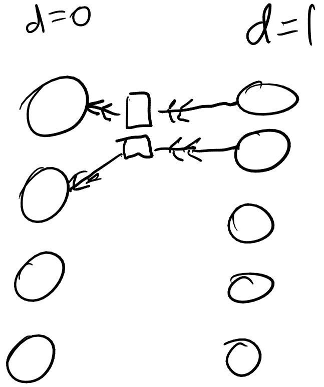

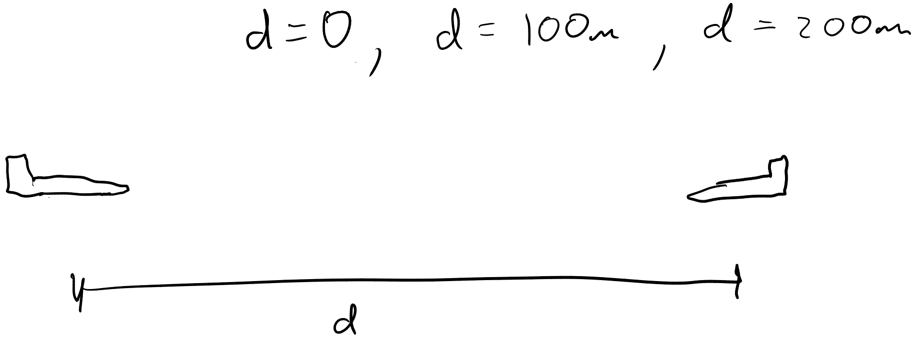

Value Iteration Convergence

Review

$Q^{\pi}(s,a)$

$V^{\pi}(s)$ $U^{\pi}(s)$

Review

Value function

- How do we reason about the **future consequences** of actions in an MDP?

Review

- How do we reason about the **future consequences** of actions in an MDP?
- What are the basic **algorithms for solving MDPs**?

Offline   Value Iteration
Policy Iteration

Online   MCTS $\leftarrow$ insensitive $|S|$
FSSS

Guiding Questions

Guiding Questions

- Does value iteration always converge?
- Is the optimal value function unique?
- Can there be multiple optimal policies?
- Is there always a deterministic optimal policy?

Value Iteration: The Bellman Operator

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Algorithm: Value Iteration

while $\|V - V'\|_\infty > \epsilon$

$\quad V \leftarrow V'$

$\quad V' \leftarrow B[V]$

return V'

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$$B[V](s) = \max_{a \in A} \left(R(s,a) + \gamma E\left[V(s')\right] \right)$$

Value Iteration Convergence

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Theorem 1: Let $\{V_1, \dots, V_\infty\}$ be a sequence of value functions for a discrete MDP generated by the recurrence $V_{k+1} = B[V_k]$. If $\gamma < 1$, then $\lim_{k\to\infty} V_k = V^*$.

Metrics

Metrics

$d(x,y)$ $x, y \in M$

Definition: Let M be a set. A *metric* on M is a function $d : M \times M \to [0, \infty)$ which satisfies the following three conditions for all $x, y, z \in M$:

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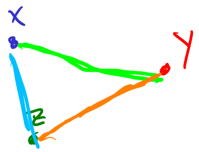
$M = \mathbb{R}^2$



$d(x,y) = \|x - y\|$

Contraction Mappings

Contraction Mappings

Definition: A *contraction mapping* on metric space (M, d) is a function $f : M \to M$ satisfying

$$d(f(x), f(y)) \leq \alpha\, d(x, y)$$

for some α, $0 \leq \alpha \leq 1$ and all x and y in M.

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$$f(x) = \begin{bmatrix} \frac{x_2}{2} + 1 \\ \frac{x_1}{2} + \frac{1}{2} \end{bmatrix}$$

Script: contraction_mapping.jl

Banach's Theorem

Banach's Theorem

If we can prove
$d(f(x), f(y)) \leq \alpha \cdot d(x,y)$
then
convergence to fixed point

Theorem (Banach): If f is a contraction mapping on metric space (M, d), then

1. f has a single, unique fixed point x^*.
2. If $\{x_k\}$ is a sequence defined by $x_{k+1} = f(x_k)$, then $\lim_{k\to\infty} x_k = x^*$.

B

Max Norm

Max Norm

Lemma 1: $(\mathbb{R}^{|S|}, \|\cdot\|_\infty)$ is a metric space.

$V \in \mathbb{R}^{|S|}$ ← size of state space

Max Norm

$\|x\|_\infty = \max |x_i|$

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Bellman Operator Contraction

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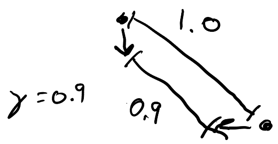

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&= \gamma \|V_1 - V_2\|_\infty \max_{s\in S, a\in A} \sum_{s'\in S} T(s'|s,a) \quad = 1 \\
&= \gamma \|V_1 - V_2\|_\infty
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Value Iteration Convergence

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Theorem 1: Let $\{V_1, \ldots, V_\infty\}$ be a sequence of value functions for a discrete MDP generated by the recurrence $V_{k+1} = B[V_k]$. If $\gamma < 1$, then $\lim_{k\to\infty} V_k = V^*$.

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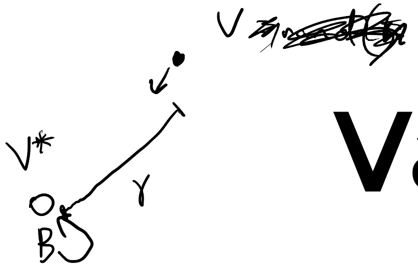
By Lemma 2 and Banach's theorem (part 2), repeated application of the Bellman operator always has a fixed point limit, $\hat{V}$.



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$\|V_k - V^*\| < (\bar{V} - \underline{V})\gamma^k$

Theorem 1: Let $\{V_1, \ldots, V_\infty\}$ be a sequence of value functions for a discrete MDP generated by the recurrence $V_{k+1} = B[V_k]$. If $\gamma < 1$, then $\lim_{k\to\infty} V_k = V^*$.

Proof:

Lemma 2: B is a γ contraction mapping on $(\mathbb{R}^{|S|}, \|\cdot\|_\infty)$.

Theorem (Banach): If f is a contraction mapping on metric space (M, d), then

1. f has a single, unique fixed point x^*.
2. If $\{x_k\}$ is a sequence defined by $x_{k+1} = f(x_k)$, then $\lim_{k\to\infty} x_k = x^*$.

By Lemma 2 and Banach's theorem (part 2), repeated application of the Bellman operator always has a fixed point limit, $\hat{V}$.

By Banach's theorem (part 1), $\hat{V} = B[\hat{V}]$. Since $\hat{V}$ satisfies Bellman's equation, it is optimal and $\hat{V} = V^*$.

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4. By (1), (2), and (3), the policy will improve until it finds the optimal policy, and it will always find the optimal policy.

Is there always a deterministic optimal policy?

<u>Thm</u>

For any finite MDP, there is always at least 1 deterministic optimal policy

Suppose $\pi^*(a|s)$ is a nondeterministic optimal policy. Then

$$V^*(s) = \sum_{a \in \text{support}(\pi^*(a|s))} \pi^*(a|s) \underbrace{\left(R(s,a) + \gamma \sum_{s' \in S} T(s'|s,a) V^*(s') \right)}_{Q^*(s,a)}$$

Lemma 3: If $a' \in \text{support}(\pi^*(a|s))$ then $Q^*(s,a') = V^*(s)$

Proof: (by contradiction) w.l.o.g. suppose that $Q^*(s,a^1) > Q^*(s,a^2)$ and $\text{support}(\pi(a|s)) = \{a^1, a^2\}$

Let π' be same as π^* except that $\pi'(s) = a^1$

$$V^{\pi'}(s) = Q^*(s,a^1) > \pi^*(a^1|s)\, Q^*(s,a^1) + \pi^*(a^2|s)\, Q^*(s,a^2) = V^*(s)$$

Let π'' be a deterministic policy with $\pi''(s) \in \text{support}(\pi^*(a|s))$.

By Lemma 3, $Q^*(s,\pi''(s)) = V^*(s)$ at every state so π'' is optimal.

Guiding Questions

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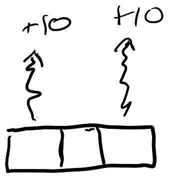
B is a contraction mapping

- Does value iteration always converge? Yes
- Is the value function unique? Yes
- Can there be multiple optimal policies? Yes
- Is there always a deterministic optimal policy? Yes



Break

Conservation MDP