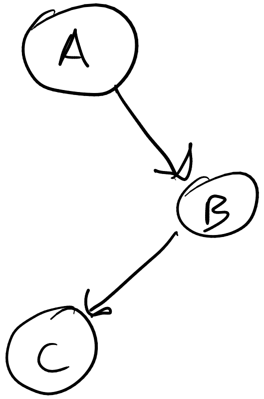
Stochastic Processes and Simple Decisions

Review



Chain Rule

$$P(\underbrace{\cancel{A,B,C}}_{X_{1:n}}) = \prod P(X_i \mid pa(X_i))$$

Structure ← sometimes prove conditional independence

Parameters : Θ determine values of conditional dist

$$P(B=1 \mid A=3) = \Theta_{1,3}$$

$G = \{B\}$

$A \perp C \mid B$

$$P(C=1 \mid A=0, B=2)$$
$$= P(C=1 \mid B=2)$$

Evidence

$$P(B=1 \mid C=3, D=0)$$

"evidence variables"

G

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?

Stochastic Process

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$$x' = x + v$$

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In a *stationary* stochastic process (all in this class), this relationship does not change with time

Example:

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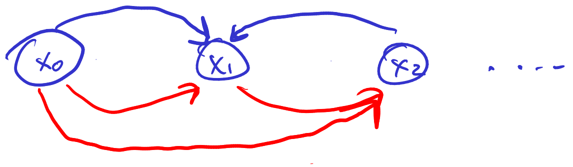
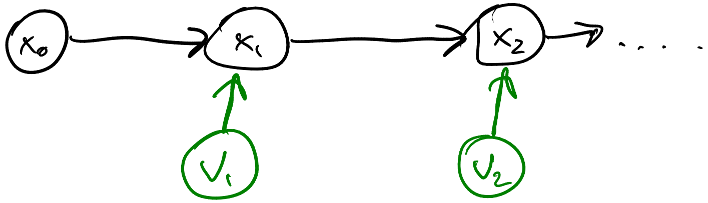
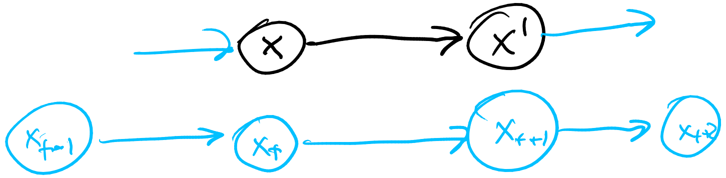
In general



for this particular s.p.



dynamic Bayesian network



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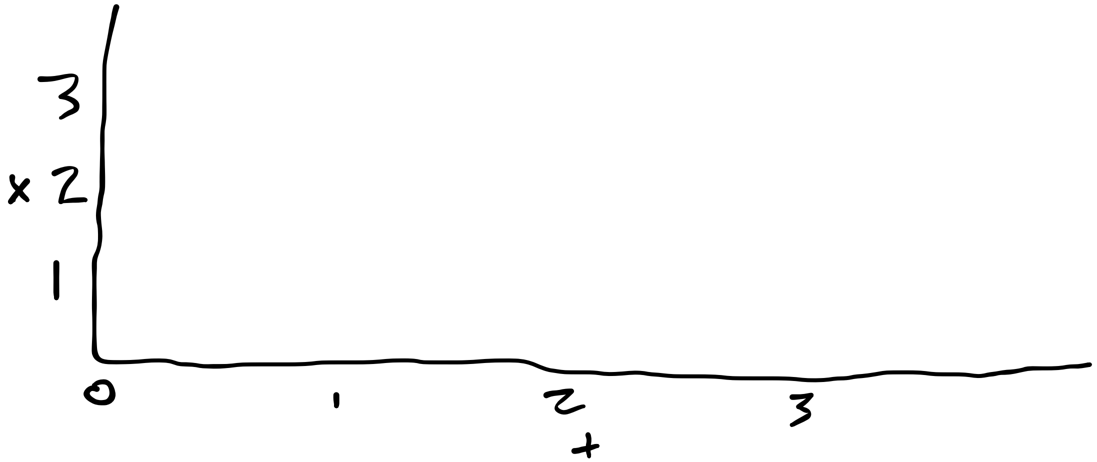

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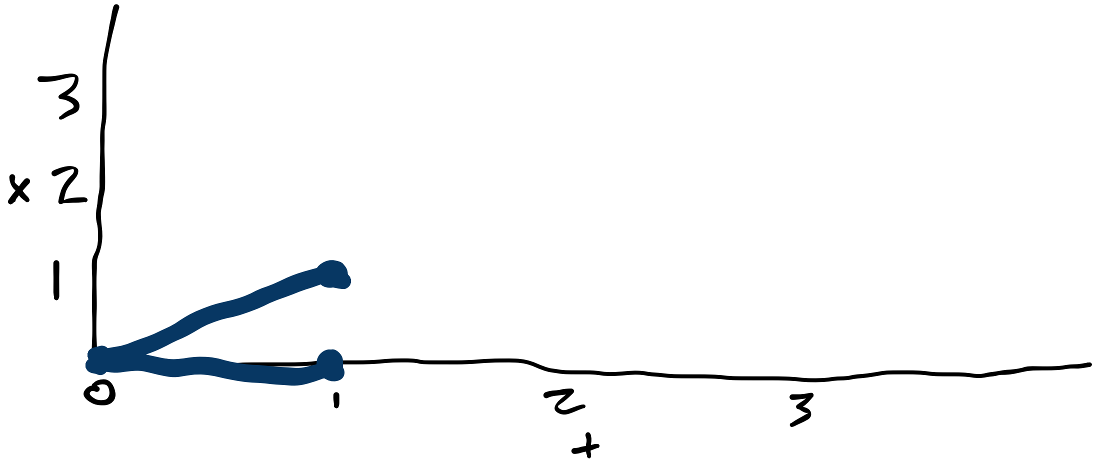

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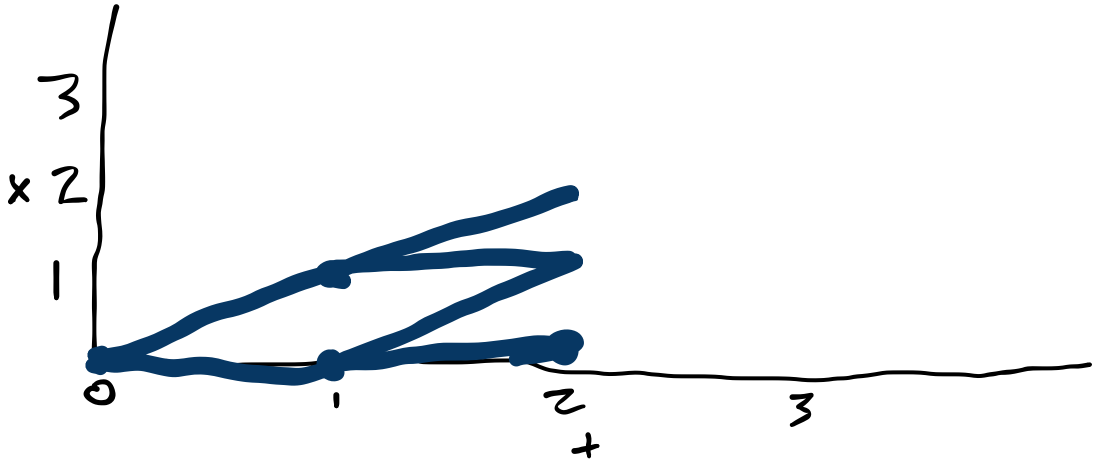

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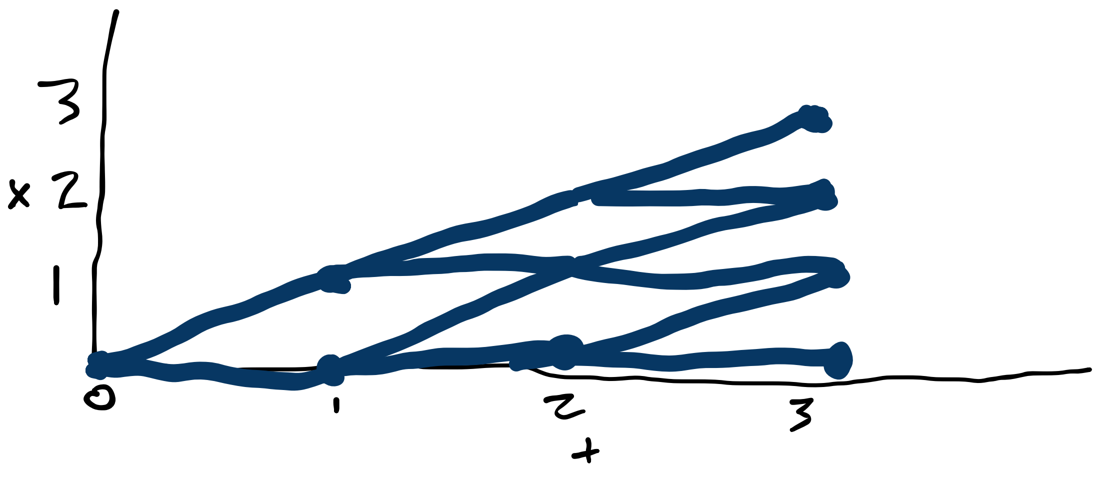

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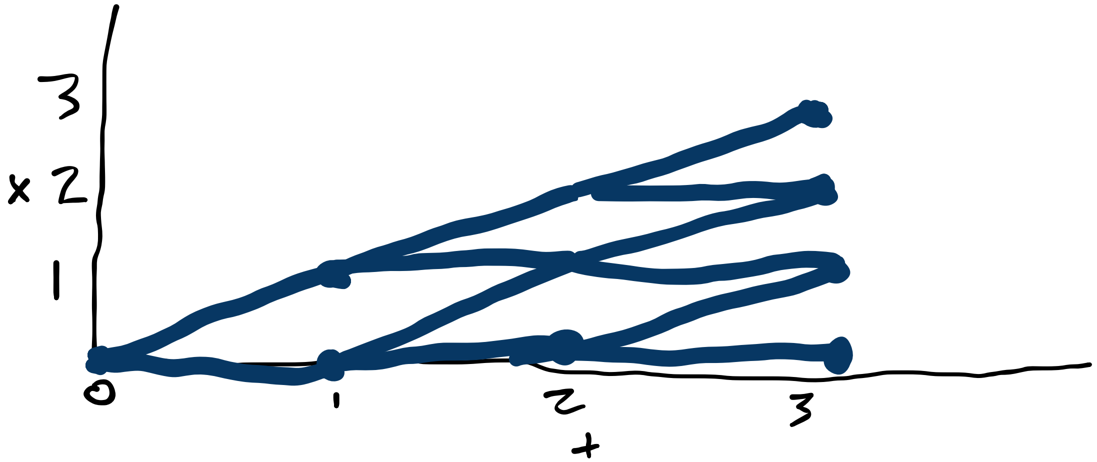

$$P(x_{1:n}) = \prod_{t=1}^{n} P(x_t \mid \mathrm{pa}(x_t))$$

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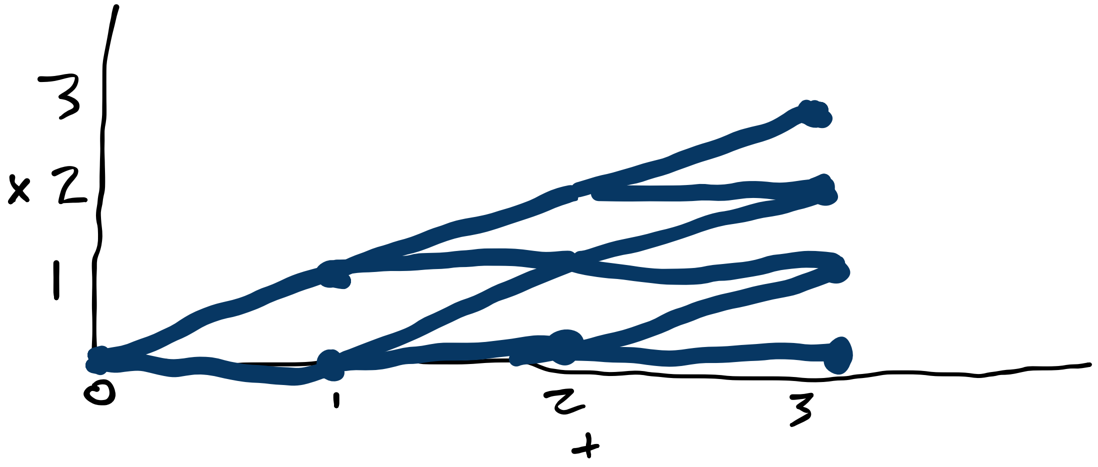

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For this particular process,

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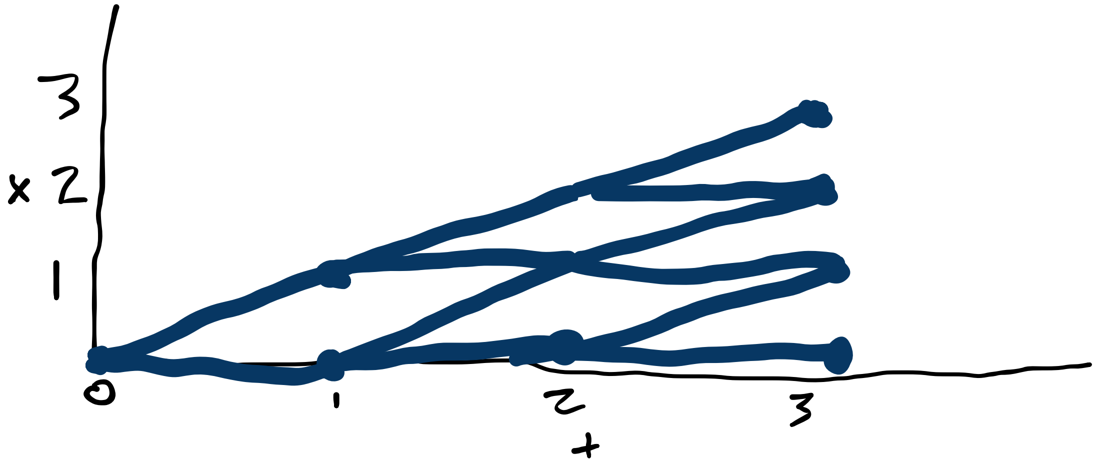

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Joint

x0	x1	x2	P(x1, x2, x3)
0	0	0	0.25
0	0	1	0.25
0	1	1	0.25
0	1	2	0.25

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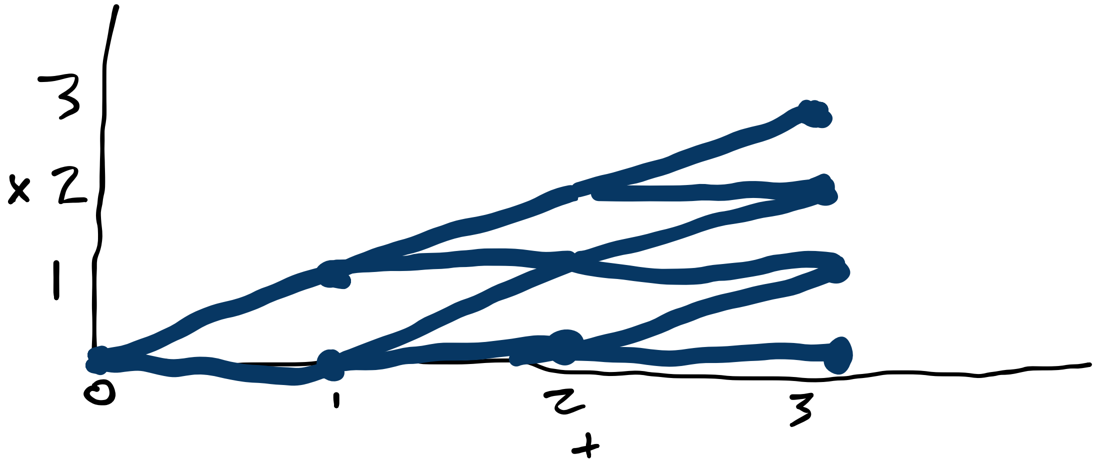
$v_t \sim \mathcal{U}(\{0,1\})$ (i.i.d.)



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Joint

$P(x_2)$

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0	0	1	0.25
0	1	1	0.25
0	1	2	0.25

Marginal $\qquad$ $P(x_t)$

For this particular process, since $\mathrm{pa}(x_t) = x_{t-1}$, if $P(x_{t-1})$ is known,

$$P(x_t) = \sum_{k \in x_{t-1}} P\left(x_t \mid x_{t-1} = k\right) P(x_{t-1} = k)$$

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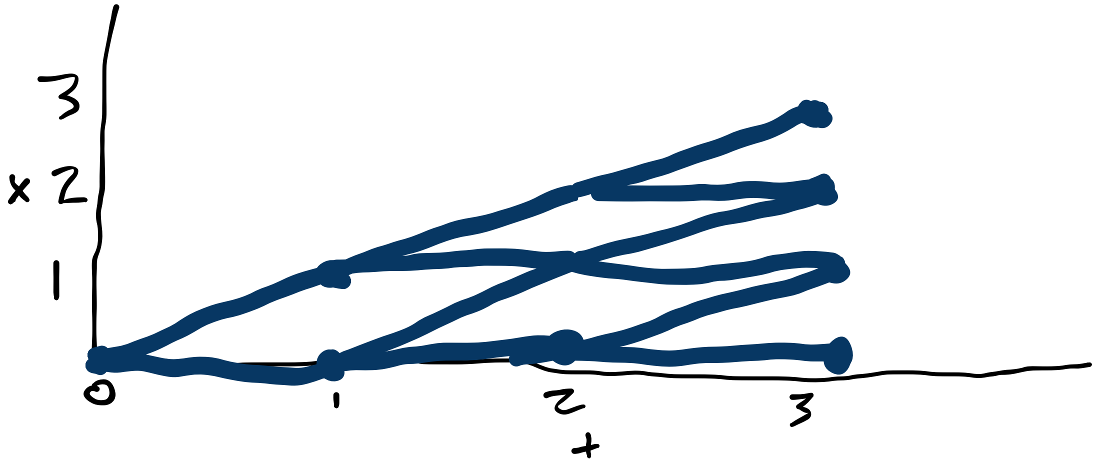

$$P(x_{1:n}) = \prod_{t=1}^{n} P(x_t \mid \mathrm{pa}(x_t))$$

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$$= 0.5\, P(x_{t-1} = x_t - 1) + 0.5\, P(x_{t-1} = x_t)$$

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Expectation

$$E[x_t] = \sum_{x \in x_t} x P(x_t = x)$$

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For this particular process, $x_t = \sum_{i=1}^{t} v_t$, so

$$E[x_t] = E\left[\sum_{i=1}^{t} v_t\right] = \sum_{i=1}^{t} E[v_t] = 0.5t$$

Stochastic Process

Expectation

$$E[x_t] = \sum_{x \in x_t} x P(x_t = x)$$

Expectation of a function
(such as reward)

$$E[f(x_t)] = \sum_{x \in x_t} f(x) P(x_t = x)$$

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Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

Markov Process

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- A stochastic process $\{s_t\}$ is *Markov* if
  $P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$

$s_{t+1} \perp s_i \mid s_t$

$\forall\, i \in \{0 \ldots t-1\}$

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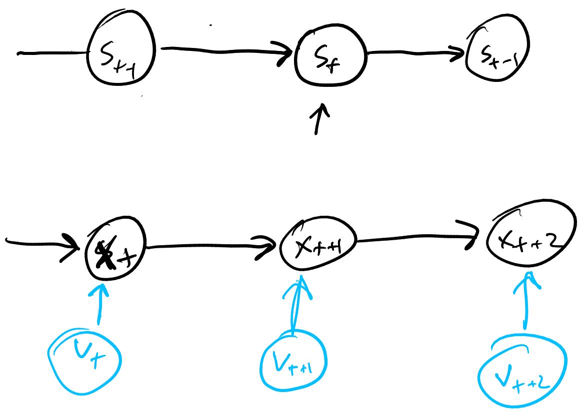
  $$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$
- s_t is called the "state" of the process

$s_{t+1} \perp s_i \mid s_t$

$\forall i < t$



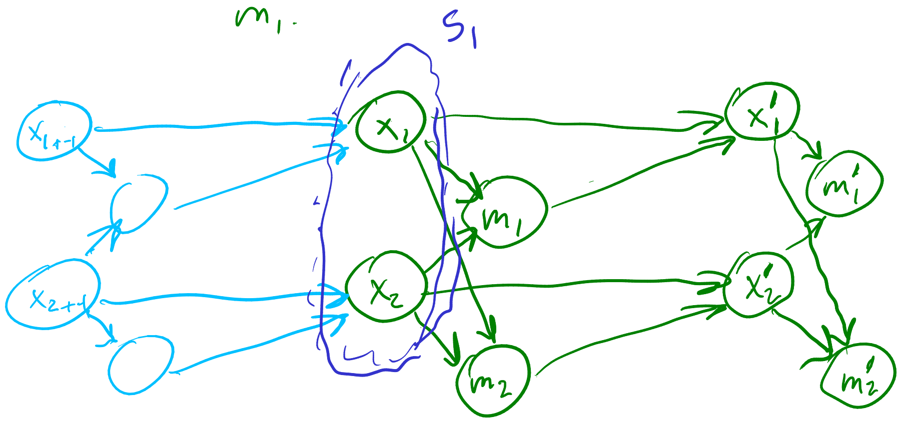
ACAS

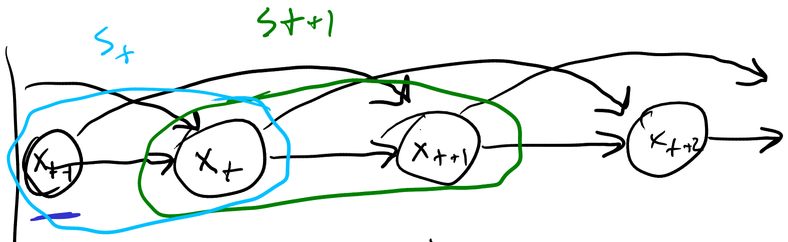
x_1 = physical state of A/C 1

x_2 = " " A/C 2

m_1 = A/C 1 maneuvers

m_1..





$x_{t+1} \perp x_{t-1} \mid x_t$ ✗

$\{x_t\}$ Not Markov

$s_t = \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix}$

$\{s_t\}$ is Markov

Break

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Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

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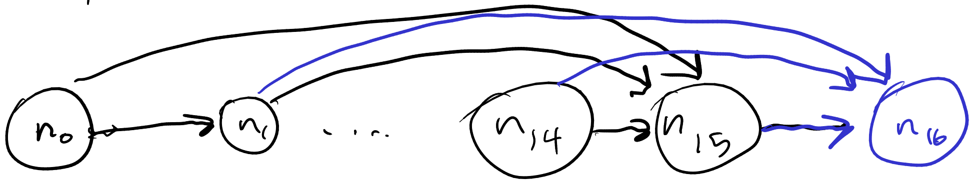
Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 14 days after they contract the disease.
- The number of people infected by each person on day d of their illness is roughly $\mathcal{N}(\mu_d, \sigma^2)$

$n_t =$ number of new infections on t



$\{n_t\}$ not Markov

$$s_t = \begin{bmatrix} n_{t-14} \\ \vdots \\ n_t \end{bmatrix}$$

$\{s_t\}$ is Markov

$$s_{t+1} = \begin{bmatrix} n_{t-13} \\ \vdots \\ n_{t+1} \end{bmatrix}$$

$n_{t+1} \sim \mathcal{N}\left(\sum n_d \mu_d, \text{some variance}\right)$

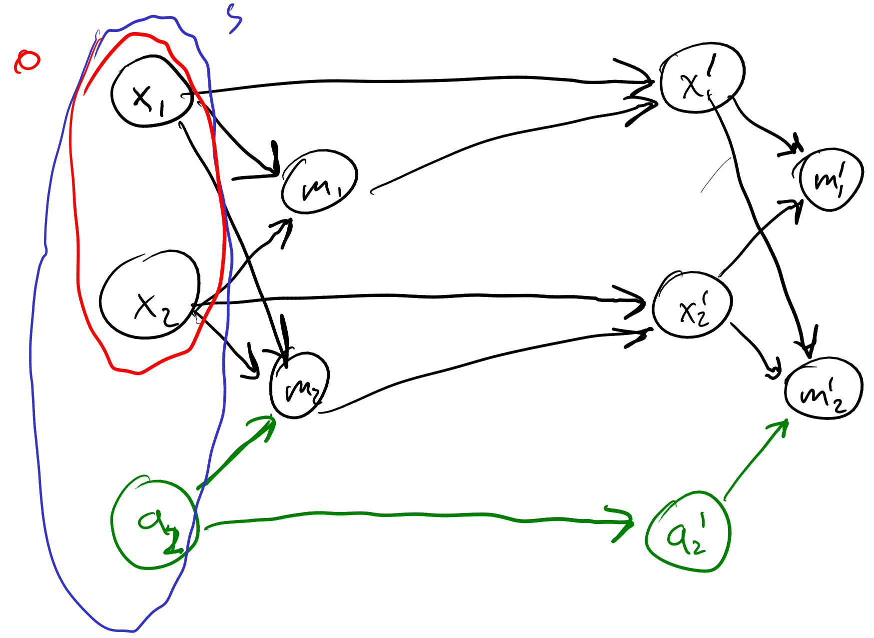
Hidden Markov Model

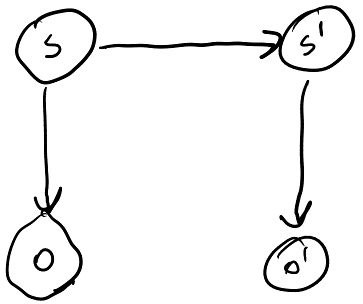
(Often you can't measure the whole state)

ACAS

a_2 = pilot paying attention

DBN





Simple Decisions

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Outcomes

$S_1 \ldots S_n$

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Probabilities

$p_1 \ldots p_n$

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Lottery

$[S_1 : p_1; \dots; S_n : p_n]$

Simple Decisions

Outcomes	Probabilities	Lottery
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- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$

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von Neumann - Morgenstern Axioms

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von Neumann - Morgenstern Axioms

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von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$

Simple Decisions

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Probabilities $p_1 \dots p_n$

Lottery $[S_1 : p_1; \dots; S_n : p_n]$

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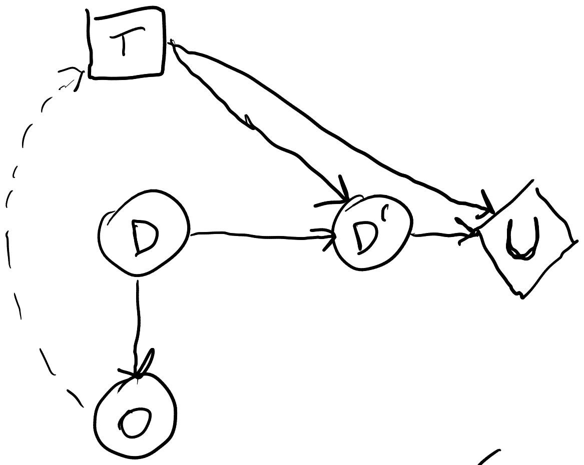
- $U(A) > U(B)$ iff $A \succ B$
- $U(A) = U(B)$ iff $A \sim B$
- $U([S_1 : p_1; \dots; S_n : p_n]) = \sum_{i=1}^{n} p_i U(S_i)$

Decision Networks

D = disease
T = treatment
U = utility
O = observation



□ Action Node
○ Chance Node
◇ Utility

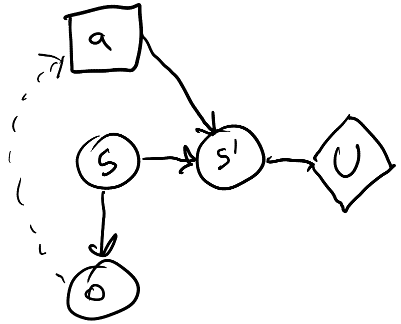
ends in ○

Conditional Edge ⟶
Functional Edge ⟶
Informational Edge - - - →

end in □

$$E[A|B] = \sum_{a \in A} a\, P(A=a|B)$$

Simple decision



$$EU(a|o) = E\left[U(s') \mid a, o\right]$$
$$= \sum_{s'} P(s'|a,o)\, U(s')$$

principle of maximum EU

$$a^* = \underset{a \in A}{\operatorname{argmax}}\, EU(a|o)$$

Markov Decision Process

Finite MDP Objectives

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1. Finite time

Finite MDP Objectives

1. Finite time $$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

Finite MDP Objectives

1. Finite time
$$\mathrm{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward
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4. Terminal States

Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?