LECTURE NOTES ON MEASURE THEORY
FALL 2020

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Contents

1. Preface.

These are the slides I used while teaching this course in 2020. I projected them (spaced out) in class, and filled in the proofs by writing over them with a tablet. Both the annotated version of these slides with handwritten proofs, and the compactified un-annotated version can be found on the class website. The LaTeXsource of these slides is also available on git.

2. Syllabus Overview

- Class website and full syllabus: http://www.math.cmu.edu/~gautam/sj/teaching/2020-21/720-measure
- TA: Lantian Xu <lxu2@andrew.cmu.edu>
- Homework Due: Every Wednesday, before class (on Gradescope)
- Midterm: Fri Oct 9th (90 mins, self proctored, can be taken any time)
- **Zoom lectures:**
  - ▷ Please enable video. (It helps me pace lectures).
  - ▷ Mute your mic when you're not speaking. Use headphones if possible. Consent to be recorded.
  - ▷ If I get disconnected, check your email for instructions.
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ Bottom 20% homework is dropped from your grade (personal emergencies, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Exams:**
  - ▷ Can be taken at any time on the exam day. Open book. Use of internet allowed.
  - ▷ Collaboration is forbidden. You may not seek or receive assistance from other people. (Can search forums; but may not post.)
  - ▷ Self proctored: Zoom call (invite me). Record yourself, and your screen to the cloud.
  - ▷ Share the recording link; also download a copy and upload it to the designated location immediately after turning in your exam.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:

  - Not writing up solutions independently and/or plagiarizing solutions
  - Turning in solutions you do not understand.
  - Seeking, receiving or providing assistance during an exam.
  - Discussing the exam on the exam day (24h). Even if you have finished the exam, others may be taking it.
- ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 40% homework, 20% midterm, 40% final.

3. Sigma Algebras and Measures

- **Motivation:** Suppose $f_n\colon [0,1] \to [0,1]$, and $(f_n) \to 0$ pointwise. Prove $\lim_{n\to\infty} \int_0^1 f_n = 0$.
  - ▷ Simple to state using Riemann integrals. Not so easy to prove. (Challenge!)
  - ▷ Will prove this using Lebesgue integration.
    - Riemann integration: partition the domain (count sequentially)
    - Lebesgue integration: partition the range (stack and sort).
- **Goal:**
  - ▷ Develop Lebesgue integration.
  - ▷ Need a notion of "measure" (generalization of volume)
  - ▷ Need "σ-algebras".
- Why σ-algebras?

Theorem 3.1 (Banach Tarski)**.** *There exists $n \in \mathbb{N}$, sets $A_1, \ldots, A_n \subseteq B(0,1) \subseteq \mathbb{R}^3$ such that:*

(1) $A_1, \ldots, A_n$ partition $B(0,1)$.

(2) There exist isometries R_i such that $R_1(A_1), \ldots, R_n(A_n)$ partition $B(0,2)$.

• How do you explain this?

Definition 3.2 (σ-algebra)**.** Let X be a set. We say $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra on X if:

(1) Nonempty: $\emptyset \in \Sigma$
(2) Closed under compliments: $A \in \Sigma \implies A^c \in \Sigma$.
(3) Closed under countable unions: $A_i \in \Sigma \implies \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

Remark 3.3. Any σ-algebra is also closed under countable intersections.

Question 3.4. *Is $\mathcal{P}(X)$ is a σ-algebra?*

Question 3.5. *Is $\Sigma \stackrel{\text{def}}{=} \{\emptyset, X\}$ is a σ-algebra?*

Question 3.6. *Is $\Sigma = \{A \mid |A| < \infty \text{ or } |A^c| < \infty\}$ a σ-algebra?*

Question 3.7. *Is $\Sigma = \{A \mid \text{either } A \text{ or } A^c \text{ is finite or countable}\}$ a σ-algebra?*

Proposition 3.8. *If $\forall \alpha \in \mathcal{A}$, Σ_α is a σ-algebra, then so is $\bigcap_{\alpha \in \mathcal{A}} \Sigma_\alpha$.*

Definition 3.9. If $\mathcal{E} \subseteq \mathcal{P}(X)$, define $\sigma(\mathcal{E})$ to be the intersection of all σ-algebras containing $\mathcal{E}$.

Remark 3.10. $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$.

Definition 3.11. Suppose X is a topological space. The *Borel σ-algebra on X* is defined to be the σ-algebra generated by all open subsets of X. Notation: $\mathcal{B}(X)$.

Question 3.12. *Can you get $\mathcal{B}(X)$ by taking all countable unions / intersections of open and closed sets?*

Question 3.13. *Is $\mathcal{B}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$?*

Definition 3.14. Let Σ be a σ-algebra on X. We say μ is a (positive) measure on (X, Σ) if:

(1) $\mu\colon \Sigma \to [0, \infty]$
(2) $\mu(\emptyset) = 0$
(3) (Countable additivity): $E_1, E_2, \dots \in \Sigma$ are (countably many) pairwise disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Question 3.15. *Is the second assumption necessary?*

Question 3.16. *Let $\mu(A) =$ cardinality of A. Is μ a measure?*

Question 3.17. *Fix $x_0 \in X$. Let $\mu(A) = 1$ if $x_0 \in A$, and 0 otherwise. Is μ a measure?*

Theorem 3.18. *There exists a measure λ on $\mathcal{B}(\mathbb{R}^d)$ such that $\lambda(I) = \mathrm{vol}(I)$ for all cuboids I.*

- **Goal:** Define $\int_X f\, d\mu$ (the Lebesgue integral).
- **Idea:**
  - ▷ Say $s : X \to \mathbb{R}$ is such that $s = \sum_1^N a_i \mathbf{1}_{A_i}$, for some $a_i \in \mathbb{R}$, $A_i \in \Sigma$. (Called *simple functions*.)
  - ▷ Define $\int_X s\, d\mu = \sum_1^N a_i \mu(A_i)$.
  - ▷ If $f \geqslant 0$, define $\int_X f\, d\mu = \sup_{s \leqslant f} \int_X s\, d\mu$.
- Will do this after constructing the Lebesgue measure.

4. Lebesgue Measure

4.1. Lebesgue Outer Measure.

Definition 4.1. We say $I \subseteq \mathbb{R}$ is a *cell* if I is a finite interval. Define $\ell(I) = \sup I - \inf I$.

Definition 4.2. We say $I \subseteq \mathbb{R}^d$ is a *cell* if it is a product of cells. If $I = I_1 \times \cdots \times I_d$, then define $\ell(I) = \prod_{i=1}^{d} \ell(I_i)$.

Remark 4.3. $\ell(I) = \ell(\mathring{I}) = \ell(\bar{I})$.

Remark 4.4. $\emptyset = \prod_1^d (a, a)$, and so $\ell(\emptyset) = 0$.

Remark 4.5. For all $\alpha \in \mathbb{R}^d$, $\ell(I) = \ell(I + \alpha)$.

Theorem 4.6. *There exists a (unique) measure λ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\lambda(I) = \ell(I)$ for all cells I.*

Question 4.7. *How do you extend ℓ to other sets?*

Definition 4.8 (Lebesgue outer measure)**.** Given $A \subseteq \mathbb{R}^d$, define $\lambda^*(A) = \inf\left\{\sum_1^\infty \ell(I_k) \,\middle|\, A \subseteq \bigcup_1^\infty I_k, \text{where } I_k \text{ is a cell}\right\}$.

Remark 4.9. Some authors use m^* instead of λ^*.

Remark 4.10. λ^* is defined on $\mathcal{P}(\mathbb{R}^d)$; but only "well behaved" on a σ-algebra.

Question 4.11. *What is $\lambda^*(\emptyset)$? What is $\lambda^*(\mathbb{R}^d)$?*

Proposition 4.12. *If $E \subseteq F$, then $\lambda^*(E) \leqslant \lambda^*(F)$.*

Proposition 4.13. *If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, then $\lambda^*(\cup_1^\infty E_i) \leqslant \sum_1^\infty \lambda^*(E_i)$.*

Proposition 4.14. *Let $A, B \subseteq \mathbb{R}^d$, and suppose $d(A, B) > 0$. Then $\lambda^*(A \cup B) = \lambda * (A) + \lambda * (B)$.*

Proof: Only need to show $\lambda^*(A \cup B) \geqslant \lambda^*(A) + \lambda^*(B)$. If $\lambda^*(A \cup B) = \infty$, we are done, so assume $\lambda^*(A \cup B) < \infty$.

Proposition 4.15. *If $I \subseteq \mathbb{R}^d$ is a cell, then $\lambda^*(I) = \ell(I)$.*

Lemma 4.16. *If $\{I_k\}$ divide I by hyperplanes, then $\sum \ell(I_k) = \ell(I)$.*

Lemma 4.17. $\lambda^*(A) = \inf\{\sum \ell(I_i) \mid A \subseteq \cup I_k,$ *and* I_k *are all* open *cells*$\}$.

Proof of Proposition 4.15: Suppose first I is closed (hence compact). Pick $\varepsilon > 0$.

Proposition 4.18 (Translation invariance)**.** *For all* $A \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{R}^d$, $\lambda^*(A) = \lambda^*(\alpha + A)$.

4.2. **Carathéodory Extension.** Our goal is to start with an *outer measure*, and restrict it to a *measure*.

Definition 4.19. We say μ^* is an outer measure on X if:

(1) $\mu^* : \mathcal{P}(X) \to [0, \infty]$, and $\mu^*(\emptyset) = 0$.
(2) If $A \subseteq B$ then $\mu^*(A) \leqslant \mu^*(B)$.
(3) If $A_i \subseteq X$ (not necessarily disjoint), then $\mu^*(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \mu^*(A_i)$.

Example 4.20. Any measure is an outer measure.

Example 4.21. The Lebesgue outer measure is an outer measure.

Theorem 4.22 (Carathéodory extension)**.** *Let* $\Sigma \stackrel{\text{def}}{=} \{E \subseteq X \mid \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\ \forall A \subseteq X\}$. *Then* Σ *is a* σ*-algebra, and* μ^* *is a measure on* (X, Σ).

Remark 4.23. Clearly $\mu^*(A) \leqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all E, A.

Intuition: Suppose $\mu^* = \lambda^*$. In order to show $\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$, cover A by cells so that $\mu^*(A) \geqslant \sum \ell(I_k) - \varepsilon$. Split this cover into cells that intersect E and cells that intersect E^c. If E is nice, hopefully the overlap is small.

Proof of Theorem 4.22

(1) $\emptyset \in \Sigma$.
(2) $E \in \Sigma \implies E^c \in \Sigma$.
(3) $E, F \in \Sigma \implies E \cup F \in \Sigma$. (Hence $E_1, \dots, E_n \in \Sigma \implies \cup_1^n E_i \in \Sigma$.)

(4) If $E_1, \ldots, E_n \in \Sigma$ are pairwise disjoint, $A \subseteq X$, then $\mu^*(A \cap (\cup_1^n E_i)) = \sum_1^n \mu^*(A \cap E_i)$.
(5) Σ is closed under countable *disjoint* unions, and μ^* is countably additive on Σ.
Proof: Let $E_1,\, E_2,\, \ldots, \in \Sigma$ be pairwise disjoint, and $A \subseteq X$ be arbitrary.

Remark 4.24. Note, the above shows $\mu^*(A \cap (\cup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i)$.

Definition 4.25. Define the *Lebesgue σ-algebra* by $\mathcal{L}(\mathbb{R}^d) = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) \cap \lambda^*(A \cap E^c)\ \forall A \subseteq \mathbb{R}^d\}$.

Definition 4.26. Define the *Lebesgue measure* by $\lambda(E) = \lambda^*(E)$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

Remark 4.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a σ-algebra, and λ is a measure on $\mathcal{L}$.

Question 4.28. *Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?*

Proposition 4.29. *If $I \subseteq \mathbb{R}^d$ is a cell, then $I \in \mathcal{L}(\mathbb{R}^d)$.*

Proof:

Proposition 4.30. $\mathcal{L}(\mathbb{R}^d) \supseteq \mathcal{B}(\mathbb{R}^d)$.

Remark 4.31. We will show later that $\mathcal{L}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) + \mathcal{N}$, where $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$.

Here are two results that will be proved later:

Theorem 4.32. $\mathcal{L}(\mathbb{R}^d) \supsetneq \mathcal{B}(\mathbb{R}^d)$. *(In fact the cardinality of $\mathcal{L}(\mathbb{R}^d)$ is larger than that of $\mathcal{B}(\mathbb{R}^d)$.)*

Theorem 4.33. $\mathcal{L}(\mathbb{R}^d) \subsetneq \mathcal{P}(\mathbb{R}^d)$.

Theorem 4.34 (Uniqueness)**.** *If μ is any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu(I) = \lambda(I)$ for all cells, then $\mu(E) = \lambda(E)$ for all $E \in \mathcal{B}(\mathbb{R}^d)$.*

Question 4.35. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and suppose μ, ν are two measures which agree on $\mathcal{E}$. Must they agree on $\sigma(E)$?*

5. Abstract measures

5.1. Dynkin systems.

Question 5.1. *Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?*

▷ Clearly need Π to be closed under intersections.

Question 5.2. *Let $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. Must Λ be a σ-algebra?*

▷ If $A, B \in \Lambda$, must $A \cup B \in \Lambda$?
▷ If $A \subseteq B$, $A, B \in \Lambda$, must $B - A \in \Lambda$?
▷ If $A_i \subseteq A_{i+1} \in \Lambda$, must $\cup_1^\infty A_i \in \Lambda$?

Definition 5.3. We say $\Lambda \subseteq \mathcal{P}(X)$ is a λ-system if:

(1) $X \in \Lambda$
(2) If $A \subseteq B$ and $A, B \in \Lambda$ then $B - A \in \Lambda$.
(3) If $A_n \in \Lambda$, $A_n \subseteq A_{n+1}$ then $\cup_1^\infty A_n \in \Lambda$.

Definition 5.4. We say $\Pi \subseteq \mathcal{P}(X)$ is a π-system if whenever $A, B \in \Pi$, we have $A \cap B \in \Pi$.

Lemma 5.5 (Dynkin system lemma)**.** *If Π is a π-system, and $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \sigma(\Pi)$.*

Corollary 5.6. *If μ, ν are finite measures such that $\mu = \nu$ on Π, and Π is closed under intersections, then $\mu = \nu$ on $\sigma(\Pi)$.*

Proof of Lemma 5.5

(1) The arbitrary intersection of λ-systems is a λ-system. So it make sense to talk about $\lambda(\Pi)$.
(2) If $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \lambda(\Pi)$.

(3) If Λ is *both* a π-system and a λ-system, then Λ is a σ-algebra.
(4) To finish the proof, we only need to show $\lambda(\Pi)$ is closed under intersections.
(5) Let $C \in \lambda(\Pi)$, and define $\Lambda_C = \{B \in \lambda(\Pi) \mid B \cap C \in \lambda(\Pi)\}$. Then Λ_C is a λ-system.
(6) If $B, C \in \lambda(\Pi)$, then $B \cap C \in \lambda(\Pi)$.
- ▷ Suppose first $D \in \Pi$. Then $D \cap B \in \lambda(\Pi)$ for all $B \in \lambda(\Pi)$.
- ▷ For all $B \in \lambda(\Pi)$, we must have $\Lambda_B \supseteq \lambda(\Pi)$.

5.2. Regularity of measures.

Definition 5.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

(1) For all compact sets K, we have $\mu(K) < \infty$.
(2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ is compact}\}$.
(3) For all Borel sets A we have $\mu(A) = \inf\{\mu(U) \mid U \supseteq A,\ U \text{ open}\}$.

Motivation:

- ▷ Approximation of measurable functions by continuous functions
- ▷ Differentiation of measures
- ▷ Uniqueness in the Riesz representation theorem

Question 5.8. *If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$ for all Borel sets A?*

Remark 5.9. (1) If $X = \mathbb{R}^d$, and μ is regular, then $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) Further, for any $\varepsilon > 0$ there exists an open set $U \supseteq A$ and a closed set $C \subseteq A$ such that $\mu(U - C) < \varepsilon$.
(3) If $\mu(A) < \infty$, then can make C above compact.

Proof. Will return and prove it using the next theorem. □

Theorem 5.10. *Suppose X is a compact metric space, and μ is a finite Borel measure on X. Then μ is regular. Further, for any $\varepsilon > 0$, there exists $U \supseteq A$ open and $K \subseteq A$ closed such that $\mu(U - K) < \varepsilon$.*

Proof:

(1) Let $\Lambda = \{A \in \mathcal{B}(X) \mid \forall \varepsilon > 0,\ \exists K \subseteq A \text{ compact},\ U \supseteq A \text{ open, such that } \mu(U - K) < \varepsilon\}$.
(2) Λ contains all open sets.
(3) Λ is a λ-system. (In this case it's easy to directly show that Λ is a σ-algebra.)
(4) Dynkin's Lemma implies $\Lambda \supseteq \mathcal{B}(X)$, finishing the proof.

Corollary 5.11. *Let X be a metric space and μ a Borel measure on X. Suppose there exists a sequence of sets $B_n \subset X$ such that $\bar{B}_n \subset \mathring{B}_{n+1}$, $\bar{B}_n$ is compact, $X = \cup_1^\infty B_n$ and $\mu(B_n) < \infty$. Then μ is regular. Further:*

(1) For any Borel set A, $\mu(A) = \sup\{\mu(K) \mid K \subseteq K \text{ is compact}\}$.
(2) For any $\varepsilon > 0$, there exists $U \supseteq A$ open and $C \subseteq A$ closed such that $\mu(U - C) < \varepsilon$.

Proof. On homework. □

Theorem 5.12. *Let $A \in \mathcal{L}(\mathbb{R}^d)$, $\lambda(A)$.*

(1) $\lambda(A) = \inf\{\lambda(U) \mid U \supseteq A,\ U \text{ open}\} = \sup\{\lambda(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) There exists $\varepsilon > 0$, $C \subseteq A$ closed and $U \supseteq A$ open such that $\lambda(U - C) < \varepsilon$.

5.3. **Non-measurable sets.**

Theorem 5.13. *There exists $E \subseteq \mathbb{R}$ such that $E \notin \mathcal{L}(R)$.*

Proof:

(1) Let $C_\alpha = \{\beta \in \mathbb{R} \mid \beta - \alpha \in \mathbb{Q}\}$. (This is the coset of $\mathbb{R}/\mathbb{Q}$ containing α.)

(2) Let $E \subseteq \mathbb{R}$ be such that $|E \cap C_\alpha| = 1$ for all α.
(3) Note if $q_1, q_2 \in \mathbb{Q}$ with $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$.
(4) Suppose for contradiction $E \in \mathcal{L}(\mathbb{R})$.
(5) $\lambda(E) > 0$
(6) $\lambda(E) = 0$ (contradiction).

Theorem 5.14. *Let $A \subseteq \mathbb{R}^d$. Every subset of A is Lebesgue measurable if and only if $\lambda(A^*) = 0$.*

Proof. One direction is immediate. The other direction is accessible with what we know so far, but we won't do the proof in the interest of time. □

5.4. **Completion of measures.**

Theorem 5.15. *$A \in \mathcal{L}(\mathbb{R}^d)$ if and only if there exist $F, G \in \mathcal{B}(\mathbb{R}^d)$ such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$.*

Corollary 5.16. *Let $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$. Then $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if $A = B \cup N$ for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $N \in \mathcal{N}$.*

Definition 5.17. Let (X, Σ, μ) be a measure space. We define the completion of Σ with respect to the measure μ by

$$\Sigma_\mu \stackrel{\text{def}}{=} \{A \subseteq X \mid \exists F, G \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G - F) = 0\}$$

For every $A \in \Sigma_\mu$, find F, G as above and define $\bar{\mu}(A) = \mu(F)$.

Definition 5.18. Let $\mathcal{N} = \{A \subseteq X \mid \exists E \in \Sigma,\ E \supseteq A,\ \mu(E) = 0\}$. We say (X, Σ, μ) is complete if $\mathcal{N} \subseteq \Sigma$.

Theorem 5.19. *Σ_μ is a σ-algebra, $\bar{\mu}$ is a measure on Σ_μ, and $(X, \Sigma_\mu, \bar{\mu})$ is complete.*

Theorem 5.20. *Σ_μ is the smallest μ-complete σ-algebra containing Σ.*

Corollary 5.21. $\Sigma_\mu = \sigma(\Sigma \cup \mathcal{N})$.

Corollary 5.22. $\mathcal{L}(\mathbb{R}^d) = \sigma(\mathcal{B}(\mathbb{R}^d) \cup \mathcal{N})$.

Remark 5.23. There could exist μ-null sets that are not in Σ.

6. Measurable Functions

6.1. Measurable functions.

Definition 6.1. Let (X, Σ, μ) be a measurable space, and (Y, τ) a topological space. We say $f\colon X \to Y$ is measurable if $f^{-1}(\tau) \subseteq \Sigma$.

Remark 6.2. Y is typically $[-\infty, \infty]$, $\mathbb{R}^d$, or some linear space.

Remark 6.3. Any continuous function is Borel measurable, but not conversely.

Question 6.4. *Say $f\colon X \to Y$ is measurable. For every $B \in \mathcal{B}(Y)$, must $f^{-1}(B) \in \Sigma$?*

Theorem 6.5. *Say $f\colon X \to Y$ is measurable. Then, for every $B \in \mathcal{B}(Y)$, we must have $f^{-1}(B) \in \Sigma$.*

Lemma 6.6. *Let $f\colon X \to Y$ be arbitrary, and Σ be a σ-algebra on X. Then $\Sigma' = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$ is a σ-algebra (on Y).*

Corollary 6.7. *Let $f\colon X \to [-\infty, \infty]$. Then f is measurable if and only if for all $a \in \mathbb{R}$, we have $\{f < a\} \in \Sigma$.*

Lemma 6.8. *If $f\colon X \to \mathbb{R}^m$ is measurable, and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ is Borel, then $g \circ f\colon X \to \mathbb{R}^n$ is measurable.*

Question 6.9. *Is the above true if g was Lebesgue measurable?*

Theorem 6.10. *Let $f_n\colon X \to \mathbb{R}$ be a sequence of measurable functions. Then $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ and $\lim f_n$ (if it exists) are all measurable.*

Theorem 6.11. *Let $f, g\colon X \to \mathbb{R}$. The function $(f, g)\colon X \to \mathbb{R}^2$ is measurable if and only if both f and g are measurable.*

Corollary 6.12. *If $f, g\colon X \to \mathbb{R}$ are measurable, then so is $f + g$, fg and f/g (when defined).*

6.2. **Cantor Function.**

Definition 6.13 (Cantor function)**.** Let C be the Cantor set, and $\alpha = \log 2/\log 3$ be the Hausdorff dimension of C. Let $f(x) = H_\alpha(C \cap [0,x])/H_\alpha(C)$.

(1) $f(0) = 0$, $f(1) = 1$ and f is increasing. (In fact, f is differentiable exactly on C, and $f' = 0$ wherever defined.)
(2) f is continuous everywhere. (In fact f is Hölder continuous with exponent $\alpha = \log 2/\log 3$.)
(3) Let $g = f^{-1}$. That is, $g(x) = \inf\{y \mid f(y) = x\}$ (Note, since f is continuous $f(g(x)) = x$)).

Proposition 6.14. *The function $g\colon [0,1] \to C$ is a strictly injective Borel measurable function.*

Theorem 6.15. $\mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$.

Theorem 6.16. *There exists $h_1, h_2\colon \mathbb{R} \to \mathbb{R}$ such that h_1 is $\mathcal{L}(\mathbb{R})$-measurable, h_2 is $\mathcal{B}(\mathbb{R})$ measurable, but $h_1 \circ h_2$ is not $\mathcal{L}(\mathbb{R})$ measurable.*

Remark 6.17. Note $h_2 \circ h_1$ has to be $\mathcal{B}(\mathbb{R})$-measurable.

6.3. **Almost Everywhere.**

Definition 6.18. Let (X, Σ, μ) be a measure space. We say a property P holds almost everywhere if there exists a null set N such that P holds on N^c.

Example 6.19. If f, g are two functions, we say $f = g$ almost everywhere if $\{f \neq g\}$ is a null set.

Example 6.20. Almost every real number is irrational.

Example 6.21. If $A \in \mathcal{L}(\mathbb{R})$, then $\displaystyle\lim_{h \to 0} \frac{\lambda(A \cap (x, x+h))}{h} = \mathbf{1}_A(x)$ for almost every x. (Contrast with HW3, Q3b)

Example 6.22. Let $x \in (0,1)$, and p_n/q_n be the n^{th} convergent in the continued fraction expansion of x. Then $\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12\log 2}$.

Assume hereafter (X, Σ, μ) is complete.

Proposition 6.23. *If $f = g$ almost everywhere and f is measurable, then so is g.*

Proposition 6.24. *If $(f_n) \to f$ almost everywhere, and each f_n is measurable, then so is f.*

6.4. **Approximation.**

Definition 6.25. A function $s\colon X \to \mathbb{R}$ is called *simple* if s is measurable, and has finite range (i.e. $s(\mathbb{R}) = \{a_1, \dots a_n\}$).

Question 6.26. *Why bother with simple functions?*

Theorem 6.27. *If $f \geqslant 0$ is a measurable function, then there exists a sequence of simple functions (s_n) which increases to f.*

Corollary 6.28. *If $f\colon X \to \mathbb{R}$ is measurable, then there exists a sequence of simple functions (s_n) such that $(s_n) \to f$ pointwise, and $|s_n| \leqslant |f|$.*

Theorem 6.29 (Lusin)**.** *Let μ be a finite regular measure on a metric space X. Let $f\colon X \to \mathbb{R}$ be measurable. For any $\varepsilon > 0$ there exists a continuous function $g\colon X \to \mathbb{R}$ such that $\mu\{f \neq g\} < \varepsilon$.*

Lemma 6.30 (Tietze's extension theorem)**.** *If $C \subseteq X$ is continuous, and $f\colon C \to \mathbb{R}$ is continuous, then there exist $\bar{f}\colon X \to \mathbb{R}$ such that $\bar{f} = f$ on C.*

Lemma 6.31. *Let $f\colon X \to \mathbb{R}$ be measurable. For every $\varepsilon > 0$, there exists $C \subseteq X$ closed such that $\mu(X - C) < \varepsilon$ and $f\colon C \to \mathbb{R}$ is continuous.*

Proof of Lusin's theorem. Previous two lemmas. □

Proof of Lemma 6.31. □

7. Integration

7.1. **Construction of the Lebesgue integral.** Recall, $s\colon X \to \mathbb{R}$ is *simple* if s is measurable and has finite range.

Definition 7.1. Let $s \geqslant 0$ be a simple function. Let $\{a_1, \dots, a_n\} = s(X)$, and set $A_i = s^{-1}(a_i)$. Define $\int_X s\,d\mu = \sum_{i=1}^n a_i A_i$.

Remark 7.2. Always use the convention $0 \cdot \infty = 0$.

Remark 7.3. Other notation: $\int_X s\,d\mu = \int_X s(x)\,d\mu(x)$.

Proposition 7.4. *If $0 \leqslant s \leqslant t$ are simple, then $\int_X s\,d\mu \leqslant \int_X t\,d\mu$.*

Proposition 7.5. *If $s, t \geqslant 0$ are simple, then $\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu$.*

Definition 7.6. Let $f\colon X \to [0,\infty]$ be measurable. Define $\int_X f\,d\mu = \sup\{\int_X s\,d\mu \mid 0 \leqslant s \leqslant f,\ s \text{ simple.}\}$.

Definition 7.7. Let $f\colon X \to [-\infty,\infty]$ be measurable. We say f is *integrable* if $\int_X f^+\,d\mu < \infty$ and $\int_X f^-\,d\mu < \infty$. In this case we define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

Definition 7.8. We let $L^1(X) = L^1(X, \Sigma, \mu)$ be the set of all integrable functions on X. (Note $f \in L^1 \iff |f| \in L^1$.)

Definition 7.9. We say f is *integrable in the extended sense* if *either* $\int_X f^+\,d\mu < \infty$ or $\int_X f^-\,d\mu < \infty$. In this case we still define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

Remark 7.10. If both $\int_X f^+\,d\mu = \infty$ and $\int_X f^-\,d\mu = \infty$, then $\int_X f\,d\mu$ is not defined.

Question 7.11. *Do we have linearity?*

Proposition 7.12 (Consistency). *If $s = \sum_1^n a_i \mathbf{1}_{A_i} \geqslant 0$ is simple, then $\sum a_i \mu(A_i) = \sup\{\int_X t\,d\mu \mid 0 \leqslant t \leqslant s,\ \text{simple}\}$.*

Theorem 7.13 (Monotone convergence). *Say $(f_n) \to f$ almost everywhere, $0 \leqslant f_n \leqslant f_{n+1}$, then $(\int_X f_n\,d\mu) \to \int_X f\,d\mu$.*

Theorem 7.14. *If f, g are integrable, then $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$.*

7.2. **Dominated convergence.** When does $\lim \int_X f_n \, d\mu \neq \int_X f \, d\mu$? Two typical situations where it fails:

(1) Mass escapes to infinity
(2) Mass clusters at a point

Theorem 7.15 (Dominated convergence)**.** *Say (f_n) is a sequence of measurable functions, such that $(f_n) \to f$ almost everywhere. Moreover, there exists $F \in L^1(X)$ such that $|f_n| \leqslant F$ almost everywhere. Then $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.*

Lemma 7.16 (Fatou)**.** *Suppose $f_n \geqslant 0$, and $(f_n) \to f$. Then $\liminf \int_X f_n \, d\mu \geqslant \int_X f \, d\mu$.*

Proof of Theorem 7.15

Theorem 7.17 (Beppo-Levi)**.** *If $f_n \geqslant 0$, then $\sum_1^\infty \int_X f_n \, d\mu = \int_X (\sum_1^\infty f_n) \, d\mu$.*

Theorem 7.18. *If $f \colon \mathbb{R}^d \to \mathbb{R}$ is Riemann integrable, then the Riemann integral of f is the same as the Lebesgue integral.*

Proof. IOU $\square$

Question 7.19. *Let $f \colon [0, \infty) \to \mathbb{R}$ be measurable, and define the* Laplace transform *of f by $F(s) = \int_0^\infty e^{-st} f(t) \, dt$. Is F continuous? Is F differentiable?*

Question 7.20. *Let φ be a bump function, and (q_n) be an enumeration of the rationals. Define $f(x) = \sum_{n=1}^\infty \varphi(2^n(x - q_n))$. Is f finite almost everywhere?*

7.3. **Push forward measures.**

Definition 7.21. Say $f \colon X \to \mathbb{R}^d$ is integrable, then define $\int_X f \, d\mu = (\int_X f_1 \, d\mu, \ldots, \int_X f_d \, d\mu$, where $f = (f_1, \ldots, f_d)$.

Theorem 7.22. *Let (X, Σ, μ) be a measure space, $f \colon X \to Y$ be arbitrary. Define $\tau = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$, and define $\nu(A) = \mu(f^{-1}(A))$. Then ν is a measure on (Y, τ) and $\int_Y g \, d\nu = \int_X g \circ f \, d\mu$.*

Remark 7.23. The measure ν is called the *push forward* of μ and denoted by $f^*(\mu)$, or $\mu_{f^{-1}}$. This is used often to define Laws of random variables. (We will use it to prove the change of variable formula.)

Corollary 7.24. *If $\alpha \in \mathbb{R}^d$, then $\int_{\mathbb{R}^d} f(x+\alpha)\, d\lambda(x) = \int_{\mathbb{R}^d} f(x)\, d\lambda(x)$.*

8. Convergence

8.1. Modes of convergence.

Definition 8.1. We say $(f_n) \to f$ almost everywhere if for almost every $x \in X$, we have $(f_n(x)) \to f(x)$.

Definition 8.2. We say $(f_n) \to f$ in measure (notation $(f_n) \xrightarrow{\mu} f$) if for all $\varepsilon > 0$, we have $(\mu\{|f_n - f| > \varepsilon\}) \to 0$.

Definition 8.3. Let $p \in [1, \infty)$. We say $(f_n) \to f$ in L^p if $(\int_X |f_n - f|^p \, d\mu) \to 0$.

Question 8.4. *Why $p > 1$? How about $p = \infty$?*

(1) $(f_n) \to f$ almost everywhere implies $(f_n) \to f$ in measure if $\mu(X) < \infty$.
(2) $(f_n) \to f$ in measure implies $(f_n) \to f$ almost everywhere along a subsequence.
(3) $(f_n) \to f$ in L^p implies $(f_n) \to f$ in measure (for $p < \infty$), and hence $(f_n) \to f$ along a subsequence.
(4) Convergence almost everywhere or in measure don't imply convergence in L^p.

Theorem 8.5. *If $(f_n) \to f$ almost everywhere and $\mu(X) < \infty$, then $(f_n) \to f$ in measure.*

Lemma 8.6 (Egorov). *If $(f_n) \to f$ almost everywhere and $\mu(X) < \infty$, for every $\varepsilon > 0$ there exists A_ε such that $\mu(A_\varepsilon^c) < \varepsilon$ and $(f_n) \to f$ uniformly on A_ε.*

Question 8.7. *Does this imply $(f_n) \to f$ uniformly almost everywhere?*

Proof of Theorem 8.5

Proposition 8.8. *If $(f_n) \to f$ in measure then (f_n) need not converge to f almost everywhere.*

Proposition 8.9. *If $(f_n) \to f$ in measure, then there exists a subsequence (f_{n_k}) such that $(f_{n_k}) \to f$ almost everywhere.*

8.2. L^p spaces.

Definition 8.10. A *Banach space* is a normed vector space that is complete under the metric induced by the norm.

Example 8.11. $\mathbb{C}$, $\mathbb{R}^d$, $C(X)$, etc. are all Banach spaces.

Definition 8.12. For $p \in (0, \infty)$, define $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$.

Definition 8.13. For $p = \infty$, define $\|f\|_\infty = \operatorname{ess\,sup}|f| = \inf\{C \geqslant 0 \mid |f| \leqslant C \text{ almost surely}\}$

Definition 8.14. Let (X, Σ, μ) be a measure space, and assume Σ is μ-complete. Define $\mathcal{L}^p(X) = \{f \colon X \to \mathbb{R} \mid \|f\|_p < \infty\}$.

Question 8.15. *Is $\mathcal{L}^p(X)$ a Banach space?*

Definition 8.16. Define an equivalence relation on $\mathcal{L}^p$ by $f \sim g$ if $f = g$ almost everywhere.

Definition 8.17. Define $L^p(X) = \mathcal{L}^p(X)/\sim$.

Remark 8.18. We will always treat elements of $L^p(X)$ as functions, implicitly identifying a function with its equivalence class under the relation $\sim$. In order to be logically correct, however, we need to ensure that every operation we perform on functions respects the equivalence relation $\sim$.

Theorem 8.19. *For $p \in [1, \infty]$, $L^p(X)$ is a Banach space.*

Theorem 8.20 (Hölder's inequality)**.** *Say $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $|\int_X fg \, d\mu| \leqslant \|f\|_p \|g\|_q$.*

Remark 8.21. The relation between p and q can be motivated by dimension counting, or scaling.

Brute force proof of Theorem 8.20
Proof of Theorem 8.20 using Young's inequality.

Theorem 8.22 (Young's inequality)**.** *If $x, y \geqslant 0$, $1/p + 1/q = 1$ then $xy \leqslant x^p/p + y^q/q$.*

Lemma 8.23 (Duality)**.** *If $p \in [1, \infty)$, $1/p + 1/q = 1$, then $\|f\|_p = \sup_{g \in L^q - 0} \frac{1}{\|g\|_q} \int_X fg\, d\mu = \sup_{\|g\|_q = 1} \int_X fg\, d\mu = 1$*

Remark 8.24. For $p = \infty$ this is still true if X is σ-finite.

Theorem 8.25 (Minkowski's inequality)**.** *If $f, g \in L^p$, then $f + g \in L^p$ and $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$.*

Theorem 8.26 (Jensen's inequality)**.** *If $\mu(X) = 1$, $f \in L^1(X)$, $a < f < b$ almost everywhere, and $\varphi : (a, b) \to \mathbb{R}$ is convex, then*

$$\varphi\Big(\int_X f\, d\mu\Big) \leqslant \int_X \varphi \circ f\, d\mu\,.$$

Proof of Theorem 8.19: Only remains to show L^p is complete.

Lemma 8.27. *Suppose $p < \infty$, $f_n \in L^p$ and $\sum \|f_n\|_p < \infty$. Let $f = \sum f_n$. Then $f \in L^p$, and $\sum f_n \to f$ in L^p and $\sum f_n \to f$ almost everywhere.*

Proof of Theorem 8.19:

Proposition 8.28. *If $p \in [1, \infty)$, $(f_n) \to f$ in L^p, then $(f_n) \to f$ in measure.*

Lemma 8.29 (Chebychev's inequality)**.** *For any $\lambda > 0$, we have $\mu(\{|f| > \lambda\}) \leqslant \frac{1}{\lambda}\|f\|_1$*

Proof of Proposition 8.28

8.3. **Uniform integrability.**

Question 8.30. *When does convergence in measure imply L^1 convergence?*

Definition 8.31. We say $\{f_\alpha \alpha \in \mathcal{A}\}$ is uniformly integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$ we have $\int_E |f| \, d\mu < \varepsilon$.

Proposition 8.32. *If $|f_\alpha| \leqslant F$ for all $\alpha \in \mathcal{A}$, and $F \in L^1$, then $\{f_\alpha \mid \alpha \in A\}$ is uniformly integrable.*

Theorem 8.33 (Vitali)**.** *Let $(f_n) \in L^1(X)$. The sequence (f_n) is convergent in L^1 if and only if*

(1) *(f_n) converges in measure,*
(2) *(f_n) is uniformly integrable,*
(3) (tightness) *there exists $F \in \Sigma$ with $\mu(F) < \infty$ such that $\int_{F^c} |f_n| \, d\mu < \varepsilon$ for all n.*

Proof:

Theorem 8.34. *If $\displaystyle\lim_{\lambda \to \infty} \sup_n \int_{\{|f_n| > \lambda\}} |f_n| \, d\mu = 0$, then (f_n) is uniformly integrable.*

Theorem 8.35. *If there exists an increasing function $\varphi \colon [0, \infty) \to [0, \infty)$ such that $\displaystyle\lim_{x \to \infty} \frac{\varphi(x)}{x} = \infty$, and $\displaystyle\sup_n \int_X \varphi(|f_n|) \, d\mu < \infty$, then (f_n) is uniformly integrable.*

Remark 8.36. The hypothesis in both the above theorems are equivalent.

Remark 8.37. If additionally $\sup_n \int_X |f_n| \, d\mu < \infty$, then the converse of both the above theorems are true.

Proof:

Corollary 8.38. *If $(f_n) \to f$ in measure, $\mu(X) < \infty$, and $\sup_n \|f\|_p < \infty$ for any $p > 1$, then $(f_n) \to f$ in L^q for every $q \in [1, p)$.*

9. Signed Measures

9.1. Hanh and Jordan Decomposition Theorems.

Definition 9.1. We say $\mu\colon \Sigma \to [-\infty, \infty]$ is a *signed measure* if:

(1) The range of μ doesn't contain *both* $+\infty$ and $-\infty$.
(2) $\mu(\emptyset) = 0$
(3) If $A_i \in \Sigma$ are countably many pairwise disjoint sets then $\mu(\cup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$.

Example 9.2. Let $f \in L^1(X, \mu)$, and define ν by $\nu(A) = \int_A f\, d\mu$. Then ν is a signed measure, and we write $d\nu = f\, d\mu$.

Example 9.3. If μ, ν are two (positive) measures such that either one is finite, then $\mu - \nu$ is finite.

Theorem 9.4 (Jordan Decomposition)**.** *Any signed measure can be written (uniquely) as the difference of two mutually singular positive measures.*

Definition 9.5. We say $A \in \Sigma$ is a *negative set* if $\mu(B) \leqslant 0$ for all measurable sets $B \subseteq A$.

Proposition 9.6. *If $\mu(A) \in (-\infty, \infty)$ then there exists $B \subseteq A$ such that B is negative and $\mu(B) \leqslant \mu(A)$.*

Theorem 9.7 (Hanh decomposition)**.** *If μ is a signed measure on X, then $X = P \cup N$ where P is positive and N is negative.*

Remark 9.8. The decomposition is unique up to null sets.

Definition 9.9. We say two positive measures μ, ν are *mutually singular* if there exists $C \subseteq X$ such that for every $A \in \Sigma$ we have $\mu(A \cap C) = \nu(A \cap C^c) = 0$.

Proof of Theorem 9.4

Definition 9.10. Let μ be a signed measure with Jordan decomposition $\mu = \mu^+ - \mu^-$. Define the variation of μ to be the (positive) measure $|\mu| \stackrel{\text{def}}{=} \mu^+ + \mu^-$.

Definition 9.11. Define the *total variation* of μ by $\|\mu\| = |\mu|(X)$.

Proposition 9.12. *Let $\mathcal{M}$ be the set of all finite signed measures on X. Then $\mathcal{M}$ is a Banach space under the total variation norm.*

9.2. **Absolute Continuity.**

Definition 9.13. Let μ, ν be two measures. We say ν is absolutely continuous with respect to μ (notation $\nu \ll \mu$) if whenever $\mu(A) = 0$ we have $\nu(A) = 0$.

Example 9.14. Let $g \geqslant 0$ and define $\nu(A) = \int_A g\, d\mu$. (Notation: Say $d\nu = g\, d\mu$.)

Theorem 9.15 (Radon-Nikodym)**.** *If μ, ν are two positive σ-finite measures such $\nu \ll \mu$, then there exists a unique measurable function g such that $0 \leqslant g < \infty$ almost everywhere and $d\nu = g\, d\mu$.*

Theorem 9.16. *Let μ, ν be positive measures such that ν is σ-finite. There exists a unique pair of measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$, and $\nu = \nu_{ac} + \nu_s$.*

Corollary 9.17. *Let μ be a positive measure, and ν be a finite signed measure. There exists a unique pair of signed measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$ and $\nu = \nu_{ac} + \nu_s$.*

Corollary 9.18. *Let μ, ν be σ-finite positive measures. There exists a unique positive measure ν_s and nonnegative measurable function g such that $\mu \perp \nu_s$ and $d\nu = d\nu_s + g\, d\mu$.*

9.3. **Dual of L^p.**

Proposition 9.19. *Let U, V be Banach spaces, and $T\colon U \to V$ be linear. Then T is continuous if and only if there exists $c < \infty$ such that $\|Tu\|_V \leqslant c\|u\|_U$ for all $u \in U$, $v \in V$.*

Definition 9.20. We say $T\colon U \to V$ is a bounded linear transformation if T is linear and there exists $c < \infty$ such that $\|Tu\|_V \leqslant c\|u\|_U$ for all $u \in U$, $v \in V$.

Definition 9.21. The *dual* of U is defined by $U^* = \{u^* \mid u^*\colon U \to \mathbb{R}$ is bounded and linear.$\}$ Define a norm on U^* by

$$\|u^*\|_{U^*} \stackrel{\text{def}}{=} \sup_{u \in U - 0} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U = 1} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U = 1} \frac{1}{\|u\|_U} |u^*(u)| \,.$$

Proposition 9.22. *The dual of a Banach space is a Banach space.*

Proposition 9.23. *Let $1/p + 1/q = 1$, $g \in L^q(X)$. Define $T_g\colon L^p \to \mathbb{R}$ by $T_g f = \int_X f g \, d\mu$. Then $T_g \in (L^p)^*$.*

Proposition 9.24. *The map $g \mapsto T_g$ is a bounded linear map from $L^q \to (L^p)^*$.*

Theorem 9.25. *Let (X, Σ, μ) be a σ-finite measure space, $p \in [1, \infty)$, $1/p + 1/q = 1$. The map $g \mapsto T_g$ is a bijective linear isometry between L^q and $(L^p)^*$.*

Remark 9.26. For $p \in (1, \infty)$ the above is still true even if X is not σ-finite.

Remark 9.27. For $p = \infty$, the map $g \mapsto T_g$ gives an *injective* linear isometry of $L^1 \to (L^\infty)^*)$. It is not surjective in most cases.

9.4. **Riesz Representation Theorem.**

Theorem 9.28 (Riesz Representation Theorem)**.** *Let X be a compact metric space, and $\mathcal{M}$ be the set of all finite signed measures on X. Define $\Lambda\colon \mathcal{M} \to C(X)^*$ by $\Lambda_\mu(f) = \int_X f \, d\mu$ for all $\mu \in \mathcal{M}$ and $f \in C(X)$. Then Λ is a bijective linear isometry.*

Remark 9.29. In particular, for every $I \in C(X)^*$, there exists a unique finite regular Borel measure μ such that $I(f) = \int_X f \, d\mu$ for every $f \in C(X)$.

10. **Product measures**

10.1. **Fubini and Tonelli theorems.** Let (X,Σ,μ) and (Y,τ,ν) be two measure spaces. Define $\Sigma\times\tau=\{A\times B\mid A\in\Sigma,\ B\in\tau\}$, and $\Sigma\otimes\tau=\sigma(\Sigma\times\tau)$.

Theorem 10.1. *Let μ,ν be two σ-finite measures. There exists a unique measure π on $\Sigma\otimes\tau$ such that $\pi(A\times B)=\mu(A)\nu(B)$ for every $A\in\Sigma$, $B\in\tau$.*

Theorem 10.2 (Tonelli). *Let $f\colon X\times Y\to[0,\infty]$ be $\Sigma\otimes\tau$-measurable. For every $x_0\in X$, $y_0\in Y$ the functions $x\mapsto f(x,y_0)$ and $y\mapsto f(x_0,y)$ are measurable. Moreover,*

$$\int_{X\times Y} f(x,y)\,d\pi(x,y)=\int_{x\in X}\Big(\int_{y\in Y} f(x,y)\,d\nu(y)\Big)\,d\mu(x)=\int_{y\in Y}\Big(\int_{x\in X} f(x,y)\,d\mu(x)\Big)\,d\nu(y)\,. \tag{10.1}$$

Theorem 10.3 (Fubini). *If $f\in L^1(X\times Y,\pi)$ then for almost every $x_0\in X$, $y_0\in Y$, the functions $x\mapsto f(x,y_0)$ and $y\mapsto f(x_0,y)$ are integrable in x and y respectively. Moreover,(10.1) holds.*

Lemma 10.4. *For every $E\subseteq X\times Y$, $x\in X$, $y\in Y$ define the horizontal and vertical slices of E by $H_y(E)=\{x\in X\mid (x,y)\in E\}$ and $V_x(E)=\{y\in Y\mid (x,y)\in E\}$.*

1. *For every $x\in X$, $y\in Y$ we have $H_y(E)\in\Sigma$ and $V_x(E)\in\tau$.*
2. *The functions $x\mapsto\nu(V_x(E))$ and $y\mapsto\mu(H_y(E))$ are measurable.*

Proof of Theorem 10.1
Proof of Theorem 10.2
Proof of Theorem 10.3

Theorem 10.5 (Layer Cake). *If $f\colon X\to[0,\infty]$ is measurable then $\displaystyle\int_X f\,d\mu=\int_0^\infty \mu(f>t)\,dt$.*

Proposition 10.6. *If* $(a_{m,n})$ *are such that* $\sum_{m,n=0}^{\infty} |a_{m,n}| < \infty$, *then* $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}$.

Theorem 10.7 (Minkowski's inequality)**.** *If* $f\colon X \times Y \to \mathbb{R}$ *is measurable, then*

$$\Big(\int_X \Big|\int_Y f(x,y)\,d\nu(y)\Big|^p \,d\mu(x)\Big)^{1/p} \leqslant \int_Y \Big(\int_X |f(x,y)|^p \,d\mu(x)\Big)^{1/p} \,d\nu(y)$$

10.2. Convolutions.

Definition 10.8. If $f, g \in L^1(\mathbb{R}^d)$ define the *convolution* by $f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy$.

Remark 10.9. If $f, g \in L^1(\mathbb{R}^d)$, then $f * g < \infty$ almost everywhere.

Theorem 10.10 (Young)**.** *If* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *then* $f * g \in L^r(\mathbb{R}^d)$, *and* $\|f * g\|_{L^r} \leqslant \|f\|_{L^p}\|g\|_{L^q}$.

Remark 10.11. One can show $\|f * g\|_r \leqslant C_{p,q}\|f\|_p\|g\|_q$ for some constant $C_{p,q} < 1$. The optimal constant can be found by choosing f, g to be Gaussian's.

Definition 10.12. (φ_n) is an *approximate identity* if: (1) $\varphi_n \geqslant 0$, (2) $\int_{\mathbb{R}^d} \varphi_n = 1$, and (3) $\forall \varepsilon > 0$, $\lim_{n\to\infty} \int_{\{|y|>\varepsilon\}} \varphi_n(y)\,dy = 0$.

Example 10.13. Let $\varphi \geqslant 0$ be any function with $\int_{\mathbb{R}^d} \varphi = 1$, and set $\varphi_\varepsilon = \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})$.

Example 10.14. $G_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$, for $x \in \mathbb{R}^d$.

Proposition 10.15. *If* $p \in [1, \infty)$, $f \in L^p$, *and* (φ_n) *is an approximate identity, then* $\varphi_n * f \to f$ *in* L^p.

Remark 10.16. For $p = \infty$ the above is still true at points where f is continuous.

10.3. **Fourier Series.** Let $X = [0,1]$ with the Lebesgue measure. For $n \in \mathbb{Z}$ define $e_n(x) = e^{2\pi i n x}$, and given $f, g \in L^2(X, \mathbb{C})$ define $\langle f, g \rangle = \int_X f \bar{g} \, d\lambda$. This defines an *inner product* on $L^2(X)$, and $\|f\|_{L^2}^2 = \langle f, f \rangle$.

Definition 10.17. If $f \in L^2$, $n \in \mathbb{Z}$, define the n^{th} Fourier coefficient of f by $\hat{f}(n) = \langle f, e_n \rangle$.

Definition 10.18. For $N \in \mathbb{N}$, let $S_N f = \sum_{-N}^{N} \hat{f}(n) e_n$, be the N-th partial sum of the *Fourier Series* of f.

Question 10.19. *Does $S_N f \to f$? In what sense?*

Lemma 10.20. $\langle e_n, e_m \rangle = \delta_{n,m}$.

Corollary 10.21. *Let $p \in \operatorname{span}\{e_{-N}, \dots, e_N\}$. Then $\langle f - S_N f, p \rangle = 0$. Consequently, $\|f - S_N f\|_2 \leqslant \|f - p\|_2$.*

Proposition 10.22. *$S_N f = D_N * f$, where $D_N = \dfrac{\sin(2\pi(N + \frac{1}{2})x)}{\sin(\pi x)}$. The functions D_N are called the* Dirichlet Kernels.

Proposition 10.23. *Define the* Cesàro sum *by $\sigma_N f = \frac{1}{N} \sum_0^{N-1} S_n f$. Then $\sigma_N f = F_N * f$, where $F_N = \dfrac{1}{N} \Big(\dfrac{\sin(N\pi x)}{\sin(\pi x)} \Big)^2$.*

Remark 10.24. The functions F_N are called the *Fejér Kernels.*

Proposition 10.25. *The Fejér kernels are an approximate identity, but the Dirichlet kernels are not.*

Corollary 10.26. *If $p \in [1, \infty)$ and $f \in L^p$, then $\sigma_N f \to f$ in L^p.*

Corollary 10.27. *If $f \in L^2$ then $S_N f \to f$ in L^2.*

Theorem 10.28. *If $p \in (1, \infty)$, $f \in L^p$ then $S_N f \to f$ in L^p.*

Proof. The proof requires boundedness of the Hilbert transform and is beyond the scope of this course. $\square$

Theorem 10.29. *If $f \in L^\infty$ and is Hölder continuous at x with any exponent $\alpha > 0$, then $S_n f(x) \to x$.*

Proof. On homework. □

Remark 10.30. If f is simply continuous at x, then certainly $\sigma_n f(x) \to f(x)$, but $S_n f(x)$ need not converge to $f(x)$. In fact, for almost every continuous periodic function function, $S_N f$ diverges on a dense G_δ.

The next few results establish a connection between the regularity (differentiability) of a function and decay of its Fourier coefficients.

Theorem 10.31 (Riemann Lebesgue)**.** *Let μ be a finite measure and set $\hat{\mu}(n) = \int_0^1 \bar{e}_n \, d\mu$. If $\mu \ll \lambda$, then $(\hat{\mu}(n)) \to 0$ as $n \to \infty$.*

Theorem 10.32 (Parseval's equality)**.** *If $f \in L^2([0,1])$ then $\|\hat{f}\|_{\ell^2} = \|f\|_{L^2}$.*

Question 10.33. *What are the Fourier coefficients of f'?*

Definition 10.34. We say g is a *weak derivative* of f if $\langle f, \varphi' \rangle = -\langle g, \varphi \rangle$ for all $\varphi \in C^\infty_{per}([0,1])$.

Proposition 10.35. *If $f \in L^1$ has a weak derivative $f' \in L^1$, then $(f')^\wedge(n) = 2\pi i n \hat{f}(n)$.*

Corollary 10.36. *If $f \in L^2$ has a weak derivative $f' \in L^2$, then $\sum [(1+|n|)|\hat{f}(n)|]^2 < \infty$.*

Definition 10.37. For $s \geqslant 0$, let $H^s_{per} \stackrel{\text{def}}{=} \{f \in L^2 \mid \|f\|_{H^s} < \infty\}$, where $\|f\|^2_{H^s} = \sum (1+|n|)^{2s} |\hat{f}(n)|^2$.

Remark 10.38. H^s is essentially the space of L^2 functions that also have s "weak derivatives" in L^2.

Theorem 10.39 (1D Sobolev Embedding)**.** *If $s > \frac{1}{2}$ and $H^s_{per} \subseteq C_{per}([0,1])$ and the inclusion map is continuous.*

Remark 10.40. Need $s > \frac{1}{2}$. The theorem is false when $s = 1/2$.

Remark 10.41. In d dimensions the above is still true if you assume $s > d/2$.

Remark 10.42. More generally one can show for $\alpha \in (0,1)$, $s = \frac{1}{2} + n + \alpha$, $H^s_{per} \subseteq C^{n,\alpha}$.

Theorem 10.43 (1D Sobolev embedding). *If $s > \frac{1}{2} - \frac{1}{2n}$, then $H^s_{per} \subseteq L^{2n}$ and the inclusion map is continuous.*

Remark 10.44. The above is true for $s = \frac{1}{2} - \frac{1}{p}$, for some $p \in [1, \infty)$ but our proof won't work.

11. Differentiation

11.1. Lebesgue Differentiation.

Theorem 11.1 (Fundamental theorem of Calculus 1). *If f is continuous and $F(x) = \int_0^x f(t)\,dt$, then F is differentiable and $F' = f$.*

Theorem 11.2 (Fundamental theorem of Calculus 2). *If f is Riemann integrable, and $F' = f$, then $\int_a^b f = F(b) - F(a)$.*

Our goal is to generalize these to Lebesgue integrable functions.

Theorem 11.3 (Lebesgue Differentiation). *If $f \in L^1(\mathbb{R}^d)$, then for almost every $x \in \mathbb{R}^d$, $\lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f\,d\lambda = f(x)$.*

Lemma 11.4 (Vitali Covering Lemma). *Let $W \subseteq \cup_1^N B(x_i, r_i)$. There exists $S \subseteq \{1, \dots, N\}$ such that:*

1. *$\{B(x_i, r_i) \mid i \in S\}$ are pairwise disjoint.*
2. *$W \subseteq \cup_{i \in S} B(x_i, 3r_i)$ and hence $|W| \leqslant 3^d \sum_{i \in S} B(x_i, r_i)$.*

Definition 11.5 (Maximal function). Let μ be a finite (signed) Borel measure on $\mathbb{R}^d$. Define the *maximal function* of μ by

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{|B(x,r)|}$$

Proposition 11.6. *$M\mu \in L^{1,\infty}$, and $|M\mu > \alpha| \leqslant \frac{3^d}{\alpha}\|\mu\|$.*

Corollary 11.7. *If $f \in L^1(\mathbb{R}^d)$, then $|\{Mf > \alpha\}| \leqslant \frac{3^d}{\alpha}\|f\|_{L^1}$.*

Proposition 11.8. *If $f \in L^1(\mathbb{R}^d)$, then $\lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{|y-x|<r} |f(y) - f(x)|\, dy = 0$ almost everywhere.*

Remark 11.9. This immediately implies Theorem 11.3.

Corollary 11.10. *If $\mu \ll \lambda$ is a finite signed measure, then the Radon-Nikodym derivative is given by $\frac{d\mu}{d\lambda} = \lim_{r\to 0} \frac{\mu(B(x,r))}{|B(x,r)|}$.*

Remark 11.11. Will use this to prove the change of variables formula.

11.2. **Fundamental theorem of calculus.**

Question 11.12. *Does $f\colon [0,1] \to \mathbb{R}$ differentiable almost everywhere imply $f' \in L^1$?*

Question 11.13. *Does $f\colon [0,1] \to \mathbb{R}$ differentiable almost everywhere, and $f' \in L^1$ imply $f(x) = \int_0^x f'$?*

Definition 11.14. We say $f\colon \mathbb{R} \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_1^N |x_i - y_i| < \delta \implies \sum_1^N |f(x_i) - f(y_i)| < \varepsilon$.

Remark 11.15. Any absolutely continuous function is continuous, but not conversely.

Theorem 11.16. *Let $f\colon [a,b] \to \mathbb{R}$ be measurable. Then f is absolutely continuous if and only if f is differentiable almost everywhere, $f' \in L^1$, and $f(x) - f(a) = \int_a^x f'$ everywhere.*

Proof of the reverse implication of Theorem 11.16

Lemma 11.17. *If f is absolutely continuous, monotone and injective, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ everywhere.*

Lemma 11.18. *If f is absolutely continuous and monotone, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ almost everywhere.*

Lemma 11.19. *If f is absolutely continuous then there exist g, h increasing such that $f = g - h$.*

Proof of the forward implication of Theorem 11.16. Follows immediately from the previous lemmas. □

11.3. **Change of variables.**

Theorem 11.20. *Let $U, V \subseteq \mathbb{R}^d$ be open and $\varphi\colon U \to V$ be C^1 and bijective. If $f \in L^1(V)$, then $\displaystyle\int_V f\, d\lambda = \int_U f \circ \varphi |\det \nabla\varphi|\, d\lambda$.*

The main idea behind the proof is as follows: Let $\mu(A) = \lambda(\varphi(A))$.

Lemma 11.21. *μ is a Borel measure and $\int_U f \circ \varphi\, d\mu = \int_V f\, d\lambda$.*

Lemma 11.22. *$\mu \ll \lambda$*

Lemma 11.23. *$D\mu = |\det \nabla\varphi|$, where $D\mu(x) = \displaystyle\lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|}$.*

Proof of Theorem 11.20. Follows immediately from the above Lemmas. □

Proof of Lemma 11.21
Proof of Lemma 11.22
Proof of Lemma 11.23

12. **Fourier Transform**

12.1. **Definition and Basic Properties.**

(1) Recall if $f \in L^2_{per}([0,1])$, we set $e_n(x) = e^{2\pi i n x}$, $a_n = \int_0^1 f(x) e^{-2\pi i n x}\, dx$ and got $f = \sum a_n e_n$ in L^2.
(2) Suppose now $f \in L^2_{per}([-L/2, L/2])$. Can we rescale and send $L \to \infty$?

Definition 12.1. If $f \in L^1(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, define the *Fourier transform* of f (denoted by $\hat{f}$) by $\hat{f}(\xi) = \displaystyle\int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx$

Remark 12.2. More generally, if μ is a finite (signed) Borel measure, then can define $\hat{\mu}(\xi) = \displaystyle\int_{\mathbb{R}^d} e^{-2\pi i \langle x, \xi \rangle}\, d\mu(x)$.

Analogous to Fourier series, we will show that $\hat{f}$ is defined even for $f \in L^2$, and prove $f(x) = \displaystyle\int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle}\, d\xi$.

Lemma 12.3 (Linearity). *If $f, g \in L^1$, $\alpha \in \mathbb{R}$ then $(f + \alpha g)^\wedge = \hat{f} + \alpha \hat{g}$.*

Lemma 12.4 (Translations). *Let $\tau_y f(x) = f(x - y)$. Then $(\tau_y f)^\wedge(\xi) = e^{-2\pi i \langle y, \xi \rangle} \hat{f}(\xi)$.*

Lemma 12.5 (Dilations). *Let $\delta_\lambda f(x) = \frac{1}{\lambda^d} f(\frac{x}{\lambda})$. Then $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda \xi)$.*

Lemma 12.6. *If $f, g \in L^1$, then $(f * g)^\wedge = \hat{f}\hat{g}$.*

Lemma 12.7. *If $(1 + |x|) f(x) \in L^1(\mathbb{R}^d)$ then $\partial_j \hat{f}(\xi) = (-2\pi i x_j f(x))^\wedge(\xi)$.*

Lemma 12.8. *If $f \in C^1_0$, $\partial_j f \in L^1$, then $(\partial_j f)^\wedge(\xi) = 2\pi i \xi_j \hat{f}(\xi)$.*

Theorem 12.9 (Riemann-Lebesgue Lemma). *If $f \in L^1$, then $\hat{f} \in C_0$ and $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$.*

12.2. **Fourier Inversion.**

Theorem 12.10 (Inversion)**.** *If $f, \hat{f} \in L^1$, then* $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} \, d\xi.$

Direct proof attempt:

Lemma 12.11. *If* $G(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$, *then* $\hat{G}(\xi) = e^{-|2\pi\xi|^2/2}$, *and hence* $\hat{\hat{G}} = G$.

Lemma 12.12. *If* $f, g \in L^1$ *then* $\int_{\mathbb{R}^d} f \hat{g} = \int_{\mathbb{R}^d} \hat{f} g$.

Lemma 12.13. *If* $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *and* $\hat{f} \in L^1(\mathbb{R}^d)$, *then* $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} \, d\xi$.

Proof of Theorem 12.10.

Remark 12.14. If $f, \hat{f} \in L^1$, then $\| f - \varphi_\varepsilon * f \|_{L^\infty} \leqslant \| \hat{f} - (\varphi_\varepsilon * f)^\wedge \|_{L^1} \to 0$

Remark 12.15. If $f, \hat{f} \in L^1$ then $\hat{\hat{f}}(x) = f(-x)$.

12.3. L^2**-theory.**

Theorem 12.16 (Plancherel)**.** *The Fourier transform extends to a bijective linear isometry on* $L^2(\mathbb{R}^d; \mathbb{C})$.

Definition 12.17. Define the *Schwartz space*, $\mathcal{S}$, to be the set of all smooth functions such that $\sup_x (1 + |x|^n) |D^\alpha f(x)| < \infty$ for all $n \in \mathbb{N}$ and multi-indexes α.

Remark 12.18. Note $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}$, and so $\mathcal{S}$ is a dense subset of $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

Lemma 12.19. *If* $f, g \in \mathcal{S}$, *then* $\int_{\mathbb{R}^d} f \, \bar{g} \, dx = \int_{\mathbb{R}^d} \hat{f} \, \bar{\hat{g}} \, d\xi$.

Proof of Theorem 12.16

Definition 12.20. Let $s \geqslant 0$ and define the *Sobolev space of index s* by

$$H^s = \{f \in L^2(\mathbb{R}^d) \mid \|f\|_{H^s} < \infty\}, \qquad \text{where} \qquad \|f\|_{H^s} = \Big(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi\Big)^{1/2}.$$

Remark 12.21. A function $f \in H^1$ if and only if f and all first order *weak derivatives* are in L^2.

Remark 12.22. For $s < 0$, one needs to define H^s as the completion of $\mathcal{S}$ under the H^s norm.

Proposition 12.23. *Let $s \in (0,1)$. Then $f \in H^s$ if and only if $\int_0^\infty \Big(\frac{\|f - \tau_{hv} f\|_{L^2}}{|h|^s}\Big)^2 \frac{dh}{h^d} < \infty$ for all $v \in \mathbb{R}^d$.*

Remark 12.24. For $s = 1$, we instead need $\sup_{h>0} \frac{1}{h} \|f - \tau_{hv} f\|_{L^2} < \infty$.

Remark 12.25. If $s \in (0,1]$, then there exists $C = C(s)$ such that $\|f - \tau_h f\|_{L^2} \leqslant C|h|^s \|f\|_{L^2}$ for all $f \in H^s$, $h \in \mathbb{R}^d$.

Theorem 12.26 (Sobolev embedding)**.** *If $s > d/2$ then $H^s(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d)$, and the inclusion map is continuous.*

Corollary 12.27. *If $s > n + d/2$, then $H^s(\mathbb{R}^d) \subseteq C_b^n(\mathbb{R}^d)$ and the inclusion map is continuous.*

Proposition 12.28 (Elliptic regularity)**.** *Say $f \in \mathcal{S}(\mathbb{R}^d)$, $u \in H^2(\mathbb{R}^d)$ is such that $\lim_{|x|\to\infty} |x|^d |\nabla u(x)| = 0$ and $-\Delta u = f$, then $u \in \mathcal{S}$.*

Appendix A. The d-dimensional Hausdorff measure in $\mathbb{R}^d$

Let (X, d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and $H^*_{\alpha,\delta}$ be the outer measure defined by

$$H^*_{\alpha,\delta}(A) = \inf\Big\{\sum_1^\infty \rho_\alpha(E_i) \,\Big|\, \operatorname{diam}(E_i) < \delta\,, \text{ and } A \subset \bigcup_1^\infty E_j\Big\}, \quad \text{where} \quad \rho_\alpha(A) = \frac{\pi^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}\Big(\frac{\operatorname{diam}(A)}{2}\Big)^\alpha.$$

Remark A.1. The function ρ_α above are chosen so that if $A = B(0, r) \subseteq \mathbb{R}^d$, then $\rho_d(A) = |A|$.

Definition A.2. Let $H^*_\alpha = \lim_{\delta \to 0} H^*_\alpha$.

Proposition A.3 (From homework 2)**.** *The outer measure H^*_α restricts to a measure on the Borel σ-algebra.*

Theorem A.4. *If $X = \mathbb{R}^d$, and $\alpha = d$ then $H_\alpha = \lambda$ (the Lebesgue measure).*

Lemma A.5 (Infinite version of Vitali's Covering Lemma)**.** *Let $W \subseteq \cup_{\alpha \in A} B(x_\alpha, r_\alpha)$, with $\sup r_\alpha < \infty$. There exists a countable set $\mathcal{I} \subseteq A$ such that:*

(1) $\{B(x_i, r_i) \mid i \in \mathcal{I}\}$ are pairwise disjoint.
(2) $W \subseteq \cup_{i \in \mathcal{I}} B(x_i, 5r_i)$ and hence $|W| \leqslant 5^d \sum_{i \in S} B(x_i, r_i)$.

Lemma A.6. *Let $U \subseteq \mathbb{R}^d$ be open and $\delta > 0$. There exists countably many $x_i \in U$, $r_i \in (0, \delta)$ such that $\overline{B(x_i, r_i)} \subseteq U$, are pairwise disjoint, and $|U - \cup \overline{B(x_i, r_i)}| = 0$.*

Lemma A.7. *$H_d \leqslant \lambda$.*

Theorem A.8 (Isodiametric inequality)**.** $|A| \leqslant |B(0, 1/2)| \operatorname{diam}(A)^d = |B(0, \operatorname{diam}(A)/2)|$.

Remark A.9. Note A need not be contained in a ball of radius $\operatorname{diam}(A)/2$.

Proof of Theorem A.4.

Proposition A.10 (Steiner Symmetrization)**.** *Let $P \subseteq \mathbb{R}^d$ be a hyperplane with unit normal $\hat{n}$. Let $A \in \mathcal{L}(\mathbb{R}^d)$. There exists $S_P(A) \in \mathcal{L}(\mathbb{R}^d)$ such that:*

(1) $S_P(A)$ is symmetric about P (i.e. for any $x \in P$, $t \in \mathbb{R}$, we have $x + t\hat{n} \in S_P(A) \iff x - t\hat{n} \in S_P(A)$).
(2) $\operatorname{diam}(S_P(A)) \leqslant \operatorname{diam}(A)$.

(3) $|S_P(A)| = |A|$.

Proof of Theorem A.8