30th oct Limit function

$E \subseteq \mathbb{R}$ or E is metric space

$n \in \mathbb{N}$ $(n \in \mathbb{N} = \{0\} \cup \mathbb{N})$

$f_n : E \to \mathbb{R}$ $(f_n)_{n\in\mathbb{N}}$ is sequence of function on E

$\mathbb{N} \to \text{Func}(E, \mathbb{R})$

$n \to f_n$

Suppose for each $x \in E$, $(f_n(x))_{n\in\mathbb{N}}$ is convergent

$x \xmapsto{f} \lim_{n\to\infty} f_n(x)$

Define $f(x) = \lim_{n\to\infty} f_n(x)$

f is called the limit function of the sequence (f_n)

Defination: (Uniform convergence) A sequence $f_n : E \to \mathbb{R}$ is said to be uniformly convergent to a function $f : E \to \mathbb{R}$

If $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in E \ \& \ \forall n \geq n_0$$

1) $f_n(x) = x^n$ $x \in [0,1]$

$$f(x) = \begin{cases} 0 & ; x \in [0,1) \\ 1 & ; x = 1 \end{cases}$$

$f_n \longrightarrow f$ pointwise but not uniformly

2) $h_n(x) = \dfrac{x^{2n}}{x^{2n}+1}$; $x \in \mathbb{R}$; $n \in \mathbb{N}$

$$\lim_{n\to\infty} h_n(x) = \begin{cases} 0 & \text{if } |x|<1 \\ 1/2 & \text{if } |x|=1 \\ 1 & \text{if } |x|>1 \end{cases}$$

$h_n \to h$ pointwise on $\mathbb{R}$. but convergence is not uniform

Weierstrass Approximation theorem

Every continous $f : [a,b] \to \mathbb{R}$, can be uniformly approximated by polynomial function.

Consider a series $\sum_{n=1}^{\infty} f_n$ of function on E

We say that the series $\sum_{n=1}^{\infty}$ is uniformly convergent on E, If the sequence $S_n = \sum_{i=1}^{n} f_i$ of partial sums converges uniformly on E

Theorem (Cauchy's criterian for uniform converge)

Let $(f_n)_{n\in\mathbb{E}}$ be a sequence functions on E. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $E \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ (St)

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in E \ \& \ \forall m,n \geq n_0$$

Proof $\Rightarrow$ let $f = \lim f_n$ suppose $f_n \to f$ uniformly on E

Given $\varepsilon > 0$. $\exists n_0 \in \mathbb{N}$ s.t

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall x \in E \ \& \ \forall n \geq n_0$$

Cla $m, n \geq n_0$, then from ① (for every $x \in E$) we get

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Suppose condition hold. Then for $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t we have

$$|f_n(x) - f_m(x)| \leq \varepsilon/2 \quad \forall x \in E \ \& \ \forall n, m \geq n_0 \quad -(2)$$

Claim, the sequence $(f_n(x))$ is a cauchy sequence of real number thus it is convergent. Hence we can define the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x \in E$ $-(3)$

Letting $m \to \infty$ in (2) we get

$$|f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon \quad , \forall x \in E, \text{ when } n \geq n_0$$

$\Rightarrow f_n \to f$ uniformly on E

Corollary: consider a sequence $(f_n)_{n\in\mathbb{N}}$ of function on E. Let $f: E \to \mathbb{R}$ be function and

$$r_n = \sup\{|f_n(x) - f(x)| : x \in E\}. \ \forall n \in \mathbb{N}$$

Then

$f_n \to f$ uniformly on E $\Leftrightarrow$ $r_n \to 0$ as $n \to \infty$

Corollary

Let $\sum_{n=1}^{\infty} f_n$ be a series of function on E. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E if and only if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$

$$\text{s.t } \left|\sum_{k=n+1}^{m} f_k(x)\right| < \varepsilon \quad \forall x \in E \ \& \ \forall m > n > n_0$$

Weierstrass M-test. Let $\sum_{n=1}^{\infty} f_n$ be a series of function on E. Suppose $|f_n(x)| \leq M_n$. $\forall x \in E$ & $\forall n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} M_n$ is convergent, then the series, $\sum_{n=1}^{\infty} f_n$ of function is uniformly convergent

36

Theorem (uniformly convergence and continuity)

Let E be a metric space and let $f_n: E \to \mathbb{R}$ be functions for each $n \in \mathbb{N}$: Suppose f_n converges uniformly to a function $f. E \to \mathbb{R}$ Then f is continous on E.

proof: ($\varepsilon/3$ - trick) let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly on E $\exists n_0 \in \mathbb{N}$ s.t

$$|f_n(x) - f(x)| < \varepsilon/3 \quad \forall x \in E \text{ and } n \geq n_0 \quad (1)$$

Let $x_0 \in E$. we shall show that f is continuous at x_0. As f_n is continuous at x_0; $\exists$ an open ball $B(x_0, \delta)$ such that whenever

$$y \in B(x_0, \delta), \text{ we have } |f_n(y) - f_{n_0}(x_0)| < \varepsilon/3 \quad -(2)$$

Clearly, from (1) & (2) we see that

$$\begin{aligned}|f(y) - f(x_0)| &= |f(y) - f_n(y) + f_n(y) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0)| \\ &\leq |f(y) - f_{n_0}(y)| + |f_{n_0}(y) - f_{n_0}(x)| + |f_n(x) - f(x_0)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

31 oct

$f_n : E \to \mathbb{R}$; $n \in \mathbb{N}$

$f_n \to f$ uniformly on E if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$

st $|f_n(x) - f(x)| < \varepsilon$ $\forall x \in E$ & $\forall n \geq n_0$

Cauchy's crieterion $(f_n)_{n \in \mathbb{N}}$ converges uniformly on E if

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t $|f_n(x) - f_m(x)| < \varepsilon$ $\forall x \in E$ & $\forall m, n \geq n_0$

Weierstrass M-test : $\sum_{n=1}^{\infty} f_n$ convergent on E

If $|f_n(x)| \leq M_n$. $\forall x \in E$ and $\sum_{n=1}^{\infty} M_n < \infty$

Ex: Suppose $f_n : E \to \mathbb{R}$ is function and $f = \lim f_n$. If $f_n \to f$ unimorily on E

$x \in E$ is an accumulation point of E and $\lim_{t \to x} f_n(t) = A_n$, then

the sequence $(A_n)_{n \in \mathbb{N}}$ is convergent and

$$\lim_{n \to \infty} A_n = \lim_{t \to x} f(t)$$

$$\lim_{n \to \infty} \left(\lim_{t \to x} f_n(t) \right) = \lim_{t \to x} \left(\lim_{n \to \infty} f_n(t) \right)$$

Theorem: Suppose $f_n \in \mathcal{R}[a,b]$ for $n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[a,b]$

then $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f\,dx = \lim_{n \to \infty} \int_a^b f_n\,dx$$

$$\int_a^b \left(\lim_{n \to \infty} f_n \right) dx = \lim_{n \to \infty} \left(\int_a^b f_n\,dx \right)$$

$$f(x) - f(a) = \int_a^x f'(t)\,dt$$

Theorem let $f_n : [a,b] \to \mathbb{R}$ be differentiable function. If $n \in \mathbb{N}$ and $x_0 \in [a,b]$

Suppose $(f_n')_{n \in \mathbb{N}}$ converges uniformly on $[a,b]$ and $(f_n(x_0))_{n \in \mathbb{N}}$ is convergent

Then f_n converges uniformly to a function f on $[a,b]$ and $f'(x) = \lim_{n \to \infty} f_n'(x)$

$\forall x \in [a,b]$

$$\frac{d}{dx} \left(\lim_{n \to \infty} f_n(x) \right) = \lim_{n \to \infty} \left(\frac{d}{dx} f_n(x) \right)$$

$r_n = \sup\{ |f_n(x) - f(x)| : x \in [a,b] \}$

$f_n \to f$ uniformly $[a,b]$ $\iff r_n \to 0$ as $n \to \infty$

Clearly, we see that $|f_n(x) - f(x)| \leq r_n$

$\Rightarrow f_n(x) - r_n \leq f(x) \leq f_n(x) + r_n$

$$\int_a^b (f_n(x) - r_n) \leq \underline{\int} f \leq \overline{\int} f \leq \int (f_n(x) + r_n)$$

$\Rightarrow 0 \leq \overline{\int} f - \underline{\int} f \leq 2 r_n (b-a) \to 0$ as $n \to \infty$

$\Rightarrow \underline{\int} f = \overline{\int} f \Rightarrow f \in \mathcal{R}[a,b]$

consider $\left| \int_a^b (f - f_n)\,dx \right| \leq \int_a^b |f - f_n|\,dx$

$\leq \left(\sup\{ |f_n(x) - f(x)| : x \in [a,b] \} \right) (b-a)$ $= \int_a^x f'(t)\,dt$

$\to 0$ as $n \to \infty$

Sketch of proof

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t

$$|f_n'(x) - f_m'(x)| < \varepsilon/2(b-a) \quad \forall x \in [a,b]$$

and $|f_n(x_0) - f_m(x_0)| < \varepsilon/2$

for $n, m \geq n_0$

Apply LMVT to $f_n - f_m$

$$\left|(f_n - f_m)(x) - (f_n - f_m)(t)\right| = \left|(f_n - f_m)'(s)\right| (x-t) < \frac{\varepsilon(x-t)}{2(b-a)} \leq \varepsilon/2 \Big\} \quad ②$$

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n - f_m)(x) - (f_n - f_m)(x_0) + (f_n - f_m)(x_0)| \\ &\leq |(f_n - f_m)(x) - (f_n - f_m)(x_0)| + |f_n(x_0) - f_m(x_0)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{if } n, m \geq n_0 \end{aligned}$$

By Cauchy's criterion for uniform converges

(f_n) is uniformly convergent

Let $f = \lim_{n \to \infty} f_n$

Let $x \in [a,b]$ and $t \neq x$ in $[a,b]$

Define

$$\varphi_n(t) = \frac{f_n(x) - f_n(t)}{x - t} \; ; \; t \neq x$$

and

$$\varphi(t) = \frac{f(x) - f(t)}{x - t} \quad t \neq x$$

Then $(\varphi_n)_{n \in \mathbb{N}}$ is sequence of continuous function

we see that

$$\varphi_n(s) - \varphi_n(t) = \frac{(f_n(x) - f_n(t)) - (f_m(x) - f_m(t))}{x - t}$$

$$\leq \frac{\varepsilon(x-t)}{2(b-a)(x-t)} \leq \frac{\varepsilon}{2(b-a)}$$

If $n, m \geq n_0$

$\Rightarrow (\varphi_n)_{n \in \mathbb{N}}$ converges uniformly on $[a,b] \setminus \{x\}$

Since $f_n \to f$ uniformly, we see that $\varphi_n \to \varphi$ uniformly on $[a,b] - \{x\}$

For uniform convergence interchange of limit is valid

Thus $\lim_{n \to \infty} (\lim_{t \to x} \varphi_n(t)) = \lim_{t \to x} \lim_{n \to \infty} \varphi_n(t)$

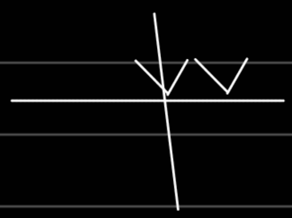
$$\Rightarrow \lim_{n \to \infty} f_n'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x)$$

Theorem: there exists a continuous function on $\mathbb{R}$ which is nowhere differentiable

Proof: $\varphi(x) = |x|$ if $x \in [-1, 1]$

$\varphi(x+2) = \varphi(x)$ for any $x \in \mathbb{R}$

$|\varphi(s) - \varphi(t)| \leq |s - t| \quad \forall s, t \in \mathbb{R}$



$$f(x) = \sum_{n=1}^{\infty} (3/4)^n \varphi(4^n x)$$

$x \in \mathbb{R}$ $m \in \mathbb{N}$

$4^m x \cdot 4^m(x + \delta_m)$

$\delta_m = \pm \frac{1}{2} 4^{-m} \to 0$

$$\frac{f(x+\delta_m) - f(x)}{\delta_m}$$

NOV 2

$(f_n)_{n\in\mathbb{N}}$ $f_n : E \to \mathbb{R}$

$f_n \to f$ uniformly if $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$

$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in E \ \& \ \forall n \geq n_0.$

$r_n = \sup\{|f_n(x) - f(x)| : x \in E\}$

Existence of nowhere differentiable continous function on $\mathbb{R}$

$\varphi(x) = |x| \quad ; x \in [-1,1]$

$\varphi(x+2) = \varphi(x) \quad \forall x \in \mathbb{R}$

$\sup |\varphi(x)| = 1$

$x \in \mathbb{R} \quad |\varphi(s) - \varphi(t)| \leq |s-t|$

$f(x) = \sum_{n=1}^{\infty} (3/4)^n \varphi(4^n x)$

$|3/4 \, \varphi(4^n x)| \leq (3/4)^n = M_n$

$x \in \mathbb{R}, m \in \mathbb{N} \quad |4^m(x+\delta_m) - 4^m x| = 1/2$

$\delta_m = \pm \frac{1}{2} 4^{-m} \quad |\varphi(4^m(x+\delta_m)) - \varphi(4^m x)| = 4^m|\delta_m| = 1/2$

$n \in \mathbb{N}, n > m \quad |\varphi(4^n(x+\delta_m)) - \varphi(4^n x)| = 0$

$|\varphi(4^n(x+\delta_m)) - \varphi(4^n x)| \leq 4^n |\delta_m|$

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \sum_{n=1}^{m} (3/4)^n \left(\frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}\right)$$

$$\geq (3/4)^m 4^m - \sum_{n=1}^{m-1} (3/4)^n 4^n = 3^m - \sum_{n=1}^{m-1} 3^n = 1/2(3^m + 1)$$

to get function where a function is differentiable, but not double differentiable

$F(x) = \int_0^x f(t)\,dt.$

Spaces of continous function

$C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continous on } [a,b]\}$

$|f(x)| \leq M_f \quad \forall x \in [a,b]$

observation 1) $C[a,b]$ is vector space over $\mathbb{R}$

2) sup norm $\|f\|_2 = \sup\{|f(x)| : x \in [a,b]\} < \infty$

$d_\infty(f,g) = \|f-g\|_\infty \quad ; f,g \in C[a,b]$

theorem: $C[a,b]$ is complete metric space (under the metric induced by sup norm)

P.f. Every $(f_n)_{n\in\mathbb{N}}$ This a sequence in $C[a,b]$

$f_n \to f$ in $C[a,b] \iff f_n \to f$ uniformly on $[a,b]$

$\sup\{|f_n(x) - f(x)| : x \in [a,b]\} = \|f_n - f\|_\infty \to 0$ as $n \to \infty$

$(X, d) \rightarrow$ compact metric space

$C(x) = C(x, \mathbb{R}) = \{f : x \to \mathbb{R} \mid f \text{ continous}\}$

$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$

$(C(X), \|\cdot\|_\infty)$ is a metric spaces

Thm: $C(x)$ is a complete metric space.

Aim: To characterize compact subsets of $C[a,b]$ or in general $C(X)$
$\hookrightarrow$ X is compact

Definition: Let (X, d) be a compact metric space and $C(x)$ be the metric space of real-valued continuous function on X with sup metric

Let $\mathcal{E} \subseteq C(x')$. We say that $\mathcal{E}$ is an equivalent continuous family of function if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ st $x, y \in$ with $d(x,y) < \delta$, we have $f(x) - f(y) < \varepsilon$ $\forall f \in \mathcal{E}$

NOV:3

Spaces Of Function.

$X \rightarrow$ a compact $\qquad X = [a,b]$

$C(x)$ = set of real valued continous function on X

$= \{f : X \to \mathbb{R} \mid f \text{ is continous}\}$

$\forall\ f, g \in C(x)$, $(f+g)x = f(x) + g(x)$

$\alpha \in \mathbb{R}$; $(\alpha f)(x) = \alpha f(x) \quad \forall x \in X$

$C(x)$ is vectorspace over $\mathbb{R}$.

$\|f\|_\infty = \sup\{|f(x)| ; x \in X\} < \infty$

Q verify that $\|\cdot\|_\infty\ \alpha(x) \to \mathbb{R}$ is a norm function

$C(x)$ is metric space under sub-metric given by

$d_\infty(f, g) = \|f - g\|_\infty$; $f, g \in C(x)$

Proposition

$C(x)$ is a complete metric space under sub-metric

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be cauchy sequence in $C(x)$

Given $\varepsilon > 0$ $\exists\ n_0 \in \mathbb{N}$ s.t

$\|f_n - f_m\|_\infty < \varepsilon \qquad \forall n, m \geq n_0$

$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon \qquad \forall n, m \geq n_0 \quad \forall x \in E$

$\therefore$ By cauchy Criterion for uniform convergence the sequence $(f_n)_{n \in \mathbb{N}}$ of function is uniformly convergent on X

Let $f(x) : \lim\limits_{n \to \infty} f_n(x)$; $x \in X$ be the limit function

We shall show that the uniform limit of a sequence of continues function is continous (Check This?)

There fore $f \in C(x)$

Aim: characterize compact subset of $C(X)$

$X - \{1, 2, \cdots n\}$ $\quad C(X) = (\mathbb{R}^n, \|\cdot\|_\infty)$

$X = \bar{N} = \mathbb{N} \cup \{\infty\}$

$\bar{N} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$

Arzela - Ascoli theorem

Let (X,d) be a compact metric space. consider the space $C(X)$ of real-valued continous function on X with sup-metric. Then a subset of $E \subseteq C(X)$ is compact if and only if E is closed, E is bounded and E is an equi continous family of function

Defn Let $E \subseteq C(X)$ be a subset

We say that E is on equicontinous family of function on X, if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that for $x, y \in X$ with $d(x,y) < \delta$. we have

$$\|f(x) - f(y)\| < \varepsilon \quad \forall f \in E$$

Observation

1) $E_0 = \{f_0\}$ is an equicontinous family
2) $E = \{f_1, f_2 \cdots f_n\}$ is an equicontinous family
3) let $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}$ $\quad x \in [0,1] = X$ $\quad n \in \mathbb{N}$

let $E = \{f_n : n \in \mathbb{N}\} \subseteq \mathcal{E}[0,1]$

fix $\varepsilon = 1/2$ Assume it work, $\forall \delta > 0$

let $\delta = 1/m$

then $|f_m(0) - f_m(1/m)| = 1$

$1 \nleq 1/2 = \varepsilon$

$\Rightarrow |f_m(0) - f_m(1/m)| > \varepsilon$

$\rightarrow\leftarrow$ then E is not equicontinous

Definition

Let $E \subseteq C(X)$ where X is a compact

A bounded subset E of $C(X)$ is called uniformly bounded. In other words, E is uniformly bounded if $\exists M \in \mathbb{N}$ s.t

$$\|f\|_\infty \leq M \quad \forall f \in E$$

- A subset E of $C(X)$ is called pointwise bounded. if for every $x \in X$ there is $M_x \in \mathbb{N}$ s.t $|f(x)| \leq M_x$ $\forall f \in E$

(Check)

Theorem:

Let E be compact subset of $C(X)$ [(X,d) is compact metric space].

Then i) E is closed

ii) E is uniformly bounded

iii) E is equicontinous family

Proof: i) & ii) ar obvious (proved earlier)

iii) let $\varepsilon > 0$ be given

since E is compact,

$\exists$ a ε/ε not of $\in$

That is there are functions $g_1, g_2, \dots g_m \in E$

S.t $E \subseteq \bigcup_{i=1}^{m} B(g_i, \varepsilon/3) \rightarrow ①$

Since $g_1, g_2 \dots g_m$ are continous function on a compact metric space (X,d)

$\exists\, \delta > 0$ such that

$|g_i(x) - g_i(y)| < \varepsilon/3$ whenever $x, y \in X$

with $d(x,y) < \delta$

Let $f \in E$ Then by ① $\exists\, 1 \leq k \leq m$ such that

$\| f - g_k \| < \varepsilon/3 \rightarrow ③$

Clearly for $x, y \in X$ with $d(x,y) < \delta$

from ①, ②, & ③ we have

$$|f(x) - f(y)| = |f(x) - g_k(x) + g_k(x) - g_k(y) + g_k(y) - f(y)|$$
$$\leq |f(x) - g_k(x)| + |g_k(x) - g_k(y)| + |g_k(y) - f(y)|$$
$$\leq \|f - g_k\|_\infty + \varepsilon/3 + \|f - g_k\|_\infty$$
$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$\Rightarrow |f(x) - f(y)| < \varepsilon$