Theorem: π is irrational.

Proof (after Niven): 1. Suppose f is any polynomial of degree $2n$, so $f^{(2n+1)}(x) \equiv 0$. Repeated integration by parts starting with $u = f(x)$, $dv = \sin x\, dx$ yields, I claim,

$$\begin{aligned} \int f(x)\sin x\, dx &= -f(x)\cos x + f'(x)\sin x + f''(x)\cos x + \ldots \\ &= -F(x)\cos x + F'(x)\sin x \end{aligned}$$

where $F(x) := f(x) - f''(x) + \ldots + (-1)^n f^{(2n)}(x)$. Note

$$F(x) + F''(x) = f(x) + (-1)^n f^{(2n+2)}(x) = f(x),$$

so indeed by a product rule calculation we verify that

$$(-F(x)\cos x + F'(x)\sin x)' = F(x)\sin x + F''(x)\sin x = f(x)\sin x.$$

Then in particular,

$$\int_0^\pi f(x)\sin x\, dx = (-F(x)\cos x + F'(x)\sin x)|_0^\pi = F(\pi) + F(0). \tag{1}$$

If $0 < f(x) < M$ for $0 < x < \pi$, we conclude that

$$0 < F(\pi) + F(0) < \int_0^\pi M \sin x\, dx = 2M. \tag{2}$$

2. Suppose now that $\pi = \frac{a}{b}$ where a, b are positive integers. With n an integer to be determined, let

$$f(x) = \frac{b^n}{n!} x^n (\pi - x)^n = \frac{x^n}{n!}(a - bx)^n. \tag{3}$$

Then f is a polynomial of degree $2n$, and for $0 < x < \pi$, $0 < a - bx < a$ and

$$0 < f(x) < \frac{\pi^n}{n!}(a)^n = \frac{(\pi a)^n}{n!}.$$

We can select n so that $M = \frac{(\pi a)^n}{n!} < \frac{1}{2}$, and do so. Since

$$\frac{d^k}{dx^k}\left(\frac{x^n}{n!}\right) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

and $(a-bx)^n$ is a polynomial with integer coefficients, it is clear from the product rule that

$$f^{(k)}(0) \quad \text{is an integer, for every integer } k \geq 0.$$

Since $f(x) = f(\pi - x)$, $f^{(k)}(\pi) = (-1)^k f^{(k)}(0)$ is an integer also.

Hence $F(\pi) + F(0)$ is an integer. But by part 1,

$$0 < F(\pi) + F(0) < 2M < 1.$$

This is a contradiction. We infer that π is irrational.