VIP Cheatsheet: First-order ODE

Afshine Amidi and Shervine Amidi

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Introduction

❒ **Differential Equations** – A differential equation is an equation containing derivatives of a dependent variable y with respect to independent variables x. In particular,

- Ordinary Differential Equations (ODE) are differential equations having one independent variable.
- Partial Differential Equations (PDE) are differential equations having two or more independent variables.

❒ **Order** - An ODE is said to be of order n if the highest derivative of the unknown function in the equation is the n^{th} derivative with respect to the independent variable.

❒ **Linearity** – An ODE is said to be linear only if the function y and all of its derivatives appear by themselves. Thus, it is of the form:

$$\boxed{a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y+b(x)=0}$$

Direction Field Method

❒ **Implicit form** – The implicit form of an ODE is where y' is not separated from the remaining terms of the ODE. It is of the form:

$$\boxed{F(x,y,y')=0}$$

Remark: Sometimes, y' cannot be separated from the other terms and the implicit form is the only one that we can write.

❒ **Explicit form** – The explicit form of an ODE is where y' is separated from the remaining terms of the ODE. It is of the form:

$$\boxed{y'=f(x,y)}$$

❒ **Direction field method** – The direction field method is a graphical representation for the solution of ODE $y'=f(x,y)$ without actually solving for $y(x)$. Here is the procedure:

- Determine the values (x_i,y_i) that form the grid.
- Compute the slope $f(x_i,y_i)$ for each point of the grid.
- Report the associated vector for each point of the grid.

Separation of variables

❒ **Separable** – An ODE is said to be separable if it can be written in the form:

$$\boxed{f(x,y)=g(x)h(y)}$$

❒ **Reduction to separable form** – The following table sums up the variable changes that allow us to change the ODE $y'=f(x,y)$ to $u'=g(x,u)$ that is separable.

Original form	Change of variables	New form
$y'=f\left(\frac{y}{x}\right)$	$u\triangleq\frac{y}{x}$	$u'x+u=f(u)$
$y'=f\left(ax+by+c\right)$	$u\triangleq ax+by+c$	$\frac{u'-a}{b}=f(u)$

Equilibrium

❒ **Characterization** – In order for an ODE to have equilibrium solutions, it must be (1) autonomous and (2) have a value y^* that makes the derivative equal to 0, i.e:

$$(1)\quad\boxed{\frac{dy}{dt}=f(\cancel{t},y)=f(y)}\quad\text{and}\quad(2)\quad\boxed{\exists\ y^*,\ \frac{dy^*}{dt}=f(y^*)=0}$$

❒ **Stability** – Equilibrium solutions can be classified into 3 categories:

- Unstable: solutions run away with any small change to the initial conditions.
- Stable: any small perturbation leads the solutions back to that solution.
- Semi-stable: a small perturbation is stable on one side and unstable on the other.

Linear first-order ODE technique

❒ **Standard form** – The standard form of a first-order linear ODE is expressed with $p(x),r(x)$ known functions of x, such that:

$$\boxed{y'+p(x)y=r(x)}$$

Remark: If $r=0$, then the ODE is homogenous, and if $r\neq0$, then the ODE is inhomogeneous.

❒ **General solution** – The general solution y of the standard form can be decomposed into a homogenous part y_h and a particular part y_p and is expressed in terms of $p(x),r(x)$ such that:

$$\boxed{y=y_h+y_p}\quad\text{with}\quad\boxed{y_h=Ce^{-\int pdx}}\quad\text{and}\quad\boxed{y_p=e^{-\int pdx}\times\int\left[re^{\int pdx}\right]dx}$$

Remarks: Here, for any function p, the notation $\int pdx$ denotes the primitive of p without additive constant. Also, the term $e^{-\int pdx}$ is called the basis of the ODE and $e^{\int pdx}$ is called the integrating factor.

❒ **Reduction to linear form** – The one-line table below sums up the change of variables that we apply in order to have a linear form:

Name, setting	Original form	Change	New form
Bernoulli, $n\in\mathbb{R}\backslash\{0,1\}$	$y'+p(x)y=q(x)y^n$	$u\triangleq y^{1-n}$	$u'+(1-n)p(x)u=(1-n)q(x)$

Existence and uniqueness of an ODE

Here, we are given an ODE $y' = f(x,y)$ with initial conditions $y(x_0) = y_0$.

❒ **Existence theorem** – If $f(x,y)$ is continuous at all points in a rectangular region containing (x_0,y_0), then $y' = f(x,y)$ has at least one solution $y(x)$ passing through (x_0,y_0).
Remark: If the condition does not apply, then we cannot say anything about existence.

❒ **Uniqueness theorem** – If both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous at all points in a rectangular region containing (x_0,y_0), then $y' = f(x,y)$ has a unique solution $y(x)$ passing through (x_0,y_0).
Remark: If the condition does not apply, then we cannot say anything about uniqueness.

Numerical methods for ODE - Initial value problems

In this section, we would like to find $y(t)$ for the interval $[0,t_f]$ that we divide into $N+1$ equally-spaced points $t_0 < t_1 < ... < t_N = t_f$, such that:

$$\frac{dy}{dt} = f(t,y) \quad \text{with} \quad y(0) = y_0$$

❒ **Error** – In order to assess the accuracy of a numerical method, we define its local and global errors $\epsilon_{\textrm{local}}, \epsilon_{\textrm{global}}$ as follows:

$$\boxed{\epsilon_{\textrm{local}} = |y^{\textrm{exact}}(t_n) - y^{\textrm{numerical}}(t_n)|} \quad \text{and} \quad \boxed{\epsilon_{\textrm{global}} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}|y^{\textrm{exact}}(t_n) - y^{\textrm{numerical}}(t_n)|^2}}$$

Remarks: If $\epsilon_{local} = O(h^k)$, then $\epsilon_{global} = O(h^{k-1})$. Also, when we talk about the 'error' of a method, we refer to its global error.

❒ **Taylor series** – The Taylor series giving the exact expression of y_{n+1} in terms of y_n and its derivatives is:

$$\boxed{y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + ... = \sum_{k=0}^{+\infty}\frac{h^k}{k!}y_n^{(k)}}$$

We can also have an expression of y_n in terms of y_{n+1} and its derivatives:

$$\boxed{y_n = y_{n+1} - hy_{n+1}' + \frac{h^2}{2}y_{n+1}'' - \frac{h^3}{6}y_{n+1}''' + ... = \sum_{k=0}^{+\infty}\frac{(-h)^k}{k!}y_{n+1}^{(k)}}$$

❒ **Stability** – The stability analysis of any ODE solver algorithm is performed on the model problem, defined by:

$$\boxed{y' = \lambda y \quad \text{with} \quad y(0) = y_0 \quad \text{and} \quad \lambda < 0}$$

which gives $y_n = y_0\sigma^n$, for which h verifies the condition $|\sigma(h)| < 1$.

❒ **Euler methods** – The Euler methods are numerical methods that aim at estimating the solution of an ODE:

Type	Update formula	Error	Stability condition
Forward Euler	$y_{n+1} = y_n + hf(t_n,y_n)$	$O(h)$	$h < \frac{2}{\lvert\lambda\rvert}$
Backward Euler	$y_{n+1} = y_n + hf(t_{n+1},y_{n+1})$	$O(h)$	None

❒ **Runge-Kutta methods** – The table below sums up the most commonly used Runge-Kutta methods:

Type	Method	Update formula	Error	Stability condition
RK1	Euler's	$y_{n+1} = y_n + hk_1$ where $k_1 = f(t_n,y_n)$	$O(h)$	$h < \frac{2}{\lvert\lambda\rvert}$
RK2	Heun's	$y_{n+1} = y_n + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)$ where $k_1 = f(t_n,y_n)$ and $k_2 = f(t_n+h,y_n+hk_1)$	$O(h^2)$	$h < \frac{2}{\lvert\lambda\rvert}$

System of linear ODEs

❒ **Definition** – A system of n first order linear ODEs

$$\begin{cases} y_1' = a_{11}y_1 + ... + a_{1n}y_n \\ \vdots \\ y_n' = a_{n1}y_1 + ... + a_{nn}y_n \end{cases}$$

can be written in matrix form as:

$$\boxed{\vec{y}\,' = A\vec{y}}$$

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

❒ **System of homogeneous ODEs** – The resolution of the system of 2 homogeneous linear ODEs $\vec{y}\,' = A\vec{y}$ is detailed in the following table:

Case	Eigenvalues $\leftrightarrow$ Eigenvectors	Solution
Real distinct eigenvalues	$\lambda_1 \leftrightarrow \vec{\eta}_{\lambda_1}$ $\lambda_2 \leftrightarrow \vec{\eta}_{\lambda_2}$	$\vec{y} = C_1\vec{\eta}_{\lambda_1}e^{\lambda_1 t} + C_2\vec{\eta}_{\lambda_2}e^{\lambda_2 t}$
Double root eigenvalues	$\lambda \leftrightarrow \vec{\eta}$ $\vec{\rho}$ s.t. $(A-\lambda I)\vec{\rho} = \vec{\eta}$	$\vec{y} = [(C_1 + C_2 t)\vec{\eta} + C_2\vec{\rho}]e^{\lambda t}$
Complex conjugate eigenvalues	$\alpha + i\beta \leftrightarrow \vec{\eta}_R + i\vec{\eta}_I$ $\alpha - i\beta \leftrightarrow \vec{\eta}_R - i\vec{\eta}_I$	$\vec{y} = C_1(\cos(\beta t)\vec{\eta}_R - \sin(\beta t)\vec{\eta}_I)e^{\alpha t}$ $+C_2(\cos(\beta t)\vec{\eta}_I + \sin(\beta t)\vec{\eta}_R)e^{\alpha t}$

VIP Cheatsheet: Second-order ODE

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General case

❒ **General form** – The general form of a second-order ODE can be written as a function F of x, y, y' and y'' as follows:

$$\boxed{F(x,y,y',y'')=0}$$

❒ **Methods of resolution** – The table below summarizes the general tricks to apply when the ODE has the following classic forms:

Old form	Trick	New form
$F\left(x,y',y''\right)=0$	$y'\triangleq u,\quad y''=\frac{du}{dx}$	$G\left(x,u,\frac{du}{dx}\right)=0$
$F\left(y,y',y''\right)=0$	$y'\triangleq u,\quad y''=u\frac{du}{dy}$	$G\left(y,u,\frac{du}{dy}\right)=0$
$F\left(y',y''\right)=0$	$y'\triangleq u,\quad y''=\frac{du}{dx}$	Missing-y approach $G\left(u,\frac{du}{dx}\right)=0$
	$y'\triangleq u,\quad y''=u\frac{du}{dy}$	Missing-x approach $G\left(u,\frac{du}{dy}\right)=0$

❒ **Standard form of a linear ODE** – The standard form of a second-order linear ODE is expressed with p, q and r known functions of x such that:

$$\boxed{y''+p(x)y'+q(x)y=r(x)}$$

for which the total solution y is the sum of a homogeneous solution y_h and a particular solution y_p:

$$\boxed{y=y_h+y_p}$$

Remark: if $r=0$, then the ODE is homogeneous (and we have $y_p=0$). If $r\neq 0$, then the ODE is said to be inhomogeneous.

❒ **Linear dependency** – Two functions y_1, y_2 are said to be *linearly dependent* if $\frac{y_2}{y_1}=C$ constant. Conversely, they are *linearly independent* if $\frac{y_2}{y_1}\neq C$.

Linear homogeneous – Variable coefficients

❒ **Method of reduction of order** – Let y_1 be a solution to the equation $y''+p(x)y'+q(x)y=0$. By noting C_1, C_2 constants, the global solution y_h is written as:

$$\boxed{y_h=C_1y_1+C_2y_1\int\frac{e^{-\int pdx}}{y_1^2}dx}$$

Remark: Here, for any function p, the notation $\int pdx$ denotes the primitive of p without additive constant.

Linear homogeneous – Constant coefficients

❒ **General form** – The general form of a linear homogeneous second-order ODE with a,b,c constant coefficients is:

$$\boxed{ay''+by'+cy=0}$$

❒ **Resolution** – Based on the types of solution of the characteristic equation $\boxed{a\lambda^2+b\lambda+c=0}$, and by noting $\boxed{\Delta=b^2-4ac}$ its discriminant, we distinguish the following cases:

Name	Case	Roots	Solution
Two distinct real roots	$\Delta>0$	$\lambda_1=\frac{-b+\sqrt{\Delta}}{2a}$ $\lambda_2=\frac{-b-\sqrt{\Delta}}{2a}$	$y_h=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$
Double real root	$\Delta=0$	$\lambda=-\frac{b}{2a}$	$y_h=[C_1+C_2x]e^{\lambda x}$
Complex conjugate roots	$\Delta<0$	$\lambda_1=\alpha+i\beta$ $\lambda_2=\alpha-i\beta$ where $\alpha=-\frac{b}{2a}$ and $\beta=\frac{\sqrt{\lvert\Delta\rvert}}{2a}$	$y_h=[C_1\cos(\beta x)+C_2\sin(\beta x)]\,e^{\alpha x}$

A special case: the Euler-Cauchy equation

❒ **General form** – The Euler-Cauchy equation is a special case of linear homogeneous ODEs and has the following general form, where each $a_i\in\mathbb{R}$ is a constant coefficient:

$$\boxed{a_nx^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+...+a_1xy'+a_0y=0}$$

❒ **Second-order case** – For $n=2$, by noting $y=x^m$, the ODE provides the indicial equation:

$$\boxed{am^2+(b-a)m+c=0}$$

with discriminant $\boxed{\Delta=(b-a)^2-4ac}$ and where the resolution of the ODE depends on the cases summarized in the table below.

Name	Case	Roots	Solution
Two distinct real roots	$\Delta > 0$	$m_1 = \dfrac{-b+a+\sqrt{\Delta}}{2a}$ $m_2 = \dfrac{-b+a-\sqrt{\Delta}}{2a}$	$y_h = C_1 x^{m_1} + C_2 x^{m_2}$
Double real root	$\Delta = 0$	$m = -\dfrac{b-a}{2a}$	$y_h = [C_1 + C_2 \ln\|x\|]x^m$
Complex conjugate roots	$\Delta < 0$	$m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$ where $\alpha = -\frac{b-a}{2a}$ and $\beta = \frac{\sqrt{\|\Delta\|}}{2a}$	$y_h = \Big[C_1 \cos(\beta \ln\|x\|)$ $+C_2 \sin(\beta \ln\|x\|)\Big]x^\alpha$

Linear inhomogeneous – Variable coefficients

❒ **Wronskian** – Given y_1 and y_2 the two solutions of the homogeneous equation, we define the Wronskian W as follows:

$$\boxed{W = y_1 y_2' - y_2 y_1'}$$

❒ **Method of Variation of Parameters** – The particular solution y_p of the inhomogeneous ODE is given by:

$$\boxed{y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$$

Linear inhomogeneous – Constant coefficients

❒ **Undetermined coefficients method** – The particular solution y_p of the inhomogeneous ODE $ay'' + by' + cy = r(x)$ is determined from the correspondance table below:

Form of r	Form of y_p
C	A
x^n, $n \in \mathbb{N}^*$	$A_0 + A_1 x + ... + A_n x^n$
$e^{\gamma(x)}$	$Ae^{\gamma x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
$x^n e^{\gamma x}\cos(\omega x)$ or $x^n e^{\gamma x}\sin(\omega x)$	$(A_0 + A_1 x + ... + A_n x^n)\cos(\omega x)e^{\gamma x} +$ $(B_0 + B_1 x + ... + B_n x^n)\sin(\omega x)e^{\gamma x}$

Remark: all new constants are determined after plugging back y_p into the ODE.

❒ **Modification rule** – If the particular solution y_p picked from the above table matches either y_1 or y_2, then has to be multiplied by the lowest power of x such that it is no more the case.

❒ **Sum rule** – If $r(x)$ is a sum of functions of the first column of the above table, then y_p is the sum of its associated particular solutions.

VIP Cheatsheet: Applications

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Physics Laws

❒ **Gravitational force** – A mass m is subject to the gravitational force $\vec{F}_g$, which is expressed with respect to $\vec{g}$ of magnitude 9.81 m $\cdot$ s^{-2} and directed towards the center of the Earth, as follows:

$$\boxed{\vec{F}_g = m\vec{g}}$$

❒ **Spring force** – A spring of constant k and of relaxed position $\vec{x}_0$ attached a mass m of position $\vec{x}$ has a force $\vec{F}_s$ expressed as follows:

$$\boxed{\vec{F}_s = -k(\vec{x} - \vec{x}_0)}$$

❒ **Friction force** – The friction force F_f of constant coefficient β applied on a mass of velocity $\vec{v}$ is written as:

$$\boxed{\vec{F}_f = -\beta\vec{v}}$$

❒ **Mass moment of inertia** – The mass moment of inertia of a system of mass m_i located at distance r_i from point O, expressed in point O is written as:

$$\boxed{J_0 = \sum_i m_i r_i^2}$$

❒ **Torque** – The torque $\vec{T}$ of a force $\vec{F}$ located at $\vec{r}$ from the reference point O is written as:

$$\boxed{\vec{T} = \vec{r} \times \vec{F}}$$

❒ **Newton's second law** – A mass m of acceleration $\vec{a}$ to which forces $\vec{F}_i$ are applied verifies the following equation:

$$\boxed{m\vec{a} = \sum_i \vec{F}_i}$$

– In the 1-D case along the x axis, we can write it as $mx'' = \sum_i F_i$.

– In the rotationary case, around point O, we can write it as $J_0\theta'' = \sum_i T_i$.

Spring-mass system

❒ **Free undamped motion** – A free undamped spring-mass system of mass m and spring coefficient k follows the ODE $x'' + \frac{k}{m}x = 0$, which can be written as a function of the natural frequency ω as:

$$\boxed{x'' + \omega^2 x = 0} \quad \text{with} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

❒ **Free damped motion** – A free damped spring-mass system of mass m, of spring coefficient k and subject to a friction force of coefficient β follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$, which can be written as a function of the damping parameter λ and the natural frequency ω as:

$$\boxed{x'' + 2\lambda x' + \omega^2 x = 0} \quad \text{with} \quad \boxed{\lambda = \frac{\beta}{2m}} \quad \text{and} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

which has the following cases summed up in the table below:

Condition	**Type of motion**
$\lambda > \omega$	Over damped
$\lambda = \omega$	Critically damped
$\lambda < \omega$	Under damped

❒ **Forcing frequency** – A forcing function $F(t)$ is often modeled with a periodic function of the form $F(t) = F_0 \sin(\gamma t)$, where γ is called the forcing frequency.

❒ **Forced undamped motion** – A forced undamped spring-mass system of mass m and spring coefficient k follows the ODE $x'' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the natural frequency ω as:

$$\boxed{x'' + \omega^2 x = F_0 \sin(\gamma t)} \quad \text{with} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

which has the following cases summed up in the table below:

Condition	**Type of motion**
$\gamma \neq \omega$	General response
$\gamma \approx \omega$	Beats
$\gamma = \omega$	Resonance

❒ **Forced damped motion** – A forced damped spring-mass system of mass m, of spring coefficient k and subject to a friction force of coefficient β follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the damping parameter λ and the natural frequency ω as:

$$\boxed{x'' + 2\lambda x' + \omega^2 x = F_0 \sin(\gamma t)} \quad \text{with} \quad \boxed{\lambda = \frac{\beta}{2m}} \quad \text{and} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

Boundary Value Problems

❒ **Types of boundary conditions** – Given a numerical problem between 0 and L, we distinguish the following types of boundary conditions:

Name	Boundary values
Dirichlet	$y(0)$ and $y(L)$
Neumann	$y(0)$ and $y'(L)$
Robin	$y(0)$ and $\alpha y(L)+\beta y'(L)$

❒ **Numerical differentiation** – The table below sums up the approximation of the derivatives of y at point x_j, knowing the values of y at each point of a uniformly spaced set of grid points.

Order of derivative	Name	Formula	Order of error
First derivative	Forward difference	$y_j'=\dfrac{y_{j+1}-y_j}{h}$	$O(h)$
	Backward difference	$y_j'=\dfrac{y_j-y_{j-1}}{h}$	$O(h)$
	Central difference	$y_j'=\dfrac{y_{j+1}-y_{j-1}}{2h}$	$O(h^2)$
Second derivative	Central difference	$y_j''=\dfrac{y_{j+1}-2y_j+y_{j-1}}{h^2}$	$O(h^2)$

❒ **Direct method** – The direct method can solve linear ODEs by reducing the problem to the resolution of a linear system $Ay=f$, where A is a tridiagonal matrix.

❒ **Shooting method** – The shooting method is an algorithm that can solve ODEs through an iterative process. It uses a numerical scheme, such as Runge-Kutta, and converges to the right solution by iteratively searching for the missing initial condition $y'(0)$.

Remark: in the linear case, the shooting method converges after the first two initial guesses.