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Curve Fitting Based on Interactive Fuzzy Data

Ajuste de Curvas para Dados Fuzzy Interativos

Campinas

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"To teach effectively a teacher must develop a feeling for his subject: he cannot make his students sense its vitality if he does not sense it himself"

(George Pólya)

"Nunca deixe que lhe digam que não vale a pena acreditar no sonho que se tem"

(Renato Russo)

Resumo

Esta tese se dedica ao estudo de problemas de ajuste de curvas fuzzy. Os dados considerados possuem entradas *crisp* e saídas fuzzy, e os métodos propostos fornecem uma aproximação, que se trata de uma função a valores fuzzy. Primeiramente, considera-se que os dados fuzzy possuem memória do tipo linear, que na proposta deste trabalho, serão modelados por números fuzzy linearmente interativos. Para esse tipo de dados o método proposto consiste em estender a solução clássica de quadrados mínimos utilizando uma generalização da Extensão de Zadeh, a Extensão Sup-J, em que J é a distribuição que relaciona os dados e dá origem a relação de interatividade. São apresentados vários exemplos, nos quais são produzidas aproximações para dados fuzzy triangulares, trapezoidais e gaussianos, e também são exibidos exemplos de aproximações não-lineares. Em seguida, considerando a classe de números fuzzy A-linearmente interativos, prova-se que estender a solução clássica de fato produz a solução que melhor aproxima o conjunto de dados. Também define-se uma nova métrica baseada em uma imersão no espaço de funções quadrado integráveis. Com isso, introduz-se a noção de números fuzzy quasi linearmente interativos. Considerando tal classe de dados, usa-se o método de quadrados mínimos fuzzy apresentado anteriormente para aproximar o número de novos infectados por HIV. Posteriormente se faz uma discussão sobre equações lineares com coeficientes *crips* e variáveis fuzzy, na qual se discute o papel da interatividade para a existência de soluções. Por fim, assumindo que os dados são números fuzzy triangulares interativos segundo a distribuição de possibilidade conjunta J_0, propõe-se a aproximação dos dados pela resolução de sistemas lineares.

Palavras-chave: ajuste de curvas fuzzy. números fuzzy linearmente interativos. dados fuzzy interativos. método de quadrados mínimos fuzzy.

Abstract

This thesis addresses the problem of curve fitting fuzzy data. Data has crisp inputs and fuzzy outputs and the proposed method produces an approximation for data, which is given by a fuzzy-valued function. Firstly, data has linear memory, which in this thesis will be modeled by linearly correlated fuzzy numbers. For this typo of data, the curve fitting method consists of extending the classical solution of least square problems by the Sup-J Extension, a generalization of Zadeh's Extension, where J is the joint possibility distribution that gives raise to the linear interactivity. Several examples are provided, including triangular, trapezoidal and gaussian fuzzy numbers, and non-linear approximations. Next, the class of A-linearly interactive fuzzy numbers is presented, whereby it was possible to prove that the first proposed method produces the best approximation, indeed. Also, a new metric is introduced. The metric is based on the norm of square-integrable functions. From this notion, quasi linearly interactive fuzzy numbers are defined. Data of people newly infected with HIV is considered as quasi linearly interactive fuzzy data and the curve fitting method is applied. After that, fuzzy linear equations are reviewed, highlighting the main role of interactivity. Finally, data is considered as being triangular and related with the J_0 joint possibility distribution, and the approximation is calculated by means of fuzzy linear systems.

Keywords: fuzzy curve fitting. linearly interactive fuzzy numbers. interactive fuzzy data. fuzzy least square method.

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List of abbreviations and acronyms

FLSM	Fuzzy Least Square Method
FSLE	Fuzzy Systems of Linear Equations
JPD	Joint Possibility Distribution
LI	Linearly Interactive Fuzzy Numbers
A-LI	A-Linearly Interactive Fuzzy Numbers
QLI	Quasi Linearly Interactive Fuzzy Numbers
SLI	Strongly Linearly Interactive Fuzzy Numbers
WHO	World Health Organization
COVID-19	Coronavirus Disease 2019
HIV	Human Immunodeficiency Virus
FOP	Fuzzy Optimization Problem
ETS	Emissions Trading System
MAC	Marginal Abatement Cost
SCC	Social Cost Curve

List of symbols

A, B, C	Capital letters for fuzzy sets
μ_A	Membership function of the fuzzy set A
χ_A	Characteristic function of the set A
a_α^-, a_α^+	Endpoints of the fuzzy number A
$\widehat{f}$	Zadeh extension of the function f
$\mathcal{F}(X)$	Class of fuzzy sets of X
$\mathbb{R}_\mathcal{F}$	Class of fuzzy numbers
$\mathbb{R}_{\mathcal{F}_C}$	Class of fuzzy numbers with continuous endpoints
cl	Closure
conv	Convex hull
supp	Support
$\bigwedge$ (or $\wedge$)	Infimum
$\bigvee$ (or $\vee$)	Supremum
$\langle \cdot, \cdot \rangle$	Inner product

Contents

Introduction

Mathematics field always had problems to determine nature phenomena precisely. In general, the systems are much too complex to draw conclusion, or much too simple but far from reality. In both cases mathematical models fall short [29]. The Fuzzy Set Theory arouse in this context.

In 1965, Lofti A. Zadeh introduced the definition of fuzzy sets [127]. Since then, applications have been made in various different areas, such as fuzzy neural network [80], fuzzy differential equations [4] and fuzzy optimization problems [16].

Fuzzy least squares method (FLSM) is part of curve fitting problem. The objective is to minimize the distance between the fitting function ϕ and the data D. Fitting function has fuzzy image and data is given by pairs $D = \{(x_1, Y_1), \ldots, (x_m, Y_m)\}$, where each Y_i is a fuzzy set, $i = 1, \ldots, m$. In this context, all the outputs are defuzzified, and after that the least square method is applied.

The first study in fuzzy least squares method is due to Tanaka *et al.* [106] who proposed a fuzzy least squares method based on fuzzy regression models. They minimize the spread between the approximation and triangular and symmetrical data. This method was used to find fuzzy parameters of a fuzzy linear function from a fuzzy data. However, this approach converts the problem to a classical linear programming problem which may lead to lack of the notion of distance between the fuzzy data and the obtained solution.

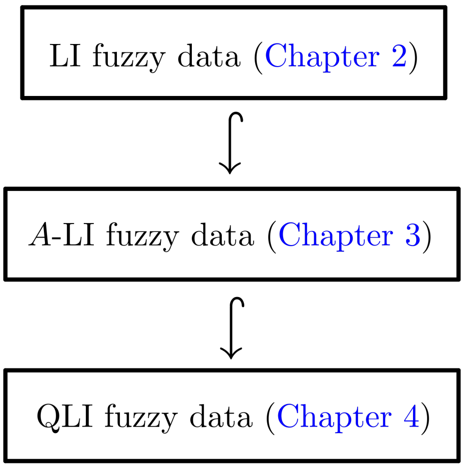
Celmiņš [18] proceeded with the same methodology of Tanaka *et al.* [106] but considered an intrinsic relation among the dataset based on conical membership functions that are (geometrically) similar to joint possibility distributions. In addition, the concept of interactive fuzzy numbers [27, 17, 43] was only considered by Tanaka and Ishibuchi [104], which improved the approach presented by Celmiņš [18].

In contrast to these previous methods, Diamond [23] proposed a fuzzy least squares method based on distance between functions. He proposed a metric based on the space of square-integrable continuous functions. Also, he used projection theorems for cones in Banach spaces to find the fuzzy linear function that best fit data.

It is worth noting that all these approaches were developed only for data given by triangular fuzzy numbers and only providing fuzzy linear functions as solutions. Nevertheless, these methods can be used to model many phenomena, for example, in economy [123], psychology [103], medicine [95], and logistics [108].

This thesis focuses on least squares problems. The data to be fitted is considered being interactive. This study contains two points of view. First, the classical method is extended by a generalization of Zadeh's extension principle. Different types of interactivity are considered in this process, as one can see in Figure 1, namely, linearly interactive (LI) data, A-linearly interactive (A-LI) data and quasi linearly interactive (QLI) data.

Figure 1 – Diagram of the types of the addressed interactivity in Chapters 2, 3 and 4



Types of interactivity considered in Chapters 2, 3, 4 and 5. Hook arrow symbol stands for inclusion and usual arrow represents an existent relation between linearly interactive fuzzy numbers and J_0-interactive fuzzy data. Source: Author.

Chapter 1 presents the mathematical background. In particular, it addresses the necessary topics on linear interactivity required to comprehend the following chapters. Chapter 2 introduces a method for fuzzy curve fitting. The Sup-J Extension is applied to the classical solution of least square problems for linearly interactive (LI) fuzzy data. This method is exemplified by several cases, including nonlinear approximations for triangular fuzzy data and ozone data from longitudinal study on air pollution.

Chapter 3 considers the structure of A-linearly interactive (A-LI) fuzzy numbers to prove that the approximation suggested in Chapter 2 is indeed the solution that best fits the dataset. In fact, the method presented in Chapter 3 is proved to be an extension of the method proposed in Chapter 2. Moreover, several examples are presented, in order to show the advantages of the method. The proposed method models the trajectory sketched by the data as well as the uncertainty of the data itself. Moreover, it can be applied not only for triangular fuzzy numbers, but for any type of completely correlated fuzzy numbers. In addition, this approach can be applied to adjust data by nonlinear functions. For these reasons, the method introduces new approach to FLSM.

In contrast to the usual Hausdorff metric, Chapter 4 establishes a metric for the class of fuzzy numbers with continuous endpoints. This pseudonorm is defined by the inner product for square-integrable continuous functions. The purpose of this pseudonorm is to introduce the notion of quasi linearly interactive (QLI) fuzzy numbers. This new concept allows one to address a broader type of data in FLSM.

The second approach attempts to fit interactive data by means of fuzzy linear systems, considering operations given by a broader type of interactivity, given by the relation J_0. Chapter 5 changes the point of view towards the second approach. Fuzzy linear equations are discussed, highlighting the role of interactivity. The arithmetic operations of the corresponding fuzzy linear systems are given by the Sup-J_0 Extension of the classical arithmetic operations. Based on these ideas, a fuzzy squared linear system that is motivated by the normal equation from the classical least square method is solved.

Finally, Apêndice A contains a study about the fuzzy set approach to optimization problems. In particular, it has a discussion about the state of art, the meaning of fuzzy inequalities and constraints, and the definition of a solution to the optimization problem. Furthermore, it shows a model to the carbon market policy viewed as a constraint in a marginal abatement cost problem.

1 Fuzzy Set Theory

The Fuzzy Set Theory [127] was introduced in order to express subjectivity and uncertainty presented in phenomena of nature, in mathematical terms. It is used to describe linguistic variables, such as “high”, “low”, “around”, “close to”, which are difficult to determinate with the classical set theory. It can be also used to model uncertain quantities, when one has only a piece of information and, therefore, one can not attach a precise value or quantity.

This first chapter is dedicated to introduce some concepts on Fuzzy Set Theory, allowing the reader to go forward in the next chapters. It is based on references [4, 10, 58, 80], which are good guides for novice readers. This chapter is organized as follows. Section 1.1 and Section 1.2 contain basic concepts of fuzzy sets and fuzzy numbers. Section 1.3 establishes the basis for Section 1.4 which explains the definition and the interpretation of interactive fuzzy numbers. Section 1.5 and Section 1.6 are dedicated to build the arithmetic on interactive fuzzy numbers, in particular, on linear interactive fuzzy numbers. Finally, Section 1.7 briefly discusses fuzzy norms.

1.1 Fuzzy Sets

A fuzzy subset A of a universe X is characterized by a membership function

$$\mu_A : X \to [0,1].$$

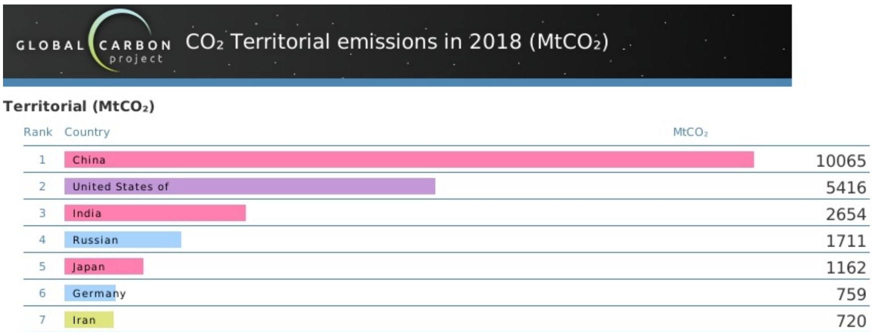
Each value $\mu_A(x) \in [0,1]$ or, simply, $A(x) \in [0,1]$, represents the degree that the element x belongs the set A, for all $x \in X$. Note that the universe X can be seen as a fuzzy set whose membership function is given by $\mu_X(x) = 1$ for all $x \in X$. Similarly, the empty set $\varnothing$ can be seen as the fuzzy set defined by $\mu_\varnothing(x) = 0$, for all $x \in X$.

For example, one can consider A as the set of countries that emitted many million tons of carbon dioxide ($MtCO_2$) in 2018. According to Global Carbon Atlas [44], as can be seen in Figure 2, China definitely is in the set A, because it is the country that most emitted $MtCO_2$.

On the other hand, other countries, such as Germany and Iran, may or may not be in the set A. They emitted much less CO_2 in 2018, in comparison with China, but still being the sixth and seventh in the list. So it is possible to assign degrees to which they are in the set A. Germany has degree bigger than Iran, but smaller than the top five countries.

Classical subsets A of X are particular fuzzy sets whose membership functions are given by their characteristics function $\chi_A : X \to \{0,1\}$, where $\chi_A(x) = 1$ if x belongs to A

Figure 2 – Emissions of CO_2 in 2018 by countries



Top seven countries that most emitted CO_2 in 2018. Source: Global Carbon Atlas [44].

and $\chi_A(x) = 0$ if x is not an element of A. Classical subsets are also called *crisp* subsets. The class of fuzzy sets of X is represented by $\mathcal{F}(X)$.

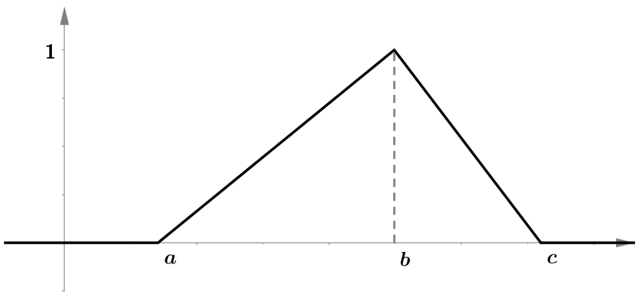
Triangular, trapezoidal and Gaussian fuzzy sets are commons types of fuzzy sets.

Definition 1.1 (Triangular fuzzy set)**.** *A fuzzy set A of $\mathbb{R}$ is said to be fuzzy triangular, and is denoted by $A = (a; b; c)$ for $a \leqslant b \leqslant c$, if its membership function is given by*

$$A(x) = \begin{cases} 0, & \text{if } x < a \\ \dfrac{x-a}{b-a}, & \text{if } a \leqslant x \leqslant b \\ \dfrac{x-c}{b-c}, & \text{if } b < x \leqslant c \\ 0, & \text{if } x > c \end{cases}. \tag{1.1}$$

This type of fuzzy set is depicted in Figure 3.

Figure 3 – Triangular fuzzy set $A = (a; b; c)$



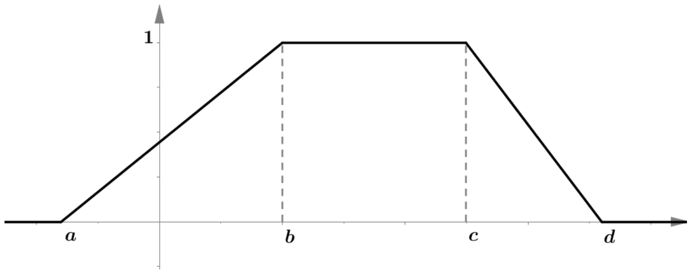
A triangular fuzzy set $A = (a; b; c)$ with membership function (1.1). Source: Author.

Definition 1.2 (Trapezoidal fuzzy set)**.** *A fuzzy set A of $\mathbb{R}$ is said to be fuzzy trapezoidal,* *and is denoted by* $A = (a; b; c; d)$ *for* $a \leqslant b \leqslant c \leqslant d$*, if its membership function is given by*

$$A(x) = \begin{cases} 0, & \text{if } x < a \\ \dfrac{x-a}{b-a}, & \text{if } a \leqslant x < b \\ 1, & \text{if } b \leqslant x \leqslant c \\ \dfrac{x-d}{c-d}, & \text{if } c < x \leqslant d \\ 0, & \text{if } x > d \end{cases} . \tag{1.2}$$

This type of fuzzy set is depicted in Figure 4.

Figure 4 – Trapezoidal fuzzy set $A = (a; b; c; d)$



A trapezoidal fuzzy set $A = (a; b; c; d)$ with membership function (1.2). Source: Author.

Note that a triangular fuzzy set is a particular case of a trapezoidal fuzzy set, for $b = c$. A trapezoidal fuzzy set can be also viewed as a fuzzyfication of an interval $[b, c]$, where the boundaries are not well stated. Indeed, for $a = b$ and $c = d$, the trapezoidal fuzzy number $A = (b; b; c; c)$ corresponds to the interval $[b, c]$.

Definition 1.3 (Gaussian fuzzy set)**.** *A Gaussian fuzzy set* $A(x; a; \sigma)_G$ *depends on the parameters* a*, the modal value, and* σ*, the dispersion, which in probability theory correspond to the average and the standard deviation, respectively. Its membership function is defined on* $\mathbb{R}$ *as follows:*

$$A(x; a; \sigma)_G = \exp\left(-\left(\frac{x-a}{\sigma}\right)^2\right), \tag{1.3}$$

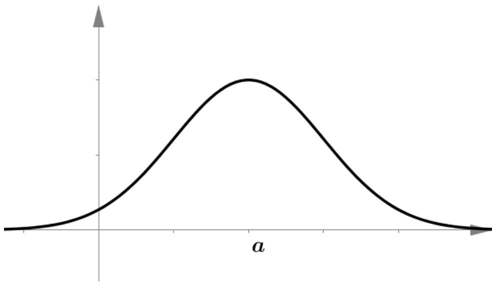
but without the requirement that $\displaystyle\int_{-\infty}^{+\infty} A(x; a; \sigma)_G dx = 1$*. This type of fuzzy set is depicted in Figure 5, for arbitrarily* a *and* σ*.*

Intersection, union, complement and inclusion of fuzzy subsets can be defined as follows.

Definition 1.4 (Intersection and Union)**.** *Let* A *and* B *be fuzzy subsets of* X*. The intersection* $A \cap B$ *and the union* $A \cup B$ *are defined by*

$$(A \cap B)(x) = \min\{A(x), B(x)\},$$
$$(A \cup B)(x) = \max\{A(x), B(x)\},$$

Figure 5 – Gaussian fuzzy set $A(x; a; \sigma)_G$



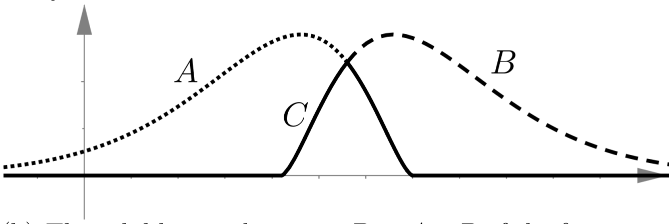
A Gaussian fuzzy set $A = (x; a; \sigma)_G$ with membership function (1.3). Source: Author.

for all $x \in X$*.*

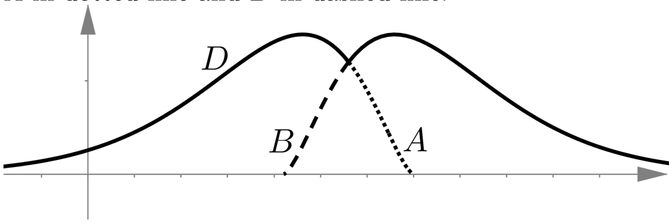
Let A and B be two fuzzy numbers as in Figure 6. Subfigures 6(a) and 6(b) exhibit respectively, the intersection $C = A \cap B$ and the union $D = A \cup B$.

Figure 6 – Intersection and union of fuzzy sets

(a) The solid line is the intersection $C = A \cap B$ of the fuzzy sets A in dotted line and B in dashed line.



(b) The solid line is the union $D = A \cup B$ of the fuzzy sets A in dotted line and B in dashed line.



Intersection and union of two fuzzy sets A and B. Source: Author.

Definition 1.5 (Inclusion and Equality)**.** *Let* A, B *be fuzzy sets of* X*. The inclusion* $A \subseteq B$ *holds true if*

$$A(x) \leqslant B(x),$$

for all $x \in X$*.*

The fuzzy sets are equal $A = B$*, if* $A \subseteq B$ *and* $B \subseteq A$*, that is, if* $A(x) = B(x)$*, for all* $x \in X$*. On the other hand, if* $A(x^*) \neq B(x^*)$ *for some* $x^* \in X$*, then* A *differs from* B*, that is,* $A \neq B$*.*

Definition 1.6 (Specificity)**.** *The fuzzy set $A \in \mathcal{F}(X)$ is said to be more specific than the fuzzy set $B \in \mathcal{F}(X)$ if $A \subseteq B$. If $B \subseteq A$, then A is said to be less specific than B.*

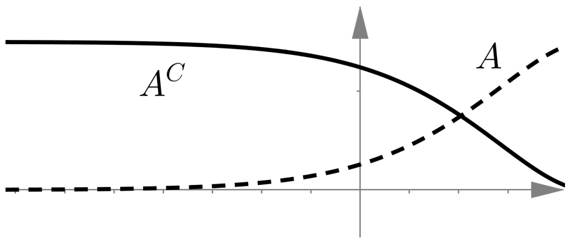
Definition 1.7 (Complement)**.** *Let $A, B \in \mathcal{F}(X)$ be such that $A \subseteq B$. The complement $C \in \mathcal{F}(X)$ of A in B is defined by*

$$C(x) = B(x) - A(x),$$

for all $x \in X$. If $B = X$, then C is called by the standard complement of A and is denoted by A^c. In this case, we have

$$A^c(x) = 1 - A(x).$$

Figure 7 – Complement A^c of a fuzzy set A



The solid curve represents the complement A^c of A (dashed curve) a subset fuzzy. Source: Author.

One can note that the intersection $A \cap A^c$ may differ from $\varnothing$ and the union $A \cup A^c$ may differ from universe X. In fact, $A \cap A^c = \varnothing$ and $A \cup A^c = X$ if, and only if, A is a crisp subset of X.

An important tool to deal with fuzzy set are the α-cuts, whereby it is possible to associate a class of fuzzy set to a family of crisp subsets.

Definition 1.8 (α-cuts [4])**.** *The $\alpha-$cut of a fuzzy set A of X, denoted by $[A]^\alpha$, is defined as*

$$[A]_\alpha = \{x \in X : A(x) \geqslant \alpha\},$$

for all $\alpha \in (0, 1]$. If X is also a topological space[1]*, then the $0-$cut of A is defined by*

$$[A]_0 = \mathrm{cl}(\mathrm{supp}(A)),$$

where $\mathrm{supp}(A) = \{x \in X : A(x) > 0\}$ *is the support of A and* $\mathrm{cl}(Y)$*, $Y \subseteq X$, denotes the closure of Y.*

[1] A topological space is a pair (X, ω) with a set X and a topology ω, which has three properties: (i) $\varnothing, X \in \omega$; (ii) $w_1 \cap w_2 \in \omega$, for any $w_1, w_2 \in \omega$; and (iii) $\bigcup_{i \in I} w_i \in \omega$, for I a non-empty sub-collection of ω. The real line $\mathbb{R}$ with all its subsets is a topological space [15].

Example 1.1. *The α-cuts of a triangular fuzzy set $A = (a; b; c)$, such as in Definition 1.1, are given by*

$$[A]_\alpha = [a + \alpha(b - a), c - \alpha(c - b)],$$

for all $\alpha \in [0, 1]$.

The α-cuts of a trapezoidal fuzzy set $A = (a; b; c; d)$, such as in Definition 1.2, are given by

$$[A]_\alpha = [a + \alpha(b - a), d - \alpha(d - c)],$$

for all $\alpha \in [0, 1]$.

The core of a fuzzy set A is the crisp set $[A]_1$ and A is said to be normal if its core is a non-empty set [81].

One can prove that $A = B$ if, and only if, $[A]_\alpha = [B]_\alpha$, for all $\alpha \in [0, 1]$ [58].

1.2 Fuzzy Numbers

If the universe of discourse is $X = \mathbb{R}$, then it is possible to discuss the natural extension of real numbers to fuzzy environment. A fuzzy number can be used whenever one does not have enough information to specify a number in real line.

Definition 1.9 (Fuzzy number)**.** *A fuzzy set A of $\mathbb{R}$ is said to be a fuzzy number if the following properties are satisfied:*

- *A is normal;*
- *A has a convex membership function, that is, $A(\lambda x + (1 - \lambda)y) \geqslant \min\{A(x), A(y)\}$, for all $\lambda \in [0, 1]$;*
- *A is upper continuous, that is, for all $\epsilon > 0$ and $x^* \in \mathbb{R}$, there is $\delta > 0$ such that $A(x) - A(x^*) < \epsilon$, for all $|x - x^*| < \delta$; and*
- *A has bounded support.*

The class of fuzzy numbers is denoted by the symbol $\mathbb{R}_{\mathcal{F}}$.

The next theorem indicates when a family of subsets can be uniquely associated with a fuzzy number.

Theorem 1.1 (Negoita-Ralescu's characterization theorem or Stacking theorem [70, 10])**.** *Given a family of subsets $\{A_\alpha : \alpha \in [0, 1]\}$ that satisfies the following conditions*

(a) A_α is a non-empty, closed, and bounded interval for any $\alpha \in [0, 1]$;

(b) $A_{\alpha_2} \subseteq A_{\alpha_1}$, *for all* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$;

(c) *For any sequence* α_n *which converges from below to* $\alpha \in (0,1]$ *it follows*

$$\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha;$$

(d) *For any sequence* α_n *which converges from above to* 0 *it follows*

$$A_0 = cl\left(\bigcup_{n=1}^{\infty} A_{\alpha_n}\right).$$

Then there exists a unique $A \in \mathbb{R}_{\mathcal{F}}$, *such that* $[A]_\alpha = A_\alpha$, *for each* $\alpha \in [0,1]$.

Conversely, let $A \in \mathbb{R}_{\mathcal{F}}$, *if* $A_\alpha = [A]_\alpha$ *for all* $\alpha \in [0,1]$ *then the family of subsets* $\{A_\alpha : \alpha \in [0,1]\}$ *satisfies the conditions (a)-(d).*

Proposition 1.1. *[3, 34] A fuzzy set* A *of* $\mathbb{R}$ *is a fuzzy number if, and only if, all its* α*-cuts are non-empty bounded closed intervals of* $\mathbb{R}$.

Since each $\alpha-$cut of a fuzzy number A is a closed interval, one can write $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$, where a_α^- and a_α^+ are called endpoints of $[A]_\alpha$.

Definition 1.10 (Symmetry [32])**.** *A fuzzy number* A *is symmetric, with respect to* $x \in \mathbb{R}$, *if*

$$A(x-y) = A(x+y),$$

for all $y \in \mathbb{R}$. *If there is no* $x \in \mathbb{R}$ *such that this property is satisfied, then* A *is non-symmetric or asymmetric.*

Definition 1.11 (Width *or* Diameter)**.** *The width of an interval* $I = [a,b]$ *is given by* $\mathrm{width}(I) = b - a$. *The width of a fuzzy number* A *is given by* $\mathrm{width}(A) = a_0^+ - a_0^-$. *It is also called diameter and denoted by* $\dim(A) = a_0^+ - a_0^-$.

One can easily conclude that triangular and trapezoidal fuzzy sets (see Equations (1.1) and (1.2)) are fuzzy numbers. Note that a real number a is a particular case of triangular fuzzy number since $a \equiv (a;a;a)$.

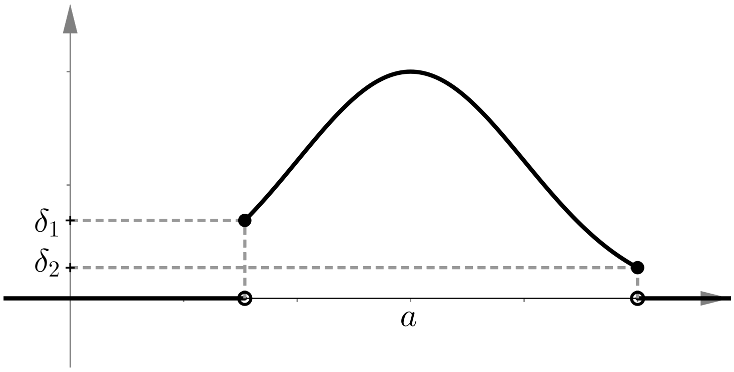
On the other hand, Gaussian fuzzy set in Definition 1.3 does not have bounded support. In order to convert a Gaussian fuzzy set A in a fuzzy number, it is necessary to modify the membership function as follows

$$A(x;a,\sigma,\delta_1,\delta_2)_G = \begin{cases} \exp\left(-\left(\dfrac{x-a}{\sigma}\right)^2\right), & \text{if } x \in \left[a - \sigma\sqrt{\ln\dfrac{1}{\delta_1}}, a + \sigma\sqrt{\ln\dfrac{1}{\delta_2}}\right] \\ 0, & \text{otherwise} \end{cases}, \quad (1.4)$$

where $\delta_1, \delta_2 \in (0, 1]$.

Figure 8 depicts the Gaussian fuzzy number (in contrast to Figure 5) with the limitation of the original membership function by the values of δ_1 and δ_2.

Figure 8 – Gaussian fuzzy number $A(x; a, \sigma, \delta_1, \delta_2)_G$



The Gaussian fuzzy number given in Equation (1.4) with $\delta_1 \geqslant \delta_2$. Source: Author.

The Gaussian fuzzy number is, therefore, written simply by the quadruple $A = (a, \sigma, \delta_1, \delta_2)_G$. If $\delta_1 = \delta_2$, then the Gaussian number A is symmetric with respect to δ_1 (or δ_2), otherwise, A is non-symmetric.

The α-cuts of A Gaussian fuzzy numbers depend on δ_1 and δ_2. If $\delta_1 \geqslant \delta_2$, then

$$[A]_\alpha = \begin{cases} \left[a - \sigma\sqrt{\ln\dfrac{1}{\alpha}}, a + \sigma\sqrt{\ln\dfrac{1}{\alpha}}\right], & \text{if } \delta_1 \leqslant \alpha \leqslant 1 \\ \left[a - \sigma\sqrt{\ln\dfrac{1}{\delta_1}}, a + \sigma\sqrt{\ln\dfrac{1}{\alpha}}\right], & \text{if } \delta_2 \leqslant \alpha < \delta_1 \\ \left[a - \sigma\sqrt{\ln\dfrac{1}{\delta_1}}, a + \sigma\sqrt{\ln\dfrac{1}{\delta_2}}\right], & \text{if } 0 \leqslant \alpha < \delta_2 \end{cases}. \tag{1.5}$$

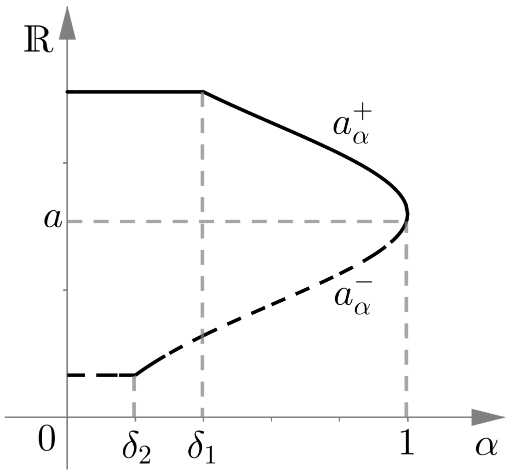
Note that $[A]_\alpha = [A]_{\delta_2}$, for all $\alpha \in [0, \delta_2]$. On the other hand, if $\delta_2 \geqslant \delta_1$, then

$$[A]_\alpha = \begin{cases} \left[a - \sigma\sqrt{\ln\dfrac{1}{\alpha}}, a + \sigma\sqrt{\ln\dfrac{1}{\alpha}}\right], & \text{if } \delta_2 \leqslant \alpha \leqslant 1 \\ \left[a - \sigma\sqrt{\ln\dfrac{1}{\alpha}}, a + \sigma\sqrt{\ln\dfrac{1}{\delta_2}}\right], & \text{if } \delta_1 \leqslant \alpha < \delta_2 \\ \left[a - \sigma\sqrt{\ln\dfrac{1}{\delta_1}}, a + \sigma\sqrt{\ln\dfrac{1}{\delta_2}}\right], & \text{if } 0 \leqslant \alpha < \delta_1 \end{cases}. \tag{1.6}$$

Note that $[A]_\alpha = [A]_{\delta_1}$, for all $\alpha \in [0, \delta_1]$.

An interesting subclass of fuzzy numbers is the class $\mathbb{R}_{\mathcal{F}_C}$, which contains all the fuzzy numbers with continuous endpoints with respect to α. All trapezoidal fuzzy numbers belong to $\mathbb{R}_{\mathcal{F}_C}$. Although Gaussian fuzzy numbers does not have continuous membership functions, they have continuous endpoints with respect to α as can be seen in Figure 9.

Figure 9 – Endpoints of Gaussian fuzzy number



Right endpoint a_α^+ is the solid curve and left endpoint a_α^- is the dashed curve, for $\delta_1 \geqslant \delta_2$. Source: Author.

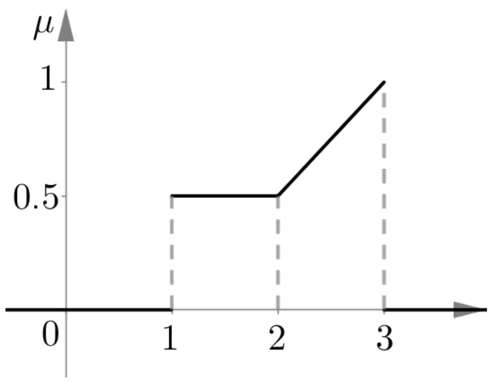
An example of fuzzy number which is not in $\mathbb{R}_{\mathcal{F}_C}$ is given below.

Example 1.2. *Let A be the fuzzy number given by*

$$A(x) = \begin{cases} 0.5, & \text{if } 1 \leqslant x \leqslant 2 \\ 0.5x - 0.5, & \text{if } 2 < x \leqslant 3 \\ 0, & \text{otherwise} \end{cases}, \tag{1.7}$$

A is depicted in Figure 10.

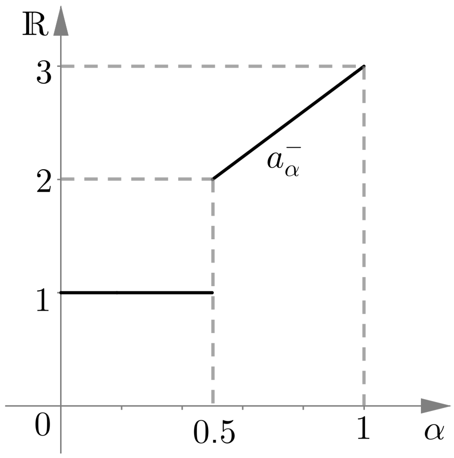
Figure 10 – Fuzzy number not contained in $\mathbb{R}_{\mathcal{F}_C}$



Membership of $A \notin \mathbb{R}_{\mathcal{F}_C}$. Source: Author.

The left endpoint a_α^- is not continuous at 0.5, as we can see in Figure 11. Note that, the intervals where the membership function is constant corresponds to the discontinuity of left endpoint.

Figure 11 – Left endpoint of a fuzzy number not contained in $\mathbb{R}_{\mathcal{F}_C}$



Left endpoint a_α^- of $A \notin \mathbb{R}_{\mathcal{F}_C}$. Source: Author.

1.3 Fuzzy Relations

This section reviews the definition of interactive fuzzy numbers [128, 27, 57, 112].

Definition 1.12 (Fuzzy Relation)**.** *A fuzzy relation R over $X = X_1 \times \ldots \times X_n$ is a fuzzy subset of $X_1 \times \ldots \times X_n$. Thus, a fuzzy relation R is associated with a membership function $R : X_1 \times \ldots \times X_n \to [0,1]$, where $R(x_1, \ldots, x_n) \in [0,1]$ represents the degree of relationship among $x_1, \ldots, x_n$ with respect to R [4].*

Note that a fuzzy relation is, in particular, a fuzzy set over $X_1 \times \ldots \times X_n$, that is, $R \in \mathcal{F}(X_1 \times \ldots \times \ldots X_n)$.

Example 1.3. *[25] Let $A_i \in \mathcal{F}(X_i)$, for $i = 1, \ldots, n$. The Cartesian product $A_1 \times \ldots \times A_n$ over $X_1 \times \ldots \times X_n$ is defined by*

$$(A_1 \times \ldots \times A_n)(x_1, \ldots, x_n) = A_1(x_1) \wedge \ldots \wedge A_n(x_n),$$

where $\wedge$ represents the infimum operation.

Another fuzzy relation is the cylindrical extension.

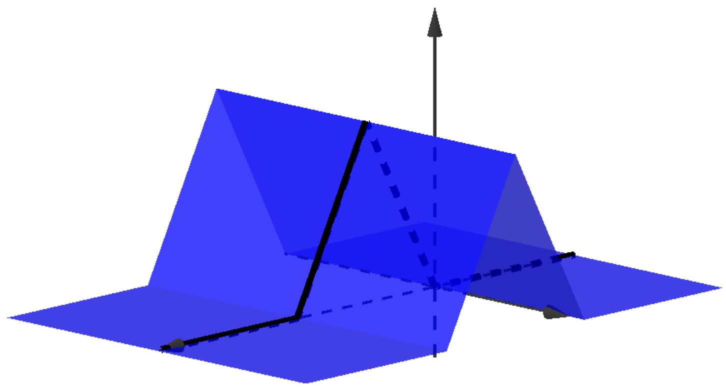
Definition 1.13 (Cylindrical Extension [80])**.** *The cylindrical extension of $A \in \mathcal{F}(X_1)$ over $X_1 \times X_2$ is the fuzzy relation $\operatorname{cyl}(A) \in \mathcal{F}(X_1 \times X_2)$ given by*

$$(\operatorname{cyl}(A))(x_1, x_2) = A(x_1),$$

for all $x_1 \in X_1$ and $x_2 \in X_2$.

An example of a cylindrical extension of the fuzzy number $A = (0; 1; 2)$ is depicted in Figure 12.

Figure 12 – Cylindrical extension



The surface represents the $\mathrm{cyl}(A)$ and $A = (0; 1; 2)$ is represented by the black line. Source: Author.

Another interesting concept that is related to the notion of fuzzy relation is the projection.

Definition 1.14 (Projection [81]). *The projection of fuzzy relation $R \in \mathcal{F}(X_1 \times \ldots \times X_n)$ onto X_i, for $i \in \{1, \ldots, n\}$, is the fuzzy set Π_R^i of X_i given by*

$$\Pi_R^i(y) = \bigvee_{x \in X : x_i = y} R(x_1, \ldots, x_n),$$

where $X = X_1 \times \ldots \times X_n$ and $\bigvee$ stands for supremum operation.

In general, cylindrical extension and projections can be viewed as opposite operations, since $\Pi^1_{\mathrm{cyl}(A)}(A) = A$, for any $A \in \mathcal{F}(X)$.

Definition 1.15 (t-norm [62, 56]). *A t-norm is an operation $\triangle : [0,1] \times [0,1] \to [0,1]$ with the following properties, for all $x, y, z \in [0,1]$,*

1. *Commutativity: $\triangle(x, y) = \triangle(y, x)$;*
2. *Associativity: $\triangle(x, \triangle(y, z)) = \triangle(\triangle(x, y), z)$;*
3. *Monotonicity: if $y \leqslant z$, then $\triangle(x, y) \leqslant \triangle(x, z)$;*
4. *$\triangle(x, 1) = x$ and $\triangle(x, 0) = 0$.*

For the sake of simplicity, we write $\triangle(x, y) = x \triangle y$ and $\triangle(x, y, z) = \triangle(x, \triangle(y, z)) = x \triangle y \triangle z$.

Definition 1.16 (s-norm [62, 56]). *A s-norm is an operation $\triangledown : [0,1] \times [0,1] \to [0,1]$ with the following properties, for all $x, y, z \in [0,1]$,*

1. *Commutativity:* $\bigtriangledown(x, y) = \bigtriangledown(y, x)$;

2. *Associativity:* $\bigtriangledown(x, \bigtriangledown(y, z)) = \bigtriangledown(\bigtriangledown(x, y), z)$;

3. *Monotonicity: if* $y \leqslant z$, *then* $\bigtriangledown(x, y) \leqslant \bigtriangledown(x, z)$;

4. $\bigtriangledown(x, 1) = 1$ *and* $\bigtriangledown(x, 0) = x$.

For notational convenience, we write $\bigtriangledown(x, y) = x \bigtriangledown y$ and $\bigtriangledown(x, y, z) = \bigtriangledown(x, \bigtriangledown(y, z)) = x \bigtriangledown y \bigtriangledown z$.

Definition 1.17 (Negation [81])**.** *A negation is a continuous application* $\eta : [0, 1] \to [0, 1]$ *with the following properties for all* $x \in [0, 1]$

1. *Monotonicity: if* $x \leqslant y$, *then* $\eta(x) \geqslant \eta(y)$;

2. *Involution:* $\eta(\eta(x)) = x$;

3. $\eta(0) = 1$ *and* $\eta(1) = 0$.

The standard negation is given by $\eta(x) = \neg(x) = 1 - x$.

The only difference between t-norm and s-norm is the fourth property. Actually, there exists a strong connection between them. A t-norm $\bigtriangleup$ and an s-norm $\bigtriangledown$ are said to be duals with respect to a negation η if the following equalities hold true for all $x, y \in [0, 1]$

$$\eta(x) \bigtriangleup \eta(y) = \eta(x \bigtriangledown y),$$
$$\eta(x) \bigtriangledown \eta(y) = \eta(x \bigtriangleup y).$$

Examples of t-norm and s-norm are the minimum norm $\bigtriangleup = \wedge$ and the maximum norm $\bigtriangledown = \vee$. For more examples of t and s-norms the reader may refer to [56, 80].

1.4 Interactive Fuzzy Numbers

Interactivity among fuzzy numbers resembles dependence between random variables in probability theory. According to Zadeh [128], "fuzzy variables are interactive if the assignment of a value to one affects the fuzzy restrictions placed on the others". The concept of interactivity is associated with the notion of joint possibility distribution (JPD), as well as the dependence relation is associated with joint probability distribution. JPD could also be named as joint membership distribution (JMD).

Definition 1.18 (Joint Possibility Distribution [27])**.** *A fuzzy relation* $J \in \mathcal{F}(\mathbb{R}^n)$ *is said to be a joint possibility distribution (JPD) of* $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$ *if*

$$A_i(y) = \Pi^i_J(y) = \bigvee_{x \in X : x_i = y} J(x_1, \ldots, x_n),$$

for all $y \in \mathbb{R}$ *and for all* $i = 1, \ldots, n$.

In other words, given a JPD J of $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$, each A_i is a projection of J (see Definition 1.14). Each A_i is called a marginal possibility distribution of J.

A fuzzy relation $J_{\triangle}$ given by

$$J_{\triangle}(x_1, \ldots, x_n) = A_1(x_1) \triangle \ldots \triangle A_n(x_n) \tag{1.8}$$

is said to be a t-norm-based joint possibility distribution of $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$ [27].

The notion of interactivity between fuzzy numbers is given by means of joint possibility distributions as follows.

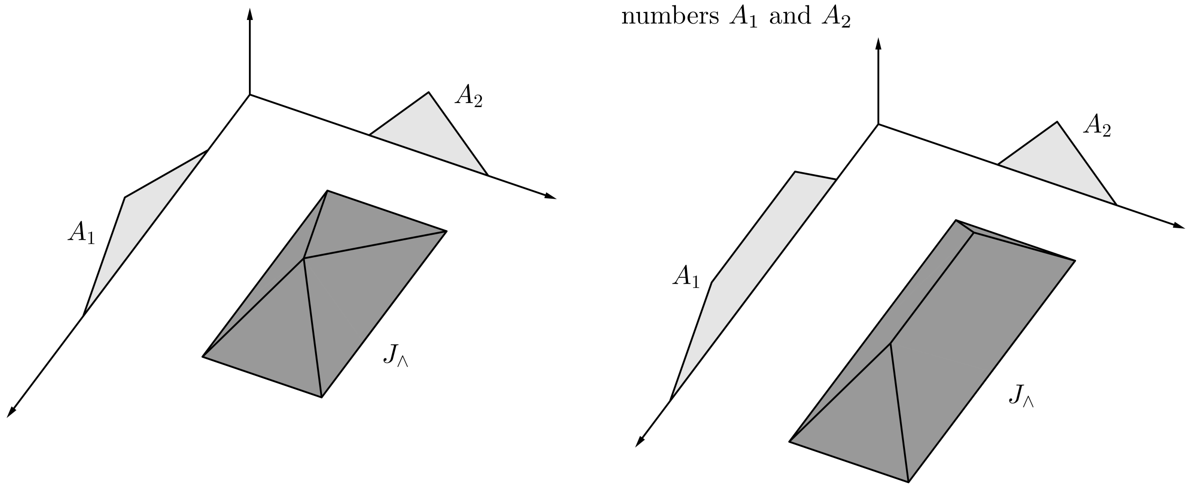
Definition 1.19 (Interactive and Non-Interactive Fuzzy Numbers [128, 17, 27])**.** *Given a JPD of* $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$*, the fuzzy numbers* $A_1, \ldots, A_n$ *are said to be non-interactive when the underlying JPD is given as in* (1.8) *with* $\triangle = \wedge$*, that is* $J = J_{\wedge}$*. Otherwise, if* $J \neq J_{\wedge}$*, then the fuzzy numbers* $A_1, \ldots, A_n$ *are said to be interactive.*

Examples of non-interactive fuzzy numbers and their respective JPDs $J_{\wedge}$ are depicted in Figure 13.

Figure 13 – Joint possibility distribution for two non-interactive fuzzy numbers

(a) JPD $J_{\wedge}$ of triangular fuzzy numbers A_1 and A_2

(b) JPD $J_{\wedge}$ of triangular and trapezoidal fuzzy numbers A_1 and A_2



Joint possibility distributions $J_{\wedge}$ for different pairs of fuzzy numbers. Source: Author.

The α-cuts of a JPD $J_{\wedge}$ for non-interactive fuzzy numbers $A_1 \times \ldots \times A_n$ can be easily calculated since

$$[J_{\wedge}]_{\alpha} = [A_1]_{\alpha} \times \ldots \times [A_n]_{\alpha},$$

for all $\alpha \in [0, 1]$.

Esmi *et al.* [33] proposed a family of JPDs that is not based on t-norm. They introduced a family of JPD J_γ index by a parameter $\gamma \in [0,1]$, which controls the specificity, as we can see in Figure 14. As γ approaches to 1, the JPD has more points in the domain with $J_\gamma(x,y) > 0$. Otherwise, as γ gets closer to 0, the smaller becomes the support of J_γ. The construction of this JPD can be found in [33]. Here, we focus on the JPD J_0.

Definition 1.20 (Joint Possibility Distribution J_0 [33])**.** *Let A_1 and A_2 be two fuzzy numbers in $\mathbb{R}_{\mathcal{F}_C}$ and g^i be the function*

$$g^i(z,\alpha) = \bigwedge_{w \in [A_{3-i}]_\alpha} |w+z|,$$

for all $z \in \mathbb{R}$, $\alpha \in [0,1]$ and $i \in \{1,2\}$.

Also, consider the sets R^i_α and $L^i(z,\alpha)$,

$$R^i_\alpha = \begin{cases} \{a^-_{i_\alpha}, a^+_{i_\alpha}\}, & \text{if } 0 \leqslant \alpha < 1 \\ [A_i]_1 & \text{if } \alpha = 1 \end{cases}$$

and

$$L^i(z,\alpha) = [A_{3-i}]_\alpha \cap [-g^i(z,\alpha) - z, g^i(z,\alpha) - z].$$

Finally, J_0 is defined by

$$J_0(x_1,x_2) = \begin{cases} A_1(x_1) \wedge A_2(x_2), & \text{if } (x_1,x_2) \in P \\ 0 \quad , & \text{otherwise} \end{cases}, \tag{1.9}$$

with $P = \bigcup_{i=1}^{2} \bigcup_{\alpha \in [0,1]} \{(x_1,x_2) : x_i \in R^i_\alpha \text{ and } x_{3-i} \in L^i(x_i,\alpha)\}$.

Note that the set P contains all the pairs $(x_1,x_2) \in \mathbb{R}^2$ that satisfy $J_0(x_1,x_2) > 0$. Since P is a proper subset of $\mathbb{R}^2$, in general, we have that $J_0 \neq J_\wedge$ [118]. The equality holds if A_1 or A_2 are real numbers.

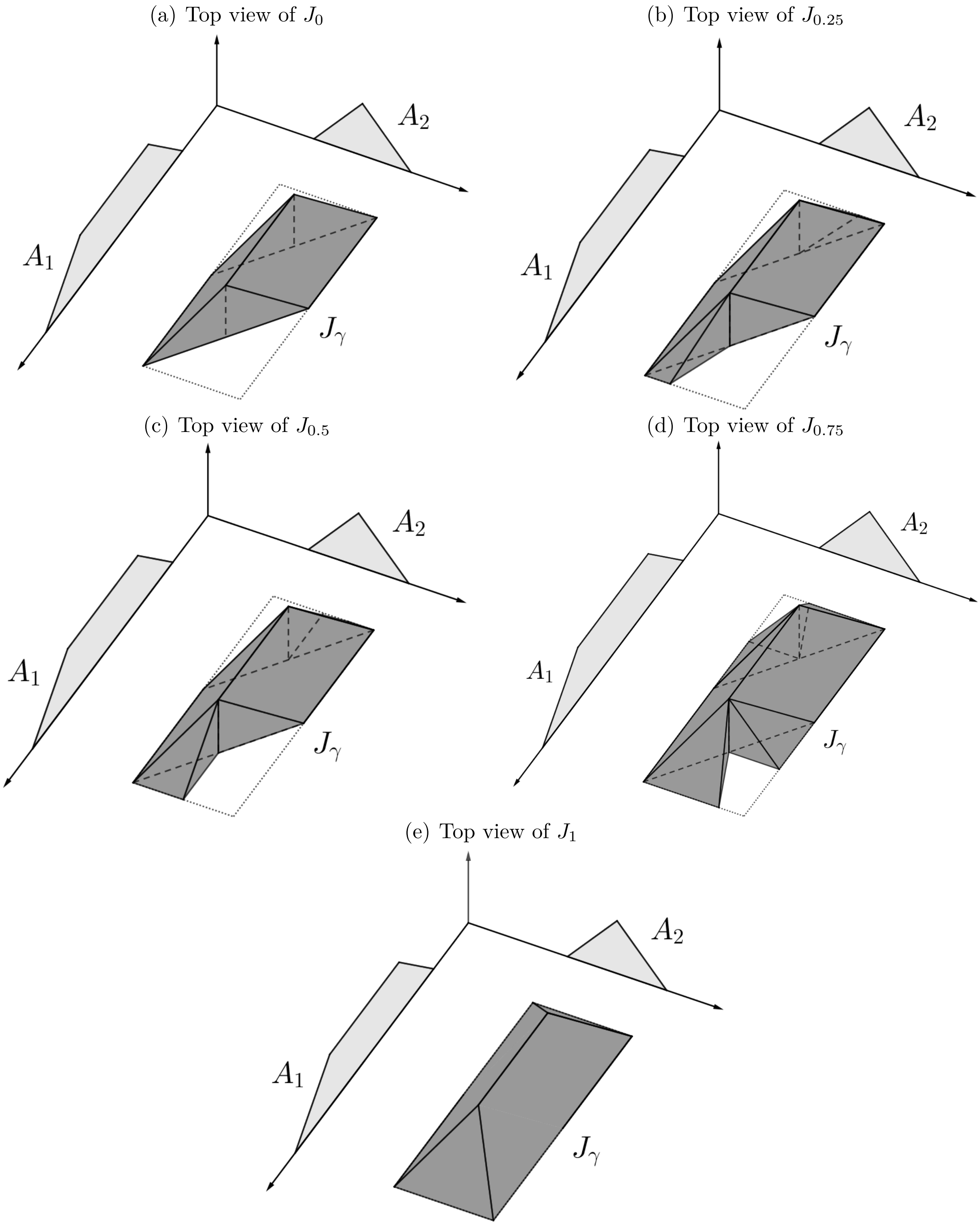
Definition 1.21 (Translated fuzzy number)**.** *Let $A \in \mathbb{R}_\mathcal{F}$. The translation of A by $k \in \mathbb{R}$ is defined as the fuzzy number*

$$\widetilde{A}(x) = A(x+k),$$

for all $x \in \mathbb{R}$.

Definition 1.21 gives raise to a new JPD, which incorporates the translation of fuzzy numbers [102].

Figure 14 – Joint possibility distribution J_γ for different values of γ



Joint possibility distributions J_γ for different parameters γ. Source: Wasques [112].

Theorem 1.2. *[102] Given $A_1, A_2 \in \mathbb{R}_{\mathcal{F}}$ and $c = (c_1, c_2) \in \mathbb{R}^2$. Let $\tilde{A}_i \in \mathbb{R}_{\mathcal{F}}$ be such that $\tilde{A}_i(x) = A_i(x + c_i)$, for all $x \in \mathbb{R}$ and $i = 1, 2$. Let $\tilde{J}_0$ be the joint possibility distribution of* *fuzzy numbers* $\tilde{A}_1, \tilde{A}_2 \in \mathbb{R}_{\mathcal{F}}$ *defined as Equation* (1.9)*. The fuzzy relation* J_0^c *given by*

$$J_0^c(x_1, x_2) = \tilde{J}_0(x_1 - c_1, x_2 - c_2), \tag{1.10}$$

for all $(x_1, x_2) \in \mathbb{R}^2$*, is a joint possibility distribution of* A_1 *and* A_2*.*

In particular, comparing Subfigure 14(e) and Subfigure 13(b) it is possible to note that $J_1 = J_\wedge$. Esmi *et al.* [33] proved that the JPD J_0 controls the norm of arithmetic addition of fuzzy numbers with continuous endpoints. Sussner *et al.* [102] showed that by a translation to origin, the J_0^c also controls the diameter of the sum.

Despite the many advantages of JPD J_γ, its construction is not simple and requires a reasonable computational effort [119]. As a consequence, it is difficult to write the J_γ in terms of α-cuts. In order to avoid these types of problems, another JPD can also be considered.

Carlsson *et al.* [17] introduced a possible type of interactivity relation between two fuzzy numbers that is also not based on t-norms. This relation araises from the assumption that two fuzzy numbers can be related by a real line. It is called completely correlated fuzzy numbers and is similar to longitudinal data in statistical analysis.

Data correlations arise naturally in longitudinal datasets. A dataset is said to be longitudinal if it contains the same type of information on the same items at multiple points in time. Therefore, longitudinal data is characterized by the fact that repeated observations are correlated [129]. In this work, the supposition is that this correlation is given by the notion of completely correlated fuzzy numbers [17, 9].

Definition 1.22 (Complete Correlation [17])**.** *The fuzzy numbers* A_1 *and* A_2 *are said to be completely correlated if there exist* $q, r \in \mathbb{R}$ *with* $q \neq 0$ *such that the corresponding joint possibility distribution is given by*

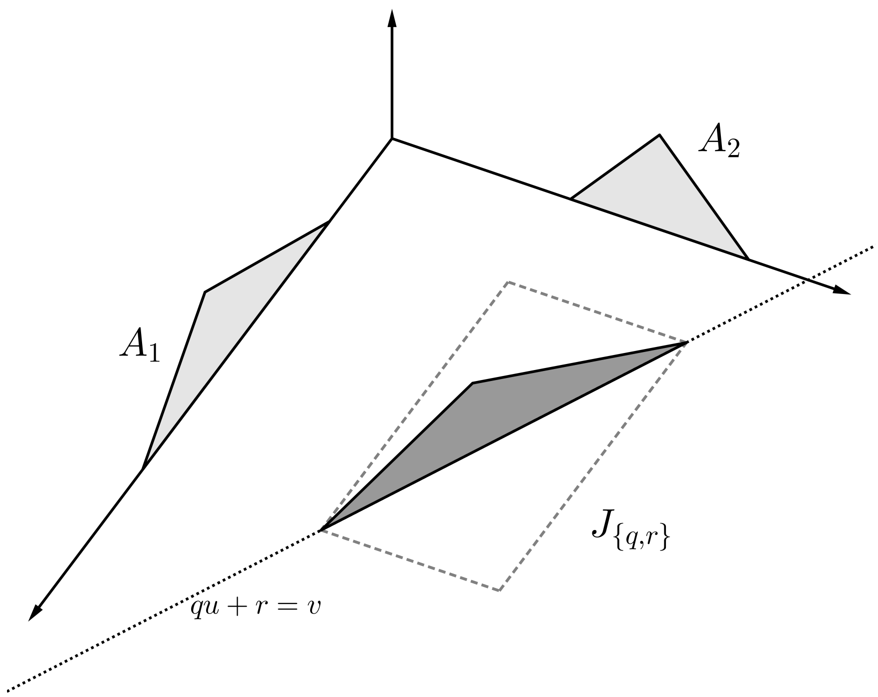
$$\begin{aligned} J_{\{q,r\}}(x_1, x_2) &= A_1(x_1)\chi_{\{qu+r=v\}}(x_1, x_2) \\ &= A_2(x_2)\chi_{\{qu+r=v\}}(x_1, x_2), \end{aligned} \tag{1.11}$$

where $\chi_{\{qu+r=v\}}$ *stands for the characteristic function of the set* $L = \{(u, v) \in \mathbb{R}^2 : qu + v = r,\ u \in \mathbb{R}\} \subset \mathbb{R}^2$.

The requirement in (1.11) ensures that the marginals A_1 and A_2 must have same shape. For example, Figure 15 depicts the JPD $J_{\{q,r\}}$ for triangular fuzzy numbers A_1 and A_2. It is not possible to exist such line L if A_1 and A_2 are different types of fuzzy numbers.

In addition, the JPDs $J_{\{q,r\}}$ are classified according to the value of q. If $q > 0$ $(q < 0)$, then A_1 and A_2 are said to be positively (negatively) completely correlated. In Figure 15, A_1 and A_2 are negatively correlated.

Figure 15 – Joint possibility distribution $J_{\{q,r\}}$



A point possibility distribution $J_{\{q,r\}}$ for triangular fuzzy numbers A_1 and A_2 with $q < 0$.
Source: Author.

Since $q \neq 0$ in Equation (1.11), the membership function of A_2 can be written in terms of A_1 as follows

$$A_2(qx + r) = A_1(x),$$

for all $x \in \mathbb{R}$. Consequently,

$$[A_2]_\alpha = q[A_1]_\alpha + \{r\}, \tag{1.12}$$

for all $\alpha \in [0, 1]$. Moreover, for each $\alpha \in [0, 1]$, the α-cut of the joint possibility distribution $J_{\{q,r\}}$ is given by [17]

$$[J_{\{q,r\}}]_\alpha = \{(x, qx + r) \ : \ x \in [A_1]_\alpha\}.$$

Note that if A_1 is symmetric (see Definition 1.10) and if A_1 and A_2 are completely correlated by the line $qu + r_1 = v$, then there exists another line $-qu + r_2 = v$, with $r_2 = q(a_1^- + a_1^+) + r_1$, such that A_1 and A_2 are also completely correlated. Therefore, the distribution $J_{\{q,r\}}$ is not unique for symmetric fuzzy numbers. On the other hand, if A_1 is non-symmetric and A_1 and A_2 are completely correlated, then $J_{\{q,r\}}$ is unique [32].

Example 1.4. *Let* $A_1 = (1; 2; 3)$ *and* $A_2 = (-1; 0; 1)$ *be symmetric triangular fuzzy numbers, with respect to* 2 *and* 0*, respectively. Note that they may be completely correlated by two possible JPDs:* $J_{\{1,-2\}}$ *and* $J_{\{-1,2\}}$*. Thus the* α*-cuts of* A_2 *can be written in two* *different ways in terms of A_1 α-cuts:*

$$[A_2]_\alpha = [A_1]_\alpha - \{2\},$$
$$[A_2]_\alpha = -[A_1]_\alpha + \{2\}.$$

Example 1.5. *The triangular fuzzy numbers $A_1 = (1;2;4)$ and $A_2 = (-1;0;1)$ are not completely correlated, because it is impossible to find q and r such that (1.11) is satisfied.*

Example 1.6. *The non-symmetric triangular fuzzy numbers $A_1 = (1;2;4)$ and $A_2 = (-2;0;1)$ are uniquely completely correlated by $J_{\{-1,2\}}$, because there is no other pair $\{q,r\}$ satisfying (1.11).*

The next proposition ensures that the completely correlation is a transitive relation of interactivity between fuzzy numbers.

Proposition 1.2. *[79] Let A_1, A_2, $A_3 \in \mathbb{R}_\mathcal{F}$. If A_1 and A_2 are completely correlated with respect to $J_{\{q_1,r_1\}}$ and A_2 and A_3 are completely correlated with respect to $J_{\{q_2,r_2\}}$, then there exist real numbers q_3 and r_3 such that A_1 and A_3 are completely correlated with respect to $J_{\{q_3,r_3\}}$.*

Based on the previous propositions, the notion of complete correlation can be extended to n fuzzy numbers as follows.

Definition 1.23 (Linear Interactivity [116, 49, 88])**.** *The fuzzy numbers $A_1, \ldots, A_n \in \mathbb{R}_\mathcal{F}$ are said to be linearly interactive if their joint possibility distribution J_L is given by*

$$\begin{aligned} J_L(x_1,\ldots,x_n) &= \chi_L(x_1,\ldots,x_n)A_1(x_1) \\ &= \chi_L(x_1,\ldots,x_n)A_2(x_2) \\ &\vdots \\ &= \chi_L(x_1,\ldots,x_n)A_n(x_n), \end{aligned} \tag{1.13}$$

where $L = \{(u, q_2u + r_2, \ldots, q_nu + r_n) : u \in \mathbb{R}\}$, $q_i, r_i \in \mathbb{R}$, with $q_i \neq 0$, $\forall i = 1, \ldots, n$.

From Equations (1.11) and (1.13), one can see that, in particular, A_1 and A_i, $i > 1$, are also completely correlated with JPD $J_{\{q_i,r_i\}}$. Thus, the α-cuts of each A_i, $i > 1$, can be written in terms of the α-cuts of A_1 as follows:

$$[A_i]_\alpha = q_i[A_1]_\alpha + \{r_i\},$$

for all $i = 2, \ldots, n$ and $\alpha \in [0,1]$. Moreover, for each $\alpha \in [0,1]$, the α-cut of J_L is given by

$$[J_L]_\alpha = \{(x, q_2x + r_2, \ldots, q_nx + r_n) \ : \ x \in [A_1]_\alpha\}. \tag{1.14}$$

Equation (1.14) means that the α-cuts of the joint possibility distribution J_L can be expressed in terms of α-cuts of A_1 and the parameters q_i and r_i, for all $i = 2, \ldots, n$.

Observe that, for A_1 symmetric, there exist 2^{n-1} different joint possibility distributions that satisfy (1.13). On the other hand, if A_1 is non-symmetric, J_L is unique [112].

1.5 Extension Principle

A function $f : X \to Y$ can be extended to $\widehat{f} : \mathcal{F}(X) \to F(Y)$ by means of the Zadeh's extension principle [128]. Given $A \in \mathcal{F}(X)$, the Zadeh's extension principle of f at A yields $\widehat{f}(A) \in \mathcal{F}(Y)$.

Definition 1.24 (Zadeh's extension principle [128, 4])**.** *Let be $X = X_1 \times \ldots \times X_n$ and $f : X \to Y$. The Zadeh's extension of f at $A = A_1 \times \ldots \times A_n \in \mathcal{F}(X_1 \times \ldots \times X_n)$ is the fuzzy set $\widehat{f}(A) \in \mathcal{F}(Y)$ whose membership function is given by*

$$\widehat{f}(A_1, \ldots, A_n)(y) = \bigvee_{(x_1, \ldots, x_n) \in f^{-1}(y)} A_1(x_1) \wedge \ldots \wedge A_n(x_n), \tag{1.15}$$

for all $y \in Y$, with $f^{-1}(y) = \{(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n : f(x_1, \ldots, x_n) = y\}$. The symbol $\bigvee$ stands for supremum operation, and, by definition, $\bigvee \varnothing = 0$.

For $f : X \to Y$, with an uni-dimensional set X, the Zadeh's extension of f at $A \in \mathcal{F}(X)$ boils down to

$$\widehat{f}(A)(y) = \bigvee_{x \in X : f(x) = y} A(x).$$

A simple example of this extension is given below.

Example 1.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) = \dfrac{x^2}{10}$. Figure 16 illustrates the Zadeh's extension of f at $A = (1; 3; 4)$ given by*

$$\widehat{f}(A)(y) = \begin{cases} \dfrac{\sqrt{10y} - 1}{2}, & \text{if } 0.1 \leqslant y \leqslant 0.9 \\ 4 - \sqrt{10y}, & \text{if } 0.9 < y \leqslant 1.6 \\ 0, & \text{if } y < 0.1 \text{ or } 1.6 < y \end{cases}.$$

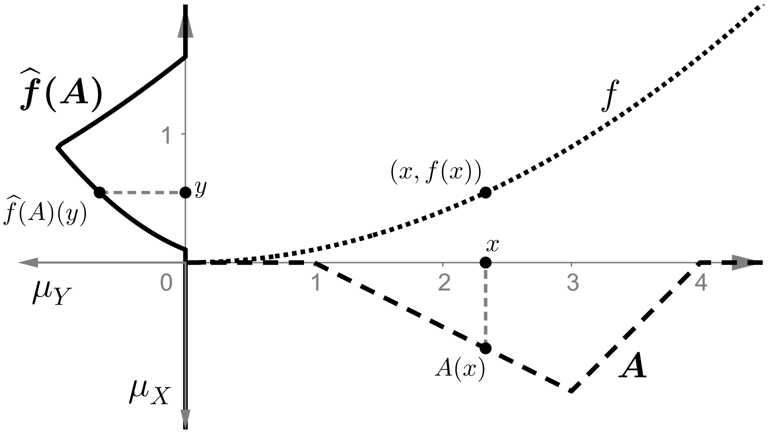
Note that A is a triangular fuzzy number and $\widehat{f}(A)$ is a non-triangular fuzzy number.

Theorem 1.3. *[72, 5] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $A \in \mathbb{R}_{\mathcal{F}}$. The Zadeh's extension of f at A has α-cuts given by*

$$[\widehat{f}(A)]_\alpha = f([A]_\alpha),$$

for all $\alpha \in [0, 1]$.

Figure 16 – The Zadeh's extension of f at A given in Example 1.7



The fuzzy set $\widehat{f}(A)$ is represented by the solid curve. The fuzzy number A is represented by dashed curve and the function f is represented by the dotted curve. Source: Author.

Remark 1.1. *[60] If $A \in \mathbb{R}_{\mathcal{F}}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then the Zadeh's extension of f at A is also a fuzzy number.*

Remark 1.1 reveals that $A \in \mathbb{R}_{\mathcal{F}}$ implies $\widehat{f}(A) \in \mathbb{R}_{\mathcal{F}}$ if f is continuous. The next example highlights that the continuity plays a key role in Theorem 1.3.

Example 1.8. *Let f be a non-continuous function defined by*

$$f(x) = \begin{cases} \dfrac{x^2}{10}, & \text{if } x < 2.5 \\ \dfrac{x}{2}, & \text{if } 2.5 \leqslant x \end{cases}.$$

The Zadeh's extension of f at $A = (1; 3; 4)$ is given by

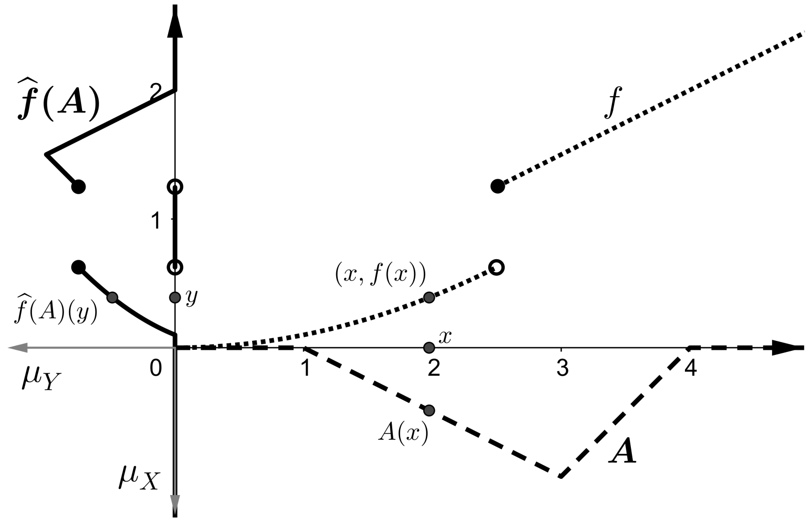
$$\widehat{f}(A)(y) = \begin{cases} \dfrac{\sqrt{10y} - 1}{2}, & \text{if } 0.1 \leqslant y \leqslant 0.625 \\ y - \dfrac{1}{2}, & \text{if } 1.25 \leqslant y \leqslant 1.5 \\ 4 - 2y, & \text{if } 1.5 < y \leqslant 2 \\ 0, & \text{if } y < 0.1 \text{ or } 0.625 < y < 1.25 \text{ or } 2 < y \end{cases},$$

which is not a fuzzy number, as it can be viewed in Figure 17.

In Equation (1.15), the cartesian product $A_1 \times \ldots \times A_n$ can be replaced by any joint possibility distribution. Thus one obtains the notion of $\sup -J$ extension principle, given as follows.

Definition 1.25 (Sup-J Extension Principle [43])**.** *Let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$ and let $f : \mathbb{R}^n \to \mathbb{R}$. The $\sup -J$ extension of f at*

Figure 17 – The Zadeh's extension of a non-continuous function f at A given in Example 1.8



The fuzzy set $\widehat{f}(A)$ is represented by the solid curve. The fuzzy number A is represented by dashed curve and the function f is represented by the dotted curve. Source: Author.

$(A_1, \ldots, A_n)$ *is defined by*

$$\widehat{f}_J(A_1, \ldots, A_n)(y) = \widehat{f}(J)(y) = \bigvee_{(x_1,\ldots,x_n)\in f^{-1}(y)} J(x_1, \ldots, x_n),$$

where $f^{-1}(y) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : f(x_1, \ldots, x_n) = y\}$.

Theorem 1.4. *[72, 5] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and a JPD $J \in \mathcal{F}(\mathbb{R}^n)$ of $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$. If $[J]_\alpha$ is a compact set for all $\alpha \in [0, 1]$, then the* sup $-J$ *extension of f at $(A_1, \ldots, A_n)$ has α-cuts given by*

$$[\widehat{f}_J(A_1, \ldots, A_n)]_\alpha = f([J]_\alpha),$$

for all $\alpha \in [0, 1]$.

Proposition 1.3. *[60] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and a JPD $J \in \mathcal{F}(\mathbb{R}^n)$ of $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$. If $[J]_\alpha$ is a connected compact non-empty set for all $\alpha \in [0, 1]$, then $\widehat{f}_J(A_1, \ldots, A_n)$ is a fuzzy number.*

Under the conditions of Proposition 1.3, the $\alpha-$cuts of $\widehat{f}_J(A_1, \ldots, A_n) = \widehat{f}(J)$ can be written as follows:

$$[\widehat{f}_J(A_1, \ldots, A_n)]_\alpha = \left[\bigwedge_{(x_1,\ldots,x_n)\in[J]_\alpha} f(x_1, \ldots, x_n) \quad , \quad \bigvee_{(x_1,\ldots,x_n)\in[J]_\alpha} f(x_1, \ldots, x_n)\right]. \quad (1.16)$$

In particular, if the JPD J of $A_1, \ldots, A_n$ fuzzy numbers is $J = J_\wedge$ or $J = J_L$, then the sup $-J$ extension $\widehat{f}_J(A_1, \ldots, A_n)$ is also a fuzzy number and its α-cuts can be calculated by Equation (1.16).

A combination of the JPD J_L and Equation (1.16) will be used to extend and characterize fuzzy functions in the proposed fuzzy least square method in Chapter 2.

The next section presents two types of arithmetic for fuzzy numbers: interactive and non-interactive arithmetic.

1.6 Arithmetic on Fuzzy Numbers

Usual (or standard) operations $\circledast$, such as addition, subtraction, multiplication and division, are defined through Zadeh's extension principle (see Definition 1.24), of the function $f(x_1, \ldots, x_n) = x_1 \circledast \ldots \circledast x_n$, where $\circledast \in \{+, -, \times, \div\}$. The α-cuts of the fuzzy set $A \circledast B$ are given by [42, 81]

$$[A \circledast B]_\alpha = [A]_\alpha \circledast [B]_\alpha,$$

for all $\alpha \in [0, 1]$. More precisely, the α-cuts of standard operations are calculated as follows.

Proposition 1.4. *[4] Let $A, B \in \mathbb{R}_\mathcal{F}$, with α-cuts $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$ and $[B]_\alpha = [b_\alpha^-, b_\alpha^+]$. The α-cuts of the standard operations of A and B are given by*

1. *Addition: $[A + B]_\alpha = [a_\alpha^- + b_\alpha^-, a_\alpha^+ + b_\alpha^+]$;*
2. *Subtraction: $[A - B]_\alpha = [a_\alpha^- - b_\alpha^+, a_\alpha^+ - b_\alpha^-]$;*
3. *Multiplication: $[A \times B]_\alpha = [\min\{a_\alpha^- b_\alpha^-, a_\alpha^- b_\alpha^+, a_\alpha^+ b_\alpha^-, a_\alpha^+ b_\alpha^+\},\ \max\{a_\alpha^- b_\alpha^-, a_\alpha^- b_\alpha^+, a_\alpha^+ b_\alpha^-, a_\alpha^+ b_\alpha^+\}]$;*
4. *Division: $[A \div B]_\alpha = \left[\min\left\{\frac{a_\alpha^-}{b_\alpha^-}, \frac{a_\alpha^-}{b_\alpha^+}, \frac{a_\alpha^+}{b_\alpha^-}, \frac{a_\alpha^+}{b_\alpha^+}\right\}, \max\left\{\frac{a_\alpha^-}{b_\alpha^-}, \frac{a_\alpha^-}{b_\alpha^+}, \frac{a_\alpha^+}{b_\alpha^-}, \frac{a_\alpha^+}{b_\alpha^+}\right\}\right]$, if $0 \notin [B]_0$.*

Furthermore, the product of a fuzzy number A by a scalar λ is also a fuzzy number, denoted by λA, whose α-cuts are given by

$$[\lambda A]_\alpha = \lambda [A]_\alpha = \begin{cases} [\lambda a_\alpha^-, \lambda a_\alpha^+], & \text{if } \lambda \geqslant 0 \\ [\lambda a_\alpha^+, \lambda a_\alpha^-], & \text{if } \lambda < 0 \end{cases}.$$

Example 1.9. *Let $A = (0; 1; 2)$, with α-cuts $[A]_\alpha = [\alpha, 2 - \alpha]$. For $\lambda = -1$, we have $B = \lambda A = -A$, where $[-A]_\alpha = [\alpha - 2, -\alpha]$. The sum $A + B = A - A$ has α-cuts given by $[A - A]_\alpha = [2\alpha - 2, 2 - 2\alpha]$. Thus $A - A = (-2, 0, 2)$.*

As we can see in Example 1.9, the usual subtraction of fuzzy numbers $(-)$ does not extend the property $a - a = 0$, for all $a \in \mathbb{R}$, to the class of fuzzy numbers. In Fuzzy Set Theory there are several attempts to define a difference to which, for any $A \in \mathbb{R}_\mathcal{F}$, the following equality holds true

$$A - A = 0,$$

where 0 is the crisp number zero represented by $\chi_{\{0\}}$.

Definition 1.26 (Hukuhara Difference [48, 91]). *Let $A, B \in \mathbb{R}_{\mathcal{F}}$. The Hukuhara difference of A and B is defined by*

$$A -_H B = C \Leftrightarrow A = B + C,$$

where + stands for standard addition. If there exists such $C \in \mathbb{R}_{\mathcal{F}}$, then the difference is called H-difference.

The existence of Hukuhara difference requires that $\text{width}([A]_\alpha) \geqslant \text{width}([B]_\alpha)$ (see Definition 1.11), for all $\alpha \in [0, 1]$. If it exists, the α-cuts of $A -_H B$ are given by [91]

$$[A -_H B]_\alpha = [a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+],$$

for all $\alpha \in [0, 1]$. Note that the Hukuhara difference satisfies $A -_H A = 0$, for all $A \in \mathbb{R}_{\mathcal{F}}$.

Example 1.10. *Let $A = (2; 4; 6)$ and $B = (1; 2; 3)$. The Hukuhara difference $A -_H B$ exists and $A -_H B = (1; 2; 3)$. Nonetheless, the Hukuhara difference of B and A does not exist.*

The generalized Hukuhara difference was proposed by Bede, Gal and Stefanini [11, 101, 100] in order to extend the Hukuhara difference to a broad class of pairs of fuzzy numbers.

Definition 1.27 (Generalized Hukuhara Difference [11, 101, 100]). *Let $A, B \in \mathbb{R}_{\mathcal{F}}$. The generalized Hukuhara difference is given by*

$$A -_{gH} B = C \Leftrightarrow \begin{cases} A = B + C, & or \\ B = A - C \end{cases},$$

where + and − stand for the standard addition and subtraction, respectively. If there exists such difference, it is called, for short, gH-difference.

Note that the first case in Definition 1.27 coincides with the definition of H-difference. Hence, if there exists the H-difference of two fuzzy numbers, then gH-difference of these fuzzy numbers also exists and coincides with the H-difference.

Example 1.11. *Consider A and B given as in Example 1.10. Even though the H-difference of B and A does not exist, we have gH-difference $B -_{gH} A = (-3; -2; -1)$.*

If there exists the gH-difference of $A, B \in \mathbb{R}_{\mathcal{F}}$, then its α-cut are given by [12]

$$[A -_{gH} B]_\alpha = [\min\{a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+\}, \max\{a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+\}],$$

for all $\alpha \in [0, 1]$.

It is important to observe that the gH-difference of triangular fuzzy numbers does not always exist. Example 1.12 illustrates this fact.

Example 1.12. *Let $A = (0; 1; 3)$ and $B = (2; 4; 5)$. If $A -_{gH} B$ exists, then its α-cuts are given by $[\min\{\alpha - (2 + 2\alpha), 3 - 2\alpha - (5 - \alpha)\}, \max\{\alpha - (2 + 2\alpha), 3 - 2\alpha - (5 - \alpha)\}]$, which implies that $[A -_{gH} B]_\alpha = [-2 - \alpha, -2 - \alpha]$, for all $\alpha \in [0, 1]$. Since $[A -_{gH} B]_0 = [-2, -2]$ and $[A -_{gH} B]_1 = [-3, -3]$, the Stacking theorem (see Theorem 1.1) is not satisfied. Thus, $A -_{gH} B$ is not a fuzzy number. Hence, the gH-difference of A and B does not exist.*

From Example 1.12, one can note that gH-difference may not exist. On the other hand, there is another difference which always exists. It is known as generalized difference (g-difference for short) and was proposed by Bede and Stefanini [12]. Later Gomes and Barros [46] updated the definition as follows.

Definition 1.28 (Generalized Difference [12, 46])**.** *Let $A, B \in \mathbb{R}_\mathcal{F}$. The g-difference is defined by its α-cuts:*

$$[A -_g B]_\alpha = \mathrm{cl}\left\{\mathrm{conv}\left(\bigcup_{\beta \geqslant \alpha}([A]_\beta -_{gH} [B]_\beta)\right)\right\},$$

where $\mathrm{conv}(X)$ is the convex hull of X and $\mathrm{cl}(X)$ is the closure of X.

The α-cuts of $A -_g B$ are given by [10]

$$[A -_g B]_\alpha = \left[\bigwedge_{\beta \geqslant \alpha} \min\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\}, \bigvee_{\beta \geqslant \alpha} \max\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\}\right].$$

Example 1.13. *For A and B as in Example 1.12, the g-difference exists and its α-cuts are given by $[-3, 1]$, for all $\alpha \in [0, 1]$.*

If the gH-difference of A and B exists, then g-difference of A and B exists and they are equal, that is, for $A, B \in \mathbb{R}_\mathcal{F}$,

$$A -_H B = A -_{gH} B = A -_g B.$$

Under some weak conditions, Barros *et al.* [6] indicated the existence of a joint possibility distribution that provides the standard addition and the Hukuhara difference, using the sup-J extension principle (see Definition 1.25). Wasques *et al.* [118, 112] proved that the generalized difference is equal to an interactive difference, that is defined through sup-J extension principle, with the JPD given in Definition 1.20. Therefore, it is worth studying interactive arithmetic. From Definition 1.25, arithmetic operations among n fuzzy numbers can be defined by taking the sup-J extension of the corresponding arithmetic operator. In this case, not only the fuzzy numbers are given, but also the associated JPD.

Let $\circledast$ be an arithmetic operator and JPD $J_\wedge$ for $A_1, \ldots, A_n$, i.e., $A_1, \ldots, A_n$ are non-interactive. The $\sup -J$ extension $\widehat{f}_{J_\wedge}(A_1, \ldots, A_n)$ boils down to Zadeh's extension of

f at $(A_1, \ldots, A_n)$, i.e.,

$$\widehat{f}_{J_\wedge}(A_1, \ldots, A_n)(y) = \bigvee_{(x_1,\ldots,x_n)\in f^{-1}(y)} A_1(x_1) \wedge \ldots \wedge A_n(x_n), \quad \forall\, y \in \mathbb{R},$$

where $y = f(x_1, \ldots, x_n) = x_1 \circledast \ldots \circledast x_n$and $\circledast \in \{+, -, \times, \div\}$.

For any JPD J of $A_1, \ldots, A_n$, the interactive arithmetic with respect to J is defined by $\widehat{f}_J(A_1, \ldots, A_n)$

$$\widehat{f}_J(A_1, \ldots, A_n)(y) = \bigvee_{(x_1,\ldots,x_n)\in f^{-1}(y)} J(x_1, \ldots, x_n), \quad \forall\, y \in \mathbb{R},$$

where $y = f(x_1, \ldots, x_n) = x_1 \circledast \ldots \circledast x_n$ and $\circledast \in \{+, -, \times, \div\}$.

From this point of view, the standard arithmetic can be obtained from the sup-J extension principle when $J = J_\wedge$. In this case, the standard arithmetic is called non-interactive arithmetic. Any other arithmetic obtained from the sup-J extension principle such that $J \neq J_\wedge$ is called interactive arithmetic. Moreover, it follows that $A \circledast_J B \subseteq A \circledast_{J_\wedge} B$, for all $A, B \in \mathbb{R}_{\mathcal{F}_C}$ and $\circledast \in \{+, -, \times, \div\}$ [115].

For J_0 given in Definition 1.20, the resulting arithmetic operations have been used in fuzzy differential equations [112, 117, 114, 113] and in fuzzy linear equations with triangular fuzzy numbers [120], by using Theorem 1.5. In addition, Wasques et al. proved that the generalized difference $(-_g)$ is actually the interactive difference with respect to J_0^c, which is denoted by $-_I$ [118].

Theorem 1.5. *[118] Let $A = (a; b; c)$ and $B = (d; e; f)$ be triangular fuzzy numbers. Let J_0 be the joint possibility distribution between A and B given by (1.10). Thus,*

$$A +_0 B = \begin{cases} ((a+f) \wedge (b+e); b+e; (b+e) \vee (c+d)), & \textit{if } \operatorname{width}(A) \geqslant \operatorname{width}(B) \\ ((c+d) \wedge (b+e); b+e; (b+e) \vee (a+f)), & \textit{if } \operatorname{width}(A) \leqslant \operatorname{width}(B) \end{cases}. \tag{1.17}$$

A particular but interesting case arises for linearly interactive fuzzy numbers, whose arithmetic is defined as follows.

Definition 1.29 (Arithmetic Operations [17])**.** *Let $A, B \in \mathbb{R}_{\mathcal{F}}$ linearly interactive fuzzy numbers, with respect to a JPD J_L. The interactive arithmetic operations are given by*

1. *Addition:*
$$(A +_L B)(y) = \sup_{y=x_1+x_2} A(x_1)\chi_{\{qx_1+r=x_2\}}(x_1, x_2); \tag{1.18}$$

2. *Subtraction:*
$$(A -_L B)(y) = \sup_{y=x_1-x_2} A(x_1)\chi_{\{qx_1+r=x_2\}}(x_1, x_2); \tag{1.19}$$

3. *Multiplication:*

$$(A \times_L B)(y) = \sup_{y = x_1 \times x_2} A(x_1)\chi_{\{qx_1 + r = x_2\}}(x_1, x_2); \tag{1.20}$$

4. *Division:*

$$(A \div_L B)(y) = \sup_{y = x_1 \div x_2} A(x_1)\chi_{\{qx_1 + r = x_2\}}(x_1, x_2). \tag{1.21}$$

Clearly, the operations based on J_L do not always exist but, when they exist, their α-cuts can be easily calculated, by the following proposition.

Proposition 1.5. *[17, 50] If the addition, subtraction, multiplication or division of the linearly interactive fuzzy numbers A and B exist, then its α-cuts are given by*

1. *Addition:*

$$[A +_L B]_\alpha = (q + 1)[A]_\alpha + r; \tag{1.22}$$

2. *Subtraction:*

$$[A -_L B]_\alpha = (1 - q)[A]_\alpha + r; \tag{1.23}$$

3. *Multiplication:*

$$[A \times_L B]_\alpha = \{qx_1^2 + rx_1 \in \mathbb{R} \ : \ A(x_1) \geqslant \alpha\}; \tag{1.24}$$

4. *Division:*

$$[A \div_L B]_\alpha = \left\{ \frac{x_1}{qx_1 + r} \in \mathbb{R} \ : \ A(x_1) \geqslant \alpha \right\}. \tag{1.25}$$

In view of Proposition 1.5, the α-cut of the interactive addition and subtraction can be written in terms of parameter q as follows:

$$[A +_L B]_\alpha = \begin{cases} [a_\alpha^- + b_\alpha^-, a_\alpha^+ + b_\alpha^+], \text{ if } q > 0 \\ [a_\alpha^- + b_\alpha^+, a_\alpha^+ + b_\alpha^-], \text{ if } -1 \leqslant q < 0 \\ [a_\alpha^+ + b_\alpha^-, a_\alpha^- + b_\alpha^+], \text{ if } q < -1 \end{cases}, \tag{1.26}$$

$$[A -_L B]_\alpha = \begin{cases} [b_\alpha^- - a_\alpha^-, b_\alpha^+ - a_\alpha^+], \text{ if } q \geqslant 1 \\ [b_\alpha^+ - a_\alpha^+, b_\alpha^- - a_\alpha^-], \text{ if } 0 < q < 1 \\ [b_\alpha^- - a_\alpha^+, b_\alpha^+ - a_\alpha^-], \text{ if } q < 0 \end{cases}. \tag{1.27}$$

Example 1.14. *Recall that the fuzzy numbers $A = (1; 2; 3)$ and $B = (-1; 0; 1)$ from Example 1.4, which can be associated by two different JPDs J_L. From (1.22) and (1.23) it is possible to obtain the interactive sum and difference using these two JPDs:*

- *For $J_1 = J_{\{1,-2\}}$, we have $A +_{J_1} B = (0; 2; 4)$ and $A -_{J_1} B = \{-2\}$;*

- *For* $J_2 = J_{\{-1,2\}}$*, we have* $A +_{J_2} B = \{2\}$ *and* $A -_{J_2} B = (-4; -2; 0)$*.*

Applications considering arithmetic based on J_L can be found in fuzzy optimization problems [82, 83] and in fuzzy differential equations [112, 49, 116].

1.7 Norm for Fuzzy Numbers

Let $\mathcal{K}$ be the class of the compact sets a normed space $(X, ||\cdot||)$. The distance between $a \in X$ and $B \in \mathcal{K}$ is given by

$$d(a, B) = \bigwedge_{b \in B} ||a - b||, \tag{1.28}$$

where $\bigwedge$ stands for minimum operation.

Definition 1.30 (Hausdorff Separation [47])**.** *Let* $A, B \in \mathcal{K}$*. The Hausdorff separation is given by*

$$d_H^*(A, B) = \bigvee_{a \in A} d(a, B), \tag{1.29}$$

where $d(a, B)$ *is given in Equation* (1.28)*.*

Definition 1.31 (Pompeiu-Hausdorff metric [47])**.** *Let* $A, B \in \mathcal{K}$*. The Pompeiu-Hausdorff metric is given by*

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\}, \tag{1.30}$$

where d_H^* *is given by Equation* (1.29)*.*

Proposition 1.6. *[24] The Pompeiu-Hausdorff metric is a metric in* $\mathcal{K}$*.*

It is possible to define a metric on $\mathbb{R}_{\mathcal{F}}$. For this class, the metric is obtained in terms of the levelwise metric d_H as follows.

Definition 1.32 (Pompeiu-Hausdorff distance [24])**.** *Let* $A, B \in \mathbb{R}_{\mathcal{F}}$*. The Pompeiu-Hausdorff distance* d_F *is given by*

$$d_F(A, B) = \bigvee_{0 \leqslant \alpha \leqslant 1} d_H([A]_\alpha, [B]_\alpha), \tag{1.31}$$

where d_H *is given by Equation* (1.30)*.*

In other words, for any $A, B \in \mathbb{R}_{\mathcal{F}}$, the Pompeiu-Hausdorff is given by

$$d_F(A, B) = \bigvee_{0 \leqslant \alpha \leqslant 1} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}. \tag{1.32}$$

Remark 1.2. *Note that the distance in Equation (1.32) has a supremum operation, thus it is not easily computable distance. More than that, to minimize the Hausdorff distance between two fuzzy numbers is a hard task, both from mathematical and computational implementation point of view. For this reason, Hausdorff norm is not a great choice to compare fuzzy numbers and, therefore, to adjust curves to fuzzy data.*

The class of fuzzy numbers $(\mathbb{R}_{\mathcal{F}}, +)$ is not a vector space, consequently, is not normed space. Nevertheless, a quasi-norm based on Pompeiu-Hausdorff metric can be defined:

$$||A||_{\mathcal{F}} = d_F(A, 0), \tag{1.33}$$

where d_F is given by (1.32) and 0 is given by characteristic function $\chi_{\{0\}}$. The quasi-norm $||\cdot||_{\mathcal{F}}$ in (1.33) is also called, by language abuse, fuzzy norm.

Consider $A \in \mathbb{R}_{\mathcal{F}}$. The α-cuts of $\{0\}$ are given by $[0]_\alpha = \{0\}$, for all $\alpha \in [0,1]$. From Theorem 1.1, $a_0^- \leqslant a_\alpha^- \leqslant a_\alpha^+ \leqslant a_0^+$, for all $\alpha \in [0,1]$, hence. Thus,

$$\begin{aligned} ||A||_{\mathcal{F}} &= \bigvee_{0 \leqslant \alpha \leqslant 1} \max\{|a_\alpha^- - 0|, |a_\alpha^+ - 0|\} \\ &= \max\{|a_0^-|, |a_0^+|\}. \end{aligned}$$

Note that the fuzzy norm of $A = (a; b; c)$ is $||A||_{\mathcal{F}} = \max\{|a|, |c|\}$, and the fuzzy norm of $B = (a; b; c; d)$ is $||B||_{\mathcal{F}} = \max\{|a|, |d|\}$.

It is worth to emphasize that Hausdorff norm measures only the support of a fuzzy number, thus it can not properly distinguish fuzzy numbers with the same support but different cores. In Chapter 4 (see Section 4.1) we present some examples in which we criticize the use of Pompeiu-Hausdorff metric. Also, we propose a new metric for fuzzy numbers with continuous endpoints.

1.8 Conclusion

This chapter provided the background of Fuzzy Sets Theory that is necessary for this thesis. The definitions of fuzzy sets and fuzzy numbers were provided. This chapter focused on the relation of interactivity and two particular types of interactivity were presented: one is associated with J_L and the other with associated with J_0^c. The arithmetic for linearly interactive fuzzy numbers was also established. At the end, we briefly discussed fuzzy norms.

2 Fuzzy Least Square Method for Linearly Interactive Fuzzy Numbers

This chapter discusses the curve fitting problem with uncertainty in the outputs. In general these problems consist in finding fuzzy parameters for a fuzzy curve that fits some available data.

Least squares methods are used, in general, to obtain a function in the linear spam of a given family of functions $\{g_1, \ldots, g_n\}$, $g_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, n$, that best fits pairs of data in a dataset $D = \{(x_1, y_1), \ldots, (x_m, y_m)\} \subseteq \mathbb{R}^m \times \mathbb{R}^m$, in some sense [14]. The fuzzy least squares method arises when the dataset is composed by fuzzy numbers as outputs, that is, $D = \{(x_1, Y_1), \ldots, (x_m, Y_m)\} \subseteq \mathbb{R}^m \times \mathbb{R}^m_{\mathcal{F}}$. In this case the goal is to find a fuzzy-valued function $\phi : \mathbb{R}^n \to \mathbb{R}_{\mathcal{F}}$ by the form $\phi(x) = A_1 g_1(x) + \ldots + A_n g_n(x)$, such that $\phi(x_i) \approx Y_i$, for each $i = 1, \ldots, m$.

The chapter is divided as follows. Section 2.1 presents the classical method, with crisp dataset. Section 2.2 shows how to extend the classical method via Zadeh Extension Principle (see Definition 1.24) for linearly interactive data.

2.1 Classical Least Square Method

This section presents the classical least squares method [14], which can be used for curve fitting.

Let $f : [c, d] \to \mathbb{R}$ be a continuous function. Given n functions $g_1, \ldots, g_n$, where $g_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, n$, the goal is to find n coefficients $a_1, \ldots, a_n \in \mathbb{R}$ such that the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = a_1 g_1(x) + \ldots + a_n g_n(x) \tag{2.1}$$

is the best approximation of the function f, *i.e.*, $\varphi \approx f$.

The function φ is obtained by minimizing the distance between f and φ. More precisely, let $||\cdot||_2$ be the $\mathcal{L}^2$-norm defined on the class of the continuous functions from $[c, d]$ to $\mathbb{R}$ (denoted by $C([c, d])$) given by $||h||_2 = \left(\int_c^d |h(s)|^2 ds\right)^{1/2}$, for all $h \in C([c, d])$. The coefficients $a_1, \ldots, a_n$ of the function φ which produces the best fit to f are obtained by solving the following minimization problem:

$$\min_{a_1,\ldots,a_n \in \mathbb{R}} {}^1\!/_2 ||\varphi - f||_2^2.$$

In the case where some values of f are known, say $D = \{(x_1, y_1), \ldots, (x_m, y_m)\}$, where $x_i \in \mathbb{R}$ and $y_i = f(x_i)$, for all $i = 1, \ldots, m$. The function φ must fit the data D, that is, $\varphi(x_i) \approx y_i$, for all $i = 1, \ldots, m$. Therefore the following minimization problem must be solved.

$$\min_{a_1,\ldots,a_n \in \mathbb{R}} {}^1\!/_2 \|(\varphi(x_1) - y_1, \ldots, \varphi(x_m) - y_m)\|_2^2. \tag{2.2}$$

The real coefficients $a_1, \ldots, a_n$ that minimize the problem (2.2), *i.e.*, that produces the best approximation φ of f, are obtained by solving the following minimization problem which corresponds to the matricial form of (2.2).

$$\min_{a \in \mathbb{R}^n} \|Ga - y\|_2^2, \tag{2.3}$$

where

$$G = \begin{bmatrix} g_1(x_1) & \ldots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_m) & \ldots & g_n(x_m) \end{bmatrix}, \; a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \; \text{and} \; y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

The solution of (2.3) can be obtained by solving the associated normal equation, which is given by

$$(G^T G)a = b, \tag{2.4}$$

where $b = G^T y$.

If the matrix $G^T G$ is non-singular, say $P = (G^T G)^{-1} = [p_{ij}]$, then the vector a is obtained by

$$a = Pb \text{ or } a = Cy, \tag{2.5}$$

where

$$C = (G^T G)^{-1} G^T. \tag{2.6}$$

Thus, each parameter a_i is given by

$$\begin{aligned} a_i &= p_{i1} b_1 + p_{i2} b_2 + \ldots + p_{in} b_n \\ &= p_{i1} \left(\sum_{k=1}^{m} y_k g_1(x_k) \right) + \ldots + p_{in} \left(\sum_{k=1}^{m} y_k g_n(x_k) \right) \\ &= \left(\sum_{j=1}^{n} p_{ij} g_j(x_1) \right) y_1 + \ldots + \left(\sum_{j=1}^{n} p_{ij} g_j(x_m) \right) y_m \\ &= c_{i1} y_1 + \ldots + c_{im} y_m, \end{aligned}$$

where $c_{ik} = \sum_{j=1}^{n} p_{ij} g_j(x_k)$, for $i = 1, \ldots, n$ and $k = 1, \ldots, m$. In general case, the matrix C stands for the pseudoinverse of G, denoted by $G^\dagger$.

Note that Equation (2.4) always have a solution, since $Im(G^T G) = Im(G^T)$, where $Im(G)$ stands for the image of the linear transformation G. For the case Equation (2.4) has more than one solution, the solution has minimum Euclidian norm [121]. For the purpose of this thesis, it is enough to consider that G has full rank, that is, $G^T G$ is positive definite, and this assumption does not cause loss of generality.

Since the parameters of the function φ can be obtained by computing the matrix product (2.5), the function φ can be rewritten in terms of $y_1, \ldots, y_m$ as follows:

$$\begin{aligned}
\varphi(x) &= a_1 g_1(x) + \ldots + a_n g_n(x) \qquad (2.7)\\
&= (c_{11}y_1 + \ldots + c_{1m}y_m)g_1(x) + \ldots + (c_{n1}y_1 + \ldots + c_{nm}y_m)g_n(x)\\
&= \left(\sum_{j=1}^{n} g_j(x)c_{j1}\right) y_1 + \ldots + \left(\sum_{j=1}^{n} g_j(x)c_{jm}\right) y_m\\
&= s_1(x)y_1 + \ldots + s_m(x)y_m, \qquad (2.8)
\end{aligned}$$

where

$$s_i(x) = \left(\sum_{j=1}^{n} g_j(x)c_{ji}\right)$$

for each $i = 1, \ldots, n$.

Note that, defining $S(x) = (s_1(x), \ldots, s_m(x))$, the function φ in (2.7) can be simply written by

$$\varphi(x) = \langle S(x), y\rangle, \qquad (2.9)$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product of $\mathbb{R}^m$.

This classical approach can not be applied directly to the case where dataset is given by fuzzy sets or fuzzy numbers. The next section provides the first contribution of this thesis, which is a generalization of this classical method for fuzzy data.

2.2 Least Squares Method for Interactive Fuzzy Data via Extension Principle

This section deals with least squares method to fit uncertain data given by interactive fuzzy numbers [88]. In particular, we focus on the case where these fuzzy numbers are linearly interactive. A typical example of correlated data are the well-known longitudinal data, which are widely studied in the statistical area [129].

Let $D = \{(x_1, Y_1), \ldots, (x_m, Y_m)\} \subset \mathbb{R} \times \mathbb{R}_{\mathcal{F}}$ be such that $Y_1, \ldots, Y_m$ are linearly interactive fuzzy numbers, with respect to a joint possibility distribution J as in (1.13), and let $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a function that satisfies $F(x_i) = Y_i$ for $i = 1, \ldots, m$. The function $\Phi : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ that approximates F is obtained by means of the sup-J extension principle of a function $\varphi : \mathbb{R} \to \mathbb{R}$ of the form

$$\varphi(x) = a_1 g_1(x) + \ldots + a_n g_n(x),$$

where $a_1, \ldots, a_n \in \mathbb{R}$ and $g_1, \ldots, g_n$ are real-valued-functions. More precisely, the function Φ is defined in terms of the sup-J extension principle of (2.7) at $(Y_1, \ldots, Y_m)$. Since (2.7) is continuous with respect to $y_1, \ldots, y_m$, from Theorem 1.4 and Equation (1.14), the α-cuts of the fuzzy number $\Phi(x)$ are given by

$$\begin{aligned}[\Phi(x)]_\alpha &= \{s_1(x)y_1 + \ldots + s_m(x)y_m : (y_1, \ldots, y_m) \in [J]_\alpha\} \\ &= \{s_1(x)y + s_2(x)(q_2 y + r_2) + \ldots + s_m(x)(q_m y + r_m) : y \in [Y_1]_\alpha\}.\end{aligned} \tag{2.10}$$

Since the interval $[Y_1]_\alpha = [y_{1\alpha}^-, y_{1\alpha}^+]$ can be rewritten as the set of all convex combination of $y_{1\alpha}^-$ and $y_{1\alpha}^+$, that is, $[Y_1]_\alpha = \{(1-\lambda)y_{1\alpha}^- + \lambda y_{1\alpha}^+ : \lambda \in [0,1]\}$, the α-cut of J can also be expressed in terms of a parameter $\lambda \in [0,1]$ as follows:

$$[J]_\alpha = \{(1-\lambda)Y_\alpha^- + \lambda Y_\alpha^+ : \lambda \in [0,1]\},$$

where $Y_\alpha^- = (y_{1\alpha}^-, q_2 y_{1\alpha}^- + r_2, \ldots, q_m y_{1\alpha}^- + r_m)$ and $Y_\alpha^+ = (y_{1\alpha}^+, q_2 y_{1\alpha}^+ + r_2, \ldots, q_m y_{1\alpha}^+ + r_m)$. Thus, Equation (2.10) can be expressed as

$$[\Phi(x)]_\alpha = \{(1-\lambda)B_1(x,\alpha) + \lambda B_2(x,\alpha) : \lambda \in [0,1]\} \tag{2.11}$$

where $S(x) = (s_1(x), s_2(x), \ldots, s_m(x))$, $x \in \mathbb{R}$, $B_1(x,\alpha) = \langle S(x), Y_\alpha^- \rangle$, and $B_2(x,\alpha) = \langle S(x), Y_\alpha^+ \rangle$. Therefore,

$$\begin{aligned}[\Phi(x)]_\alpha &= \{h(x,\alpha,\lambda) : \lambda \in [0,1]\} \\ &= \left[\bigwedge_{\lambda \in [0,1]} h(x,\alpha,\lambda), \bigvee_{\lambda \in [0,1]} h(x,\alpha,\lambda)\right],\end{aligned} \tag{2.12}$$

with $h(x,\alpha,\lambda) = (1-\lambda)B_1(x,\alpha) + \lambda B_2(x,\alpha)$.

Note that if $B_1(x,\alpha) \leqslant B_2(x,\alpha)$, then the function $h(x,\alpha,\cdot)$ assumes the minimum and the maximum values at $\lambda = 0$ and $\lambda = 1$, respectively. On the other hand, if $B_1(x,\alpha) > B_2(x,\alpha)$ then the minimum and maximum values of $h(x,\alpha,\cdot)$ are achieved at $\lambda = 1$ and $\lambda = 0$, respectively. In other words, the global minimizer and maximizer of $h(x,\alpha,\lambda)$ for $\lambda \in [0,1]$ are given at $\lambda = 0$ or $\lambda = 1$. Therefore, for each $x \in \mathbb{R}$, the α-cuts of the fuzzy solution φ are given by

$$[\Phi(x)]_\alpha = [\min\{h(x,\alpha,0), h(x,\alpha,1)\}, \max\{h(x,\alpha,0), h(x,\alpha,1)\}], \tag{2.13}$$

where

$$h(x,\alpha,0) = B_1(x,\alpha) = \langle S(x), Y_\alpha^- \rangle$$

and

$$h(x,\alpha,1) = B_2(x,\alpha) = \langle S(x), Y_\alpha^+ \rangle.$$

Proposition 2.1. *Let $Y_1, \ldots, Y_m$ be linearly interactive fuzzy numbers outputs of data D. The sup-J extension Φ of the classical solution φ in $(Y_1, \ldots, Y_m)$ is a fuzzy number, with α-cuts given by Equation (2.13).*

Proof. This result follows from Proposition 1.3. If $Y_1, \ldots, Y_m$ are linearly interactive fuzzy numbers, then the underlying JDP J_L has closed intervals $[J_L]_\alpha$ for all $\alpha \in [0, 1]$. In particular, all the α-cuts of J_L are connected compact nonempty sets. Hence the sup-J_L extension of the continuous function $\varphi(x, y) = \langle S(x), y\rangle$ in $(Y_1, \ldots, Y_m)$ is the fuzzy number $\Phi(x, Y_1, \ldots, Y_m)$ with α-cuts given by Equation (2.13). □

Proposition 2.1 guarantees that the function Φ is well-defined, in terms of $Y_1, \ldots, Y_m$. Moreover, it assures that the α-cuts of Φ given in Equation (2.13) are nested.

Example 2.1. *Consider the data in Table 1. The first data is $Y_1 = (1; 2; 3)$ and the coefficients are $p = (1, 2, 3)$ and $r = (0, 1, 2)$. That is, Y_1, Y_2 and Y_3 are linearly interactive with J_L, where $L = \{px + r, x \in \mathbb{R}\}$.*

Table 1 – Dataset for Example 1

x	Y
1	$(1; 2; 3)$
2	$(3; 5; 7)$
3	$(5; 8; 11)$

Source: Author.

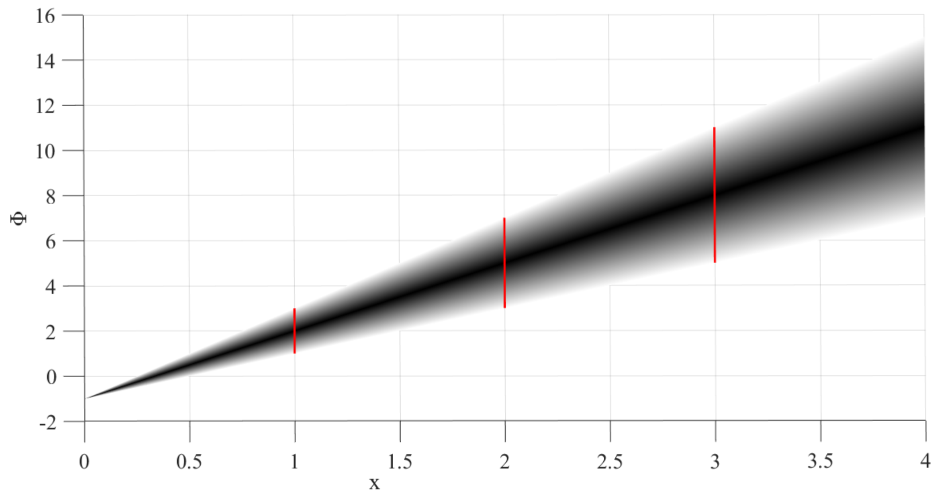
Considering the basis functions given by $g_1(x) = x^2$, $g_2(x) = x$ and $g_3(x) = 1$, the solution is given by

$$\Phi_1(x) = x - 1 + x \cdot (1; 2; 3). \tag{2.14}$$

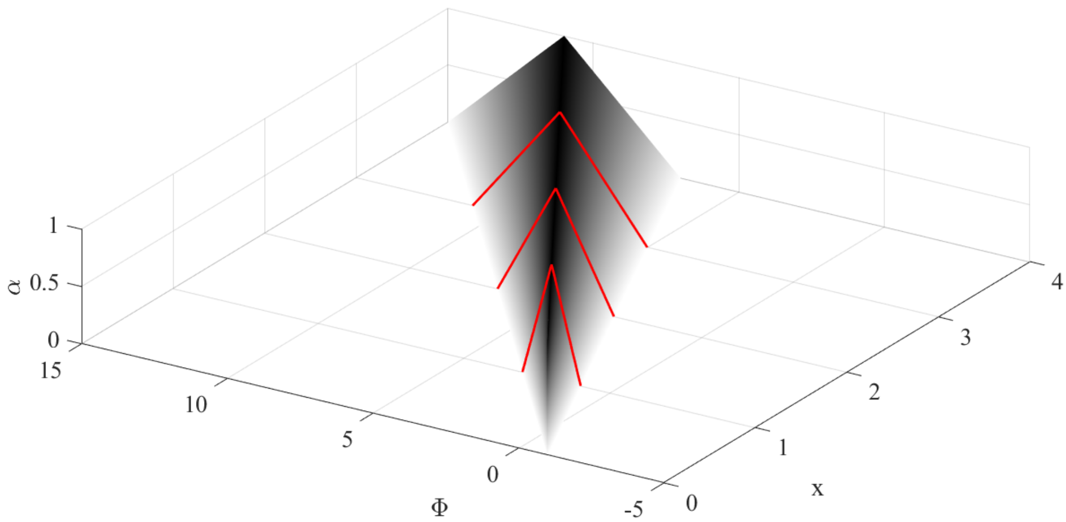
The solution Φ_1 is depicted in Figure 18. As we can see in Figure 18, the function Φ_1 fits the data perfectly. In fact, $\Phi_1(1) = Y_1$, $\Phi_1(2) = Y_2$ and $\Phi_1(3) = Y_3$. The 1-cut of function Φ is linear, and the specificity (see Definition 1.6) increases linearly with respect to x.

Figure 18 – Approximation Φ_1 for data in Example 2.1

(a) Top view.



(b) Tridimensional view.



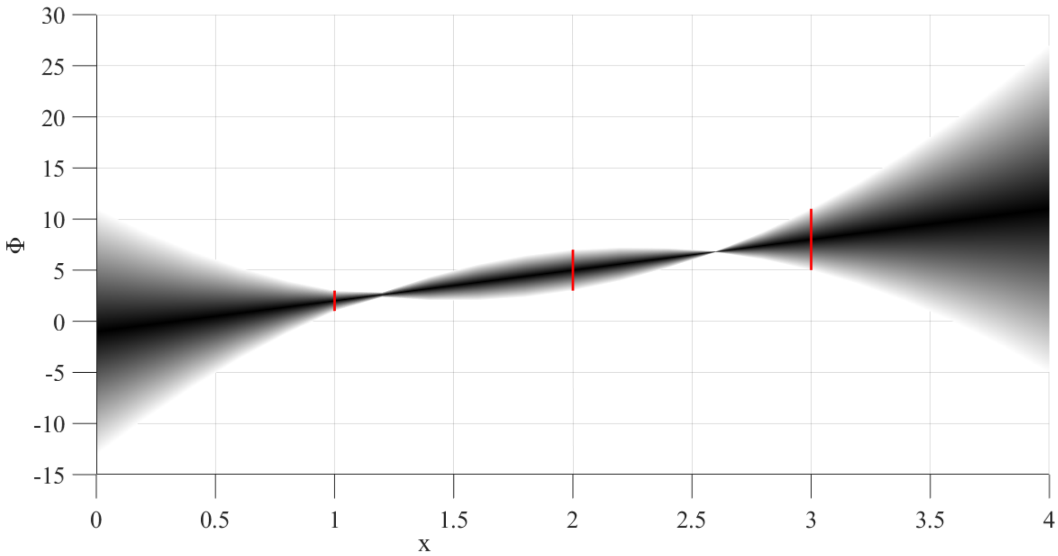
The top and tridimensional views of the fuzzy function Φ_1, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 1. Source: Author.

Since the first data is symmetric, there are three more JPD that one can consider for the data in Table 1. Let consider the JPD J_{L_2} given by $L_2 = \{px + r, x \in \mathbb{R}\}$, where $p = (1, -2, 3)$ and $r = (0, 9, 2)$. Using J_{L_2} we obtain a solution Φ_2 that is quite different of Φ_1, as we can observe in Figure 19. More precisely, the solution is given by

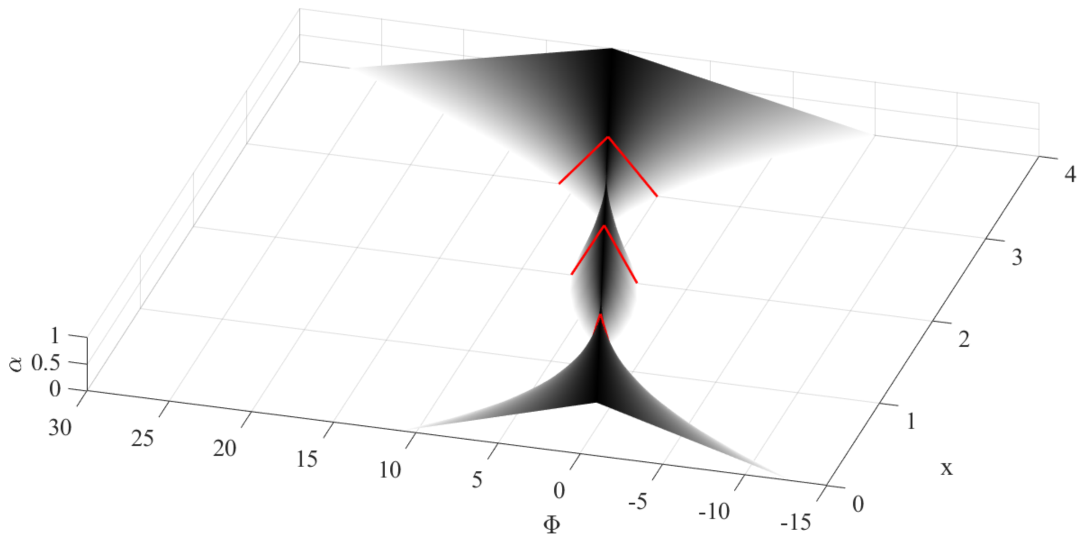
$$\Phi_2(x) = (4x^2 - 15x + 12) \cdot (1; 2; 3) + (-8x^2 + 33x - 25).$$

Figure 19 – Approximation Φ_2 for data in Example 2.1

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function Φ_2, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 1. Source: Author.

Note that for $x < 1$ and $x > 3$ the approximation Φ_2 is less specific than the expected. In the intervals $(1, 2)$ and $(2, 3)$ the specificity increases.

Since the basis functions are continuous and $p_1 > 0$, $p_2 < 0$ and $p_3 > 0$, the function $\Phi_2(x)$ must have two switch points, that is, the changing of type of correlation, from positively correlation to negatively correlation, or vice versa. Hence the approximation Φ_2 fits perfectly the dataset, but it does not represent the data behavior.

The next example shows capacity of the proposed method to model data with quadratic behavior.

Example 2.2. *Consider data in Table 2. The first data is* $Y_1 = (10; 11; 12)$ *and the coefficients of* J_L*, with* $L = \{px + r, x \in \mathbb{R}\}$*, are* $p = (1, 0.9, 0.75, 0.5)$ *and* $r = (0, -0.9, 0.2, 3)$.

Table 2 – Dataset for Example 2

x	Y
1	$(10; 11; 12)$
2	$(8.1; 9; 9.9)$
3	$(7.7; 8.45; 9.2)$
4	$(8; 8.5; 9)$

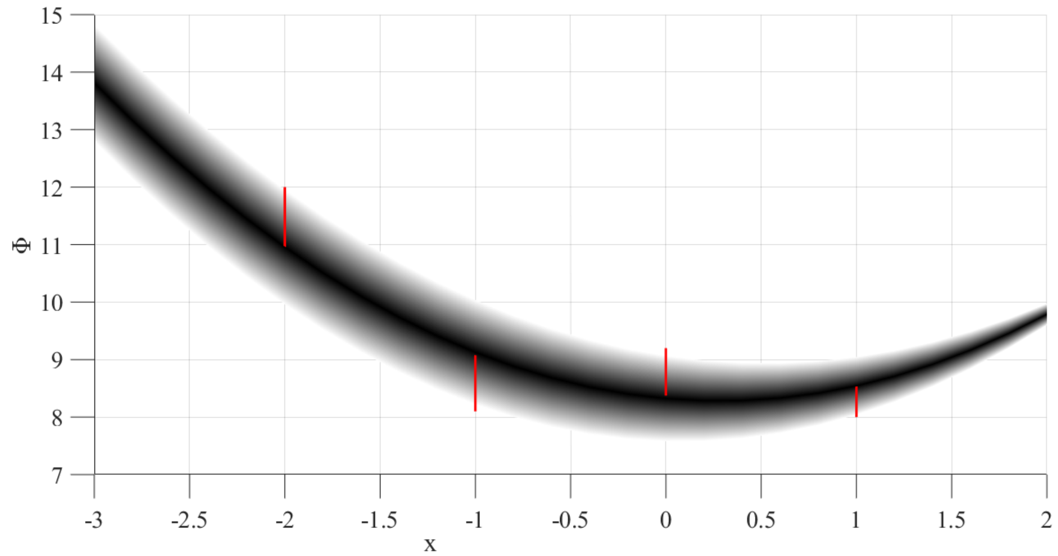
Source: Author.

Considering the basis functions $g_1(x) = x^2$, $g_2(x) = x$ *and* $g_3(x) = 1$*, the solution is given by*

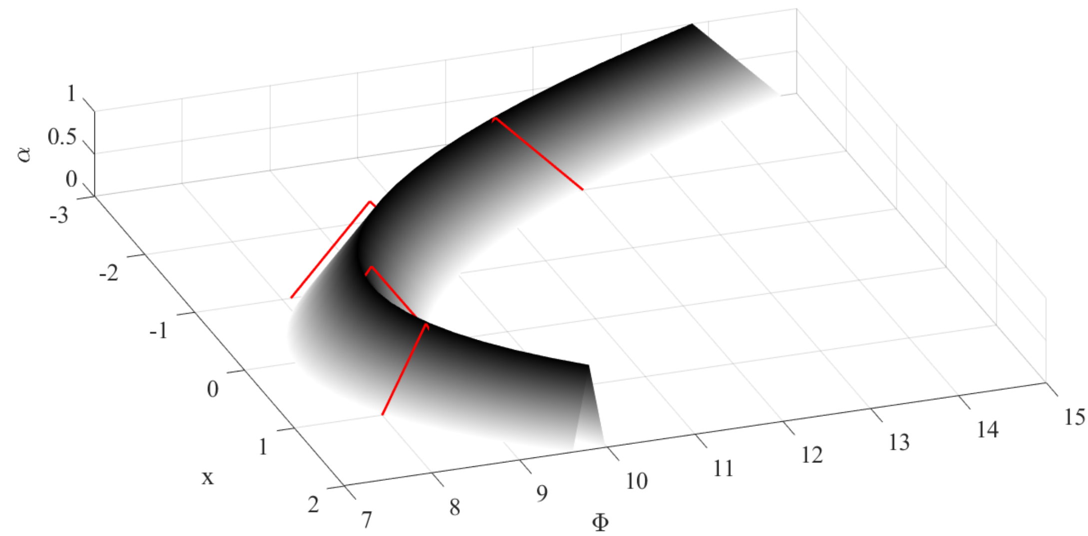
$$\Phi(x) = (-0.0375x^2 - 0.2025x + 0.7425) \cdot (10; 11; 12) + (0.155x^2 + 1.935x + 0.925). \quad (2.15)$$

Figure 20 – Approximation for data in Example 2.2

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function Φ, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 2. Source: Author.

The solution Φ *given in* (2.15) *is depicted in Figure 20. As we can see in Figure 20, the function* Φ *models the data behavior. The* 1*-cut of the fuzzy approximation is quadratic, and the specificity increases with respect to* $x > 0$.

Example 2.3. *Consider the data in Table 3. One of the associated JPDs is* J_L *where* $L = \{px + r, x \in \mathbb{R}\}$, $p = (1, 2, 1, 2)$ *and* $r = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$*. Note that the diameter oscillates as well as the* 1*-cut.*

Table 3 – Dataset for Example 3

x	Y
1	$(1;2;3)$
2	$\left(\frac{5}{2};\frac{9}{2};\frac{13}{2}\right)$
3	$(1;2;3)$
4	$\left(\frac{5}{2};\frac{9}{2};\frac{13}{2}\right)$

Source: Author.

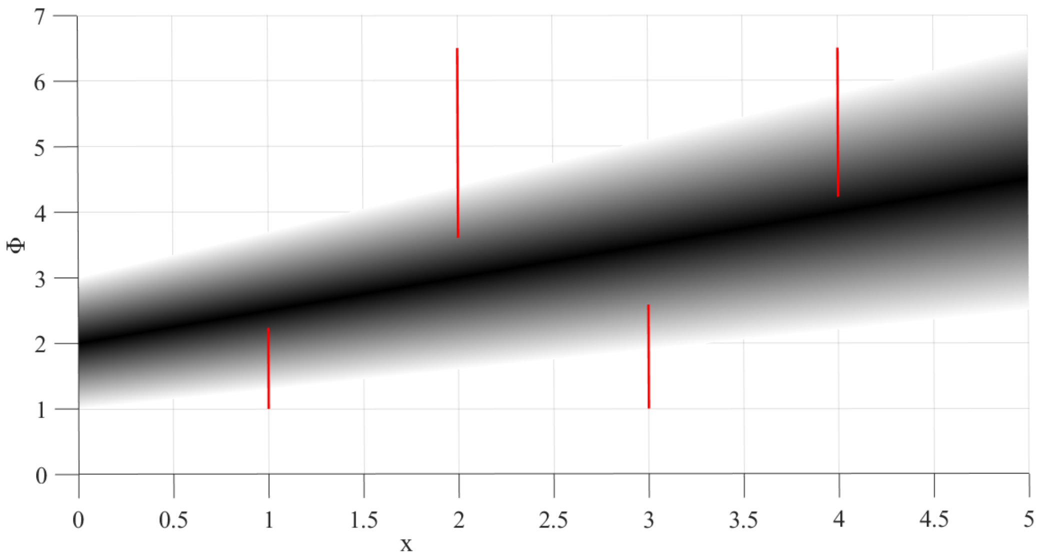
First, consider the basis functions $g_1(x) = x^2$, $g_2(x) = x$ *and* $g_3(x) = 1$*. The obtained approximation is given by*

$$\Phi_1(x) = (x^2 + 0.2x) \cdot (1;2;3) + 0.1x.$$

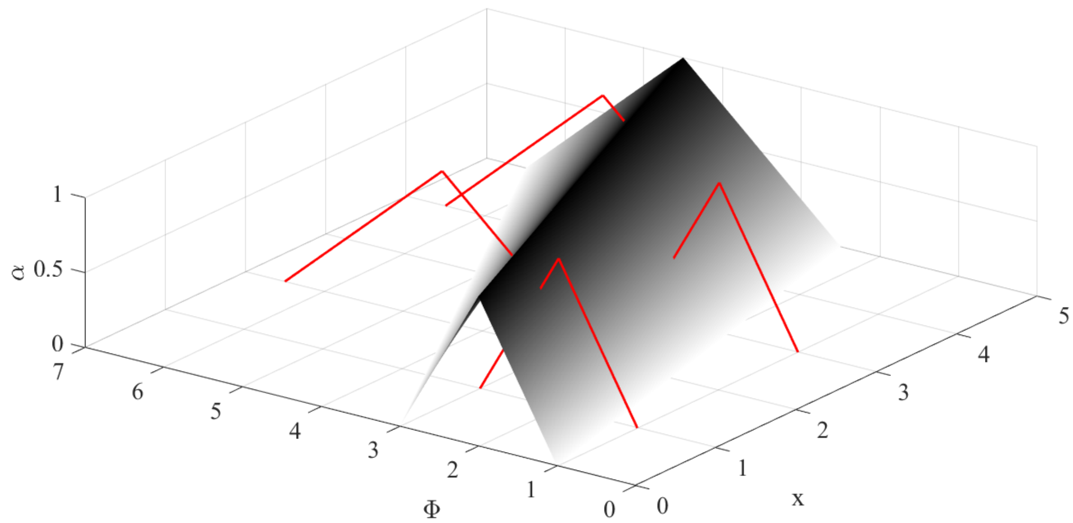
As we can observe in Figure 21, Φ_1 *does not represent properly the data behavior. The function* Φ_1 *has increasing diameter, as well as increasing* 1*-cut. As a consequence, only* Y_1 *and* Y_4 *are approximated, whereas* Y_2 *and* Y_3 *are not well approximated by* Φ_1*.*

Figure 21 – Approximation Φ_1 for data in Example 2.3

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function Φ_1, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 3. Source: Author.

In order to obtain a better fitting of the data in Table 3, a different basis functions can be considered. In particular, let us use the Fourier basis with period $\frac{\pi}{4}$, *that is,* $g_1(x) = \frac{\sqrt{2}}{4}$, $g_2(x) = \frac{\sin(\frac{\pi}{4}x)}{2}$ *and* $g_3(x) = \frac{\cos(\frac{\pi}{2}x)}{2}$.

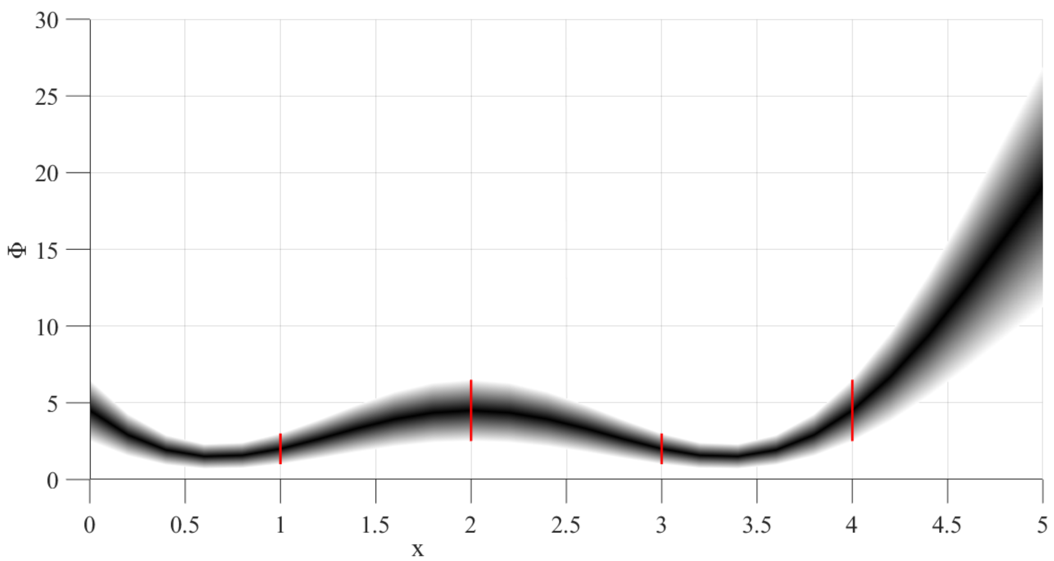
The resulting function Φ_2 *is given by*

$$\Phi_2(x) = \left(-4.8285\frac{\sin(\frac{\pi}{4}x)}{2} - 2.4142\frac{\cos(\frac{\pi}{2}x)}{2} + 3.1213\sqrt{2}\right) \cdot (1;2;3)+$$
$$+\left(-2.4142\frac{\sin(\frac{\pi}{4}x)}{2} - 1.2071\frac{\cos(\frac{\pi}{2}x)}{2} + 1.2071\sqrt{2}\right).$$

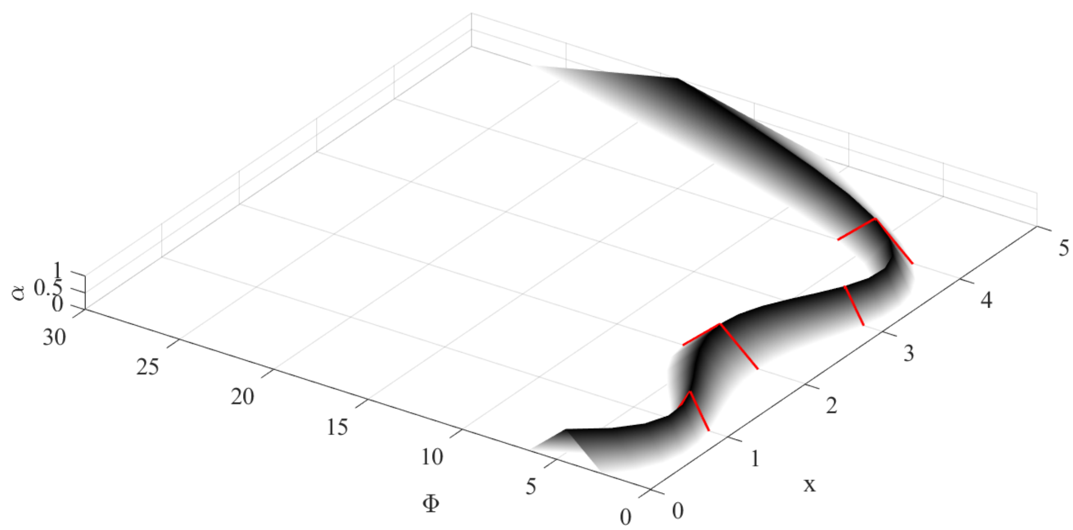
The fuzzy approximation $\Phi_2(x)$ *is depicted in Figure 22. As we can observe, the data is perfectly fitted. Both* 1*-cut and specificity are well fitted as well. For the region where* $0 \leqslant x \leqslant 1$ *the function generalizes the data behavior. On the other hand, in the region where* $4 \leqslant x \leqslant 5$ *the function* Φ_2 *increases, whereas the specificity decreases.*

Figure 22 – Approximation Φ_2 for data in Example 2.3

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function Φ_2, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 3. Source: Author.

2.2.1 Application

The proposed method is applied to determine a function that fits the longitudinal data available in [20]. In [20] the authors discussed the association between children mortality and air pollution in São Paulo, Brazil, from 1994 to 1997. In their study, it was collected longitudinal data of sulfur dioxide (SO_2), carbon monoxide (CO), inhalable particulate (PM_{10}) and ozone (O_3). Here, we only consider ozone dataset.

For the sake of simplicity, suppose that the longitudinal data are given by linearly interactive triangular fuzzy numbers of the form $(M - \sigma; M; M + \sigma)$, where M and σ are the mean and the standard deviation of the collected data in each year, respectively. Note that the fuzzy numbers $(M_1 - \sigma_1; M_1; M_1 + \sigma_1)$ and $(M_2 - \sigma_2; M_2; M_2 + \sigma_2)$, with $\sigma_1, \sigma_2 \neq 0$, are linearly interactive with coefficients $q_1 = \dfrac{\sigma_1}{\sigma_2}$ and $r_1 = M_1 - \dfrac{\sigma_1}{\sigma_2} M_2$, and also $q_2 = \dfrac{\sigma_2}{\sigma_1}$ and $r_2 = M_2 - \dfrac{\sigma_2}{\sigma_1} M_1$. Recall that the proposed method is not restricted to triangular fuzzy numbers, thus, other types of fuzzy number can be considered.

Let $D = \{(x_1, Y_1), (x_2, Y_2), (x_3, Y_3), (x_4, Y_4)\} \subset \mathbb{R} \times \mathbb{R}_{\mathcal{F}}$ be the fuzzy dataset given in Table 4. The values $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$ represent respectively the years 1994, 1995, 1996, and 1997. The fuzzy numbers $Y_1 = (17.6; 57; 96.4)$, $Y_2 = (25.3; 60.7; 96.1)$, $Y_3 = (34.8; 76.3; 117.8)$, and $Y_4 = (29.5; 63; 96.5)$ are linearly interactive with respect to joint possibility distribution J, whose membership function is given by

$$J(v_1, v_2, v_3, v_4) = \chi_U(v_1, v_2, v_3, v_4) Y_1(v_1),$$

for all $(v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, where

$$U = \{(u, 0.8985u + 9.4855, 1.0533u + 16.2619, 0.8502u + 14.5386) \ : \ u \in \mathbb{R}\}. \tag{2.16}$$

Table 4 – Ozone fuzzy dataset

Year	x	Y
1994	1	(17.6; 57; 96.4)
1995	2	(25.3; 60.7; 96.1)
1996	3	(34.8; 76.3; 117.8)
1997	4	(29.5; 63; 96.5)

Source: Based on ozone dataset from [20].

Consider the functions $g_1(x) = x^2$, $g_2(x) = x$ and $g_3(x) = 1$. From (2.13), for each $\alpha \in [0, 1]$ and $x \in [1, 4]$, the fuzzy function Φ is given by

$$[\Phi(x)]_\alpha = [\min\{h(x, \alpha, 0), h(x, \alpha, 1)\}, \max\{h(x, \alpha, 0), h(x, \alpha, 1)\}],$$

where

$$h(x, \alpha, 0) = -3.24x^2 + 20.76x - 0.75 + \alpha(-x^2 + 3.84x + 35.34)$$

and

$$h(x, \alpha, 1) = -5.24x^2 + 28.44x + 69.93 - \alpha(-x^2 + 3.84x + 35.34).$$

For $x \in [e, f]$, where $e \approx -4.327$ and $f \approx 8.167$, the α-cuts of Φ are given by

$$\begin{aligned} [\Phi(x)]_\alpha = [&-3.24x^2 + 20.76x - 0.75 + \alpha(-x^2 + 3.84x + 35.34), \\ &-5.24x^2 + 28.44x + 69.93 - \alpha(-x^2 + 3.84x + 35.34)], \end{aligned} \tag{2.17}$$

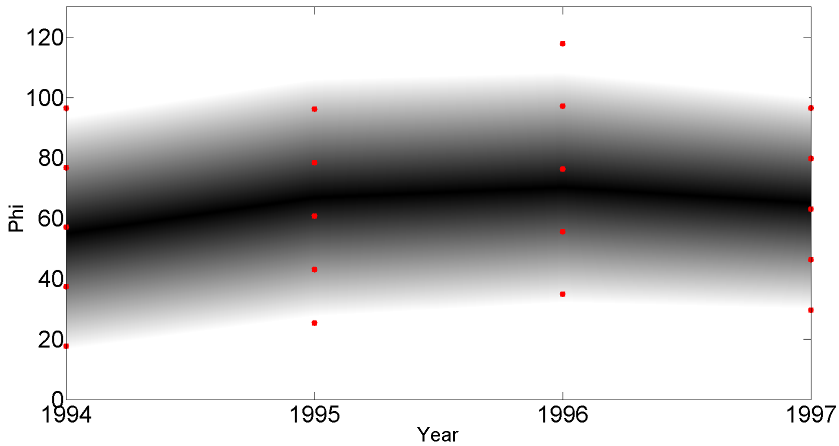
and if $x \notin [e, f]$, the α-cuts are

$$[\Phi(x)]_\alpha = [-5.24x^2 + 28.44x + 69.93 - \alpha(-x^2 + 3.84x + 35.34),$$
$$-3.24x^2 + 20.76x - 0.75 + \alpha(-x^2 + 3.84x + 35.34)].$$

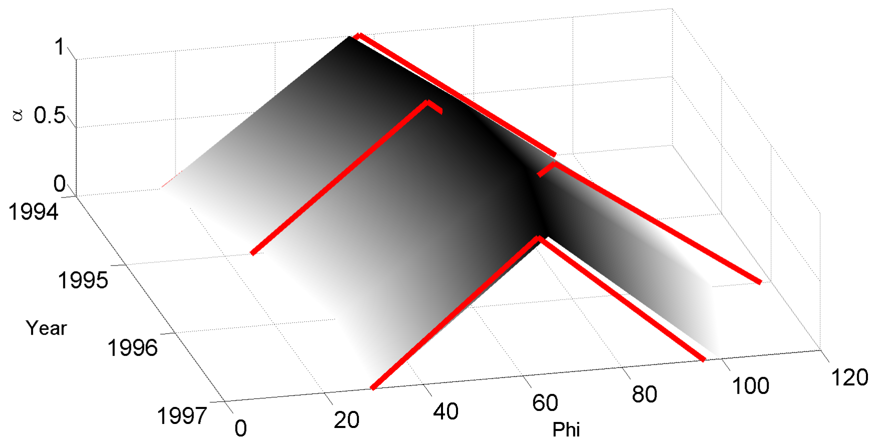
Figure 23 exhibits the fuzzy function Φ produced by our proposal. One can observe in Subfigure 23(a) that the function Φ fits the data of Table 4 which varies from 1994 to 1997. The red triangles and the gray-scale surface depicted in Subfigure 23(b) correspond to the membership functions of fuzzy data Y_i, $i = 1, \ldots, 4$, and fuzzy solution, respectively.

Figure 23 – Approximation for ozone data

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function Φ, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. In Subfigure 23(a) the red dots represent the endpoints of the α-cuts of the fuzzy data Y_i for $\alpha = 0, 0.5, 1$ and $i = 1, \ldots, 4.$, given in Table 4. In Subfigure 23(b) data is represented by red triangles. Source: Author [88].

Note that $Y_1, \ldots, Y_4$ may be considered linearly interactive with respect to 2^3 different JPDs, since Y_1 is symmetric [34]. Thus, one can obtain 2^3 fuzzy functions Φ. However, in general, the choice of a joint possibility distribution is not arbitrary and depends on the context. For example, if each object is measured m times with the same n measuring devices then one can assume that the obtained values depend only on the calibration of each equipment and not on the objects. This type of assumption induces the choice of specific parameters q_i and r_i in (1.14).

2.3 Conclusion

This chapter introduced a new method for solving fuzzy least squares problem. The method can be used to fit fuzzy data by polynomial functions with degree more than one. Recall that the most approaches to fuzzy least square methods assume that the fuzzy function being adjusted is linear. In fact, the data can be approximated by more general basis functions, as shown in Example 2.3.

3 Fuzzy Least Square Method for A-Linearly Interactive Fuzzy Numbers

Esmi *et al.* [32, 30] developed a theory for the class of A-linearly interactive (A-LI) fuzzy numbers, denoted by $\mathbb{R}_{\mathcal{F}(A)}$, where A is a non-symmetric fuzzy number. They proved that there is an isomorphism between $\mathbb{R}^2$ and $\mathbb{R}_{\mathcal{F}(A)}$. For symmetric fuzzy number A, Shen [96] developed an analogous theory, considering the quotient space $\mathbb{R}^2/\equiv_A$, where the equivalence relation identifies all the fuzzy sets that are linearly interactive.

For the set $\mathbb{R}_{\mathcal{F}(A)}$ a broader curve fitting method can be proposed. This is the focus of this chapter, which is divided as follows. First, the class of A-linearly interactive fuzzy numbers (A-LI) is presented in Section 3.1. Thereafter, the curve fitting method for A-LI data is proposed in Section 3.2. Section 3.3 is dedicated to prove that the method suggested in previous chapter is, in fact, a particular case. Finally, Section 3.4 provides and analyzes several examples.

3.1 The Set of A-Linearly Interactive Fuzzy Numbers

Esmi *et al.* [32] introduce the space of A-linearly interactive fuzzy numbers, denoted by $\mathbb{R}_{\mathcal{F}(A)}$, for any $A \in \mathbb{R}_{\mathcal{F}}$. Let $\Psi_A : \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ be the map given by

$$[\Psi_A(p, r)]_\alpha = p[A]_\alpha + r,$$

for all $\alpha \in [0, 1]$ and $p, r \in \mathbb{R}$. The set $\mathbb{R}_{\mathcal{F}(A)}$ is defined as being the image of Ψ, that is, $\mathbb{R}_{\mathcal{F}(A)} = \{pA + r \mid p, q \in \mathbb{R}\}$, for a fixed $A \in \mathbb{R}_{\mathcal{F}}$. If A is non-symmetric, then $\mathbb{R}_{\mathcal{F}(A)}$ is isomorphic to $\mathbb{R}^2$ via the linear isomorphism Ψ_A [32], where the vector addition and scalar product of $\mathbb{R}_{\mathcal{F}(A)}$ is defined as follows. In this chapter, unless otherwise stated, we assume that $A \in \mathbb{R}_{\mathcal{F}}$ is non-symmetric.

Let be $B, C \in \mathbb{R}_{\mathcal{F}(A)}$, that is, $B = pA + r$ and $C = qA + s$, for some $p, q, r, s \in \mathbb{R}$, and a scalar $\beta \in \mathbb{R}$. The sum $B +_A C$ is defined as

$$B +_A C = \Psi_A(\Psi_A^{-1}(B) + \Psi_A^{-1}(C)) = (p + q)A + (r + s), \tag{3.1}$$

and the scalar product $\beta \cdot_A B$ is given by

$$\beta \cdot_A B = \Psi_A(\beta \Psi_A^{-1}(B)) = \beta pA + \beta r. \tag{3.2}$$

Furthermore, for A non-symmetric, $\mathbb{R}_{\mathcal{F}(A)}$ is isometric to $\mathbb{R}^2$, with norm given by

$$\|B\|_\Psi = \|\Psi^{-1}(B)\|_2, \tag{3.3}$$

for all $B \in \mathbb{R}_{\mathcal{F}(A)}$. Note that, Equation (3.3) is given by $\|B\|_\Psi = (p^2 + r^2)^{\frac{1}{2}}$.

There is a slight difference between linearly correlated fuzzy numbers and A-linearly correlated fuzzy numbers. Any real number $a \in \mathbb{R}$, represented by the membership function $\chi_{\{a\}}$, may be written as $a = 0A + a$. Therefore $a \in \mathbb{R}$ also belongs to $\mathbb{R}_{\mathcal{F}(A)}$. On the other hand, a and A are not linearly correlated. As a consequence, the set $\mathbb{R}_{\mathcal{F}(A)}$ is broader than the set of linearly interactive fuzzy numbers with respect to A. In addition, the class $\mathbb{R}_{\mathcal{F}(A)}$ is isomorphic to the Euclidian space $\mathbb{R}^2$, which does not occurs with linearly correlated fuzzy numbers. Consequently, in $\mathbb{R}_{\mathcal{F}(A)}$, it is possible to define the best solution that fits a dataset.

Here, we define a product of $B \in \mathbb{R}_{\mathcal{F}(A)}$ by a pair of scalars by

$$(\beta, \gamma) \cdot_A B = \beta p A + \gamma r, \tag{3.4}$$

which is an extension of the product of B by a scalar since

$$\beta \cdot_A B = (\beta, \beta) \cdot_A B = \beta p A + \beta r.$$

We may also define the isomorphism $\Psi_A : \mathbb{R}^{2n} \to \mathbb{R}^n_{\mathcal{F}(A)}$, for $A \in \mathbb{R}_\mathcal{F}$ non-symmetric, in terms of

$$\left[\Psi_A \begin{pmatrix} (p_1, r_1) \\ \vdots \\ (p_n, r_n) \end{pmatrix}\right]_\alpha = \begin{pmatrix} [\Psi_A(p_1, r_1)]_\alpha \\ \vdots \\ [\Psi_A(p_n, r_n)]_\alpha \end{pmatrix} = \begin{pmatrix} p_1[A]_\alpha + r_1 \\ \vdots \\ p_n[A]_\alpha + r_n \end{pmatrix}, \tag{3.5}$$

for all $\alpha \in [0, 1]$. Thus, the set given by the image of Ψ_A is

$$\mathbb{R}^n_{\mathcal{F}(A)} = \{(B_1, \ldots, B_n) = (p_1 A + r_1, \ldots, p_n A + r_n) \mid (p_1, \ldots, p_n), (r_1, \ldots, r_n) \in \mathbb{R}^n\},$$

for a fixed $A \in \mathbb{R}_\mathcal{F}$.

Analogously, using the addition in (3.1) and product by a scalar in (3.2) to define pointwise operations, the set $\mathbb{R}^n_{\mathcal{F}(A)}$ is isomorphic to $\mathbb{R}^{2n}$ via (3.5). For $B \in \mathbb{R}^n_{\mathcal{F}(A)}$ the norm is given by

$$\|B\|_\Psi = (\sum_{i=1}^{n} (p_i^2 + r_i^2))^{\frac{1}{2}}.$$

3.2 Least Square Method for A-Linearly Interactive Fuzzy Numbers

In what follows, we assume that the data is given by $D = \{(x_1, Y_1), \ldots, (x_m, Y_m)\}$, such that $Y_i \in \mathbb{R}_{\mathcal{F}(A)}$ with A non-symmetric, that is, $Y_i = p_i A + r_i$, with $p_i, r_i \in \mathbb{R}$, $i = 1, \ldots, m$.

A basic hypothesis of the proposed method is the non-symmetry of the fuzzy number A. Nonetheless Esmi *et al.* [30] showed that the class of non-symmetric fuzzy numbers is dense in $\mathbb{R}_{\mathcal{F}}$ with the metric d_F. Moreover, any function of the form $\widetilde{f}(x) = \widetilde{q}(x)\widetilde{A} + \widetilde{r}(x)$, $x \in [a,b]$, can be approximated as closely as desired by a function $f : [a,b] \to \mathbb{R}_{\mathcal{F}(A)}$ with A as symmetric, that is an immediate consequence of Theorem 3 from [30]. For this reason, it is sufficient to consider A as a non-symmetric fuzzy number [7]. As a consequence, data Y_i must be non-symmetric for all $i = 1, \dots, m$.

Our objective is to find a function $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ such that $F(x_i) \approx Y_i$ for all $i = 1, \dots, m$. This function F is given by a linear combination

$$F(x) = (\beta_1, \gamma_1) \cdot_A G_1(x) +_A \dots +_A (\beta_n, \gamma_n) \cdot_A G_n(x),$$

where each $G_i : \mathbb{R} \to \mathbb{R}_{\mathcal{F}(A)}$, is a basis function, that is, $G_i(x) = q_i(x)A + s_i(x)$, $i = 1, \dots, n$.

Consider the vectors $q(x) = (q_1(x), \dots, q_n(x))^T$, $s(x) = (s_1(x), \dots, s_n(x))^T$, $\beta = (\beta_1, \dots, \beta_n)^T$, $\gamma = (\gamma_1, \dots, \gamma_n)^T$ and $Y = (Y_1, \dots, Y_m)^T$. The function F can be written as

$$\begin{aligned} F(x) &= (\beta_1, \gamma_1) \cdot_A [q_1(x)A + s_1(x)] +_A \dots +_A (\beta_n, \gamma_n) \cdot_A [q_n(x)A + s_n(x)] && (3.6)\\ &= [\beta_1 q_1(x)A + \gamma_1 s_1(x)] +_A \dots +_A [\beta_n q_n(x)A + \gamma_n s_n(x)] \\ &= [\beta_1 q_1(x) + \dots + \beta_n q_n(x)]A + [\gamma_1 s_1(x) + \dots + \gamma_n s_n(x)] \\ \Rightarrow F(x) &= q(x)^T \beta A + s(x)^T \gamma. && (3.7) \end{aligned}$$

Therefore the problem is to find scalars (β, γ) that are the solutions of the following problem

$$\min_{(\beta,\gamma)\in\mathbb{R}^n\times\mathbb{R}^n} \frac{1}{2}\|Y -_A F\|_\Psi, \tag{3.8}$$

where each data is given by $Y_i = p_i A + r_i$, with $p_i, r_i \in \mathbb{R}$ for $i = 1, \dots, m$ and $A \in \mathbb{R}_{\mathcal{F}}$ fixed. This problem is equivalent to

$$\min_{(\beta,\gamma)\in\mathbb{R}^{2n}} \frac{1}{2}\left\| \begin{pmatrix} (p_1 A + r_1) -_A \left(\sum_{i=1}^n \beta_i q_i(x_1)A + \gamma_i s_i(x_1)\right) \\ \vdots \\ (p_m A + r_m) -_A \left(\sum_{i=1}^n \beta_i q_i(x_m)A + \gamma_i s_i(x_m)\right) \end{pmatrix} \right\|_\Psi^2,$$

that is,

$$\min_{(\beta,\gamma)\in\mathbb{R}^{2n}} \frac{1}{2}\left\| \begin{pmatrix} \left(p_1 - \sum_{i=1}^n \beta_i q_i(x_1)\right)A + \left(r_1 - \sum_{i=1}^n \gamma_i s_i(x_1)\right) \\ \vdots \\ \left(p_m - \sum_{i=1}^n \beta_i q_i(x_m)\right)A + \left(r_m - \sum_{i=1}^n \gamma_i s_i(x_m)\right) \end{pmatrix} \right\|_\Psi^2.$$

Thus, we have that

$$\min_{(\beta,\gamma)\in\mathbb{R}^{2n}} \frac{1}{2}\left[\sum_{j=1}^m \left(p_j - \sum_{i=1}^n \beta_i q_i(x_j)\right)^2 + \sum_{j=1}^m \left(r_j - \sum_{i=1}^n \gamma_i s_i(x_j)\right)^2 \right].$$

The solution of the above problem is obtained solving the following linear systems

$$Q^T Q\beta = Q^T p$$
$$S^T S\gamma = S^T r,$$

where $q_{ij} = q_j(x_i)$ and $s_{ij} = s_j(x_i)$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and $p = (p_1, \ldots, p_m)^T$, $r = (r_1, \ldots, r_m)^T$. Thus, assuming that Q and S have full hank, we have

$$\beta = (Q^T Q)^{-1} Q^T p$$
$$\gamma = (S^T S)^{-1} S^T r.$$

Therefore, from Equation (3.7), the best approximation for data D is given by

$$F(x) = q^T(x)(Q^T Q)^{-1} Q^T p \cdot A + s^T(x)(S^T S)^{-1} S^T r. \tag{3.9}$$

In general form, this discussion may be encapsulated in the next theorem.

Theorem 3.1. *Let $D = \{(x, pA + r)\}$ be the dataset with $x = (x_1, \ldots, x_m) \in \mathbb{R}^m, A \in \mathbb{R}_{\mathcal{F}}, p = (p_1, \ldots, p_m) \in \mathbb{R}^m$ and $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$. The least square method for D, with basis functions $q(x) = (q_1(x), \ldots, q_n(x)) \in \mathbb{R} \times \mathbb{R}^n$ and $r(x) = (r_1(x), \ldots, r_l(x)) \in \mathbb{R} \times \mathbb{R}^l$, generates a function $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ given by*

$$F(x) = q^T(x) Q^\dagger p \cdot A + s^T(x) S^\dagger r, \tag{3.10}$$

where $Q^\dagger$ and $S^\dagger$ denote respectively the pseudoinverse of Q and S.

One may note that the solution (3.9) does not depend on the shapes of the fuzzy data Y_i. The solution depends only on A, the coefficient vectors p and r, and the basis functions $q_j(x)$ and $s_j(x)$. Regarding the calculation of β and γ, the solution in Equation (3.10) requires the double of operations in comparison with numerical solution of classical least square method.

Next section proves that the solution presented in Chapter 2 is a particular case of the solution in Equation (3.9).

3.3 Method of Chapter 2 as a Particular Case

In Chapter 2, we described a least square method to fit fuzzy data. From a visual inspection, the approximation seems to be a good one, but there no discussion about how good it is. In the last section, we proved that the solution in Equation (3.9) is the best fitting function for for data in $\mathbb{R}_{\mathcal{F}(A)}$. In this section it is shown that the solution presented in Chapter 2 is a particular case of (3.9), for specific choices of the functions $q_i(x)$ and

$s_i(x)$. Such families of functions q_i and s_i will fit the diameter and the 1-cut curve of data, respectively.

First, it is necessary to analyze the essence of the method of Chapter 2. In classical theory, the data set is $D = \{(x_1, y_1), \ldots, (x_m, y_m)\}$, and the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\varphi(x) = a_1 g_1(x) + \ldots + a_n g_n(x)$. From Equation (2.5), $a = Cy$, where the $C = G^\dagger$, and from Equation (2.7),

$$\varphi(x) = g(x)Cy.$$

For fuzzy data $Y_1, \ldots, Y_m$ linearly interactive by the JPD J_L with coefficients $p_1, \ldots, p_m, r_1, \ldots, r_m$, the sup-$J$ extension of φ is denoted by Φ. As a consequence of Theorem 1.4, the α-cuts of the extension of Φ are given by

$$[\Phi(x)]_\alpha = \{g(x)Cy \mid y \in [J_L]_\alpha\},$$

where $[J_L]_\alpha = \{(p_1 y + r_1, \ldots, p_m y + r_m), y \in [A]_\alpha\}$, that is, $[J_L]_\alpha = \{yp + r, y \in [A]_\alpha\}$ and $A = Y_1$. Thus, we have

$$[\Phi(x)]_\alpha = \{g(x)Cpy + g(x)Cr \mid y \in [A]_\alpha\}. \tag{3.11}$$

In Equation (3.9), consider the following basis functions given by

$$q_j(x_i) = s_j(x_i) = g_j(x_i) = g_{ij}. \tag{3.12}$$

Thus we obtain

$$\begin{aligned} \beta &= (G^T G)^{-1} G^T p \\ \gamma &= (G^T G)^{-1} G^T r. \end{aligned}$$

In this case, from Equation (3.7), the α-cuts of F are given as follows

$$[F(x)]_\alpha = \{q^T(x)\beta y + s^T(x)\gamma \mid y \in [A]_\alpha\},$$

that is,

$$[F(x)]_\alpha = \{g(x)Mpy + g(x)Mr \mid y \in [A]_\alpha\}, \tag{3.13}$$

where $g(x) = [g_1(x), \ldots, g_n(x)]^T$ and $M = G^\dagger$.

Equations (3.13) and (3.11) are equal. Hence, the solution given in Equation (3.11) is a particular case of the solution given in Equation (3.13). That is, the method of Section 3.2 can be viewed as an extension of the method proposed in Chapter 2, where $A = Y_1$ is non-symmetric. As a consequence, the solution founded in Chapter 2 is the best fitting function for the case where the basis functions q_i and s_i are the same (see Equation (3.12)) and A is a non-symmetric fuzzy number.

Summarizing the reasoning, the following corollary holds true.

Corollary 3.1. *Let D be the dataset with linearly interactive fuzzy numbers as outputs. The least square method generates a function given by*

$$F(x) = g(x)G^{\dagger}p \cdot A + g(x)G^{\dagger}r.$$

3.4 Applications

This section presents several data sets, in order to exemplify the method.

3.4.1 Approximation for Ozone Data

The first example illustrates the fact that the method given in Chapter 2 is a particular case of the method presented in this chapter, under some weak conditions. Consider the dataset in Table 5, which contains the data from Table 4 with small perturbations, to guarantee that Y_1 is non-symmetric.

Table 5 – Perturbed ozone fuzzy dataset

x	Y
1	$(17.6; 57; 96.41)$
2	$(25.3; 60.7; 96.11)$
3	$(34.8; 76.3; 117.81)$
4	$(29.5; 63; 96.51)$

Source: Author.

Here, we consider $A = Y_1$. Thus the outputs can be written as the vector $Y = (Y_1, Y_2, Y_3, Y_4)$, where $Y_1 = 1A + 0$, $Y_2 = 0.8985A + 9.4855$, $Y_3 = 1.0533A + 16.2619$ and $Y_4 = 0.8502A + 14.5386$.

Let be $q_1(x) = s_1(x) = 1$, $q_2(x) = s_2(x) = x$, $q_3(x) = s_3(x) = x^2$. The approximation is given by the function

$$F(x) = (-0.0254x^2 + 0.0975x + 0.8972) \cdot Y_1 + (-2.802x^2 + 19.049x - 16.5364), \quad (3.14)$$

and it is depicted in Figure 24.

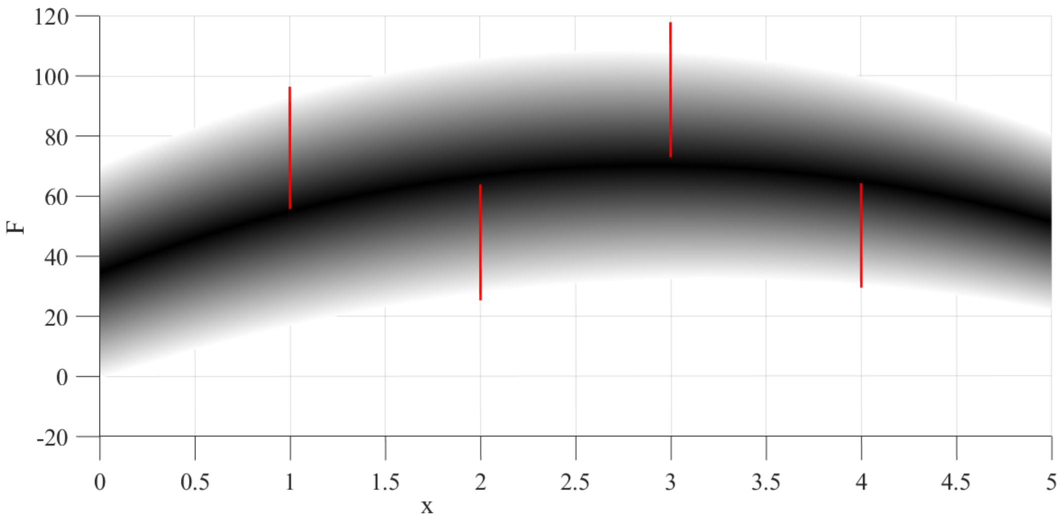
The α-cuts of F are given by

$$\begin{aligned}[F(x)]_\alpha = [&-3.249x^2 + 20.765x - 0.7457 + \alpha(-x^2 + 3.8415x + 35.3497),\\ &-5.2506x^2 + 28.448x + 69.9537 - \alpha(-x^2 + 3.8415x + 35.3497)].\end{aligned}$$

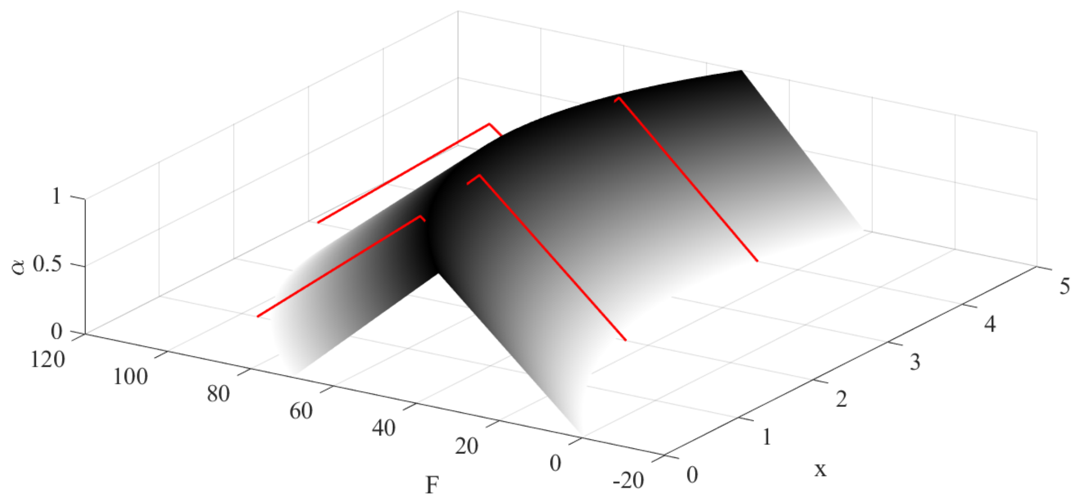
Note that the function F has almost the same α-cuts of the function ϕ founded in Chapter 2 (see Equation (3.14)), with slight differences due to the perturbations inserted in data from Table 5.

Figure 24 – Approximation for ozone perturbed fuzzy data

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function F, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 5. Source: Author.

Function F in Equation (3.14) is a good approximation if we consider the 1-cuts of the data. The uncertainty is fitted satisfactorily, except for data Y_3, as can be seen in Figure 24.

3.4.2 Approximation for Underreporting COVID-19 Mortality Data in Brazil

COVID-19 mortality data has uncertainty of fuzzy type due to underreporting. Underreporting is calculated by means of the average (expected) number of deaths due to respiratory problems and other natural causes. Howsoever the underreporting is not precisely known. Data has epistemic uncertainty, therefore it is epistemic fuzzy data [28, 59].

In Brazil, researches estimate that the average underreporting of 40,7% for deaths related to COVID-19 [98]. Moreover, so far COVID-19 epidemiological dynamic is described by, at least, four ordinary differential equations [90, 69, 37, 1]. As a consequence, it is not easy to consider fuzzy uncertainty into COVID-19 mortality data by means of epidemiological dynamics.

Here, the number of deaths due to COVID-19 is considered by epidemiological weeks (EPI weeks). EPI week starts on Sunday and ends on Saturday. The fuzzy sets are given by $Y = (y_i; y_i; 1.407y_i)$, for $i = 1, \ldots, 6$, where 1.407 represents the uncertainty due to underreporting data and each y_i is extracted from Our World in Data [26]. Data are given in Table 6, where epidemiological weeks are taken from 2021.

Table 6 – COVID-19 Mortality fuzzy dataset

From ___ to ___	EPI week	Y	width(Y)
07 Mar - 13 Mar	10	$(12777; 12777; 17977)$	5200
14 Mar - 20 Mar	11	$(15650; 15650; 22020)$	6370
21 Mar - 27 Mar	12	$(17798; 17798; 25042)$	7244
28 Mar - 03 Apr	13	$(19643; 19643; 27637)$	7994
04 Apr - 10 Apr	14	$(21141; 21141; 29745)$	8604
11 Apr - 17 Apr	15	$(20344; 20344; 28624)$	8280
18 Apr - 24 Apr	16	$(17814; 17814; 25064)$	7250

Source: Our World in Data [26].

The coefficients are given by the vectors $p = (1, 1.22, 1.3931, 1.5373, 1.5923, 1.3942)$ and $r = (0, -1.825, -1.3438, 0.8196, -0.9154, -0.0865)$. The basis functions are $q_1(x) = s_1(x) = 1$, $q_2(x) = s_2(x) = x$, $q_3(x) = s_3(x) = x^2$. The function that fits data in Table 6 is written as

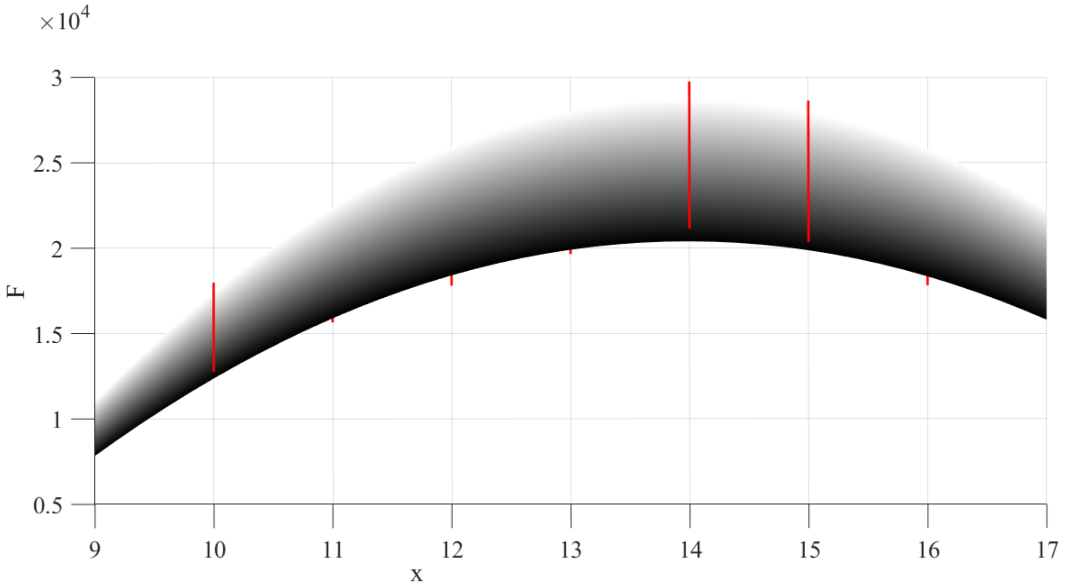
$$F(x) = (-3.4044x^2 + 0.6426x - 0.202) \cdot Y_1 + (126.6518x^2 - 22.6393x + 0.9971), \quad (3.15)$$

where $Y1$ is the data from 10^{th} EPI week: $Y1 = (12777; 12777; 17977)$. The approximation is depicted in Figure 25.

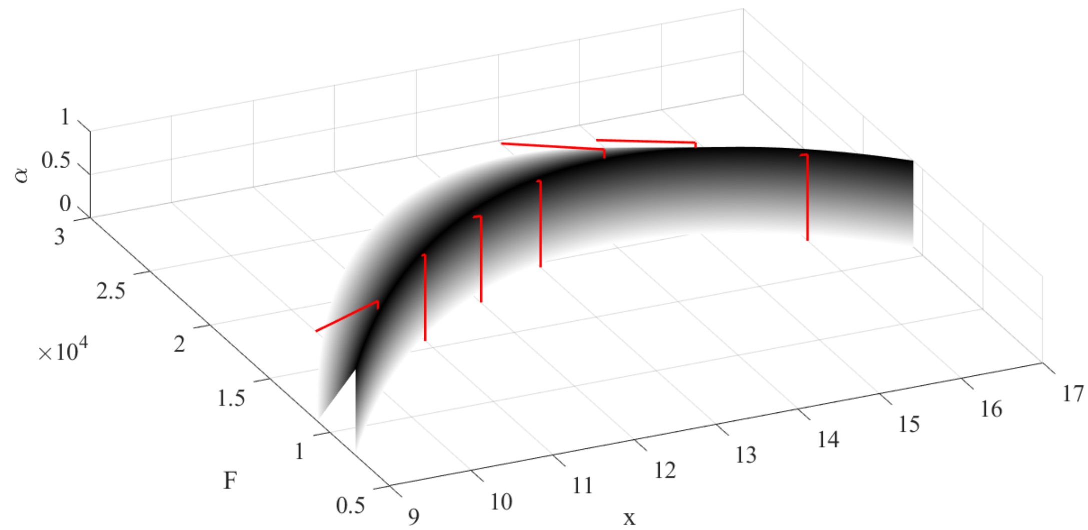
As we can observe in both top and tridimensional views, the curve seems to be a good data approximation. The solution in Equation (3.15) describes the behavior of increasing mortality number in the 10^{th}, 11^{th}, 12^{th}, 13^{th} and 14^{th} EPI weeks as well as the uncertainty (less specificity) in the process. On the other hand, for EPI weeks 15 and 16, the mortality numbers decreases, as well as increases the specificity.

Figure 25 – Approximation for COVID-19 mortality fuzzy data

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function F, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 6. Source: Author.

3.4.3 Fitting the Diameter

The proposed method in Section 3.2 finds two vectors of parameters β and γ. This example shows the meaning of each parameter. Consider the data is given in Table 7, where the coefficients are $p = (1, 1.5, 1, 0.5)$ and $r = (0, 1, 2, 3)$ with $A = Y_1$. Since $p_2 = 1.5$, $p_3 = 1$ and $p_4 = 0.5$, the diameter of data oscillates. Polynomial basis function may not be suitable to model the diameter in this case. For example, if the basis functions are

$q_1(x) = s_1(x) = 1$, $q_2(x) = s_2(x) = x$, $q_3(x) = s_3(x) = x^2$, then the method produces the vectors $\beta_{pol} = (0.25, 1.05, -0.25)$ and $\gamma_{pol} = (-1, 1, -1.4717 \times 10^{-15})$. Therefore, the function that fits the data is given by

$$F_1(x) = (-0.25x^2 + 1.05x + 0.25) \cdot (0; 1; 2.01) + (-x^2 + x - 1.4717 \times 10^{-15}).$$

Table 7 – Fuzzy data with diameter variations

x	Y	$\dim(Y)$
1	$(0; 1; 2.01)$	2.01
2	$(1; 2.5; 4.15)$	3.15
3	$(2; 3; 4.01)$	2.01
4	$(3; 3.5; 4.005)$	1.005

Source: Author.

One can observe in Figure 26 that between $x = 4$ and $x = 4.5$, the diameter almost collapses and after that the diameter grows back. This occurs because that diameter of the third and fourth data indicate a tendency of decrement, but this seems to be not satisfactorily captured by the fuzzy curve F_1.

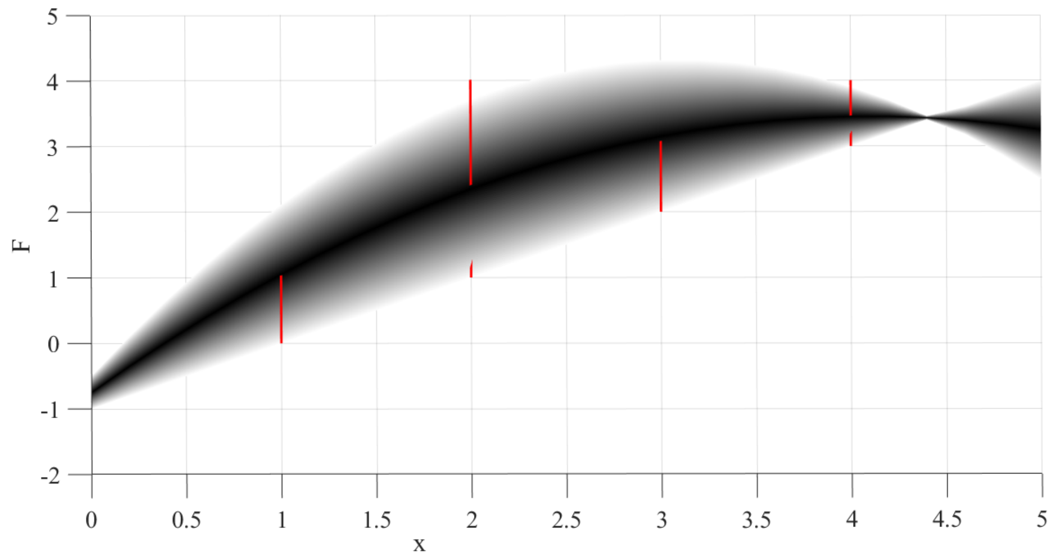
In order to model the diameter variation, the basis functions $\{q_1, q_2, q_3\}$, that multiples A in the solution (see Equation (3.9)), may be changed. Consider the Fourier basis functions $q_1(x) = \dfrac{\sqrt{2}}{4}$, $q_2(x) = \dfrac{\sin\left(\frac{\pi}{4}x\right)}{2}$ and $q_3(x) = \dfrac{\cos\left(\frac{\pi}{4}x\right)}{2}$. This Fourier basis has period $\dfrac{\pi}{4}$. The approximation obtained with these basis functions is depicted in Figure 27, with vectors $\beta_{sc} = (2.8284, -1, -4.2829 \times 10^{-15})$ and $\gamma_{sc} = (-1, 1, 0)$. Note that $\gamma_{pol} = \gamma_{sc}$, and the solution is

$$\begin{aligned} F_2(x) = \left(-4.2829 \times 10^{-15} \frac{\sin\left(\frac{\pi}{4}x\right)}{2} - \frac{\cos\left(\frac{\pi}{4}x\right)}{2} + 2.8284\right) \cdot (0; 1; 2.01) + \\ +(-x^2 + x - 1.4717 \times 10^{-15}). \end{aligned}$$

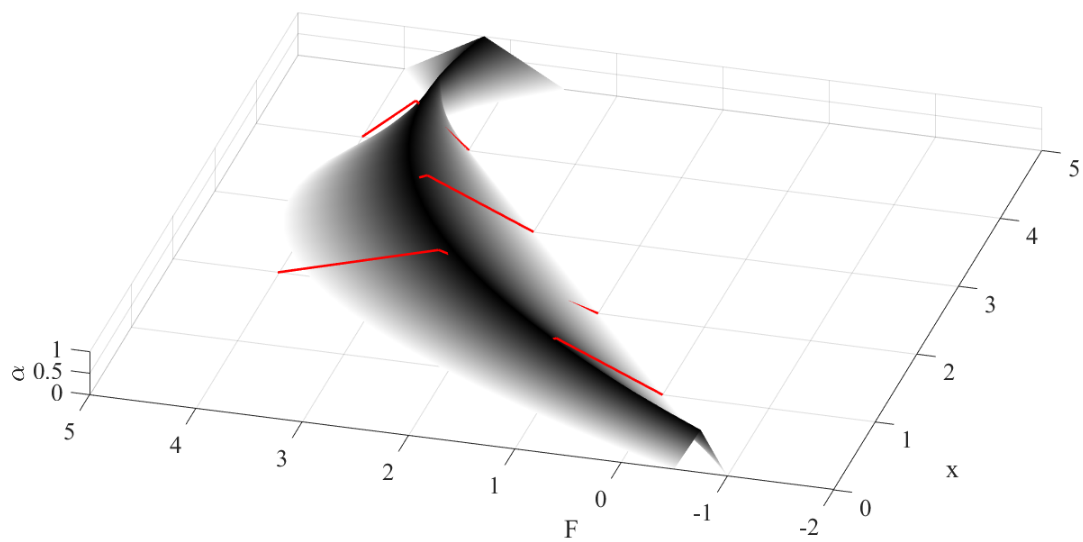
From a visual comparison of Subfigures 26(a) and 26(b) and Subfigures 27(a) and 27(b), one can observe that Fourier basis was able to better capture the data behavior, since the data is precisely fitted. The 1-cuts are well fitted in both cases, but the uncertainty is better fitted by Fourier basis Subfigures 27(a) and 27(b).

Figure 26 – Approximation F_1 for fuzzy data with diameter variations

(a) Top view.



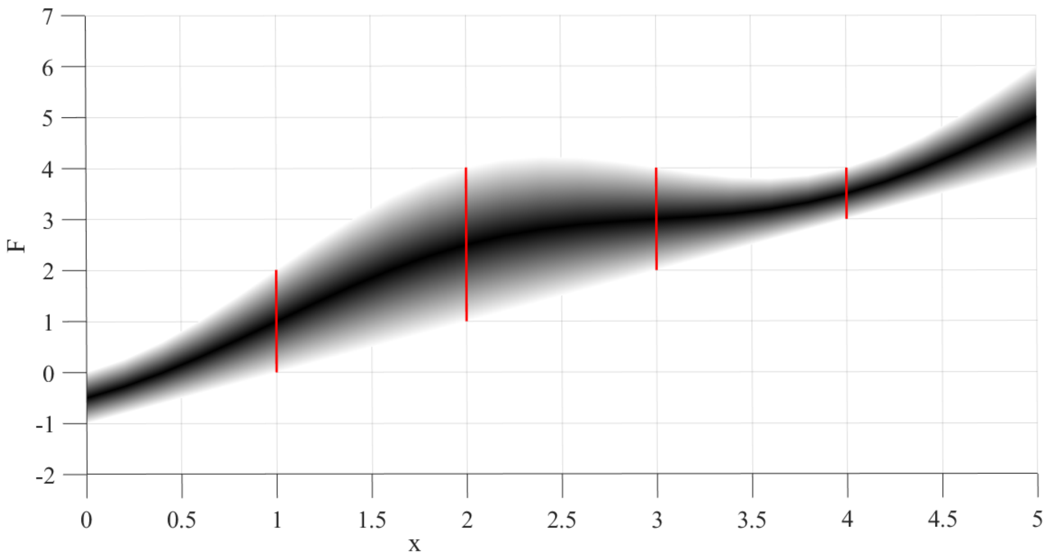
(b) Tridimensional view.



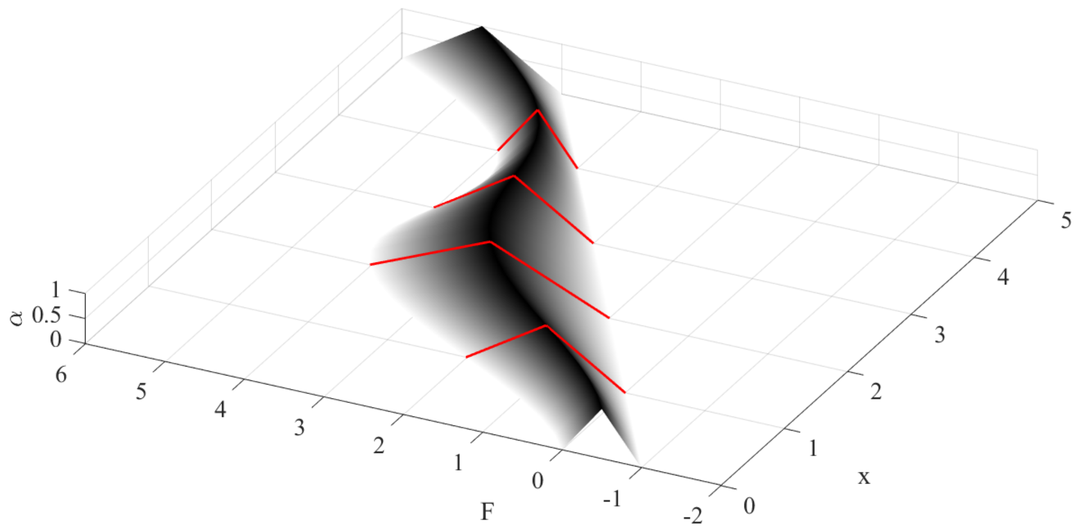
The top and tridimensional views of the fuzzy function F_1, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 7. Source: Author.

Figure 27 – Approximation F_2 for fuzzy data with diameter variations

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function F_2, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 7. Source: Author.

3.4.4 Trapezoidal Fuzzy Data

The proposed method can also be applied to trapezoidal dataset, since there is no constrains on fuzzy data shapes. Consider data given in Table 8.

From Table 8, the vectors of coefficients are $p = (1, 1.2, 2, 2.2)$ and $r = (0, 1, 2, 1)$ with $A = Y_1$. Thus, the solution, depicted in Figure 28, is given by the function $F(x) = (-4.0617 \cdot 10^{-15}x^2 + 0.44x - 0.82) \cdot (1; 2; 3; 4) + (-0.5x^2 + 5.9x - 15.7)$, where the basis

Table 8 – Trapezoidal fuzzy dataset

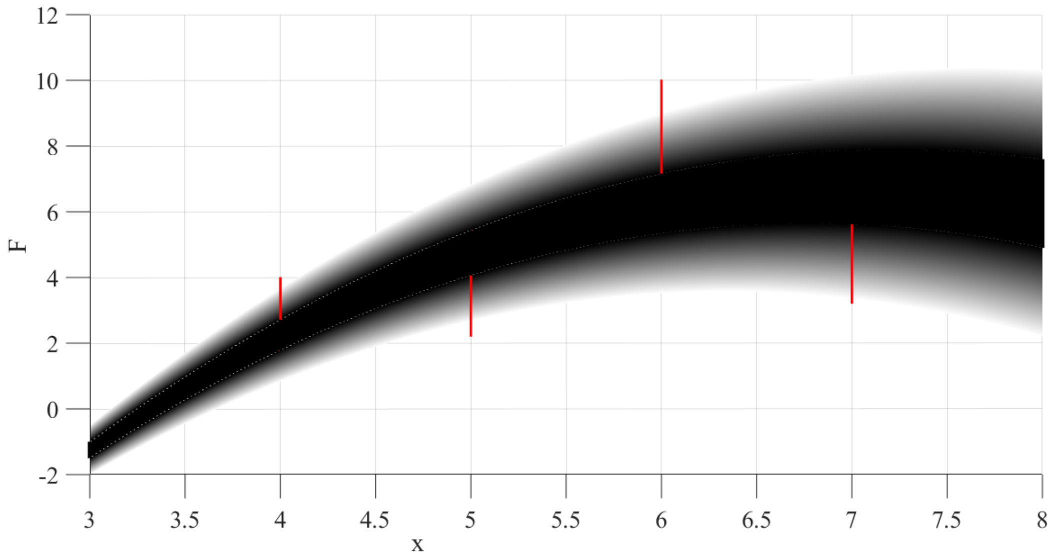
x	Y	width(Y)
4	$(1; 2; 3; 4.01)$	3.01
5	$(2.2; 3.4; 4.6; 5.812)$	3.612
6	$(4; 6; 8; 10.02)$	6.02
7	$(3.2; 5.4; 7.6; 9.822)$	6.622

Source: Author.

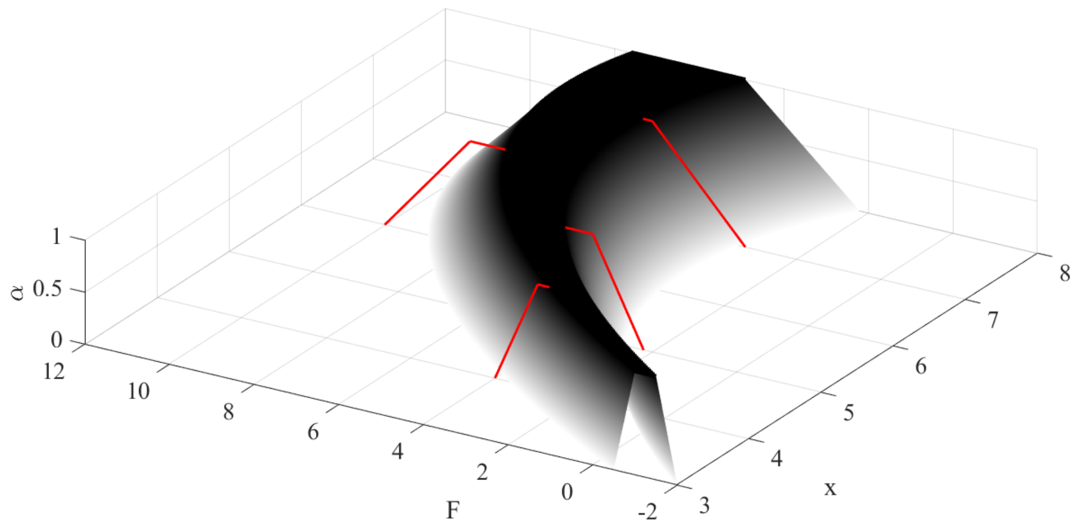
functions are $q_1(x) = s_1(x) = 1$, $q_2(x) = s_2(x) = x$ and $q_3(x) = s_3(x) = x^2$.

Figure 28 – Approximation for trapezoidal fuzzy data

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function F, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 8. Source: Author.

3.4.5 Gaussian Fuzzy Data

The Gaussian fuzzy data has form $Y_i = (\mu_i; \sigma_i; \delta_1; \delta_2)_G$, for $i = 1, \ldots, m$. In order to model the data by Gaussian fuzzy number, we must to state a limitation to obtain a bounded support. Here the same limitation equal to 0.1 is going to be used for all data.

Consider the data given in Table 9. The vectors of coefficients are $p = (1, 1.2, 2, 2.2)$ and $r = (0, 1, 2, 1)$, with $A = Y_1$, and the basis functions are $q_1(x) = s_1(x) = 1$, $q_2(x) =$

$s_2(x) = x$ and $q_3(x) = s_3(x) = x^2$. The solution, depicted in Figure 29, is given by the function $F(x) = (-4.0617 \cdot 10^{-15}x^2 + 0.44x - 0.82) \cdot (10; 2; 0.1, 0.2)_G + (-0.5x^2 + 5.9x - 15.7)$.

Table 9 – Gaussian fuzzy dataset

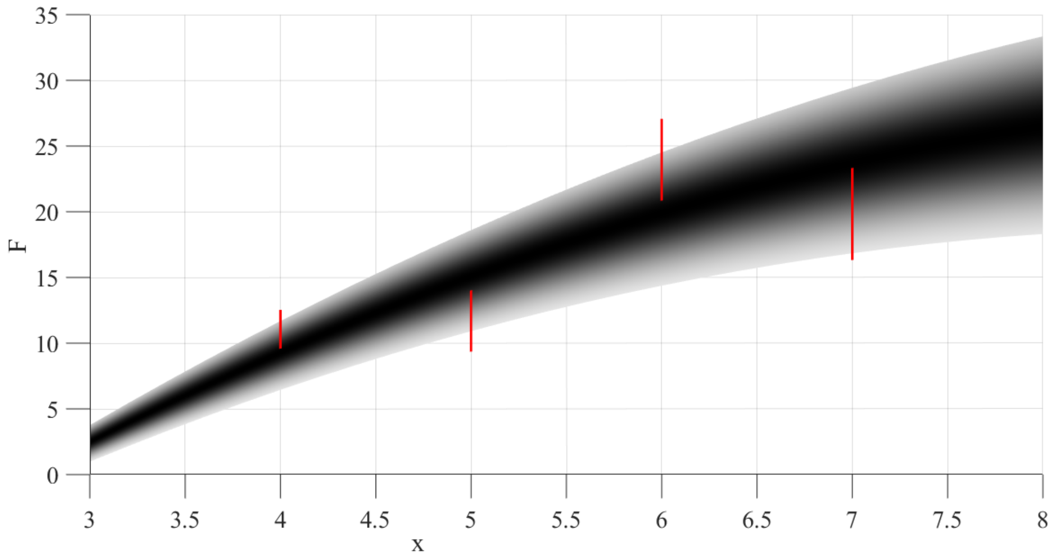
x	Y	$\dim(Y)$
4	$(10; 2; 0.1; 0.2)_G$	5.5721
5	$(13; 2.4; 0.1; 0.2)_G$	6.6866
6	$(22; 4; 0.1; 0.2)_G$	11.1443
7	$(23; 4.4; 0.1; 0.2)_G$	12.2587

Source: Author.

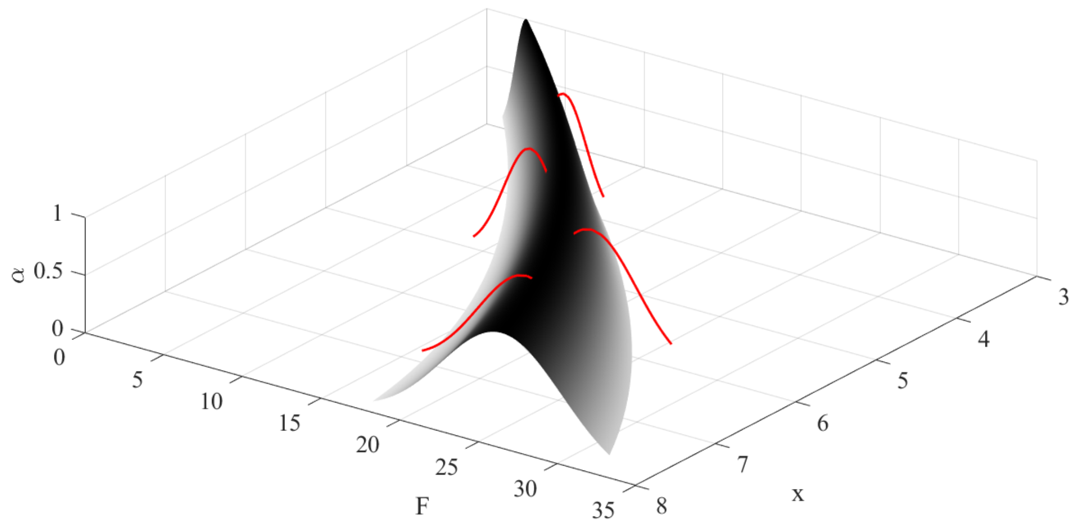
It is worth noting that the examples in Subsections 3.4.4 and 3.4.5 have the same vectors of coefficients p and r, hence the obtained parameters β and γ are the same.

Figure 29 – Approximation for Gaussian fuzzy data

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function F, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 9. Source: Author.

3.5 Conclusion

This chapter developed a least square method for data belonging to the class of A-linearly interactive fuzzy numbers. The novelty of this method is the capacity to fitting separately the trajectory and the specificity of the uncertain data. It was shown that the method presented in Chapter 2 is a particular case.

4 Fuzzy Least Square Method for Quasi Linearly Interactive Fuzzy Numbers

The hypothesis of linearly interactive fuzzy data is restrictive. If one of data outputs $Y_{\widehat{i}}$ is non-linearly interactive with the others, for some $\widehat{i} \in \{1, \ldots, m\}$, then the method presented in Chapter 2 and amplified in Chapter 3 is no longer useful.

In order to overcome this limitation, this chapter introduces the class of quasi linearly interactive fuzzy numbers (QLI), which expands the frontier of linearly interactive fuzzy numbers. The components of QLI class are given by a projection of a fuzzy number in the LI class.

The objective of this chapter is to provide a norm for fuzzy numbers based on the class of continuous and square-integrable functions. This norm will be used to define a new class of fuzzy numbers: the class of quasi linearly interactive fuzzy numbers.

The chapter is divided as follows. Section 4.1 defines the proposed norm. Section 4.2 establishes the definition of quasi linearly interactive fuzzy numbers (QLI). The relation between LI and QLI fuzzy numbers is discussed in Section 4.3. The QLI concept is generalized for n-quasi linearly interactive fuzzy numbers in Section 4.4. Finally, Section 4.5 provides some examples of curve fitting method for QLI data.

4.1 Norm for Fuzzy Numbers

The fuzzy norm based on Pompeiu-Hausdorff metric (see Section 1.7) considers only the information in the 0-cut of the fuzzy number and, therefore, may not distinguish very different fuzzy numbers, as we can see in the next example.

Example 4.1. *Let $A = (0; 0; 1)$ and $B = (0; 1; 1)$. Thus $\|A\|_{\mathcal{F}} = 1 = \|B\|_{\mathcal{F}}$.*

Although in Example 4.1 B is intuitively farther from the 0 than A, their fuzzy norm are equal. This occurs because in this case $\|\cdot\|_{\mathcal{F}}$ only consider the 0-cut.

In Section 4.2 it will be necessary, however, to distinguish fuzzy numbers by its norm. For this reason, this thesis proposes a norm, which is provided as follows.

Consider $\mathcal{L}^2$ the vector space of square-integrable continuous functions from $[c, d]$ to $\mathbb{R}$, denoted by $\mathcal{L}^2([c, d])$. The inner product between $f, g \in \mathcal{L}^2([c, d])$ is defined as

$$\langle f, g \rangle = \int_c^d f(x)g(x)dx,$$

and the norm $\|\cdot\|_2$ is defined as $\|f\|_2 = \sqrt{\langle f, f\rangle}$, for all $f \in \mathcal{L}^2([c,d])$.

Inspired by this definition, let be $A, B \in \mathbb{R}_{\mathcal{F}}$, and denote $[A]_\alpha = [a^-(\alpha), a^+(\alpha)]$ and $[B]_\alpha = [b^-(\alpha), b^+(\alpha)]$, with $a^-, a^+, b^-, b^+ : [0,1] \to \mathbb{R}$. Since a^-, a^+, b^-, b^+ are both bounded monotonic functions, with respect to α, it follows $a^-, a^+, b^-, b^+ \in \mathcal{L}^2([0,1])$. We define the operator

$$\langle A, B\rangle_2 = \int_0^1 a^-(\alpha)b^-(\alpha)d\alpha + \int_0^1 a^+(\alpha)b^+(\alpha)d\alpha, \tag{4.1}$$

Theorem 4.1. *In the space of fuzzy numbers the operator $\langle \cdot, \cdot\rangle_2$ defined in Equation (4.1) has the following properties:*

(i) $\langle A, A\rangle_2 \geqslant 0$;

(ii) $\langle A, A\rangle_2 = 0$ *if, and only if,* $A = 0$, *where* 0 *stands for* $\chi_{\{0\}}$;

(iii) $\langle A, B\rangle_2 = \langle B, A\rangle_2$;

(iv) $\langle \lambda A, B\rangle_2 = \langle A, \lambda B\rangle_2 = \lambda\langle A, B\rangle_2$, *if* $\lambda \geqslant 0$, *and* $\langle \lambda A, B\rangle_2 = \langle A, \lambda B\rangle_2$, *if* $\lambda < 0$; *and*

(v) $\langle A, B + C\rangle_2 = \langle A, B\rangle_2 + \langle A, C\rangle_2$.

Proof. Let $A, B, C \in \mathbb{R}_{\mathcal{F}}$, $a^-, a^+, b^-, b^+, c^-, c^+ \in \mathcal{L}^2([0,1])$. Since $\langle A, A\rangle_2 = \int_0^1 (a^-_\alpha)^2 d\alpha + \int_0^1 (a^+_\alpha)^2 d\alpha$, it is true that $\langle A, A\rangle_2 \geqslant 0$. Moreover, if $\langle A, A\rangle_2 = 0$, then $a^-_\alpha = a^+_\alpha = 0$, for all $\alpha \in [0,1]$, which is only possible if $A = \chi_{\{0\}}$, because A is a fuzzy number. It is obvious that $\langle A, B\rangle_2 = \langle B, A\rangle_2$.

For any $\lambda \geqslant 0$,

$$\begin{aligned}
\langle \lambda A, B\rangle_2 &= \int_0^1 [(\lambda a^-_\alpha)b^-_\alpha + (\lambda a^+_\alpha)b^+_\alpha]d\alpha \\
&= \int_0^1 [a^-_\alpha(\lambda b^-_\alpha) + a^+_\alpha(\lambda b^+_\alpha)]d\alpha && = \langle A, \lambda B\rangle_2 \\
&= \lambda \int_0^1 (a^-_\alpha b^-_\alpha + a^+_\alpha b^+_\alpha)d\alpha && = \lambda\langle A, B\rangle_2.
\end{aligned}$$

On the other hand, for any $\lambda < 0$,

$$\begin{aligned}
\langle \lambda A, B\rangle_2 &= \int_0^1 [(\lambda a^+_\alpha)b^-_\alpha + (\lambda a^-_\alpha)b^+_\alpha]d\alpha \\
&= \int_0^1 [a^-_\alpha(\lambda b^+_\alpha) + a^+_\alpha(\lambda b^-_\alpha)]d\alpha \\
&= \langle A, \lambda B\rangle_2.
\end{aligned}$$

Finally,

$$\begin{aligned}\langle A, B + C\rangle_2 &= \int_0^1 [a_\alpha^-(b_\alpha^- + c_\alpha^-)]d\alpha + \int_0^1 [a_\alpha^+(b_\alpha^+ + c_\alpha^+)]d\alpha \\ &= \int_0^1 (a_\alpha^- b_\alpha^- + a_\alpha^+ b_\alpha^+)d\alpha + \int_0^1 (a_\alpha^- c_\alpha^- + a_\alpha^+ c_\alpha^+)d\alpha \\ &= \langle A, B\rangle_2 + \langle A, C\rangle_2.\end{aligned}$$

□

Note that the operator $\langle \cdot, \cdot \rangle_2$ does not have the property $\langle \lambda A, B\rangle = \lambda\langle A, B\rangle$, for any $\lambda \in \mathbb{R}$. For example, if $\lambda = -1$ and $A = (0;0;1)$, then $\langle -A, A\rangle_2 = 0$, whereas $-\langle A, A\rangle_2 = -\dfrac{1}{3}$. For this reason, the operator is not an inner product for $\mathbb{R}_{\mathcal{F}}$.

Although $\mathbb{R}_{\mathcal{F}}$ is not a vector space, it is reasonable to define a norm considering this idea associating $\mathbb{R}_{\mathcal{F}}$ with $(\mathcal{L}^2 \times \mathcal{L}^2, +, ||\cdot||_2)$. Hence, the proposed norm for $(\mathbb{R}_{\mathcal{F}}, +, ||\cdot||_2)$, that is, $\mathcal{L}^2$-norm $\|\cdot\|_2$ is defined as

$$d_2(A, 0) = \|A\|_2,$$

where 0 stands for the crisp number zero and

$$\|A\|_2 = \sqrt{\langle A, A\rangle_2} = \left(\int_0^1 (a_\alpha^-)^2 d\alpha + \int_0^1 (a_\alpha^+)^2 d\alpha\right)^{\frac{1}{2}}.$$

Definition 4.1. *The $\mathcal{L}^2$-norm of a fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ is given by*

$$\|A\|_2 = \left(\|a_\alpha^-\|_2^2 + \|a_\alpha^+\|_2^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Moreover, the definition of $d_2(A, B)$ is given by

$$d_2(A, B) = \left(\|a_\alpha^- - b_\alpha^-\|_2^2 + \|a_\alpha^+ - b_\alpha^+\|_2^2\right)^{\frac{1}{2}}. \tag{4.3}$$

It is important to observe that the distance $d_2(A, B)$ is coherent with the norm induced by the space $\mathcal{L}^2$:

$$d_2(A, B) = \|(a^-, a^+) - (b^-, b^+)\|_2.$$

In addiction, $\|A + B\|_2 = \left(\displaystyle\int_0^1 \|a_\alpha^- + b_\alpha^-\|^2 + \|a_\alpha^+ + b_\alpha^+\|^2\right)^{\frac{1}{2}}$.

Based on the fact that 2-norm $\|\cdot\|_2$ is a norm for the space $\mathcal{L}^2([0,1])$ the following theorem holds true.

Theorem 4.2. *The $\mathcal{L}^2$-norm satisfies:*

(i) $\|A\|_2 \geqslant 0$*;*

(ii) $\|A\|_2 = 0$ *if, only if,* $A = 0$*;*

(iii) $\|\lambda A\|_2 = |\lambda|.\|A\|_2$*, for all* $\lambda \in \mathbb{R}$*;*

(iv) $\|A + B\|_2^2 = \|A\|_2^2 + \|B\|_2^2 + 2\langle A, B\rangle_2$*;*

(v) $\begin{cases} \|A +_L B\|_2^2 = \|A\|_2^2 + \|B\|_2^2 + 2\langle A, B\rangle_2, & \text{if } q > 0 \\ \|A +_L B\|_2^2 = \|A\|_2^2 + \|B\|_2^2 - 2\langle A, -B\rangle_2, & \text{if } q < 0 \end{cases}$*, whenever* $A, B \in \mathbb{R}_{\mathcal{F}}$ *are linearly interactive with respect to* J_L *with* $L = \{qx + r, x \in \mathbb{R}\}$ *and* $q \neq 0$*, that is,* $B = qA + r$*.*

Proof. Items *(i)* and *(ii)* follow directly from Theorem 4.1.

Let A be a fuzzy number. For any $\lambda \in \mathbb{R}$, $\|\lambda A\|_2^2 = \int_0^1 [(\lambda a_\alpha^-)^2 + (\lambda a_\alpha^+)^2] d\alpha = \lambda^2 \int_0^1 [(a_\alpha^-)^2 + (a_\alpha^+)^2] d\alpha$, then $\|\lambda A\|_2 = |\lambda|\|A\|_2$.

Given A and B,

$$\begin{aligned} \|A + B\|_2^2 &= \int_0^1 [(a_\alpha^- + b_\alpha^-)^2 + (a_\alpha^+ + b_\alpha^+)^2] d\alpha \\ &= \int_0^1 [(a_\alpha^-)^2 + (a_\alpha^+)^2] d\alpha + \int_0^1 [(b_\alpha^-)^2 + (b_\alpha^+)^2] d\alpha + 2\int_0^1 (a_\alpha^- b_\alpha^- + a_\alpha^+ b_\alpha^+) d\alpha \\ &= \|A\|_2^2 + \|B\|_2^2 + 2\langle A, B\rangle_2. \end{aligned}$$

For $q > 0$, linearly interactive sum $+_L$ coincides with stand sum $+$, nonetheless for $q < 0$, the interactive sum $A +_L B$ has norm given by

$$\begin{aligned} \|A +_L B\|_2^2 &= \int_0^1 [(a_\alpha^- + b_\alpha^+)^2 + (a_\alpha^+ + b_\alpha^-)^2] d\alpha \\ &= \int_0^1 [(a_\alpha^-)^2 + (a_\alpha^+)^2] d\alpha + \int_0^1 [(b_\alpha^-)^2 + (b_\alpha^+)^2] d\alpha + 2\int_0^1 (a_\alpha^- b_\alpha^+ + a_\alpha^+ b_\alpha^-) d\alpha. \end{aligned}$$

Note that $-\langle A, -B\rangle = -\int_0^1 a_\alpha^-(-b_\alpha^+) d\alpha - \int_0^1 a_\alpha^+(-b_\alpha^-) d\alpha = \int_0^1 (a_\alpha^- b_\alpha^+ + a_\alpha^+ b_\alpha^-) d\alpha$. Hence, item *(*v*)* is proved. □

Remark 4.1. *One could suggest to define the norm based on integral of membership function. Suppose that for a fuzzy number* A *the norm was given by the integral of* $A(x)$ *over the real line, that is,* $\|A\| = \int_{a_0^-}^{a_0^+} A(x)dx$*. Note that if* $\int_{a_0^-}^{a_0^+} A(x)dx = 0$*, then* $A(x)$ *is not necessarily equal to* $\chi_{\{0\}}$*. For instance, consider* $A(x) = 1$ *for* $x = 2$ *and* $A(x) = 0$*, otherwise. For this fuzzy number, it follows that* $\int_{a_0^-}^{a_0^+} A(x)dx = 0$*, but* $A \neq \chi_{\{0\}}$*.*

For $A = (a; b; c)$, the $\mathcal{L}^2$-norm can be calculated by solving the integral in (4.2):

$$\begin{aligned} \|A\|_2 &= \left(\int_0^1 [a + \alpha(b-a)]^2 d\alpha + \int_0^1 [c - \alpha(c-b)]^2 d\alpha \right)^{\frac{1}{2}} \\ &= \left(\frac{a^2 + ab + 2b^2 + bc + c^2}{3} \right)^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

In particular, for a real number $a \equiv (a; a; a)$, its fuzzy norm and $\mathcal{L}^2$-norm are respectively, $\|(a; a; a)\|_{\mathcal{F}} = |a|$ and $\|(a; a; a)\|_2 = |a|\sqrt{2}$. For symmetric fuzzy numbers with respect to the origin, that is, $A = (-a; 0; a)$, with positive a, the $\mathcal{L}^2$-norm is simply $\|A\|_2 = \dfrac{\sqrt{6}|a|}{3}$.

Example 4.2. *Let $A = (0; 0; 1)$. Its $\mathcal{L}^2$-norm is $\|A\|_2 = \dfrac{\sqrt{3}}{3} \approx 0.58$.*

Let $B = (0; 1; 1)$. Its $\mathcal{L}^2$-norm is $\|B\|_2 = \dfrac{2\sqrt{3}}{3} \approx 1.15$.

From Example 4.1 and Example 4.2 one can observe that 2-norm reveals more information about how far A and B are from zero, in comparison with the fuzzy norm $|| \cdot ||_{\mathcal{F}}$, since the first one takes into account all α-cuts of A and B.

For trapezoidal fuzzy number $A = (a; b; c; d)$ the $\mathcal{L}^2$-norm in (4.2) boils down to

$$\begin{aligned} \|A\|_2 &= \left(\int_0^1 [a + \alpha(b-a)]^2 d\alpha + \int_0^1 [d - \alpha(d-c)]^2 d\alpha \right)^{\frac{1}{2}} \\ &= \left(\frac{a^2 + ab + b^2 + c^2 + cd + d^2}{3} \right)^{\frac{1}{2}}. \end{aligned}$$

Example 4.3. *Let $A = (-1; -1; 1; 1)$. The norms are $\|A\|_{\mathcal{F}} = 1$ and $\|A\|_2 = \sqrt{2}$. Let $B = (0; 0; 1; 4)$ and $C = (0; 3; 4; 4)$. The fuzzy norms coincide, in contrast to the $\mathcal{L}^2$-norm:*

$$\begin{aligned} &\|B\|_{\mathcal{F}} = \|C\|_{\mathcal{F}} = 4, \\ &\|B\|_2 = \sqrt{7} \approx 2.65 \text{ and } \|C\|_2 = \frac{\sqrt{171}}{3} \approx 4.36. \end{aligned}$$

The $\mathcal{L}^2$-norm for Gaussian fuzzy number $A(x; a, \sigma, \delta)_G$ is given by the following integral

$$\begin{aligned} \|A\|_2 = \Bigg[&\int_0^\delta \left(a - \sigma\sqrt{\ln\frac{1}{\delta}} \right)^2 d\alpha + \int_\delta^1 \left(a - \sigma\sqrt{\ln\frac{1}{\alpha}} \right)^2 d\alpha + \\ &+ \int_0^\delta \left(a + \sigma\sqrt{\ln\frac{1}{\delta}} \right)^2 d\alpha + \int_\delta^1 \left(a + \sigma\sqrt{\ln\frac{1}{\alpha}} \right)^2 d\alpha \Bigg]^{\frac{1}{2}}, \end{aligned}$$

whose result is

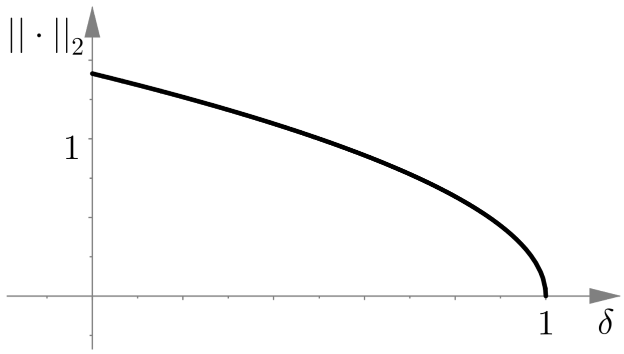
$$\|A\|_2 = \sqrt{2(a^2 + \sigma^2(1-\delta))}.$$

Example 4.4. *Let $A(x; 0, 1, \delta)_G$ be a Gaussian fuzzy number based on normal distribution with mean 0, standard deviation 1 and truncation at $\alpha = \delta$ (see Equation (1.4)). The $\mathcal{L}^2$-norm of A in terms of δ is $\|A\|_2 = \sqrt{2(1-\delta)}$ and it is depicted in Figure 30.*

In particular, for $\delta = 0.1$, the $\mathcal{L}^2$-norm $\|A\|_2 = 1.34$, whereas the $\|A\|_{\mathcal{F}} = 1.52$.

One can observe that the norm approaches the value $\sqrt{2}$ as δ approaches the 0. On the other hand, the fuzzy norm (based on Pompeiu-Hausdorff metric) of the same Gaussian fuzzy number A tends to infinity as the δ tends to 0.

Figure 30 – $\mathcal{L}^2$-Norm of a Gaussian fuzzy number in terms of truncation δ



The $\mathcal{L}^2$-norm of a Gaussian fuzzy number $A(x; 0, 1, \delta)_G$ in terms of the truncation δ.
Source: Author.

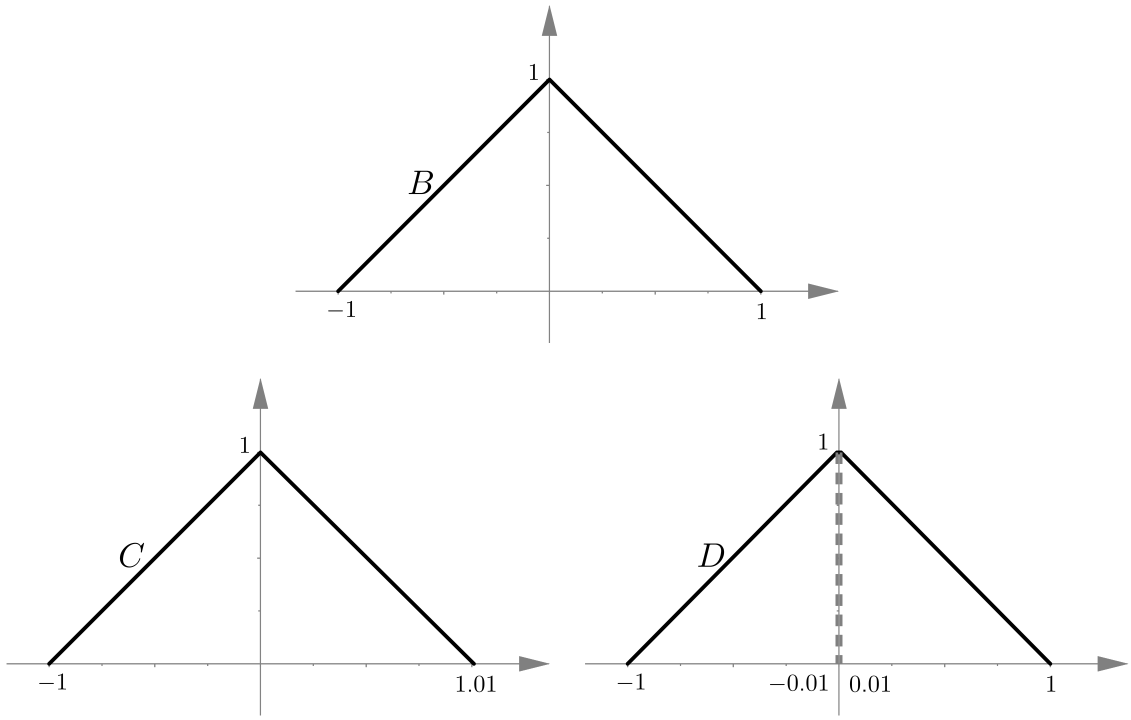
In the next section this $\mathcal{L}^2$-norm will be a useful tool to define a new class of fuzzy numbers.

4.2 Quasi Linearly Interactive Fuzzy Numbers

The linearly interactive fuzzy numbers can be used in several problems, for example, in fitting a longitudinal dataset such as was did in previous chapters. However, in general, the uncertainty data may not be precisely described by this type of interactivity. Our purpose is to relax the definition of linear interactivity.

For example, for data $B = (-1; 0; 1)$, $C = (-1; 0; 1.01)$ and $D = (-1; -0.1; 0.1; 1)$ in Figure 31. There is no $A \in \mathbb{R}_{\mathcal{F}}$ such that B, C and D are A-linearly interactive. The least square methods presented in previous chapters would not be applied to outputs $\{B, C, D\}$. To consider this type of data, the concept of quasi linearly interactive fuzzy numbers is introduced.

Figure 31 – Motivation for Quasi Linearly Interactive Fuzzy Numbers



Fuzzy numbers B, C and D. Source: Author.

Definition 4.2. *Let $A, B \in \mathbb{R}_{\mathcal{F}}$. For given $\epsilon > 0$, the fuzzy numbers A and B are said to be ϵ-quasi linearly interactive or, for short QLI, if there exist $q, r \in \mathbb{R}$, with $q \neq 0$, such that*

$$d_2((qA + r), B) < \epsilon, \tag{4.5}$$

where the $\mathcal{L}^2$-norm is given by Equation (4.3).

The above definition stands a concept which consists in finding parameters $q, r \in \mathbb{R}$ such that $qA + r$ is the closest to B, with respect to the $\mathcal{L}^2$-norm and ϵ is the tolerance error.

Remark 4.2. *If A and B are linearly interactive fuzzy numbers with parameters q and r, that is, $B = qA + r$, then the $\mathcal{L}^2$-norm is $\|(qA + r) - B\|_2 = 0$. In other words, if two fuzzy numbers are linearly interactive, then they are quasi linearly interactive fuzzy numbers. For A and B non-linearly interactive fuzzy numbers, the $\mathcal{L}^2$-norm $\|(qA + r) - B\|_2$ will not be equal to zero. Therefore, A and B will be ϵ-quasi linearly interactive, if $\|(qA + r) - B\|_2 < \epsilon$.*

The objective here is to minimize the $\mathcal{L}^2$-norm in (4.3). Let $f, g : [0, 1] \to \mathbb{R}$ be real functions and the residual function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ given by

$$h(q, r) = \int_0^1 [(qf(\alpha) - r) - g(\alpha)]^2 d\alpha.$$

Since $\langle f, g\rangle = \int_0^1 f(\alpha)g(\alpha)d\alpha$, the h function is given by

$$h(q,r) = q^2||f||_2^2 + 2qr\langle f,1\rangle + r^2 - 2q\langle f,g\rangle - 2r\langle g,1\rangle + ||g||_2^2,$$

where 1 is the constant function $1(x) = 1$ for all $x \in [0,1]$.

The values (q,r) that minimize the function h satisfy the following conditions:

$$\begin{cases} \dfrac{\partial h}{\partial q}(q,r) = -2\langle f,g\rangle + 2q||f||_2^2 + 2r\langle f,1\rangle = 0 \\ \dfrac{\partial h}{\partial r}(q,r) = -2\langle g,1\rangle + 2q\langle f,1\rangle + 2r = 0 \end{cases}.$$

Thus q and r are obtained by solving the following system

$$\begin{pmatrix} ||f||_2^2 & \langle f,1\rangle \\ \langle f,1\rangle & 1 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \langle f,g\rangle \\ \langle g,1\rangle \end{pmatrix}. \tag{4.6}$$

From the definition in (4.3), we define functions $h^1, h^2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ as follows

$$d_2((qA+r),B) = \begin{cases} h^1(q,r) = \displaystyle\int_0^1 [qa^+(\alpha) + r - b^+(\alpha)]^2 d\alpha + \int_0^1 [qa^-(\alpha) + r - b^-(\alpha)]^2 d\alpha, \\ \hfill \text{if } q \geqslant 0, \\ h^2(q,r) = \displaystyle\int_0^1 [qa^-(\alpha) + r - b^+(\alpha)]^2 d\alpha + \int_0^1 [qa^+(\alpha) + r - b^-(\alpha)]^2 d\alpha, \\ \hfill \text{if } q < 0, \end{cases} \tag{4.7}$$

where $[A]_\alpha = [a^-(\alpha), a^+(\alpha)]$, $[B]_\alpha = [b^-(\alpha), b^+(\alpha)]$, for all $\alpha \in [0,1]$.

Repeating the process used for h, we obtain the following expressions:

$$\begin{aligned} h^1(q,r) &= q^2(\|a_\alpha^-\|_2^2 + \|a_\alpha^+\|_2^2) + 2qr(\langle a_\alpha^-,1\rangle + \langle a_\alpha^+,1\rangle) + 2r^2 - 2q(\langle a_\alpha^-, b_\alpha^-\rangle + \langle a_\alpha^+, b_\alpha^+\rangle) \\ &\quad - 2r(\langle b_\alpha^-,1\rangle + \langle b_\alpha^+,1\rangle) + (\|b_\alpha^-\|_2^2 + \|b_\alpha^+\|_2^2), \\ h^2(q,r) &= q^2(\|a_\alpha^-\|_2^2 + \|a_\alpha^+\|_2^2) + 2qr(\langle a_\alpha^-,1\rangle + \langle a_\alpha^+,1\rangle) + 2r^2 - 2q(\langle a_\alpha^-, b_\alpha^+\rangle + \langle a_\alpha^+, b_\alpha^-\rangle) \\ &\quad - 2r(\langle b_\alpha^-,1\rangle + \langle b_\alpha^+,1\rangle) + (\|b_\alpha^-\|_2^2 + \|b_\alpha^+\|_2^2). \end{aligned}$$

Therefore

$$h^1(q,r) = q^2\|A\|_2^2 + 2qr\langle A,1\rangle_2 + 2r^2 - 2q\langle A,B\rangle_2 - 2r\langle B,1\rangle_2 + \|B\|_2^2, \tag{4.8}$$

$$h^2(q,r) = q^2\|A\|_2^2 + 2qr\langle A,1\rangle_2 + 2r^2 - 2q(-\langle A,-B\rangle) - 2r\langle B,1\rangle_2 + \|B\|_2^2. \tag{4.9}$$

The pairs (q,r) that minimize the functions h^1 and h^2 are the solutions to $\nabla h^1(q,r) = (0,0)$ and $\nabla h^2(q,r) = (0,0)$, respectively, that is, the solutions for the following systems:

$$\begin{pmatrix} \|A\|_2^2 & \langle A,1\rangle_2 \\ \langle A,1\rangle_2 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \langle A,B\rangle_2 \\ \langle B,1\rangle_2 \end{pmatrix} \tag{4.10}$$

and

$$\begin{pmatrix} \|A\|_2^2 & \langle A,1\rangle_2 \\ \langle A,1\rangle_2 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} -\langle A,-B\rangle_2 \\ \langle B,1\rangle_2 \end{pmatrix}. \tag{4.11}$$

In other words, the values q and r that minimize the functions in (4.7) are given respectively by the systems

$$Nu = v^1, \text{ if } q \geqslant 0, \tag{4.12}$$

$$Nu = v^2, \text{ if } q < 0, \tag{4.13}$$

where

$$N = \begin{pmatrix} ||A||_2^2 & \langle A,1\rangle_2 \\ \langle A,1\rangle_2 & 2 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}$$

and

$$v^1 = \begin{pmatrix} \langle A,B\rangle_2 \\ \langle B,1\rangle_2 \end{pmatrix}, \; v^2 = \begin{pmatrix} -\langle A,-B\rangle_2 \\ \langle B,1\rangle_2 \end{pmatrix}.$$

Let u^1 and u^2 be the solutions of systems (4.12) and (4.13), and will be chosen the pair that produces the minimum value of residual function h, that is,

$$u = (q,r) = \text{argmin}\{h^1(u^1), h^2(u^2)\}. \tag{4.14}$$

If $u_1^1 < 0$ or $u_1^2 > 0$, then consider the solution as being $u^3 = \begin{pmatrix} 0 \\ \langle B,1\rangle_2 \end{pmatrix}$.

Hence $qA+r$ is the fuzzy number that best approximates B, according to $\mathcal{L}^2$-norm.

4.2.1 Triangular Quasi Linearly Interactive Fuzzy Numbers

For a triangular fuzzy number $A = (a;b;c)$ the entries of matrix N are given by

$$\|A\|_2^2 = \frac{a^2+ab+2b^2+bc+c^2}{3}$$

and

$$\begin{aligned} \langle A,1\rangle_2 &= \int_0^1 [a+\alpha(b-a)]d\alpha + \int_0^1 [c-\alpha(c-b)]d\alpha \\ &= \frac{1}{2}(a+c)+b. \end{aligned}$$

For $A = (a_1;b_1;c_1)$ and $B = (a_2;b_2;c_2)$, the entries of vectors v^1 and v^2 are given by

$$\begin{aligned} \langle A,B\rangle_2 &= \int_0^1 [c_1-\alpha(c_1-b_1)][c_2-\alpha(c_2-b_2)] + [a_1+\alpha(b_1-a_1)][a_2+\alpha(b_2-a_2)]d\alpha \\ &= \frac{2b_1b_2}{3} + \frac{a_1a_2}{3} + \frac{c_1c_2}{3} + \frac{a_1b_2}{6} + \frac{a_2b_1}{6} + \frac{c_1b_2}{6} + \frac{b_1c_2}{6} \end{aligned}$$

and

$$\begin{aligned}
-\langle A, -B\rangle_2 &= \int_0^1 [a_1 + \alpha(b_1 - a_1)][c_2 - \alpha(c_2 - b_2)]d\alpha + \\
&+ \int_0^1 [c_1 - \alpha(c_1 - b_1)][a_2 + \alpha(b_2 - a_2)]d\alpha \\
&= \frac{2b_1b_2}{3} + \frac{a_1c_2}{3} + \frac{a_2c_1}{3} + \frac{a_1b_2}{6} + \frac{a_2b_1}{6} + \frac{b_1c_2}{6} + \frac{c_1b_2}{6}.
\end{aligned}$$

Therefore, the systems to be solved are

$$\begin{pmatrix} \frac{a_1^2 + a_1b_1 + 2b_1^2 + b_1c_1 + c_1^2}{3} & \frac{a_1 + c_1}{2} + b_1 \\ \frac{a_1 + c_1}{2} + b_1 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \\
= \begin{pmatrix} \frac{2b_1b_2 + a_1a_2 + c_1c_2}{3} + \frac{a_1b_2 + a_2b_1 + b_1c_2 + c_1b_2}{6} \\ \frac{a_2 + c_2}{2} + b_2 \end{pmatrix}$$

for $q \geqslant 0$, and

$$\begin{pmatrix} \frac{a_1^2 + a_1b_1 + 2b_1^2 + b_1c_1 + c_1^2}{3} & \frac{a_1 + c_1}{2} + b_1 \\ \frac{a_1 + c_1}{2} + b_1 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \\
= \begin{pmatrix} \frac{2b_1b_2 + a_1c_2 + a_2c_1}{3} + \frac{a_1b_2 + a_2b_1 + b_1c_2 + c_1b_2}{6} \\ \frac{a_2 + c_2}{2} + b_2 \end{pmatrix}$$

for $q < 0$.

Example 4.5. *Let $A = (0; 0; 1)$ and $B = (0; 0; 1.01)$ be triangular fuzzy numbers. Note that $B = 1.01A$. The first system to be solved is*

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \frac{1.01}{3} \\ \frac{1.01}{2} \end{pmatrix}.$$

The first possible solution is $u^1 = \begin{pmatrix} 1.01 \\ 0 \end{pmatrix}$.

The second system to be solved is

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1.01}{2} \end{pmatrix}.$$

The second possible solution is $u^2 = \begin{pmatrix} \dfrac{-3.03}{5} \\ \\ \dfrac{17.17}{20} \end{pmatrix}$.

From Equation (4.14)*, the pair* (q, r) *that makes* B *QLI with* A *is given by argmin*$\{h^1(u^1), h^2(u^2)\}$*, where* $h^1(u^1) = 0$ *and* $h^2(u^2) = \dfrac{323}{600}(1.01)^2$*. Hence the solution is* $u = \begin{pmatrix} 1.01 \\ 0 \end{pmatrix}$*, with error* $\epsilon = 0$*. Indeed,* $1.01A + 0 = (0; 0; 1.01) = B$*.*

In Example 4.5, $qA + r$ is QLI with B, but it is also possible to find q and r such that, $B + r$ is QLI with A, as it is shown in next example.

Example 4.6. *Let* $A = (0; 0; 1.01)$ *and* $B = (0; 0; 1)$ *be triangular fuzzy numbers. The first system to be solved is*

$$\begin{pmatrix} \dfrac{1.01^2}{3} & \dfrac{1.01}{2} \\ \\ \dfrac{1.01}{2} & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \dfrac{1.01}{3} \\ \\ \dfrac{1}{2} \end{pmatrix}.$$

The first possible solution is $u^1 = \begin{pmatrix} \dfrac{1}{1.01} \\ \\ 0 \end{pmatrix}$.

The second system to be solved is

$$\begin{pmatrix} \dfrac{1.01^2}{3} & \dfrac{1.01}{2} \\ \\ \dfrac{1.01}{2} & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \dfrac{1}{2} \end{pmatrix}.$$

The second possible solution is $u^2 = \begin{pmatrix} \dfrac{-3}{5.05} \\ \\ \dfrac{2}{5} \end{pmatrix}$.

From Equation (4.14)*, the pair* (q, r) *that makes* B *QLI with* A *is given by argmin*$\{h^1(u^1), h^2(u^2)\}$*, where* $h^1(u^1) = 0$ *and* $h^2(u^2) = 8.92$*. Hence the solution is* $u = \begin{pmatrix} \dfrac{1}{1.01} \\ \\ 0 \end{pmatrix}$*, with error* $\epsilon = 0$*. Indeed,* $\dfrac{1}{1.01}A + 0 = (0; 0; 1) = B$*.*

Example 4.7. *Let $A = (-1; 0; 1)$ and $B = (-1; 0; 1.01)$ be triangular fuzzy numbers. The first system to be solved is*

$$\begin{pmatrix} \frac{2}{3} & 0 \\ & \\ 0 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \frac{2.01}{3} \\ \\ \frac{0.01}{2} \end{pmatrix}.$$

The first possible solution is $u^1 = \begin{pmatrix} \frac{2.01}{2} \\ \\ \frac{0.01}{4} \end{pmatrix}$.

The second system to be solved is

$$\begin{pmatrix} \frac{2}{3} & 0 \\ & \\ 0 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \frac{-2.01}{3} \\ \\ \frac{0.01}{2} \end{pmatrix}.$$

The second possible solution is $u^2 = \begin{pmatrix} \frac{-2.01}{2} \\ \\ \frac{0.01}{4} \end{pmatrix}$.

From Equation (4.14), the pair (q, r) that makes B QLI with A is given by argmin$\{h^1(u^1), h^2(u^2)\}$, where $h^1(u^1) = h^2(u^2) = 0.00001\overline{6}$. Hence both u^1 and u^2 are solutions.

In fact, $\frac{-2.01}{2}(-1; 0; 1) + \frac{0.01}{4} = \frac{2.01}{2}(-1; 0; 1) + \frac{0.01}{4} = (-1.0025; 0; 1.0075)$. *The fuzzy numbers A and B are $0.00001\overline{6}$-QLI.*

4.2.2 Triangular and Trapezoidal Quasi Linearly Interactive Fuzzy Numbers

Let $A = (a; b; c)$ and $B = (d; e; f; g)$. In order to find q, r such that $qA + r$ is QLI with B, the entries of matrix N are given by

$$\begin{aligned} \langle B, 1\rangle &= \int_0^1 [d + \alpha(e - d)]d\alpha + \int_0^1 [g + \alpha(f - g)]d\alpha \\ &= d + g + \frac{1}{2}(e + f - d - g), \end{aligned}$$

$$\begin{aligned} \langle A, B\rangle &= \int_0^1 [a + \alpha(b - a)][d + \alpha(e - d)]d\alpha + \int_0^1 [c + \alpha(b - c)][g + \alpha(f - g)]d\alpha \\ &= \frac{1}{6}(ae + bd + bg + cf) + \frac{1}{3}(ad + be + bf + cg) \end{aligned}$$

and

$$-\langle A, -B\rangle = \int_0^1 [a+\alpha(b-a)][g+\alpha(f-g)]d\alpha + \int_0^1 [c+\alpha(b-c)][d+\alpha(e-d)]d\alpha$$
$$= \frac{1}{6}(af+bd+bg+ce) + \frac{1}{3}(ag+be+bf+cd).$$

In comparison with the case that A and B are triangular fuzzy numbers, the only change is done in vectors v^1 and v^2, with the entries described above.

The systems to be solved are

$$\begin{pmatrix} \dfrac{a^2+ab+2b^2+bc+c^2}{3} & \dfrac{a+c}{2}+b \\ \dfrac{a+c}{2}+b & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} =$$
$$= \begin{pmatrix} \dfrac{1}{6}(ae+bd+bg+cf) + \dfrac{1}{3}(ad+be+bf+cg) \\ d+g+\dfrac{1}{2}(e+f-d-g) \end{pmatrix}$$

and

$$\begin{pmatrix} \dfrac{a^2+ab+2b^2+bc+c^2}{3} & \dfrac{a+c}{2}+b \\ \dfrac{a+c}{2}+b & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} =$$
$$= \begin{pmatrix} \dfrac{af+bd+bg+ce}{6} + \dfrac{ag+be+bf+cd}{3} \\ d+g+\dfrac{e+f-d-g}{2} \end{pmatrix}.$$

Example 4.8. *Let $A=(-1;0;2)$ and $B=(-1;-0.1;0.1;2)$ be respectively triangular and trapezoidal fuzzy numbers. The first system to be solved is*

$$\begin{pmatrix} \dfrac{5}{3} & \dfrac{1}{2} \\ \dfrac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \dfrac{10.3}{6} \\ \dfrac{1}{2} \end{pmatrix}.$$

The first possible solution is $u^1 = \begin{pmatrix} \dfrac{38.2}{37} \\ \dfrac{-0.3}{37} \end{pmatrix}$.

The second system to be solved is

$$\begin{pmatrix} \dfrac{5}{3} & \dfrac{1}{2} \\ \dfrac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \dfrac{-8.3}{6} \\ \dfrac{1}{2} \end{pmatrix}.$$

The second possible solution is $u^2 = \begin{pmatrix} \dfrac{-28.2}{37} \\ \\ \dfrac{16.3}{37} \end{pmatrix}$.

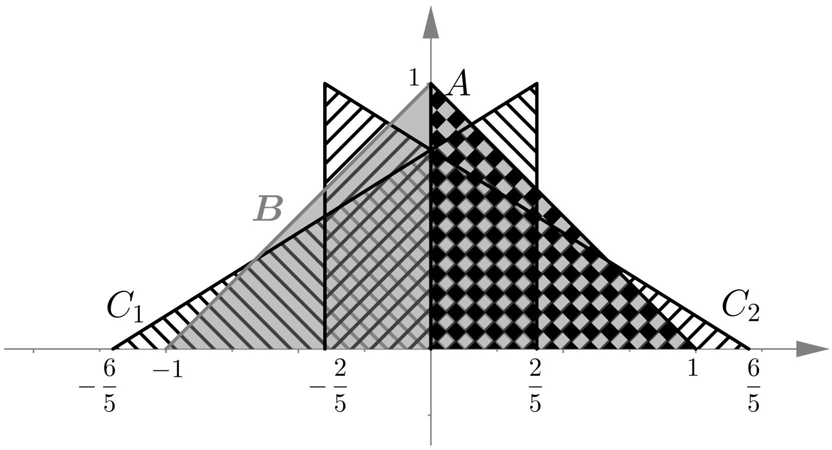
From Equation (4.14), the pair (q, r) that makes B QLI with A is given by argmin$\{h^1(u^1), h^2(u^2)\}$, where $h^1(u^1) = 0.00\overline{504}$ and $h^2(u^2) = 0.2\overline{468}$. Hence u^1 is the solution.

The fuzzy number $\dfrac{38.2}{37}(-1; 0; 2) - \dfrac{0.3}{37} = (-1.0\overline{405}; -0.0\overline{081}; 2.0\overline{567})$ *is QLI with B, with an error of $h^1(u^1)$.*

The next example shows that, although it is possible to find pairs $\{q, r\}$ for many pairs of triangular fuzzy numbers $\{A, B\}$, such that A and B are ϵ-QLI, this approach is coherent with respect to the error tolerance ϵ only if ϵ is close to zero. Otherwise, the error ϵ is big and the intuition fails.

Example 4.9. *Consider the fuzzy numbers $A = (0; 0; 1)$ and $B = (-1; 0; 1)$. It is possible to calculate* $\|A\|_2^2 = \dfrac{1}{3}$, $\|B\|_2^2 = \dfrac{2}{3}$, $\langle A, 1\rangle = \dfrac{1}{2}$, $\langle B, 1\rangle = 0$, $\langle A, B\rangle = \dfrac{1}{3}$ *and* $-\langle A, -B\rangle = -\dfrac{1}{3}$. *Thus, the pairs u^1 and u^2 obtained are* $u^1 = \begin{pmatrix} \dfrac{8}{5} \\ \\ -\dfrac{2}{5} \end{pmatrix}$ *and* $u^2 = \begin{pmatrix} -\dfrac{8}{5} \\ \\ \dfrac{2}{5} \end{pmatrix}$. *Moreover,* $h^1(u^1) = h^2(u^2) = 0, 1\overline{3}$. *The fuzzy numbers are* $C_1 = \left(-\dfrac{6}{5}; \dfrac{2}{5}; \dfrac{2}{5}\right)$ *and* $C_2 = \left(-\dfrac{2}{5}; -\dfrac{2}{5}; \dfrac{6}{5}\right)$, *as shown in Figure 32.*

Figure 32 – Non-QLI Fuzzy Numbers



The fuzzy A is painted with squares, B is painted in gray, C_1 and C_2 are painted with solid lines in $\dfrac{3\pi}{4}$ and $\dfrac{\pi}{4}$ directions, respectively. Source: Author.

It was possible to find two pairs $\{q, r\}$, nevertheless $q_iA + r_i$, for $i = 1, 2$, it is not *close to B, since $h^1(u^1)$ and $h^2(u^2)$ are not small enough. In Figure 32 one may observe that both C_1 and C_2 are not similar with B.*

Although the definition of QLI fuzzy numbers is more general than the definition of LI fuzzy numbers, the above example shows that not every pair of fuzzy numbers is QLI.

4.3 Quasi Linearly Interactive Fuzzy Numbers as an Extension of Linearly Interactive Fuzzy Numbers

In Section 4.2.2, Examples 4.5 and 4.6 indicate that the class of LIFN is somehow contained in the class of QLIFN. The following results corroborate this assumption under some conditions.

Lemma 1. *Let be the matrix N in the QLI systems in Equations (4.12) and (4.13),*

$$N = \begin{pmatrix} ||A||_2^2 & \langle A, 1\rangle_2 \\ \langle A, 1\rangle_2 & 2 \end{pmatrix}. \tag{4.15}$$

The determinant of the matrix N is invariant with respect to translations of $A \in \mathbb{R}_{\mathcal{F}}$.

Proof. Consider the translation $\widetilde{A} = A + k$ of $A \in \mathbb{R}_{\mathcal{F}}$ by $k \in \mathbb{R}$, and the translated version of N given by $\widetilde{N} = \begin{pmatrix} ||A+k||_2^2 & \langle A+k, 1\rangle_2 \\ \langle A+k, 1\rangle_2 & 2 \end{pmatrix}$.

Note that $\langle A, 1\rangle_2 = \int_0^1 (a_\alpha^- + a_\alpha^+)d\alpha$, where 1 stands for $\chi_{\{1\}}$. For any $k \in \mathbb{R}$, $[k]_\alpha = [k, k]$ for all $\alpha \in [0, 1]$, it follows that:

- $\langle k, 1\rangle_2 = \int_0^1 2k d\alpha = 2k$,
- $\|k\|_2^2 = \int_0^1 2k^2 d\alpha = 2k^2$, and
- $\langle A, k\rangle_2 = \int_0^1 k(a_\alpha^- + a_\alpha^+)d\alpha = k\langle A, 1\rangle_2$.

As a consequence, for any $k \in \mathbb{R}$, it holds that $\langle A+k, 1\rangle_2 = \langle A, 1\rangle_2 + \langle k, 1\rangle_2 = \langle A, 1\rangle_2 + 2k$. From item *(iv)* of Theorem 4.2, $\|A+k\|_2^2 = \|A\|_2^2 + \|k\|_2^2 + 2\langle A, k\rangle_2$, hence

$$\begin{aligned} \det(\widetilde{N}) &= 2\|A+k\|_2^2 - \langle A+k, 1\rangle_2^2 \\ &= 2\|A\|_2^2 + 4k^2 + 4k\langle A, 1\rangle_2 - \langle A, 1\rangle_2^2 - 4k\langle A, 1\rangle_2 - 4k^2 \\ &= 2\|A\|_2^2 - \langle A, 1\rangle_2^2 \\ &= \det(N). \end{aligned}$$

□

Remark 4.3. *If $A \in \mathbb{R}$ and $B \in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}$, then $A = k$ for some $k \in \mathbb{R}$ which implies $\langle A, B\rangle_2 = -\langle A, -B\rangle_2 = k\langle B, 1\rangle_2 \in \mathbb{R}$. The systems in Equations (4.12) and (4.13) are equal and given by*

$$\begin{pmatrix} k^2 & k \\ k & 1 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \frac{\langle B, 1\rangle_2}{2} \begin{pmatrix} k \\ 1 \end{pmatrix},$$

thus the solution is $r = \dfrac{\langle B, 1\rangle_2}{2} - kq$. It means that $B = kq + r \in \mathbb{R}$, which is a contradiction. The conclusion is that a real number (crisp fuzzy number) and a non-crisp fuzzy number are not quasi linearly interactive.

Lemma 2. *The matrix N in Equation (4.15) is nonsingular for trapezoidal fuzzy numbers $A \in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}$.*

Proof. From Lemma 1 it is reasonable to consider A centered in the origin, without loss of generality. If $A \in \mathbb{R}_{\mathcal{F}}$ is trapezoidal, then there exists $c_1, c_2 \in \mathbb{R}^*$ and $d_1, d_2 \in \mathbb{R}$ such that its α-cuts are given by $[A]_\alpha = [c_1\alpha + d_1, c_2\alpha + d_2]$ for all $\alpha \in [0, 1]$. Moreover, $c_1 > 0$ and $c_2 < 0$, $d_1 < 0$ and $d_2 > 0$. The determinant of N is given by

$$\begin{aligned}
\det(N) &= 2\|A\|_2^2 - \langle A, 1\rangle_2^2 \\
&= 2\int_0^1 \left[(c_1\alpha + d_1)^2 + (c_2\alpha + d_2)^2\right] d\alpha - \left[(c_1 + c_2)\int_0^1 \alpha d\alpha + (d_1 + d_2)\right]^2 \\
&= \frac{2(c_1^2 + c_2^2)}{3} + 2(c_1 d_1 + c_2 d_2) + 2(d_1^2 + d_2^2) - (c_1 + c_2)^2 \\
&\quad - (c_1 + c_2)(d_1 + d_2) - (d_1 + d_2)^2 \\
&= \frac{c_1^2 + c_2^2}{6} - (c_1 + d_1)(c_2 + d_2) + \frac{d_1^2 + d_2^2}{2}.
\end{aligned}$$

Since A is not a real number, consider $\alpha = 1$ to conclude that $c_1 + d_1 \leqslant 0$ and $c_2 + d_2 \geqslant 0$. Thus $-(c_1 + d_1)(c_2 + d_2) \geqslant 0$ and $\det(N) > 0$. In other words, N is a nonsingular matrix. □

Proposition 4.1. *Let $A, B \notin \mathbb{R}$ be trapezoidal linearly interactive fuzzy numbers such that $B = \widehat{q}A + \widehat{r}$. The pair (q, r) given by the Equation (4.14) is equal to $(\widehat{q}, \widehat{r})$.*

Proof. On one hand, if $\widehat{q} > 0$, then $\langle A, B\rangle_2 = \widehat{q}\|A\|_2^2 + \widehat{r}\langle A, 1\rangle_2$. On the other hand, if $\widehat{q} < 0$, then $-\langle A, -B\rangle_2 = \widehat{q}\|A\|_2^2 + \widehat{r}\langle A, 1\rangle_2$. In both cases, $\langle B, 1\rangle_2 = \widehat{q}\langle A, 1\rangle_2 + 2\widehat{r}$. Moreover, the systems in Equations (4.12) and (4.13) are given by

$$\begin{pmatrix} ||A||_2^2 & \langle A, 1\rangle_2 \\ \langle A, 1\rangle_2 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} ||A||_2^2 & \langle A, 1\rangle_2 \\ \langle A, 1\rangle_2 & 2 \end{pmatrix} \begin{pmatrix} \widehat{q} \\ \widehat{r} \end{pmatrix}. \tag{4.16}$$

From Lemma 2, the solution to the system in Equation (4.16) is unique, hence $(q, r) = (\widehat{q}, \widehat{r})$. □

Corollary 4.1. *Let $A, B \notin \mathbb{R}$ be triangular linearly interactive fuzzy numbers such that $B = \widehat{q}A + \widehat{r}$. The pair (q, r) given by the Equation (4.14) is equal to $(\widehat{q}, \widehat{r})$.*

Proposition 4.1 and Corollary 4.1 guarantee that, given A and B non-crisp trapezoidal (or triangular) linearly interactive fuzzy numbers, Equation (4.14) is a simple method to find the pairs (q, r) such that $B = qA + r$. A natural question to ask is if there are other shapes of fuzzy numbers, for that matter may help to consider symmetry.

Lemma 3. *The matrix N in Equation (4.15) is nonsingular for symmetrical fuzzy numbers $A \in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}$.*

Proof. Consider the translation $m = \dfrac{-a_1^- - a_1^+}{2}$ such that $\widetilde{A} = A + m$ is symmetrical with respect to the origin. Thus the membership function of $\widetilde{A}$ is an even function, that is, $\widetilde{A}(-x) = \widetilde{A}(x)$ for every $x \in \mathbb{R}$, and $\widetilde{a}_\alpha^- = -\widetilde{a}_\alpha^+$ for all $\alpha \in [0, 1]$. For matrix N, it holds that

$$\det(N) = \det(\widetilde{N}) = 4\int_0^1 (\widetilde{a}_\alpha^+)^2 d\alpha.$$

Hence $\det(N) = 0$ if, and only if, $\displaystyle\int_0^1 (\widetilde{a}_\alpha^+)^2 d\alpha = 0$. For all $\alpha \in [0, 1]$, we have $\widetilde{a}_\alpha^+ \geqslant 0$ and $\widetilde{A} \in \mathbb{R}_{\mathcal{F}}$, so $\det(N) \neq 0$ if, and only if, $\widetilde{A} \neq 0$, that is, $A \notin \mathbb{R}$. □

Proposition 4.2. *Let $A, B \notin \mathbb{R}$ be symmetrical linearly interactive fuzzy numbers such that $B = \widehat{q}A + \widehat{r}$. The pair (q, r) given by Equation (4.14) is equal to $(\widehat{q}, \widehat{r})$.*

Proof. Analogously to the proof of Proposition 4.1. From Lemma 3, the solution to the system in Equation (4.16) is unique, hence $(q, r) = (\widehat{q}, \widehat{r})$. □

Proposition 4.2 guarantees that, given A and B non-crisp symmetrical linearly interactive fuzzy numbers, Equation (4.14) is a simple method to find the pairs (q, r) such that $B = qA + r$.

The results of this section show that if two fuzzy numbers are linearly interactive, then they are also quasi-linearly interactive fuzzy numbers. QLI fuzzy numbers are broader than LI fuzzy numbers. For this reason, it is correct to claim that QLI fuzzy numbers are an extension of LI fuzzy numbers, which is summarized in Figure 33.

Next section addresses the definition of n-quasi linearly interactive fuzzy numbers as the natural extension of QLI fuzzy numbers.

4.4 n-Quasi Linearly Interactive Fuzzy Numbers

The methods given in Section 2.2 and Section 3.2 will be adapted for QLI fuzzy numbers. To this end, consider the dataset $D = \{(x_1, Y_1), \ldots, (x_m, Y_m)\} \subset \mathbb{R} \times \mathbb{R}_{\mathcal{F}}$. In

Figure 33 – LI is a subclass of QLI

$$\boxed{\text{LI}} \subseteq \boxed{\text{QLI}}$$

for triangular, trapezoidal and
symmetrical fuzzy numbers

The class of linearly interactive fuzzy numbers is contained in the class of quasi-linearly interactive fuzzy numbers, for pairs of triangular, trapezoidal and symmetrical fuzzy numbers. Source: Author.

order to guarantee that the pairs (Y_1, Y_i) , $i = 2, \ldots, m$, are QLI the following minimization problem must be solved

$$\min_{q_i, r_i} \|(q_i Y_1 + r_i) - Y_i\|_2^2, \tag{4.17}$$

where $q_i \neq 0$ for all $i = 2, \ldots, m$.

From Section 4.2, for $i = 2, \ldots, m$, the systems to be solved are

$$\begin{aligned} N u_i &= v_i^1, \text{ if } q_i > 0, \\ N u_i &= v_i^2, \text{ if } q_i < 0, \end{aligned}$$

where

$$N = \begin{pmatrix} \|Y_1\|_2^2 & \langle Y_1, 1 \rangle_2 \\ \langle Y_1, 1 \rangle_2 & 2 \end{pmatrix}, u = \begin{pmatrix} q_i \\ r_i \end{pmatrix}$$

and

$$v_i^1 = \begin{pmatrix} \langle Y_1, Y_i \rangle_2 \\ \langle Y_i, 1 \rangle_2 \end{pmatrix}, v_i^2 = \begin{pmatrix} -\langle Y_1, -Y_i \rangle \\ \langle Y_i, 1 \rangle_2 \end{pmatrix}.$$

The solutions, for each $i = 2, \ldots, m$, are given by

$$u_i = \begin{pmatrix} q_i \\ r_i \end{pmatrix} = \operatorname{argmin}\{h_i^1(u_i^1), h_i^2(u_i^2)\}, \tag{4.18}$$

where

$$\begin{aligned} h_i^1(q_i, r_i) =& q_i^2 \|Y_1\|_2^2 + 2 q_i r_i \langle Y_1, 1 \rangle_2 + 2 r_1^2 - 2 q_1 (\langle Y_{1_\alpha}^-, Y_{i_\alpha}^+ \rangle + \langle Y_{1_\alpha}^+, Y_{i_\alpha}^- \rangle) + \\ & - 2 r_i \langle Y_i, 1 \rangle_2 + \|Y_i\|_2^2, \end{aligned} \tag{4.19}$$

$$\begin{aligned} h_i^2(q_i, r_i) =& q_i^2 \|Y_1\|_2^2 + 2 q_i r_i \langle Y_1, 1 \rangle_2 + 2 r_i^2 - 2 q_i \langle Y_1, Y_i \rangle_2 + \\ & - 2 r_i \langle Y_i, 1 \rangle_2 + \|Y_i\|_2^2. \end{aligned} \tag{4.20}$$

That is, the pair $\{q_i, r_i\}$ is calculated for each pair $\{Y_1, Y_i\}$, hence is reasonable to establish the following definition.

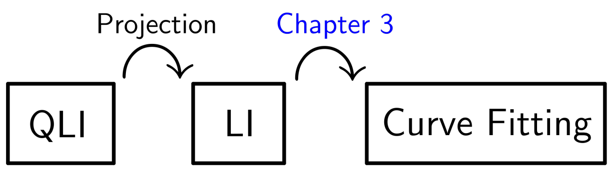
Definition 4.3. *A n-uple of fuzzy numbers $\{Y_1, \ldots, Y_n\}$ is said to be ϵ-nQLI if*

$$\epsilon = \max\{h_1(u_1), \ldots, h_n(u_n)\},$$

where $h_i(u_i)$ is i-th error given by $h_i(u_i) = \min\{h_i^1(u_i^1), h_i^2(u_i^2)\}$.

From the discussion on Section 4.3, the class of nQLI fuzzy numbers is well-defined. The construction of the pair of parameters for QLI fuzzy numbers is similar to a projection of a pair of fuzzy numbers in the space of linearly interactive fuzzy numbers. Once the pairs q_i, r_i are calculated, the method discussed in previous chapters is suitable for the QLI data. This scheme is depicted in Figure 34.

Figure 34 – Connection between LI and QLI fuzzy numbers, and curve fitting model



Data is projected in A-LI space, finding parameters q and r. Those parameters are used for cur fitting the data. Source: Author.

Next section provides an example of a n-QLI fuzzy numbers and shows the curve fitting method proposed in the previous chapters.

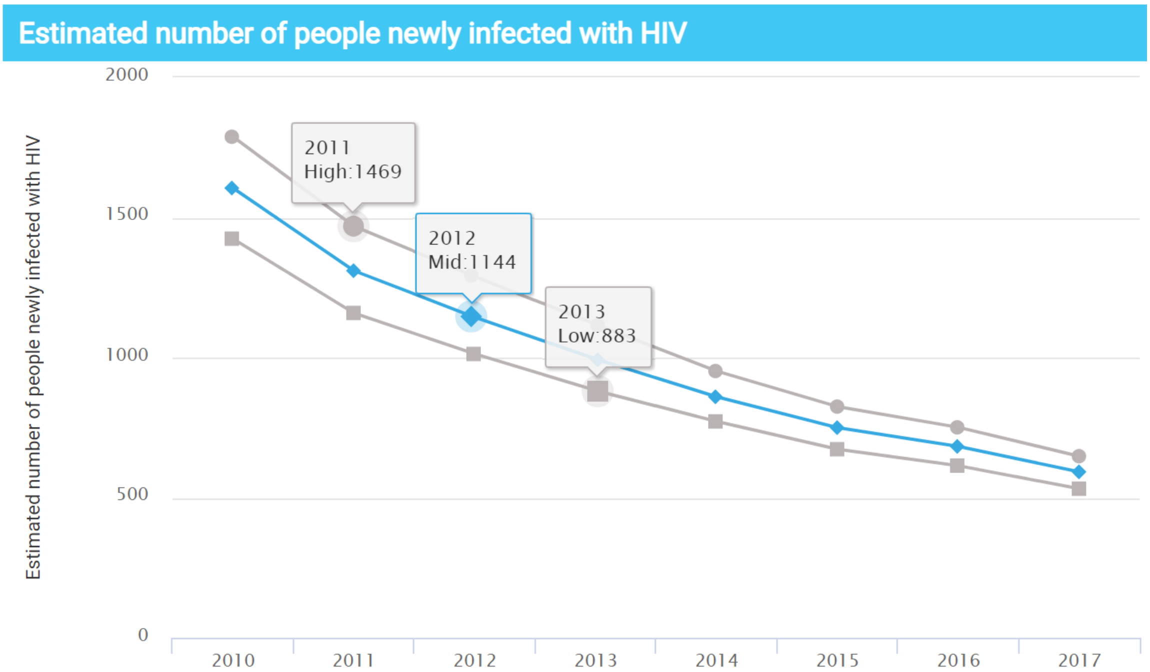
4.5 Application to the HIV Dataset

This subsection presents an application of the methods given in Section 2.2 and Section 4.2 combined. A HIV dataset from World Health Organization (WHO) [73] is fitted. More precisely, the data is the estimated number of people newly (from 2010 to 2017) infected with HIV and one may estimate the number of infected individuals for the year of 2018. The data which considered have different behaviors and they have been taken from three countries from different continents: Cambodia (Asia), Zimbabwe (Africa) and Venezuela (South America).

WHO reports the HIV profiles by countries. These data, as we can observe in Figure 35, have intrinsically characteristics of uncertainty. When looking at the graph it is possible to notice that each year has information of low, mid and high numbers, which represents the lack of precise information about newly infected people over the years in each country.

The data are modeled by the triangular fuzzy numbers of the form $Y = (a; b; c)$, where a represents an approximation for "low" data, b is an approximation for "medium"

Figure 35 – World Health Organization data sample about newly infected people with HIV by country



WHO data about number of HIV newly infected people in Cambodia over time. Source: World Health Organization [73].

data, and c stands for an approximation of "high" data. For all simulations the functions considered are $g_1(x) = x^2$, $g_2(x) = x$, and, $g_3(x) = 1$, for all $x \in \mathbb{R}$. Here, we only present the Cambodia dataset (see Table 10). The dataset of the others countries can be found in [73].

Table 10 – Dataset of approximations of the estimated number of people newly infected with HIV in Cambodia per year and the respectively parameters q and r

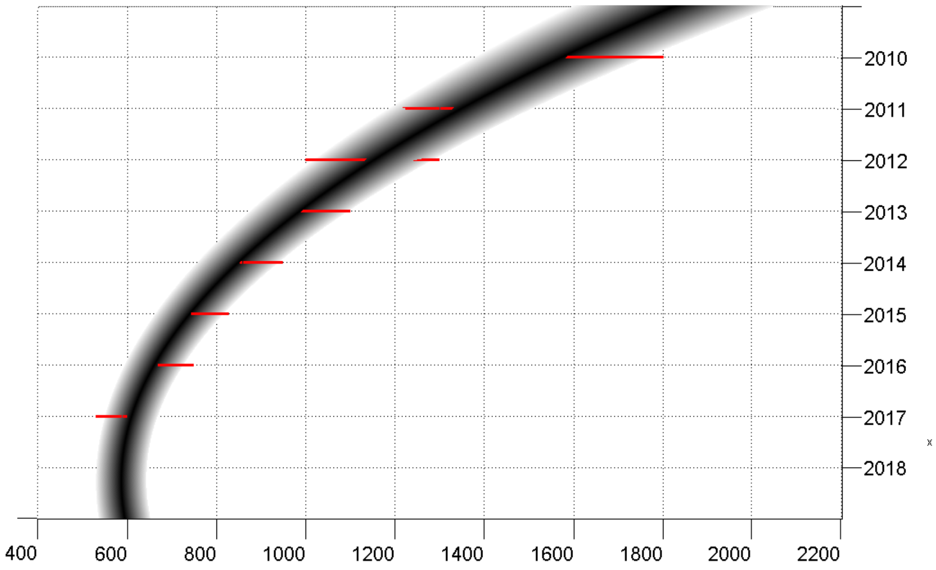
Year	Infected	(q, r)
2010	(1400; 1600; 1800)	(1,0)
2011	(1200; 1300; 1500)	(0.75,125)
2012	(1000; 1100; 1300)	(0.75,-75)
2013	(880; 1000; 1100)	(0.55,115)
2014	(770; 860; 950)	(0.45,140)
2015	(670; 750; 830)	(0.40,110)
2016	(610; 680; 750)	(0.35,120)
2017	(530; 590; 650)	(0.30,110)

Source: Based on WHO data [73].

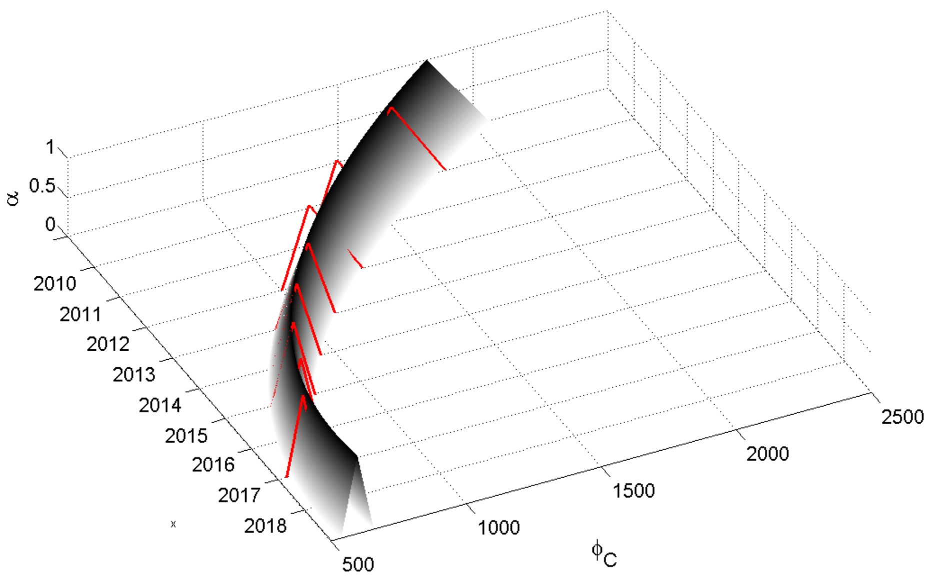
Figures 36 to 38 depict the fuzzy functions Φ_C, Φ_Z and Φ_V that approximates the dataset of Cambodia, Zimbabwe and Venezuela, respectively. Table 11 exhibits the data of Cambodia and Zimbabwe from 2016 to 2017, and for Venezuela in 2016 [73].

Figure 36 – Approximation for Cambodia fuzzy data

(a) Top view.



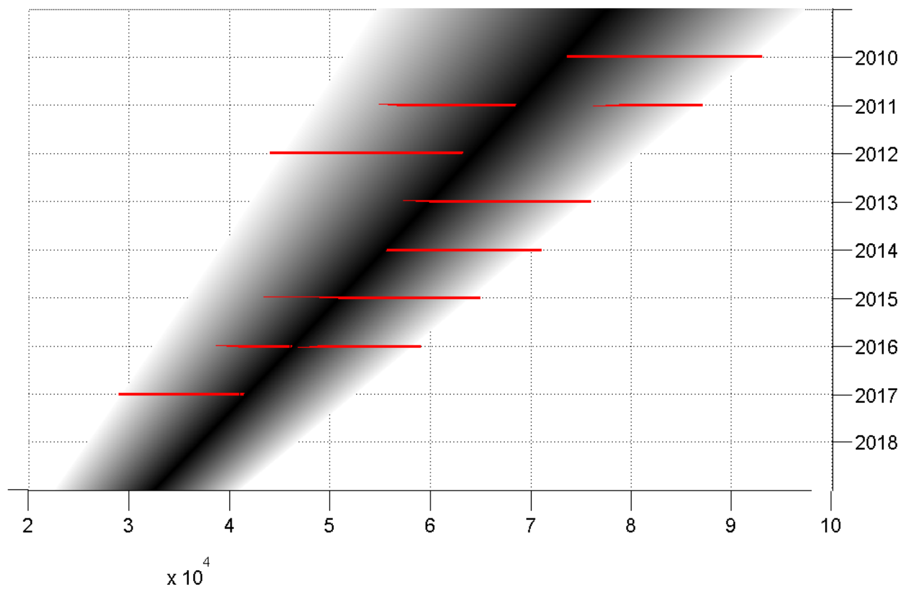
(b) Tridimensional view.



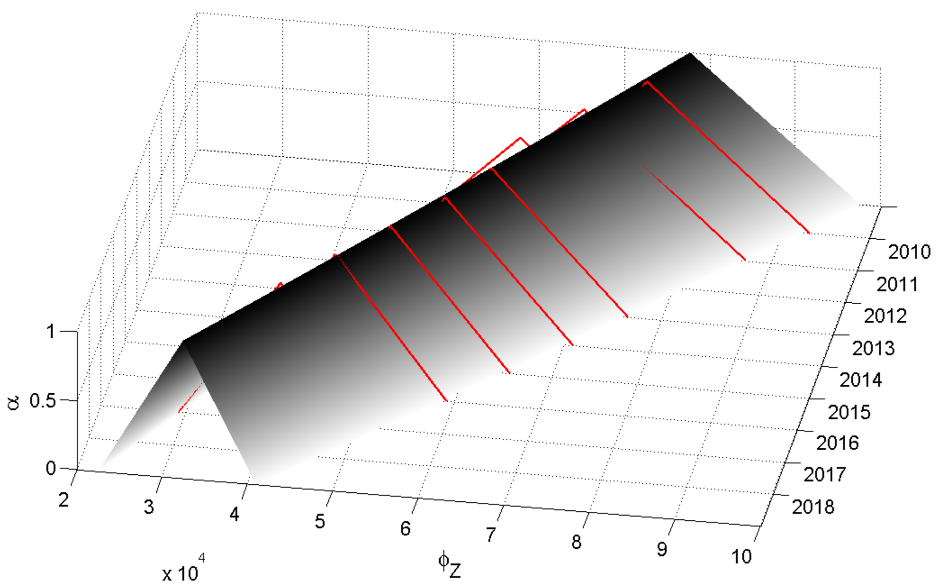
The top and tridimensional views of the fuzzy function Φ_C, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 10. Source: Author [87].

Figure 37 – Approximation for Zimbabwe fuzzy data

(a) Top view.



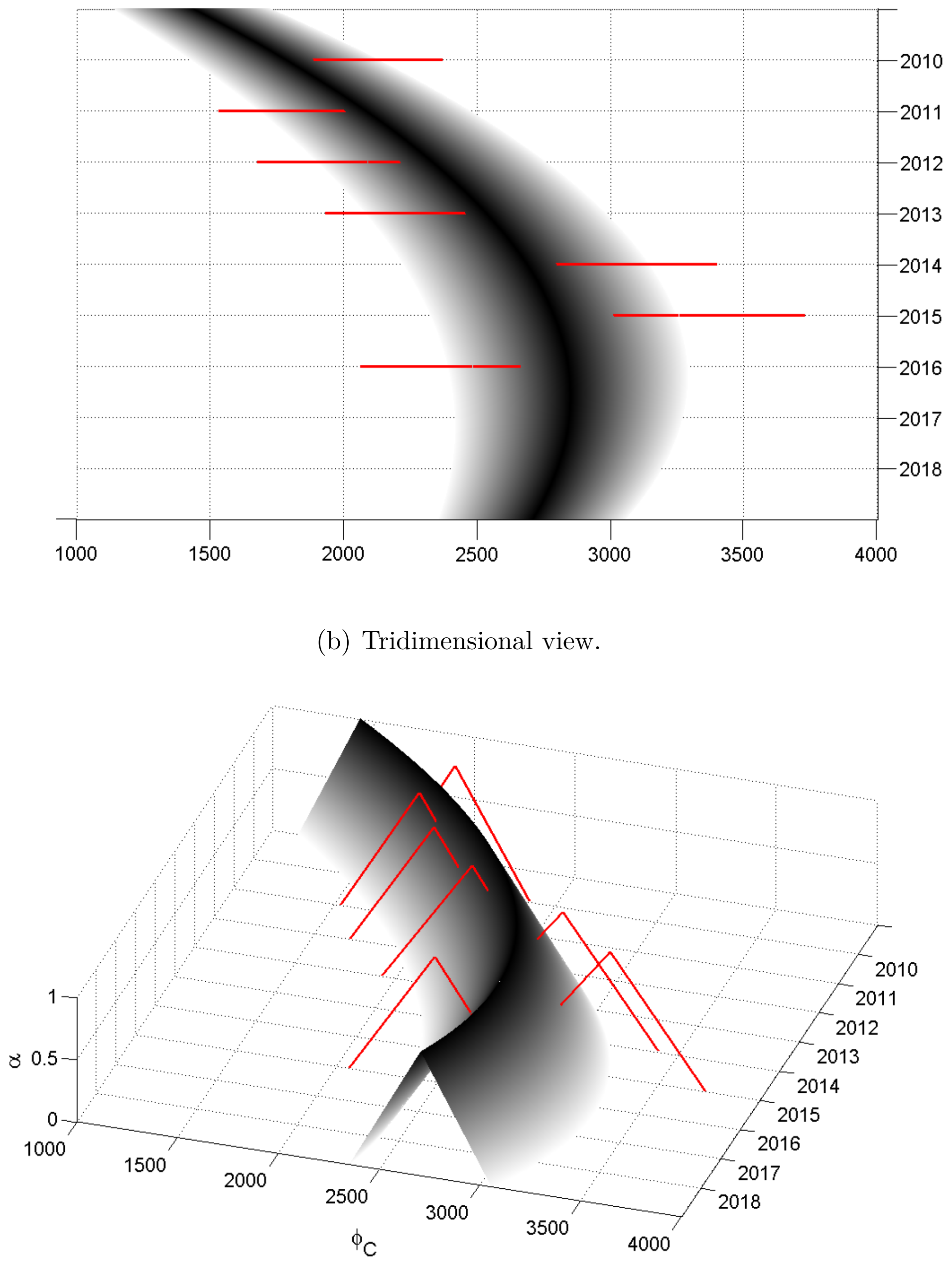
(b) Tridimensional view.



The top and tridimensional views of the fuzzy function Φ_Z, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data. Source: Author [87].

Figure 38 – Approximation for Venezuela fuzzy data

(a) Top view.



(b) Tridimensional view.

The top and tridimensional views of the fuzzy function Φ_V, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data. Source: Author [87].

Outputs of Table 11 were obtained via fuzzy least squares methods based on sup-J extension principle for quasi linearly interactive fuzzy data. According to Table 11, the newly infected individuals in Cambodia and Zimbabwe in 2018 decreases with respect to the year of 2017. For the case of Venezuela, one can predict that the newly infected individuals of 2017 increases with respect to 2016 and, in 2018, decreases with respect to 2017.

Table 11 – Dataset of the estimated number of people newly infected with HIV in Cambodia, Zimbabwe and Venezuela per years 2016, 2017 and 2018.

Year	Cambodia	Zimbabwe	Venezuela
2016	(610; 680; 750)	(33000; 46000; 59000)	(2060; 2483; 2920)
2017	(530; 590; 650)	(29000; 41000; 52000)	(2413; 2844; 3272)
2018	(530; 589; 647)	(25806; 37053; 46766)	(2401; 2799; 3194)

Source: Author.

Data of newly infected with HIV by year by country is, typically, a set of quasi-linearly interactive fuzzy numbers. Considering the first year as Y_1, pairs (q_i, r_i), for $i = 2, \ldots, m$, were calculated. Moreover, data in form of $Y_1, (q_2, r_2), \ldots, (q_m, r_m)$ were used as entries for the least square method established in this thesis.

4.6 Conclusion

This chapter was dedicated to three complementary topics. The first one was the definition of a pseudo inner product and a semi-norm for the set of fuzzy numbers with continuous square-integrable endpoints. Both definitions are based on the immersion of space $\mathbb{R}_{\mathcal{F}_C}$ in the space of square-integrable continuous functions $\mathcal{L}^2$. From these definitions, quasi linearly interactive fuzzy numbers were defined, along with a broad set of examples. Also, a n-uple of quasi linearly interactive fuzzy numbers was defined. At the end, the method of least square method provided in Chapter 2 and Chapter 3 was applied to QLI data of newly infected people with HIV over years 2010 to 2017 for three countries: Cambodia, Zimbabwe and Venezuela.

5 Least Squares Method for Interactive Fuzzy Data via Linear Systems

In this chapter, we deal with fuzzy linear systems or fuzzy systems of linear equations (FSLE) of the form $Ax = b$, with $A \in \mathbb{R}^{n\times n}$ and $x, b \in \mathbb{R}^n_{\mathcal{F}}$. FSLE were first studied by Friedman *et al.* [41] by converting a $n \times n$ FSLE in a classical linear system $2n \times 2n$.

Carlsson, Fúller and Majlender [17] showed that the sum between two fuzzy numbers may result in a real number. This occurs due to the interactivity between fuzzy numbers. Esmi *et al.* [31] showed that existence and uniqueness solutions for equation with fuzzy numbers may depend on the choice of the JPD between the fuzzy numbers involved.

This chapter develops the fuzzy least square problem through FSLE. To this end, we first study fuzzy linear equations in Section 5.1 in order to comprehend the role of interactivity for fuzzy linear equations. After that, Section 5.2 presents least square method via linear systems solutions.

5.1 Fuzzy Linear Equation

Before studying fuzzy linear systems, it is necessary to analyse fuzzy linear equations. Here, the importance of interactivity on fuzzy linear equations will be highlighted from both modeling and algebraic point of views.

5.1.1 Crisp Equation

First, let us consider the equation

$$x_1 + x_2 = r, \text{ with } x_1, x_2, r \in \mathbb{R}, \tag{5.1}$$

where $x_1, r \in \mathbb{R}$ are given and $x_2 \in \mathbb{R}$ is a variable that we wish to determine.

From algebraic point of view, Equation (5.1) implies that there is just one JPD which relates x_1 and x_2, which is $\chi_{\{(x_1,x_2)\}}$. For a fixed r, the solution is $x_2 = r - x_1$, which means that for each x_1 there is only one x_2 that solves the Equation (5.1).

From modeling point of view, the parameters x_1 and r described in Equation (5.1) are precisely known. However, in general, this may represent a simplification of phenomena. For example, in Epidemiology field, one can consider $s + i = n$, where s represents the susceptible population, i denotes the infected population and n corresponds to the total population. Since there is no uncertainty on susceptible and infected populations, it could model non-contagious disease, such as cancer, where there is no uncertainty about the number of infected people. This limitation is the motivation to include uncertainty in linear equations.

5.1.2 Partial Fuzzy Linear Equation

The second possible case is

$$\begin{aligned} &x_1 + X_2 = r, \ \text{ with } x_1, r \in \mathbb{R} \text{ and } X_2 \in \mathbb{R}_{\mathcal{F}} \text{ or} \\ &X_1 + x_2 = r, \ \text{ with } x_2, r \in \mathbb{R} \text{ and } X_1 \in \mathbb{R}_{\mathcal{F}}. \end{aligned} \tag{5.2}$$

From algebraic point of view, these type of equations do not make sense. Let us consider, without loss of generality, the first case in Equation (5.2). There is only one JPD between x_1 and X_2, hence there is only one possible sum – the usual one. The usual sum between a fuzzy number and a real number always results in a fuzzy number. Therefore Equation (5.2) has no solution.

From modeling point of view, Equation (5.2) represents that only one variable has uncertainty while the other has no uncertainties. In Epidemiology, one has $s + I = n$ when there is uncertainty about the number of infected individuals $I \in \mathbb{R}_{\mathcal{F}}$. This models disease with long incubation periods or asymptomatic infected population. In both cases uncertainty on infected people implies uncertainty on susceptible population, hence it is not possible to have $s \in \mathbb{R}$.

In cutting stock problems uncertainty means maladjusted machines. The machine cuts the objects with little mistakes. In order to maintain the mean and the standard deviation, the machine will be erroneous again in the next piece. If the defective piece is the penultimate piece in the roll, then there is a good chance that the last piece also is defective. This process implies that uncertainty in one piece size is connected with error in the others pieces. One more time, variables in Equation (5.2) do not represent uncertainty. The consistent equation would be with the two variables $X_1, X_2 \in \mathbb{R}_{\mathcal{F}}$.

5.1.3 Fuzzy Linear Equation

Consider the following equation

$$X_1 + X_2 = r, \ \text{ with } r \in \mathbb{R} \text{ and } X_1, X_2 \in \mathbb{R}_{\mathcal{F}}. \tag{5.3}$$

From modeling point of view, Equation (5.3) represent uncertainty in both operands. In cutting stock problems, uncertainty in pieces sizes cut implies in uncertainty on total size (roll size). It is possible to consider the total amount of pieces with certain size, but it is not realistic. In Epidemiology, for example, the susceptible S and infected I

population have uncertainty, but the total population n is certain. Equation (5.3) may model populations without vital dynamics, which is a valid approach in a short period of time, but it is not in a long period of time.

From an algebraic point of view, such addition $+$ may not generate a real number as a result. For instance, let $+$ be the standard sum and $S + I = n$. One can suggest that the solution is $S = n - I$. Since $n = (n; n; n)$ and $\text{width}(n) \leqslant \text{width}(I)$, the $-_{gH}$ difference exists [112]. In this case, if $I = (a; b; c)$, then the susceptible population is $S = (n; n; n) -_{gH} (a; b; c)$, that is, $S = (n - c; n - b; n - a)$. From Equation (5.3), we have that $S + I = (n - c + a; n; n - a + c)$ is equal to $(n; n; n)$ if, and only if, $a = b = c$ (which is the case in Subsection 5.1.2).

In particular, Esmi *et al.* [31] investigated this topic considering that the arithmetic operation is given by an interactive sum. This equation is given by

$$X +_J B = C, \tag{5.4}$$

where $B, C \in \mathbb{R}_{\mathcal{F}}$ are given, X is the free variable and J is some joint possibility distribution between X and B.

The existence and uniqueness of a solution for Equation (5.4) depends on the choice of a JPD J, since the variable X correspond to a marginal of J. This means that the only independent variable is in fact the joint possibility distribution J. They showed that an interactive addition of two fuzzy numbers with equal shapes may result in a fuzzy number with a different shape. For example, an interactive sum of two Gaussian fuzzy numbers may result in a triangular fuzzy number. This fact corroborates that it is possible to obtain a real number as result of an interactive addition of two fuzzy numbers and this consequence is associated with the choice of the JPD [17, 31].

For completely correlated fuzzy numbers, we have the following linear equation

$$A +_{J_{\{q,r\}}} B = C, \tag{5.5}$$

where $J_{\{q,r\}}$ is the JPD given by (1.11). In this case, the fuzzy numbers A and B must satisfy $[B]_\alpha = q[A]_\alpha + r$, for all $\alpha \in [0, 1]$. Thus, the interactive sum of A and B can be written in terms of α-cuts as follows

$$[A +_{J_{\{q,r\}}} B]^\alpha = [(q + 1)a_\alpha^- + r, (q + 1)a_\alpha^+ + r], \tag{5.6}$$

for all $\alpha \in [0, 1]$.

Thus, $C = r$ if, and only if, $q = -1$. This means that Equation (5.5) may result in a real number and, in this case, the fuzzy number B is given by

$$B = -A + r. \tag{5.7}$$

The joint possibility distribution $J_{\{q,r\}}$ ensures that the problem investigated in this work can be solved and it exhibits the solution. However, as we mentioned before, the JPD $J_{\{q,r\}}$ is restricts the possible shapes of the fuzzy numbers A and B. In (5.3), a priori, one does not know the shapes of the solution. So, the use of $J_{\{q,r\}}$ is not appropriate.

On the other hand, the JPD J_0 gives rise to arithmetic operations with no restrictions on the shapes of operands given by fuzzy numbers. Moreover, for triangular fuzzy number the addition $+_0$ can easily be done by Theorem 1.5. Thus, in Equation (5.3), the variables X_1 and X_2 will be considered as triangular fuzzy numbers, say $X_1 = (a; b; c)$ and $X_2 = (d; e; f)$. Furthermore, let us consider the following equation:

$$\alpha_1 X_1 +_0 \alpha_2 X_2 = r, \tag{5.8}$$

where $\alpha_1, \alpha_2, r \in \mathbb{R}$, with $\alpha_1, \alpha_2 \neq 0$, are given, and the triangular fuzzy variables are $X_1 = (a; b; c)$ and $X_2 = (d; e; f)$ are the free variables [120].

In Equation (5.8) there are four combinations that depend on signals of α_1 and α_2, which boil down in two cases.

1. If $\alpha_1, \alpha_2 > 0$ (or if $\alpha_1, \alpha_2 < 0$), then the sum is

$$(\alpha_1 a; \alpha_1 b; \alpha_1 c) +_0 (\alpha_2 d; \alpha_2 e; \alpha_2 f) = r$$
$$(\text{or } (\alpha_1 c; \alpha_1 b; \alpha_1 a) +_0 (\alpha_2 f; \alpha_2 e; \alpha_2 d) = r).$$

Since $r \in \mathbb{R}$, from Theorem 1.5 the fuzzy variables $(a; b; c)$ and $(d; e; f)$ must satisfy

$$\begin{cases} \alpha_1 a + \alpha_2 f = r \\ \alpha_1 b + \alpha_2 e = r \\ \alpha_1 c + \alpha_2 d = r \end{cases}.$$

Therefore, given b and e such that $\alpha_1 b + \alpha_2 e = r$, from Equations (5.9) and (5.10) a, f, c and d can be derived:

$$\alpha_1(b - a) = \alpha_2(f - e) \tag{5.9}$$
$$\alpha_1(c - b) = \alpha_2(e - d). \tag{5.10}$$

The choices of a, b, c, d, e, f are not arbitrarily. Instead, they are constructed according to the following algorithm.

1a. From a given value b (or e) in Equation (5.8), the other value e (or b) is determined by

$$e = \frac{r}{\alpha_2} - \frac{\alpha_1}{\alpha_2} b. \tag{5.11}$$

1b. The choice of $a < b$ implies in the determination of f by

$$f = \frac{\alpha_1}{\alpha_2}(b - a) + e.$$

1c. The choice of $c > b$ implies in the determination of d by

$$d = \frac{\alpha_1}{\alpha_2}(b - c) + e.$$

Note that the choices are attached in pairs a and f, b and e, c and d.

2. If $\alpha_1 > 0$ and $\alpha_2 < 0$ (or if $\alpha_1 > 0$ and $\alpha_2 < 0$), then the sum in (5.8) is

$$(\alpha_1 a; \alpha_1 b; \alpha_1 c) +_0 (\alpha_2 f; \alpha_2 e; \alpha_2 d) = r$$
$$(\text{or } (\alpha_1 c; \alpha_1 b; \alpha_1 a) +_0 (\alpha_2 d; \alpha_2 e; \alpha_2 f) = r).$$

Since $r \in \mathbb{R}$, from Theorem 1.5 it follows that

$$\begin{cases} \alpha_1 a + \alpha_2 d = r \\ \alpha_1 b + \alpha_2 e = r \\ \alpha_1 c + \alpha_2 f = r \end{cases}.$$

In other words,

$$\alpha_1(b - a) = -\alpha_2(e - d) \tag{5.12}$$
$$\alpha_1(c - b) = -\alpha_2(f - e). \tag{5.13}$$

The algorithm to find a solution is the following.

2a. To choose $b \in \mathbb{R}$, so the value of e is calculated by (5.11).

2b. To choose a, which implies in the determination of d by

$$d = \frac{\alpha_1}{\alpha_2}(b - a) + e.$$

2c. The choice of $c > b$, which implies in the determination of f by

$$f = \frac{\alpha_1}{\alpha_2}(b - c) + e.$$

In this second case, the pairs attached changed: a is now connected with d, and c is connected with f.

Summarizing, it is possible to write X_2 in terms of $r, \alpha_1\alpha_2$ and X_1:

$$X_2 = \frac{r}{\alpha_2} - \frac{\alpha_1}{\alpha_2}X_1.$$

Next, two examples are provided in order to illustrate the construction of this method.

Example 5.1. *Consider the fuzzy linear equation given by* $X_1 +_0 (-X_2) = 2$*. This equation enters in Case 2, since* $\alpha_1 = 1$ *and* $\alpha_2 = -1$*. The solution is* $X_1 = (a; b; c)$ *and* $X_2 = (a - 2; b - 2; c - 2)$*, therefore* $X_2 = X_1 - 2$*, where* 2 *stands for the real number* 2*.*

Example 5.2. *Consider the fuzzy linear equation given by* $3X_1 +_0 2X_2 = 10$*. This equation enters in Case 1, since* $\alpha_1 = 3$ *and* $\alpha_2 = 2$*. The solution is* $X_1 = (a; b; c)$ *and* $X_2 = \left(5 - \frac{3}{2}c; 5 - \frac{3}{2}b; 5 - \frac{3}{2}a\right)$*. That is,* $X_2 = 5 - \frac{3}{2}X_1$*.*

Note that in Example 5.1, the fuzzy number X_2 is given by a translation of the given fuzzy number X_1. On the other hand, in Example 5.2, the fuzzy number X_2 is given by performing a reflection and a translation on X_1. Also note that once the fuzzy number X_1 is fixed, the fuzzy number X_2 is unique determined.

It is interesting to observe the relation of symmetry of the fuzzy numbers in this algorithm. Examples 5.1 and 5.2 reveal that, if X_1 is (non) symmetric, then X_2 must be (non) symmetric as well. This fact can be verified directly from the proposed algorithm.

The next example illustrates the restrictions imposed by this algorithm in modeling.

Example 5.3. *Consider a population with ten individuals and a contagious disease such as the HIV. Suppose that there are "around" three infected individuals, which implies that there are "around" seven susceptible individuals. If one describes the linguistic variable "around" three by the fuzzy number* $(2; 3; 4)$*, then the algorithm establishes that "around" seven must be modeled by the fuzzy number* $(6; 7; 8)$*, in order to guarantee that the initial condition* $(2; 3; 4) +_0 (6; 7; 8) = 10$ *be satisfied.*

On the other hand, if one chooses to describe "around" three by the fuzzy number $(2; 3; 5)$*, then the algorithm ensures that "around" seven must be* $(5; 7; 8)$*.*

From the consistence (or algebraic) point of view, it was shown here that there are criteria to consider triangular fuzzy numbers as fuzzy initial conditions in biological models.

5.1.4 Fully Fuzzy Linear Equation

The fully fuzzy linear equation is

$$X_1 + X_2 = Y, \tag{5.14}$$

where $X_1, X_2, Y \in \mathbb{R}_{\mathcal{F}}$ and Y is given.

Example 5.4. *Equation* $(1; 2; 3) +_{\wedge} (-X_2) = (1; 2; 3)$ *has unique solution, given by Hukuhara difference, which is* $X_2 = (0; 0; 0)$*.*

For equation $(1;2;3) +_{J_{\{q,r\}}} (-X_2) = (1;2;3)$, *the value* X_2 *could be obtained by* $-X_2 = (1;2;3) -_{J_{\{q,r\}}} (1;2;3)$. *The result would be* $X_2 = (0;0;0)$, *since* $(1;2;3)$ *and* $(1;2;3)$ *are related by* $J_{\{1,0\}}$. *Nonetheless* $(1;2;3)$ *and* $(0;0;0)$ *are not completely correlated, hence the equation does not have solution for any* $J_{\{q,r\}}$.

Remark 5.1. *Example 5.4 shows that the choice of sum operator leads to different solutions scenario. For* $+_{J_{\{q,r\}}}$ *the equation may have no solution; for* $+_\wedge$ *the equation may have unique solution, and for* $+_0$ *the equation may have at most two solutions, as will be seen in Example 5.5.*

From modeling point of view, Equation (5.14) represents phenomena under uncertainty in both variables and in total process. In Epidemiology, this equation models contagious diseases with asymptomatic (infected) population and vital dynamics in total population. In cutting stock problems, Equation (5.14) models leftovers and remains parts in roll during the cut process.

From algebraic point of view, one could use the standard arithmetic operation $+_\wedge$ (non-interactive sum). In this case, the solution would be $X_2 = Y -_\wedge X_1$. For instance, for $Y = (r;s;t)$ and $X_1 = (a;b;c)$ triangular fuzzy numbers, $X_2 = (r-c; s-b; t-a)$. On the other hand, $X_1 +_\wedge X_2 = (a+r-c; s; c+t-a)$ is equal to Y if, and only if, $a = b = c$, which is the case in Subsection 5.1.2.

Other possible solutions are the differences $Y -_H X_1$ (or $Y -_{gH} X_1$ or $Y -_g X_1$). Since the difference $-_H$ (and $-_{gH}$ and $-_g$) is interactive [118], to be consistent it is necessary to use interactive sum as well. Therefore Equation (5.14) boils down to $X_1 +_J X_2 = Y$, with $X_1, X_2, Y \in \mathbb{R}_\mathcal{F}$, where $J \neq J_\wedge$.

As mentioned in Subsection 5.1.3, $J_{\{q,r\}}$ is not a good choice of JPD, because it takes into account that both variables X_1 and X_2 have same shapes. Since JPD J_0 does not have this restriction, this will be, again, the chosen one.

More generally, let be the following equation with real coefficients

$$\alpha_1 X_1 +_0 \alpha_2 X_2 = Y, \tag{5.15}$$

where $X_1 = (a;b;c), X_2 = (d;e;f)$ and $Y = (r;s;t)$, with $r < s < t$, and $\alpha_1, \alpha_2 \in \mathbb{R}$.

There are four possibilities to consider, taking into account the signals of α_i.

If $\alpha_1, \alpha_2 > 0$, then Equation (5.15) can be rewritten as

$$(\alpha_1 a; \alpha_1 b; \alpha_1 c) +_0 (\alpha_2 d; \alpha_2 e; \alpha_2 f) = (r;s;t)$$

According to Theorem 1.5, if $\text{width}(\alpha_1 X_1) \geqslant \text{width}(\alpha_2 X_2)$, then

$$((\alpha_1 a + \alpha_2 f) \wedge (\alpha_1 b + \alpha_2 e); \alpha_1 b + \alpha_2 e; (\alpha_1 b + \alpha_2 e) \vee (\alpha_1 c + \alpha_2 d)) = (r;s;t).$$

As a consequence

$$\begin{cases} \alpha_1 b + \alpha_2 e = s \\ (\alpha_1 a + \alpha_2 f) \wedge s = r \\ s \vee (\alpha_1 c + \alpha_2 d) = t \end{cases}.$$

Since $r < s < t$, $\alpha_1 a + \alpha_2 f = r$ and $\alpha_1 c + \alpha_2 d = t$.

On the other hand, if $\text{width}(\alpha_1 X_1) \leqslant \text{width}(\alpha_2 X_2)$, then

$$((\alpha_1 c + \alpha_2 d) \wedge (\alpha_1 b + \alpha_2 e); \alpha_1 b + \alpha_2 e; (\alpha_1 b + \alpha_2 e) \vee (\alpha_1 a + \alpha_2 f)) = (r; s; t).$$

As a consequence

$$\begin{cases} \alpha_1 b + \alpha_2 e = s \\ (\alpha_1 c + \alpha_2 d) \wedge s = r \\ s \vee (\alpha_1 a + \alpha_2 f) = t \end{cases}.$$

Since $r < s < t$, $\alpha_1 a + \alpha_2 f = t$ and $\alpha_1 c + \alpha_2 d = r$.

The three other cases follow analogously. The algorithm to find the solutions is the following.

Inputs $Y = (r; s; t)$, α_1 and α_2.

Step 1 To choose the triple $(a; b; c)$ such that $a \leqslant b \leqslant c$.

Step 2 The value of e is obtained in terms of α_1 and α_2 by

$$e = \frac{1}{\alpha_2}(s - \alpha_1 b).$$

Step 3 If $\alpha_1, \alpha_2 > 0$ or $\alpha_1, \alpha_2 < 0$, then the values of d and f are related with c and a respectively, and there are at most two different solutions.

3a. The values of d and f are determined by

$$d = \frac{1}{\alpha_2}(t - \alpha_1 c)$$
$$f = \frac{1}{\alpha_2}(r - \alpha_1 a).$$

If $d \leqslant e \leqslant f$, then the pair (X_1, X_2) is a solution for Equation (5.15). Otherwise, this pair is not a solution for (5.15), because X_2 is not a fuzzy number.

3b. The values of d and f are determined by

$$d = \frac{1}{\alpha_2}(r - \alpha_1 c)$$
$$f = \frac{1}{\alpha_2}(t - \alpha_1 a).$$

If $d \leqslant e \leqslant f$, then the pair (X_1, X_2) is a solution for Equation (5.15). Otherwise, this pair is not a solution for (5.15).

Step 4 If a_1 and a_2 have different signals, then the values of d and f are related with a and c respectively, and there are at most two different solutions.

4a. The values of d and f are determined by

$$d = \frac{1}{\alpha_2}(r - \alpha_1 a)$$
$$f = \frac{1}{\alpha_2}(t - \alpha_1 c).$$

If $d \leqslant e \leqslant f$, then the pair (X_1, X_2) is a solution for Equation (5.15). Otherwise, this pair is not a solution for (5.15).

4b. The values of d and f are determined by

$$d = \frac{1}{\alpha_2}(t - \alpha_1 a)$$
$$f = \frac{1}{\alpha_2}(r - \alpha_1 c).$$

If $d \leqslant e \leqslant f$, then the pair (X_1, X_2) is a solution for Equation (5.15). Otherwise, this pair is not a solution for (5.15).

Outputs $X_1 = (a; b; c)$ and $X_2 = (d; e; f)$ solutions for Equation (5.15).

Example 5.5. *Let be the equation* $X_1 +_0 (-X_2) = (1; 2; 3)$. *For* $X_1 = (1; 2; 3)$, *the algorithm produces (in Step 4a and 4b) two different solutions* $X_2^1 = (0; 0; 0)$ *and* $X_2^2(-2; 0; 2)$. *Pairs* (X_1, X_2^1) *and* (X_1, X_2^2) *satisfy Equation* (5.15), *therefore both pairs are solutions.*

On the other hand, for $X_1 = (2; 2; 2)$ *the algorithm produces as a result only one* $X_2 = (-1; 0; 1)$ *such that the pair* (X_1, X_2) *is a solution for Equation* (5.15).

This sections discussed the relation among fuzzy linear equations and interactivity. Next section considers a system of two fully fuzzy linear equations.

5.2 Least Squares Method for Interactive Fuzzy Data via Linear Systems

Our intention in this section is to fitting data using a linear function, that is, the function

$$\varphi(x) = a_1 g_1(x) + a_2 g_2(x), \tag{5.16}$$

where $g_1(x) = k \in \mathbb{R}$ and $g_2(x) = x$.

Suppose that φ is adjusted by means of least square method for some available fuzzy data. In this context, the associated matrix in (2.4) is 2×2, and the normal equation to be solved has the following formula:

$$UX = V, \tag{5.17}$$

where $U = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ with X_1 and X_2 being triangular fuzzy numbers, that is, $X_1 = (a; b; c)$ and $X_2 = (d; e; f)$, and $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ with V_1 and V_2 being also triangular fuzzy numbers, that is, $V_1 = (r; s; t)$ and $V_2 = (u; v; w)$. Therefore, it is necessary to solve fuzzy linear systems in Equation (5.17) in order to find first degree approximations for fuzzy data. Here, all the operations are given by sup-J_0 extension such as in Definition 1.20.

5.2.1 Fuzzy Linear Systems

Consider the following fuzzy linear system:

$$\begin{cases} \alpha_{11}X_1 +_0 \alpha_{12}X_2 = V_1 \\ \alpha_{21}X_1 +_0 \alpha_{22}X_2 = V_2 \end{cases}, \tag{5.18}$$

where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \neq 0$, $X_1 = (a; b; c)$, $X_2 = (d; e; f)$, $V_1 = (r; s; t)$ and $V_2 = (u; v; w)$, with $r < s < t$ and $u < v < w$.

The algorithm below is based on Theorem 1.5 and aims to solve (5.18). Since it is motivated by Theorem 1.5, it is necessary to know the width of fuzzy sets $\alpha_{11}X_1$, $\alpha_{21}X_1$, $\alpha_{12}X_2$ and $\alpha_{22}X_2$. Therefore, the solution of the system (5.18) will be provided in terms of signal of the entries α_{ij} as follows.

For $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} > 0$, there are four subcases.

1. If $\text{width}(\alpha_{11}X_1) \geqslant \text{width}(\alpha_{12}X_2)$ and $\text{width}(\alpha_{21}X_1) \geqslant \text{width}(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}a + \alpha_{12}f) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}c + \alpha_{12}d)) = (r; s; t) \\ ((\alpha_{21}a + \alpha_{22}f) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}c + \alpha_{22}d)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix}, \quad U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} r \\ u \end{bmatrix} \text{ and } U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}.$$

2. If width$(\alpha_{11}X_1) \geqslant$ width$(\alpha_{12}X_2)$ and width$(\alpha_{21}X_1) \leqslant$ width$(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}a + \alpha_{12}f) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}c + \alpha_{12}d)) = (r; s; t) \\ ((\alpha_{21}c + \alpha_{22}d) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}a + \alpha_{22}f)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix}, \quad U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} r \\ w \end{bmatrix} \text{ and } U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} t \\ u \end{bmatrix}.$$

3. If width$(\alpha_{11}X_1) \leqslant$ width$(\alpha_{12}X_2)$ and width$(\alpha_{21}X_1) \geqslant$ width$(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}c + \alpha_{12}d) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}a + \alpha_{12}f)) = (r; s; t) \\ ((\alpha_{21}a + \alpha_{22}f) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}c + \alpha_{22}d)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix}, \quad U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} t \\ u \end{bmatrix} \text{ and } U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} r \\ w \end{bmatrix}.$$

4. If width$(\alpha_{11}X_1) \leqslant$ width$(\alpha_{12}X_2)$ and width$(\alpha_{21}X_1) \leqslant$ width$(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}c + \alpha_{12}d) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}a + \alpha_{12}f)) = (r; s; t) \\ ((\alpha_{21}c + \alpha_{22}d) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}a + \alpha_{22}f)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix}, \quad U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix} \text{ and } U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} r \\ u \end{bmatrix}.$$

For $\alpha_{11}, \alpha_{12}, \alpha_{21} > 0$ and $\alpha_{22} < 0$, there are four subcases.

1. If width$(\alpha_{11}X_1) \geqslant$ width$(\alpha_{12}X_2)$ and width$(\alpha_{21}X_1) \geqslant$ width$(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}a + \alpha_{12}f) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}c + \alpha_{12}d)) = (r; s; t) \\ ((\alpha_{21}a + \alpha_{22}d) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}c + \alpha_{22}f)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix} \text{ and } \begin{bmatrix} \alpha_{11} & 0 & 0 & \alpha_{12} \\ \alpha_{21} & 0 & \alpha_{22} & 0 \\ 0 & \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{21} & 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} a \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} r \\ u \\ t \\ w \end{bmatrix}.$$

2. If width$(\alpha_{11}X_1) \geqslant$ width$(\alpha_{12}X_2)$ and width$(\alpha_{21}X_1) \leqslant$ width$(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}a + \alpha_{12}f) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}c + \alpha_{12}d)) = (r; s; t) \\ ((\alpha_{21}c + \alpha_{22}f) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}a + \alpha_{22}d)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix} \text{ and } \begin{bmatrix} \alpha_{11} & 0 & 0 & \alpha_{12} \\ 0 & \alpha_{21} & 0 & \alpha_{22} \\ 0 & \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & 0 & \alpha_{22} & 0 \end{bmatrix} \begin{bmatrix} a \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} r \\ u \\ t \\ w \end{bmatrix}.$$

3. If $\text{width}(\alpha_{11}X_1) \leqslant \text{width}(\alpha_{12}X_2)$ and $\text{width}(\alpha_{21}X_1) \geqslant \text{width}(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}c + \alpha_{12}d) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}a + \alpha_{12}f)) = (r; s; t) \\ ((\alpha_{21}a + \alpha_{22}d) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}c + \alpha_{22}f)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & 0 & \alpha_{22} & 0 \\ \alpha_{11} & 0 & 0 & \alpha_{12} \\ 0 & \alpha_{21} & 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} a \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} r \\ u \\ t \\ w \end{bmatrix}.$$

4. If $\text{width}(\alpha_{11}X_1) \leqslant \text{width}(\alpha_{12}X_2)$ and $\text{width}(\alpha_{21}X_1) \leqslant \text{width}(\alpha_{22}X_2)$, then

$$\begin{cases} ((\alpha_{11}c + \alpha_{12}d) \wedge (\alpha_{11}b + \alpha_{12}e); \alpha_{11}b + \alpha_{12}e; (\alpha_{11}b + \alpha_{12}e) \vee (\alpha_{11}a + \alpha_{12}f)) = (r; s; t) \\ ((\alpha_{21}c + \alpha_{22}f) \wedge (\alpha_{21}b + \alpha_{22}e); \alpha_{21}b + \alpha_{22}e; (\alpha_{21}b + \alpha_{22}e) \vee (\alpha_{21}a + \alpha_{22}d)) = (u; v; w). \end{cases}$$

Thus,

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{21} & 0 & \alpha_{22} \\ \alpha_{11} & 0 & 0 & \alpha_{12} \\ \alpha_{21} & 0 & 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} a \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} r \\ u \\ t \\ w \end{bmatrix}.$$

The other twelve cases are analogous to these four. The all cases can be summarized in following algorithm.

Inputs $V_1 = (r; s; t)$, $V_2 = (u; v; w)$ and matrix $U = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$.

Step 1 Solve the system

$$U\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} s \\ v \end{bmatrix}. \tag{5.19}$$

Step 2 If $\alpha_{ij} \geqslant 0, i, j = 1, 2$, or $\alpha_{ij} \leqslant 0, i, j = 1, 2$, or $\alpha_{11}, \alpha_{12} \geqslant 0$ and $\alpha_{21}, \alpha_{22} \leqslant 0$, or $\alpha_{21}, \alpha_{22} \geqslant 0$ and $\alpha_{11}, \alpha_{12} \leqslant 0$, or $\alpha_{11}, \alpha_{21} \geqslant 0$ and $\alpha_{12}, \alpha_{22} \leqslant 0$, or $\alpha_{11}, \alpha_{22} \geqslant 0$ and $\alpha_{12}, \alpha_{21} \leqslant 0$, or $\alpha_{12}, \alpha_{21} \geqslant 0$ and $\alpha_{11}, \alpha_{22} \leqslant 0$, or $\alpha_{12}, \alpha_{22} \geqslant 0$ and $\alpha_{11}, \alpha_{21} \leqslant 0$, then solve the systems $Ux_1 = b_1$ and $Ux_2 = b_2$, with x_1, x_2 and b_1, b_2 given according to Table 12.

Table 12 – Systems on Step 2

Signal of each α_{ij}	Systems to be solved
$\alpha_{ij} \geqslant 0, i,j = 1,2$ or $\alpha_{ij} \leqslant 0, i,j = 1,2$ or $\alpha_{11}, \alpha_{12} \geqslant 0$ and $\alpha_{21}, \alpha_{22} \leqslant 0$ or $\alpha_{21}, \alpha_{22} \geqslant 0$ and $\alpha_{11}, \alpha_{12} \leqslant 0$	$U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} r \\ u \end{bmatrix}$ and $U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}$; and $U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} r \\ w \end{bmatrix}$ and $U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} t \\ u \end{bmatrix}$; and $U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} t \\ u \end{bmatrix}$ and $U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} r \\ w \end{bmatrix}$; and $U\begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}$ and $U\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} r \\ u \end{bmatrix}$
$\alpha_{11}, \alpha_{21} \geqslant 0$ and $\alpha_{12}, \alpha_{22} \leqslant 0$ or $\alpha_{11}, \alpha_{22} \geqslant 0$ and $\alpha_{12}, \alpha_{21} \leqslant 0$ or $\alpha_{12}, \alpha_{21} \geqslant 0$ and $\alpha_{11}, \alpha_{22} \leqslant 0$ or $\alpha_{12}, \alpha_{22} \geqslant 0$ and $\alpha_{11}, \alpha_{21} \leqslant 0$	$U\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} r \\ u \end{bmatrix}$ and $U\begin{bmatrix} c \\ f \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}$; and $U\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} r \\ w \end{bmatrix}$ and $U\begin{bmatrix} c \\ f \end{bmatrix} = \begin{bmatrix} t \\ u \end{bmatrix}$; and $U\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} t \\ u \end{bmatrix}$ and $U\begin{bmatrix} c \\ f \end{bmatrix} = \begin{bmatrix} r \\ w \end{bmatrix}$; and $U\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}$ and $U\begin{bmatrix} c \\ f \end{bmatrix} = \begin{bmatrix} r \\ u \end{bmatrix}$

Source: Author [89].

Note: Systems to be solved in Step 2, according to the matrix U coefficient signals.

Step 3 If just one element α_{ij} has different signal from the others, then one need to solve four linear systems of type $Z_k x = z$, in which $x = [a, c, d, f]^t$, $z = [r, u, t, w]^t$ and the matrix Z_k is one of four matrices given in Table 13.

Step 4 If $a \leqslant b \leqslant c$ and $d \leqslant e \leqslant f$, then the vector $[(a; b; c), (d; e; f)]^t$ is the solution for (5.17). Otherwise the triples $(a; b; c)$ and $(d; e; f)$ do not represent a solution for the system (5.17).

Outputs The solutions $X_1 = (a; b; c)$ and $X_2 = (d; e; f)$ for Equations (5.17).

If U is a non-singular matrix, then Equation (5.17) has a unique solution. Nevertheless, even though U is singular, the system (5.17) could have multiples solutions. The next examples illustrate this discussion.

Example 5.6. *Consider the system*

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1; 2; 3) \\ (3; 4; 5) \end{bmatrix}. \tag{5.20}$$

Table 13 – Matrices on Step 3

Signal of each α_{ij}	Matrices from the systems to be solved
$\alpha_{11}, \alpha_{12}, \alpha_{21} \geqslant 0$ and $\alpha_{22} \leqslant 0$ or $\alpha_{11}, \alpha_{12}, \alpha_{22} \geqslant 0$ and $\alpha_{21} \leqslant 0$ or $\alpha_{21} \geqslant 0$ and $\alpha_{12}, \alpha_{21}, \alpha_{22} \leqslant 0$ or $\alpha_{22} \geqslant 0$ and $\alpha_{11}, \alpha_{21}, \alpha_{22} \leqslant 0$	$Z_1 = \begin{bmatrix} \alpha_{11} & 0 & 0 & \alpha_{12} \\ \alpha_{21} & 0 & \alpha_{22} & 0 \\ 0 & \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{21} & 0 & \alpha_{22} \end{bmatrix}$, $Z_2 = \begin{bmatrix} \alpha_{11} & 0 & 0 & \alpha_{12} \\ 0 & \alpha_{21} & 0 & \alpha_{22} \\ 0 & \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & 0 & \alpha_{22} & 0 \end{bmatrix}$, $Z_3 = \begin{bmatrix} 0 & \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & 0 & \alpha_{22} & 0 \\ \alpha_{11} & 0 & 0 & \alpha_{12} \\ 0 & \alpha_{21} & 0 & \alpha_{22} \end{bmatrix}$, $Z_4 = \begin{bmatrix} 0 & \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{21} & 0 & \alpha_{22} \\ \alpha_{11} & 0 & 0 & \alpha_{12} \\ \alpha_{21} & 0 & \alpha_{22} & 0 \end{bmatrix}$.
$\alpha_{11}, \alpha_{21}, \alpha_{22} \geqslant 0$ and $\alpha_{12} \leqslant 0$ or $\alpha_{12}, \alpha_{21}, \alpha_{22} \geqslant 0$ and $\alpha_{11} \leqslant 0$ or $\alpha_{11} \geqslant 0$ and $\alpha_{12}, \alpha_{21}, \alpha_{22} \leqslant 0$ or $\alpha_{12} \geqslant 0$ and $\alpha_{11}, \alpha_{21}, \alpha_{22} \leqslant 0$	$Z_5 = \begin{bmatrix} 0 & \alpha_{11} & 0 & \alpha_{12} \\ \alpha_{21} & 0 & 0 & \alpha_{22} \\ \alpha_{11} & 0 & \alpha_{12} & 0 \\ 0 & \alpha_{21} & \alpha_{22} & 0 \end{bmatrix}$, $Z_6 = \begin{bmatrix} 0 & \alpha_{11} & 0 & \alpha_{12} \\ 0 & \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{11} & 0 & \alpha_{12} & 0 \\ \alpha_{21} & 0 & 0 & \alpha_{22} \end{bmatrix}$, $Z_7 = \begin{bmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 \\ \alpha_{21} & 0 & 0 & \alpha_{22} \\ 0 & \alpha_{11} & 0 & \alpha_{12} \\ 0 & \alpha_{21} & \alpha_{22} & 0 \end{bmatrix}$, $Z_8 = \begin{bmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 \\ 0 & \alpha_{21} & \alpha_{22} & 0 \\ 0 & \alpha_{11} & 0 & \alpha_{12} \\ \alpha_{21} & 0 & 0 & \alpha_{22} \end{bmatrix}$.

Source: Author [89].

Note: Matrices of systems to be solved in Step 3, according to the matrix U coefficient signals.

First, the values b and e are given by solving the system (5.19). The values are $b = 0$ and $e = 2$. Since all the entries of matrix are positives, all the four systems from row 1 in Table 12 need to be solved. Three of the solutions do not satisfy $a \leqslant b \leqslant c$ and $d \leqslant e \leqslant f$, thus the only solution for the problem (5.20) is given by $X_1 = (-1; 0; 1)$ and $X_2 = (1; 2; 3)$.

Example 5.7. *Consider the system*

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1/3; 2; 7) \\ (2/3; 1; 2) \end{bmatrix}. \tag{5.21}$$

Note that the matrix remains the same from (5.20)*, the only change was the vector V on the right side. Solving the system at* (5.19) *one obtains $b = 1$ and $e = 0$. Again, solving the systems from row 1 in Table 12, two valid solutions are $X = [(0; 1; 4), (-1; 0; 1/3)]^t$ and $X = [(4/9; 1; 40/9), (17/9; 0; 11/9)]^t$.*

Comparing Examples 5.6 and 5.7 one can note that although the matrix U is the same and is non-singular, the systems may have unique solution, or not. The uniqueness of the solution also depends on the right vector V. This behavior occurs on the cases described at Step 3 as well (see Examples 5.8 and 5.9).

Example 5.8. *Consider the system*

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1; 2; 3) \\ (-5; -4; -3) \end{bmatrix} \tag{5.22}$$

From Equation (5.19) *one obtains $b = 0$ and $e = 0$. Since $\alpha_{22} < 0$ and the others α_{ij} are positives, the case boils down to first row of Table 13. Solving the four systems $Z_k x = z$ only one valid solution is found, which is $X_1 = (-1; 0; 1)$ and $X_2 = (1; 2; 3)$. The other solutions do not satisfy $a \leqslant b \leqslant c$ and $d \leqslant e \leqslant f$.*

Example 5.9. *Consider the system*

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1/3; 2; 23/3) \\ (2/3; 1; 10/3) \end{bmatrix} \tag{5.23}$$

Again, $b = 2$ and $e = 1$. To solve the four systems from first row in Table 13 leads to two valid solutions, they are given by $X^1 = [(0; 1; 4), (-1/3; 0; 1/3)]^t$ and $X^2 = [(8/9; 1; 44/9), (-19/9; 0; 19/9)]^t$.

Note, one more time, that the matrix in Examples 5.8 and 5.9 is the same and is non-singular. The solution uniqueness also depends, however, on the right vector.

Regarding the Table 12 first row, one can note that there exist at most two solutions. It occurs because solutions must satisfy $a \leqslant b \leqslant c$ and $d \leqslant e \leqslant f$, and only four of the systems will lead to a fuzzy solution.

Another comment that can be made is about the symmetry of X_1 and X_2. The distances $b - a$ and $c - b$ sometimes are related with $f - e$, or with $e - d$. In other words, the use of J_0 always interferes in the symmetry of fuzzy solutions.

Next subsection addresses the curve fitting problem by using the steps presented here.

5.2.2 Linear Approximation for Interactive Data

Let $D = \{(x_1, Y_1), \ldots, (x_m, Y_m)\} \subset \mathbb{R} \times \mathbb{R}_{\mathcal{F}}$ be a fuzzy data set, with triangular fuzzy numbers $Y_1, \ldots, Y_m$. This time the fuzzy data set will be considered as non-interactive. The approximation will be linear, that is, the function ϕ that fits the data has form

$$\phi(x) = C_1 x +_0 C_2,$$

with $C_1, C_2 \in \mathbb{R}_{\mathcal{F}}$, denoted by $C_1 = (a; b; c)$ and $C_2 = (d; e; f)$.

The associated system is given by

$$WC = Y,$$

with $W = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ and $y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}$.

Proceeding analogously to the classical case, one can multiply both sides by W^t. Since data is non-interactive, the associated normal equations are the following

$$\begin{bmatrix} x_1^2 + \ldots + x_m^2 & x_1 + \ldots + x_m \\ x_1 + \ldots + x_m & m \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_1 Y_1 +_\wedge \ldots +_\wedge x_m Y_m \\ Y_1 +_\wedge \ldots +_\wedge Y_m \end{bmatrix}. \tag{5.24}$$

The right side of Equation (5.24) could use interactive fuzzy data, and, consequently, interactive sum, however it is not the case analyzed here.

The new system in Equation (5.24) is written in $UC = V$ form, in which $U = W^t W$ and $V = W^t y$. Hence the algorithm will be used.

Example 5.10. *Consider the data from Table 1 in Chapter 2. The system in (5.24) is*

$$\begin{bmatrix} 1^2 + 2^2 + 3^2 & 1 + 2 + 3 \\ 1 + 2 + 3 & 3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1(1; 2; 3) +_\wedge 2(3; 5; 7) +_\wedge 3(5; 8; 11) \\ (1; 2; 3) +_\wedge (3; 5; 7) +_\wedge (5; 8; 11) \end{bmatrix}.$$

The system to be solved is given by

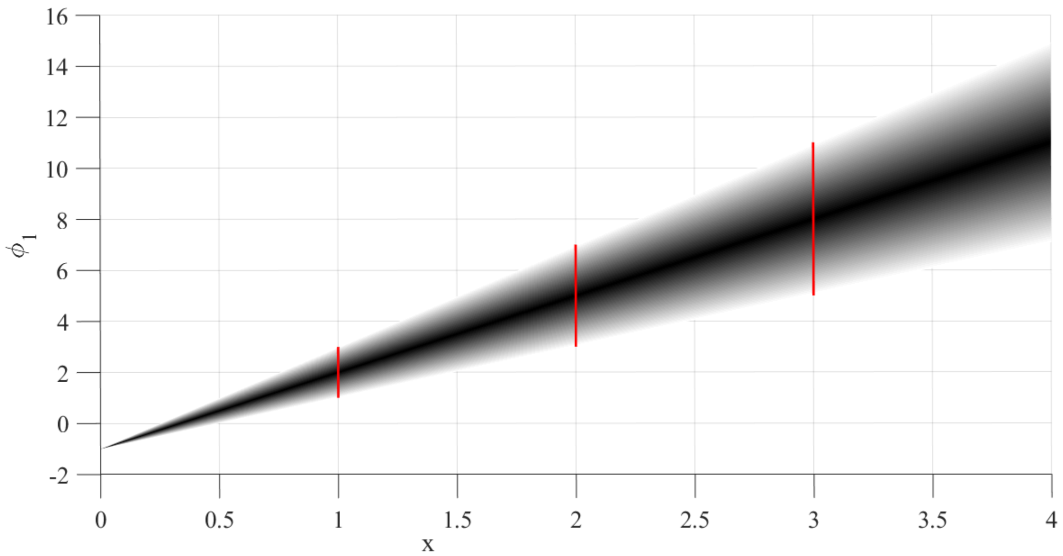
$$\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} (22; 36; 50) \\ (9; 15; 21) \end{bmatrix}.$$

According to the algorithm there are two solutions $C^1 = \begin{bmatrix} (2; 3; 4) \\ -1 \end{bmatrix}$ *and* $C^2 = \begin{bmatrix} (-10; 3; 16) \\ (-29; -1; 27) \end{bmatrix}$. *Hence the approximations are ϕ_1 and ϕ_2 given by*

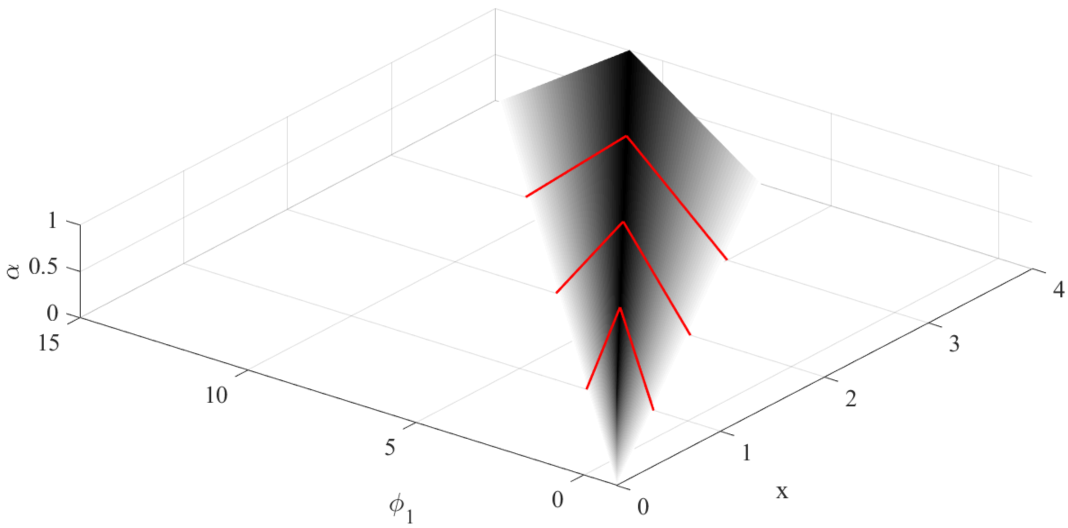
$$\phi_1(x) = (2; 3; 4)x - 1,$$
$$\phi_2(x) = (-10; 3; 16)x +_0 (-29; -1; 27).$$

Figure 39 – Approximation ϕ_1 for data in Example 5.10

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function ϕ_1, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 1. Source: Author.

The approximations are depicted in Figures 39 and 40, respectively.

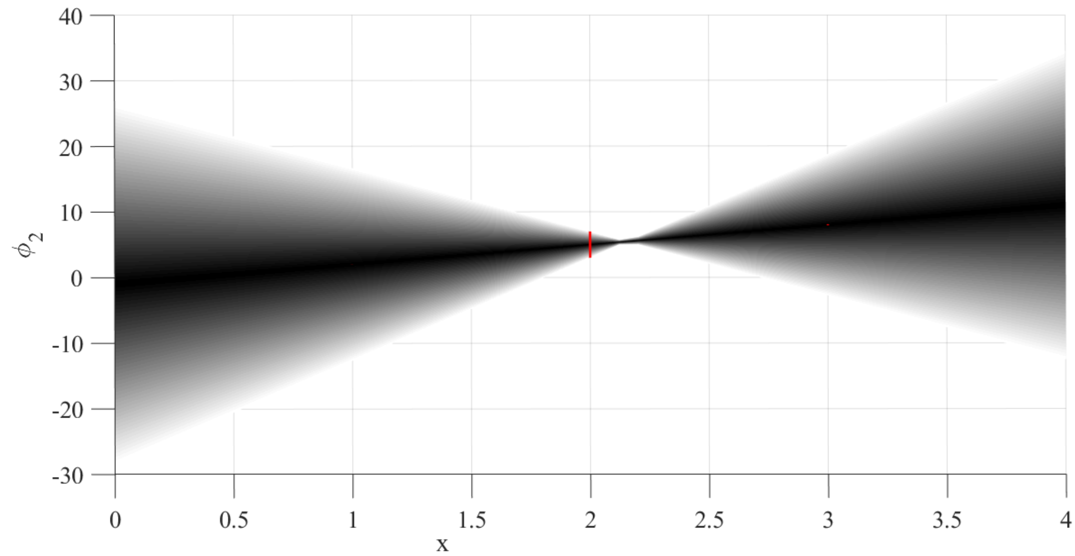
The approximation ϕ_1 coincides with Φ_1 in Equation (2.14). In contrast, from a visual inspection, the function ϕ_2 does not characterize the data.

5.2.3 Approximation for Ozone Data

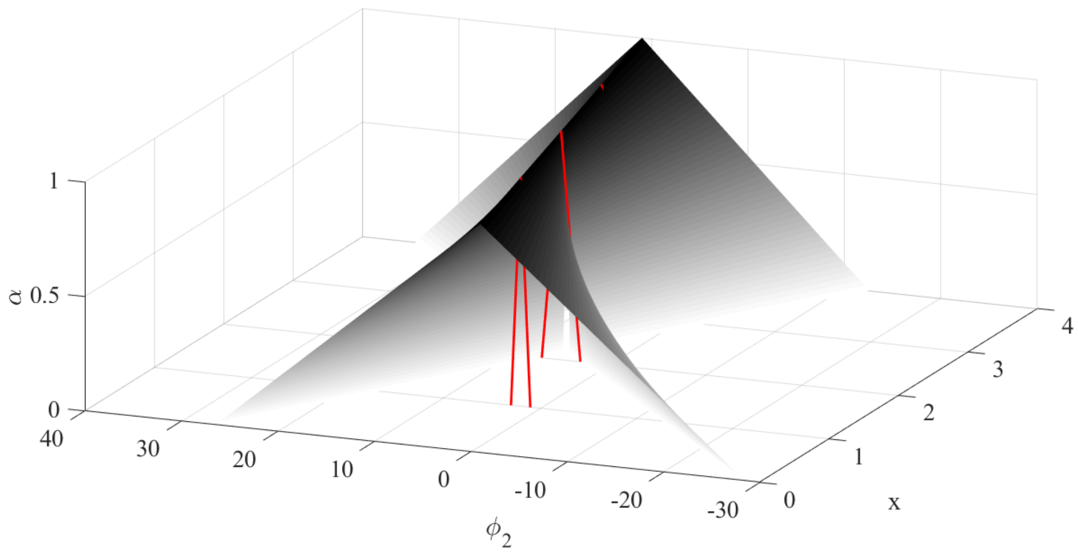
The considered data set is the same Ozone data from Table 4.

Figure 40 – Approximation ϕ_2 for data in Example 5.10

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function ϕ_2, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 1. Source: Author.

The matrix U of the system is

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} (290.6; 659.3; 1028.2) \\ (107.2; 257; 406.8) \end{bmatrix}. \tag{5.25}$$

The values of b and e are, respectively 3.36 e 55.85. The case boils down to first row case in Table 12. By solving the four systems, one obtain two solutions C, which are the vectors $C^1 = \begin{bmatrix} (2.24; 3.36; 4.52) \\ (15.5; 55.85; 96.1) \end{bmatrix}$ and $C^2 = \begin{bmatrix} (145.28; 3.36; 152.04) \\ (-353.3; 55.85; 464.9) \end{bmatrix}$.

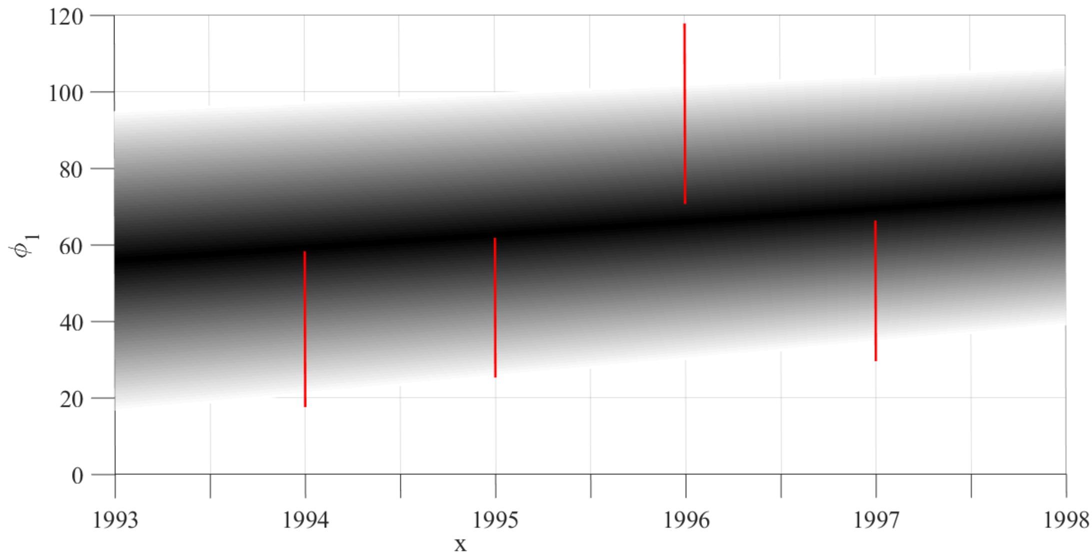
In other words, two functions ϕ_1 and ϕ_2 are approximations for the data. Their formulas are

$$\phi_1(x) = (2.24; 3.36; 4.52)x +_0 (15.5; 55.85; 96.1),$$
$$\phi_2(x) = (145.28; 3.36; 152.04)x +_0 (-353.3; 55.85; 464.9).$$

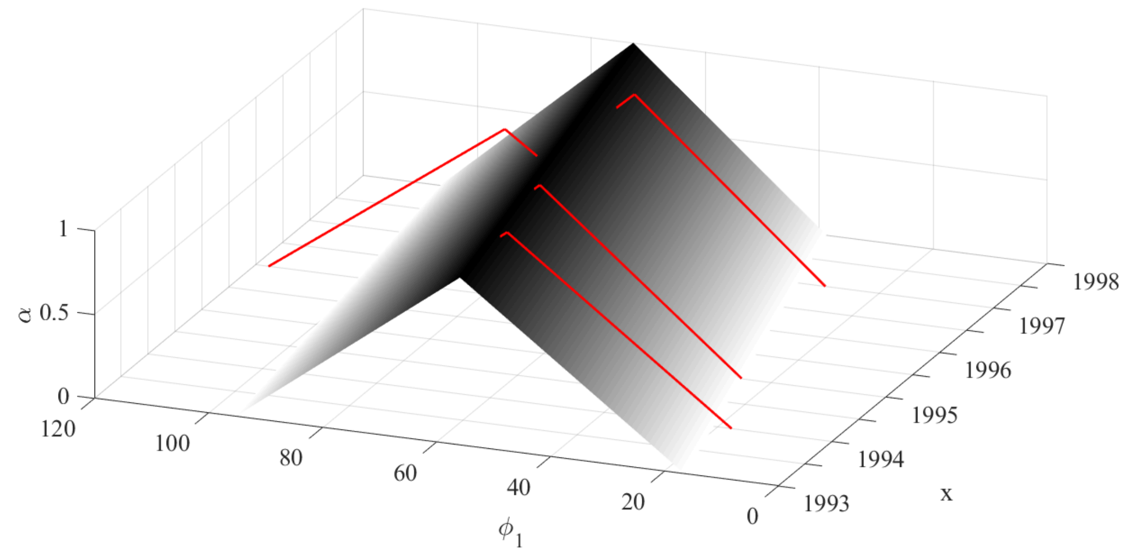
Regarding Figures 41 and 42 one can note that ϕ_1 is a good approximation, whereas ϕ_2 is not.

Figure 41 – Approximation ϕ_1 for data ozone data

(a) Top view.



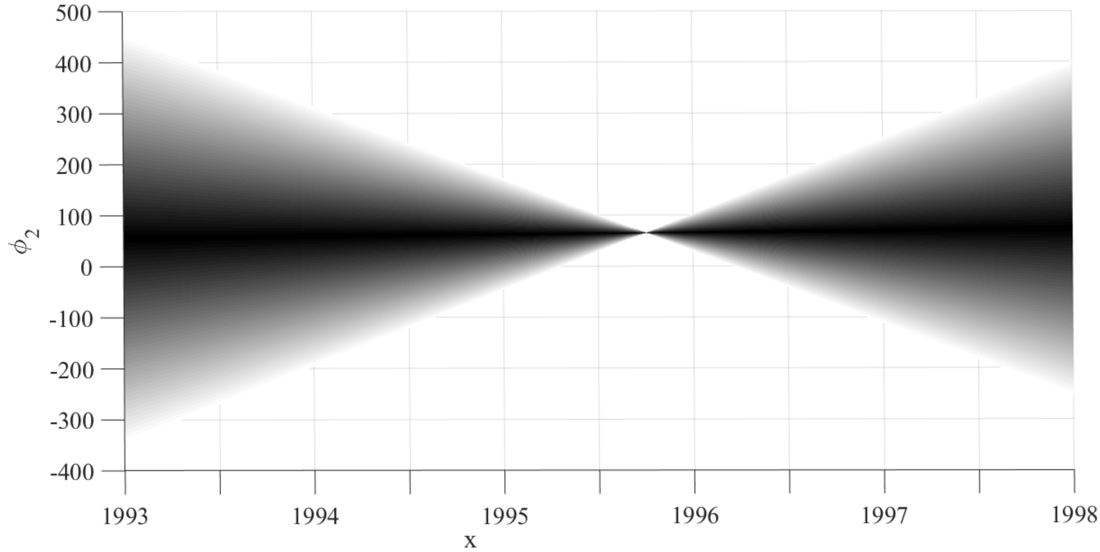
(b) Tridimensional view.



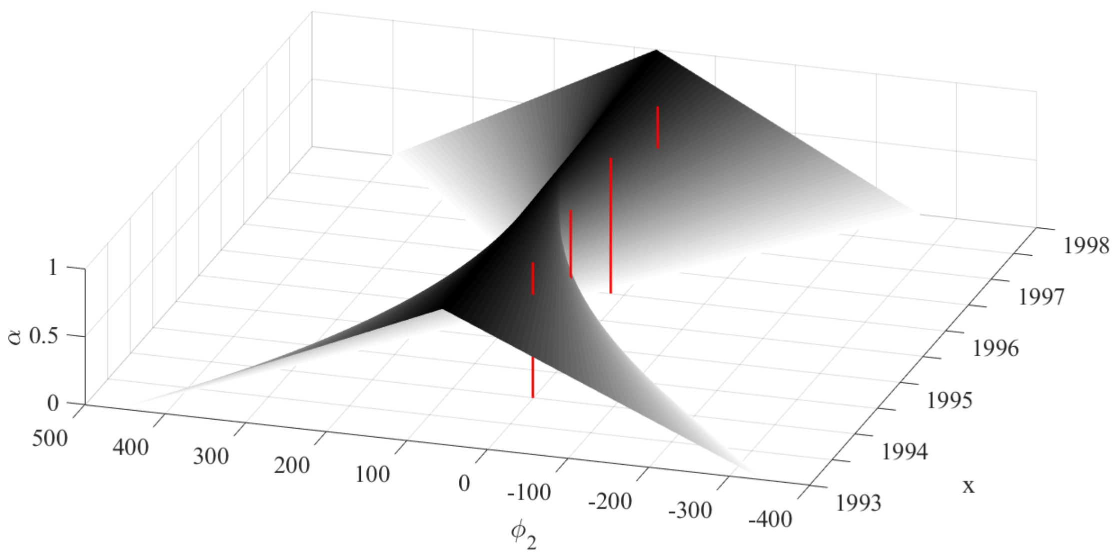
The top and tridimensional views of the fuzzy function ϕ_1, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 4. Source: Author.

Figure 42 – Approximation ϕ_2 for data ozone data

(a) Top view.



(b) Tridimensional view.



The top and tridimensional views of the fuzzy function ϕ_2, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. The red lines represent the fuzzy data given in Table 4. Source: Author.

5.3 Conclusion

This chapter approached the least squares method for fuzzy input from the second point of view. The approach was done through fuzzy linear systems. First linear equations with fuzzy variables were discussed, in order to understand the importance of interactivity to find solutions to those equations. Then 2×2 systems of fuzzy linear equations were considered and their solutions were founded taking J_0 interactivity into account in the operations of the system. Finally, it was considered non-interactive data JPD J_0 and normal equations were solved. This procedure may produces, for two basis functions, more than one solution.

Conclusion

This thesis is divided in two parts. The first part contains curve fitting methods for three types of interactive fuzzy data. All methods are based on sup-J extension of the classical least square method via normal equations. The second part investigates the normal equations via interactive arithmetic.

Chapter 2 considered linearly interactive fuzzy data. In this approach, the basic functions of the classical least square method are extended by means of sup-J extension. This method provides linear and non-linear approximation for fuzzy numbers. Moreover, the method can be applied to triangular and non-triangular fuzzy numbers.

Chapter 3 studied A-linearly interactive fuzzy data. It is shown that the method via sup-J extension provides the best approximation to A-LI data. In addition, the method is able to adjust both 1-cut and specificity behavior. Furthermore, there is no restrictions on data shapes, whereas, to the best of our knowledge, in the current literature only triangular fuzzy numbers are considered.

Chapter 4 brought the third contribution. A new distance is proposed, which is based on projections on square-integrable continuous function space. Using this distance, the class of quasi linearly interactive fuzzy numbers is defined. The curve fitting method from previous chapters are used to approximate HIV newly infected people in different countries.

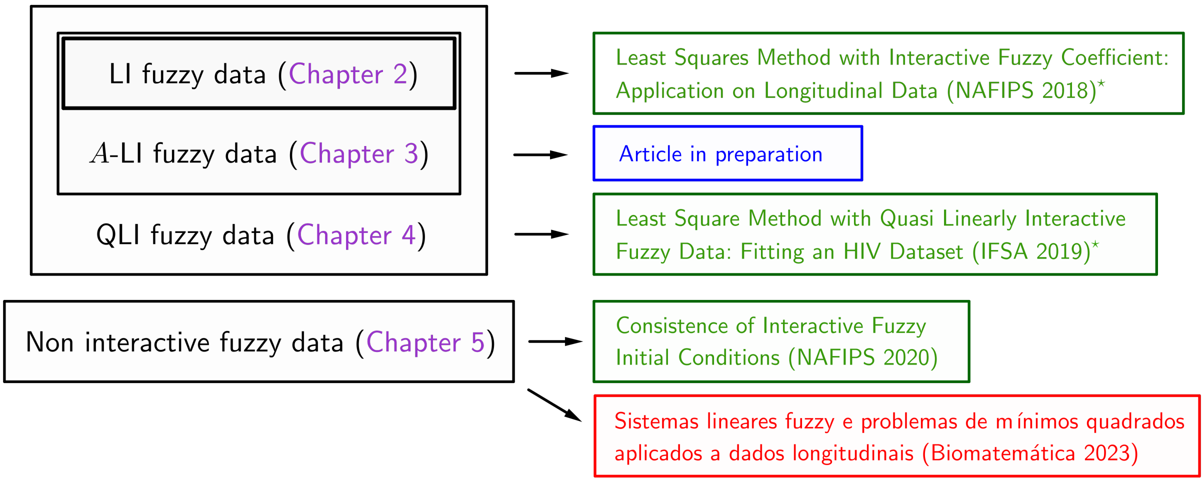
Chapter 5 presented the last contribution. Here the discussion is devoted to yield solutions for fully fuzzy linear systems associated with least square method. In fact, these solutions are not restrict to solve normal equation, they can solve any fully fuzzy linear system where variables non-interactive. In this case operations are interactive via the sup-J extension principle with joint possibility distribution J_0.

This thesis provides, at least, four contributions to the literature on fuzzy set theory and fuzzy optimization problems. The achievements are from the theoretical point of view, and also from modeling point of view. Several examples and applications were provided throughout the chapters, and some of them were presented by the author, as can be observed in Figure 43.

Further Works

Recently was defined in the literature the class of strongly linearly interactive (SLI) fuzzy numbers [30]. Such class is a generalization of A-LI. One further work is to consider SLI data and find approximations via sup-J extension of normal equations solutions.

Figure 43 – Contributions of this thesis



The contributions written in green represent book chapters, the one written in red is a journal article. Source: Author [88, 87, 120, 89].

Hopefully, such method will give rise to the best approximation, as found in Chapter 3.

The next steps for the study of normal equations is to consider another interactive arithmetic. For example, it may be considered another family of joint possibility distributions proposed by Esmi *et al.* [35, 97]. The purpose is to provide a study of higher order approximations for interactive data.

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APPENDIX A – Some Notes on Fuzzy Optimization and Climate Change Policy

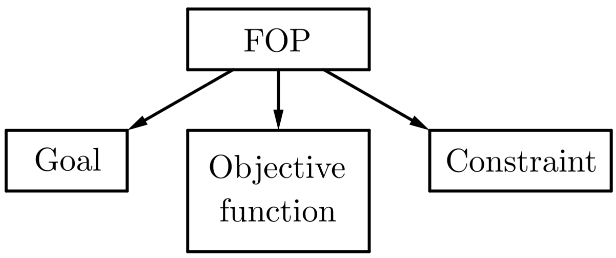
This appendix contains a complementary study, made during the doctorate time, about fuzzy optimization field.

As Fuzzy Set Theory is an extension of Classical Set Theory, fuzzy optimization problem (FOP) is an extension of classical optimization problem. However it is important to understand how this extension is made.

It begins by noting that an optimization problem contains three parts: goal, objective function and constraints. The first one, the goal, is usually described by commands of type *minimize* or *maximize*, which involves some ordering sense. The second is the objective function, which can be more than one. It has a domain and a codomain which is submitted to the command. Lastly, the problem can be constrained or unconstrained. In general the constraints represent a limitation on domain of the objective function.

In order to put a classical optimization problem in a fuzzy environment, that means, to consider a fuzzy optimization problem it is needed to take into account all the three parts explicit in Figure 44. Each one of them can be somehow fuzzyfied and it is possible to consider a combination of the fuzzyfied parts. That is, one can consider a problem with only the objective function to be fuzzy, but the rest to be crisp; or takes the objective function and the constraints to be fuzzy, maintaining the crisp goal.

Figure 44 – Diagram of parties of Fuzzy Optimization Problem



Each block is a subject discussed in Fuzzy Optimization Problems. Source: Author.

Jointly with the mathematical proposal to put uncertainty in optimization problems it is important to give meaning to the new fuzzy parts. Different arrangements of FOPs are linked to different interpretations and different algorithms to solve the problem.

This chapter is dedicated to understanding the foundations on fuzzy optimization. Section A.1 presents the state of art in Fuzzy Optimization field. Section A.2 contains a brief discussion about previews main works, pointing out what are the inconsistencies until the current time. In order to fill this blanks, Section A.3 begins with simple examples about how the process of finding a solution for a fuzzy optimization works. Section A.4 introduces a new definition for inequalities. Section A.5 completes the argument, showing how it is possible to put more mathematical rigor into the methods analyzed in Section A.2. Finally, Section A.6 shows an application on Climate Change, specifically on Carbon Market Problem, and Section A.7 analyses the marginal cost abatement curve, which is a tool to achieve the goal of climate change.

A.1 State of Art

Fuzzy optimization problems were firstly analyzed in the linear case, and after that in the non-linear case. Furthermore the interpretations changed over the years with advance of studies: the form to consider uncertainties of the fuzzy type has been changed. For more details the reader can refer to the surveys [58, 80, 107].

The FOPs were first introduced by Bellman and Zadeh in 1970 [13]. In that work the authors tried to express the elements of FOP in fuzzy terms, setting a real objective function with fuzzy constrains and the goal was described by a fuzzy set, named decision set. The solution for this problem was constructed by maximizing membership degree of all the possible solutions in the decision set.

Subsequently several authors study these problems, such as, Zimmermann [132, 130], Verdegay [111], Werners [122], Hommelfanger [94] and Mohamed [64]. Thereafter were considered possibilistic approach [53], multiple objective functions, and still type-2 fuzzy constraints [38, 39].

Applications have been made in several areas, such as agronomic planning [63, 77], financial market [133], environment [99], industrial process [109, 110], among others.

The second approach arises with Wu [124, 125, 126], where the objective function is given by a non-linear fuzzy function. For that type of FOPs were established optimality conditions similar to Karush-Kuhn-Tucker optimality conditions for classical optimization problem, by using fuzzy derivatives [19, 75, 76]. Also, there is a version with constraints given by fuzzy-valued functions [78].

This work focuses on a discussion of the most applied method, from to Bellman and Zadeh [13], in order to describe the method with more mathematical rigor. The purpose is not to establish new methods, but instead to provide theoretical background according to the well-founded tools available in fuzzy set theory. This is made by means of Zadeh's extension principle.

In optimization problems the standard form is used in order to simplify the study.

That is,

$$\begin{aligned} &\min \ f \\ &\text{subject to } x \in \Omega, \end{aligned}$$

which is equivalent to the problem

$$\begin{aligned} &\max \ -f \\ &\text{subject to } \neg x \notin \Omega^C, \end{aligned}$$

where Ω^C is the complement of set Ω. Solutions of both problems are the same.

Here the case considered is the simplest one:

$$\begin{aligned} &\min \ f \\ &\text{s.t. } x \geqslant b, \end{aligned} \tag{OP}$$

with $f : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$.

In the next section the main studies to solve (OP) are presented.

A.2 Previews Works

The first papers appear in the 80's and recent articles continuous to use the principles introduced by Bellman, Zadeh, Tanaka, Okuda, Asai and Zimmermann [13, 105, 130, 131]. The following subsections presents these main works.

A.2.1 Bellman-Zadeh Decision Principle

In 1970, two experts gathered in the article "Decision-Making in a Fuzzy Environment" [13] and have started the Fuzzy Optimization area of study. This paper established the well-known Bellman-Zadeh Decision Principle, which has been used in every article about this issue.

They first described the three elements of the decision process: the set of alternatives, the set of constraints on the choice between different alternatives, and the performance function, which associates each choice of an alternative the corresponding gain or loss. This last one is a function that represents how much the goal is going to be attained if an alternative is picked. In this sense, performance function and goal are the same object.

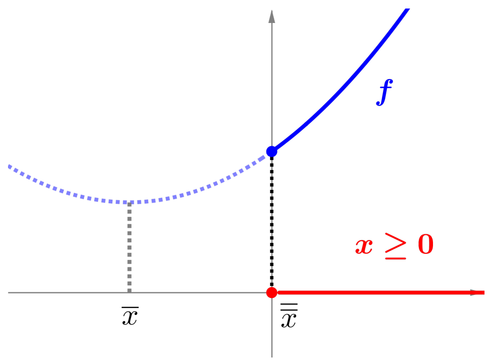
The authors considered a decision "basically a choice or a set of choices drawn from the available alternatives", then decision is just a confluence of goals and constraints, because there is no difference between goals and constraints in terms of importance in the decision process. Actually, the opposite occurs: there is a symmetry with respect to goals and constraints.

The simplest case is the one in which the goals G and the constraints C are fuzzy sets defined in the same space of alternatives X. In Fuzzy Set Theory terms, the "confluence" can be interpreted as a conjunction between constraints and goals, that means, the decision D is given by the intersection between the sets G and C:

$$D = G \cap C. \tag{A.1}$$

This statement is very reasonable, even though the goal seems to be more important. The example in classical problem is depicted in Figure 45, where the goal is to minimize $f(x) = (x+1)^2$ and the constraint is $x \geqslant 0$, illustrating that the searching for the goal is guided by restriction, which in turn changes the decision from $\overline{x} = -1$ to $\overline{\overline{x}} = 0$. As a conclusion, the decision is made by taking into account the goals *and* the constraints, which is in accordance with the Equation (A.1).

Figure 45 – Symmetry with respect to goals and constraints



The function f in blue represents the goal and the set in red represents the constraint. Source: Author [8].

Returning to the fuzzy environment, it is necessary to assign membership functions to C and G, then take the t-norm between them. In general, it is considered minimum t-norm, so the decision is the red set depicted in Figure 46.

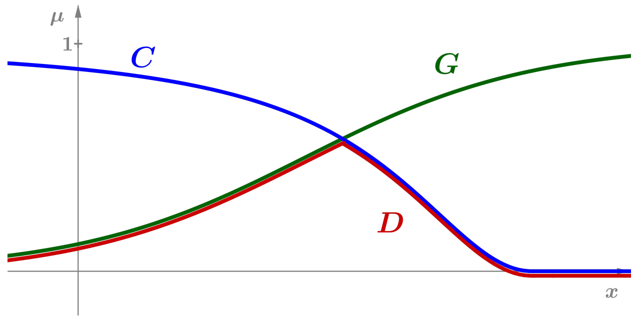
The lack of an $x \in X$ such that $D(x) = 1$, means, according to the authors [13], "that the specified goals and constraints conflict with one another, ruling out the existence of an alternative which fully satisfies all of them". More than that, the optimal decision is obtained by looking for the points x that attained the maximum in D, i.e., the maximizing decision is given by argument from

$$\max_{x \in X} D(x) = \max_{x \in X} \min\{G(x), C(x)\}. \tag{A.2}$$

That means, $\overline{x}$ is the maximizing decision when satisfies

$$\overline{x} = \arg\max_{x \in X} D(x). \tag{A.3}$$

Figure 46 – Decision as a confluence between goal and constraint



The goal set is in green, the constraint set is in blue and the decision set is in red. Source: Author.

In the case where there are n goals $G_1, \ldots, G_n$ and m constraints $C_1, \ldots, C_m$, then the maximizing decision is obtained as a generalization of Equation (A.2),

$$\max_{x \in X} D(x) = \max_{x \in X} \min\{G_1(x), \ldots, G_n(x), C_1(x) \ldots, C_m(x)\}. \tag{A.4}$$

If some goals or constraints are of greater importance than others, the decision might be a weighted convex combination of goals and constraints, with weights reflecting the importance given by the decision-maker $\alpha_i(x)$, for $i = 1, \ldots, n$ and $\beta_j(x)$, for $j = 1, \ldots, m$, for each goal G_i and constraint C_j, respectively. In this case the decision set is defined as

$$D(x) = \sum_{i=1}^{n} \alpha_i(x) G_i(x) + \sum_{j=1}^{m} \beta_j(x) C_j(x),$$

and the weights must satisfy

$$\sum_{i=1}^{n} \alpha_i(x) + \sum_{j=1}^{m} \beta_j(x) = 1.$$

Bellman and Zadeh also proved that for $X = \mathbb{R}^n$ there is a unique maximizing decision if D is strongly convex fuzzy set, in other words, if D is convex and have a unimodal membership function.

On the other hand, constraints C and goals G can be fuzzy sets in different spaces X and Y, respectively. For instance, if the goal is $f : X \to Y$, then the goal $f(x)$ lies in Y. In this case it is necessary to redefine G by the set $\overline{G}$ in the universe X, induced by the function f. The procedure is given by

$$\overline{G}(x) = G(f(x)). \tag{A.5}$$

In this context the decision set is given by

$$D(x) = \min_{x \in X}\{\overline{G}(x), C(x)\}. \tag{A.6}$$

And the maximizing decision $\overline{x}$ is given in Equation (A.3).

The authors also mentioned some linguistics examples, such as the goal "x should be substantially larger than 10" and the constraint "x should be approximately between 2 and 10". Nonetheless they do not specify how to build the sets G and C, in particular they did not propose how C and G should looks like when they lie in different sets of alternatives.

A.2.2 Writting the Problem in Terms of α-Cuts

The second important work was made by Tanaka, Okuda and Asai in 1974 [105], where they considered X as a normed space and the α-cuts of the constraint C as

$$C_\alpha = \{x \in X, C(x) \geqslant \alpha\}.$$

For C_α a compact set, they proved that the maximizing decision from Bellman and Zadeh in Equation(A.2) is equivalent to

$$\sup_x D(x) = \sup_\alpha [\alpha \wedge \max_{x \in C_\alpha} G(x)].$$

They determined that, from the fact that the performance function f normalized in the closure of C, $cl(C) = \{x, C(x) > 0\}$, have image in $[0,1]$, then it can be treated as a fuzzy set. In this context, the fuzzy mathematical programming boils down to search for a optimal pair (α^*, x^*) such that

$$\alpha^* \wedge f(x^*) = \sup_\alpha [\alpha \wedge \max_{x \in C_\alpha} f(x)].$$

The authors suggested that the decision-maker first has to determine the optimal α, corresponding to the level of rigor intended, and then the problem boils down to a classical mathematical programming, "since it remains only to determine the optimal x^* such that it maximizes $f(x)$ in the set $C_{\alpha *}$". In other words, the following equivalence is valid.

$$\alpha^* \text{ optimal } \Leftrightarrow \alpha^* = \max_{x \in C_{\alpha *}} f(x).$$

The authors created an algorithm that first search the optimal level α^*, then solves the classical programming problem of finding x^* such that $f(x^*) = \max_{x \in C_{\alpha *}} f(x)$.

One more time was not specified how the constrains and goals sets are derived from the problem information. More than that, although the normalized objective function f is associated with a fuzzy set, this new set does not precisely describe the original problem. This normalized function is useless in practical sense.

A.2.3 Zimmermann's Results

Two papers written by Zimmermann in 1976 and 1978 [130, 131] provided ideas to calculate constrains and goal sets for linear programming problems.

The problem studied was

$$\begin{aligned} &\min\ z = cx \\ &\text{subject to } Ax \leqslant b \\ &x \geqslant 0, \end{aligned} \tag{LP}$$

with $A \in \mathbb{R}^{m\times n}, x, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The problem was adopted to a fuzzy version

$$\begin{aligned} cx &\lesssim z_0 \\ Ax &\lesssim b \\ x &\lesssim 0, \end{aligned} \tag{A.7}$$

where z_0 means an aspiration level of the decision maker and $\lesssim$ represents a relaxation in the constrains, which means that a constraint can be violated just a little, or in Zimmermann words this symbols means "essentially smaller than or equal to".

Putting as in Equation (A.7), the problem can be rewritten as

$$Bx \lesssim b', \tag{A.8}$$

where $B = \begin{bmatrix} c \\ A \end{bmatrix}$ and $b' = \begin{bmatrix} z_0 \\ b \end{bmatrix}$.

Each line i, which corresponds to an inequality, has a membership function given in terms of how much the inequality is violated. If it is strongly violated, then it membership degree is zero; if it is satisfied, has membership degree equal to one; and if it is partially violated, has membership degree between zero and one. For each line i in (A.8) the decision-maker has to choose a limit to the admissible violation, denoted by d_i, which determine the membership function of ith line, as in Equation (A.9),

$$(Bx)_i(y) = \begin{cases} 0,\ \text{if } (Bx)_i(y) > b'_i + d_i \\ 0 < (Bx)_i(y) < 1,\ \text{if } b'_i < (Bx)_i(y) \leqslant b'_i + d_i \\ 1,\ \text{if } (Bx)_i(y) \leqslant b'_i \end{cases}, \tag{A.9}$$

The solution set, following the Bellman-Zadeh Decision Principle, is given by $\min\limits_i (Bx)_i$ and the maximizing decision decision can be found by

$$\max_{y \geqslant 0} \min_i (Bx)_i(y). \tag{A.10}$$

Writting in terms of α-cuts, as Tanaka *et al.* [105] did, the problem (A.10) turns into

$$\begin{aligned}&\max \lambda\\ &\text{s. t. } \lambda \leqslant (Bx)_i,\ i = 0, \dots, m.\\ &x \geqslant 0\end{aligned}$$

This approach has the purpose of choose a plausible good value z_0. In general z_0 is chosen as a solution for the crisp problem in Equation (LP), which means the adopted problem (A.8) can have solution not smaller than crisp problem. In this case, the fuzzy approach is no longer useful.

As conclusion, the goal set is not precisely defined. The following sections discuss this subject.

A.3 Introduction to Inequalities

To understand decision process in optimization it is important to comprehend the notion of inequality. It is possible to define different types of fuzzy inequalities.

One is the $\lesssim$ inequality, which means "almost less than or equal to", and represents that a real inequality is partially satisfied, or in other words, it is a relaxation in the constraint $\leqslant$. For example, the set $A = \{x \in \mathbb{R}, x \lesssim 5\}$ with membership function

$$A(x) = \begin{cases} 1, \text{ if } x < 5 \\ 6 - x, \text{ if } 5 \leqslant x < 6 \\ 0, \text{ if } 6 \leqslant x \end{cases}$$

represents the set of real numbers almost less than or equal to 5, with 1 unit of relaxation.

Another case is the inequality $x \leqslant B$, with B a fuzzy set, $[B]_\alpha = [b_\alpha^-, b_\alpha^+]$. This case will be studied here, since it will be useful to analyse the goal set.

Let L be the set $L = \{x \in \mathbb{R}, x \leqslant B\}$, then L is a fuzzy set. For $x, y \in \mathbb{R}$, if $x \leqslant y$, then the membership degree of x in L is at least equal to membership degree of y in L. Intuitively this analysis means that L has membership function given by

$$\mu_L(x) = \sup_{x \leqslant y} B(y) = \begin{cases} 1, \text{ if } x \leqslant b_1^+ \\ B(x), \text{ if } b_1^+ < x \leqslant b_0^+ \\ 0, \text{ if } x > b_0^+ \end{cases}.$$

This definition of L coincides with the definition for $x \lesssim b_1^+$, that can be found in [58, 81]. In this sense we can say that there is no distinction between the set $A = \{x \in$

$\mathbb{R}, x \lesssim b_1^+\}$ and $L = \{x \in \mathbb{R}, x \leqslant B\}$, as long as the decision maker models $\lesssim$ such that the uncertainty in the interval $[b_1^+, b_0^+]$ in given by the right side of B.

Part of constrains are given by $g(x) \leqslant B$. In this case we must consider the Zadeh's extension of g at A, denoted by $\widehat{g}$, and defined by

$$\widehat{g}(A)(y) = \sup_{x \in g^{-1}(y)} A(x).$$

If g is a bijection, then

$$\widehat{g}(A)(y) = A(g^{-1}(y)),$$

and for g increasing we can define the set referring to $g(x) \leqslant B$, denoted by L_g,

$$L_g = \{x \in \mathbb{R}, x \leqslant \widehat{g}^{-1}(B)\}.$$

From Nguyen-Barros Theorem (see Theorem 1.3), the α-cuts of $\widehat{g}(A)$ are

$$[\widehat{g}(A)]_\alpha = g([A]_\alpha) = g([a_\alpha^-, a_\alpha^-]) = \begin{cases} [g(a_\alpha^-), g(a_\alpha^+)], \text{ if } g \text{ is increasing} \\ [g(a_\alpha^+), g(a_\alpha^-)], \text{ if } g \text{ is decreasing} \end{cases}.$$

For L_g – g increasing –, the membership function is then

$$\mu_{L_g}(x) = \sup_{x \leqslant y} B(y) = \begin{cases} 1, \text{ if } x \leqslant g^{-1}(b_1^+) \\ \mu_{\widehat{g}(B)}(x), \text{ if } g^{-1}(b_1^+) < x \leqslant g^{-1}(b_0^+) \\ 0, \text{ if } x > g^{-1}(b_0^+) \end{cases}.$$

Which can be written as

$$L_g(x) = \sup_{x \leqslant y} B(y) = \begin{cases} 1, \text{ if } g(x) \leqslant b_1^+ \\ B(g(x)), \text{ if } b_1^+ < g(x) \leqslant b_0^+ \\ 0, \text{ if } g(x) > b_0^+ \end{cases}.$$

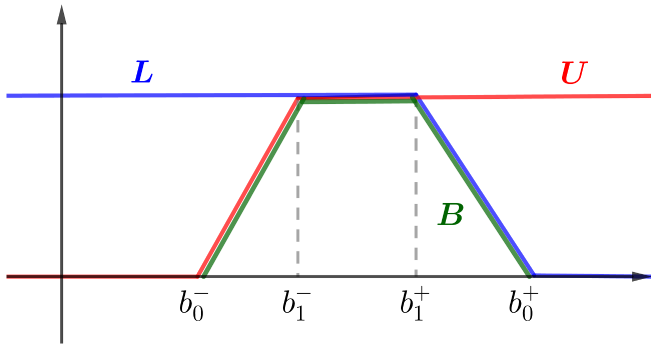
Supposing now g decreasing, the membership function of L_g is calculated by

$$L_g(x) = \sup_{x \leqslant y} B(y) = \begin{cases} 1, \text{ if } g(x) \geqslant b_1^+ \\ B(g(x)), \text{ if } b_1^+ > g(x) \geqslant b_0^+ \\ 0, \text{ if } g(x) < b_0^+ \end{cases}.$$

This argument can be used similarly to produce the set $U = \{x \in \mathbb{R}, x \geqslant B\}$.

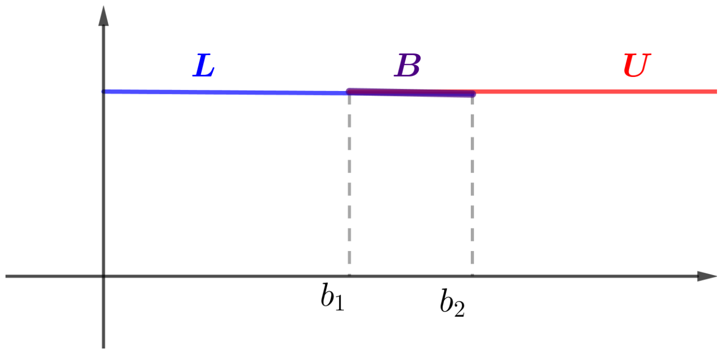
$$U(x) = \sup_{x \geqslant y} B(y) = \begin{cases} 0, \text{ if } x \leqslant b_0^- \\ B(x), \text{ if } b_0^- < x \leqslant b_1^- \\ 1, \text{ if } x > b_1^- \end{cases}.$$

Figure 47 – Membership functions of L, U and B



The inequalities $x \leqslant B$ (in blue) and $x \geqslant B$ (in red) for B a trapezoidal fuzzy number (in green). Source: Author [8].

Figure 48 – Recovering classical notion of inequality



The inequalities $x \leqslant B$ (in blue) and $x \geqslant B$ (in red) simultaneous give back to the crisp set $B = [b_1, b_2]$ (in purple). Source: Author [8].

For B a trapezoidal fuzzy number, the L and U sets are depicted in Figure 47.

This notion, naturally, extends the classical notion of inequality, because it preserves the expected $x = b \Leftrightarrow x \leqslant b$ and $x \geqslant b$, as can be observed in Figure 48.

Considering g increasing $U_g = \{x \in \mathbb{R}, g(x) \geqslant B\}$ is defined as

$$U_g(x) = \sup_{x \geqslant y} B(y) = \begin{cases} 0, \text{ if } g(x) \leqslant b_0^- \\ B(g(x)), \text{ if } b_0^- < g(x) \leqslant b_1^- \\ 1, \text{ if } g(x) > b_1^- \end{cases}.$$

Now, considering g decreasing, U_g is given by

$$U_g(x) = \sup_{x \geqslant y} B(y) = \begin{cases} 1, \text{ if } g(x) \geqslant b_1^- \\ B(g(x)), \text{ if } b_0^- \leqslant g(x) < b_1^- \\ 0, \text{ if } g(x) < b_0^- \end{cases}.$$

The question from this moment is how to consider this definition of constrains L_{g_i} to solve the optimization problem

$$\begin{aligned} &\min \; f \\ &\text{subject to } x \in L_{g_i}, \text{ for } i = 1, \dots, m. \end{aligned}$$

According to Bellman and Zadeh [13], the goal is not influenced by the constrains, so the decision came simply from normalized function f, given rise to G, and taking

$$\max \bigcap_{i=1}^{n} L_{g_i} \cap G.$$

Nevertheless, it is natural to expected that a choice of constraints influence the goal, and in this case normalize the objective function is not sufficient to build the goal fuzzy set. In order to formalize the notion of fuzzy inequality and, consequently, the notion of minimum of function in a fuzzy environment, the next sections show how it is possible to use Zadeh's Extension Principle to define goal and constrains sets.

A.4 Extension of Constraints

Note that the following equivalence set

$$\Omega = \{x \in \mathbb{R}; x \geqslant b\} \Leftrightarrow \Omega = \{x \in \mathbb{R}; \max\{x, b\} = x\}. \tag{A.11}$$

Besides that, in terms of fuzzy set theory, $x \in \Omega$ is the same as x is Ω. Here the intention is to extend this crisp set to a fuzzy set, that is, to extend classical constrain in (OP).

Let $B \in \mathcal{F}_{\mathbb{R}}$ with membership function $B(x), \forall\ x \in \mathbb{R}$, with 0 and 1-cuts given by $[B]_0 = [b_0^-, b_0^+]$ and $[B]_1 = [b_1^-, b_1^+]$, respectively. A fuzzy set Ω can be produced as an extension of inequality $x \geqslant b$. How is defined the membership function of the set $\Omega = \{x \in \mathbb{R}; x \geqslant B\}$?

First of all, the cylindrical extension of B (see Definition 1.13), denoted by $U = \mathbb{R} \times B$, is taken in order to compare $x \in \mathbb{R}$, with $B \in \mathcal{F}_{\mathbb{R}}$,

$$U(x, y) = B(y), \forall\ (x, y) \in \mathbb{R} \times \mathbb{R}. \tag{A.12}$$

Due to equivalence (A.11), the set Ω will be constructed by taking the Zadeh extension of the function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by $\varphi(x, y) = \max\{x, y\}$, where $y \in B$, fixed. This extension, with respect to the set U, is given by

$$\widehat{\varphi}(U)(z) = \sup_{(x,y) \in \varphi^{-1}(z)} U(x, y), \forall\ z \in \mathbb{R}. \tag{A.13}$$

The inverse image of z by φ is given by

$$\varphi^{-1}(z) = \underbrace{\{(z,y); z \geqslant y\}}_{(I)} \cup \underbrace{\{(x,z); z \geqslant x\}}_{(II)}, \tag{A.14}$$

so we separated in to cases:

1. If $(x,y) \in (I)$, then $U(x,y) = U(z,y) = B(y)$. In this case, the extension in (A.13) becomes

$$\begin{aligned}\widehat{\varphi}(U)(z) &= \sup_{(x,y)\in(I)} B(x) = \sup_{(z,y),z\geqslant y} B(y) = \begin{cases} 1, \text{ if } z \geqslant b_1^- \\ B(y), \text{ if } b_0^- < z \leqslant b_1^- \\ 0, \text{ if } z < b_0^- \end{cases} \\ &= U_\Omega(z).\end{aligned} \tag{A.15}$$

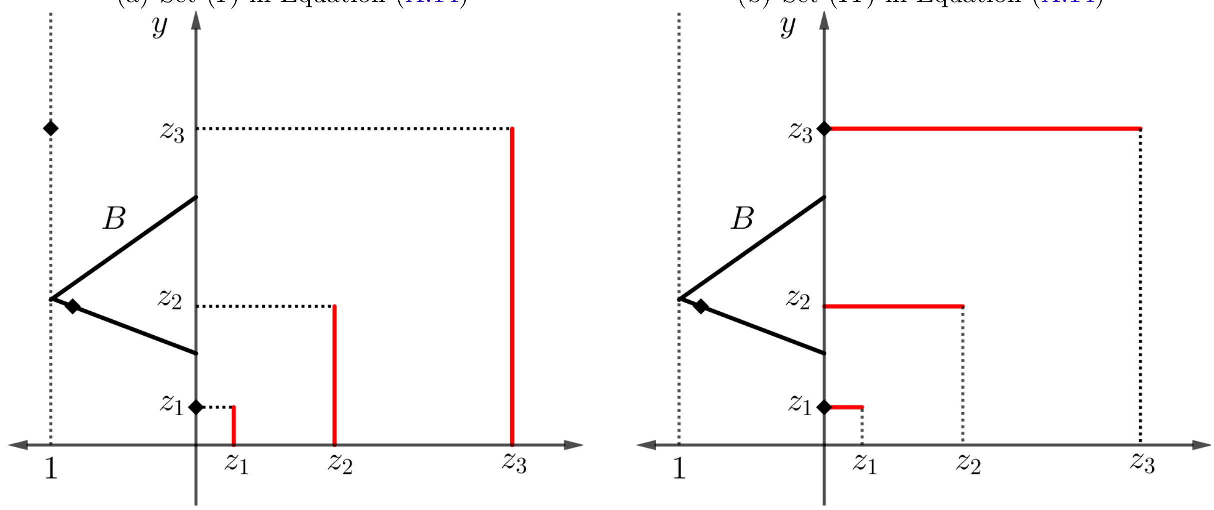
2. If $(x,y) \in (II)$, then $U(x,y) = U(x,z) = B(z)$. The extension in (A.13) becomes

$$\widehat{\varphi}(U)(z) = \sup_{(x,y)\in(II)} B(z) = \sup_{(x,z),z\geqslant x} B(z) = B(z). \tag{A.16}$$

This process is depicted in Figure 49 for the case B is a triangular fuzzy number.

Figure 49 – Graphical representation of sets (I) and (II) in Equation (A.14)

(a) Set (I) in Equation (A.14) (b) Set (II) in Equation (A.14)

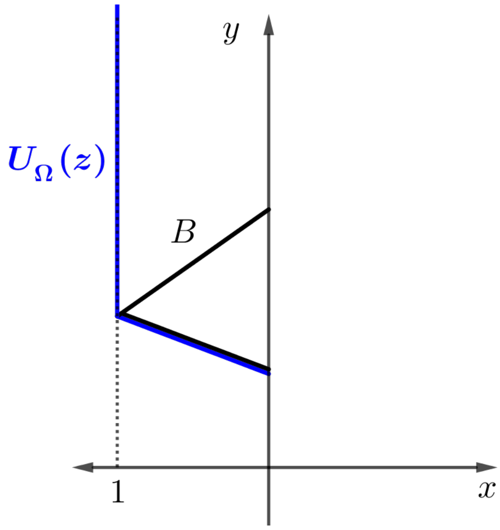


For each fixed point z_1, z_2 and z_3, the red line is the set (I) in Subfigure(a) and its membership degree (see Equation (A.15)) are the rhombus points. For each fixed point z_1, z_2 and z_3, the red line is the set (II) in Subfigure(b) and its membership degree (see Equation (A.16)) are the rhombus points. Source: Author.

It is possible to gather these two cases because for two any bounded sets $A, C \subseteq \mathbb{R}$ it holds true $\sup(A \cup C) = \sup\{\sup(A), \sup(C)\}$. Hence,

$$\widehat{\varphi}(U)(z) = B(z) \vee U_\Omega(z) = U_\Omega(z).$$

Figure 50 – Graphical representation of U_Ω



Blue curve represents the set U_Ω for the case where B is a triangular fuzzy number. Source: Author.

Consequently $\widehat{\varphi}(U) = U_\Omega$ and the final set U_Ω is depicted in Figure 50.

On the other hand, the constrain set could be

$$\Omega = \{x \in \mathbb{R}; x \leqslant b\} \Leftrightarrow \Omega = \{x \in \mathbb{R}; \min\{x, b\} = x\}.$$

Similarly, the fuzzy set Ω can be produced as an extension of inequality $x \leqslant B$. The cylindrical extension U remains the same as in (A.12). But here the function $\phi(x, y) = \min\{x, y\}$, with $y \in B$, is considered.

The Zadeh extension of ϕ with respect to U is given by

$$\widehat{\phi}(U)(z) = \sup_{(x,y)\in\phi^{-1}(z)} U(x, y), \forall\ z \in \mathbb{R}.$$

The set $\phi^{-1}(z)$ can be written as

$$\phi^{-1}(z) = \underbrace{\{(z, y); z \leqslant y\}}_{(III)} \cup \underbrace{\{(x, z); z \leqslant x\}}_{(IV)}, \tag{A.17}$$

and we analyse each case.

1. If $(x, y) \in (III)$, then $U(x, y) = U(z, y) = B(y)$. The extension in (A.17) can be rewritten as

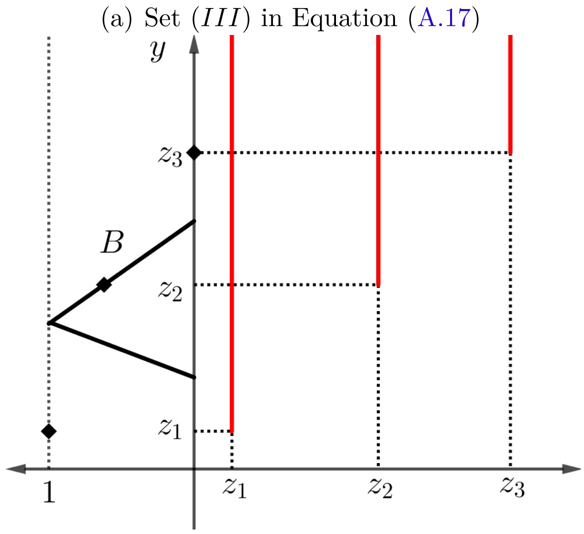
$$\begin{aligned} \widehat{\phi}(U)(z) &= \sup_{(x,y)\in(III)} U(x, y) = \sup_{(z,y), z\leqslant y} B(y) = \begin{cases} 1, \text{ if } z < b_1^+ \\ B(z), \text{ if } b_1^+ \leqslant z < b_0^+ \\ 0, \text{ if } z \geqslant b_0^+ \end{cases} \\ &= L_\Omega(z). \end{aligned} \tag{A.18}$$

2. If $(x, y) \in (IV)$, then $U(x, y) = U(x, z) = B(z)$. In this case the extension (A.17) is

$$\widehat{\phi}(U)(z) = \sup_{(x,y)\in(IV)} U(x, y) = \sup_{(x,z),z\leqslant x} B(z) = B(z). \tag{A.19}$$

This process is depicted in Figure 51.

Figure 51 – Graphical representation of sets (III) and (IV) in Equation (A.17)



For each fixed point z_1, z_2 and z_3, the red line is the set (III) in Subfigure(a) and its membership degree (see Equation (A.18)) are the rhombus points. For each fixed point z_1, z_2 and z_3, the red line is the set (IV) in Subfigure(b) and its membership degree (see Equation (A.19)) are the rhombus points. Source: Author.

Therefore $\widehat{\phi}(U)(z) = B(z) \vee L_\Omega(z) = L_\Omega(z)$, or yet, $\widehat{\phi}(U) = L_\Omega$. This analysis is illustrated in Figure 52.

Remark A.1. *From classical set theory, $x = b$ if, and only if, $x \leqslant b$ and $x \geqslant b$. Therefore it is expected that $x = B$ if, and only if, $x \leqslant B$ and $x \geqslant B$; indeed it follows that $U_\Omega \cap L_\Omega = B$, so this notion of inequality extends the classical notion.*

One last equivalence remains to guarantee the existence of standard form, i.e. $x \leqslant B \Leftrightarrow -x \geqslant -B$.

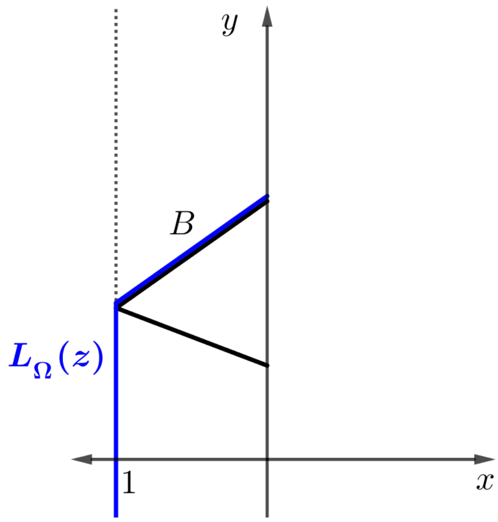
Note that $-x \geqslant -b = \{x \in \mathbb{R}; \max\{-x, -b\} = -x\}$ and $(-B)(y) = B(-y), \forall\, y \in \mathbb{R}$.

Moreover, $\varphi(x, y) = \max\{-x, -y\} = -\min\{x, y\} = -\phi(x, y)$. Considering the cylindrical extension given by (A.12), but U_1 for B and U_2 for $-B$, the Zadeh extension of φ and ϕ obey

$$\widehat{\varphi}(U)(z) = -\widehat{\phi}(-U)(z), \forall\; z \in \mathbb{R}.$$

Thus $\widehat{\varphi} = \widehat{\phi}$, hence $x \leqslant B$ is equivalent to $-x \geqslant -B$. Furthermore, $L_\Omega = -U_\Omega$.

Figure 52 – Graphical representation of L_Ω



Blue curve represents the set L_Ω for the case where B is a triangular fuzzy number. Source: Author.

Since it is possible to interchange the L_Ω and U_Ω by simply multiplying one of them by (-1), the fuzzy set of constrains will be denoted by $\Omega \in \mathcal{F}_\mathbb{R}$, with membership function $\Omega(x), \forall\ x \in \mathbb{R}$.

The same arguments can be used to consider more general constrain set. Let be $g_i : \mathbb{R} \to \mathbb{R}$, $b_i \in \mathbb{R}$ for $i = 1, \ldots, k$ and $\Omega = \{x \in \mathbb{R}; g_i(x) \leqslant b_i\}$ the constrains set.

For each $i = 1, \ldots, k$ we intend to extend the Ω to fuzzy environment by taking $B_i \in \mathcal{F}_\mathbb{R}$, and setting inequalities in the form $g_i(x) \leqslant B_i$.

Consider the cylindrical extension U_i of B_i and the comparison function φ_i given by

$$U_i(x, y) = B_i(x), \forall\ (x, y) \in \mathbb{R} \times \mathbb{R}$$

and

$$\varphi_i(x, y) = \min\{g_i(x), y\}.$$

The Zadeh extension of φ_i with respect to U is, $\forall\ z \in \mathbb{R}$,

$$\widehat{\varphi}_i(U)(z) = \sup_{(g_i(x),y)\in\varphi^{-1}(z)} U_i(g_i(x), y). \tag{A.20}$$

The set $\varphi^{-1}(z)$ can be written in terms of the union

$$\varphi_i^{-1}(z) = \underbrace{\{z, y); z = g_i(x) \text{ and } y \geqslant z\}}_{(I)} \cup \underbrace{\{(g_i(x), z); z = y \text{ and } z \leqslant g_i(x)\}}_{(II)}.$$

If $(g_i(x), y) \in (I)$, then $U(g_i(x), y) = U(z, y) = B(y)$ and $\widehat{\varphi}_i$ in (A.20) can be rewritten as

$$\begin{aligned}\widehat{\varphi}_i(U)(z) &= \sup_{(z,y), z=g_i(x)\leqslant y} B(y) = \begin{cases}1, \text{ if } g_i(x) < b_1^+ \\ B(g_i(x)), \text{ if } b_1^+ \leqslant g_i(x) < b_0^+ \\ 0, \text{ if } g_i(x) \geqslant b_0^+\end{cases} \\ &= L_\Omega^i(g_i(x)).\end{aligned}$$

On the other hand, if $(g_i(x), y) \in (II)$, then $U(g_i(x), y) = U(g_i(x), z) = B(z)$, and from (A.20) we have

$$\begin{aligned}\widehat{\varphi}_i(U)(z) &= \sup_{(g_i(x),z), z\leqslant g_i(x)} B(z) \\ &= B(z).\end{aligned}$$

Therefore, $\widehat{\varphi}_i(U)(z) = B(z) \vee L_\Omega^i(g_i(x)) = L_\Omega^i(g_i(x))$, hence $\widehat{\varphi}_i(U) = L_\Omega^i$.

Since x in Ω must satisfies all the k inequalities, we can state the membership function of Ω:

$$\Omega(x) = \bigcap_{i=1}^{k} L^i(g_i(x)).$$

Example A.1. *The inequality* $x \geqslant (0, 0, 1, 2)$ *gives raise to* $\Omega(x) = 1, \forall\ x \geqslant 0$, *thus is a good extension for* $x \geqslant 0$.

Example A.2. *The inequalities* $1 \leqslant x \leqslant 2$ *can be extended considering the fuzzy numbers instead of* 1 *and* 2*:* $(0, 1, 2) \leqslant x \leqslant (1, 2, 3)$.

The constrain set given by this inequalities has membership function

$$\Omega(x) = \begin{cases}x, \ \text{if } 0 \leqslant x \leqslant 1 \\ 1, \ \text{if } 1 < x \leqslant 2 \\ 3 - x, \ \text{if } 2 < x \leqslant 3 \\ 0, \ \text{if } x < 0 \text{ or } x > 3\end{cases}.$$

Example A.3. *If* Ω *is the set of real numbers* x *such that* $x \leqslant 1$ *and* $x \geqslant 2$. *In classical sense,* $\Omega = \varnothing$, *but considering* $B_1 = (0, 1, 2)$, $B_2 = (1, 2, 4)$ *and* $\Omega = \{x \in \mathbb{R}; x \leqslant B_1 \text{ and } x \geqslant (1, 2, 4)\}$, *we obtain*

$$\Omega(x) = \begin{cases}x - 1, \ \text{if } 1 \leqslant x \leqslant 1.5 \\ 2 - x, \ \text{if } 1.5 < x \leqslant 2 \\ 0, \ \text{otherwise}\end{cases},$$

which implies that $\Omega \neq \varnothing$.

Example A.4. *For $\Omega : g(x) \leqslant B_1$, with g an increasing function, its membership function is*

$$\Omega(x) = \begin{cases} 1, \ \text{if } x \leqslant g^{-1}(b_1^+) \\ B(g(x)), \ \text{if } g^{-1}(b_1^+) < x \leqslant g^{-1}(b_0^+) \\ 0, \ \text{if } x > g^{-1}(b_0^+) \end{cases}.$$

For g a decreasing function, we have

$$\Omega(x) = \begin{cases} 1, \ \text{if } x \geqslant g^{-1}(b_1^-) \\ B(g(x)), \ \text{if } g^{-1}(b_1^-) > x \geqslant g^{-1}(b_0^-) \\ 0, \ \text{if } x < g^{-1}(b_0^-) \end{cases}.$$

Example A.5. *Consider an example of $\Omega \subset \mathcal{F}_{\mathbb{R}^2}$, such that $\Omega : \begin{cases} g_1(x) \leqslant B_1 \\ g_2(x) \leqslant B_2 \end{cases}$, where $g_1(x) = x_1 + 2x_2$, $g_2(x) = -x_1 + 2x_2$, $B_1 = (0, 4, 8)$ and $B_2 = (1, 2, 3)$.*

In this cases $\Omega(x) = C_1(x) \cap C_2(x)$, where

$$C_1(x) = \begin{cases} 1, \ \text{if } x_1 + 2x_2 \leqslant 4 \\ \dfrac{8 - (x_1 + 2x_2)}{4}, \ \text{if } 4 < x_1 + 2x_2 \leqslant 8 \\ 0, \ \text{if } x_1 + 2x_2 > 8 \end{cases}$$

and

$$C_2(x) = \begin{cases} 1, \ \text{if } -x_1 + 2x_2 \leqslant 2 \\ 3 - (-x_1 + 2x_2), \ \text{if } 2 < -x_1 + 2x_2 \leqslant 3 \\ 0, \ \text{if } -x_1 + 2x_2 > 3 \end{cases}.$$

The set $\Omega = C_1 \cap C_2$ is a fuzzy polytope, in the sense of [54], since each α-cut of Ω is a polytope. The set Ω is depicted in Figure 53.

Classically the polytope would be defined only by the pairs with membership degree 1 in Ω, but here the boundaries are extended and have membership degree between 0 and 1.

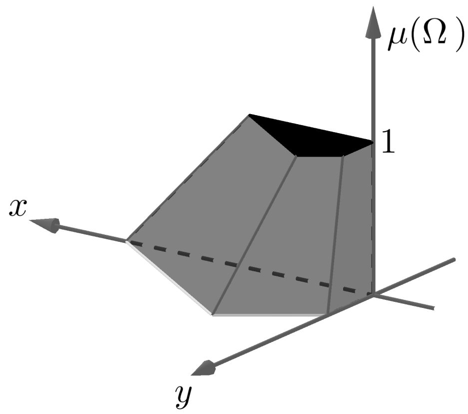
A.5 Extension of Minimum Definition

Let be, in (OP), $f : \Omega \to \mathbb{R}$, where Ω is the domain. Classically $\overline{x}$ is the global minimum of (OP) if

$$f(\overline{x}) \leqslant f(x), \forall\ x \in \Omega. \tag{A.21}$$

Indeed, $\overline{x}$ can be seen as a point such that $f(\overline{x}) = \min f(\Omega)$.

Figure 53 – Graphical representation of a fuzzy polytope



Black region represents the membership degree 1 of Ω and the gray region represents membership degree between 0 and 1.

Since Ω is a fuzzy set, we need to compare $f(\overline{x})$ with $\widehat{f}(\Omega)$, then we take the cylindrical extension $V = A \times \mathbb{R}$ with $A = \widehat{f}(\Omega)$:

$$V(f(x), f(y)) = A(f(x)) = \widehat{f}(\Omega)(w),$$

where $(f(x), f(y)) \in \mathbb{R} \times \mathbb{R}$, and $w = f(x)$.

The inequality (A.21) in minimum definition can be rewritten as $f(\overline{x}) \leqslant f(x) \Leftrightarrow \max\{f(\overline{x}), f(x)\} = f(x)$, hence we consider the function $m(f(x), f(y)) = \max\{f(x), f(y)\}$ that represents the set of all $x \in \mathbb{R}$ such that $f(x)$ is greater than or equal to the minimum value $f(\overline{x})$. We extend m function by Zadeh extension principle:

$$\widehat{m}(V)(w) = \sup_{(f(x),f(y)) \in m^{-1}(w)} V(f(x), f(y)), \forall\ z \in \mathbb{R}. \tag{A.22}$$

Resembling what was done with $\varphi^{-1}(z)$ in (A.14), we have

$$\begin{aligned} m^{-1}(w) = \quad & \underbrace{\{(w, f(y)); w = f(x) \text{ and } f(y) \leqslant w\}}_{(I)} \\ & \cup \underbrace{\{(f(x), w); w = f(y) \text{ and } f(x) \leqslant w\}}_{(II)}. \end{aligned}$$

If $(f(x), f(y)) \in (I)$, then $V(f(x), f(y)) = V(w, f(y)) = A(w) = A(f(x))$. From (A.22),

$$\begin{aligned} \widehat{m}(V)(w) &= \sup_{(f(x),f(y)) \in m^{-1}(w)} V(f(x), f(y)) \\ &= \sup_{(w,y), w=f(x), w \geqslant f(y)} A(w) \\ &= \sup_{w=f(x), w \geqslant f(y)} \widehat{f}(\Omega)(w) \\ &= \widehat{f}(\Omega)(w). \end{aligned}$$

Considering $(f(x), f(y)) \in (II)$, then $V(f(x), f(y)) = V(f(x), w) = A(f(x))$, and the extension in (A.22) becomes

$$\begin{aligned}\widehat{m}(V)(w) &= \sup_{(x,w), w=f(y), f(x)\leqslant w} \widehat{f}(\Omega)(w) \\ &= \begin{cases} 1, \text{ if } f(x) \geqslant (\widehat{f}(\Omega))_1^- \\ \widehat{f}(\Omega)(z), \text{ if } (\widehat{f}(\Omega))_1^- > f(x) \geqslant (f(\Omega))_0^- \\ 0, \text{ if } f(x) < (f(\Omega))_0^- \end{cases} \\ &= O(f(x)).\end{aligned}$$

Therefore $\widehat{m}(z) = \widehat{f}(\Omega)(z) \vee U_O(f(x)) = U_O(f(x))$, as shown in Figure 54. Note that $\widehat{f}(\Omega)$ is contained in $\widehat{m}$.

In classical case the value $f(\overline{x})$ is given by the intersection $m \cap \overline{m^C}$, where $m = \{\overline{x} \in \mathbb{R}; f(\overline{x}) \leqslant f(x), \forall\ x \in \mathbb{R}\}$:

$$f(\overline{x}) = m \cap \overline{m^C}. \tag{A.23}$$

Then it is necessary to search for $\overline{x}$ such that (A.23) is satisfied for $\widehat{m}$ a fuzzy set, that is,

$$f(\overline{x}) = \widehat{m} \cap \overline{\widehat{m}^C}. \tag{A.24}$$

We consider the set

$$O(x) = \overline{\widehat{m}^C}(f(x)),$$

this means, the membership degree of x in O is calculated by taking the its image $f(x)$ and considering its membership degree in fuzzy set $\overline{\widehat{m}^C}$. This process is depicted in Figure 54.

Thus the decision set can be defined by

$$D(x) = O(x) \cap \Omega(x).$$

From Bellman-Zadeh decision principle [13], minimum $\overline{x}$ is given by

$$\overline{x} = \text{argmax } D(\overline{x}) = \text{argmax}(O(x) \cap \Omega(x)).$$

From this it is possible to redefine the notion of minimum.

Definition A.1 (Global minimum of (OP)). *A point $\overline{x}$ is the global minimum of (OP) if it satisfies*

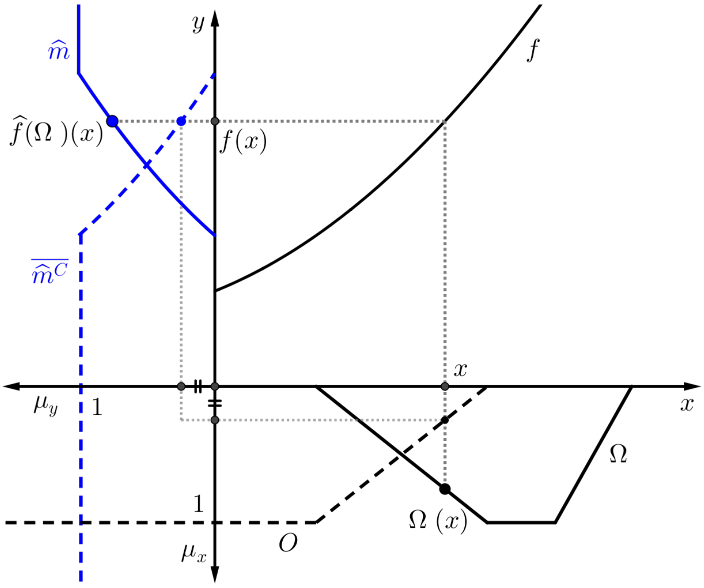
$$D(\overline{x}) > 0 \textit{ and } D(\overline{x}) \geqslant D(x), \forall\ x \in \mathbb{R}^n. \tag{A.25}$$

Definition A.2 (Local minimum of (OP)). *A point $\overline{x}$ is the local minimum if there is $\delta > 0$ such that*

$$D(\overline{x}) > 0 \textit{ and } D(\overline{x}) \geqslant D(x), \forall\ x \in \mathcal{V}(\overline{x}, \delta), \tag{A.26}$$

where $\mathcal{V}(\overline{x}, \delta)$ is the δ-neighborhood of $\overline{x}$.

Figure 54 – Membership function of $\widehat{m}$



Membership function of $\widehat{m}$ in blue. In dashed blue, the fuzzy set $\overline{\widehat{m}^C}$. Sets Ω and O in black and dashed black, respectively. Source: Author [84].

Definition in (A.25) means that $\overline{x}$ must satisfy

$$\begin{cases} \Omega(\overline{x}) \cap O(\overline{x}) > 0 \\ \Omega(\overline{x}) \cap O(\overline{x}) \geqslant \Omega(x) \cap O(x), \forall\ x \in \mathbb{R}^n \end{cases}.$$

And definition in (A.26) means that $\overline{x}$ must satisfy

$$\begin{cases} \Omega(\overline{x}) \cap O(\overline{x}) > 0 \\ \Omega(\overline{x}) \cap O(\overline{x}) \geqslant \Omega(x) \cap O(x) \cap B(\overline{x}, \delta) \end{cases}.$$

Example A.6. *Let be the following optimization problem*

$$\begin{aligned} &\min\ f \\ &s.\ t.\ g(x) \leqslant B, \end{aligned} \tag{A.27}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ *are strictly increasing and* $B \in \mathcal{F}_{\mathbb{R}}$*.*

The membership function of Ω *is given in Example A.4. It is necessary to compute* $\widehat{f}(\Omega)$*.*

$$\widehat{f}(\Omega)(x) = \begin{cases} 1,\ \text{if } f(x) \geqslant f(g^{-1}(b_1^+)) \\ \widehat{f}(\Omega)(x),\ \text{if } f(g^{-1}(b_1^+)) > f(x) \geqslant f(g^{-1}(b_0^+)) \\ 0,\ \text{if } f(x) < f(g^{-1}(b_1^+)) \end{cases},$$

given that g *is increasing. Since* f *is also increasing,* $\widehat{m} = \widehat{f}(\Omega) = \Omega(f^{-1})$*.*

The set $\widehat{\overline{m}^C}$ *is then given by*

$$\overline{\widehat{m}^C}(f(x)) = \begin{cases} 1, \text{ if } f(x) \leqslant f(g^{-1}(b_0^+)) \\ 1 - \widehat{m}(f(x)), \text{ if } f(g^{-1}(b_1^+)) < f(x) \leqslant f(g^{-1}(b_0^+)) \\ 0, \text{ if } f(x) < f(g^{-1}(b_1^+)) \end{cases} .$$

Then

$$O(x) = \begin{cases} 1, \text{ if } x \leqslant g^{-1}(b_0^+) \\ 1 - \Omega(x), \text{ if } g^{-1}(b_0^+) < x \leqslant g^{-1}(b_0^+) \\ 0, \text{ if } x < g^{-1}(b_1^+) \end{cases} .$$

In particular, for $f, g : \mathbb{R} \to \mathbb{R}$ *and* B *a fuzzy number, we have* $\widehat{m}(x) = 1, \forall\ x \in \mathbb{R}$, *then* $\overline{\widehat{m}^C}(y), \forall\ y \in \mathbb{R}$, *thus* $O = \varnothing$ *and there is no* $\overline{x}$ *that minimizes the problem* (A.27).

Example A.7. *Let be the problem* (A.27), *but with* f *strictly increasing and* g *strictly decreasing.*

From Example A.4 we have the constraint set Ω *and the objective set is built by*

$$\overline{\widehat{m}^C}(f(x)) = \begin{cases} 1, \text{ if } f(x) \leqslant f(g^{-1}(b_0^-)) \\ 1 - \widehat{m}(f(x)), \text{ if } f(g^{-1}(b_0^-)) < f(x) \leqslant f(g^{-1}(b_1^-)) \\ 0, \text{ if } f(x) > f(g^{-1}(b_1^-)) \end{cases} .$$

Obtaining

$$O(x) = \begin{cases} 1, \text{ if } x \leqslant g^{-1}(b_0^-) \\ 1 - \Omega(x), \text{ if } g^{-1}(b_0^-) < x \leqslant g^{-1}(b_1^-) \\ 0, \text{ if } x > g^{-1}(b_1^-) \end{cases} .$$

Considering (A.27), *where* f *is strictly decreasing and* g *strictly decreasing, we have*

$$\overline{\widehat{m}^C}(f(x)) = \begin{cases} 1, \text{ if } f(x) \leqslant f(g^{-1}(b_1^+)) \\ 1 - \widehat{m}(f(x)), \text{ if } f(g^{-1}(b_1^+)) < f(x) \leqslant f(g^{-1}(b_0^+)) \\ 0, \text{ if } f(x) > f(g^{-1}(b_0^+)) \end{cases} .$$

Then

$$\Omega(x) = \begin{cases} 1, \text{ if } x \geqslant g^{-1}(b_1^+) \\ 1 - \Omega(x), \text{ if } g^{-1}(b_1^+) < x \leqslant g^{-1}(b_0^+) \\ 0, \text{ if } x \geqslant g^{-1}(b_0^+) \end{cases} .$$

Now considering (A.27) with f and g strictly decreasing,

$$\overline{\widehat{m}^C}(f(x)) = \begin{cases} 1, \ \text{if } f(x) \leqslant f(g^{-1}(b_1^+)) \\ 1 - \widehat{m}(f(x)), \ \text{if } f(g^{-1}(b_1^+)) < f(x) \leqslant f(g^{-1}(b_0^+)) \\ 0, \ \text{if } f(x) > f(g^{-1}(b_0^+)) \end{cases} .$$

And

$$O(x) = \begin{cases} 1, \ \text{if } x \leqslant g^{-1}(b_1^+) \\ 1 - \Omega(x), \ \text{if } g^{-1}(b_1^+) < f(x) \leqslant g^{-1}(b_0^+) \\ 0, \ \text{if } x > g^{-1}(b_0^+) \end{cases} .$$

In particular, for $f, g : \mathbb{R} \to \mathbb{R}$ and B a fuzzy number, we have $\overline{\widehat{m}^C}(y) = 1, \forall\, y \in \mathbb{R}$ and then $O = \varnothing$. Thus there is no x that solves (A.27) for f and g strictly decreasing.

One more time, in order to retrieve the standard form, we need to show that minimize the objective function f is the same as maximize $-f$.

A.6 Application on Climate Change Policy

Climate change is nowadays one of the most studies subject and the cause of amount of international negotiations [67]. Kyoto protocol (1995), signed by 192 parties, is one result from these arrangements. This pact created the Emissions Trading Systems (ETS), which converts greenhouse gas emissions in a new commodity: a mean to facility countries to reach the target of reducing greenhouse gas emissions [68].

Despite the efforts, CO_2 emissions continuous to grow [74], so countries on United Nations Framework Convention on Climate Change (UNFCCC) still have to work on solutions to reduce the gap between the reduction target and the actual emissions. Furthermore pricing carbon still being a strategy to be used by policy makers and companies [51].

In Brazil there are discussions about Carbon Emissions Trading [21], specially to escape taxation from countries that already adopt such regulations [40]. There are also studies over feasibility of such policies [66, 93].

Reducing carbon emissions does not depend only on aiming from policy makers. There are other obstacles, such as energy transition to post-fossil fuel and cultural changes [55, 61]. Moreover it is necessary to find an equilibrium between gains in climate policies and losses in financial stability [22]. For this reason fuzzy approach may be used in order to model subjectivity and duality in decision process [65].

The purpose is to represent the social cost of carbon [71] in terms of fuzzy variables; in other words, the intention is modelling decision process for policy makers by a fuzzy optimization problem, according to Bellman-Zadeh decision scheme [13], assigning welfare (utility) as the objective function and carbon trading as a constraint.

From extending the classical notion of maximum by Zadeh's extension principal, which leads to Zimmermann's method [130, 131], it is possible to estimate which decision is better. For instance: 1) to maximize the welfare, but increasing CO_2 emissions and with zero profit in carbon trading; 2) to lost in welfare, but earning a lot of in carbon market; 3) or an intermediate choice between 1) and 2).

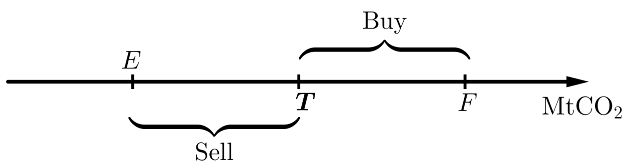
A.6.1 Carbon Market

Kyoto's protocol and Paris Agreement put nations on movement towards preserve the environment. Many research groups are forecasting and arguing about solutions for carbon emission - one of the greenhouse gas emissions. Carbon market or carbon trading arouse in the context of pricing carbon.

> "To achieve the large-scale emission reductions required under the Paris Agreement, the international community needs to find ways to rapidly decarbonize the economy. Putting a price on carbon pollution is one of the most effective and efficient strategies that governments, companies, and other actors can use to reduce carbon emissions and combat climate change." [52]

Emission Trading System (ETS) is based on emission targets, say T (in $MtCO_2$). If an agent pollutes less than T, say E, then the agent may sell the difference $T - E$ in the Carbon Market in form of carbon credit. On the other hand, if an agent pollutes a quantity F more than T, then it is possible to buy the difference $F - T$ in Carbon Market, in order to avoid greater sanctions. The scheme is depicted in Figure 55.

Figure 55 – Scheme of Emission Trading System

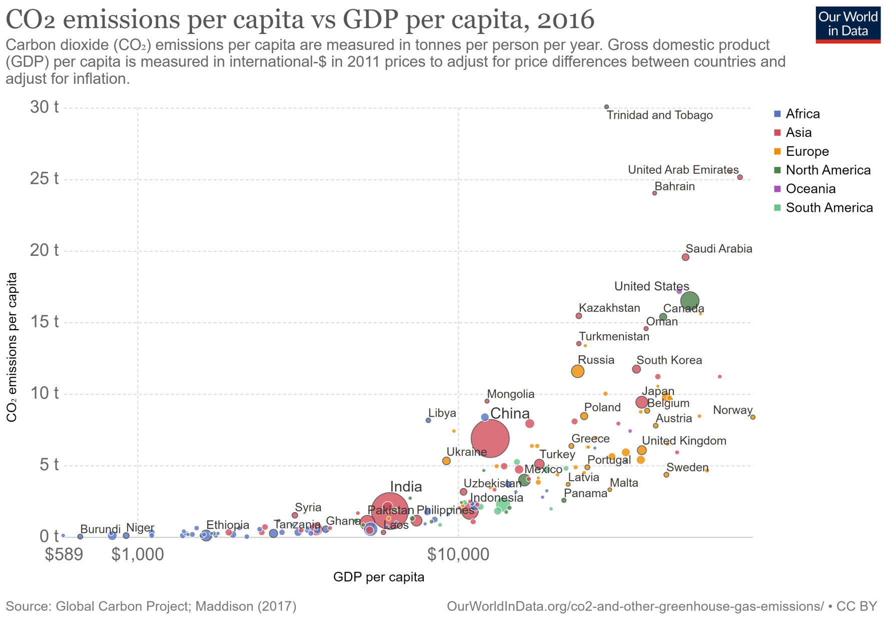


Scheme of ETS showing $F - T$ as a difference that must be bought and the difference $T - E$ that may be sold in Carbon Market. Source: Author [84].

The carbon market became a lucrative market, since the objective is to decrease the target T, increasing the price of carbon credit. Nevertheless there are costs in embedding less carbon emissions and a fair deliberation is been made during the last decades about how much this ETS will decrease the social welfare, or in other words, what is the benefit-cost analysis on the social cost of carbon [71].

[Figure 56](#) portrays the emissions in 2016. As can be seen, as much as gross domestic product (GDP) per capita increases, also the CO_2 emissions per capita increases, which endorses the dichotomy mentioned before.

Figure 56 – Emissions of CO_2 per capita *versus* gross domestic product per capita in 2016 by countries



Source: Our World in Data [92].

The problem consists in finding an equilibrium between losing welfare and reducing emissions. There are lots of works dealing with this subject from the mathematical point of view [71], however here we include the subjectivity in the decision of reducing emissions [65]. Furthermore, we describe the economical gain and losses in terms of fuzzy optimization objects.

A.6.2 Carbon Market as a Constraint

In terms of forecasting, policy makers may impose a country - Germany, for example - to emit at most 760 $MtCO_2$. This target T depends on the region/ sector and the decision-makers must choose how much less than T will be really emitted. This "how much" is uncertain and we model this linguistic term by a triangular fuzzy number $B = (L; E; T)$, where L can be $-\infty$. So we create the constraint on emissions: $x \leqslant B$, where x is a measure of CO_2 emission.

We consider the logarithmic utility function [45] as the objective function in the problem. Our focus is only on the relation emissions *versus* utility, and not in the utility format function. Hence, we consider the utility as a function that only depends on x, the emission. Therefore, the goal is to maximize $f(x) = \log(x)$ subject to $x \leqslant B$. The problem boils down to (A.28).

$$\begin{aligned} &\max\ \log(x) \\ &\text{s. t. }\ x \leqslant (L; E; T), \end{aligned} \tag{A.28}$$

The objective O and constraint Ω sets have respectively, membership functions

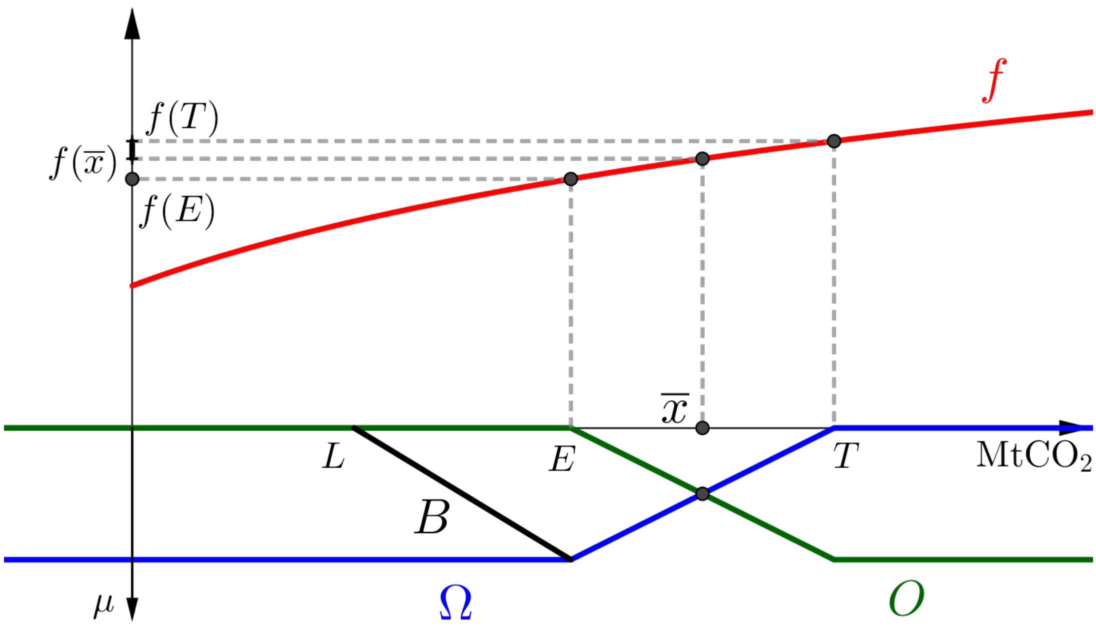
$$O(x) = \begin{cases} 0, \text{ if } x \leqslant E \\ \dfrac{x-E}{T-E}, \text{ if } E < x \leqslant T \\ 1, \text{ if } x > T \end{cases}$$

and

$$\Omega(x) = \begin{cases} 1, \text{ if } x \leqslant E \\ \dfrac{T-x}{T-E}, \text{ if } E < x \leqslant T \\ 0, \text{ if } x > T \end{cases} \quad .$$

The solution, represented in Figure 57, is given by $\max O \cap \Omega$, which corresponds to the point $\overline{x}$.

Figure 57 – Decision of problem (A.28)



The membership function of $B = (L; E; T)$. The utility function f is represented by the red curve. The constraint set Ω is represented by the blue curve. The goal O is represented by the green curve. The decision scheme leads to $\overline{x}$ as a solution for problem (A.28). Source: Author [84].

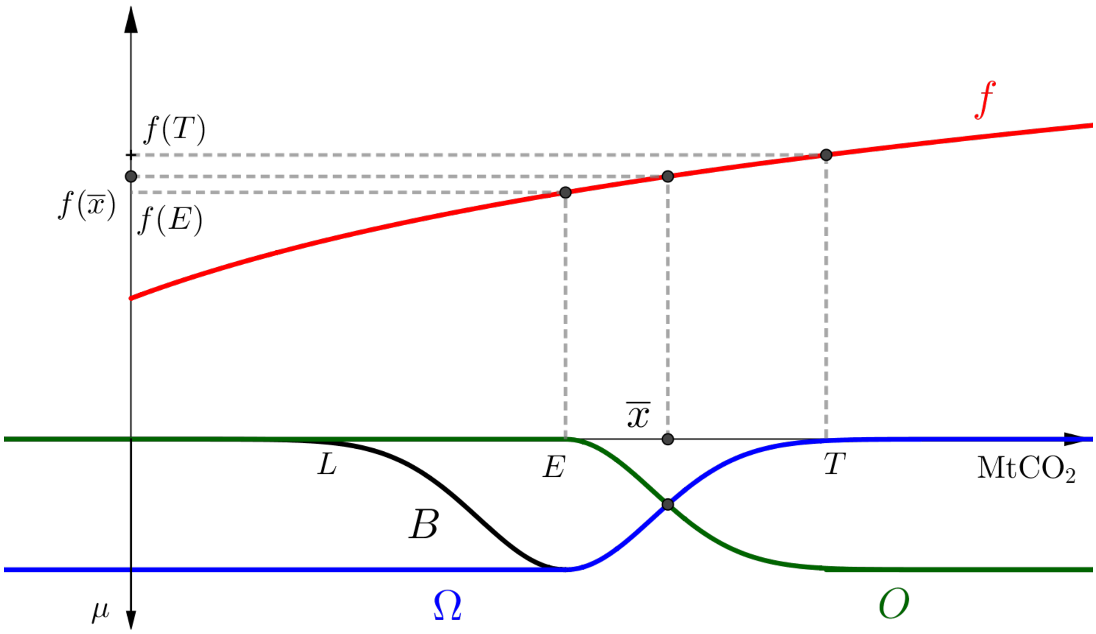
The lack $T - \overline{x}$ is the chosen quantity to be sell in carbon market and $f(T) - f(\overline{x})$ is the lost in utility (objective) function. Even though E would be the best decision in terms of reduction of emissions, $f(E)$ may not be the best decision in terms of utility decision. Depending on the value of $\mathrm{MtCO_2}$ in carbon market, it is desirable to sell more and more tons of carbon, and the lost $f(T) - f(\overline{x})$ causes less in the economy.

By choosing different forms of B, it is possible to put more importance on less emission. For B gaussian fuzzy set in form

$$B(x) = e^{-\frac{(x-E)^2}{\sigma^2}}, \tag{A.29}$$

with $\sigma = \dfrac{T-E}{2}$, for example. The decision scheme is represented in Figure 58.

Figure 58 – Decision of problem (A.29)



The membership function of $B = (L; E; T)$.The utility function f is represented by red curve. The constraint set Ω is represented by the blue curve. The goal O is represented by the green curve. The decision scheme leads to $\overline{x}$ as a solution for problem (A.28) with B given in (A.29). Source: Author [84].

In this case, σ became a parameter and could be any fraction of $T - E$. By choosing smaller σ, B would be narrower to E, modelling a mayor intention to attain E $\mathrm{MtCO_2}$ emission.

Comparing the two cases, the first solution led to more emission, with less gain in carbon market, but with little lost in the utility function. On the other hand, the second led to a solution with less emission and with more gain in carbon market and more lost in the utility function.

Carbon market arouse in a context of attempt to decelerate climate changes and impacts. Here we explained briefly the dynamics of carbon market and the subjective decision between pushing economy towards growing and slowing down the pollution in the whole world. We translated the emission target as a fuzzy constraint, and applied the Bellman-Zadeh scheme in order to find a solution for the problem. We extended the notion of solution by Zadeh's extension principle. By this incipient approach it is possible to estimate numerically the intention to emits less carbon and maintain good levels of welfare state. This solution not only ensures low emissions of CO_2, but also shows how many millions tons of carbon could be sold in carbon market. In terms of plan ahead policies on reduction of carbon emissions the next step is adopt this method to marginal abatement cost curves, the tool used to estimate the tax price on emissions of carbon dioxide.

A.7 Marginal Abatement Cost

One of the strategies to reduce climate change is to charge carbon. It is done by Marginal Abatement Cost Curve (MAC curve). Using the estimation MAC curve, the allowances of carbon emissions are calculated as well as the tax price over emissions above the limits. The carbon taxes are imposed over agents that emit more than the allowed, that is, they are penalties applied in order to try to reduce carbon emissions. Emissions quotas below the limits predicted by MAC curve, called carbon credits, may be sold to he other companies. These organizations, on the other hand, must purchase the credits to avoid carbon taxes.

In MAC study the efforts of entities to emits lower quantities of CO_2 are characterized. The measure is monetary unit per ton of carbon equivalent ($\$/tCO_2e$) and the MAC curve behavior is shown in Figure 59. High carbon emissions are related to low costs – efforts – to reduce emissions, while low carbon emissions implies in high production costs.

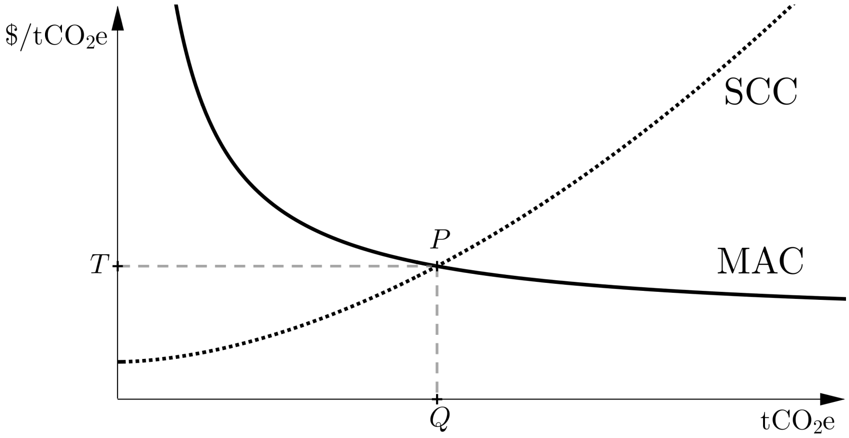
In order to obtain the limit to carbon emissions of an economic sector, the MAC curve is superimposed to the Social Cost Curve (SCC). This last one is the estimation of necessary expenses to compensate for the harmful effects of carbon emission – for the society. SCC is increasing, with increasing derivative, as can be seen in Figure 59.

The equilibrium point is $P = (Q, T)$ and represents the equilibrium between the social cost and the effort applied in reducing carbon emissions. The equilibrium point reveals the allowance quota $Q tCO_2e$ and the carbon price $T \$/tCO_2e$ that will be used as the carbon tax.

The hypothesis here considered is that entities are ambitious and intend to profit from emissions trading system. Therefore first the organization needs to guarantee that accumulated emissions do not overtake the maximum quota Q units of carbon dioxide. Besides that, the company will try to sell as much as possible of carbon credits in the carbon market.

The model is based in minimize the cost function MAC with the intention of

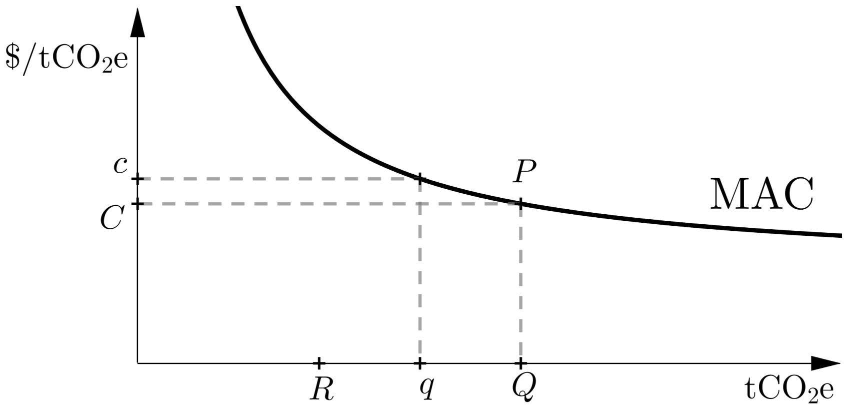
Figure 59 – Marginal Abatement Cost Curve *versus* Social Cost Curve



The solid curve is the marginal abatement cost, the dashed curve is the social cost curve and $P = (Q, T)$ is the equilibrium point. Source: Author [85].

reducing carbon emissions, one more time, *as much as possible*. To this fuzzy intention will the attached a fuzzy constraint. A health middle point q (see Figure 60) will be taken between the ambitious intention R and the maximum quota Q; the quantity $Q - q$ may be sell in carbon market with current price [36]; and the extra cost to reach q will be $\$c - C$. In the fuzzy decision scheme one have to choose the most appropriate shape of constraint in order to obtain the best relative profit $\$(Q - q) - \$(c - C)$.

Figure 60 – Reducing emissions scheme



The solid curve is the marginal abatement cost, Q is the maximum quota at cost C, R is the ambitious intention and the middle q, to be found, at cost c. Source: Author [85].

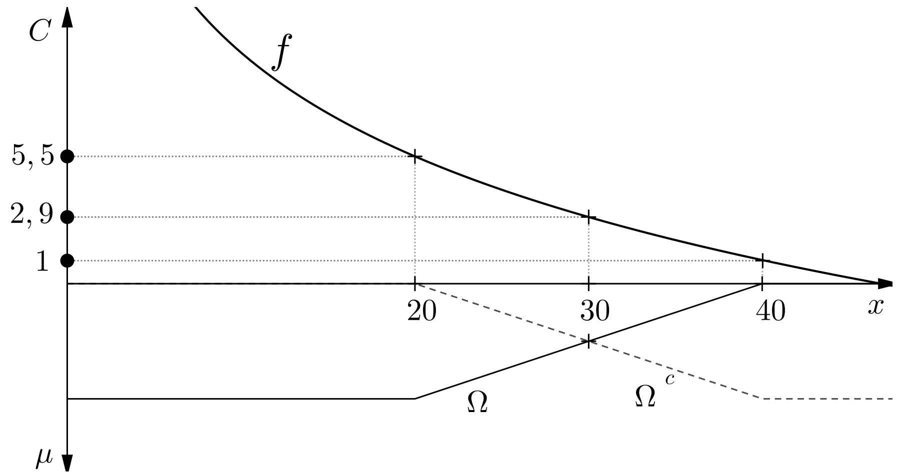
A.7.1 Application

To exemplify the MAC model in fuzzy terms, it will be considered data from BR MALLS PARTICIPAÇÕES S.A., a Brazilian company which have shares on Brazilian stock exchange called Brasil Bolsa Balcão [2]. In 2017 this company emitted approximately 40000 tons of CO_2e, with approximately 1.3 billions of reais.

The maximum quota for the next year will be considered as 40000 tCO_2e. Moreover, the budget earmarked for internal changes with respect to reducing emissions will be considered as 5 millions of reais. The MAC curve for this company will be considered as $f(x) = 25 - 6,5\log(x)$, since at least 20000 tCO_2 will be emitted during the year.

The intention will be described by the fuzzy number $B = (10; 20; 40)$ (in millions tons of dioxide carbon), since the fuzzy set Ω will be as in Figure 61. From these assumptions the optimal point is $\overline{x} = 30000$ and the quantity of 10000 tons of $C0_2$ can be sold at the ETS.

Figure 61 – MAC curve in an application



The MAC curve is f. Axis μ exhibits the membership function of Ω and Ω^c; axis C exhibits the cost (in millions of reais); and the in axis x is the total emissions of CO_2 during the year. Source: Author [85].

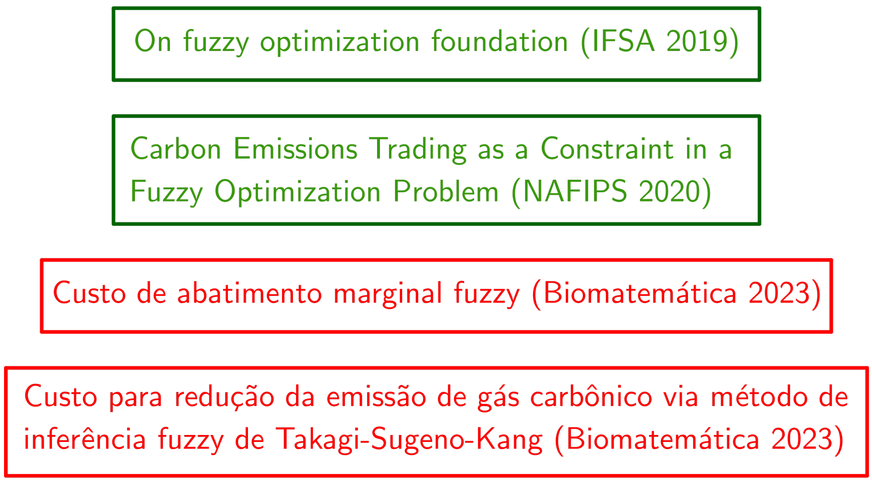
Since in Brazil the emissions trading system has not been implanted yet, one can only suppose values for allowances. One can choose the current value in dollar [36], which is US\$26.75/tCO_2, but in real currency, that is R\$26, 75. From this, the profit in ETS would be R\$267500, whereas the investment made by company was $f(30) - f(40) = 1.9$ millions of reais. In this case the company would have a loss. Nevertheless, the price of each carbon ton will increase more and more during the year, while costs f tends to become flatter over time.

This study can be done jointly with regression models to predict the profit/loss and be an auxiliary study to assist decision makers in planning.

A.8 Conclusion

This appendix built a foundation for optimization in Zimmermann's sense. The constraints in inequality form were rewritten by applying Zadeh's extension principle for the minimum (maximum) function, as well as minimum definition was rewritten using the same process. Moreover, it was given a new interpretation for fuzzy constraints: in order to model intention. It was proposed an application on environment issue, in particular, in climate change policy. In this field the decision to be make is to minimize cost subject to constraints on carbon dioxide emissions. Figure 62 shows the contributions already published. These works are mainly described here, with the exception of the article entitled "Custo para redução da emissão de gás carbônico via método de inferência fuzzy de Takagi-Sugeno-Kang". It used fuzzy inference, which is not included in this text.

Figure 62 – Contributions of the appendix



The contributions written in green represent book chapters, in red represent journal articles.
Source: Author [8, 84, 85, 86].