Q.) $A = \{1/n : n \in \mathbb{N}\}$

ans) A' be the derived set of

$0 \in A'$, since for $\forall r > 0$, $B(0,r) = (-r, r)$

$\Rightarrow B(0,r) \cap (A - \{0\}) \neq \phi$

conside and $a \in A$, $a \neq 0$,

then $r > 0$, $B(a,r) = (a-r, a+r)$

for any n, $\frac{1}{r-a} < n$

$B(a,r) \cap (A - \{0\}) \neq \phi$

$\Rightarrow A' = 0$

$\frac{1}{x} < a-r$

$1 < xa - xr$

$x(r-a) < 1$

$x \leq \frac{1}{r-a}$

$n \in \mathbb{N}$

$\frac{1}{r-a} < n$

then

ii) $\{1/n + 1/m : n, m \in \mathbb{N}\}$

Idea of proof: we can easily deduce that derived set $D = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. To prove that this the case, we need to show ① $1/n$ & $\{0\}$ indeed a limit point ② any other point is not a limit point

② we know the set is bounded above by 1 and below by zero so any negative number and $a \in \mathbb{R}$, $a > 1$ is not a limit point.

Let x be a positive number other than 0 or $\frac{1}{n}$, $n \in \mathbb{N}$.

Claim: b cant be a limit point.

then 2 case here, ① $b > 1$ or

(ii) $\frac{1}{p+1} < b < \frac{1}{p}$ $\quad p \in \mathbb{N}$

(i) this case already proven.

(ii) let $\varepsilon = \min\{1/q - b,\ b - 1/(q+1)\}$

there only finitely many numbers of the form $1/m + 1/n$ @ distance less than $\varepsilon/2$ from b.

i.e $B(b, \varepsilon/2)$ contain finitely number of the form $1/m + 1/n$

because for any constant $1/n$, the $\frac{1}{n} + \frac{1}{m}$ converges to $1/n$ so

To get within $\varepsilon/2$, we must are m & $n \in \mathbb{Z}$ such the each $\geq q+1$. Let m be smaller, $m < 2(q+1)$

i.e

$$\frac{1}{q} \leq \frac{1}{m} + \frac{1}{n} \leq \frac{1}{2(q+1)} + \frac{1}{2(q+1)}$$

So m has only finite many possibilities.

for any such m, there are only finitely many n s.t $\frac{1}{m} + \frac{1}{n} > \frac{1}{q+1} + \varepsilon/2$. So finitely many within $\varepsilon/2$ of b.

This completes the argument for case (ii).

So $D = \{1/n : n \in \mathbb{N}\} \cup \{0\}$

iii) $\{\frac{a}{2^n} : n \in \mathbb{N}$ an $0 \le a < 2^n ; a \in \mathbb{N}\}$

ans) In this set points, we need to identify the limit point.
considering as n become larger than any finite behavior.
for any $n \in \mathbb{N}$ the set contains the element $0/2^n, \frac{1}{2^n} \ldots \frac{n}{2^n}$.
and this become zero. . This means limits point is zero.

iv) Cantor set

Cantor set C is defined as

$A_0 := [0,1]$, $A_{n+1} := 1/3 (A_n \cup (2 + A_n))$

$C := \bigcap_{n \in \mathbb{N}_0} A_n$

derived set of C is C itself

proof:

$x \in D$ & $\varepsilon > 0$ be arbitrary, chose $n \in \mathbb{N}$ large enough such that $3^{-n} < \varepsilon$. C_n ,the n^{th} stage is cantor set is union of 2^n pairwise disjoint closed interval, each of lengths 3^{-n}. let I be the some of the intervals

C_n, its disjoint closed interval are I_0 + , such I
I_0 be the one that contain x $D \cap E$ is nonempty and then
$y \in D \cap E$, $y \in$ is ε ke

$x \in D$ $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon)$ using archimedean $\exists\, n_0 \in \mathbb{N}$
$\frac{1}{3^n} < \varepsilon$. let $\frac{M}{3^n} = \max\{\frac{m}{3^n} : \frac{m}{3^N} < x \quad m \in \mathbb{N}\}$

So $x \in [\frac{M}{3^N}, \frac{M+1}{3^N}] \subset (\varepsilon, \frac{m}{3})$

case1 $x \in [\frac{M}{3^n}, \frac{3M+1}{3^n}]$. $\exists\, c \in D$ $c \in [\frac{3M+2}{3^{n+1}}, \ldots)$ then

Question 2: (X, d).

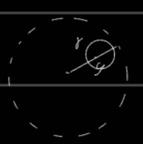
i) open ball is openset, closed ball closed set
ii) arbitary union of opensets in (X,d) open
iii) finite intersection of opensets is open

① open ball are defined as $B(a,r) = \{x \in X, d(x,a) < r\}$

a set is openset, If every point in set is interior point
i.e $\forall a \in A$, $\exists\, r > 0$ st $B(a,r) \subseteq A$

consider a arbitary point in a open ball S.
i.e $y \in B(x_0,r)$

then $\exists\, r_1$ st $B(y,r_1) \subseteq B(x_0,r)$



for this we consider
$r_1 = r - d(y, x_0)$

Then for $x \in B(y, r_1)$

$d(x, x_0) \le d(x, y) + d(y, x_0) < r_1 + d(y, x_0) = r$

so $B(y, r_1) \subseteq B(x_0, r)$.

$\Rightarrow B(x_0, r)$ is open set

ii) $\forall$ closed ball is closed set

i.e $\bar{B}(a, r) = \{x \in X : d(a, x) \le r\}$

a set is closed set if it contain all it accumulation point

consider an arbitrary closed ball $\bar{B}(x_0, r)$

there we need to show that every point accumulation point of $\bar{B}(x_0 r)$ is in $\bar{B}(x, r)$

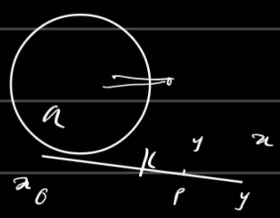
consider $y \notin \bar{B}(x_0, r)$

claim: y is not accumulation point of $\bar{B}(x_0 r)$.

If y is accumulation point then

$\forall r_1 > 0$, $B(y, r_1) \cap \{S - \{y\}\} = \phi$

let $r_1 = \dfrac{d(x_0, y) - d(x_0, r)}{2} > 0$



let $p \in B(x, r_1)$

$\| p - x_0 \| \ge \| y - x_0 \| - \| p - y \|$

$\| p - x_0 \| \ge r + r_1 - r/2 \ge r + r/2$

$\| p - x_0 \| > r$

$y \notin \bar{B}(x_0, r)$

so $B(x, r_1)$ is subset of $\overline{B(x_0, r)}^c$

this show $\overline{B(a; r)}^c$ is open, hence $B(a; r)$ is closed.

suppose $D(x_0, r)$ is closed ball. $X / D(x_0, r)$ is open.

for $\forall$ $y \in X | D(x_0, r)$, we need to find an open ball in $X | D$

$y \in X | D(x_0 r)$

$\Rightarrow d(y\, x_0) > r$

$d(y, x_0) - r > 0$

$r_1 = d(y, x_0) - r$

I claim the open ball $B(y, r_1)$ is contained in $X | D(x_0, r)$

to prove this, consider any $z \in B(y, r_1)$

$d(x_0, y) \le d(x_0, z) + d(z, y)$

$\Rightarrow d(z, x_0) \ge d(x_0, y) - d(z, y) > d(x_0, y) - r_1 = r$

$z \in X \setminus D(x_0, r)$

ii) arbitary union of open set (X, d) is open

$$U := \bigcup_{i \in I} U_i \quad \text{is open}$$

$$\forall x \in U, \exists \delta > 0 \text{ (st) } B(x, \delta) \subseteq U$$

proof: $x \in U$, by definition, $\exists i \in I$ $x \in U_i$, Since U_i is open take $\delta > 0$ st $B_\delta(x) \subseteq U_i$ since $U_i \subseteq U$ we have $B_\delta(x) \subseteq U$

this simplify the proof we have one pol : The logic

iii) finite Intersection of open set

iv) $f \subseteq X$ is closed $\Leftrightarrow$ $X \setminus f$ is open

a set X is closed if $\forall$ limit point of X is point of x.

Suppose S^c is closed. choose $x \in S$, then $x \notin S^c$ and x is not limit point S^c, because S^c already contain all its limit points. $\exists$ $B(x, r)$ such that $S^c \cap N$ is empty. an $B(x, r) \subseteq S$. Thus x interior point of S. Since x was chosen arbitrarily $x \in S$ is an interior point of S. by definition. $\forall$ point of S is an interior $\Rightarrow$ open S

Suppose S is open. Since S is open. Let x be any limit point of S^c. Then $\forall$ $B(x, r)$ x contain a point in S^c, and x is not interior point of S. then $x \in S^c$. Thus $\forall$ limit point is point S^c.

v) $\{x_1, x_2 .. x_n\} \subseteq X$ then $X - \{x_1 .. x_n\}$ is open

consider $X \setminus \{x_1 ... x_n\}$

Let $\delta = \min \{|x - x_1|, |x - x_2| .. (x - x_n|\}$ distance to x to closest,

Since $x, x_1 .. x_n$ is finite set

Then $(x - \delta/2, x + \delta/2)$ contain x and is completely contain inside $X \setminus \{x_1 .. x_n\}$. Since $x \in X \setminus \{x_1 .. x_n\}$ arbitrary

$X \setminus \{x_1 .. x_n\}$ is open

A set U is open in topological space X. if $\forall x$ in U.

$\exists$ open ball of x.

In $X \setminus \{x_1 ... x_n\}$. for any x, it mean x not in $\{x_1 .. x_n\}$

$\therefore$

now consider $B(a, d(x, a)/2)$