Differentiation

$f: [a,b] \to \mathbb{R}$

$x \in (a,b)$ f is differentiable at if $\exists\, l \in \mathbb{R}$ s.t

$$\lim_{t \to x} \quad f(t) - f(x) = l = f'(x)$$

$$f'(t) = Rf'(a) = \lim_{t \to x^+} \frac{f(t) - f(x)}{t - x}$$

f is diff on $[a b]$ if $f'(x)$ exist $\forall x \in [a,b]$

f' is called derivative of f

$f'(x)$ exists $\Rightarrow$ $f(x)$ is continous at x

- properties of derivatives

$$(f+g)'(x) = f'(x) + g'(x)$$

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

Assume $g(x) \neq 0$ & $g'(x) \neq 0$

$$\left(f/g\right)(x) = \frac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$$

Chain rule:

$$f: [a,b] \xrightarrow{\text{continous fn}} [m, M] \xrightarrow{g} \mathbb{R}$$

$\forall x \in [a,b]$ s.t $f'(x)$ exists

IIIly for $y = f(x)$, $g'(y)$ exist

Given $h = g \circ f$

$$\Rightarrow h'(x) = g'(f(x)) \cdot f'(x)$$

$$g(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$g'(x) = 2x \sin 1/x + x^2 \cos(1/x) \log(x)$$

$$\frac{g(t) - g(a)}{t-a} = t \sin 1/t$$

local maxima or local minima.

(X,d) be a metric space, and $f: X \to \mathbb{R}$. A point $p \in X$ is s.t b a point of local maxima or [local minima] of f if $\exists\, \delta > 0$ s.t $f(t) \leq f(p)$ $\forall t \in B(p,\delta)$ $f(t) \geq f(p)$

proposition

Let $f: [a,b] \to \mathbb{R}$ be a function. Suppose f has a local maximum or local minimum at $x \in (a,b)$. If f is differentiable at x, then $f'(x) = 0$

Proof: Since f is differentiable at x, i.e $f'(x)$ exists

$$f'(x) = Rf'(x) = Lf'(x)$$

Since x is point of local maxima of f;

$\exists\ \delta > 0$ s.t $a < x - \delta < x < x + \delta < b$ and

$$f(x) \geq f(t) \quad \forall t \in (x, \delta, x+\delta$$

Thus $\quad L f'(x) = \lim\limits_{t \to x^-} \dfrac{f(t) - f(x)}{t - x} \geq 0 \quad$ &

$R f'(x) = \lim\limits_{t \to x^+} \dfrac{f(t) - f(x)}{t - x} \geq 0$

$\Rightarrow f'(x) = 0$

Generalised Mean - value Thm [Cauchy's MVT]

Let $f, g : [a,b] \to \mathbb{R}$ be continous functions. Suppose f and g are differentiable in (a,b)

Then $\exists x \in (a,b)$ such that

$$(f(b) - f(a))\, g'(x) = (g(b) - g(a))\, f'(x)$$

Proof: consider

$$h(t) = (f(b) - f(a))\, g(t) - (g(b) - g(a))\, f(t)$$

then h is continous on $[a,b]$ and

h is differentiable on $(a,b]$

also $h(a) = h(b)$

If h is constant, then $h'(x) = 0 \quad \forall x \in (a,b)$

otherwise $\exists\, t \in (a,b)$ s.t

$h(t) > h(a)$ [or $h(t) < h(a)$]

Since this continous function on compact set (a,b)

It has a local maxima at point $x \in (a,b)$

[respectively local minima]

Clearly $h'(x) = 0$

$\Rightarrow \quad 0 = h'(x) = (f(b) - f(a))\, g'(x) + (g(b) - g'(a))\, f'(x)$

Corollary Lagrange's MVT

$f : [a,b] \to \mathbb{R}$ is continous and f is differentiable on (a,b)

Then $\exists x \in (a,b)$ s.t

$f(b) - f(a) = (b-a) f'(x)$

Proof: $C^\circ_0\ g(x) = x$

Corollary Rolle's theorem

$\exists x \in (a,b)$ s.t $f'(x) = 0$

using MVT, we can prove the following;

Let f be a real valued on (a,b). Suppose f is differentiable in (a,b)

Then the following statement hold:

* If $f'(x) \geq 0 \quad \forall x \in (a,b)$, then f is monotonically $\uparrow$ strictly
$f(x) > 0$

i) Converse does not hold

ii) If $f'(x) = 0 \quad \forall x \in (a,b)$, Then f is constant

iii) If $f'(x) \leq 0 \quad \forall x \in (a,b)$, then f is montonically $\downarrow$ strictly
$f'(x) < 0$

Note:

mean value theorem do not hold for complex valued function

eg $f(t) = e^{it}$ $t \in [0, 2\pi]$

$f'(t) = ie^{it}$

$f(2\pi) - f(0) = 1 - 1 = 0$

but $(2\pi - 0)\, f'(t) \neq 0$

$\Rightarrow f(b) - f(a) \neq (b-a)\, f'(t)$

MVT doesn't holds

Darboux Continuity of f'

Let f be differentiable function on (a,b). Suppose $a < x_1 < x_2 < b$ such that $f'(x_1) < \lambda < f'(x_2)$ / $f'(x_1) > \lambda > f'(x_2)$ then $\exists\, c \in (x_1, x_2)$ s.t $f'(c) = \lambda$

Proof: Suppose $g(t) = f(t) - \lambda t$ $t \in (a,b)$

Then $g'(t) = f'(t) - \lambda$

$g'(x_1) = f'(x_1) - \lambda$

If $f'(x_1) < \lambda < f'(x_2) \Rightarrow g'(x_1) < 0 < g'(x_2)$

$\Rightarrow g'(x_1) < 0$

$g'(x_2) > 0$ $(\because g'(x_2) = f'(x_2) - \lambda$

Thus $\exists\, t_1, t_2 \in (x_1, x_2)$ s.t

$g(t_1) < g(x_1)$, $g(t_2) < g(x_2)$

$g(t_1) < g(x_1)$, $g(t_2) < g(x_2)$

$\Rightarrow g$ has a local minimum at a point (x_1, x_2)

If $c \in (x_1, x_2)$ be the point of local minim of g

Then $g'(c) = 0$

$\Rightarrow f'(c) - \lambda = 0$

$f'(c) = \lambda$

corollary

The derivative of a differentiable function in an open interval has discontinuity

L' Hospital's Rule

Rule I: let $-\infty \leq a < b \leq \infty^+$, suppose f, g are differentiable functions on (a,b); suppose $g'(t) \neq 0$ $\forall (a,b)$

Suppose $\lim\limits_{t \to t} f(t) = \lim\limits_{t \to a^+} g(t) = 0$

Then; If $\lim\limits_{t \to a^+} \dfrac{f'(t)}{g'(t)} = l \in \mathbb{R}$, then $\lim\limits_{t \to a^+} \dfrac{f(t)}{g(t)} = l$.

Rule 2 : Suppose $\lim_{t\to a^+} g(t) = \pm\infty$

Then

If $\lim_{t\to a^+} \frac{f'(t)}{g'(t)} = l \in \mathbb{R}$ then

$$\lim_{t\to a^+} \frac{f(t)}{g(t)} = l$$

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* If f is differentiable at a point x, then f is continous in a neighbhour hood of x.
* If g is differentiable in on interval, then what can be say about continuity off ?

$$g(x) = \begin{cases} x^2 \sin(1/x) ; & x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$$g'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

or $g(x) = \begin{cases} x^2 ; & x \in \mathbb{Q} \\ 0 & ; x \in \mathbb{R} - \mathbb{Q} \end{cases}$

L'Hos pital's Rule

$-\infty \leq a < b \leq \infty$ $f, g : (a,b) \to \mathbb{R}$ be differentiable fn

$(g(t) \neq 0 \ \forall t \in (a,b), g'(t) \neq 0)$

such that $\lim_{t\to a^+} f(t) = \lim_{t\to a^+} g(t) = 0$

Rule 1: $\lim_{t\to a^+} \frac{f'(t)}{g'(t)} = l \in \mathbb{R} \Rightarrow \lim_{t\to a^+} \frac{f(t)}{g(t)} = l$

Rule 2 $\lim_{t\to a^+} g(t) = \pm\infty$ then, also similar holds.

Proof:

let $a < x < y < b$. Then since $g(t) \neq 0$

and $g'(t) \neq 0 \ \forall t \in (a,b)$

By MVT ; $g(y) - g(x) \neq 0$

also by Cauchy's MVT , we have

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(u)}{g'(u)} \quad \text{fr some } u \in (x,y)$$

We have $\lim_{t\to a^+} \frac{f'(t)}{g'(t)} = l$.

i.e $\forall \varepsilon > 0$, $\exists$ $c \in (a,b)$ s.t

$l - \varepsilon < \frac{f'(t)}{g'(t)} < l + \varepsilon$ $\forall t \in (a,c)$

i.e $\left| \frac{f'(t)}{g'(t)} - l \right| < \varepsilon$

If $a < x < y \le c$ then from ① & ②

$$1-\varepsilon < \frac{f(y)-f(x)}{g(y)-g(x)} < 1+\varepsilon \quad -③$$

Letting $x \to a^+$ ③ by given hypothesis, we have

$$1-\varepsilon \le \frac{f(y)}{g(y)} \le 1+\varepsilon \quad \forall y \in (a,c)$$

$$\Rightarrow \lim_{t \to a^+} \frac{f(y)}{g(y)} = 1$$

* $\lim_{x \to 0^+} \frac{1-\cos x}{x^2} = \lim_{x \to 0^+} \frac{\sin x}{2x} = \lim_{x \to 0^+} \frac{\cos x}{2} = \frac{1}{2} \lim_{x \to 0^+} \cos$

* $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$

* $\lim_{x \to \pi/2} \frac{\tan x}{\sec x} = \lim_{x \to \pi/2} \frac{\sec x}{\tan x}$

* $f(x) = \begin{cases} e^{-1/x} & ; x > 0 \\ 0 & ; x \le 0 \end{cases}$ $\quad f'(x) = \begin{cases} 1/x \, e^{-1/x} & ; x \ge 0 \\ 0 & ; x \le 0 \end{cases}$

Taylor's Thm

Suppose $f : [a,b] \to \mathbb{R}$ in n-times differentiable function. For $a \le \alpha < \beta < b$

$\exists$ $x \in (\alpha, \beta)$ s.t

$$f(\beta) = f(\alpha) + \frac{f'(\alpha)}{1!}(\beta-\alpha) + \frac{f''(\alpha)}{2!}(\beta-\alpha)^2 + \cdots \frac{f^n(\alpha)}{n!}(\beta-\alpha)^n$$

If we put $P(t) = f(\alpha) + \frac{f'(\alpha)}{1!} + \frac{f^{n-1}(\alpha)}{(n-1)}(\beta-\alpha)^{n-1}$

$$f(\beta) = P(\beta) + \frac{f^{(n)}(\alpha)}{n!}(\beta-\alpha)^n$$

Proof:

Consider the function $g(t) = f(t) - P(t) - M(t-\alpha)^n \quad t \in [a,b]$

where $M \in \mathbb{R}$ s.t $g(\beta) = 0$

Then theorem is established if we find $x \in (\alpha, \beta)$

s.t $M = \frac{f^n(x)}{n!}$

We observe that $P(t) = \sum_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(\beta-\alpha)^k$

$$\Rightarrow P(\alpha) = f(\alpha), \quad P'(\alpha) = f'(\alpha) \quad \ldots \quad P^k(\alpha) = f^k(\alpha)$$

Now $g'(t) = f'(t) - P'(t) - nM(t-\alpha)^{n-1}$

In general $g^k(t) = f^k(t) - P^k(t) - n(n-1) \ldots (n+k-1)(t-\alpha)^{n-k}$

$\Rightarrow g^k(\alpha) = 0$ if $0 \le k \le n-1$

Also $g^n(t) = f^n(t) - n!M$

Now we use Lagrange's MVT:

we have

$g(\alpha) = 0$ & $g(B) = 0$ by choice

$\Rightarrow \exists\, x_1 \in (\alpha, B)$ s.t $g'(x_1) = 0$

Now $g'(\alpha) = 0$ & $g'(x_1) = 0$

LMVT $\exists\, x_2 \in (\alpha, x_1)$ s.t $g''(x_2) = 0$

continuing this manner, we get

$x_{n-1} \in (\alpha, x_{n-2})$ s.t $g^{n-1}(x_{n-1}) = 0$

finally $g^{n-1}(\alpha) = 0 = g^{n-1}(x_{n-1})$

Thus by LMVT

$\exists\, x_n \in (\alpha, x_{n-1})$ s.t $g^n(x_n) = 0$

Put $x = x_n \in (\alpha, B)$

Now $g^n(t) = f^n(t) - M n!$

and $g^n(x) = 0 \Rightarrow m = \dfrac{f^n(x)}{n!}$

thus $g(B) = 0 = f(B) - P(B) - \dfrac{f^n(x)}{n!}(B-\alpha)^n$

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Riemann Integration.

Let $f:[a,b] \to \mathbb{R}$ be a function ; $a<b$ where $a,b \in \mathbb{R}$.

Q1) what do we mean by f being Integrable

Q2) If f is Integrable, then what is $\int_a^b f(x)dx$?

Partition : A finite set of $P = \{x_0, x_1 \cdots x_n\} \in [a,b]$ is called partion of $[a,b]$ If $a = x_0 < x_1 < x_2 < x_3 \cdots < x_n = b$

$P = \{x_0, x_1 \dots x_n\}$ divide the interval $[a,b]$ into smaller non-overlapping subinterval.

$[x_0, x_1]$ $[x_1, x_2]$ $[x_2, x_3]$ ... $[x_{n-1}, x_n]$

Mesh of $P = \mu(P) = \max\{\Delta x_i : 1 \le i \le n\}$

where $\Delta x_i = x_i - x_{i-1}$

A subset $T = \{t_0, t_1 \dots t_n\} \subseteq [a,b]$

is said to be tag for partition $P = \{x_0, x_1 \dots x_n\}$

if $t_i \in [x_{i-1}, x_i] = I_i$ $(1 \le i \le n)$

Sometimes P together with a tag T is called a tagged partition (P,T)

For $f:[a,b] \to \mathbb{R}$, $P \in Par[a,b]$ and T is a tag for P

then number

$$R(f,P,T) = \sum_{i=1}^{n} f(i)\,\Delta x_i$$

is called Riemann sum of over a tagged partition (P,T)

If P, P' are partition with $P \subseteq P'$, then

we say P' is a refinement of P

$\mu(P') \le \mu(P)$

Def : A function $f:[a,b] \to \mathbb{R}$ is said to be Riemann integreable over $[a,b]$ If there exist $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$

we have the following

For $\forall$ partition P of $[a,b]$ with $\mu(P) < \delta$,

$|R(f,P,T) - L| < \varepsilon$ for any choice of tag T for P

In this case, we say that

$R(f,P,T) \to L$ as $\mu(P) \to 0$

Equivalent, we write

$$L = \lim_{\mu(P) \to 0} R(f,P,T)$$

further, L is called the integral of f over $[a,b]$ and we write
$$L = \int_a^b f(x)\,dx = \left(\int_a^b f\,dx = \int_a^b f \right)$$

Let $R[a,b]$ denotes the set of all Riemann Integrable function over $[a,b]$

Observation

① If $f \in \mathcal{R}[a,b]$, then L is unique

② If $f \in \mathcal{R}[a,b]$, then f is a bounded function.

$\exists P_0 \in Par[a,b]$ s.t
$$|R(f, P_0, T) - L| < 1 \qquad \varepsilon = 1$$
suppose f is unbounded. Then s.t f unbounded over $[x_{t_0-1}, x_{t_0}]$
$$|f(t_{t_0})(x_{t_0} - x_{t_0-1}) - f(t'_{t_0})(x_{t_0} - x_{t_0-1})| > 2$$
$$|R(f, P, T) - R(f, P_0, T')| > 2$$
$$|R(f, P_0, T) - R(f, P_0, T')| \le |R(f, P, T) - L| + |R(f, P_0, T') - L| < 2$$

③ If $f(x) = P \ \forall x \in [a,b]$, then $f \in \mathcal{R}[a,b]$ & $\int_a^b f\,dx = P(b-a)$

Linearity property of Riemann Intepretation.

Let $f, g \in \mathcal{R}[a,b]$ and $c_1, c_2 \in \mathbb{R}$. Then $c_1 f + c_2 g \in \mathcal{R}[a,b]$ and
$$\int_a^b (c_1 f + c_2 g)\,dx = c_1 \int f\,dx + c_2 \int g\,dx$$

Proof: For any tagged partition (P,T) of $[a,b]$ we have
$$R[c_1 f + c_2 g] = c_1 R(f, P, T) + c_2 R[g, P, T]$$

$$\sum^n (c_1 f + c_2 g)(t_i)\,\Delta x_i$$
$$= \sum (c_1 f(t_i) + c_2 g(t_i))\,\Delta x_i$$

② Suppose $f, g \in \mathcal{R}[a,b]$ $f(x) = g(x) \ \forall x \in [a,b]$
then
$$\int_a^b f\,dx \le \int_a^b g\,dx$$

Darboux Integrability:

let $f : [a,b] \to \mathbb{R}$ be a bounded function.
$$P = \{x_0, x_1, x_n \dots x_n\}$$
bounded function $\exists M > 0$ s.t $|f(x)| \le M \ \forall x \in [a,b]$

Let $m_i = \inf\{f(t) : t \in [x_{i-1}, x_i)\}$ and $M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}$

$m_i(f)$

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{and} \quad U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

lower sum of f w.r.t P | upper sum of f w.r.t P.

$$\underline{\int_a^b} f = \sup_{P \in Par(a,b)} L(f, P) \leq \overline{\int_a^b} f = \inf_{P \in Par(a,b)} U(f, P)$$