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$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}$ (n-copies)

$a = (a_1, a_2, \ldots, a_n)$ $\quad b = (b_1, b_2, \ldots b_n)$

$\|a\|_2 = \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}$ Euclidean room of a

$\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$: $\langle a, a \rangle = \|a_i\|^2$

Cauchy - Schwarz Inequality

proof: $0 \le \langle a+tb, a+tb \rangle = \langle a,a \rangle + 2t\langle a,b \rangle + t^2 \langle b\, b \rangle$

Largrange's Identity

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \sum_{1 \le i < j \le n} (a_i b_j - a_j b_i)^2$$

Holden's inequality

$$|\langle a, b \rangle| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q} = \|a\|_p \|b\|_q$$

$\|a\|_p = l^p$ - norm of a $\left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p}$

Ex: Show that $\| \; \|$ is norm on $\mathbb{R}^n$ $(1 \le p < \infty)$

$\lim_{p \to \infty} \|a\|_p = \|a\|_\infty$

$\|a+b\|_p \le \|a\|_p + \|b\|_p$ minkesskis' inequality.

$\|a\|_1 = \sum_{i=1}^{n} |a_i|$

$\|a\|_\infty = \max\{|a_i| ; 1 \le i \le n\}$

$X \longrightarrow Y$

we need a notion distance function / proximite with 2 element in X and Y

$x - a$ $\quad f(x), f_a$

Distance function (metric) on $\mathbb{R}^n$

Every norm $\| \cdot \|$ on $\mathbb{R}^n$ induces a metric on $\mathbb{R}^n$, which is given by: $\quad d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$d(a,b) = \|a-b\|$; $a, b \in \mathbb{R}^n$

There are basic properties to create a distance functions

$d_E(a,b) = \|a-b\|_2 = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$; Euclidean metric

$d_{sum}(a,b) = \|a-b\|_1 = \sum_{i=1}^{n} |a_i - b_i|$: Taxi-cab, Manhattan.

$d_{max}(a,b) - \|a-b\|_\infty - \max\{|a_i \cdot b_i| \; 1 \le i \le n\}$

Is all norm are equivalent, and give same topology.

Metric space:

Let X be a non empty set

A finite $d : X \times X \longrightarrow \mathbb{R}$ is said to be a metric (or distance function) on X if.

i) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$

ii) $d(x,y) = d(y,x)$

iii) $d(x,y) \leq d(x,z) + d(z,y)$

The set together with metric d is called a metric space and we write as a pair (X,d)

Example: 1) let X be a nonempty set. Then $d: X \times X \to \mathbb{R}$ given by $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

is a metric on X called the discrete metric on X^n

2) $\mathbb{R}^n$ is metric space with distance function induced by norm

Remark: $\mathbb{R}^n$, $\|\cdot\|_p$ $1 \leq p < \infty$ and $\|\cdot\|_\infty$

$n = 1$, then all these norms on $\mathbb{R}^1$

Considers with the Euclidean absolute value

$$\|a\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$$

$$\|a\|_p = \left(|a_1|^p\right)^{1/p} = |a_1|$$

Other metrics space in $\mathbb{R}$

Let $\varphi: \mathbb{N} \longrightarrow \mathbb{R}$ be real sequence

$\mathbb{N} \to \mathbb{R}$

Then $\varphi \equiv (a_n)_{n\in\mathbb{N}}$ $\varphi(n) = a_n \ \forall n \in \mathbb{N}$

$\varphi(n) \cong (a_n)$

Let l^∞ be the set of all bounded sequence of real numbers

In

$$l^\infty = \left\{ (a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{R} \ \forall n \in \mathbb{N} \text{ and } |a_n| \leq k \text{ for } k \in \mathbb{R} \right\}$$

Ex: Show that l^∞ is vector space over $\mathbb{R}$

① addition $\Rightarrow$ for $(a_n), (b_n)$

$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$

② Scalar multiplication

$\alpha (a_n)_{n\in\mathbb{N}} = (\alpha a_n)_{n\in\mathbb{N}}$

$\|(a_n)_{n\in\mathbb{N}}\|_\infty = \sup\{|a_n| : 1 \leq n < \infty\}$

$1 \leq p < \infty$

$l^p = \{(a_k)_{k\in\mathbb{N}} : \sum |a_k|^p < \infty\} \subseteq l^\infty$

$\|(a_k)_{k\in\mathbb{N}}\|_p = \left[\sum_{k=1}^{\infty} |a_k|^p\right]^{1/p}$

Proof:

Let $+: l^\infty \times l^\infty \longrightarrow l^\infty$

$\cdot : \mathbb{R} \times l^\infty \longrightarrow l^\infty$

defined as $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$

$\alpha \cdot (a_n)_{n\in\mathbb{N}} = (\alpha a_n)_{n\in\mathbb{N}}$

1) associativity: $\left((a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}\right) + (c_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}} + (c_n)_{n\in\mathbb{N}}$

$= ((a_n + b_n) + c_n)_{n\in\mathbb{N}} = (a_n + (b_n + c_n))_{n\in\mathbb{N}} = (a_n)_{n\in\mathbb{N}} + ((b_n)_{n\in\mathbb{N}} + (c_n)_{n\in\mathbb{N}})$

2) identity in $(+,\cdot)$

$(a_n)_{n\in\mathbb{N}} + (0)_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}} + (a_n)_{n\in\mathbb{N}}$

3) inverse in $(+,\cdot)$

$(a_n)_{n\in\mathbb{N}} + (-a_n)_{n\in\mathbb{N}} = (-a_n)_{n\in\mathbb{N}} + (a_n)_{n\in\mathbb{N}}$

4) distributivity ...

l^∞ sequence space whose elements are bounded sequence

$\|x\|_\infty = \sup_n |x_n|$

also a banach space (complete normed vector space)

Ex: Show that $\| \ \|$ is norm on $\mathbb{R}^n$ $(1 \le p < \infty)$

$$\lim_{p \to \infty} \|a\|_p = \|a\|_\infty$$

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Metric space

X with distance function (or metric)

$d: X \times X \longrightarrow \mathbb{R}$

i) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$

ii) $d(x,y) = d(y,x)$

iii) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$

Example / Exercises

1) $X \neq \phi$, discrete metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

2) If V is vector space over $\mathbb{R}$ and there is a norm function on V. $\|\cdot\| : V \to \mathbb{R}$

Every norm generate metric on V.

$$d(v,w) = \|v - w\|$$

On $\mathbb{R}^n$, there are various norm

$a = (a_1, a_2, \ldots a_n) \quad \mathbb{R}^n$

$$\|a\|_1 = \sum_{i=1}^{n} |a_i|$$

$$\|a\|_2 = \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2}$$

$$\|a\|_\infty = \max\{|a_1|, |a_2| \ldots |a_n|\}$$

$1 \leq p < \infty$

$$\|a\|_p = \left(\sum^{n} |a_k|^p\right)^{1/p}$$

3) If (X,d) is metric space and S is any nonempty subset of X

then $d|_{S \times S} \quad S \times S \to \mathbb{R}$

metric on S. This metric space $(S, d|_{S\times S})$ is called a metric subspace of (X,d)

Example $X = \mathbb{R}^3 \quad d = d_E$

$S = S^2 = \{(x_1, x_2, x_3) \quad x_1^2 + x_2^2 + x_3^2 = 1\}$.

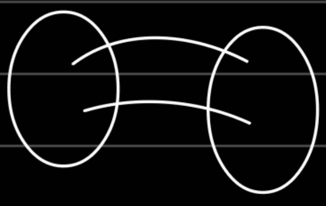
4) If (X,d) is metric space, then $f(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ $\quad h(x,y) = \max\{d(x,y), 1\}$ $\quad$ what is this

$\forall x,y \in X$

are both metric on X

5) Let (X,d) be a metric space and $\varphi : X \longrightarrow X$ be an injective map. Then

$$\rho(x,y) = d(\varphi(x), \varphi(y)) \quad \forall x,y \in X$$



Particular examples of metrics on $\mathbb{R}$

$$\rho(x,y) = \frac{|x-y|}{1+|x-y|} \quad ; \quad n(x,y) = \max\{|x-y|, 1\}$$

$$\rho(x,y) = |e^x - e^y|$$

$$d_1(x,y) = \sqrt{|x-y|} \quad \text{is a metric on } \mathbb{R}$$

$$d_2(x,y) = \begin{cases} |x-y|+1 & \text{If only of } x \& y \text{ is positive} \\ |x-y| & \text{otherwise} \end{cases}$$

To check a give distance function is metric.

1. $d(x,y) = 0 \iff x = y$
2. $d(x,y) \geq 0$
3. $d(x,y) \leq d(x,z) + d(z,y)$

$$\varphi : \mathbb{R} \to \mathbb{R} \qquad \varphi(x) = \begin{cases} x ; & x<0 \\ x+1 ; & x \geq 0 \end{cases}$$

l^∞ = the set of bounded real sequences

$= \{ (a_k)_{k\in\mathbb{N}} : a_k \in \mathbb{R}$ and $\exists M>0$ s.t $|a_k| \leq M \ \forall k \in \mathbb{N} \}$

$$\|(a_k)\|_\infty = \sup\{|a_k| : k \in \mathbb{N}\}$$

$$\|(a_k) + (b_k)\| = \|(a_k + b_k)\|_\infty$$

$(1, 1, \ldots\ldots)$

$1 \leq p < \infty$

l^p = the set of p^{th} power summable real sequence

$$= \{ (a_k)_{k\in\mathbb{N}} \ \sum_{k=1}^{\infty} |a_k|^p < \infty \} \subseteq l^\infty$$

$$\|(a_k)\|_p = \left[\sum_{k=1}^{\infty} |a_k|^p \right]^{1/p}$$

Holder inequality $\quad p \to q$ is conjugate exponent of p

$$1/p + 1/q = 1 \qquad (\text{If } p=1, \text{ then } q=\infty$$

If $(a_k) \in l^p$; $(b_k) \in l^q$, then $(a_k b_k)$

$$\|(a_k b_k)\| \leq \|a_k\|_p \ \|(b_k)\|_q$$

$$\sum |a_k b_k| \leq \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |b_k|^q\right)^{1/q}$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$

$$a^\alpha b^\beta \leq \alpha a + \beta b \quad \text{and} \quad \alpha + \beta = 1$$

Inequality is in convex.

Convex functions

$$f((1-t)a + tb) \leq (1-t) f(a) + t f(b)$$

Young's Inequality

If $a, b \geq 0$, $p, q > 0$ & $1/p + 1/q = 1$

then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

proof:

let $p = \frac{m}{m+n}$ $q = \frac{n}{m+n}$ where $m, n \in \mathbb{Z}$

write $a = x^{1/p}$ $b = y^{1/q}$

then $\frac{x^p}{p} + \frac{y^q}{q} = \frac{a}{\left(\frac{m}{m+n}\right)} + \frac{b}{\left(\frac{n}{m+n}\right)}$

$= \frac{ma + nb}{m+n}$

by AM-GM inequality

$\frac{ma + nb}{m+n} \geq (a^m \cdot b^n)^{\frac{1}{m+n}} = a^{1/p} b^{1/q} = xy$

thus $\frac{x^p}{p} + \frac{y^q}{q} \geq xy$

Holder's Inequality for n tuple

$x := (x_1 \ldots x_n)$ $y = (y_1, \ldots, y_n)$

$\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$

then $\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \|y\|_q$

Proof: for $x = 0$, $y = 0$ → holds

for $x \neq 0$, $y \neq 0$ → $\|x\|_p \neq 0$ $\|y\|_q \neq 0$

by young's Inequality

$a_i = \frac{|x_i|}{\|x\|_p}$ $b_j = \frac{|y_j|}{\|y\|_q}$

$a_i b_i \leq \frac{a_i^p}{p} + \frac{b_i^q}{q}$

$\frac{|x_i y_i|}{\|x_i\|_p \|y_i\|_q} \leq \frac{1}{p} \frac{|x_i|^p}{\sum |x_i|^p} + \frac{1}{q} \frac{|y_i|^q}{\sum |y_i|^q}$

adding above inequalities

$\frac{\sum_{i=1}^{n} |x_i y_i|}{\|x\|_p \|y\|_q} = \frac{1}{p} \frac{\sum_{i=1}^{n} |x_i|^p}{\sum_{i=1}^{n} |x_i|^p} + \frac{1}{q} \frac{\sum_{i=1}^{n} |y_i|^q}{\sum_{i=1}^{n} |y_i|^q} = \frac{1}{p} + \frac{1}{q} = 1$

$\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \cdot \|y\|_q$

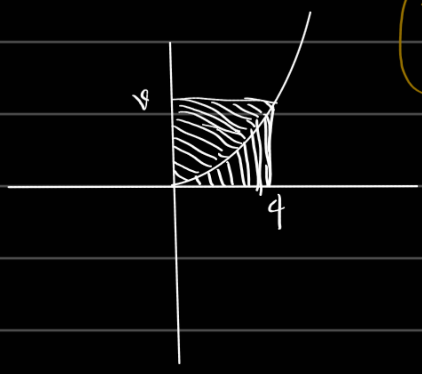
$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$

Proposition: If $\alpha, \beta \geq 0$ then

$1 < p < \infty$

$1 < q < \infty$ $\quad 1/p + 1/q = 1$

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$



$y = x^{p-1}$ for $x \geq 0$

$$uv \leq \int_0^u x^{p-1}\,dx + \int_0^v y^{q-1}\,dy$$

$$\frac{u^p}{p} + \frac{v^q}{q}$$

$$\frac{|a_k b_k|}{\|a_k\|_p \|b_k\|_q} = \left(\frac{|a_k|}{\|a_k\|_p}\right)\left(\frac{|b_k|}{\|b_k\|_q}\right) = \frac{|a_k|^p}{p\|a_k\|_p^p} + \frac{|b_k|^q}{q\|b_k\|_q^q}$$

$$\frac{\sum_{k=1}^{n} |a_k b_k|}{\|a_k\| \|b_k\|} \leq \frac{\left(\sum_{k=1}^{\infty} |a_k|^p\right)}{p\|a_k\|_p^p} + \frac{\left(\sum_{k=1}^{\infty} |b_k|^q\right)}{q\|b_k\|_q^q}$$

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Basic concepts of metric space

Let (X,d) be a metric space

For $a \in X$ and $r > 0$, the subset

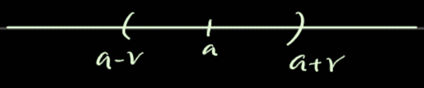
$$B(a,r) = \{x \in X : d(x,a) < r\}$$

is called open ball with centre 'a' and radius 'r'

The subset $\hat{B}(a,r) = \{x \in X : d(x,a) \leq r\}$

is called a closed with center a and radius r

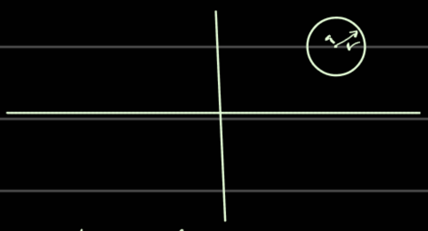
Ex: $\mathbb{R}$ with Euclidean dt



$$B(a;r) = \{x \in \mathbb{R} : |x-a| < r\} = (a-r, a+r)$$

$$\{x \in \mathbb{R} : a-r < x < a+r\}$$

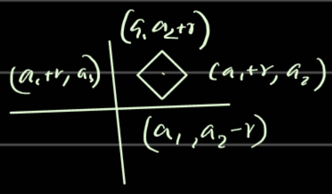
2) $\mathbb{R}^2 \longrightarrow$ Euclidean plane



$\mathbb{R}^2$ with taxi-cab metric

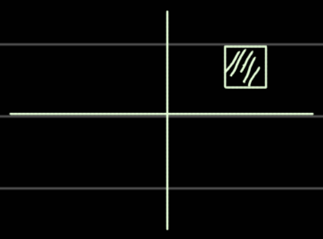
$$d(x,a) = |x_1 - a_1| + |x_2 - a_2|$$

$$B_1(a;r) = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - a_1| + |x_2 - a_2| < r\}$$



$\mathbb{R}^2$ with supremum

$d(x,a) = \max\{|x_1-a_1|, |x_2-a_2|\}$



$B_{dsc}(a;r) = \{x \in \mathbb{R}, d(x,a) < r\}$

$= \begin{cases} \{a\} & \text{if } r \le 1 \\ \mathbb{R} & \text{if } r > 1 \end{cases}$

Let (X,d) be a metric space

Let S be subset of X

2) A point $x \in S$ is said to be an interior point of S if there $\exists\ r>0$ such that $B(x;r) \subseteq S$

The set of all interior points of S is denoted by $Int(S)$ or S°.

We say S is open set in (X,d) if $S = S^\circ$

In other words S is open set if every point of S is an interior point of S

$\Leftrightarrow \forall x \in S, \exists\ r>0$ s.t $B(x,r) \subseteq S$

Ex: open ball are open set

4) Let $x \in X$. we say that x is an accumulation (limit point) point of S

If $\forall$ open ball contains a point

$B(x;r)$; $r>0$ contain a point S different from x.

In other words, x is an accumulation point of S if $\forall r>0$,

$B(x;r) \cap (S \setminus \{x\}) \neq \emptyset$

The set of accumulation points of S is denoted by $D(S)$ and called the derived set of S

A subset S of (X,d) is called closed subset if $D(S) \subseteq S$, In other words, S is said to be closed if it contain all its accumulation points

6) A convergent sequence in a metric space (X,d)

A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to converge to a point $p \in X$ if $\forall \varepsilon > 0$. $\exists\ n_0 \in \mathbb{N}$ s.t

$$d(x_n, p) < \varepsilon \quad \forall n \geq n_0$$

In this case p is unique and it is called the limit of the sequence (x_n)

7) A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called Cauchy sequence $\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}$

$d(x_n, x_m) < \varepsilon \quad \forall\, n, m \geq n_0$

8) A metric space (X, d) is called complete if $\forall$ Cauchy seq in X is convergent