in\mathbb{R}-\mathbb{Q}\end{cases}$

Discontinue @ $\forall$ point. Discontinuity of 2^{nd} kind i.e left & right limit doesn't exist ?

But $g(x)\begin{cases}x ; & x\in\mathbb{Q}\\ 0 & x\in\mathbb{R}-\mathbb{Q}\end{cases}$

at only @ point $x=0$

$x=0$; $\forall\varepsilon>0$ $\exists\delta>0$ s.t

$t\in(-\delta,\delta)$ with $|f(t)|<\varepsilon$ take $\delta=\varepsilon$ or $\delta<\varepsilon$

26/9/23 monotonic function :

$f : (a,b) \longrightarrow \mathbb{R}$

f is monotonically increasing if $a < x < y < b \Rightarrow f(x) \le f(y)$

Thm :

f is monotonically increasing if $x \in (a,b)$, Then

$\sup \{ f(t) : t \in (a,x) \} = f(x-) \le f(x) \le f(x^+)$

$= \inf \{ f(t) : t \in (x,b) \}$

Proof : consider the subset

$\{ f(t) : t \in (a,x) \}$ since f increasing, we see that

$f(t) \le f(x) \quad \forall t \in (a,x)$

Then $f(x)$ is an upper bound for the subset;

$\{ f(t) : t \in (a,x) \}$

$\therefore \sup \{ f(t) : t \in (a,x) \} = A \in \mathbb{R}$ exists

Also $A \le f(x)$

We need to show that $A = f(x^-)$

Let $\varepsilon > 0$, be given, Then $\exists$ $t_0 \in (a,x)$ s.t

$A - \varepsilon < f(t_0) \le A$

Take $\delta = x - t_0 > 0$ If $(x-\delta) - t_0 < t < x$

Then as $f(t_0) \le f(t) \le f(x)$

$\Rightarrow A - \varepsilon < f(t) < A \quad \forall t \in (x-\delta, x) = (t_0, x)$

$(A - f(t)) < \varepsilon \quad \forall t \in (x-\delta, x)$

Hence $A = f(x^-)$

Similarly we shall show that

$f(x) \le B$ For $\varepsilon > 0$ $\exists t_1 \in (x,b)$ s.t

$B \le f(t_1) < B + \varepsilon$

Let $\delta = t_1 - x > 0$

claim ; as $f \uparrow$, we see that $f(t) \le f(t_1)$

for $t \in (x, t_1) = (x, x+\delta)$

$\Rightarrow B \le f(t) < B + \varepsilon \quad \forall t \in (x, x+\delta)$

$\Rightarrow B = f(x+)$

Observation

- monotonic function has only simple discontinuity (Jump discontinuties).
- If $a < x < y < b$ then for $f \uparrow$

  $f(x^+) \le f(y^-)$

doubt?

In fact ;

$f(x^+) = \inf \{ f(t) ; t \in (x,b) \} \le \inf \{ f(t) ; t \in (x,b) \}$

$f(y^-) = \sup \{ f(t) ; t \in (a,y) \} \ge \sup \{ f(t) ; t \in (a,y) \}$

But since $\inf A \le \sup A$

$\inf \{ f(t) ; t \in (x,b) \} \le \sup \{ f(t) ; t \in (x,b) \}$

Theorem

$f:(a,b) \to \mathbb{R}$, A monotonic function has almost acntably many jump discontinuities

proof: Let $E = \{x \in (a,b),\ f$ is discontinous at $x\}$

Since fn monotonic, we have seen that

$\sup\{f(t);\ t \in (a,x)\} \le f(x^-) \le f(x) \le f(x^+) \le \inf f(t);\ t \in (x,b)$

$\Rightarrow$ provided $f\uparrow$

[If $f\downarrow$; then $\sup\{f(t)\ t \in (x,b)\} = f(x^+) \le f(x) \le f(x^-) \le \inf\{f(t)\ t \in (a,x)\}$

Claim If $f(x^-) = f(x^+)$ then cts at x

Then for $x \in E \iff f(x^-) \neq f(x^+)$

Suppose $f\uparrow$, Then $f(x) < f(x^+)$

choose $r_x \in \mathbb{Q}$ s.t $f(x^-) < r_x < f(x^+)$

Now define $\phi : E \to \mathbb{Q}$ by

$\phi(x) = r_x$; $x \in E$

we shall show that $\phi : E \to \mathbb{Q}$ is injective

let $x, y \in E$ with $x < y$

since $f(x) < r_x < f(x) < f(y^-) < r_y < f(y^+)$

$r_x < r_y$ Hence $\phi : E \to \mathbb{Q}$ is injective

and thus E is countable set

Observation

Let f and g be a real valued function If $x \neq 0$ and f and g agree in neighbourhood of x, then both are continous at x simultanously

Example

$f(x) = \lfloor x \rfloor$ = floor of x

= greatest Integer $\le x$

Then $x \notin \mathbb{Z}$

then f is constant function

f is continous

If $x \in \mathbb{Z}$ then $f(x^+) = n$

$f(n^-) = n-1$

$\Rightarrow$ f is discontinous $x \in \mathbb{Z}$

$$f(x) = \begin{cases} 1 & ; x \ge 0 \\ 0 & ; x < 0 \end{cases}$$

discontinuty point $\in 0$

i.e a jump from ϕ at for $f(x)$

Let $a < b$ and $E \subseteq (a,b)$ be a dense subset

[For example $E = \mathbb{Q} \cap (a,b)$]

Consider a convergent series $\sum_{n=1}^{\infty} C_n$ of positive term

i.e $C_n > 0\ \forall n \in \mathbb{N}$

Suppose $E = \{P_1\ P_2 \ldots$

(we could had taken $C_n = 1/n$)

Define $g : (a,b) \to \mathbb{R}$ by

$$g(x) = \sum_{\substack{n \\ p_n < x}} c_n$$

Claim: (i) g is monotonically strictly increasing

(ii) g is continous @ $x \in (a,b) \setminus E$

(i) this is where as for any $x < y$ $\exists\, r$ s.t $x < r < y$

$\Rightarrow f(x), f(r), g(y)$ all exist

since $f(x) = \sum_{p_n < x} c_n \quad c_n > 0$

$g(x) < g(r) < g(y)$ (clearly)

(ii) for $x = p_m \in E$, we have

$$\underbrace{g(p_m^+)}_{=B} - \underbrace{g(p_m^-)}_{=A} = c_m > 0$$

since $\sum_{n=1}^{m} c_n < \infty$

For $\sum_{n=n_0} c_n < \varepsilon/2$

And $n < n_0 \quad \sum_{n=0}^{n-1} c_n$

retrictay $c_n > 0$

Dirichlet function

$$f(x) = \begin{cases} 1 & ; \text{ if } x \in \mathbb{Q} \\ 0 & ; \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

f is discontinous of sound kind at enemy $x \in \mathbb{R}$

Thomae function

$$h(x) = \begin{cases} 1/q & : x = p/q \quad \gcd(p,q) = 1, q \geq 1 \\ 0 & : x \in \mathbb{R} / \mathbb{Q} \end{cases}$$