Tutorial Questions (Set–I)

Q1. Let $<$ be a strict order on a field $\mathbb{K}$, i.e., $(\mathbb{K},<)$ be an ordered field. Let $\mathbb{P}=\{x\in\mathbb{K}: x>0\}$ be the set of positive elements of $\mathbb{K}$. Then show that i) $x,y\in\mathbb{P} \Rightarrow x+y\in\mathbb{P}$ and $xy\in\mathbb{P}$
ii) $\forall x\in\mathbb{K}$, one and only one of the following

$$x\in\mathbb{P}, \text{ or } x=0, \text{ or } -x\in\mathbb{P} \text{ hold.}$$

Conversely, if $\mathbb{P}$ is a subset of a field $\mathbb{K}$ satisfying (i) and (ii), then show that there is a strict order '$<$' on $\mathbb{K}$ given by $x<y$ if $y-x\in\mathbb{P}$. Further, show that $(\mathbb{K},<)$ is an ordered field and $\mathbb{P}=\mathbb{K}^+$, the set of positive elements.

Q2. Let $(\mathbb{K},<)$ be an ordered field. Show that

a) If $x\neq 0$ in $\mathbb{K}$, then $x^2>0$. b) If $0<x<y$, then $0<y^{-1}<x^{-1}$.

c) The field $\mathbb{Q}$ of rational numbers is isomorphic to an ordered subfield of $\mathbb{K}$. (i.e. $\mathbb{Q}$ is the smallest ordered field)

Q3. A field isomorphism $f:\mathbb{K}\to\mathbb{K}$ is called an automorphism of $\mathbb{K}$.

a) Show that an automorphism of $\mathbb{Q}$ (also $\mathbb{R}$) must be the identity automorphism.

b) What are automorphisms of the field $\mathbb{Q}[\sqrt{2}]$ or $\mathbb{C}$?

Q4. Show that the field $\mathbb{R}(x)$ of rational functions over $\mathbb{R}$ is an ordered field. Show that $\mathbb{R}(x)$ is not Dedekind complete. Also, $R(x)$ does not have the Archimedean property.

Q5. Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence of real numbers.

a) Show that $(a_n)_{n\in\mathbb{N}}$ is bounded and a Cauchy sequence.

b) Every subsequence of a convergent sequence is convergent and has the same limit.

Q6. Define $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ of a real sequence $(a_n)_{n\in\mathbb{N}}$.
Show that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$. Also, prove that $(a_n)_{n\in\mathbb{N}}$ is convergent if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Q7. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of real numbers. Show that a) (x_n) is bounded. b) (x_n) has a convergent subsequence.
c) every Cauchy sequence of real numbers is convergent (i.e. the field $\mathbb{R}$ of real numbers is Cauchy complete.).

Q8. Let $b>1$ be a real number. If $r=\frac{p}{q}$, $q\in\mathbb{N}$ and $p\in\mathbb{Z}$, then define $b^r=(b^p)^{1/q}$. Show that b^r is well-defined and $b^{r+s}=(b^r)(b^s)$ for $r,s\in\mathbb{Q}$. If $x\in\mathbb{R}$, set $B(x)=\{b^t : t\in\mathbb{Q} \text{ \& } t\leq x\}$.
Show that $b^r=\sup B(r)$ for $r\in\mathbb{Q}$. Define $b^x=\sup B(x)$ for $x\in\mathbb{R}$ and show that $b^{x+y}=(b^x)(b^y)$ for $x,y\in\mathbb{R}$.

(Optional)
Q9. Let $(\mathbb{K},<)$ be an ordered field. a) Show that $(\mathbb{K},<)$ is Dedekind complete if and only if $(\mathbb{K},<)$ is Archimedean and Cauchy complete. b) Let $(\mathbb{K},<)$ and $(\mathbb{K}',<')$ be complete, ordered field. Show that there is a unique isomorphism $f:\mathbb{K}\longrightarrow\mathbb{K}'$ which is order preserving.

(Optional)
Q10. (Cantor's construction of real numbers) Let $\mathcal{C}(\mathbb{Q})$ be the set of $\mathbb{Q}$-Cauchy sequences (i.e. Cauchy sequences of rational numbers).
a) Show that $\mathcal{C}(\mathbb{Q})$ is a commutative ring with identity.
b) Show that $\mathcal{J}=\{(a_n)\in\mathcal{C}(\mathbb{Q}) : \lim_{n\to\infty} a_n = 0\}$ is an ideal of $\mathcal{C}(\mathbb{Q})$.
c) Show that $\mathcal{R}=\mathcal{C}(\mathbb{Q})/\mathcal{J}$ is a field.
d) Define a strict order $<$ on $\mathcal{R}$ such that $(\mathcal{R},<)$ is an ordered field. Further, show that
e) The field $(\mathcal{R},<)$ is Dedekind complete.

Q) $(\mathbb{K}, <)$ strictly ordered field.

$$\mathbb{P} = \{x \in \mathbb{K} : x > 0\}$$

i) $x, y \in \mathbb{P} \Rightarrow x+y \in \mathbb{P}$ & xy $\mathbb{P}$

ii) $\forall x \in \mathbb{K} \Rightarrow$ one of following $x \in \mathbb{P} / x = 0 / -x \in \mathbb{P}$

if $\mathbb{P} \subset \mathbb{K}$ satisfying (i) & (ii), then strict order on $\mathbb{K}$ by $x < y$ $y - x \in \mathbb{P}$. futher show tha $(\mathbb{K}, <)$ is ordered field & $\mathbb{P} = \mathbb{K}^+$

ans) i) $(\mathbb{K}, <)$ is ordered field. and p is subset of $\mathbb{K}$.

for any 2 element $x, y \in \mathbb{P}$, it belong als, belong in $\mathbb{K}$. and we know $\mathbb{K}$ is closed in addition & multiplication

so $x+y$ & xy will be there in $\mathbb{K}$. also both will be positive since addition & multiplication of 2 positive numbers (since x, y belong P, which set of positive) is positive

$x+y$ & xy is positive which belong $\mathbb{K}$.

$\Rightarrow$ they belongs to P, since $\forall$ positive element in $\mathbb{K}$ in p.

ii) By law of trichotomy, any real numbers be either positive negative or zero.

consider an element in $\mathbb{K}$, it follow law trichotomy.

so it have 3 case

case 1

$x \in \mathbb{K}$, $x = 0$

then it follows 2 one

case 2

$x \in \mathbb{K}$, x +ve number.

$\Rightarrow x \in \mathbb{P}$, since $\mathbb{P}$ have all +ve number in $\mathbb{K}$.

case 3

$x \in \mathbb{K}$, x -ve numb.

we know $(\mathbb{K}, +)$ is group. so for -ve there exist 'x', in p which can be given '$\cdot x$', so $-x \in$ will belong $\mathbb{P}$.

so it is only one of them.

consider $\mathbb{P}$, as subset of $\mathbb{K}$ which satisfying i) & ii)

then for any 2 element x, y

consider $y - x \in \mathbb{P}$

$\Rightarrow$ there exist +ve number 'a' $y - x = a$.

so $x + a = y$.

$\Rightarrow x < y$

and $\mathbb{P}$ have every +ve in $\mathbb{K}$, Q2) Let $(\mathbb{K}, <)$ be an ordered field, show that

a) If $x \neq 0$ in $\mathbb{K}$, then $x^2 > 0$

b) $0 < x < y$, then $0 < y^{-1} < x^{-1}$

c) The field $\mathbb{Q}$ of rational numbers is isomorphic to an ordered subfield of $\mathbb{K}$. ($\mathbb{Q}$ is the smallest ordered field)

ans) If $x \neq 0 \Rightarrow x \in \mathbb{P}$ or $-x \in \mathbb{P}$

x +ve $\quad$ x -ve

Case 1 $\quad x$ is +ve

$x \in P \Rightarrow x^2 \in \mathbb{P}$ (Product of 2 +ve is +ve)

$\Rightarrow x^2 > 0$

Case 2 $\quad x$ is -ve

$-x \in P$

$-x \times -x = x^2$ (+ve)

i.e. $x^2 \in \mathbb{P}$.

$x^2 > 0$

b) $0 < x < y$

we know x, y is positive.

xy is positive

then dividing xy whole inequality, there would be change

so $0 < \frac{x}{xy} < y/xy$

$0 < y^{-1} < x^{-1}$

c)

Q3) A field isomorphism $f: \mathbb{K} \to \mathbb{K}$ is called an automorphism
a) Show that an automorphism of $\mathbb{Q}$ (also $\mathbb{R}$) must be the Identity automorphism.
b) What are automorphism of the field $\mathbb{Q}[\sqrt{2}]$ or $\mathbb{C}$?

$f(1) = 1$ Identity Isomorphism

$f(2) = f(1) + f(1) = 2$

$f(n) = f(1) + f(1) \cdots f(1) = n$

$f(n + (-n)) = \underbrace{f(n)}_{n \text{ times}} + f(-n)$

$= f(0) = 0$

$\Rightarrow f(n) = -f(-n)$

Let $m/n \in Q$ $n \in \mathbb{N}$ $m \in \mathbb{Z}$

$f(1/n + \cdots 1/n) = f(1/n) \cdots f(1/n) = n f(1/n)$

$f(1/n) = 1/n$

$f(m/n) = f(m) \cdot f(1/n) = m \times 1/n = m/n$

$\text{aut}[Q] = [I]$

$\text{Aut}(\mathbb{R}) = \{1_{\mathbb{R}}\}$ Automorphism on $\mathbb{R}$ is Identity

If $f: \mathbb{R} \to \mathbb{R}$ is field isomorphism

Step 1) $f|_Q : Q \to Q$ is field automorphism

$f|_Q = 1_Q$ (Identity map)

Step 2) f is strictly increasing:

$x > 0$; $\exists\, y > 0$ s.t $x = y^2$ (i.e y is +ve root of x)

$f(x) = f(y \times y) = f(y)^2 > 0$

Now claim; If $\alpha \in \mathbb{R} - Q$ then

$f(\alpha) = \alpha$

Suppose a claim is not then

either $f(\alpha) < \alpha$ or $f(\alpha) > \alpha$

$f(\alpha) < r < \alpha$ (By denses $r \in Q$)

$r < \alpha \Rightarrow f(r) < f(\alpha) \Rightarrow r < f(\alpha)$ [$\because r \in Q$]

Contd.

To prove this, we have following claims

Consider $\phi: \mathbb{R} \to \mathbb{R}$ be an automorphism of field $\mathbb{R}$

1) $\forall x$, which is +ve, $\phi(x) > 0$
2) $\forall x, y \in \mathbb{R}$, $x > y \Rightarrow \phi(x) > \phi(y)$
3) ϕ is Identity on positive ($\phi(n) = n$)
4) ϕ is Identity on real numbers ($\phi(m/n) = m/n$, $\phi(r) = r$ r rational)

1) For any positive real number x, $\phi(x) > 0$

If x positive, we know ϕ strictly increasing

$\therefore \forall x, \exists y$ st. $x = y^2$

$$\phi(x) = \phi(y^2) = \phi(y)^2 \geq 0$$

Since $\phi(0) = 0$ and ϕ is bijection

$\Rightarrow \phi(x) > 0$

2) For any $x, y \in \mathbb{R}$, $x > y \Rightarrow \phi(x) > \phi(y)$

$x > y \Rightarrow x - y > 0$

by (1) $\phi(x-y) > 0$

$\phi(x) - \phi(y) > 0 \Rightarrow \phi(x) > \phi(y)$

3) ϕ is identity on $\mathbb{Z}^+$

$$\phi(n) = \phi(1+1+\cdots+1) = n\phi(1) = n$$

Since $\phi(1) = 1$

4) ϕ is on rational number

$$\phi(q) = \phi(\pm m/n) = \pm\frac{\phi(m)}{\phi(n)} = \frac{\pm m}{n} = q$$ //

using (3rd)

5) ϕ is identity on Real numb

Now claim; If $\alpha \in \mathbb{R} - \mathbb{Q}$ then

$f(\alpha) = \alpha$

Suppose a claim is not then

either $f(\alpha) < \alpha$ or $f(\alpha) > \alpha$

$f(\alpha) < r < \alpha$ (By denses $r \in \mathbb{Q}$)

$r < \alpha \Rightarrow f(r) < f(\alpha) \Rightarrow r < f(\alpha)$ $[\because r \in \mathbb{Q}]$

contradiction contdit.

Illy $f(\alpha) > \alpha$ contradicts

so $f(\alpha) = \alpha$.

$\phi : \mathbb{R} \to \mathbb{R}$ is identity homomorphism

Suppose φ is an automorphisms $\mathbb{Q}(\sqrt{2})$

$\varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0)$

$\Rightarrow \varphi(0) = 0$

$\varphi(1) = \varphi(1.1) = \varphi(1) \times \varphi(1.)$

$\Rightarrow \varphi(1) = 1$

$\varphi(n) = \varphi(\underbrace{1+1+1+\cdots 1}_{n}) = n\,\varphi(1)$

$= n$

$\varphi(0) = \varphi(1+(-1)) = \varphi(1) + \varphi(-1)$

$\Rightarrow \varphi(-1) = -1$

$\varphi(-m) = -m$

$\therefore \varphi(n) = n$ for $n \in \mathbb{Z}$

If $m \in \mathbb{Z}$ $n \in \mathbb{Z}$

$n \cdot \varphi(m/n) = \varphi(m) = m$

$\varphi(r) = r$ $r \in \mathbb{Q}$

$2 = \varphi(2) = \varphi(\sqrt{2}) \cdot \varphi(\sqrt{2})$

$\varphi(\sqrt{2}) = \pm\sqrt{2}$

$\Rightarrow \varphi(r+s\sqrt{2}) = \varphi(r) + \varphi(s)\varphi(\sqrt{2})$

$= r + s \cdot \varphi(\sqrt{2})$

$= r \pm \sqrt{2}$

the both choice φ are automorphs

so we have 2 automorphs

$r + s\sqrt{2} \longmapsto r + s\sqrt{2}$

$r + s\sqrt{2} \longmapsto r - s\sqrt{2}$

Ans) $\mathbb{C} = \mathbb{R} \times \mathbb{R}$

$(a,b) + (c,d) = (a+c, b+d)$

$(a,b) \cdot (c,d) = (ac-bd, bc+ad)$

$\phi : \mathbb{C} \to \mathbb{C}$

$(x,y) \to (x^*, y^*)$

$\phi(0,0) = \phi((0,0)+(0,0)) = \phi(0,0) + \phi(0,0)$

$\Rightarrow \phi((0,0)) = (0,0)$

$\phi((1,0)) = \phi((1,0) \times (1,0)) = \phi(1,0) \times \phi(1,0)$

$\Rightarrow \phi((1,0)) = (1,0)$

$\phi(n,0) = \phi((1,0) + (1,0) \cdots (1,0))$

$= n\,\phi(1,0) = n(1,0)$

$\phi(0,0) = \phi((1,0) + (-1,0)) = \phi(1,0) + \phi(-1,0)$

$\phi(-1,0) = -(1,0)$

lly $\phi(-m,0) = -(m,0)$

If $p \in \mathbb{Z}$, $q \in \mathbb{N}$

$q\,\phi(p/q, 0) = \phi(q,0)$

$\Rightarrow \phi((r,0)) = (r,0)$ for $r \in \mathbb{Q}$

$(-1,0) = \phi(-1,0) = \phi((0,1) \times (0,1))$

$\phi(0,1) = \pm(0,1)$

$\phi(a,b) = \phi(a,0) + \phi(0,b)$

$= (a,0) \pm (0,\pm b)$

both are automorphisms of $\mathbb{C}$

$(a,b) \longmapsto (a,b)$

$(a,b) \longmapsto (a,-b)$

Q) Show that the field $\mathbb{R}(x)$ of rational function over $\mathbb{R}$ is an order field. Show that $\mathbb{R}(x)$ is not Dedekind complete. Also $\mathbb{R}(x)$ doesnt have archimeden property

Ans)

consider $\phi(x) = {p(x)}/{q(x)}$

where $p(x) = a_0 + a_1 x, \cdots + a_n x^n$

$q(x) = b_0 + b_1 x_1 \cdots + b_m x^m$

$\phi > 0 \iff a_n/b_n > 0 \implies a_n b_m > 0$

to determine $\phi_1 > \phi_2$, take $\phi_1 - \phi_2$ as single rational funct and determine it's $>0, =0, <0$

by definition $\phi_1 < \phi_2$ if only if $\phi_1 \neq \phi_2$ & leading coefficients $\phi_1 - \phi_2$ is positive

assume $\mathbb{R}(x)$ is archimedian

$\forall \phi_1(x), \phi_2(x)$ $\phi_1(x) > \phi(x)$, $\exists n$ st. $\phi_1(x) < n\phi_2(x)$

But consider an $\phi_1(x) = x$ & $\phi_2(x) = 1$

$\phi(x) = (\phi_1 - n\phi_2)(x) = x - n$ have +ve coeff

$\implies \phi_1(x) > n\phi_2(x)$

$\forall$ nonempty bdd subset of $\mathbb{R}(x)$, there should exist a supremum.

existance

$A = \{ r(x) \mid r(x) < p(x) = x^7 \}$

$x' \in A \implies A \neq \phi$

$\forall \phi(x)$, $\phi(x) < x^7$ definition

for $\phi(x)$

ans) a) i) Let a_n be a convergent sequence

i.e $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ for $\forall n > n_0$

$$|a_n - a| < \varepsilon$$

and $\lim_{n \to \infty} a_n = a$.

let consider an arbitrary $\varepsilon > 0$. by the definition we know that all term of sequence after a certain number (n_0) are in the interval $(a-\varepsilon, a+\varepsilon)$.

and this set is bounded and $a+\varepsilon$ is a upper bound.

Now consider the sequence $a_0, a_1 \dots a_n$, Since this is finite numbers. there exist a minimal number and maximal number as U, L respectively.

By this claims, we deduce the sequence is bound and in the interval of

$$[\min(L, a-\varepsilon), \max(U, a+\varepsilon)]$$

ii) claim: $\forall$ convergent sequence is cauchy sequence.

convergent sequence

$\forall \varepsilon > 0$ $\varepsilon \in \mathbb{R}$ $\exists$ $n_0 \in \mathbb{N}$ st for $\forall n > n_0$

$$|a_n - a| < \varepsilon$$

Cauchy sequence

$\forall \varepsilon > 0$ $\varepsilon \in \mathbb{R}$ $\exists$ $n_0 \in \mathbb{N}$ St for $\forall$ $n, m > n_0$

$$|a_m - a_n| < \varepsilon$$

Since $(a_n) \to x$, any arbitrary $\varepsilon > 0$

$\exists$ n_1 $n_2 \in \mathbb{N}$ s.t $n > n_1$ $n' > n_2$

$$|x_n - x| < \varepsilon/2$$

$$|x_{n_2} - x| < \varepsilon/2$$

$$|x_n - x_{n_2}| = |(x_n - x) - (x_{n_2} - x)| \leq |x_{n_1} - x| + |x_{n_2} - x|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$< \varepsilon$$

b) every subsequence of convergent sequence is convergent and has the same limit.

ans) Let (a_n) be a convergent sequence, a_n converge to a

$(a_n) \to l$

Let (a_{n_k}) be a subsequence of a_n.

then $n_1 \geq 1$, $n_2 \geq 2$

$\Rightarrow n_k \geq k$

let $\lim a_n = l$.

$\Rightarrow$ let $\varepsilon > 0$ $\exists n_0$ st $\forall n > n_0$

$|a_n - l| > \varepsilon$

Now $k > n \Rightarrow n_k > N \Rightarrow$

$|a_{n_k} - l| < \varepsilon$

$\therefore \lim_{k \to \infty} a_{n_k} = l$.

Let (an)mEN be a convergent sequence of real number.

a) show that (an) newd is bounded and a Cauchy sequence.

b) Every subsequence of a convergent sequence is Convergent and has the same limit.

Q6) Define $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ of a real sequence $(a_n)_{n\in\mathbb{N}}$. Show that $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$. Also prove that $(a_n)_{n\in\mathbb{N}}$ is convergent if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$

ans sup a

o n is largest value (v) of a_n s.t for $\varepsilon>0$

wy finite numbers large than $b+\varepsilon$

$$A_n = \begin{cases} 1+1/n \\ -1+1/n \end{cases}$$

Q7) $(x_n)_{n \in \mathbb{N}}$ be a cauchy sequence of real numbers. Show that
a) (x_n) is bounded
b) (x_n) has a convergent
(the field $\mathbb{R}$ of real number is cauchy complete)

ans) (a_n) be cauchy sequence
then $\forall \varepsilon > 0 \in \mathbb{R}$; $\exists n_0$ s.t $\forall m, n \geq n_0$ $m, n \in \mathbb{N}$
$|x_m - x_n| < \varepsilon$

then By triangle Inequality
$|a_m| - |a_n| \leq |a_m - a_n| < \varepsilon$

taking $n = N$
$|a_m| - |a_{n_0}| < \varepsilon$ for $m \geq N$

$|a_m| < \varepsilon + |a_N|$

$\Rightarrow$ $|a_m| \leq \max \{ |a_0|, |a_1| \ldots |a_{N-1}|, |a_N|, \varepsilon + |a_N| \}$
for all m

$a_n \to a.$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$

$|a_n - a| < \varepsilon \quad , \forall n \geq N$

$a - \varepsilon < a_n < a + \varepsilon \quad \forall n \geq N.$

$a_n > a - \varepsilon \quad \forall n \geq N$

$\Rightarrow \inf_{k \geq n}(\{a_k\}) \geq a - \varepsilon \quad \forall n \geq N$

$\sup_{k \geq n}(\{a_k\}) \leq a + \varepsilon \quad \forall n \geq N.$

$a - \varepsilon \leq \inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k \leq a + \varepsilon$

$a - \varepsilon \leq \liminf a_n \leq \limsup a_n \leq a + \varepsilon$

$\because \varepsilon$ is arbitrary

$\liminf = \limsup = a.$

$|\liminf a_n - a| < \varepsilon \qquad |\limsup a_n - a| < \varepsilon$

$\liminf a_n = \limsup a_n = a.$

$\Downarrow$

$\lim_{n\to\infty} \inf_{k \geq n} a_k = a \qquad a_n \to a.$

$\varepsilon > 0$

$\exists N_1 \qquad \lim_{n\to\infty} \sup_{k \geq n} a_k = a \qquad \exists N_2$

$n \geq N_1 \Rightarrow \left|\inf_{k \geq n} a_k - a\right| < \varepsilon, \qquad \left|\sup_{k \geq n} a_k - a\right| < \varepsilon$

$N = \max\{N_1, N_2\}$

$n \geq N \Rightarrow \qquad \left|\inf_{k \geq n} a_k - a\right| < \varepsilon \qquad \left|\sup_{k \geq n} a_k - a\right| < \varepsilon$

$a - \varepsilon < \inf_{k \geq n} a_k \leq a_n. \qquad a - \varepsilon < a_n$

$a_n \leq \sup_{k \geq n} a_k < a + \varepsilon \qquad a_n < a + \varepsilon$

when

$|a_n - a| < \varepsilon, \ n \geq N.$

$a_n \to a.$

Rd