Probability and Random Variables

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables

Utility and Probability

Utility

$U(A) > U(B)$ indicates A is preferable to B

$U(A) = U(B)$ " indifferent

Probability

$P(A) > P(B)$ " A is more plausible than B

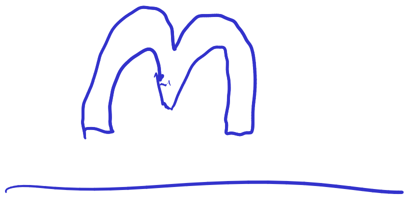
$P(A) = P(B)$ " A and B are equally plausible

Full story: https://projecteuclid.org/journals/statistical-science/volume-1/issue-3/The-Axioms-of-Subjective-Probability/10.1214/ss/1177013611.full

What is a Random Variable?

R.V. X

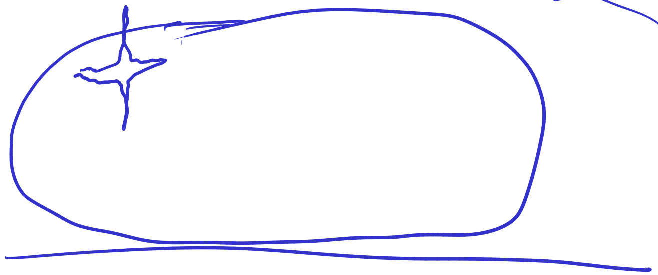
today only! capital



Variable

- finite set of values
- probability

$P(X=1)=0.5$



- Variable
- continuous / discrete
- related to other R.V.

$P(X|Y)$

(Ω, Σ, μ) ← prob space

$X: \Omega \rightarrow E$

Vocabulary/Notation

Vocabulary/Notation

Term	Definition	Coinflip Example	Uniform Example

Vocabulary/Notation

		$\mathrm{Bernoulli}(0.5)$	
Term	**Definition**	**Coinflip Example**	**Uniform Example**

Vocabulary/Notation

Term	Definition	$\text{Bernoulli}(0.5)$ Coinflip Example	$\mathcal{U}(0, 1)$ Uniform Example

Vocabulary/Notation

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support(X)			

Vocabulary/Notation

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support(X)	All the values that X can take		

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Distribution			

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X	**P(X)**
0	0.5
1	0.5

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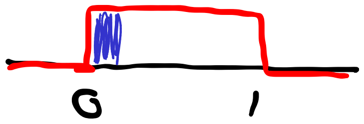
$P(0 \leq X \leq 0.2)$

$\int_0^{0.2} p(x)\,dx$



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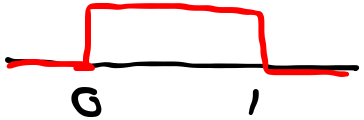
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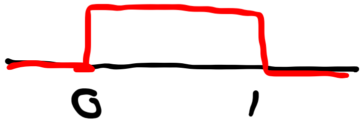

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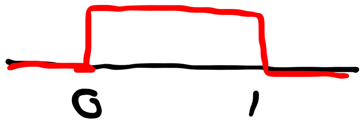
support(X) $x \in X$ $X \in [0,1]$	All the values that X can take	$\{h,t\}$ or $\{0,1\}$	$[0,1]$ 
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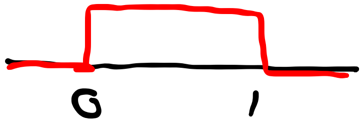 </td></tr>
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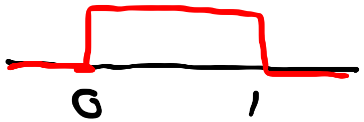 </td></tr>
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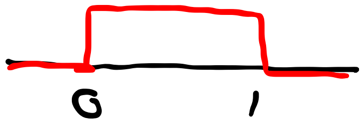 </td></tr>
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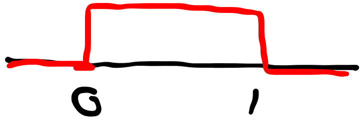 </td></tr>
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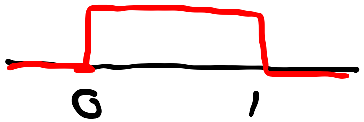 </td></tr>
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Coinflip Example

$\mathcal{U}(0,1)$

Uniform Example

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$X \in [0,1]$

All the values that X can take

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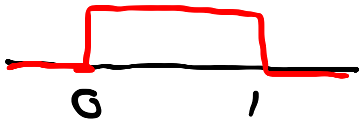

Distribution

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$$P(X = 1) = 0$$

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Expectation

$E[X]$

Single representative value of the random variable, "mean"

$$E[X] = \sum_{x \in X} xP(x)$$
$$= 0.5$$

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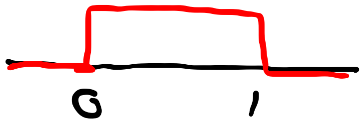 </td></tr>
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Distributions of related R.V.s

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Joint Distribution

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Joint Distribution

$$P(X, Y, Z)$$

Distributions of related R.V.s

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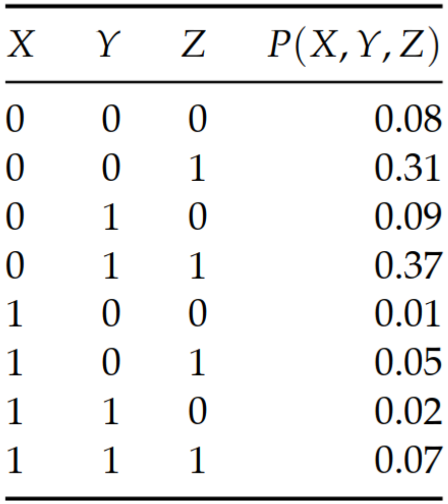
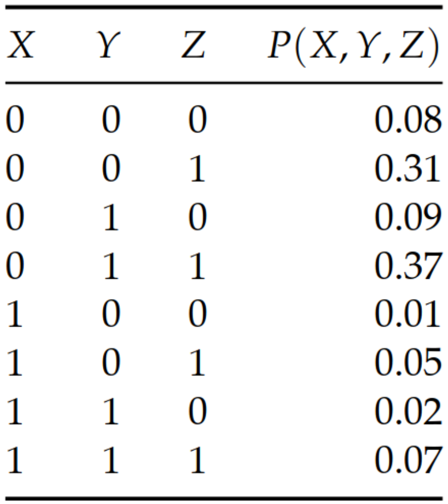


X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$



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Conditional Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

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1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

Distributions of related R.V.s

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1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y{=}1, Z{=}1)$
0	0.84
1	0.16

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
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$$P(X)\ P(Y)\ P(Z)$$

Distributions of related R.V.s

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Conditional Distribution

$P(X \mid Y, Z)$

(Distribution - valued function)

X	$P(X \mid Y{=}1, Z{=}1)$
0	0.84
1	0.16

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

X	$P(X)$
0	0.85
1	0.15

Y	$P(Y)$
0	0.45
1	0.55

Z	$P(Z)$
0	0.20
1	0.80

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

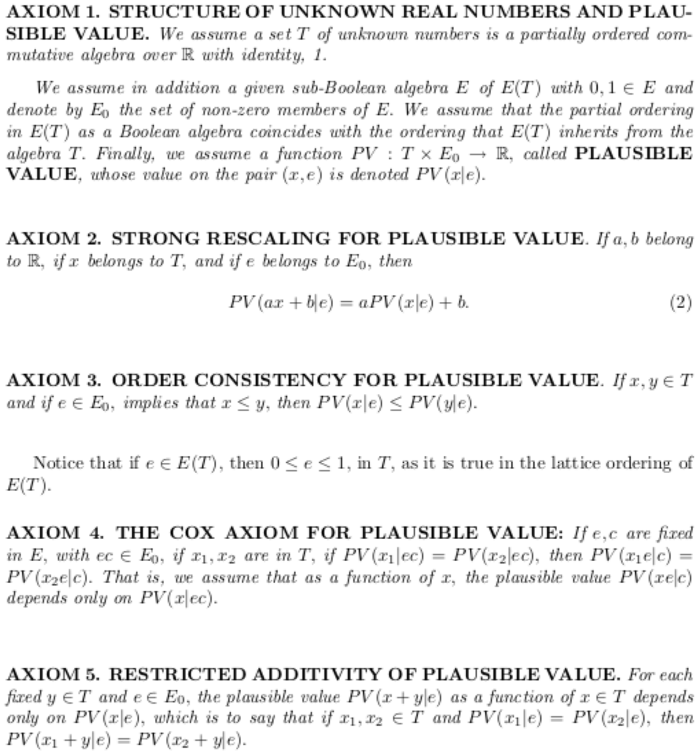
Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

(Filet Minion Level: Axioms of Probability)



AXIOM 1. STRUCTURE OF UNKNOWN REAL NUMBERS AND PLAUSIBLE VALUE. *We assume a set T of unknown numbers is a partially ordered commutative algebra over $\mathbb{R}$ with identity, 1.*

We assume in addition a given sub-Boolean algebra E of $E(T)$ with $0, 1 \in E$ and denote by E_0 the set of non-zero members of E. We assume that the partial ordering in $E(T)$ as a Boolean algebra coincides with the ordering that $E(T)$ inherits from the algebra T. Finally, we assume a function $PV : T \times E_0 \to \mathbb{R}$, called **PLAUSIBLE VALUE**, *whose value on the pair (x, e) is denoted $PV(x|e)$.*

AXIOM 2. STRONG RESCALING FOR PLAUSIBLE VALUE. *If a, b belong to $\mathbb{R}$, if x belongs to T, and if e belongs to E_0, then*

$$PV(ax + b|e) = aPV(x|e) + b. \qquad (2)$$

AXIOM 3. ORDER CONSISTENCY FOR PLAUSIBLE VALUE. *If $x, y \in T$ and if $e \in E_0$, implies that $x \leq y$, then $PV(x|e) \leq PV(y|e)$.*

Notice that if $e \in E(T)$, then $0 \leq e \leq 1$, in T, as it is true in the lattice ordering of $E(T)$.

AXIOM 4. THE COX AXIOM FOR PLAUSIBLE VALUE: *If e, c are fixed in E, with $ec \in E_0$, if x_1, x_2 are in T, if $PV(x_1|ec) = PV(x_2|ec)$, then $PV(x_1e|c) = PV(x_2e|c)$. That is, we assume that as a function of x, the plausible value $PV(xe|c)$ depends only on $PV(x|ec)$.*

AXIOM 5. RESTRICTED ADDITIVITY OF PLAUSIBLE VALUE. *For each fixed $y \in T$ and $e \in E_0$, the plausible value $PV(x + y|e)$ as a function of $x \in T$ depends only on $PV(x|e)$, which is to say that if $x_1, x_2 \in T$ and $PV(x_1|e) = PV(x_2|e)$, then $PV(x_1 + y|e) = PV(x_2 + y|e)$.*

not on exam

Distributions of related R.V.s

Joint Distribution	**Conditional Distribution**	**Marginal Distribution**
$P(X, Y, Z)$	$P(X \mid Y, Z)$	$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

1)

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

(Burrito-level)

3 Rules

1) a) $0 \leq P(X \mid Y) \leq 1$

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$
   b) $\sum_{x \in X} P(x \mid Y) = 1$

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X, y)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

(Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) "Law of total probability"

$$\underline{P(X)} = \sum_{y \in Y} \underline{P(X, y)}$$

Joint → Marginal

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$

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$$P(X) = \sum_{y \in Y} P(X, y)$$

Joint → Marginal

3) Definition of Conditional Probability

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$
   b) $\sum_{x \in X} P(x \mid Y) = 1$
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$$\underline{P(X \mid Y)} = \frac{P(X, Y)}{P(Y)}$$

Joint → Marginal

Joint + Marginal → Conditional

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\, P(Y)\, P(Z)$

3 Rules

(Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

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3) Definition of Conditional Probability

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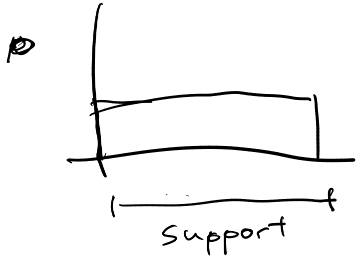
Joint → Marginal

Joint + Marginal → Conditional

Marginal + Conditional → Joint

$$P(X, Y) = P(X|Y)\, P(Y)$$

Distributions of related R.V.s



Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X, y)$$

3) Definition of Conditional Probability

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

X	Y	Z	P(X, Y, Z)
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

$$P(X{=}0) = \sum_{y \in Y,\, z \in Z} P(X{=}0, y, z)$$

X	P(X)
0	0.08 + 0.31 + 0.09 + 0.37 = 0.85
1	$\boxed{0.15}$

$$\sum_{x \in X} P(X{=}x) = 1 = \overset{P(X=0)}{0.85} + P(X{=}1)$$

$$P(X{=}1) = 1 - P(X{=}0)$$

Joint → Marginal

$$P(Y{=}1, Z{=}1 \mid X{=}1) = \frac{P(X{=}1, Y{=}1, Z{=}1)}{P(X{=}1)} = \frac{0.07}{0.15} = 0.47$$

Joint + Marginal → Conditional

Marginal + Conditional → Joint

$$P(X, Y) = P(X|Y)\, P(Y)$$

1) a) $0 \le P(X \mid Y) \le 1$

b) $\sum_{x \in X} P(\overset{X=x}{x} \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$

$$P(X,Y) = P(X|Y)\,P(Y)$$

Break

$$\overset{0.8}{P(C=1)} = \overset{0.1}{P(C=1,P=1)} + \underset{0.7}{\underline{P(C=1,P=0)}}$$

- $P \in \{0,1\}$: Powder Day
- $C \in \{0,1\}$: Pass Clear
- 1 in 5 days is a powder day $P(P=1)=0.2$
- The pass is clear 8 in 10 days $P(C=1)=0.8$
- If it is a powder day, there is a 50% chance the pass is blocked $P(C=0|P=1)$

$P(C=1|P=1) = 1-0.5 = 0.5$

1) b $\sum_{c \in C} P(C=c|P=1) = 1 = P(C=0|P=1) + P(C=1|P=1)$

P	C	
0	0	0.1
0	1	0.7
1	0	0.1
1	1	0.1

$\overset{0.5}{P(C=0|P=1)}\overset{0.2}{P(P=1)}$

$P(C=1|P=1)P(P=1)$

- Write out the joint probability distribution for P and C.
- What is the probability that the pass is blocked on a non-powder day?

$$P(C=0|P=0) = \frac{P(C=0,P=0)}{P(P=0)} = \frac{0.1}{0.8} = 0.125$$

Bayes Rule

- Know: $P(B \mid A)$, $P(A)$, $P(B)$
- Want: $P(A \mid B)$

$$P(A|B) = \frac{P(A,B)}{P(B)} \qquad P(B|A) = \frac{P(A,B)}{P(A)}$$

$$P(A|B)P(B) = P(A,B) = P(B|A)P(A)$$

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

$$P(A|B,C) = \frac{P(B|A,C)\,P(A|C)}{P(B|C)}$$

Independence

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Definition: X and Y are *independent* iff $P(X, Y) = P(X)\,P(Y)$

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$X \perp Y$

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Definition: X and Y are *independent* iff $P(X, Y) = P(X)\,P(Y)$

$X \perp Y$

$P(X|Y) = P(X)$

Independence

Definition: X and Y are *independent* iff $P(X, Y) = P(X)\,P(Y)$

$X \perp Y$

$P(X \mid Y) = P(X)$

Definition: X and Y are *conditionally independent* given Z iff $P(X, Y \mid Z) = P(X \mid Z)\,P(Y \mid Z)$

Independence

Definition: X and Y are *independent* iff $P(X,Y) = P(X)\,P(Y)$

$$X \perp Y$$

$$P(X|Y) = P(X)$$

Definition: X and Y are *conditionally independent* given Z iff $P(X,Y \mid Z) = P(X \mid Z)\,P(Y \mid Z)$

$$X \perp Y | Z$$

$X \perp Y | Z \overset{?}{\Rightarrow} X \perp Y$ No!

$X \perp Y \overset{?}{\Rightarrow} X \perp Y | Z$ No!

Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$

$$P(X, Y) = P(X \mid Y)\, P(Y)$$

Continuous

1)

Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

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3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$

   $P(X, Y) = P(X \mid Y)\, P(Y)$

Continuous

1) $0 \leq p(X \mid Y)$

Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

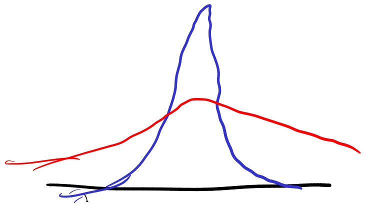
3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$

$$P(X, Y) = P(X \mid Y)\, P(Y)$$

Continuous

1) $0 \leq p(X \mid Y)$

$$\underline{\int_X p(x|Y)\, dx = 1}$$



Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$

$$P(X, Y) = P(X \mid Y)\, P(Y)$$

Continuous

1)

$$0 \leq p(X \mid Y)$$

$$\int_X p(x|Y)\, dx = 1$$

2)

$$p(X) = \int_Y p(X, y) dy$$

Rules for Continuous RVs

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$$P(X,Y) = P(X \mid Y)\, P(Y)$$

Continuous

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$$\int_X p(x|Y)\, dx = 1$$

2)

$$p(X) = \int_Y p(X, y) dy$$

3) $p(X \mid Y) = \frac{p(X,Y)}{p(Y)}$

$$p(X,Y) = p(X \mid Y)\, p(Y)$$

Multivariate Gaussian Distribution

$x = [x_1, x_2]$

$\mathcal{N}(\mu, \Sigma)$

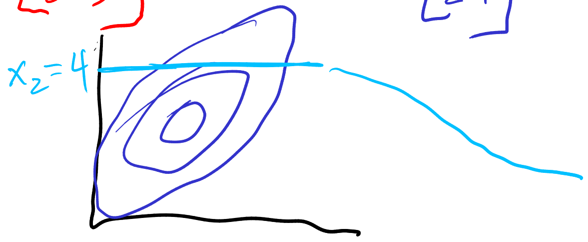
Joint Distribution

$p(x) = \mathcal{N}(x | \mu, \Sigma)$

$$= \frac{\exp\left(\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)}{(2\pi)^{n/2} \, |\Sigma|^{1/2}}$$

$\mu = [3, 3]$ $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$



Conditional Distribution

$p(x_1 | x_2)$

$= \mathcal{N}(x_1 | \bar{\mu}_1, \bar{\Sigma})$

$\bar{\mu} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2)$

$\bar{\Sigma} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$

$p(x_1 | x_2 = 4)$

Marginal Distribution

$p(x_1) = \mathcal{N}(x_1 | \mu_1, \Sigma_{11})$

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables