

# Taylor Series and Asymptotic Expansions

The importance of power series as a convenient representation, as an approximation tool, as a tool for solving differential equations and so on, is pretty obvious. What may not be so obvious is that power series can be of some use even when they diverge!

Let us start by considering Taylor series. If  $f : [-a, a] \rightarrow \mathbb{R}$  has infinitely many continuous derivatives, Taylor's theorem says that for each  $n \in \mathbb{N} \cup \{0\}$ ,

$$f(x) = P_n(x) + R_n(x)$$

$$\text{with } P_n(x) = \sum_{m=0}^n \frac{1}{m!} f^{(m)}(0) x^m$$

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(x') x^{n+1} \text{ for some } x' \text{ between } 0 \text{ and } x$$

Since  $f^{(n+1)}$  is continuous, there exists a constant  $M_n$  such that  $|R_n(x)| \leq M_n x^{n+1}$  for all  $x \in [-a, a]$ . This says that as  $x$  gets smaller and smaller, the polynomial  $P_n(x)$  becomes a better and better approximation to  $f(x)$ . The rate at which  $P_n(x)$  tends to  $f(x)$  as  $x$  tends to zero is, at worst, proportional to  $x^{n+1}$ . Thus the Taylor series  $\sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(0) x^m$  is an asymptotic expansion for  $f(x)$  where

**Definition.**  $\sum_{m=0}^{\infty} a_m x^m$  is said to be an **asymptotic expansion** for  $f(x)$ , denoted  $f(x) \sim \sum_{m=0}^{\infty} a_m x^m$  if, for each  $n \in \mathbb{N} \cup \{0\}$ ,

$$\lim_{x \rightarrow 0} \frac{1}{x^n} \left[ f(x) - \sum_{m=0}^n a_m x^m \right] = 0$$

There are several important observations to make about this definition.

- (i) The definition says that, for *each fixed*  $n$ ,  $\sum_{m=0}^n a_m x^m$  becomes a better and better approximation to  $f(x)$  as  $x$  gets smaller. As  $x \rightarrow 0$ ,  $\sum_{m=0}^n a_m x^m$  approaches  $f(x)$  faster than  $x^n$  tends to zero.
- (ii) The definition says nothing about what happens as  $n \rightarrow \infty$ . There is no guarantee that for each fixed  $x$ ,  $\sum_{m=0}^n a_m x^m$  tends to  $f(x)$  as  $n \rightarrow \infty$ .
- (iii) In particular, the series  $\sum_{m=0}^{\infty} a_m x^m$  may have radius of convergence zero. In this case,  $\sum_{m=0}^{\infty} a_m x^m$  is just a formal symbol. It does not have any meaning as a series.
- (iv) We have seen that every infinitely differentiable function has an asymptotic expansion, regardless of whether its Taylor series converges or not.

Now back to our Taylor series. There are three possibilities.

- (i) The series  $\sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(0) x^m$  has radius of convergence zero.
- (ii) The series  $\sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(0) x^m$  has radius of convergence  $r > 0$  and  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for all  $-r < x < r$ . In this case the Taylor series converges to  $f(x)$  for all  $x \in (-r, r)$ . Then  $f(x)$  is said to be analytic on  $(-r, r)$ .
- (iii) The series  $\sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(0) x^m$  has radius of convergence  $r > 0$  and the Taylor series converges to something other than  $f(x)$ .

Here is an example of each of these three types of behaviour.

**Example (i)** For this example,  $f(x) = \int_0^{\infty} \frac{e^{-t}}{1+x^2 t} dt$ . Our work on interchanging the order of differentiation and integration yields that  $f(x)$  is infinitely differentiable and that the derivatives are given by

$$\frac{d^n}{dx^n} f(x) = \int_0^{\infty} \frac{\partial^n}{\partial x^n} \left( \frac{e^{-t}}{1+x^2 t} \right) dt$$

Instead of evaluating the derivative directly, we will use a little trick to generate the Taylor series for  $f(x)$ . Recall that

$$\sum_{n=0}^N r^n = \frac{1}{1-r} - \frac{r^{N+1}}{1-r}$$

We just substitute  $r = -x^2t$  to give

$$\frac{1}{1+x^2t} = \sum_{n=0}^N (-x^2t)^n + \frac{(-x^2t)^{N+1}}{1+x^2t}$$

and

$$\begin{aligned} f(x) &= \sum_{n=0}^N \int_0^\infty (-x^2t)^n e^{-t} dt + \int_0^\infty \frac{(-x^2t)^{N+1}}{1+x^2t} e^{-t} dt \\ &= \sum_{n=0}^N (-1)^n x^{2n} \int_0^\infty t^n e^{-t} dt + (-1)^{N+1} x^{2(N+1)} \int_0^\infty \frac{t^{N+1}}{1+x^2t} e^{-t} dt \\ &= \sum_{n=0}^N (-1)^n n! x^{2n} + (-1)^{N+1} x^{2(N+1)} \int_0^\infty \frac{t^{N+1}}{1+x^2t} e^{-t} dt \end{aligned}$$

since<sup>(1)</sup>  $\int_0^\infty t^n e^{-t} dt = n!$ . To compute the  $m^{\text{th}}$  derivative,  $f^{(m)}(0)$ , we choose  $N$  such that  $2(N+1) > m$  and apply

$$\left. \frac{d^m}{dx^m} x^{2n} \right|_{x=0} = \begin{cases} 0 & \text{if } m \neq 2n \\ (2n)! & \text{if } m = 2n \end{cases}$$

Applying  $\frac{d^m}{dx^m}$  with  $m < 2(N+1)$  to  $(-1)^{N+1} x^{2(N+1)} \int_0^\infty \frac{t^{N+1}}{1+x^2t} e^{-t} dt$  gives a finite sum with every term containing a factor  $x^p$  with  $p > 0$ . When  $x$  is set to 0, every such term is zero. So

$$\left. \frac{d^m}{dx^m} f(x) \right|_{x=0} = \begin{cases} 0 & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} \left(\frac{m}{2}\right)! m! & \text{if } m \text{ is even} \end{cases}$$

So the Taylor series expansion of  $f$  is

$$\sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(0) x^m = \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{1}{m!} (-1)^{\frac{m}{2}} \left(\frac{m}{2}\right)! m! x^m \stackrel{m=2n}{=} \sum_{n=0}^{\infty} (-1)^n n! x^{2n}$$

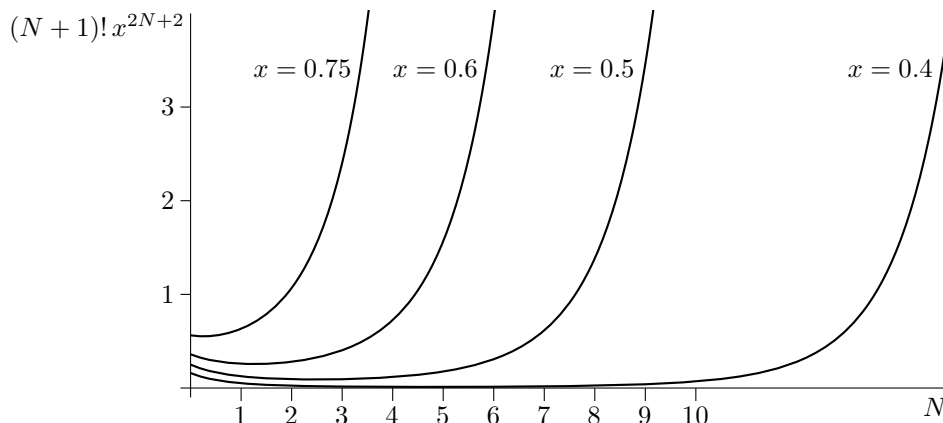
The high growth rate of  $n!$  makes the radius of convergence of this series zero. For example, one formula for the radius of convergence of  $\sum_{n=0}^{\infty} a_n x^n$  is  $\left[ \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \right]^{-1}$ . When  $a_n = (-1)^n n!$ ,  $\left[ \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \right]^{-1} = \left[ \lim_{n \rightarrow \infty} (n+1) \right]^{-1} = 0$ . Replacing  $x$  by  $x^2$  gives that  $\sum_{n=0}^{\infty} (-1)^n n! x^{2n}$  converges if and only if  $x = 0$ . Nonetheless, the error estimates

$$\left| f(x) - \sum_{n=0}^N (-1)^n n! x^{2n} \right| = x^{2(N+1)} \int_0^\infty \frac{t^{N+1}}{1+x^2t} e^{-t} dt \leq x^{2(N+1)} \int_0^\infty t^{N+1} e^{-t} dt = (N+1)! x^{2N+2}$$

shows that if we fix  $N$  and make  $x$  smaller and smaller,  $\sum_{n=0}^N (-1)^n n! x^{2n}$  becomes a better and better approximation to  $f(x)$ . On the other hand, if we fix  $x$  and let  $N$  grow, we see that the error grows dramatically for large  $N$ . Here is a graph of  $y = (N+1)! x^{2N+2}$  against  $N$ , for several different (fixed) values of  $x$ .

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<sup>(1)</sup> Denote  $\gamma_n = \int_0^\infty t^n e^{-t} dt$ . Then  $\gamma_0 = \int_0^\infty e^{-t} dt = 1$  and, by integration by parts with  $u = t^n$ ,  $dv = e^{-t} dt$ ,  $du = nt^{n-1} dt$  and  $v = -e^{-t}$ ,  $\gamma_n = n\gamma_{n-1}$  for all  $n \in \mathbb{N}$ . So, by induction,  $\gamma_n = n!$ .



**Example (ii)** For this example,  $f(x) = e^x$ . By Taylor's theorem,

$$f(x) = \sum_{m=0}^n \frac{x^m}{m!} + R_n(x)$$

with  $R_n(x) = \frac{e^{x'}}{(n+1)!}x^{n+1}$  for some  $x'$  between 0 and  $x$ . In this case,  $\sum_{m=0}^{\infty} \frac{x^m}{m!}$  has radius of convergence infinity and the error  $|R_n(x)| \leq \frac{1}{n!}e^{|x|}|x|^{n+1}$  tends to zero *both*

- (a) when  $n$  is fixed and  $x$  tends to zero and
- (b) when  $x$  is fixed and  $n$  tends to infinity. Use  $\bar{R}_n(x) = \frac{1}{n!}e^{|x|}|x|^{n+1}$  to denote our bound on the error introduced by truncating the series at  $\frac{x^n}{n!}$ . Then  $\bar{R}_{n+1}(x) = \frac{|x|}{n+1}\bar{R}_n(x)$ . So if, for example,  $n \geq 2|x|$ , then increasing the index  $n$  of  $\bar{R}_n(x)$  by one reduces the value of  $\bar{R}_n(x)$  by a factor of at least two.

**Example (iii)** For this example,

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

I shall show below that  $f$  is infinitely differentiable and  $f^{(n)}(0) = 0$  for every  $n \in \mathbb{N} \cup \{0\}$ . So in this case, the Taylor series is  $\sum_{n=0}^{\infty} 0x^n$  and has infinite radius of convergence. When we approximate  $f(x)$  by the Taylor polynomial,  $\sum_{m=0}^n 0x^m$ , of degree  $n$ , the error,  $R_n(x) = f(x)$ , is independent of  $n$ . The function  $f(x)$  agrees with its Taylor series only for  $x = 0$ . Similarly, the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

agrees with its Taylor series if and only if  $x \leq 0$ .

Here is the argument that  $f$  is infinitely differentiable and  $f^{(n)}(0) = 0$  for every  $n \in \mathbb{N} \cup \{0\}$ . For any  $x \neq 0$  and any integer  $n \geq 0$

$$e^{1/x^2} = 1 + \frac{1}{x^2} + \frac{1}{2!}\left(\frac{1}{x^2}\right)^2 + \cdots + \frac{1}{n!}\left(\frac{1}{x^2}\right)^n + \cdots \geq \frac{1}{n!}\left(\frac{1}{x^2}\right)^n$$

Consequently, for any polynomial  $P(y)$ , say of degree  $p$ , there is a constant  $C$  such that

$$\left|P\left(\frac{1}{x}\right)e^{-1/x^2}\right| = \left|\frac{P\left(\frac{1}{x}\right)}{e^{1/x^2}}\right| \leq \frac{C\frac{1}{|x|^p}}{\frac{1}{(p+1)!}\frac{1}{x^{2(p+1)}}} \leq (p+1)!C x^{2+p}$$

for all  $|x| \leq 1$ . Hence, for any polynomial  $P(y)$ ,

$$\lim_{x \rightarrow 0} P\left(\frac{1}{x}\right) e^{-1/x^2} = 0. \quad (1)$$

Applying (1) with  $P(y) = y$ , we have

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{1}{x} e^{-1/x^2} = 0 \implies f'(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{2}{x^3} e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

Applying (1) with  $P(y) = 2y^4$ , we have

$$f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \left( \frac{2}{x^3} \right) e^{-1/x^2} \implies f''(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left( -\frac{6}{x^4} + \frac{4}{x^6} \right) e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

Indeed, for any  $n \geq 1$ , if  $f^{(n-1)}(x) = P_{n-1}\left(\frac{1}{x}\right) e^{-1/x^2}$  for all  $x > 0$ , then applying (1) with  $P(y) = yP_{n-1}(y)$  gives

$$f^{(n)}(0) = \lim_{x \rightarrow 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x} = \lim_{x \rightarrow 0} \frac{1}{x} P_{n-1}\left(\frac{1}{x}\right) e^{-1/x^2} = 0$$

and hence

$$f^{(n)}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left[ -\frac{1}{x^2} P'_{n-1}\left(\frac{1}{x}\right) + \frac{2}{x^3} P_{n-1}\left(\frac{1}{x}\right) \right] e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

**Example (iv)** Stirling's formula  $\ln n! \approx (n + \frac{1}{2}) \ln n - n + \ln \sqrt{2\pi}$  actually consists of the first few terms of an asymptotic expansion

$$\ln n! \sim (n + \frac{1}{2}) \ln n - n + \ln \sqrt{2\pi} + \frac{1}{12} \frac{1}{n} - \frac{1}{360} \frac{1}{n^3} + \dots$$

If  $n > 10$ , the approximation  $\ln n! \approx (n + \frac{1}{2}) \ln n - n + \ln \sqrt{2\pi}$  is accurate to within 0.06% and the exponentiated form

$$n! \approx n^{n+\frac{1}{2}} \sqrt{2\pi} e^{-n+\frac{1}{12n}}$$

is accurate to one part in 300,000. But fixing  $n$  and taking many more terms in the expansion will in fact give you a much worse approximation to  $\ln n!$ .

## Laplace's Method

Stirling's formula may be generated by Laplace's method – one of the more common ways of generating asymptotic expansions. Laplace's method provides good approximations to integrals of the form  $I(s) = \int e^{-sf(x)} dx$  when  $s$  is very large and the function  $f$  takes its minimum value at a unique point,  $x_m$ . The method is based on the following three observations.

(i) When  $s$  is very large and  $x \neq x_m$ ,  $e^{-sf(x)} = e^{-sf(x_m)} e^{-s[f(x)-f(x_m)]} \ll e^{-sf(x_m)}$ , since  $f(x) - f(x_m) > 0$ .

So, for large  $s$ , the integral will be dominated by values of  $x$  near  $x_m$ .

(ii) For  $x$  near  $x_m$

$$f(x) \approx f(x_m) + \frac{1}{2} f''(x_m) (x - x_m)^2$$

since  $f'(x_m) = 0$ .

(iii) Integrals of the form  $\int_{-\infty}^{\infty} e^{-a(x-x_m)^2} dx$ , or more generally  $\int_{-\infty}^{\infty} e^{-a(x-x_m)^2} (x - x_m)^n dx$ , can be computed exactly.

Let's implement this strategy for the  $\Gamma$  function, defined in the lemma below. This will give Stirling's formula, because of part (d) of the lemma.

**Lemma.** Define, for  $s > 0$ , the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

(a)  $\Gamma(1) = 1$

(b)  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ .

(c) For any  $s > 0$ ,  $\Gamma(s+1) = s\Gamma(s)$ .

(d) For any  $n \in \mathbb{N}$ ,  $\Gamma(n+1) = n!$ .

**Proof:** (a) is a trivial integration.

(b) Make the change of variables  $t = x^2$ ,  $dt = 2x dx = 2\sqrt{t} dx$ . This gives

$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = 2 \int_0^\infty e^{-x^2} dx = \int_{-\infty}^\infty e^{-x^2} dx$$

We compute the square of this integral by switching

$$\Gamma(\frac{1}{2})^2 = \left( \int_{-\infty}^\infty e^{-x^2} dx \right) \left( \int_{-\infty}^\infty e^{-y^2} dy \right) = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy$$

to polar coordinates. This standard trick gives

$$\Gamma(\frac{1}{2})^2 = \int_0^{2\pi} d\theta \int_0^\infty dr r e^{-r^2} = \int_0^{2\pi} d\theta \left[ -\frac{1}{2} e^{-r^2} \right]_0^\infty = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

(c) By integration by parts with  $u = t^s$ ,  $dv = e^{-t} dt$ ,  $du = s t^{s-1} dt$  and  $v = -e^{-t}$

$$\Gamma(s+1) = \int_0^\infty t^s e^{-t} dt = t^s (-e^{-t}) \Big|_0^\infty - \int_0^\infty (-e^{-t}) s t^{s-1} dt = s \int_0^\infty t^{s-1} e^{-t} dt = s\Gamma(s)$$

(d) is obvious by induction, using parts (a) and (c). ■

To derive Stirling's formula, using Laplace's method, We first manipulate  $\Gamma(s+1)$  into the form  $\int e^{-sf(x)} dx$ .

$$\begin{aligned} \Gamma(s+1) &= \int_0^\infty t^s e^{-t} dt \\ &= s^{s+1} \int_0^\infty x^s e^{-xs} dx \quad \text{where } t = xs \\ &= s^{s+1} e^{-s} \int_0^\infty e^{-s(x - \ln x - 1)} dx \end{aligned}$$

So  $f(x) = x - \ln x - 1$  and  $f'(x) = 1 - \frac{1}{x}$ . Consequently  $f'(x) < 0$  for  $x < 1$  and  $f'(x) > 0$  for  $x > 1$ , so that  $f$  has a unique minimum at  $x = 1$  and increases monotonically as  $x$  gets farther from 1. The minimum value of  $f$  is  $f(1) = 0$ . (We multiplied and divided by  $e^{-s}$  in the last line above in order to arrange that the minimum value of  $f$  be exactly zero.)

If  $\varepsilon > 0$  is any fixed positive number,

$$\int_0^\infty e^{-sf(x)} dx \approx \int_{1-\varepsilon}^{1+\varepsilon} e^{-sf(x)} dx$$

will be a good approximation for any sufficiently large  $s$ , because, when  $x \neq 1$ ,  $f(x) > 0$  and  $\lim_{s \rightarrow \infty} e^{-sf(x)} = 0$ . We will derive, in an appendix, explicit bounds on the error introduced by this approximation as well as by the other approximations we make in the course of this development.

The remaining couple of steps in the procedure are motivated by the fact that integrals of the form  $\int_{-\infty}^\infty e^{-a(x-x_m)^2} (x-x_m)^n dx$  can be computed exactly. If we have chosen  $\varepsilon$  pretty small, Taylor's formula

$$\begin{aligned} f(x) &\approx f(1) + f'(1)(x-1) + \frac{1}{2}f^{(2)}(1)(x-1)^2 + \frac{1}{3!}f^{(3)}(1)(x-1)^3 + \frac{1}{4!}f^{(4)}(1)(x-1)^4 \\ &= \frac{1}{2} \left[ \frac{1}{x^2} \right]_{x=1} (x-1)^2 + \frac{1}{3!} \left[ -\frac{2}{x^3} \right]_{x=1} (x-1)^3 + \frac{1}{4!} \left[ \frac{2 \times 3}{x^4} \right]_{x=1} (x-1)^4 \\ &= \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{1}{4}(x-1)^4 \end{aligned}$$

will give a good approximation for all  $|x-1| < \varepsilon$ , so that

$$\int_0^\infty e^{-sf(x)} dx \approx \int_{1-\varepsilon}^{1+\varepsilon} e^{-s[\frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{1}{4}(x-1)^4]} dx = \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{s}{2}(x-1)^2} e^{[\frac{s}{3}(x-1)^3 - \frac{s}{4}(x-1)^4]} dx$$

(We shall find the first two terms an expansion for  $\Gamma(s+1)$ . If we wanted to find more terms, we would keep more terms in the Taylor expansion of  $f(x)$ .)

The next step is to Taylor expand  $e^{[\frac{s}{3}(x-1)^3 - \frac{s}{4}(x-1)^4]}$  about  $x = 1$ , in order to give integrands of the form  $e^{-\frac{s}{2}(x-1)^2} (x-1)^n$ . We could apply Taylor's formula directly, but it is easier to use the known expansion of the exponential.

$$\begin{aligned} \int_0^\infty e^{-sf(x)} dx &\approx \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{s}{2}(x-1)^2} \left\{ 1 + \left[ \frac{s}{3}(x-1)^3 - \frac{s}{4}(x-1)^4 \right] + \frac{1}{2} \left[ \frac{s}{3}(x-1)^3 - \frac{s}{4}(x-1)^4 \right]^2 \right\} dx \\ &\approx \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{s}{2}(x-1)^2} \left\{ 1 + \frac{s}{3}(x-1)^3 - \frac{s}{4}(x-1)^4 + \frac{s^2}{18}(x-1)^6 \right\} dx \\ &\approx \int_{-\infty}^\infty e^{-\frac{s}{2}(x-1)^2} \left\{ 1 + \frac{s}{3}(x-1)^3 - \frac{s}{4}(x-1)^4 + \frac{s^2}{18}(x-1)^6 \right\} dx \\ &= \int_{-\infty}^\infty e^{-\frac{s}{2}(x-1)^2} \left\{ 1 - \frac{s}{4}(x-1)^4 + \frac{s^2}{18}(x-1)^6 \right\} dx \end{aligned}$$

since  $(x-1)^3 e^{-\frac{s}{2}(x-1)^2}$  is odd about  $x = 1$  and hence integrates to zero. For  $n$  an even natural number

$$\begin{aligned} \int_{-\infty}^\infty e^{-\frac{s}{2}(x-1)^2} (x-1)^n dx &= \left(\frac{s}{2}\right)^{-\frac{n+1}{2}} \int_{-\infty}^\infty e^{-y^2} y^n dy \quad \text{where } y = \sqrt{\frac{s}{2}} (x-1) \\ &= 2 \left(\frac{s}{2}\right)^{-\frac{n+1}{2}} \int_0^\infty e^{-y^2} y^n dy \\ &= \left(\frac{s}{2}\right)^{-\frac{n+1}{2}} \int_0^\infty e^{-t} t^{\frac{n}{2}-\frac{1}{2}} dt \quad \text{where } t = y^2, \quad dt = 2y dy = 2\sqrt{t} dy \\ &= \left(\frac{s}{2}\right)^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \end{aligned} \tag{2}$$

By the lemma at the beginning of this section,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{1}{2}\sqrt{\pi} \quad \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

so that

$$\begin{aligned}\int_0^\infty e^{-s(x-\ln x-1)} dx &\approx \left(\frac{s}{2}\right)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) - \frac{s}{4}\left(\frac{s}{2}\right)^{-\frac{5}{2}}\Gamma\left(\frac{5}{2}\right) + \frac{s^2}{18}\left(\frac{s}{2}\right)^{-\frac{7}{2}}\Gamma\left(\frac{7}{2}\right) + \cdots \\ &= s^{-\frac{1}{2}}\sqrt{2\pi}\left(1 - \frac{1}{s}\frac{1}{4}4\frac{3}{2}\frac{1}{2} + \frac{1}{s}\frac{1}{18}8\frac{5}{2}\frac{3}{2}\frac{1}{2} + \cdots\right) \\ &= s^{-\frac{1}{2}}\sqrt{2\pi}\left(1 - \frac{3}{4s} + \frac{5}{6s} + \cdots\right)\end{aligned}$$

and

$$\Gamma(s+1) = s^{s+\frac{1}{2}}e^{-s}\sqrt{2\pi}\left(1 + \frac{1}{12s} + \cdots\right)$$

So we have the first two terms in the asymptotic expansion for  $\Gamma$ . We can obviously get as many as we like by taking more and more terms in the Taylor expansion for  $f(x)$  and  $e^y$ . So far, we have not attempted to keep track of the errors introduced by the various approximations. It is not hard to do so, but it is messy. So we have relegated the error estimates to the appendix.

## “Summability”

We have now seen one way in which divergent expansions may be of some use. There is another possibility. Even if we cannot take the sum of the series  $\sum_{n=0}^\infty a_n$  in the conventional sense, there may be some more general sense in which the series is “summable”. We will just briefly take a look at two such generalizations. Suppose that  $\sum_{n=0}^\infty a_n$  is a (possibly divergent) series. Let

$$S_N = \sum_{n=0}^N a_n$$

denote the series’ partial sum. Then the series is said to converge in the sense of *Cesàro* if the limit

$$S_c = \lim_{N \rightarrow \infty} \frac{S_0 + S_1 + \cdots + S_N}{N+1}$$

exists. That is, if the limit of the average of the partial sums exists. The series is said to converge in the sense of *Borel* if the power series

$$g(t) = \sum_{n=0}^\infty \frac{a_n}{n!} t^n$$

has radius of convergence  $\infty$  and the integral

$$S_B = \int_0^\infty e^{-t} g(t) dt$$

converges. As the following theorem shows, these two definitions each extends our usual definition of series convergence.

**Theorem.** *If the series  $\sum_{n=0}^\infty a_n$  converges, then  $S_C$  and  $S_B$  both exist and*

$$S_C = S_B = \sum_{n=0}^\infty a_n$$

**Proof** that  $S_C = \sum_{n=0}^\infty a_n$ . Write  $S = \sum_{n=0}^\infty a_n$  and  $\sigma_N = \frac{S_0 + S_1 + \cdots + S_N}{N+1}$  and observe that

$$S - \sigma_N = \frac{(S - S_0) + (S - S_1) + \cdots + (S - S_N)}{N+1}$$

Pick any  $\varepsilon > 0$ . Since  $S = \lim_{n \rightarrow \infty} S_n$  there is an  $M$  such that  $|S - S_m| < \frac{\varepsilon}{2}$  for all  $m \geq M$ . Also since  $S = \lim_{n \rightarrow \infty} S_n$ , the sequence  $\{S_n\}$  is bounded. That is, there is a  $K$  such that  $|S_n| \leq K$  for all  $n \geq 0$ . So if  $N > \max\{\frac{2MK}{\varepsilon/2}, M\}$

$$\begin{aligned} |S - \sigma_N| &\leq \frac{|S - S_0| + |S - S_1| + \cdots + |S - S_N|}{N+1} \\ &= \frac{|S - S_0| + |S - S_1| + \cdots + |S - S_{M-1}|}{N+1} + \frac{|S - S_M| + |S - S_{M+1}| + \cdots + |S - S_N|}{N+1} \\ &< \frac{2K + 2K + \cdots + 2K}{N+1} + \frac{\frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2}}{N+1} \\ &\leq \frac{2MK}{N+1} + \frac{N-M+1}{N+1} \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence  $\lim_{N \rightarrow \infty} \sigma_N = S$  and  $S_C = S$ .

“**Proof**” that  $S_B = \sum_{n=0}^{\infty} a_n$ . Write

$$S = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{a_n}{n!} n! = \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^{\infty} e^{-t} t^n dt = \int_0^{\infty} \overbrace{\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n}^{g(t)} e^{-t} dt$$

The last step, namely the exchange  $\sum_{n=0}^{\infty} \int_0^{\infty} = \int_0^{\infty} \sum_{n=0}^{\infty}$  can be justified rigorously. If the radius of convergence of the series  $\sum_{n=0}^{\infty} a_n \alpha^n$  is strictly greater than one, the justification is not very hard. But in general it is too involved to give here. That’s why “Proof” is in quotation marks. ■

The converse of this theorem is false of course. Consider, for example,  $\sum_{n=0}^{\infty} (-r)^n$  with  $r \neq -1$ . Then, as  $n \rightarrow \infty$ ,

$$S_n = \sum_{m=0}^n (-r)^m = \frac{1 - (-r)^{n+1}}{1+r}$$

converges to  $\frac{1}{1+r}$  if  $|r| < 1$  and diverges if  $|r| \geq 1$ . But

$$\begin{aligned} \sigma_N &= \frac{1}{N+1} \sum_{n=0}^N S_n = \frac{1}{N+1} \frac{1}{1+r} \left[ \sum_{n=0}^N 1 - \sum_{n=0}^N (-r)^{n+1} \right] = \frac{1}{N+1} \frac{1}{1+r} \left[ N+1 - \frac{(-r) - (-r)^{N+2}}{1+r} \right] \\ &= \frac{1}{1+r} + \frac{1}{N+1} \frac{r}{(r+1)^2} + \frac{1}{N+1} \frac{(-r)^{N+2}}{(r+1)^2} \end{aligned}$$

As  $N \rightarrow \infty$ , this converges to  $\frac{1}{1+r}$  if  $|r| \leq 1$  and diverges if  $|r| > 1$ . And

$$g(t) = \sum_{n=0}^{\infty} \frac{(-r)^n}{n!} t^n = e^{-rt} \implies \int_0^{\infty} e^{-t} g(t) dt = \int_0^{\infty} e^{-(1+r)t} dt = \frac{1}{1+r}$$

if  $r > -1$  and diverges otherwise. Collecting together these three results,

$$\sum_{n=0}^{\infty} (-r)^n = \frac{1}{1+r} \begin{cases} \text{in the conventional sense if } r \in (-1, 1) \\ \text{in the sense of Cesàro if } r \in (-1, 1] \\ \text{in the sense of Borel if } r \in (-1, \infty) \end{cases}$$



By way of conclusion, let me remark that these generalized modes of summation are not just mathematician's toys. The Fourier series of any continuous function converges in the sense of Cesàro. To get conventional convergence, one needs a certain amount of smoothness in addition of continuity. Also the calculation of the Lamb shift of the hydrogen atom spectrum, which provided (in addition to a Nobel prize) spectacular numerical agreement with one of the most accurate measurements in science, consisted of adding up the first few terms of a series. The rigorous status of this series is not known. But a number of similar series (arising in other quantum field theories) are known to diverge in the conventional sense but to converge in the sense of Borel.

## Appendix - Error Terms for Stirling's Formula

In this appendix, we will treat the error terms arising in our development of Stirling's formula. A precise statement of that development is

*Step 1.* Recall that  $f(x) = x - \ln x - 1$ . Write

$$\int_0^\infty e^{-sf(x)} dx = \int_{1-\varepsilon}^{1+\varepsilon} e^{-sf(x)} dx + \int_0^{1-\varepsilon} e^{-sf(x)} dx + \int_{1+\varepsilon}^\infty e^{-sf(x)} dx$$

and discard the last two terms.

*Step 2.* Write

$$\int_{1-\varepsilon}^{1+\varepsilon} e^{-sf(x)} dx = \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{1}{2}s(x-1)^2} e^{-s[f(x)-\frac{1}{2}(x-1)^2]} dx$$

Apply  $e^t = 1 + t + \frac{1}{2}t^2 + e^c \frac{1}{3!}t^3$  with  $t = -s[f(x) - \frac{1}{2}(x-1)^2]$  to give

$$\begin{aligned} e^{-s[f(x)-\frac{1}{2}(x-1)^2]} &= 1 - s[f(x) - \frac{1}{2}(x-1)^2] + \frac{1}{2}s^2[f(x) - \frac{1}{2}(x-1)^2]^2 \\ &\quad - e^c \frac{1}{3!}s^3[f(x) - \frac{1}{2}(x-1)^2]^3 \end{aligned}$$

where  $c$  is between 0 and  $-s[f(x) - \frac{1}{2}(x-1)^2]$ . Discard the last term.

*Step 3.* Taylor expand

$$f(x) - \frac{1}{2}(x-1)^2 = -\frac{1}{3}(x-1)^3 + \frac{1}{4}(x-1)^4 - \frac{1}{5}(x-1)^5 + \frac{f^{(6)}(c')}{6!}(x-1)^6$$

for some  $c' \in [1-\varepsilon, 1+\varepsilon]$ , substitute this into  $1 - s[f(x) - \frac{1}{2}(x-1)^2] + \frac{1}{2}s^2[f(x) - \frac{1}{2}(x-1)^2]^2$  and discard any terms that contain  $f^{(6)}(c')$  and any terms of the form  $s^m(x-1)^n$  with  $m - \frac{n+1}{2} \leq -\frac{5}{2}$ . (You'll see the reason for this condition shortly.) This leaves

$$\int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{1}{2}s(x-1)^2} \left\{ 1 - \frac{s}{4}(x-1)^4 + \frac{s^2}{18}(x-1)^6 \right\} dx$$

because the terms with  $n$  odd integrate to zero.

*Step 4.* Replace  $\int_{1-\varepsilon}^{1+\varepsilon}$  by  $\int_{-\infty}^\infty$

The errors introduced in each step are as follows:

*Step 1.*  $\int_0^{1-\varepsilon} e^{-sf(x)} dx$  and  $\int_{1+\varepsilon}^\infty e^{-sf(x)} dx$

*Step 2.*  $\int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{1}{2}s(x-1)^2} e^c \frac{1}{3!}s^3[f(x) - \frac{1}{2}(x-1)^2]^3 dx$

*Step 3.* A finite number of terms, each of the form a numerical constant times an integral  $\int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{1}{2}s(x-1)^2} s^m(x-1)^n dx$  with  $m - \frac{n+1}{2} \leq -\frac{5}{2}$ . Here we have used that, since  $c' \in [1-\varepsilon, 1+\varepsilon]$ , there is a constant  $\bar{M}$  such that  $|f^{(6)}(c')| \leq \bar{M}$ .

*Step 4.* A finite number of terms, each of the form a numerical constant times an integral  $\int_{|x-1| \geq \varepsilon} e^{-\frac{1}{2}s(x-1)^2} s^m (x-1)^n dx$ .

To treat these error terms, we use the following information about  $f(x) = x - \ln x - 1$ .

(F1)  $f$  increases monotonically as  $|x - 1|$  increases.

(F2) We have fixed some small  $\varepsilon > 0$ . There is a  $\delta > 0$  such that  $f(x) \geq \delta(x - 1)$  for all  $x \geq 1 + \varepsilon$ .

To see that this is the case, we first observe that

$$\frac{d}{dx}(f(x) - \delta(x - 1)) = 1 - \delta - \frac{1}{x} \geq 0$$

if  $x \geq \frac{1}{1-\delta}$ . So it suffices to pick  $\delta$  sufficiently small that  $1 + \varepsilon \geq \frac{1}{1-\delta}$  and that  $f(1 + \varepsilon) - \delta\varepsilon \geq 0$ .

The latter is always possible just because  $f(1 + \varepsilon) > 0$ .

(F3) For each  $x \in [1 - \varepsilon, 1 + \varepsilon]$  there is a  $c'' \in [1 - \varepsilon, 1 + \varepsilon]$  such that

$$f(x) - \frac{1}{2}(x - 1)^2 = \frac{1}{3!}f^{(3)}(c'')(x - 1)^3$$

Hence, there is a constant  $M$  such that

$$|f(x) - \frac{1}{2}(x - 1)^2| \leq M|x - 1|^3$$

for all  $x \in [1 - \varepsilon, 1 + \varepsilon]$ .

(F4) If  $x \in [1 - \varepsilon, 1 + \varepsilon]$  with  $\varepsilon$  small enough that  $\varepsilon M \leq \frac{1}{4}$ ,

$$|f(x) - \frac{1}{2}(x - 1)^2| \leq M|x - 1|^3 \leq \frac{1}{4}(x - 1)^2$$

We can now get upper bounds on each of the error terms. For step 1, we have

$$\int_0^{1-\varepsilon} e^{-sf(x)} dx \stackrel{\text{F1}}{\leq} e^{-sf(1-\varepsilon)} \int_0^{1-\varepsilon} dx \leq e^{-sf(1-\varepsilon)}$$

and

$$\int_{1+\varepsilon}^{\infty} e^{-sf(x)} dx \stackrel{\text{F2}}{\leq} \int_{1+\varepsilon}^{\infty} e^{-s\delta(x-1)} dx = \frac{1}{\delta s} e^{-\delta\varepsilon s}$$

For step 2, we have

$$\begin{aligned} \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{1}{2}s(x-1)^2} e^c \frac{1}{3!} s^3 |f(x) - \frac{1}{2}(x-1)^2|^3 dx &\stackrel{\text{F4}}{\leq} \frac{1}{3!} s^3 \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{s}{2}(x-1)^2} e^{\frac{s}{4}(x-1)^2} |f(x) - \frac{1}{2}(x-1)^2|^3 dx \\ &\stackrel{\text{F3}}{\leq} \frac{1}{3!} M^3 s^3 \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{s}{4}(x-1)^2} |x-1|^9 dx \\ &\leq \frac{1}{3!} M^3 s^3 \int_{-\infty}^{\infty} e^{-\frac{s}{4}(x-1)^2} |x-1|^9 dx \\ &= \frac{1}{3!} M^3 s^3 \left(\frac{s}{4}\right)^{-5} \Gamma(5) = M' s^{-2} \quad \text{as in (2)} \end{aligned}$$

For step 3, we have

$$\begin{aligned} \int_{1-\varepsilon}^{1+\varepsilon} e^{-\frac{s}{2}(x-1)^2} s^m |x-1|^n dx &\leq \int_{-\infty}^{\infty} e^{-\frac{s}{2}(x-1)^2} s^m |x-1|^n dx \\ &= s^m \left(\frac{s}{2}\right)^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \quad \text{as in (2)} \\ &\leq M'' s^{-\frac{5}{2}} \end{aligned}$$

if  $m - \frac{n+1}{2} \leq -\frac{5}{2}$ . For step 4, we have

$$\begin{aligned}
\int_{|x-1| \geq \varepsilon} e^{-\frac{s}{2}(x-1)^2} s^m |x-1|^n dx &= \int_{|x-1| \geq \varepsilon} e^{-\frac{s}{4}(x-1)^2} e^{-\frac{s}{4}(x-1)^2} s^m |x-1|^n dx \\
&\leq e^{-\frac{1}{4}\varepsilon^2 s} \int_{|x-1| \geq \varepsilon} e^{-\frac{s}{4}(x-1)^2} s^m |x-1|^n dx \\
&\leq e^{-\frac{1}{4}\varepsilon^2 s} \int_{-\infty}^{\infty} e^{-\frac{s}{4}(x-1)^2} s^m |x-1|^n dx \\
&= e^{-\frac{1}{4}\varepsilon^2 s} s^m \left(\frac{s}{4}\right)^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \quad \text{as in (2)} \\
&= M''' s^{m-\frac{n+1}{2}} e^{-\frac{1}{4}\varepsilon^2 s}
\end{aligned}$$

As promised these error terms all decay faster than  $\sqrt{2\pi}s^{-\frac{1}{2}}(1 + \frac{1}{12s})$  as  $s \rightarrow \infty$ . In particular, error terms introduced in Steps 1 and 4, by ignoring contributions for  $|x-1| \geq \varepsilon$ , decay exponentially as  $s \rightarrow \infty$ . Error terms introduced in Steps 2 and 3, by dropping part of the Taylor series approximation, decay like  $s^{-p}$  for some  $p$ , depending on how much of the Taylor series we discarded. In particular, error terms introduced in Step 3 decay like  $s^{-\frac{5}{2}}$ , because of the condition  $m - \frac{n+1}{2} \leq -\frac{5}{2}$ .