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Accuracy of Euler's γ expansion without logarithm

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Abstract. The rational approximations for the Euler–Mascheroni constant γ using the Euler asymptotic expansion are obtained. The errors of approximations are evaluated. The numerical computations reflect the accuracy of the approximations.

Keywords: Euler–Mascheroni constant, asymptotic expansion, rational approximations.

The Euler–Mascheroni constant $\gamma = 0.5772156\ldots$ is the third (after π and e) of the most important mathematical constants. It appears in analysis, number theory, probability and statistics. This constant is also the most puzzling one: even its irrationality has not been proved yet.

As a rule γ is defined as the limit

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right). \tag{1}$$

Only a few years ago it was noted [3, 4, 2] that the logarithm in (1) can be eliminated.

Indeed, denoting $H_n = \sum_{k=1}^{n} \frac{1}{k}$, relation (1) is

$$H_n - \log n \to \gamma. \tag{2}$$

This relation holds also with n^2:

$$H_{n^2} - \log n^2 \to \gamma. \tag{3}$$

Multiplying relation (2) by 2 and relation (3) by -1, and then adding the obtained ones we have

$$2H_n - H_{n^2} \to \gamma,$$

or

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{n^2}\right). \tag{4}$$

Thus, the Euler–Mascheroni constant γ can be defined by (4) which does not contain the logarithm.

The logarithm elimination procedure can be applied to other formulas for γ (see [3, 2]). We consider here the asymptotic expansion [1]

$$\begin{aligned}\gamma \sim 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4}\\ + \frac{1}{252n^6} - \frac{1}{240n^8} + \frac{1}{132n^{10}} - \frac{691}{32760n^{12}} + \frac{1}{12n^{14}} - \cdots.\end{aligned} \tag{5}$$

This series diverges but it has the Leibniz series property: the error made in replacing γ by a partial sum has the same sign as the next term of the expansion and its absolute value is less than this term. For example,

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{\theta_1}{120n^4}, \tag{6}$$

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{\theta_2}{252n^6}. \tag{7}$$

Hereinafter $0 < \theta_i < 1$ and each θ_i depends on n.

To eliminate the logarithm in (5) replace n by n^2:

$$\begin{aligned}\gamma \sim 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} + \cdots + \frac{1}{n^2-1} - \log n^2\\ + \frac{1}{2n^2} + \frac{1}{12n^4} - \frac{1}{120n^8} + \frac{1}{252n^{12}} - \cdots.\end{aligned} \tag{8}$$

Let us add the series in (5) multiplied by 2 and the series in (8) multiplied by -1. This formal addition gives the series

$$\begin{aligned}1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1}\\ - \frac{1}{3n^2} - \frac{1}{10n^4} + \frac{1}{126n^6} + \frac{0}{n^8} + \frac{1}{66n^{10}} + \cdots.\end{aligned} \tag{9}$$

Although this series is not the asymptotic expansion for γ (it does not possess Leibniz series property any more), the partial sums of it can give very good approximations for γ (see numerical computations in [4]). The error made in replacing γ by a partial sum obtained by truncating the series at the term with power n^{-k}, $k \geqslant 2$, is $O(\frac{1}{n^{k+2}})$. We give specific constants for the error.

Let us evaluate the error

$$R_n^{(2)} = \gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2}\right).$$

We have from (8)

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} + \cdots + \frac{1}{n^2-1} - \log n^2 + \frac{1}{2n^2} + \frac{\theta_3}{12n^4}.$$

Subtracting this equality from the doubled (6) gives

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{\theta_1}{60n^4} - \frac{\theta_3}{12n^4}.$$

Therefore

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} + R_n^{(2)},$$
$$-0.1n^{-4} < R_n^{(2)} < 0. \tag{10}$$

The numerical results in Table 7 of [4] well illustrate the accuracy of the approximation (10).

Let us evaluate the error

$$R_n^{(4)} = \gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{1}{10n^4}\right).$$

The asymptotic expansion (8) gives

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2-1} - \log n^2 + \frac{1}{2n^2} + \frac{1}{12n^4} - \frac{\theta_4}{120n^8}.$$

Subtracting this equality from the doubled (7) we get

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{1}{10n^4} + \frac{1}{n^6}\left(\frac{\theta_2}{126} + \frac{\theta_4}{120n^2}\right).$$

Thus,

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{1}{10n^4} + R_n^{(4)},$$
$$0 < R_n^{(4)} < 0.17n^{-6}. \tag{11}$$

To approximate γ by the next partial sum of (9) we use the following two equalities obtained respectively from (5) and (8):

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{\theta_5}{240n^8},$$
$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n^2-1} - \log n^2 + \frac{1}{2n^2} + \frac{1}{12n^4} - \frac{\theta_6}{120n^8}.$$

Subtracting the last equality from the doubled previous one we get

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{1}{10n^4} + \frac{1}{126n^6} - \frac{\theta_5}{120n^8} + \frac{\theta_6}{120n^8}.$$

Thus,

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{1}{10n^4} + \frac{1}{126n^6} + R_n^{(6)},$$
$$-0.0084n^{-8} < R_n^{(6)} < 0.0084n^{-8}.$$

The sum on the right of this equation coincides with the next partial sum of (9) since the coefficient of power n^{-8} is zero. Performing analogous procedure one can get a more precise approximation

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{1}{10n^4} + \frac{1}{126n^6} + R_n^{(8)},$$
$$-0.004n^{-10} < R_n^{(8)} < 0.015n^{-10}. \tag{12}$$

The numerical value of the error in approximation (12) for $n = 10$ presented in Table 9 of [4] is $0.014 \cdot 10^{-10}$, what is in accordance with (12).

Similarly, the approximation of γ by the next partial sum is

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \frac{1}{n+1} - \cdots - \frac{1}{n^2-1} - \frac{1}{3n^2} - \frac{1}{10n^4} + \frac{1}{126n^6} + \frac{1}{66n^{10}} + R_n^{(10)},$$

$$-0.043n^{-12} < R_n^{(10)} < 0.004n^{-12}. \tag{13}$$

The obtained equations (10)–(13) corroborate the presumption that the partial sums of the infinite series (9) well approximate the Euler–Mascheroni constant γ. The errors of the approximations are consistent with the numerical results presented in [4].

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REZIUMĖ

Oilerio konstantos γ belogaritmio skleidinio tikslumas

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Remiantis Oilerio asimptotiniu skleidiniu, tiriamas Oilerio–Maskeronio konstantos γ belogaritmio skleidinio tikslumas. Nustatyti konstantos γ aproksimacijų šio skleidinio dalinėmis sumomis paklaidų rėžiai. Skaičiavimo rezultatai visiškai atitinka šiuos rėžius.

Raktiniai žodžiai: Oilerio–Maskeronio konstanta, asimptotinis skleidinys, racionaliosios aproksimacijos.