Lecture Notes on

COMBINATORIAL STRUCTURES IN GRAPH THEORY

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Preliminaries

1.1 Notation

The sets of the real numbers, rational numbers, integers, and nonnegative integers are denoted by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$, respectively.

The **cardinality** of a finite set X, denoted by $|X|$ or by $\#X$, is the number of its elements. A set X with k elements ($|X| = k$) is called a **k-set**, and if a k-set X is a subset of Y, then it is a **k-subset** of Y. A singleton set $X = \{x\}$ is often identified with its sole member x.

Let Y and Z be subsets of a set X. They are **comparable** if $Y \subseteq Z$ or $Z \subseteq Y$. They **overlap** if they intersect, but are not comparable, i.e., Y and Z overlap if they intersect and their symmetric difference $(Y \setminus Z) \cup (Z \setminus Y)$ is nonempty.

For a set X, $2^X = \{Y \mid Y \subseteq X\}$ is its **power set**. A subset of 2^X is called a **family** (of sets). A family is usually denoted by a script letter. For a finite or infinite family $\mathscr{X} \subseteq 2^X$, we adopt the notations

$$\cup\mathscr{X} = \bigcup_{Y \in \mathscr{X}} Y = \{x \mid \exists Y \in \mathscr{X} : x \in Y\},$$
$$\cap\mathscr{X} = \bigcap_{Y \in \mathscr{X}} Y = \{x \mid \forall Y \in \mathscr{X} : x \in Y\}.$$

A family $\mathscr{X} \subseteq 2^X$ of pairwise disjoint subsets (i.e., $X \cap Y = \emptyset$ for all different $X, Y \in \mathscr{X}$) forms a **partition of** X if $X = \cup\mathscr{X}$.

Let $\mathscr{X} = \{X_i \mid i \in I\}$ be a partition of X. A subset $T \subseteq X$ is a **transversal** of $\mathscr{X}$, if $T \cap X_i$ is a singleton for all $i \in I$. Equivalently, we can say that an injective function $\tau : \mathscr{X} \to X$ is a **transversal**, if $\tau(X_i) \in X_i$ for all i.

The permutation ι_X on X with $\iota_X(x) = x$ for all $x \in X$, is the **identity function** on X.

Let $\alpha : X \to Y$ be a function. If it is a bijection, then $\alpha^{-1}\alpha = \iota_X$ and $\alpha\alpha^{-1} = \iota_Y$.

For any $\alpha : X \to Y$ and $Z \subseteq X$, we let

$$\alpha \restriction Z : Z \to Y$$

be the **restriction** of α to Z: $(\alpha \restriction Z)(x) = \alpha(x)$ for all $x \in Z$. Also, let

$$\alpha(Z) = \{\alpha(z) \mid z \in Z\} \text{ and } \alpha^{-1}(U) = \{x \mid x \in X,\ \alpha(x) \in U\}.$$

The relation

$$\ker(\alpha) = \{(x, y) \in X \times X \mid \alpha(x) = \alpha(y)\}$$

is the **kernel** of $\alpha : X \to Y$.

1.2 Edge coloured graphs

Let
$$E_2(D) = \{(x, y) \mid x, y \in D \text{ with } x \neq y\}$$
be the set of all ordered non-reflexive pairs of a set D. An **undirected graph** $g = (D, E)$ consists of a nonempty finite set D of **vertices** together with the set $E \subseteq E_2(D)$ of the **edges** satisfying

$$(u, v) \in E \iff (v, u) \in E\,.$$

for all $u, v \in D$. Hence we assume that a graph g does not have any *loops* (x, x). Also, by definition, g does not have multiple edges between the same vertices.

Example 1.1. The graph $K_D = (D, E_2(D))$ is **complete** on D. □

Each undirected graph $g = (D, E)$ can be identified with the *characteristic function* of its set of edges, i.e., with the function

$$g\colon E_2(D) \to \{0, 1\}, \quad g(e) = \begin{cases} 1 & \text{if } e \in E, \\ 0 & \text{if } e \notin E. \end{cases}$$

Motivated by this, we generalize the notion of an undirected graph for arbitrary sets of colours.

For each pair $e = (x, y) \in E_2(D)$, let $e^{-1} = (y, x)$ be its **reverse**.

Let Δ be a set of **colours** (or **labels**). A Δ-**graph** on the domain $D = D_g$ is a function

$$g\colon E_2(D) \to \Delta\,,$$

such that, for all $e_1, e_2 \in E_2(D)$,

$$g(e_1) = g(e_2) \iff g(e_1^{-1}) = g(e_2^{-1})\,. \tag{1.1}$$

We do not assume that Δ is finite, and so g need not be surjective, i.e., it may miss some colours.

Remark 1.1. The **reversibility condition** (1.1) is adopted only in order to simplify the treatment. It means that there is an **involution** $\delta\colon \Delta \to \Delta$ on the set of colours, i.e., δ is a permutation satisfying
$$\delta^2 = \iota \text{ and } g(e^{-1}) = \delta(g(e))\,.$$
Later when the set of colours has a group structure, the involution δ is assumed to satisfy $\delta(ab) = \delta(b)\delta(a)$ with respect to the given group operation.

A colour $a \in \Delta$ is said to be **symmetric**, if $a = \delta(a)$.

Example 1.2. (i) An *undirected graph* is a $\{0,1\}$-graph with $\delta(0)=0$ and $\delta(1)=1$, where the symmetric colour 1 can be interpreted as saying that the pair is an edge.

(ii) If we have three colours, say $\Delta=\{a,b,c\}$, then one of them must be symmetric, say c, and the other two can be reverse colours, $b=\delta(a)$ (and so $\delta(a)=b$). In this case a Δ-graph g is **oriented**. The edges of an undirected graph have been given orientations, say

$$g(x,y)=\begin{cases} a & \text{if } (x,y) \text{ is an edge,}\\ b & \text{if } (y,x) \text{ is an edge,}\\ c & \text{otherwise.}\end{cases}$$

(iii) A **tournament** is an oriented graph with no edges of the symmetric colour. (It is an orientation of the complete undirected graph.)

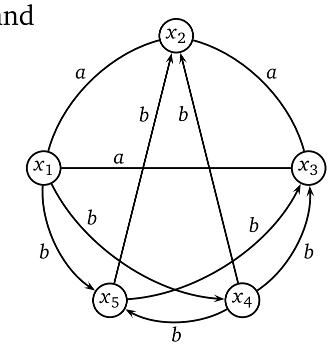
(iv) Every **directed graph** can be presented as a Δ-graph for $\Delta=\{a,b,c,d\}$, where a is a symmetric colour, denoting an edge in both directions (x,y) and (y,x) (these will be called *undirected edges*), b is also symmetric, denoting a non-edge in both direction, and $c(=\delta(d))$, denoting an oriented edge. □

Remark 1.2. As already present in the previous example, the general definition of Δ-graphs does not (necessarily) distinguish edges and non-edges without an interpretation. While using colours, we consider the colour set Δ 'up to isomorphism'.

Remark 1.3. As usual, a graph g is drawn by representing the vertices as points in the plane and the edges as connecting arcs together with the label $g(e)$. In general, the so obtained picture could be bit complicated. Reversibility makes the drawing more convenient: we can

- omit the reverse colours of the chosen non-symmetric colours,
- draw a line without arrow heads for symmetric colours, and
- omit one symmetric colour.

Example 1.3. In the figure we have a Δ-graph, where $\Delta=\{a,b,c\}$ and $g\colon E_2(\{x_1,x_2,x_3,x_4,x_5\})\to\{a,b,c\}$ with $\delta(a)=a$ and $\delta(b)=c$ (and hence $\delta(c)=b$). You do not see the reverse colour c. Also, we could have discarded the edges having the symmetric colour a, but we didn't. There you are.



□

Let g be a Δ_g-graph and h a Δ_h-graph. Then they are **isomorphic**, denoted $g\cong h$, if there are bijections $\alpha\colon D_g\to D_h$ and $\psi\colon \Delta_g\to\Delta_h$ such that

$$\psi(g(x,y))=h(\alpha(x),\alpha(y)).$$

A graph $h\colon E_2(A)\to\Delta_h$ with $A\subseteq D$ is a **subgraph** of $g\colon E_2(D)\to\Delta_g$ (**induced by** A) if $\Delta_h\subseteq\Delta_g$, and $h=g\restriction A$. In this case, we denote $h=g[A]$. In particular,

$$g[A](e_1)=g[A](e_2) \iff g(e_1)=g(e_2) \text{ for all } e_1,e_2\in E_2(A).$$

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Clans

2.1 Definition

Let $g\colon E_2(D) \to \Delta$ be a Δ-graph. We denote $y \to_g X$, if $y \notin X$ 'sees' all elements of X by the same colour, i.e., if

$$g(y, x_1) = g(y, x_2) \ \text{ for all } x_1, x_2 \in X.$$

For disjoint subsets Y and X, write $Y \to_g X$, if $g(y, x)$ is the same for all $y \in Y$ and $x \in X$.

A subset $X \subseteq D$ is a **clan** (or a **module**) if no vertex $y \notin X$ can distinguish the vertices of X by colours, i.e., for all $y \notin X$,

$$y \to_g X.$$

The set of all clans of g is denoted by

$$\mathscr{C}(g) = \{X \subseteq D_g \mid X \text{ a clan}\}.$$

- A clan $X \in \mathscr{C}(g)$ is said to be **proper** if X is a proper subset of the domain, i.e., $X \neq D_g$.
- The empty set $\emptyset$, the domain D_g, and the singletons $\{x\}$, $x \in D_g$, are always clans. These are the **trivial clans**. A **nontrivial clan** is a proper clan X with $|X| \geq 2$.
- If g has no other than the trivial clans it is called **primitive**. Moreover, it is **truly primitive**, if it has at least three three vertices. The primitive graphs are of special importance since they are the 'undecomposable graphs'.
- The subgraph $g[X]$ induced by a nonempty clan $X \in \mathscr{C}(g)$ is called a **factor** of g.

Example 2.1. The graph in Example 1.3 has only one nontrivial clan $X = \{x_2, x_3\}$. □

Example 2.2. Consider the Δ-graph g in Fig. 2.1, where the symmetric colour d is omitted (e.g., $g(x_1, x_3) = d = g(x_3, x_1)$). Here $X = \{x_1, x_2, x_3\}, Y = \{x_7, x_8\}, Z = \{x_1, x_3\} \in \mathscr{C}(g)$. On the other hand, $\{x_1, x_4\} \notin \mathscr{C}(g)$, because, e.g., x_6 distinguishes x_1 and x_4. Indeed, $g(x_6, x_1) = c$ and $g(x_6, x_4) = d$. □

Exercise 2.1. Let $h = g[X]$ for some $X \in \mathscr{C}(g)$. Show that $\mathscr{C}(h) \subseteq \mathscr{C}(g)$.

Exercise 2.2. Let $x, y \in D_g$ be different vertices of g. Show that there exists a unique $X \in \mathscr{C}(g)$ that is maximal with respect to inclusion such that $x \in X$ and $y \notin X$.

The following lemma gives the basic closure properties of clans. It is important, because it provides a set theoretic characterization of the family of clans. This explains why many proofs reduce to these closure properties.

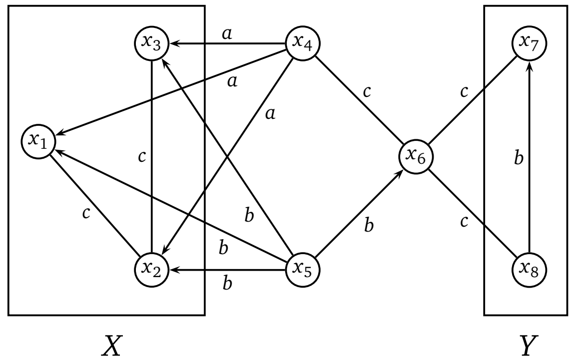


Fig. 2.1. Two clans X and Y in a reversible g. The colour d is not drawn.

Lemma 2.1. *Let g be a Δ-graph, and let $X, Y \in \mathscr{C}(g)$. Then*

(i) $X \cap Y \in \mathscr{C}(g)$;
(ii) if $X \cap Y \neq \emptyset$, then $X \cup Y \in \mathscr{C}(g)$;
(iii) if $Y \setminus X \neq \emptyset$, then $X \setminus Y \in \mathscr{C}(g)$.

Proof. For (i), let $z \notin X \cap Y$. Suppose first that $z \notin X$. Since $X \in \mathscr{C}(g)$, we have $z \to_g X$, and since $X \cap Y \subseteq X$, also $z \to_g X \cap Y$. Similarly, if $z \notin Y$, then $z \to_g X \cap Y$, and this proves that $X \cap Y \in \mathscr{C}(g)$.

For (ii), let $z \notin X \cup Y$. By the hypothesis, there exists a vertex $x_0 \in X \cap Y$. Since X and Y are clans, $z \to_g X$ and $z \to_g Y$, and hence $z \to_g X \cup Y$, because $z \to_g \{x_0, x\}$ for all $x \in X$ and $z \to_g \{x_0, y\}$ for all $y \in Y$. It follows that $X \cup Y \in \mathscr{C}(g)$.

For (iii), denote $Z = X \setminus Y$, and let $x \notin Z$, $z_1, z_2 \in Z$. By the hypothesis, there exists a vertex $y \in Y \setminus X$. If $x \notin X$, then $x \to_g X \setminus Y$, because X is a clan. On the other hand, if $x \in X$, then necessarily $x \in X \cap Y$. Because Y is a clan,

$$g(x, z_1) = g(y, z_1) = g(y, z_2) = g(x, z_2), \tag{2.1}$$

where $g(y, z_1) = g(y, z_2)$, since X is a clan. Therefore $Z = X \setminus Y$ is a clan. □

Remark 2.1. In Lemma 2.1(ii) the requirement $X \cap Y \neq \emptyset$ is essential. Indeed, if g is a truly primitive graph, then always $\{x\} \cup \{y\} \notin \mathscr{C}(g)$ for different vertices $x, y \in D_g$ although the singletons are clans of g.

Exercise 2.3. Let g be a symmetric graph, i.e., all colours are symmetric. Show that if two clans $X, Y \in \mathscr{C}(g)$ intersect, then also their symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ is a clan of g.

Corollary 2.1. *Let g be a Δ-graph. If $X, Y \in \mathscr{C}(g)$ overlap, then also $X \cap Y$, $X \cup Y$, $X \setminus Y \in \mathscr{C}(g)$.*

By the next theorem, the clans of a factor $g[X]$ are exactly the clans of g contained in X.

Theorem 2.1. *For each factor $h = g[X]$ of a Δ-graph g,*

$$\mathscr{C}(h) = \{Y \mid Y \in \mathscr{C}(g),\ Y \subseteq X\}.$$

Proof. If $Y \in \mathscr{C}(g)$ with $Y \subseteq X$, then clearly $Y \in \mathscr{C}(h)$. The converse follows from the fact $\mathscr{C}(h) \subseteq \mathscr{C}(g)$, which is clear from the definition of a clan. □

The next lemma is crucial for the definition of a quotient.

Lemma 2.2. *Let g be a Δ-graph. If $X, Y \in \mathscr{C}(g)$ with $X \cap Y = \emptyset$, then $X \to_g Y$.*

Proof. Let $X \cap Y = \emptyset$, where $X, Y \in \mathscr{C}(g)$. For all $y \in Y$, we have $y \to_g X$, since X is a clan, and $x \to_g Y$ for all $x \in X$, since Y is a clan. It follows that $X \to_g Y$ as required. □

The above situation simplifies a drawing of a Δ-graph g that has disjoint clans X and Y. Instead of drawing an arrow (or a line) between all the vertices $x \in X$ and $y \in Y$, we enclose the factors $g[X]$ and $g[Y]$ into rectangles, and then draw an arrow (or a line) between these rectangles as in Fig. 2.2. This convention reflects the relation $X \to_g Y$.

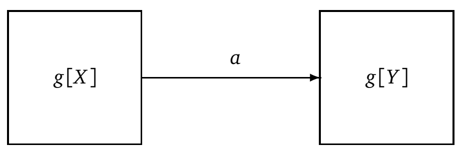


Fig. 2.2. Disjoint clans X and Y

Exercise 2.4. Let g be a Δ-graph with $X_i \in \mathscr{C}(g)$ for $i = 1, 2, \ldots, k$. Show that if for all $2 \leq j \leq k$ there is an $i < j$ such that $X_i \cap X_j \neq \emptyset$, then $\bigcup_{i=1}^{k} X_i \in \mathscr{S}$.

Example 2.3. We generalize the closure properties of Lemma 2.1 by introducing sibas (or *semi-independent boolean algebras*). Let $\mathscr{S}$ be a family of subsets of a finite set D. We say that $\mathscr{S}$ is a **siba on** D, if it satisfies the following conditions:

(S1) $\emptyset \in \mathscr{S}$, $D \in \mathscr{S}$ and $\{x\} \in \mathscr{S}$ for all $x \in D$;
(S2) if $X, Y \in \mathscr{S}$, then $X \cap Y \in \mathscr{S}$;
(S3) if $X, Y \in \mathscr{S}$ and $X \cap Y \neq \emptyset$, then $X \cup Y \in \mathscr{S}$;
(S4) if $X, Y \in \mathscr{S}$ and $Y \setminus X \neq \emptyset$, then $X \setminus Y \in \mathscr{S}$. □

Remark 2.2. By Lemma 2.1, $\mathscr{C}(g)$ forms a siba. It can be shown that also the converse is true, i.e., for any siba $\mathscr{S}$ there exists a Δ-graph g for some Δ on the same domain of elements such that $\mathscr{S} = \mathscr{C}(g)$. In this sense, Lemma 2.1 characterizes the family $\mathscr{C}(g)$ of clans. One should note, however, that *the siba $\mathscr{C}(g)$ does not determine g*, since quite different graphs may have the same family of clans. Indeed, primitive graphs g play a central role in some of our chapters, and for these $\mathscr{C}(g)$ consists of the trivial members only.

2.2 Prime clans

A nonempty clan $P \in \mathscr{C}(g)$ of g is a **prime clan**, if P does not overlap with any clan $X \in \mathscr{C}(g)$. Clearly, the nonempty trivial clans are prime. We denote the set of all prime clans of g by

$$\mathscr{P}(g) = \{P \mid P \text{ a prime clan of } g\}.$$

If P is a prime clan of g, then the factor $g[P]$ is called a **prime factor**.

Lemma 2.3. *Let $P \in \mathscr{P}(g)$. If $P \cap X \neq \emptyset$ for some $X \in \mathscr{C}(g)$, then X and P are comparable.*

Proof. Indeed, if two subsets intersect then they are either comparable or they overlap. □

Theorem 2.2. *Let $X \in \mathscr{C}(g)$. Then*

$$\mathscr{P}(g[X]) = \{P \mid P \in \mathscr{P}(g),\ P \subseteq X\} \cup \{X\}.$$

Proof. Let $h = g[X]$. First if $P \in \mathscr{P}(g)$ with $P \subseteq X$, then $P \in \mathscr{C}(h)$ and it is a prime clan of h by Theorem 2.1.

For the converse, assume that P is a nontrivial prime clan of h. Then $P \in \mathscr{C}(g)$ by Theorem 2.1. Suppose that P overlaps with some $Y \in \mathscr{C}(g)$. By assumption, $Y \notin \mathscr{C}(h)$, and hence $Y \setminus X \neq \emptyset$, and therefore $X \setminus Y \in \mathscr{C}(g)$ by Lemma 2.1. Consequently, $X \setminus Y \in \mathscr{C}(h)$. Similarly, $X \cap Y \in \mathscr{C}(h)$. Since P is a prime clan of h that intersects with both $X \setminus Y$ and $X \cap Y$, we have $X = (X \cap Y) \cup (X \setminus Y) \subseteq P$, which implies that $P = X$; a contradiction. □

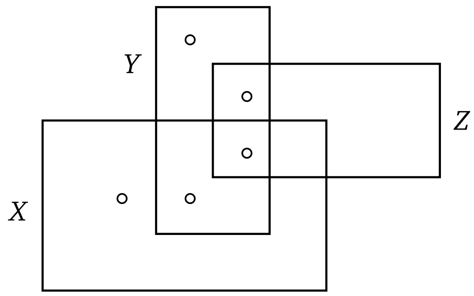
Exercise 2.5. Show that if g satisfies the equality $\mathscr{C}(g) = \mathscr{P}(g)$, then so do all its factors. Also, show that the converse holds if $|D_g| \geq 4$: if the proper factors h of g satisfy the condition $\mathscr{C}(h) = \mathscr{P}(h)$, then so does g.

If a clan $X \in \mathscr{C}(g)$ is not a prime clan, then there exists some $Y \in \mathscr{C}(g)$ that overlaps with X. The next result states that there is such a Y that is partitioned into two prime clans.

Lemma 2.4. *Let g be a Δ-graph, and assume that $X \in \mathscr{C}(g)$ is not a prime clan. Also, let $Y \in \mathscr{C}(g)$ be a minimal clan (with respect to inclusion) overlapping with X. Then $X \cap Y \in \mathscr{P}(g)$ and $Y \setminus X \in \mathscr{P}(g)$.*

Proof. Assume $Z \in \mathscr{C}(g)$ overlaps with $X \cap Y$. Since $Y \cap Z \in \mathscr{C}(g)$, the minimality of Y, implies that $Y \cap Z \subseteq Y \cap X$. Also, $Z \setminus Y \neq \emptyset$, and so $Y \setminus Z \in \mathscr{C}(g)$ overlaps with X; a contradiction.

Assume then that Z overlaps with $Y \setminus X$. Also in this case, $Y \cap Z \in \mathscr{C}(g)$ or $Y \setminus Z \in \mathscr{C}(g)$ overlaps with X; a contradiction. □



Notes on references

- Clans in the general form were introduced by EHRENFEUCHT AND ROZENBERG (1990a). Sibas were introduced by EHRENFEUCHT AND ROZENBERG (1986), and they are also used by MÖHRING (1985) with different terminology.
- The notion of a clan has been rediscovered quite many times in graph theory and related areas. Indeed, a clan is a very natural object in different areas. For undirected graphs the clans are called *closed sets* by GALLAI (1967); *stable sets* by ŠEVRIN AND FILIPPOV (1970); *autonomous sets* by KELLY (1985) and MÖHRING AND RADERMACHER (1984); *partitive sets* by SUMNER (1973) and GOLUMBIC (1980); *externally related sets* by HEMMINGER (1968) and CHEIN, HABIB AND MAURER (1981); *contractions* by ASCHBACHER (1976); *clumps* by BLASS (1978); *homogeneous sets* by CHVÁTAL AND HOANG (1985) and JAMISON AND OLARIU (1995); *condensible sets* by JAMES, STANTON AND COWAN (1972); *intervals* by FRAISSE (1984); *modules* by BIRNBAUM AND ESARY (1965) and MULLER AND SPINRAD (1989). Clans are also closely connected to *committees* of SHAPLEY (1967), see also CHEIN, HABIB AND MAURER (1981), where *partitive hypergraphs* are studied.

3

Quotients and Homomorphisms

3.1 Quotients

A **factorization** of a Δ-graph g is any partition $\mathscr{X} = \{X_1, X_2, \dots, X_k\} \subseteq \mathscr{C}(g)$ of the domain D_g into nonempty clans. The subgraphs $g[X_i]$ are called the **factors** in $\mathscr{X}$.

Example 3.1. Each graph g has the **trivial factorizations**, $\mathscr{X} = \{D_g\}$ and $\mathscr{X} = \{\{x\} \mid x \in D_g\}$. Evidently a graph is primitive just in case it has only the trivial factorizations. In the other extreme case, any partition of the domain will do as a factorization of a complete graph K_D (with only one colour). $\square$

A factorization $\mathscr{X}$ is called **proper**, if it has at least two factors, i.e., if $\mathscr{X} \neq \{D_g\}$. It is **nontrivial**, if it contains at least one nontrivial clan.

The **quotient of** g **by a factorization** $\mathscr{X}$, denoted $g/\mathscr{X}$, has the domain $D_{g/\mathscr{X}} = \mathscr{X}$ and the colouring

$$g/\mathscr{X}(X,Y) = g(x,y) \quad \text{where } x \in X, y \in Y\,.$$

By Lemma 2.2, the quotient is well defined, since $X \to_g Y$ for all disjoint clans $X, Y \in \mathscr{X}$.

The quotient $g/\mathscr{X}$ is obtained from g by contracting each clan $X \in \mathscr{X}$ into a single vertex and by inheriting the colours from g.

There is a simplified drawing of a given g for which a factorization $\mathscr{X}$ is known. Indeed, for all different $X, Y \in \mathscr{X}$, we have $X \to_g Y$, and hence we can enclose X and Y in rectangles as in Example 3.2, see Fig. 3.1, and then connect the rectangles by an edge $X \longrightarrow Y$ of the appropriate colour.

Example 3.2. The simplified drawing convention has been applied in Fig. 3.1 for a Δ-graph g. The quotient $g/\mathscr{X}$ is presented in Fig. 3.2 for the factorization $\mathscr{X} = \{X, Y, \{x_4\}, \{x_5\}, \{x_6\}\}$. $\square$

If $\mathscr{X}$ is the trivial factorization $\mathscr{X} = \{\{x\} \mid x \in D_g\}$ of g, then we may identify $g/\mathscr{X}$ with g since these are clearly isomorphic.

As stated in the next result, a quotient of a graph is isomorphic to a subgraph of the graph. *This is a rare property in algebraic and combinatorial systems.* Indeed, e.g., a quotient group need not be isomorphic to a subgroup of the group.

Theorem 3.1. *Let $\mathscr{X}$ be a factorization of a graph g. Then the quotient $g/\mathscr{X}$ is isomorphic to the subgraph $g[T]$, where T is any transversal of $\mathscr{X}$.*

Proof. Let τ be a transversal (function) of the factorization $\mathscr{X}$, and let $T = \tau(\mathscr{X})$. Now, for all different $X_1, Y_1 \in \mathscr{X}$, and different $X_2, Y_2 \in \mathscr{X}$, we have

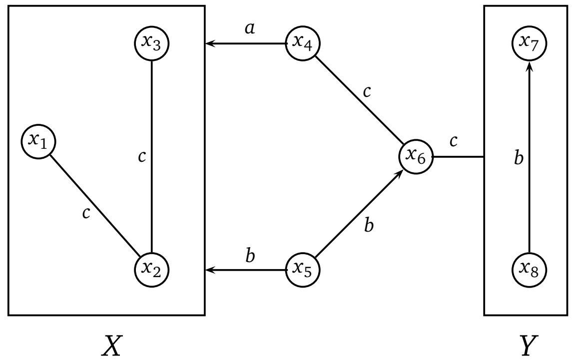


Fig. 3.1. Disjoint factors of g

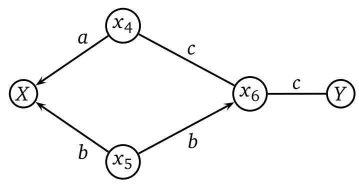


Fig. 3.2. The (primitive) quotient $g/\mathscr{X}$

$$g/\mathscr{X}(X_1, Y_1) = g/\mathscr{X}(X_2, Y_2) \iff g(\tau(X_1), \tau(Y_1)) = g(\tau(X_2), \tau(Y_2))$$

Therefore the transversal τ is an isomorphism from $g/\mathscr{X}$ onto $g[T]$. $\square$

3.2 Homomorphisms

Let $\alpha\colon X \to Y$ be a mapping. We extend it to pairs $\alpha\colon X \times X \to Y \times Y$ by setting

$$\alpha(x_1, x_2) = (\alpha(x_1), \alpha(x_2))$$

for all $x_1, x_2 \in X$. For subsets $E \subseteq X \times X$, we let

$$\alpha(E) = \{\alpha(e) \mid e \in E\}\,.$$

Let g and h be Δ-graphs. A mapping $\alpha\colon D_g \to D_h$ is a **homomorphism**, in symbols $\alpha\colon g \to h$, if for all $e_1, e_2 \in E_2(D_g)$ with $e_1, e_2 \notin \ker(\alpha)$,

$$g(e_1) = g(e_2) \iff h(\alpha(e_1)) = h(\alpha(e_2))\,.$$

A surjective homomorphism is called an **epimorphism**. An injective epimorphism is an isomorphism.

Example 3.3. Consider g and h from Fig. 3.3. Define α by $\alpha(x_1) = y_1$, $\alpha(x_2) = \alpha(x_4) = y_2$ and $\alpha(x_3) = y_3$. Then α is a homomorphism $g \to h$, where $(x_2, x_4) \in \ker(\alpha)$. By the definition of a homomorphism, $\{x_2, x_4\}$ must be a clan of g. In h the image is just the vertex y_2. In this example α is not an endomorphism, since y_4 and y_5 are not images of any vertices. □

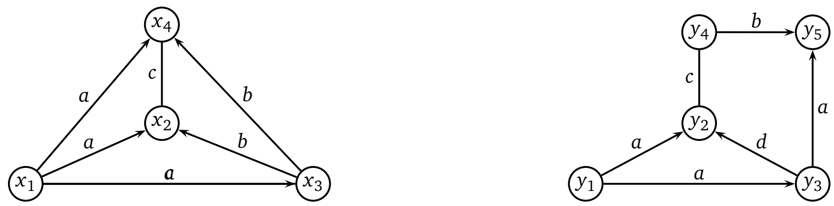


Fig. 3.3. g and h

Remark 3.1. *In general, a homomorphism need not preserves clans*. Indeed, let h be a non-complete graph and let $X \subseteq D_h$ be such that $X \notin \mathscr{C}(h)$, and let $g = h[X]$. The inclusion mapping $\iota \colon X \to D_h$, defined by $\iota(x) = x$ for all $x \in X$, is clearly a homomorphism $\iota \colon g \to h$. Now, $X = D_g$ is a trivial clan of g, but the image $\iota(X)$ is not a clan of h by assumption.

The next theorem shows that this is a typical counter-example for the preservation of clans.

Theorem 3.2. *Let $\alpha \colon g \to h$ be an epimorphism. If $X \in \mathscr{C}(g)$, then $\alpha(X) \in \mathscr{C}(h)$.*

Proof. Assume $X \in \mathscr{C}(g)$, $y \notin \alpha(X)$ and $x_1, x_2 \in \alpha(X)$. Since α is surjective, there are vertices $y' \notin X$ and $x_1', x_2' \in X$ with $\alpha(y') = y$ and $\alpha(x_i') = x_i$ for $i = 1, 2$. Since $X \in \mathscr{C}(g)$, $g(y', x_1') = g(y', x_2')$. By the definition of a homomorphism, also $h(y, x_1) = h(y, x_2)$ proving the claim. □

In order for an inverse image $\alpha^{-1}(Y)$ of a clan Y to be a clan the homomorphism α need not be surjective:

Theorem 3.3. *Let $\alpha \colon g \to h$ be a homomorphism. If $Y \in \mathscr{C}(h)$, then $\alpha^{-1}(Y) \in \mathscr{C}(g)$.*

Proof. Suppose $Y \in \mathscr{C}(h)$, and let $y \notin \alpha^{-1}(Y)$ and $x_1, x_2 \in \alpha^{-1}(Y)$. Now, $\alpha(y) \notin Y$ and $\alpha(x_1), \alpha(x_2) \in Y$ (where it can be $\alpha(x_1) = \alpha(x_2)$). Since $Y \in \mathscr{C}(h)$ is a clan, we have $h(\alpha(y), \alpha(x_1)) = h(\alpha(y), \alpha(x_2))$. By the definition of a homomorphism, also $g(y, x_1) = g(y, x_2)$. Therefore $\alpha^{-1}(Y) \in \mathscr{C}(g)$. □

We leave it as an exercise to prove that the homomorphisms are closed under compositions of mappings:

Exercise 3.1. The composition $\beta\alpha$ of two homomorphisms $\alpha \colon g \to h'$ and $\beta \colon h' \to h$ is a homomorphism $\beta\alpha \colon g \to h$.

Remark 3.2. One could start with **pre-morphisms** that are 'one-sided homomorphisms':

$$g(e_1) = g(e_2) \implies h(\alpha(e_1)) = h(\alpha(e_2)),$$

in which case the mapping can merge some of the colours. However, with this notion the inverse image of a clan need not be a clan.

Natural epimorphisms and decompositions

Let $\mathscr{X}$ be a factorization of g. Define the mapping $\kappa_{\mathscr{X}} : g \to g/\mathscr{X}$ as follows:

$$\kappa_{\mathscr{X}}(x) = X \quad \text{if } x \in X\,. \tag{3.1}$$

The mapping $\kappa_{\mathscr{X}}$ is well defined since $\mathscr{X}$ is a partition, and thus for each $x \in D_g$ there is a unique $X \in \mathscr{X}$ with $x \in X$. As usual we can extend $\kappa_{\mathscr{X}}$ to the family of all nonempty subsets of D_g by letting

$$\kappa_{\mathscr{X}}(Y) = \{\kappa_{\mathscr{X}}(y) \mid y \in Y\}\,.$$

Exercise 3.2. Let $\mathscr{X}$ be a factorization of g and let $Y \subseteq D_g$. Then, see Fig. 3.4,

$$\kappa_{\mathscr{X}}(Y) = \{X \mid X \in \mathscr{X},\ X \cap Y \neq \emptyset\}\,.$$

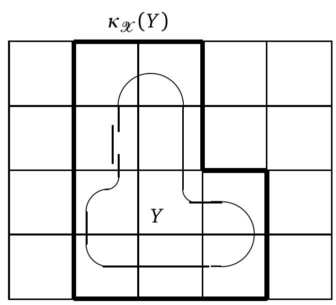


Fig. 3.4. The family $\kappa_{\mathscr{X}}(Y)$

Lemma 3.1. *Let $\mathscr{X}$ be a factorization of g. Then $\kappa_{\mathscr{X}} : g \to g/\mathscr{X}$ is an epimorphism. It is called the* ***natural epimorphism*** *onto $g/\mathscr{X}$.*

Proof. Clearly, $\kappa_{\mathscr{X}}$ is surjective. We show that it is a homomorphism.

First, if $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ are not in the kernel $\ker(\kappa_{\mathscr{X}})$, then, for $i \in \{1,2\}$, the vertices x_i and y_i belong to different clans $X_i, Y_i \in \mathscr{X}$. Hence $X_i \to_g Y_i$ holds for $i = 1,2$. Since $\kappa_{\mathscr{X}}(x_i) = X_i$ and $\kappa_{\mathscr{X}}(y_i) = Y_i$, also $g/\mathscr{X}(\kappa_{\mathscr{X}}(e_1)) = g/\mathscr{X}(\kappa_{\mathscr{X}}(e_2))$. The claim follows. □

By the definitions of an image and an inverse image of a subset, we have for the natural epimorphism $\kappa_{\mathscr{X}} : g \to g/\mathscr{X}$ and a subset $\mathscr{Z} \subseteq \mathscr{X}$ that

$$\kappa_{\mathscr{X}}(\cup\mathscr{Z}) = \mathscr{Z} \quad \text{and} \quad \kappa_{\mathscr{X}}^{-1}(\mathscr{Z}) = \cup\mathscr{Z}\,. \tag{3.2}$$

Theorem 3.4. *Let $g/\mathscr{X}$ be a quotient of g. Then for each subset $\mathscr{Z} \subseteq \mathscr{X}$,*

$$\mathscr{Z} \in \mathscr{C}(g/\mathscr{X}) \iff \cup\mathscr{Z} \in \mathscr{C}(g)\,.$$

Furthermore, $\mathscr{C}(g/\mathscr{X}) = \{\kappa_{\mathscr{X}}(Y) \mid Y \in \mathscr{C}(g)\}$.

Proof. If $\mathscr{Z} \in \mathscr{C}(g/\mathscr{X})$, then by (3.2) and Theorem 3.3, $\cup\mathscr{Z} = \kappa_{\mathscr{X}}^{-1}(\mathscr{Z}) \in \mathscr{C}(g)$. On the other hand, if $\cup\mathscr{Z} \in \mathscr{C}(g)$ for some $\mathscr{Z} \subseteq \mathscr{X}$, then by (3.2) and Theorem 3.2, $\mathscr{Z} = \kappa_{\mathscr{X}}(\cup\mathscr{Z}) \in \mathscr{C}(g/\mathscr{X})$.

For the second claim, assume first that $Y \in \mathscr{C}(g)$. Example 2.4 applied to the family $\kappa_{\mathscr{X}}(Y)$ (of intersecting sets) gives that $\cup\kappa_{\mathscr{X}}(Y) \in \mathscr{C}(g)$ which in its turn gives that $\kappa_{\mathscr{X}}(Y) \in \mathscr{C}(g/\mathscr{X})$ by the first part of the present theorem.

Conversely, if $\mathscr{Z} \in \mathscr{C}(g/\mathscr{X})$, then by (3.2), $\kappa_{\mathscr{X}}(\cup\mathscr{Z}) = \mathscr{Z}$, where $\cup\mathscr{Z} \in \mathscr{C}(g)$ by the first part of the theorem. This proves the equality. □

We say that a **decomposition**

$$g/\mathscr{X}(g[X_1],\dots,g[X_k])$$

of a Δ-graph g consists of a quotient $g/\mathscr{X}$ together with the factors $g[X_i]$ of a factorization $\mathscr{X} = \{X_1,\dots,X_k\}$.

3.3 Clans and epimorphisms

Let $\alpha\colon g \to h$ be a homomorphism. We call the equivalence class

$$x\ker(\alpha) = \{y \mid (x,y) \in \ker(\alpha)\}$$

the **kernel class of** the vertex $x \in D_g$ **modulo** α, and we let

$$\mathrm{Ker}(\alpha) = \{x\ker(\alpha) \mid x \in D_g\}$$

be the set of all kernel classes modulo $\alpha\colon g \to h$.

Exercise 3.3. Show that $x\ker(\alpha) = \alpha^{-1}(\alpha(x))$.

The following theorem is known as the **isomorphism theorem**.

Theorem 3.5. *For every epimorphism $\alpha\colon g \to h$, the family* $\mathrm{Ker}(\alpha)$ *of the kernel classes is a factorization of g, and*

$$h \cong g/\mathrm{Ker}(\alpha)\,.$$

Proof. Indeed, $x\ker(\alpha) = \alpha^{-1}(\alpha(x))$, and therefore $x\ker(\alpha) \in \mathscr{C}(g)$ by Theorem 3.3, since $\{\alpha(x)\} \in \mathscr{C}(h)$. Hence the quotient $g/\mathrm{Ker}(\alpha)$ is well defined. Define a function $\beta : h \to g/\mathrm{Ker}(\alpha)$ by $\beta(\alpha(x)) = x\ker(\alpha)$. It is straightforward to verify that β is an epimorphism, and that it is injective; therefore β is an isomorphism. □

The next **homomorphism theorem** is again a standard algebraic result for epimorphisms.

Theorem 3.6. *Let $\alpha_1 : g \to h_1$ and $\alpha_2 : g \to h_2$ be two epimorphisms such that* $\ker(\alpha_1) \subseteq \ker(\alpha_2)$. *Then there exists a unique epimorphism $\beta : h_1 \to h_2$ such that $\alpha_2 = \beta\alpha_1$, i.e., the following diagram commutes:*

$$\begin{tikzcd} & h_1 \arrow[dr, "\beta"] & \\ g \arrow[ur, "\alpha_1"] \arrow[rr, "\alpha_2"'] & & h_2 \end{tikzcd}$$

Proof. By Theorem 3.3, for each $x \in D_{h_1}$, we have $\alpha_1^{-1}(x) \in \mathscr{C}(g)$, and therefore also $\alpha_2(\alpha_1^{-1}(x)) \in \mathscr{C}(h_2)$ by Theorem 3.2. Since $\ker(\alpha_1) \subseteq \ker(\alpha_2)$, it follows that $\alpha_2(\alpha_1^{-1}(x))$ is a singleton subset of D_{h_2}. Consequently,

$$\beta = \alpha_2\alpha_1^{-1} \colon D_{h_1} \to D_{h_2}$$

is a well defined function. It is clear that β is surjective, and that $\alpha_2 = \beta\alpha_1$. We need to show that β is a homomorphism.

Let $e_1, e_2 \notin \ker(\beta)$ be in h_1. Since α_1 is surjective, there are f_1, f_2 in g such that $\alpha(f_1) = e_1$ and $\alpha_1(f_2) = e_2$. Moreover, α_1 is a homomorphism, and so $g(f_1) = g(f_2)$ if and only if $h_1(e_1) = h_1(e_2)$. Since α_2 is a homomorphism, $g(f_1) = g(f_2)$ if and only if $h_2(\alpha_2(f_1)) = h_2(\alpha_2(f_2))$, where $\alpha_2(f_i) = \alpha_2\alpha_1^{-1}(e_i) = \beta(e_i)$ for $i = 1, 2$. This shows that β is a homomorphism.

For the uniqueness of β assume that also $\beta_1 \colon h_1 \to h_2$ satisfies the condition $\alpha_2 = \beta_1\alpha_1$. Let $x \in D_{h_1}$ and $y \in D_g$ be such that $\alpha_1(y) = x$. Now $\beta_1(x) = \alpha_2(y) = \beta(x)$ from which the claim $\beta_1 = \beta$ follows. □

By Theorem 3.5, each epimorphic image of a graph g is isomorphic to a quotient of g. Also, the converse statement is true, since each quotient $g/\mathscr{X}$ is an epimorphic image of the natural epimorphism $\kappa_{\mathscr{X}}$.

Corollary 3.1. *Let g be a Δ-graph. A graph h is isomorphic to a quotient of g if and only if h is an epimorphic image of g.*

We have the following characterization of clans as kernel classes.

Theorem 3.7. *For each Δ-graph g, a nonempty subset $X \subseteq D_g$ is a clan if and only if it is a kernel class modulo an epimorphism $\alpha \colon g \to h$: $X = x\ker(\alpha)$.*

Proof. Let $X \in \mathscr{C}(g)$, and consider the factorization $\mathscr{X} = \{X\} \cup \{\{y\} \mid y \in D_g \setminus X\}$. Clearly, $X = x\ker(\kappa_{\mathscr{X}})$ for all $x \in X$. In the other direction, the claim follows from Theorem 3.5. □

By Theorem 3.1, each quotient $g/\mathscr{X}$ is isomorphic to a subgraph $g[X]$ induced by a transversal X of $\mathscr{X}$. Therefore for each epimorphism $\alpha \colon g \to h$ there corresponds an epimorphism $\alpha' \colon g \to g[X]$ such that α and α' have the same kernel.

Exercise 3.4. Let $\mathscr{X}$ be a factorization of a Δ-graph g.

(a) The quotient $g/\mathscr{X}$ is isomorphic to an image $\alpha(g)$ for a homomorphism $\alpha \colon g \to g$.

(b) If $\mathscr{Y}$ is a factorization of the quotient $g/\mathscr{X}$, then the quotient $(g/\mathscr{X})/\mathscr{Y}$ of $g/\mathscr{X}$ is isomorphic to a quotient of g, and eventually also to a subgraph of g.

Example 3.4. Let $g = g(G)$ for the symmetric graph from Fig. 3.5(a). This graph has a clan $X = \{x_1, x_3\}$, which gives the quotient $g/\mathscr{X}$ from Fig. 3.5(b), where $\mathscr{X} = \{X, \{x_2\}, \{x_4\}, \{x_5\}\}$. Also, $g/\mathscr{X}$ has a clan $Y = \{X, \{x_2\}\}$. Let $\mathscr{Y} = \{Y, \{x_4\}, \{x_5\}\}$. In Fig. 3.5(c) we have the quotient $(g/\mathscr{X})/\mathscr{Y}$, which is isomorphic to $g/\mathscr{Z}$ for $\mathscr{Z} = \{\{x_1, x_2, x_3\}, \{x_4\}, \{x_5\}\}$. □

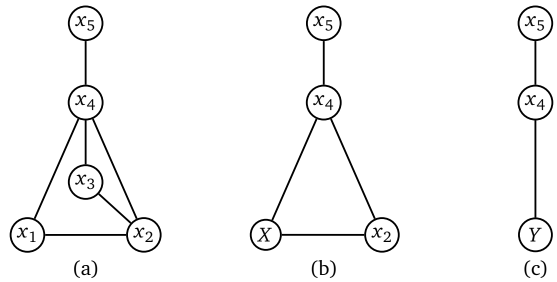


Fig. 3.5. $g = g(G)$, $g/\mathscr{X}$ and $(g/\mathscr{X})/\mathscr{Y}$

Prime clans in quotients

By Corollary 3.1, each quotient of g is isomorphic to an epimorphic image of g. Therefore we start with results on epimorphisms.

Lemma 3.2. *Let $\alpha\colon g \to h$ be an epimorphism.*

(i) If $P \in \mathscr{P}(g)$, then also $\alpha(P) \in \mathscr{P}(h)$.
(ii) If $P \in \mathscr{P}(h)$ with $|P| \geq 2$, then $\alpha^{-1}(P) \in \mathscr{P}(g)$.

Proof. For (i), let $P \in \mathscr{P}(g)$. By Theorem 3.2, $\alpha(P) \in \mathscr{C}(h)$. Let then $X \in \mathscr{C}(h)$ be a clan with $\alpha(P) \cap X \neq \emptyset$. Then also $P \cap \alpha^{-1}(X) \neq \emptyset$, where $\alpha^{-1}(X) \in \mathscr{C}(g)$ by Theorem 3.3, and therefore $\alpha^{-1}(X) \subseteq P$ or $P \subseteq \alpha^{-1}(X)$, since P is prime. But now $X \subseteq \alpha(P)$ or $\alpha(P) \subseteq X$ which shows that $\alpha(P) \in \mathscr{P}(h)$.

For (ii), assume that $P \in \mathscr{P}(h)$ is a proper prime clan of h such that $\alpha^{-1}(P) \notin \mathscr{P}(g)$. We show that P is a singleton set, and then by Theorem 3.3, $\alpha^{-1}(P)$ is a clan of g.

Let $X \in \mathscr{C}(g)$ be a minimal clan that overlaps with $\alpha^{-1}(P)$ in g. Hence $Z = X \cap \alpha^{-1}(P) \neq \emptyset$, and $Z \in \mathscr{P}(g)$ by Exercise 2.1 and by the minimality of X. (Indeed, if Z overlaps with a clan A of $g[\alpha^{-1}(P)]$, then $X \setminus A$ contradicts the minimality of X.) Now, by case (i), $\alpha(Z) \in \mathscr{P}(h)$.

By Theorem 3.2, $\alpha(X) \in \mathscr{C}(h)$, and since $X \cap \alpha^{-1}(P) \neq \emptyset$ and $X \setminus \alpha^{-1}(P) \neq \emptyset$, also $\alpha(X) \cap P \neq \emptyset$ and $\alpha(X) \setminus P \neq \emptyset$, from which it follows that $P \subset \alpha(X)$, since P is a prime clan of h, i.e., $\alpha(Z) = P$.

By the assumption, there exists a $y \in \alpha^{-1}(P) \setminus X$. Consider the set $Y = \alpha^{-1}(\alpha(y))$ $(\subseteq \alpha^{-1}(P))$. By Theorem 3.3, $Y \in \mathscr{C}(g)$, and, furthermore, $Y \cap Z \neq \emptyset$, since $\alpha(Z) = P$. Since $Z \in \mathscr{P}(g)$, it follows that $Z \subseteq Y$, which means that $P = \alpha(Z) \subseteq \alpha(Y) = \{\alpha(y)\}$. Therefore $P = \{\alpha(y)\}$ is a singleton, and this proves the claim. □

Remark 3.3. Let still $\alpha\colon g \to h$ be an epimorphism. It can occur that the inverse image $\alpha^{-1}(x)$ of a *singleton prime clan* $x \in h$ is not a prime clan of g for an epimorphism α. Indeed, suppose g has a nontrivial clan X that is not prime, and consider the factorization $\mathscr{X} = \{X\} \cup \{\{y\} \mid y \notin X\}$ of g. Now X is a singleton clan of the quotient $g/\mathscr{X}$ (and hence a prime clan there), but $\kappa_{\mathscr{X}}^{-1}(X) = X$ is not a prime clan of g.

From the previous results and Theorem 3.4 we obtain

Theorem 3.8. *Let g be a Δ-graph with a factorization $\mathscr{X}$. Then*

$$\mathscr{P}(g/\mathscr{X}) = \{\kappa_{\mathscr{X}}(P) \mid P \in \mathscr{P}(g)\}\,.$$

Proof. The inclusion from right to left follows from Lemma 3.2(i).

For the other inclusion, let $P \in \mathscr{P}(g/\mathscr{X})$. If P is a singleton clan, i.e., $P \in \mathscr{X}$, then $P = \kappa_{\mathscr{X}}(\{x\})$ for a vertex $x \in P$. On the other hand, if P is not a singleton, then by Lemma 3.2(ii), $\kappa_{\mathscr{X}}^{-1}(P) \in \mathscr{P}(g)$, and the claim follows from the equality $P = \kappa_{\mathscr{X}}(\kappa_{\mathscr{X}}^{-1}(P))$. $\square$

As shown in Lemma 3.2, for an epimorphism $\alpha\colon g \to h$ and a prime clan $P \in \mathscr{P}(h)$ if $\alpha^{-1}(P)$ is not a prime clan of g, then P must be a singleton. This gives, using (3.2), the following corollary.

Corollary 3.2. *Let g be a Δ-graph with a factorization $\mathscr{X}$.*

- *If $\mathscr{Z} \in \mathscr{P}(g/\mathscr{X})$ is not a singleton, then $\cup\mathscr{Z} \in \mathscr{P}(g)$.*
- *If $\mathscr{X} \subseteq \mathscr{P}(g)$, then for each prime clan $\mathscr{Z} \in \mathscr{P}(g/\mathscr{X})$, also $\cup\mathscr{Z} \in \mathscr{P}(g)$.*

Primitive quotients

Primitive graphs are of special importance since these have no nontrivial factorizations (and therefore they correspond to *simple algebras* in algebraic terminology).

Theorem 3.9. *Let g and h be non-singleton Δ-graphs, and let $\mathscr{X}$ be a factorization of g such that $g/\mathscr{X}$ is truly primitive. For each epimorphism $\alpha\colon g \to h$ there exists a unique epimorphism $\beta : h \to g/\mathscr{X}$ such that $\kappa_{\mathscr{X}} = \beta\alpha$, i.e., the following diagram commutes.*

$$\begin{tikzcd} & h \arrow[dr, "\beta"] & \\ g \arrow[ur, "\alpha"] \arrow[rr, "\kappa_{\mathscr{X}}"'] & & g/\mathscr{X} \end{tikzcd}$$

Proof. By the homomorphism theorem, Theorem 3.6 it is sufficient to show that $\ker(\alpha) \subseteq \ker(\kappa_{\mathscr{X}})$, i.e., for each $x \in D_g$, $x\ker(\alpha) \subseteq x\ker(\kappa_{\mathscr{X}})$.

Indeed, let $X = x\ker(\alpha)$. By Theorem 3.7, $X \in \mathscr{C}(g)$, and since h has at least two vertices, X must be a proper clan of g. By Theorem 3.2, $\kappa_{\mathscr{X}}(X) \in \mathscr{C}(g/\mathscr{X})$, and since $g/\mathscr{X}$ is primitive, $\kappa_{\mathscr{X}}(X)$ is either a singleton clan or $\kappa_{\mathscr{X}}(X) = \mathscr{X}$ $(= D_{g/\mathscr{X}})$.

Suppose first that $\kappa_{\mathscr{X}}(X) = \mathscr{X}$. In this case, $X \cap Y \neq \emptyset$ for all $Y \in \mathscr{X}$. Since $X \neq D_g$, there exists a $Y \in \mathscr{X}$ such that $Y \setminus X \neq \emptyset$, and therefore by Lemma 2.1(iv), $X \setminus Y \in \mathscr{C}(g)$. But now $\kappa_{\mathscr{X}}(X \setminus Y) = \mathscr{X} \setminus \{Y\} \in \mathscr{C}(g/\mathscr{X})$ is nontrivial, since $|\mathscr{X}| \geq 3$, and this contradicts the primitivity of $g/\mathscr{X}$.

Assume then that $\kappa_{\mathscr{X}}(X)$ is a singleton clan of $g/\mathscr{X}$. In this case, $X \subseteq Y$ for the clan $Y \in \mathscr{X}$ such that $x \in Y$, i.e., $Y = x\ker(\kappa_{\mathscr{X}})$; therefore $x\ker(\alpha) \subseteq x\ker(\kappa_{\mathscr{X}})$ as required. $\square$

The following corollary to Theorem 3.9 states that truly primitive quotients of graph are unique if they exist.

Theorem 3.10. *Each Δ-graph g can have at most one factorization $\mathscr{X}$ such that the quotient $g/\mathscr{X}$ is truly primitive.*

Proof. Assume that $\mathscr{X}_1$ and $\mathscr{X}_2$ are two factorizations of g such that both $g/\mathscr{X}_1$ and $g/\mathscr{X}_2$ are truly primitive. By applying Theorem 3.9 to the natural epimorphisms $\kappa_{\mathscr{X}_1}$ and $\kappa_{\mathscr{X}_2}$, we obtain two epimorphisms $\beta_1\colon g/\mathscr{X}_1 \to g/\mathscr{X}_2$ and $\beta_2\colon g/\mathscr{X}_2 \to g/\mathscr{X}_1$ such that $\kappa_{\mathscr{X}_2} = \beta_1\kappa_{\mathscr{X}_1}$ and $\kappa_{\mathscr{X}_1} = \beta_2\kappa_{\mathscr{X}_2}$.

Let $X \in \mathscr{X}_1$. Now $\kappa_{\mathscr{X}_2}(X) = \beta_1\kappa_{\mathscr{X}_1}(X)$, where $\kappa_{\mathscr{X}_1}(X)$ is a singleton set, and thus so is $\kappa_{\mathscr{X}_2}(X)$. This means that $X \subseteq Y$ for some $Y \in \mathscr{X}_2$. Similarly, for each $Y \in \mathscr{X}_2$, there exists some $X \in \mathscr{X}_1$ such that $Y \subseteq X$. Since $\mathscr{X}_1$ and $\mathscr{X}_2$ are partitions of D_g, it follows that they are equal. $\square$

We say that a factorization $\mathscr{X}_1$ is a **refinement** of a factorization $\mathscr{X}_2$ of a graph g if the equivalence relation induced by $\mathscr{X}_1$ is a refinement of that induced by $\mathscr{X}_2$, *i.e*, if each $X \in \mathscr{X}_2$ is a union of elements of $\mathscr{X}_1$.

The following lemma is clear from the definition and Theorem 3.6.

Lemma 3.3. *Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be factorizations of a Δ-graph g. If $\mathscr{X}_1$ is a refinement of $\mathscr{X}_2$, then* $\ker(\kappa_{\mathscr{X}_1}) \subseteq \ker(\kappa_{\mathscr{X}_2})$*, and therefore $g/\mathscr{X}_2$ is an epimorphic image of $g/\mathscr{X}_1$.*

A proper factorization $\mathscr{M}$ is called a **maximal factorization** of a graph g, if it satisfies the condition: if $\mathscr{M}$ is a refinement of a proper factorization $\mathscr{X}$ of g, then $\mathscr{X} = \mathscr{M}$.

A maximal factorization of a graph need not be unique. Indeed, a complete graph g on a domain D with $|D| \geq 3$ has $2^{|D|-1} - 1$ different maximal factorizations: $\mathscr{M}_X = \{X, D \setminus X\}$ for each nonempty $X \subset D$. In this case, each quotient $g/\mathscr{M}_X$ is a primitive and symmetric 2-vertex graph. (The quotient is primitive but not truly primitive.)

Theorem 3.11. *Let $\mathscr{X}$ be a proper factorization of a Δ-graph g. The quotient $g/\mathscr{X}$ is primitive if and only if $\mathscr{X}$ is a maximal factorization.*

Proof. By definition, if a proper factorization $\mathscr{X}$ is a refinement of a different proper factorization $\mathscr{X}_1$, then there exists an $X \in \mathscr{X}_1$ such that $X = \cup_{i=1}^{k} Y_i$, where $k \geq 2$ and each $Y_i \in \mathscr{X}$. By Theorem 3.4, $\kappa_{\mathscr{X}}(X) = \{Y_1, Y_2, \ldots, Y_k\}$ is a nontrivial clan of $g/\mathscr{X}$, and thus $g/\mathscr{X}$ is not primitive.

On the other hand, suppose that $\mathscr{X}$ is a proper factorization of g such that $g/\mathscr{X}$ is not primitive. Now for a nontrivial clan $\mathscr{Z}$ of $g/\mathscr{X}$, the set $\cup\mathscr{Z}$ is a union of two or more clans from

$\mathscr{X}$. Therefore the factorization $\mathscr{X}$ is a refinement of $\{\cup\mathscr{Z}\} \cup \{X \in \mathscr{X} \mid X \notin \mathscr{Z}\}$, and hence $\mathscr{X}$ is not a maximal factorization. □

Notes on references

- The approach to quotients by homomorphisms is standard in algebra. For the general algebraic treatment, see BURRIS AND SANKAPPANAVAR (1981) or COHN (1981). In the context of factorizations into clans of graphs and partially ordered sets an approach by quotients and homomorphisms is taken by BUER AND MÖHRING (1983).
- Prime clans of graphs were introduced by EHRENFEUCHT AND ROZENBERG (1990a). The prime clans of undirected graphs are called *strongly closed sets* by GALLAI (1967).

4

Clan Decomposition

4.1 Maximal prime clans

A prime clan $P \in \mathscr{P}(g)$ of a Δ-graph g is **maximal** if for all proper prime clans $Q \in \mathscr{P}(g)$, $P \subseteq Q$ implies $Q = P$. We denote by

$$\mathscr{P}_{\max}(g) = \{P \mid P \text{ maximal in } \mathscr{P}(g)\}$$

the family of the maximal prime clans of g. For each $P \in \mathscr{P}_{\max}(g)$, the factor $g[P]$ is called a **maximal prime factor** of g.

For completeness sake, we adopt the convention $\mathscr{P}_{\max}(g) = \{x\}$ if $D_g = \{x\}$.

Lemma 4.1. *Let g be a Δ-graph. Then $\mathscr{P}_{\max}(g)$ forms a partition of D_g. In particular, the quotient $g/\mathscr{P}_{\max}(g)$ is well defined.*

Proof. We may assume that $|D_g| \geq 2$. For each $x \in D_g$, there is at least one proper prime clan containing x, namely $\{x\}$. Let P_x be a proper prime clan that is maximal with respect to the condition $x \in P_x$. Then, $P_x = P_y$ or $P_x \cap P_y = \emptyset$ for all $x, y \in D_g$. It is also clear that the union of the maximal prime clans is equal to D_g. □

Theorem 4.1. *Let g be a Δ-graph. For each $X \in \mathscr{C}(g)$ with $|X| \geq 2$, we have*

$$X = \bigcup\{P \mid P \text{ maximal prime clan with } P \subset X\}\,.$$

Proof. By Theorem 2.2, the prime clans $P \neq X$ of $g[X]$ are exactly the prime clans of g contained in X. The claim follows by applying Lemma 4.1 to $g[X]$. □

Remark 4.1. Let g be a non-singleton Δ-graph, and consider the family $\mathscr{C}_{\max}(g)$ of all **maximal clans** of g, i.e., those proper clans M for which $M \subset X$ implies $X = D_g$ for $X \in \mathscr{C}(g)$. Then $\mathscr{C}_{\max}(g)$ need not form a factorization of g. Also, an epimorphism $\alpha\colon g \to h$ need not map $\mathscr{C}_{\max}(g)$ into $\mathscr{C}_{\max}(h)$.

The quotient $g/\mathscr{P}_{\max}(g)$ is the epimorphic image of g for the natural epimorphism $\kappa_{\mathscr{P}_{\max}(g)}$. We shall write more simply

$$\kappa_g = \kappa_{\mathscr{P}_{\max}(g)}\colon g \to g/\mathscr{P}_{\max}(g)\,,\,.$$

4.2 The clan decomposition theorem

We generalize the notion of primitivity by saying that g is **special** if it has only the trivial prime clans, i.e., g is special just in case $\mathscr{P}_{\max}(g)$ consists of the singletons of D_g.

Example 4.1. (1) All primitive Δ-graphs are special, since they do not have any nontrivial clans.

(2) Every complete (Δ-)graph g having only one colour, is special, because every subset X of the domain is a clan.

(3) A Δ-graph $g\colon E_2(D_g) \to \{a, b\}$ is said to be **linear**, if its vertices have an ordering $(x_1, x_2, \dots, x_n)$ such that

$$g(x_i, x_j) = \begin{cases} a & \text{if } i < j\,, \\ b & \text{if } i > j\,. \end{cases}$$

Every linear graph is special. (Also, if $|D_g| \le 2$, then g is primitive.) □

Theorem 4.2. *For every g, the quotient $g/\mathscr{P}_{\max}(g)$ is special.*

Proof. By Theorem 3.8, for each prime clan $\mathscr{X}$ of the quotient $g/\mathscr{P}_{\max}(g)$, there exists a prime clan $P \in \mathscr{P}(g)$ such that $\kappa_g(P) = \mathscr{X}$. If $P = D_g$, then $\mathscr{X} = \mathscr{P}_{\max}(g)$. On the other hand, if $P \neq D_g$, then there exists a $P_1 \in \mathscr{P}_{\max}(g)$ such that $P \subseteq P_1$. Because $\kappa_g(P_1)$ is a singleton of $g/\mathscr{P}_{\max}(g)$, it follows that also $\mathscr{X}$ is a singleton. In any case $\mathscr{X}$ is trivial, and so the quotient is special. □

Exercise 4.1. Let g be an undirected graph that is not complete or discrete. Show that if g is special, then it and its complement $\overline{g}$ are both connected.

In Theorem 4.3 we shall show that all special Δ-graphs are of the three special types: complete, linear, or primitive.

Exercise 4.2. Show that a Δ-graph g on the domain D. Show that

(a) g is linear with the ordering $\delta = (x_1, x_2, \dots, x_n)$, if and only if $\mathscr{C}(g)$ consists of the empty set together with the segments $\{x_i, \dots, x_{i+j}\}$, for all i and j.

(b) g is complete if and only if $\mathscr{C}(g) = 2^D$.

Lemma 4.2. *Let g be special and $X \in \mathscr{C}(g)$ a nonempty clan. Then also $g[X]$ is special. Furthermore, if $|X| \ge 3$, then $g[X]$ is not primitive.*

Proof. The first claim follows from Theorem 2.2 which says that the prime clans of factors are prime clans of g.

For the second claim, assume that $|X| \ge 2$ with $X \neq D_g$, and that $g[X]$ is primitive. Since g is special, there exists a clan $Y \in \mathscr{C}(g)$ that overlaps with X. Then also $X \cap Y$, $X \setminus Y$ are clans of g, by the closure conditions, and hence they are in $\mathscr{C}(g[X])$. From the primitivity of $g[X]$, it follows that $|X \cap Y| = 1 = |X \setminus Y|$, and therefore $|X| = 2$ which proves the claim. □

Lemma 4.3. *Let g be special with $|D_g| \geq 2$. Then for each $X \in \mathscr{C}(g)$ with $|X| \geq 2$ and for each $x \in X$, there exists a 2-element clan $Z \in \mathscr{C}(g)$ such that $x \in Z$ and $Z \subseteq X$. In particular, each vertex x of a special Δ-graph g is contained in a 2-element clan.*

Proof. We may suppose that $|D_g| \geq 3$. By Lemma 4.2, it is sufficient to show that each $x \in D_g$ belongs to a nontrivial clan $X \in \mathscr{C}(g)$, because we can then repeat the argument for $g[X]$ until we obtain a clan of two elements.

By the hypothesis, there exists a nontrivial $X \in \mathscr{C}(g)$. If $x \in X$, then the claim follows. Suppose $x \notin X$ and that X is maximal with respect to this property. Since g is special, there is a nontrivial clan $Y \in \mathscr{C}(g)$ overlapping with X. Therefore $X \cup Y \in \mathscr{C}(g)$. By the maximality of X, $x \in X \cup Y$, and thus $x \in Y$ as required. □

Every special Δ-graph g determines a new undirected graph $G(g)$ on D_g the edges of which are exactly the 2-element clans of g:

$$E_{G(g)} = \{(x, y) \mid \{x, y\} \in \mathscr{C}(g)\}.$$

It can be that $E_{G(g)}$ is empty. In that case, the special g is primitive by Lemma 4.3.

Lemma 4.4. *Let g be special with $|D_g| \geq 2$. Then $G(g)$ is either the complete graph K_D, or it is a path of length $|D|-1$.*

Proof. We show first that $G(g)$ is connected. For this, let H be a connected component of $G(g)$. By Lemma 4.3, $|H| \geq 2$, and by Lemma 2.4(ii), $H \in \mathscr{C}(g)$. Assume contrary to the present claim that H is a proper subset of D_g.

Since g is special, there exists a clan $X \in \mathscr{C}(g)$ overlapping with H. Assume that X is minimal with respect to this property. By Lemma 2.4, $H \cap X \in \mathscr{P}(g)$ and $X \setminus H \in \mathscr{P}(g)$. Since g is special, these two prime clans are singletons, and so $|X| = 2$, say $X = \{x, y\}$. But then $(x, y) \in E_{G(g)}$. It follows that also $H \cup X$ is connected; a contradiction. Thus $G(g)$ is connected.

Assume next that $G(g)$ has an element x with at least three neighbours, say y_1, y_2, y_3, so that $\{x, y_i\} \in \mathscr{C}(g)$ for all i. By the siba-condition (S3), also $\{x, y_1, y_2\} \in \mathscr{C}(g)$, and by (S4), $\{y_1, y_2\} = \{x, y_1, y_2\} \setminus \{x, y_3\} \in \mathscr{C}(g)$. This shows that the neighbourhood of x together with x, itself, is a complete subgraph of $G(g)$.

Let then Q_x be a maximal complete subgraph of $G(g)$ containing x. Hence $|Q_x| \geq 4$. If $Q_x = D$, then $G(g)$ is complete. Otherwise, since $G(g)$ is connected, there exists a 2-element clan $\{y, z\} \in \mathscr{C}(g)$ such that $y \in Q_x$ and $z \notin Q_x$. But the degree of y is at least three (since it is in Q_x), and, by the above, its neighbourhood, which contains Q_x properly, is a complete subgraph. This contradicts the maximality of Q_x, and proves that $G(g)$ is complete.

For the remaining case, assume that the degree of every element $x \in D$ is at most two. In this case, $G(g)$ is either a path containing all elements of D (this is the claim), or $G(g)$ is a Hamilton cycle containing all elements of D. The latter case occurs if each $x \in D$ has degree equal to two. Let $\{x_i, x_{i+1 (\mathrm{mod}\, n)}\} \in \mathscr{C}(g)$ for $i = 0, 1, \ldots, n-1$, where $|D| = n$. The sets $\{x_0, x_1, x_2\}$ and $\{x_2, \ldots, x_{n-1}, x_0\}$ are both in $\mathscr{C}(g)$ by (S3), and so is their intersection $\{x_0, x_2\}$ by (S2); this contradicts the assumption that the degree of x_0 is at most two. The claim follows. □

As a corollary to Lemma 4.4 we state our main theorem for special Δ-graphs.

Theorem 4.3. *A Δ-graph g is special if and only if it is complete, linear, or truly primitive.*

Proof. We need only to observe that if the Δ-graph $G(g)$ for a special g is complete, then g is complete; and if $G(g)$ is a path, then g is linear. In all other cases g is primitive. Notice that if $|D_g| \leq 2$, then g is either linear or complete (and primitive).

The converse claim is clear from Example 4.1. □

By Theorem 4.2 and Theorem 4.3, we obtain the following **clan decomposition theorem**.

Theorem 4.4. *Let g be a Δ-graph. Then the quotient $g/\mathscr{P}_{\max}(g)$ is either linear, or complete, or truly primitive.*

For a Δ-graph g with $\mathscr{P}_{\max}(g) = \{P_1, \ldots, P_m\}$, the **clan decomposition** consists of the maximal prime factors $g[P_i]$, for $i \in [1, m]$, together with the special quotient $g/\mathscr{P}_{\max}(g)$:

$$g/\mathscr{P}_{\max}(g)(g[P_1], \ldots, g[P_m]) .$$

Example 4.2. Consider g of Fig. 4.1 with five colours, a, b, c together with the reverse colours of a and b. (The colour c is symmetric.)

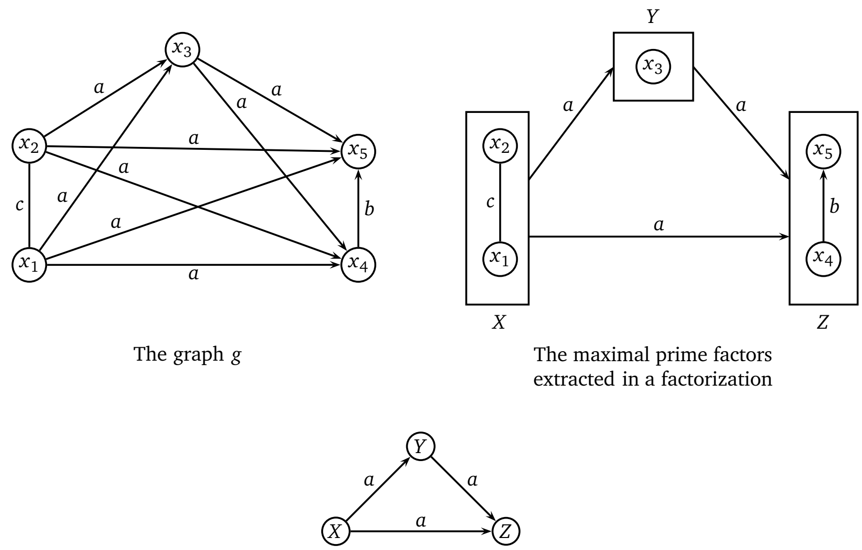
The subsets $X = \{x_1, x_2\}$, $Z = \{x_4, x_5\}$ together with the subsets $X \cup \{x_3\}$ and $Z \cup \{x_3\}$ are the nontrivial clans of g, and $\mathscr{P}_{\max}(g) = \{X, Y, Z\}$, where $Y = \{x_3\}$ is a trivial clan. In Fig. 4.1 we have extracted the maximal prime factors into rectangles, and the linear 3-vertex quotient $g/\mathscr{P}_{\max}(g)$ is also shown. □

Exercise 4.3. Let $\mathscr{X}$ be a factorization of a Δ-graph g such that $|\mathscr{X}| \geq 3$. If $g/\mathscr{X}$ is complete (respectively linear or primitive), then $g/\mathscr{P}_{\max}(g)$ is complete (respectively linear or primitive).

Theorem 4.5. *Let $\mathscr{X}$ be a factorization of a Δ-graph g such that $g/\mathscr{X}$ is truly primitive. Then $\mathscr{X} = \mathscr{P}_{\max}(g)$ is the unique maximal factorization of g.*

Remark 4.2. Let g be an undirected graph. In the **modular decomposition** the factors of g are called **modules**, and they are classified into three categories: a factor h is **parallel** if it is disconnected; **series** if h has a disconnected complement; **neighbourhood** if h and its complement are connected.

- If g is disconnected, then the connected components of g are exactly its maximal prime clans. In this case, $g/\mathscr{P}_{\max}(g)$ is complete. The same is true, of course, for the complement graph $\overline{g}$.
- If g and its complement $\overline{g}$ are both connected, then the quotient $g/\mathscr{P}_{\max}(g)$ is primitive.



The graph g

The maximal prime factors extracted in a factorization

The linear quotient $g/\mathscr{P}_{\max}(g)$

Fig. 4.1. A clan decomposition

Notes on references

- The clan decomposition theorem, Theorem 4.4, was proved in the present general form by EHRENFEUCHT AND ROZENBERG (1990a) using maximal prime clans, and by MÖHRING AND RADERMACHER (1984) in an axiomatic framework of general algebras.
- The decomposition result for undirected graphs is known as *modular decomposition*. For this theorem, see GALLAI (1967), BIRNBAUM AND ESARY (1965), BUTTERWORTH (1972). SABIDUSSI (1961) calls the modular decomposition of undirected graphs an *X-join*, see HEMMINGER (1968), SUMNER (1973) and HABIB AND MAURER (1979). For different aspects of the modular decomposition theorem, see JAMES, STANTON AND COWAN (1972) and MAURER (1977). For (directed) graphs the clan decomposition theorem was proved by CUNNINGHAM (1982).
- Related decomposition result was proven by SHAPLEY (1967) on committees of simple games, see also BILLERA (1971), who carried these ideas further for clutters on finite sets. For a (clan) decomposition of boolean functions, we refer to ASHENHURST (1959), MÖHRING AND RADERMACHER (1984) and MÖHRING (1985).
- There are other well appreciated methods to decompose graphs. We refer to CUNNINGHAM AND EDMONDS (1980) for a general decomposition theory in combinatorics; in this respect, see also WAGNER (1990).

5

Primitive Graphs

5.1 Uniformly non-primitive graphs

We study first the opposite side of primitivity – **uniformly non-primitive** having no *truly primitive subgraphs*.

Example 5.1. It may take some time to convince oneself, but the Δ-graph in Fig. 5.1 is uniformly non-primitive. □

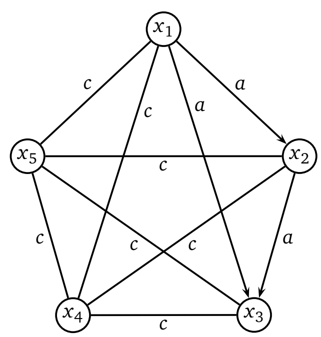


Fig. 5.1. A uniformly non-primitive graph g

It is clear that every subgraph of a uniformly non-primitive Δ-graph is itself uniformly non-primitive. The following claim for quotients follows from Theorem 3.1 by which every quotient is isomorphic to a subgraph.

Lemma 5.1. *If g is a uniformly non-primitive, then so are its subgraphs and quotients.*

Lemma 5.2. *The following statements are equivalent for a Δ-graph g:*

(i) g is uniformly non-primitive;
(ii) g has the ***2-block property****: each non-singleton subgraph has a partition into two nonempty clans;*
(iii) g has the ***doubleton clan property****: every non-singleton subgraph has a clan of cardinality two.*

Proof. (i) $\Rightarrow$ (ii): The quotient $g/\mathscr{P}_{\max}(g)$ is either linear or complete, since, by Lemma 5.1, it cannot be truly primitive.

We show that there exists a (prime) clan P of g such that $D_g \setminus P \in \mathscr{C}(g)$. From this (ii) follows. Indeed, if $g/\mathscr{P}_{\max}(g)$ is linear, we choose $P \in \mathscr{P}_{\max}(g)$ that comes first in the linear order of $g/\mathscr{P}_{\max}(g)$; and if $g/\mathscr{P}_{\max}(g)$ is complete, then any $P \in \mathscr{P}_{\max}(g)$ will do.

(ii) ⇒ (iii): Suppose g has the 2-block property. Clearly then so do all of its subgraphs. Consider a subgraph $g[X]$ with $|X| \geq 2$. We prove (iii) by induction on $|X|$. If $|X| = 2$, then $g[X]$ has a clan of cardinality two, namely X itself. Suppose then that $|X| \geq 3$. By assumption, X can be partitioned into two nonempty clans X_1 and X_2 of $g[X]$, and one of these, say X_1, has at least two vertices. By the induction hypothesis, $g[X_1]$ has the doubleton clan property, and hence $g[X_1]$ has a clan Y with $|Y| = 2$. Since $Y \in \mathscr{C}(g[X_1])$, and $X_1 \in \mathscr{C}(g[X])$, also $Y \in \mathscr{C}(g[X])$ by Lemma 2.1. This proves (iii).

(iii) ⇒ (i): This follows from the definition of uniform non-primitivity. □

Remark 5.1. The proof shows that a Δ-graph g is uniformly non-primitive if and only if every nontrivial subgraph $g[X]$ has a proper prime clan, the complement of which is a clan.

5.2 Small primitive subgraphs

Theorem 5.1. *A truly primitive Δ-graph g has a primitive subgraph of 3 or 4 vertices.*

Proof. Let g be truly primitive. For $|D_g| \leq 4$, the claim is trivial; suppose thus that $|D_g| \geq 5$.

We show that g has a proper truly primitive subgraph, which implies the claim by a reduction argument. Assume contrary to the claim that g has no proper truly primitive subgraphs.

Let $x \in D_g$ and consider the subgraph $h = g[D_g \setminus \{x\}]$. By assumption, h is uniformly non-primitive, and hence, by Lemma 5.2(ii), it has a partition into two nonempty clans $X \setminus \{x\}$ and $Z = D_g \setminus X$, where $|X| \geq 3$. By assumption, $g[X]$ is uniformly non-primitive, and hence it has a partition into nonempty clans A and B, where, say $x \in A$. Since $B, Z \in \mathscr{C}(h)$, we have $Z \to_g B$ (recall that $x \notin B$). Also, $A \to_g B$, since $A, B \in \mathscr{C}(g[X])$. Hence $B \in \mathscr{C}(g)$, and therefore it must be a singleton, say $B = \{y\}$ (and then $A = X \setminus \{y\}$).

Let us denote $g(y,x) = a$. Then $g(y,v) = a$ for all $v \in A$. Let $v \in X \setminus \{x\}$. There is a colour b such that $g(v,u) = b$ for all $u \in Z$. Also, $g(y,u) = b$, since $X \setminus \{x\}$ is a clan of h.

Since $D_g \setminus \{y\}$ is not a clan of g, we must have $a \neq b$.

Now, $X \notin \mathscr{C}(g)$ and hence there exists $z \in Z$ such that $g(x,z) \neq b$. The subgraph $g[\{x,y,z\}]$ must contain a proper clan. The only choice is $\{z,y\}$. Then $g(z,x) = a$.

Since $|X| \geq 3$ and $X \setminus \{x\}$ is not a clan of g, there exists a vertex $v \in X \setminus \{x,y\}$ with $g(v,x) \neq a$. Now, $g[\{z,x,v\}]$ must be non-primitive, and so $g(v,x) = b$. However, now the subgraph $g[\{x,y,z,v\}]$ is primitive; a contradiction. □

By Theorem 4.4, Lemma 5.2 can be extended as follows.

Exercise 5.1. The following statements are equivalent for a Δ-graph g:

(i) g is uniformly non-primitive;

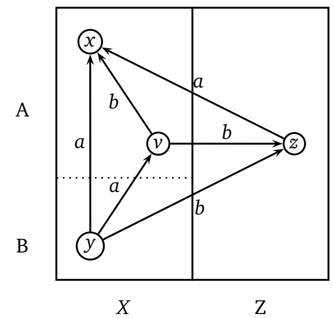


Fig. 5.2. The subgraph induced by $\{x, y, z, v\}$

(ii) g has the 2-block property;
(iii) g has the doubleton clan property;
(iv) g has no primitive subgraphs of 3 or 4 vertices.
(v) each special quotient $g[P]/\mathscr{P}_{\max}(g[P])$ with $P \in \mathscr{P}(g)$, is either linear or complete.

Exercise 5.2. An undirected graph g is a **cograph** (**complement reducible graph**) if it does not have any induced subgraph that is a path P_4 of four vertices. Show that the family of cographs is the smallest family $\mathscr{F}$ of undirected graphs that satisfies (a) or (b):

(a) $\mathscr{F}$ contains the singleton graphs and it is closed under disjoint unions and complements.

(b) $\mathscr{F}$ contains the singleton graphs and is closed under disjoint union and complete connections.

Exercise 5.3. Let g be an undirected graph. The following conditions are equivalent for g to be a cograph:

(a) g is uniformly non-primitive.

(b) Every non-singleton induced subgraph has a clan of cardinality two.

(c) Every non-singleton induced subgraph is either disconnected or its complement is disconnected.

(d) The graph g has only complete special quotients $g[P]/\mathscr{P}_{\max}(g[P])$ for $P \in \mathscr{P}(g)$.

5.3 Hereditary properties

Example 5.2. We adopt some special notations for the following primitive 4-vertex graphs from Fig. 5.3:

- P_4 is the undirected path of four vertices.
- $\vec{P}_4$ is the *oriented* version of P_4.

- $\widehat{P}_4$ is the *doubly oriented* version of P_4.

$\square$

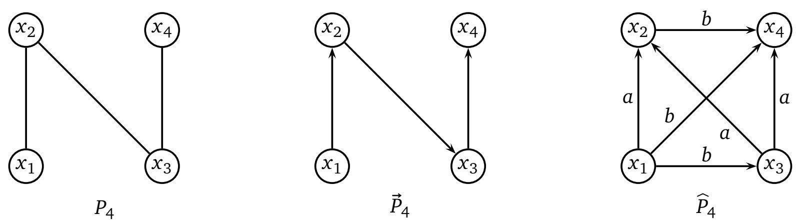


Fig. 5.3. The primitive 4-vertex graphs

In Theorem 5.2 we shall show that if g and the subgraph $h = g[X]$ are truly primitive, then we can extend h by one or two vertices to obtain a larger primitive subgraph. For this purpose, we let, for given $X \subset D_g$ and $x, y \notin X$, denote

$$h_x = g[X \cup \{x\}] \quad \text{and} \quad h_{xy} = g[X \cup \{x, y\}].$$

Lemma 5.3. *Let $h = g[X]$ be a truly primitive subgraph of g and let $y \notin X$. If h_y is not primitive, then it has a unique nontrivial clan Z_y; furthermore, either y is* ***local*** *or* ***global*** *for h:*

$$|Z_y| = 2 \ \textit{with}\ y \in Z_y \ \textit{ or } \ Z_y = X$$

but not both.

Proof. Suppose that $Z \in \mathscr{C}(h_y)$ is nontrivial. Then $Z \cap X \in \mathscr{C}(h)$, and, since h is primitive, $Z \cap X$ is trivial in h, i.e., either $Z \cap X$ is a singleton or $Z = X$:

$$Z = \{y, z\} \text{ with } z \in X \ \text{ or } \ Z = X\,. \tag{5.1}$$

In the local case, Z is unique. Indeed, if also $Z_1 \in \mathscr{C}(h_y)$ is a 2-element clan, then $y \in Z \cap Z_1$, and hence $Z \cup Z_1 \in \mathscr{C}(h_y)$; contradicting (5.1) (since h is truly primitive). Assume then that $X \in \mathscr{C}(h_y)$. Then $X \setminus Z \in \mathscr{C}(h_y)$ is nontrivial; again a contradiction. $\square$

Let $h = g[X]$ be a truly primitive subgraph of g such that h_y is not primitive for some $y \notin X$. If y is local for h, let

$$Z_y = \{y, y^h\} \in \mathscr{C}(h_y)$$

denote the unique clan of h_y.

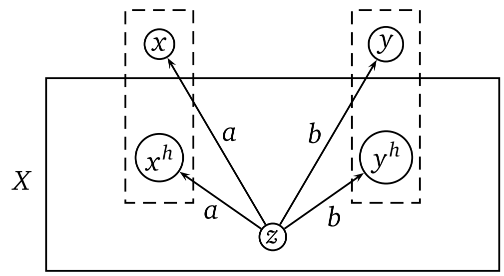
Lemma 5.4. *Let $h = g[X]$ be a truly primitive subgraph of g, and let $x, y \notin X$ be such that h_{xy} is not primitive.*

(i) If x and y are both local for h with $x^h \neq y^h$, then Z_x and Z_y are the only nontrivial clans of h_{xy}.

(ii) If x is global and y local for h, then Z_y is the unique nontrivial clan of h_{xy}.

Proof. Let $Y \in \mathscr{C}(h_{xy})$ be a nontrivial clan of h_{xy}.

In the case (i), we show first that either $x \in Y$ or $y \in Y$ but not both. Now $Y \cap X \in \mathscr{C}(h)$, and therefore either $|Y \cap X| \leq 1$ or $X \subseteq Y$, since h is primitive. The latter condition cannot hold, because x and y are local for h. Since h is truly primitive, there exists some $z \in X \setminus \{x^h, y^h\}$ with $g(z, x^h) \neq g(z, y^h)$ and so also $g(z, x) \neq g(z, y)$.



Consequently, if $\{x, y\} \subseteq Y$, then also $z \in Y$, i.e., $Y = \{x, y, z\}$ because $|Y \cap X| \leq 1$. However, now $\{z, x\} = Y \cap (X \cup \{x\}) \in \mathscr{C}(h_x)$, and so $z = x^h$; a contradiction. We conclude that either $x \in Y$ or $y \in Y$ but not both. Without loss of generality, we may assume that $x \in Y$.

Since $Y \subset X \cup \{x\}$, we have $Y \in \mathscr{C}(h_x)$ and so $Y = Z_x$ by Lemma 5.3. Hence $Z_x \in \mathscr{C}(h_{xy})$, and then $g(y, x) = g(y, x^h)$, and $g(y, x^h) = g(y^h, x^h)$, since $Z_y \in \mathscr{C}(h_y)$. Also, $g(y^h, x^h) = g(y^h, x)$, since $Z_x \in \mathscr{C}(h_x)$. We conclude that $Z_y \in \mathscr{C}(h_{xy})$. Thus the first claim follows.

For (ii), we note first that $Y \neq \{x, y\}$, for otherwise y would be global for h (since x is global for h). In particular, $Y \cap X \neq \emptyset$. We still can have that $\{x, y\} \subseteq Y$.

If $x \in Y$, then $Z = Y \cap (X \cup \{x\})$ is a nontrivial clan of h_x. But $x \in Z$ contradicts the fact that x is global. On the other hand, if $x \notin Y$, then $Y = Z_y$, because $Y \subset X \cup \{y\}$ so that $Y \in \mathscr{C}(h_y)$. This proves the second claim. □

As a consequence of Lemma 5.4 we obtain

Lemma 5.5. *Let g be primitive and $h = g[X]$ its truly primitive subgraph for which x is local. Then there exists a vertex $y \notin X$ such that $g(y, x) \neq g(y, x^h)$. Furthermore, for all such vertices y, h_y is primitive, or h_{xy} is primitive, or y is local such that $x^h = y^h$.*

Proof. Since g is primitive, there exists some $y \notin X$ that destroys $Z_x = \{x, x^h\}$, i.e., $g(y, x) \neq g(y, x^h)$, and so $Z_x \notin \mathscr{C}(h_{xy})$. Assume now that neither h_y nor h_{xy} is primitive. By Lemma 5.4(ii) and the fact that $Z_x \notin \mathscr{C}(h_{xy})$, y is not global for h. Hence y is local for h just like x. It follows from Lemma 5.4(i) and the fact $Z_x \notin \mathscr{C}(h_{xy})$ that $y^h = x^h$. This proves the claim. □

We show that every primitive graph satisfies the **upward hereditary property**: every truly primitive proper subgraph h of a primitive graph g can be extended to a primitive subgraph of g by adding one or two vertices.

Theorem 5.2. *Let g be primitive and assume that $h = g[X]$ is a proper truly primitive subgraph. There are then vertices $x, y \notin X$ (possibly $x = y$) such that the subgraph $h_{xy} = g[X \cup \{x, y\}]$ is primitive.*

Proof. We prove the claim by contradiction. For this assume that for all $x, y \in D_g \setminus X$ the subgraphs h_x, h_y and h_{xy} are not primitive.

There does exist a local vertex x for h, for otherwise X, itself, would be a clan of g. Denote

$$A = \{y \mid y \text{ local for } h \text{ with } y^h = x^h\}.$$

By Lemma 5.5, $|A| \geq 2$. Clearly, $A' = A \cup \{x^h\}$ is a clan of $g[X \cup A]$, and hence $X \cup A$ is a proper subset of D_g, see Fig. 5.4.

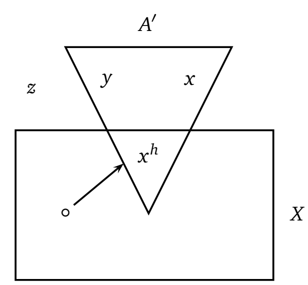


Fig. 5.4. $A' = A \cup \{x^h\}$

Since g is primitive, $A' \notin \mathscr{C}(g)$, and there is a vertex $z \notin X \cup A$ such that $g(z, y) \neq g(z, y^h)$ for some $y \in A$. (Here $y^h = x^h$.) In particular, $Z_y = \{y, y^h\} \notin \mathscr{C}(h_{zy})$. Therefore z is not global for h by Lemma 5.4(ii), and consequently, z is local for h. Due to the choice of z (i.e., $z^h \neq x^h$), this contradicts Lemma 5.4(i). $\square$

By Theorem 5.1, each truly primitive graph has a primitive subgraph with three or four vertices. Therefore, by Theorem 5.2, a truly primitive n-vertex g has at least one primitive subgraph with k or $k+1$ vertices for all $k \in [3, n-1]$. In particular, we have the following **downward hereditary property** of primitive graphs.

Theorem 5.3. *Let g be a truly primitive Δ-graph. Then there are vertices x and y (possibly $x = y$) such that $g[D_g \setminus \{x, y\}]$ is primitive.*

5.4 Critically primitive graphs

We consider critically primitive graphs through a series of exercises and examples.

A primitive and nontrivial Δ-graph g is **critically primitive** if $g[D_g \setminus \{x\}]$ is not primitive for all vertices $x \in D_g$. By Theorem 5.3, each critically primitive graph has to remove exactly two vertices in order to reach a primitive subgraph.

Example 5.3. Consider the tournament g of Fig. 5.5. It is primitive. The subgraphs $g[D_g \setminus \{x\}]$ are not primitive, and thus g is critically primitive. Also the 4-vertex graphs P_4, $\vec{P}_4$ and $\widehat{P}_4$ are critically primitive. □

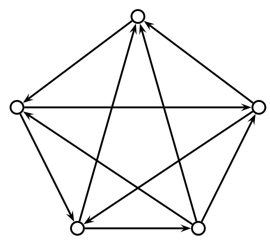


Fig. 5.5. A critically primitive tournament

Exercise 5.4. Let g be critically primitive with $n \geq 4$ vertices. Then for some k with $2 \leq k \leq 4$ there are non-primitive subgraphs $g[X_i]$, for $i = 0, 1, \ldots, k$, such that $|\cap_{i=0}^{k} X_i| = 3$ or 4 and $D_g = \cup_{i=0}^{k} X_i$.

The following **parity theorem** is an improvement of Theorem 5.2.

Theorem 5.4. *Let g be primitive, and let $h = g[X]$ be a truly primitive subgraph of g with $|X| \leq |D_g| - 2$. Then there are different vertices $x, y \notin X$ such that $g[X \cup \{x, y\}]$ is primitive.*

Proof. Hard exercise. This proves the theorem! □

Exercise 5.5. (a) Let g be a critically primitive undirected graph with $|D_g| \geq 4$. Show that $|D_g|$ is even.

(b) Show that if g is a critically primitive tournament, then $|D_g|$ is odd.

The following corollary is immediate from Theorem 5.4.

Corollary 5.1. *Let $h = g[X]$ be a primitive subgraph of a critically primitive g. If $|X| \geq 3$, then $|D_g| \equiv |X| \pmod 2$. If $|X| \geq 4$, then also h is critically primitive.*

In particular, if a critically primitive g has an even (odd) number of vertices, then its subgraphs with an odd (even) number of vertices $k \geq 3$ are non-primitive.

Remark 5.2. There are only finitely many types of critically primitive graphs.

Notes on references

- The results on the uniformly non-primitive graphs in Sec. 5.1 are from ENGELFRIET, HARJU, PROSKUROWSKI AND ROZENBERG (1996), where the graphs that can be represented by uniformly non-primitive graphs are characterized by forbidden subgraphs.
- Theorem 5.1 has been proved in this generality by EHRENFEUCHT AND ROZENBERG (1990b) and SCHMERL AND TROTTER (1993). The proof given here is from the book by EHRENFEUCHT, HARJU AND ROZENBERG (1999). For undirected graphs the related result has been stated and proved in many different contexts; see WOLK (1965), GALLAI (1967), KELLY (1985), and SEINSCHE (1974).
- Theorems 5.2 and 5.3 are due to EHRENFEUCHT AND ROZENBERG (1990b) and SCHMERL AND TROTTER (1993). Theorem 5.2 is related to a result proved in MOON (1972), ERDÖS, FRIED, HAJNAL AND MILNER (1972) and ERDÖS, HAJNAL AND MILNER (1972), which states that every tournament (not only the primitive ones) with n vertices can be embedded in a primitive tournament with $n+1$ or $n+2$ vertices.
- The parity theorem, Theorem 5.4, is proved in SCHMERL AND TROTTER (1993).
- The critically primitive graphs of the appendix were characterized by BONIZZONI (1994) and SCHMERL AND TROTTER (1993).

6

Switching over Groups

Switching serves as a transformation of Δ-graphs. Recall, page 2, that every Δ-graph presumes an involution $\delta\colon \Delta \to \Delta$ of the colours. In this part the set Δ will have a group structure that can be infinite. Now, the involution δ is assumed to satisfy that anti-homomorphism condition:

$$\delta(ab) = \delta(b)\delta(a) \quad\text{and}\quad \delta(\delta(a)) = a\,,$$

for all $a, b \in \Delta$.

We let ε_Δ, without the subscript ε if clear from the context, denote the identity element of the group Δ.

Example 6.1. (i) The most obvious involution for any group Δ is the inversion, $\delta(a) = a^{-1}$ for all $a \in \Delta$.

(ii) For the group of nonsingular $n\times n$-matrices Δ (over a field, say $\mathbb{R}$), the transposition, $\delta(M) = M^T$, is an involution. □

We shall often write a^δ instead of $\delta(a)$ for elements $a \in \Delta$.

This notational convention reflects the similarity of the involution δ to the inversion of Δ (and the transpose of matrices). In this notation, the defining properties of an involution become

$$(ab)^\delta = b^\delta a^\delta \quad\text{and}\quad (a^\delta)^\delta = a\,.$$

We shall assume that every group Δ comes with a fixed involution. This is denoted by Δ^δ.

Example 6.2. Let Δ^δ be a group with an involution δ and identity element $\varepsilon = \varepsilon_\Delta$. Then $\delta(\varepsilon) = \varepsilon$ and, for all $a \in \Delta$, $\delta(a^{-1}) = \delta(a)^{-1}$. Indeed, $\varepsilon = \delta(\varepsilon) = \delta(aa^{-1}) = \delta(a^{-1})\delta(a)$. Also, $\delta(a^n) = \delta(a)^n$ for all integers n.

Exercise 6.1. Let Δ be a group.

(a) Let the order $|\Delta|$ of the group be even. Then, by Cauchy's Theorem, it has an element a of order two: $a^2 = \varepsilon$. Show that the mapping $\delta_a(x) = ax^{-1}a$ $(x \in \Delta)$ is an involution.

(b) The group Δ is abelian if and only if the identity function $\iota_\Delta\colon \Delta \to \Delta$ is an involution.

(c) An automorphism δ of the multiplicative cyclic group $\mathbb{Z}_n$ is an involution if and only if $\delta(1)^2 \equiv 1(\operatorname{mod} n)$.

Exercise 6.2. Let Δ^δ be a group with an involution δ.

(a) Show that either there exists a fixed point $a^\delta = a$ with $a \neq \varepsilon$, or Δ is of odd order and δ is the inversion of Δ.

(b) Show that either there exists an element $a \neq \varepsilon$ such that $a^\delta = a^{-1}$, or Δ is an abelian group of odd order and δ is the identity function.

Exercise 6.3. Show that the centre $Z(\Delta) = \{a \mid ax = xa \text{ for all } x \in \Delta\}$ of a group Δ is closed under every involution of Δ: if $a \in Z(\Delta)$ and δ is an involution, then also $a^\delta \in Z(\Delta)$.

6.1 Selectors and switching classes

From now on $g: E_2(D) \to \Delta^\delta$ *means that* g *shares its involution* δ *with the involution of the group* Δ*, i.e.,* $g(e^{-1}) = g(e)^\delta$ for all pairs $e \in E_2(D)$. The set of these Δ^δ**-graphs** is denoted by

$$\Gamma_\Delta^\delta(D) = \{g \mid g: E_2(D) \to \Delta^\delta\}.$$

If the involution δ is the inversion of the group Δ, then we may write $\Gamma_\Delta(D)$ for $\Gamma_\Delta^\delta(D)$.

The group Δ of colours of $g: E_2(D) \to \Delta^\delta$ is employed by the **selectors**: $\sigma: D \to \Delta$.

For $g: E_2(D) \to \Delta^\delta$ and $\sigma: D \to \Delta$, we define a new $g^\sigma: E_2(D) \to \Delta^\delta$, called the **switch** of g by σ, as follows

$$g^\sigma(x,y) = \sigma(x) \cdot g(x,y) \cdot \sigma(y)^\delta \tag{6.1}$$

for all $(x,y) \in E_2(D)$, where the products are in Δ.

We shall prove first that g^σ is well defined.

Theorem 6.1. *Let* $g: E_2(D) \to \Delta^\delta$ *and* $\sigma: D \to \Delta$. *Then also* $g^\sigma: E_2(D) \to \Delta^\delta$.

Proof. For each $e = (x,y) \in E_2(D)$, we have $g(e^{-1}) = g(e)^\delta$, and hence

$$\begin{aligned} g^\sigma(e^{-1}) &= \sigma(y) g(e^{-1}) \sigma(x)^\delta = \sigma(y) g(e)^\delta \sigma(x)^\delta \\ &= \delta\left(\sigma(x) g(e) \sigma(y)^\delta\right) = \delta(g^\sigma(e)) = g^\sigma(e)^\delta. \end{aligned}$$

This proves the claim. □

The family

$$[g] = \{g^\sigma \mid \sigma \text{ a selector}\}$$

is called the **switching class** of g.

Example 6.3. Let $\Delta = (\mathbb{R}^+, \cdot)$ be the multiplicative group of the positive real numbers, and let δ be the inversion of Δ, $\delta(x) = x^{-1}$ for all $x \in \mathbb{R}^+$. In Fig. 6.1, we have a Δ^δ-graph, where, as usual, the reverse edges are hidden. Thus, e.g., $g(x_2, x_1) = \delta(g(x_1, x_2)) = g(x_1, x_2)^{-1} = 1$, and $g(x_3, x_2) = \frac{2}{3}$.

Define a selector $\sigma: D \to \mathbb{R}$ by $\sigma(x_n) = 5 - n$. We have then $g^\sigma(x_1, x_2) = 4 \cdot 1 \cdot \frac{1}{3} = \frac{4}{3}$. □

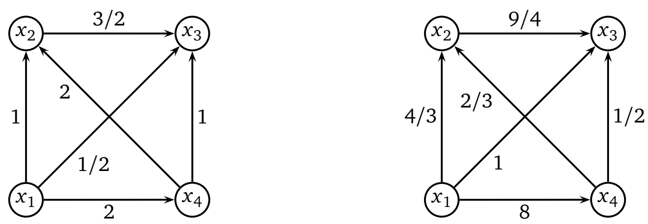


Fig. 6.1. A graph g and its switch g^σ

6.2 Examples for some special groups

For additive abelian groups (6.1) becomes

$$g^\sigma(x,y) = \sigma(x) + g(x,y) - \sigma(y).$$

Example 6.4. The cyclic group $\Delta = \mathbb{Z}_2$ of two elements has only one involution, the identity function (which is, in this case, also the inversion). In this case we have the switchings of undirected graphs that we shall treat later. □

Example 6.5. It is simple thing to show that the inversion and the identity function are the only involutions of the cyclic group $\mathbb{Z}_3$.

(1) We consider first the case, where δ is the inversion of $\mathbb{Z}_3$: $\delta(0) = 0$, $\delta(1) = 2$ and $\delta(2) = 1$. Therefore a $\mathbb{Z}_3$-graph g corresponds to an oriented graph, if $g(e) = 1$ (respectively $g(e) = 2$) is interpreted as stating that e (respectively e^{-1}) is a (directed) edge of g, and if $g(e) = 0$, then e is not an edge. Clearly, every oriented graph can be considered as a $\mathbb{Z}_3$-graph.

Let g be the directed cycle in Fig. 6.2, where an arrow denotes the value $1 \in \mathbb{Z}_3$. The second $\mathbb{Z}_3$ is obtained from g by the selector σ, for which $\sigma(x_1) = 1$ and $\sigma(x_2) = 0 = \sigma(x_3)$ (indicated in the vertices). The third oriented graph is a redrawing of g^σ, where the label 0 is hidden, and the arrow from x_1 to x_2 is reversed.

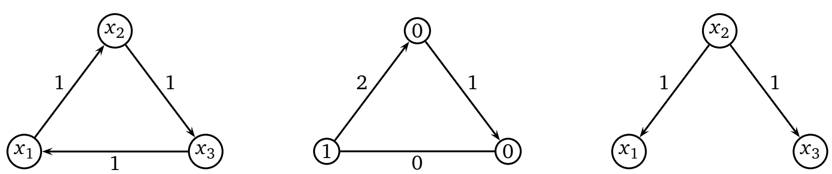


Fig. 6.2. g and g^σ with δ equal to the inversion

(2) If δ is the identity mapping of $\mathbb{Z}_3$, then all three labels are symmetric, and in this case, every graph $g\colon E_2(D) \to Z_3$ is symmetric. Let g be as in Fig. 6.3, and let σ be defined by $\sigma(x_i) = i \pmod 3$ for $i = 1, 2, 3$. Then g^σ is the second $\mathbb{Z}_3$-graph of Fig. 6.3. □

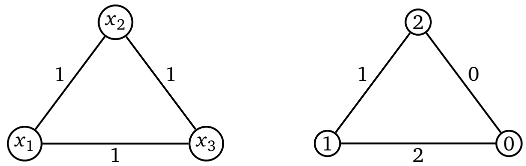


Fig. 6.3. g and g^σ with $\delta = \iota_{\mathbb{Z}_3}$

Example 6.6. Also the group $\mathbb{Z}_4$ has only the identity function and the inversion as its involutions. Let δ be the inversion of $\mathbb{Z}_4$. In this case, we have two symmetric labels 0 and 2, and the asymmetric labels 1 and 3 are reverses of each other. We can interpret here the value $g(e) = 0$ as saying that e and e^{-1} are not edges, $g(e) = 2$ that e and e^{-1} are both edges, $g(e) = 1$ that e is an edge but e^{-1} is not, $g(e) = 3$ that e^{-1} is an edge but e is not.

Therefore *all directed graphs can be represented as $\mathbb{Z}_4$-graphs.* □

Example 6.7. Let Δ^δ be an abelian group with an inversion. For each $a \in \Delta$ and $g : E_2(D) \to \Delta^\delta$, $g^\sigma = g^{\sigma_a}$, where $\sigma_a(x) = \sigma(x)+a$. This shows that the selectors do not determine uniquely the switches g^σ. □

Example 6.8. Let S_3 be the symmetric group of all permutations on $\{0,1,2\}$. This group is also known as the **dihedral group** D_3. It is the symmetry group of an equilateral triangle in the Euclidean plane. The generators are: a rotation of 60 degrees and one of the reflections of the triangle. The group S_3 consists of six elements generated by the permutations $\alpha = (0\ 1\ 2)$ and $\beta = (0\ 1)(2)$, given by their cycle structures. The elements of S_3 are:

$$\begin{array}{lll} \iota = (0)(1)(2), & \alpha = (0\ 1\ 2), & \alpha^2 = (0\ 2\ 1), \\ \beta = (0\ 1)(2), & \alpha\beta = (0\ 2)(1), & \alpha^2\beta = (0)(1\ 2)\,. \end{array}$$

Let $D = \{x, y, z\}$, and suppose that the involution δ is the inversion of S_3. There are altogether 216 S_3-graphs on D, and there are only three switching classes on a domain of three elements of 36, 72 and 108 elements, respectively.

Let g be the S_3-graph, where $g(e) = \iota$, the identity permutation, for all $e \in E_2(D)$. Let $\sigma : D \to S_3$ be defined by $\sigma(x) = \iota$, $\sigma(y) = \alpha$ and $\sigma(z) = \alpha^2$. The switch g^σ is given as the oriented graph on the left hand side of Fig. 6.4. Let also $\tau : D \to S_3$ be defined by $\tau(x) = \beta$, $\tau(y) = \alpha\beta$ and $\tau(z) = \alpha^2\beta$. Then $g^\tau = g^\sigma$.

On the other hand, if we begin with the S_3-graph h, for which $h(x, y) = h(y,z) = h(z, x) = \alpha\beta$, we obtain that $h^\sigma = g$, whereas h^τ is the oriented graph on the right hand side of Fig. 6.4. Therefore, $h^\sigma \neq h^\tau$ and $g^\sigma = g^\tau$, which explains why the switching classes have a different number of elements when the group is S_3. □

Example 6.9. Let again $D = \{x, y, z\}$. Since S_3 is not an abelian group, the identity function is not an involution of S_3. However, there is an involution of S_3 that comes close to being the

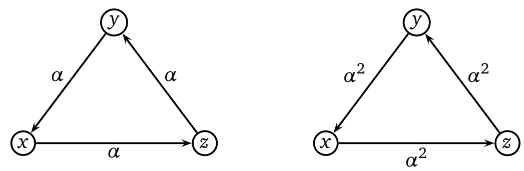


Fig. 6.4. $g^\sigma = g^\tau$ but $h^\sigma \neq h^\tau$

identity: $\delta(\alpha) = \alpha$ and $\delta(\beta) = \alpha\beta$. To see that δ is an involution one easily checks the required conditions:

$$\begin{aligned} \delta(\iota) &= \iota, & \delta(\alpha) &= \alpha, \\ \delta(\alpha^2) &= \alpha^2, & \delta(\beta) &= \alpha\beta, \\ \delta(\alpha\beta) &= \beta, & \delta(\alpha^2\beta) &= \alpha^2\beta\,. \end{aligned}$$

In this case, there are again three switching classes. □

Notes on references

- The Δ^δ-graphs, where δ is the inversion, are also known as (complete) *gain graphs*, see ZASLAVSKY (1989), where these graphs are generalized to *biased graphs* defined by their balanced cycles. A generalization for gain graphs involving involutions is considered by HAGE AND HARJU (1997). In topological graph theory Δ^δ-graphs (with the inversion as the involution) are known as (complete) *voltage graphs* in GROSS (1974) and GROSS AND TUCKER (1987).
- Our present framework for 2-structures with a group of labels comes from EHRENFEUCHT AND ROZENBERG (1994b), where the switching classes are called *dynamic labelled 2-structures*. In that article these were derived by the network oriented approach.
- Switching classes of oriented graphs have been considered in CHENG AND WELLS (1986), where the fixed point result for automorphisms, Theorem 8.5 for $\mathbb{Z}$, is partially generalized to $\mathbb{Z}_3$. The case $\mathbb{Z}_4$ for general digraphs has been considered by CHENG (1986).

7

Clans of Switching Classes

For abelian Δ, the Δ^δ-graphs on a given domain form a group. In the approach that we take, this is not true in general for non-abelian groups.

Horizons are 'isolated' vertices in the Δ^δ-graphs. They will play a central role in the treatment of switching classes of Δ^δ-graphs, since, as it turns out, each Δ^δ-graph g can be switched to g^σ that has a given vertex as its horizon.

In the graphical representations, *we shall usually omit the edges labelled by the identity element* ε. Note that ε is always a symmetric label, since $\varepsilon^\delta = \varepsilon$.

7.1 Associated groups

Let $\zeta\colon D \to \Delta$ be the constant selector having the identity as its values,

$$\zeta(x) = \varepsilon_\Delta\,,$$

for all $x \in D$.

Theorem 7.1. *Let Δ be a group, and D a nonempty set. Then the set of* <u>*all*</u> *functions $D \to \Delta$ forms a group under the operation*

$$(\alpha \cdot \beta)(x) = \alpha(x)\beta(x)\,.$$

Proof. First $\zeta \cdot \alpha = \alpha = \alpha \cdot \zeta$, and hence ζ is an identity element. Moreover, for each α let $\alpha^{-1}\colon D \to \Delta$ be defined by $\alpha^{-1}(x) = \alpha(x)^{-1}$. Then $\alpha \cdot \alpha^{-1} = \zeta = \alpha^{-1} \cdot \alpha$ giving the claim. $\square$

Theorem 7.1 yields a group structure for the selectors. For this, let

$$S_\Delta(D) = \{\sigma \mid \sigma\colon D \to \Delta\}\,.$$

Note that the selectors are independent of the involutions of the group. The **product** of selectors is defined as above: $\tau\sigma(x) = \tau(x)\sigma(x)$. The following computation show that the selectors **act on** the Δ^δ-graphs.

Theorem 7.2. *The set $S_\Delta(D)$ forms a group under the product of selectors. $S_\Delta(D)$ acts on the Δ^δ-graphs: for every $g\colon E_2(D) \to \Delta^\delta$,*

$$g^\zeta = g \qquad \text{and} \qquad g^{\tau\sigma} = (g^\sigma)^\tau\,,$$

for all $\sigma, \tau\colon D \to \Delta$. Furthermore, a switching class of a Δ^δ-graph g equals the orbit of g under the action of $S_\Delta(D)$:

$$[g] = \{g^\sigma \mid \sigma \in S_\Delta(D)\}\,. \tag{7.1}$$

Proof. By Theorem 7.1, ζ is the identity element of $S_\Delta(D)$ and the inverse of a selector σ is determined by

$$\sigma^{-1}(x) = \sigma(x)^{-1}.$$

Let then $g\colon E_2(D) \to \Delta^\delta$. For each $e = (x, y) \in E_2(D)$,

$$\begin{aligned} g^{\tau\sigma}(e) &= \tau(x)\sigma(x) \cdot g(e) \cdot (\tau(y)\sigma(y))^\delta \\ &= \tau(x)\sigma(x)g(e)\sigma(y)^\delta \tau(y)^\delta \\ &= \tau(x) \cdot g^\sigma(e) \cdot \tau(y)^\delta \\ &= (g^\sigma)^\tau(e). \end{aligned}$$

□

Exercise 7.1. Let D be a finite nonempty set, Δ a group, and $x \in D$ an element. Then Δ is isomorphic to the subgroup

$$\Sigma_x = \{\sigma \in S_\Delta(D) \mid \forall y \neq x: \ \sigma(y) = \varepsilon\}$$

of $S_\Delta(D)$.

Corollary 7.1. *Let g be a Δ^δ-graph on a domain D. Then $[g^\sigma] = [g]$ for all $\sigma\colon D \to \Delta$.*

7.2 The group of abelian switching classes

We denote by $\mathbb{O}_D$ the (discrete) Δ^δ-graph on the domain D, for which

$$\mathbb{O}_D(e) = \varepsilon$$

for all $e \in E_2(D)$. The subscript is often omitted when the group is clear from the context.

Theorem 7.3. *If Δ is abelian, then $\Gamma^\delta_\Delta(D)$ is an abelian group.*

Proof. Clearly, $\mathbb{O}_D$ is a unit in $\Gamma^\delta_\Delta(D)$. By Theorem 7.1 and the definition of the product, the set of all functions $\alpha\colon E_2(D) \to \Delta$ forms an abelian group. We still have to introduce the inversion to this, i.e., we need show that $\Gamma^\delta_\Delta(D)$ forms a subgroup of the above group. For this, let $g, h \in \Gamma^\delta_\Delta(D)$. Then

$$\begin{aligned} (g \cdot h)(e^{-1}) &= g(e^{-1})h(e^{-1}) = g(e)^\delta h(e)^\delta \\ &= (h(e)g(e))^\delta = (g(e)h(e))^\delta = (g \cdot h)(e)^\delta, \end{aligned}$$

which shows that $g \cdot h$ satisfies the reversibility condition, and thus that $g \cdot h \in \Gamma^\delta_\Delta(D)$. □

Exercise 7.2. Let δ be the inversion of Δ. Show that, for all Δ^δ-graphs g,

$$g \in [\mathbb{O}] \iff g(x, y)g(y, z) = g(x, z) \ \text{ for all different vertices } x, y, z \in D.$$

Lemma 7.1. *Let Δ be an abelian group, $g, h \colon E_2(D) \to \Delta^\delta$, and $\sigma \in S_\Delta(D)$. Then*

$$(g \cdot h)^\sigma = g \cdot h^\sigma = g^\sigma \cdot h\,.$$

Proof. Indeed,

$$\begin{aligned}(g \cdot h)^\sigma(x,y) &= \sigma(x)(g \cdot h)(x,y)\sigma(y)^\delta \\ &= \sigma(x)g(x,y)h(x,y)\sigma(y)^\delta \\ &= g(x,y)h^\sigma(x,y) \\ &= (g \cdot h^\sigma)(x,y)\,.\end{aligned}$$

Similarly, $(g \cdot h)^\sigma = g^\sigma \cdot h$. □

Theorem 7.4. *For an abelian group Δ, the class $[\mathbb{O}]$ is a subgroup of $\Gamma^\delta_\Delta(D)$, and for all $g \in \Gamma^\delta_\Delta(D)$, the class $[g]$ is a coset of the class $[\mathbb{O}]$ of the group $\Gamma^\delta_\Delta(D)$:*

$$[g] = g \cdot [\mathbb{O}].$$

Proof. Let $\sigma, \tau \colon D \to \Delta$ be two selectors. By Lemma 7.1,

$$\mathbb{O}^\sigma \cdot \mathbb{O}^\tau = (\mathbb{O}^\sigma \cdot \mathbb{O})^\tau = (\mathbb{O} \cdot \mathbb{O})^{\sigma\tau} = \mathbb{O}^{\sigma\tau}\,,$$

from which it follows that $[\mathbb{O}_D]$ is a subgroup of $\Gamma^\delta_\Delta(D)$.

By Lemma 7.1, for each $g \in \Gamma^\delta_\Delta(D)$ and $\sigma \in S_\Delta(D)$, we have

$$g^\sigma = (g \cdot \mathbb{O})^\sigma = g \cdot \mathbb{O}^\sigma \in g \cdot [\mathbb{O}]\,.$$

Hence, $[g] = g \cdot [\mathbb{O}]$ is a coset in $\Gamma^\delta_\Delta(D)$. □

Exercise 7.3. Let Δ^δ be a non-abelian group with an involution δ, and let D be a domain of at least two vertices. Then the product $g \cdot h$, as defined above, *does not* preserve reversibility, and therefore $\Gamma^\delta_\Delta(D)$ is not a group under this product.

7.3 Clans and horizons

Consider $g \colon E_2(D) \to \Delta^\delta$ together with a colour $a \in \Delta$. A vertex $x \in D$ is called an **horizon** of g, if $g(x,y) = \varepsilon$ for all $y \neq x$. A horizon can be considered as an **isolated vertex** when we interpret $g(e) = \varepsilon$ as stating that e is not an edge of g. A clan $X \in \mathscr{C}(g)$ is said to be **isolated**, if $g(x,y) = \varepsilon$ for all $x \in X$ and $y \notin X$.

Lemma 7.2. *Let g be a Δ^δ-graph, $a \in \Delta$ and $X \in \mathscr{C}(X)$ a clan. Then there exists a switch $g^\sigma \in [g]$ such that $g^\sigma(x,y) = a$ for all $x \in X$ and $y \notin X$.*

Proof. Let a selector σ be defined by

$$\sigma(y) = \begin{cases} \varepsilon & \text{if } \ y \in X\,, \\ (g(x,y)^{-1}a)^{\delta} & \text{if } \ y \notin X\,, \end{cases} \tag{7.2}$$

where $x \in X$ is a fixed vertex. Then, for all $z \in X$ and $y \notin X$,

$$g^{\sigma}(z,y) = \varepsilon \cdot g(z,y) \cdot g(x,y)^{-1} \cdot a = g(x,y) \cdot g(x,y)^{-1} \cdot a = a\,,$$

since X is a clan. □

Hence, the complement $\overline{X} = D \setminus X$ of an isolated clan X is an isolated clan of g. Clearly, if a singleton clan $X = \{x\}$ is isolated, then the vertex x is a horizon of g.

Corollary 7.2. *For each Δ^{δ}-graph g, and vertex $x \in D_g$, there is a switch $g^{\sigma} \in [g]$ such that x is a horizon of h.*

Let g be a Δ^{δ}-graph. A clan of any individual $h \in [g]$ is called a **clan** of the switching class $[g]$. We shall write

$$\mathscr{C}[g] = \bigcup_{h \in [g]} \mathscr{C}(h)$$

for the family of clans of $[g]$.

In particular, by Lemma 7.2, we have the following closure result for switching classes.

Corollary 7.3. *The set $\mathscr{C}[g]$ of clans of a switching class $[g]$ is closed under complements.*

Therefore, $\emptyset$, D, $\{x\}$ and *also* $D \setminus \{x\}$ are clans of the switching class $\mathscr{C}[g]$.

A selector $\sigma\colon D \to \Delta$ is said to be **constant on** $X \subseteq D$ if $\sigma(x) = \sigma(y)$ for all $x, y \in X$. A selector that is constant on the whole domain D is called a **constant selector**.

If Δ is abelian and σ is a constant selector, then clearly $g^{\sigma} = g$ for all g. In the general case, we almost have this also.

Lemma 7.3. *Let g be a Δ^{δ}-graph and σ a constant selector. Then g^{σ} and g are isomorphic.*

Proof. Assume that $\sigma(x) = a$ for all $x \in D_g$. The mapping ψ, defined by $\psi(b) = aba^{\delta}$, is a permutation on Δ such that

$$\psi(g(x,y)) = ag(x,y)a^{\delta} = \sigma(x)g(x,y)\sigma(y)^{\delta} = g^{\sigma}(x,y)$$

for each $(x,y) \in E_2(D)$. □

Lemma 7.4. *Let $g\colon E_2(D) \to \Delta^{\delta}$ and $\sigma\colon D \to \Delta$. Also, let $X \subset D$ be a proper subset. For $x \notin X$, if $g(x,y) = g(x,z)$ and $g^{\sigma}(x,y) = g^{\sigma}(x,z)$ for all $y,z \in X$, then σ is constant on X.*

Proof. Let $y,z \in X$ be such that $g^{\sigma}(x,y) = g^{\sigma}(x,z)$ and $g(x,y) = g(x,z)$. Then

$$\sigma(x) \cdot g(x,y) \cdot \sigma(y)^{\delta} = g^{\sigma}(x,y) = g^{\sigma}(x,z) = \sigma(x) \cdot g(x,z) \cdot \sigma(z)^{\sigma}\,,$$

which implies that $\sigma(y) = \sigma(z)$ as claimed. □

In the following lemma we conclude that the maintenance of a clan $X \in \mathscr{C}[g]$ in a switching class $[g]$ depends only on X.

Lemma 7.5. *Let $g : E_2(D) \to \Delta^\delta$ and $\sigma : D \to \Delta$. Also, let $X \in \mathscr{C}(g)$ be a proper clan.*

(i) Then $X \in \mathscr{C}(g^\sigma)$ if and only if σ is constant on X.
(ii) Let $Z \subseteq X$. If $Z \in \mathscr{C}(g^\sigma)$, then σ is constant on Z and $Z \in \mathscr{C}(g)$.

Proof. For (i), if σ is constant on $X \in \mathscr{C}(g)$, then clearly $X \in \mathscr{C}(g^\sigma)$. In the other direction, the claim follows by calculation.

For (ii), we observe, by the first claim, that σ is constant on Z, and thus also σ^{-1} is constant on Z. Now, $Z \in \mathscr{C}(g)$ by (ii), since $g = (g^\sigma)^{\sigma^{-1}}$. □

Exercise 7.4. Let g be a Δ^δ-graph. If $g[X]$ is a primitive factor of g for some $X \neq D$, then $g^\sigma[X]$ contains no proper clans of g^σ.

Theorem 7.5. *Let $[g]$ be a switching class and assume $h \in [g]$ has a horizon x. If $X \in \mathscr{C}[g]$, then*

$$\begin{aligned} X \in \mathscr{C}(h) &\ \textit{ if } \ x \notin X, \\ \overline{X} \in \mathscr{C}(h) &\ \textit{ if } \ x \in X. \end{aligned}$$

In particular,

$$\mathscr{C}[g] = \{X \mid X \in \mathscr{C}(h) \ \textit{ or } \ \overline{X} \in \mathscr{C}(h)\}.$$

Proof. Let $X \in \mathscr{C}[g]$. By Lemma 7.2, there exists a selector σ such that both X and $\overline{X}$ are isolated clans of g^σ. Let $Z \in \{X, \overline{X}\}$ be such that $x \notin Z$. Then $Z \subseteq D \setminus \{x\} \in \mathscr{C}(h)$ implies, by Lemma 7.5(iii), that $Z \in \mathscr{C}(h)$. Hence $\mathscr{C}[g] \subseteq \{X \mid X \in \mathscr{C}(h)$ or $\overline{X} \in \mathscr{C}(h)\}$.

In the other direction the claim is clear by Corollary 7.3. □

Exercise 7.5. Let Δ be a finite group of order $k \geq 2$. If g is a Δ^δ-graph having a complete factor $g[X]$ for some X with $|X| \geq k+1$, then the switching class $[g]$ has no primitive Δ^δ-graphs.

Notes on references

- For the proof of Exercise 7.1, see *e.g.* ROSE (1978).
- Horizons and clans of Δ^δ-graphs were introduced and studied by EHRENFEUCHT AND ROZENBERG (1994), and EHRENFEUCHT, HARJU AND ROZENBERG (1996).
- Special subgraphs, such as spanning trees in switching classes for undirected graphs, have occurred in many articles, see GROSS AND TUCKER (1987). These result generalizes to signed graphs and gain graphs as proved by ZASLAVSKY (1981), (1989).
- The cardinalities of the switching classes of Δ^δ-graphs for finite groups Δ have been counted by HAGE AND HARJU (1997).

8

Switching of Undirected Graphs

This is an introduction to the switching of *undirected graphs* (or **Seidel switching**). As in the more general case, switching provides a locally defined transformation of graphs. But now there are some sweet results. We identify undirected graphs with their characteristic function $g\colon E_2(D) \to \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the additive cyclic group of two elements, 0 and 1.

Example 8.1. The **complete bipartite** $\mathbb{Z}_2$**-graph** with the bipartition $D = A \cup B$ (where A or B can be empty) is defined by

$$\mathbb{O}_{AB}(x,y) = \begin{cases} 1 & \text{if } |\{x,y\} \cap A| = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$ □

If $B = \emptyset$, then $\mathbb{O}_{AB} = \mathbb{O}_A$, the discrete graph on A. (Now, $\varepsilon = 0$.)

In this chapter, $\mathbb{Z}_2$*-graphs are called simply* ***graphs****.*

8.1 Switching

Since now $\mathbb{Z}_2$ will be fixed, instead of $S_{\mathbb{Z}_2}(D)$ we write

$$S(D) = \{\sigma \mid \sigma\colon D \to \mathbb{Z}_2\}\,.$$

Also, $\mathbb{Z}_2$ is abelian, and the only involution is the identity function (which is also the inverse function). Therefore the **switch** g^σ is determined by

$$g^\sigma(x,y) = \sigma(x) + g(x,y) + \sigma(y)$$

for all $(x,y) \in E_2(D)$, since now $-\sigma(y) = \sigma(y)$.

Remark 8.1. Let $\sigma\colon D \to \mathbb{Z}_2$ be a selector. Then both g and g^σ are (undirected) graphs. Let $A = \sigma^{-1}(0)$ and $B = \sigma^{-1}(1)$. Then g^σ is obtained by removing the existing connections e (with $g(e) = 1$) between A and B, and by creating the non-existing connections (with $g(e) = 0$) between A and B. The edges inside A and B are left intact. □

If $x \in D$ is a fixed vertex $x \in D$, the **elementary selector** at x is defined by

$$\sigma(y) = \begin{cases} 1 \text{ if } & y = x\,, \\ 0 \text{ if } & y \neq x\,. \end{cases}$$

For selectors $\sigma, \tau\colon D \to \mathbb{Z}_2$, their **sum** $\sigma + \tau$ is computed componentwise: for all $x \in D$,

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x)\,.$$

Exercise 8.1. Show the following claims.

(a) $S(D)$ forms an abelian group under addition. In this group $\sigma(x)+\sigma(x)=0$ for all $x \in D$. The zero element of $S(D)$ is the selector satisfying

$$\zeta(x)=0$$

for all $x \in D$. Each selector is its own inverse.

(b) Let $\sigma\colon D \to \mathbb{Z}_2$ be a selector, and assume that $\sigma^{-1}(1)=\{x_0,x_1,\ldots,x_k\}$ for some $k \geq 0$. Denote by σ_i the elementary selector at x_i. Then

$$\sigma(x)=\sum_{i=0}^{k}\sigma_i(x). \tag{8.1}$$

Therefore every selector is a sum of elementary selectors.

For $g\colon E_2(D) \to \mathbb{Z}_2$, the family

$$[g]=\{g^\sigma \mid \sigma\colon D \to \mathbb{Z}_2\}$$

is again the **switching class** of g.

Example 8.2. Consider the graph $g\colon E_2(D) \to \mathbb{Z}_2$, i.e., P_4, of Fig. 8.1 (where as usual a line denotes the value $1 \in \mathbb{Z}_2$).

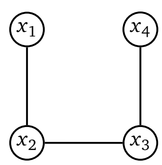
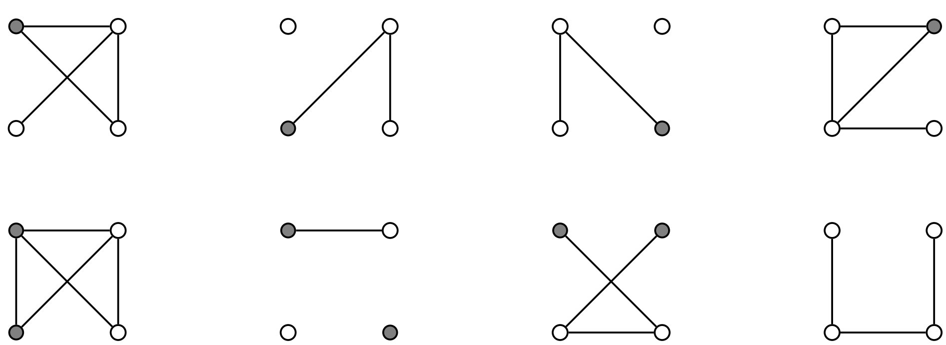


Fig. 8.1. P_4

There are $2^{|D|}=16$ different selectors $\sigma\colon D \to \mathbb{Z}_2$, but some of them share their switches. In fact, there are eight different switches g^σ. The eight graphs in the switching class $[g]$ are given in the following figure. Note that these are different graphs, although some of them are isomorphic. The grey vertices indicate the selectors that are applied to P_4. □



Let $\Gamma(D) = \Gamma^{\iota}_{\mathbb{Z}_2}(D)$, i.e.,

$$\Gamma(D) = \{g \mid g\colon E_2(D) \to \mathbb{Z}_2 \ \text{ undirected}\}\,.$$

The **sum** of the graphs $g, h\colon E_2(D) \to \mathbb{Z}_2$ is defined by

$$(g+h)(e) = g(e) + h(e)\,,$$

for all e.

Theorem 8.1. *Let D be a finite set.*

(i) The set $\Gamma(D)$ forms an abelian group under the operation of sum.
(ii) The zero element of this group is the discrete graph $\mathbb{O}_D$.

The **complement** $\overline{\sigma}$ of a selector $\sigma\colon D \to \mathbb{Z}_2$ is the selector

$$\overline{\sigma}(x) = 1 - \sigma(x)$$

for $x \in D$. Hence $\overline{\sigma}$ interchanges the values:

$$\overline{\sigma}(x) = 1 \iff \sigma(x) = 0\,.$$

Lemma 8.1. *Let $g\colon E_2(D) \to \mathbb{Z}_2$ and $\sigma, \tau\colon D \to \mathbb{Z}_2$. Then*

(i) $g^{\sigma} = g^{\overline{\sigma}}$;
(ii) $(g^{\sigma})^{\tau} = g^{\sigma+\tau}$;
(iii) $g^{\sigma} + g^{\tau} = \bar{K}^{\sigma+\tau}$;
(iv) $g + g^{\sigma} = K_{OI}$, *where* $O = \sigma^{-1}(0)$ *and* $I = \sigma^{-1}(1)$;
(v) $(g^{\sigma})^{\sigma} = g$.

Proof. We prove (i), the other cases are exercises.

For all $(x, y) \in E_2(D)$,

$$\begin{aligned} g^{\overline{\sigma}}(x,y) &= \overline{\sigma}(x) + g(x,y) + \overline{\sigma}(y) \\ &= 1 - \sigma(x) + g(x,y) + 1 - \sigma(y) \\ &= \sigma(x) + g(x,y) + \sigma(y) \\ &= g^{\sigma}(x,y)\,, \end{aligned}$$

since the addition is modulo 2. □

As in the general case, *the group $S(D)$ of selectors* ***acts on*** $\Gamma(D)$, i.e., for all $\sigma, \tau \in S(D)$ and $g \in \Gamma(D)$,

$$g^{\zeta} = g \quad \text{and} \quad (g^{\sigma})^{\tau} = g^{\tau+\sigma}.$$

By Lemma 7.1, we have

Lemma 8.2. *Let $g, h\colon E_2(D) \to \mathbb{Z}_2$ be graphs and σ a selector. Then*

$$(g+h)^{\sigma} = g + h^{\sigma} = g^{\sigma} + h.$$

Exercise 8.2. Consider $g\colon E_2(D) \to \mathbb{Z}_2$. If $|D| \geq 3$, then $\overline{g} \notin [g]$ for the complement graph $\overline{g}$, defined by $\overline{g}(e) = 1 - g(e)$ for all $e \in E_2(D)$.

In the following theorem we allow that a bipartition contains an empty set in order to include the discrete graphs as complete bipartite graphs. Recall that $\mathbb{O}_D$ is the discrete graph on D, and that $\mathbb{O}_{AB}$ is the complete bipartite graph with bipartition (A, B) (where B can be empty).

Theorem 8.2. *We have*

(i) $[g] = [g^{\sigma}]$ *for all* $g\colon E_2(D) \to \mathbb{Z}_2$ *and* $\sigma\colon D \to \mathbb{Z}_2$.
(ii) $[\mathbb{O}_D] = \{\mathbb{O}_{AB} \mid D = A \cup B \text{ a bipartition}\}$.

Proof. The first claim follows from Lemma 8.1(iv), since $(g^{\sigma})^{\sigma} = g$ for all g and σ. Hence $g \in [g^{\sigma}]$.

For (ii), we have $g \in [\mathbb{O}]$ if and only if there exists a selector σ such that $g = \mathbb{O}^{\sigma}$, which, by Lemma 8.1, implies that there exists a bipartition $D = A \cup B$ such that $g = \mathbb{O}_{AB}$. On the other hand, if $g = \mathbb{O}_{AB}$, then let σ be the characteristic function of A. Then $g^{\sigma} = \mathbb{O}$. □

By the following theorem, the class $[\mathbb{O}]$ has a central role among the switching classes.

By Theorem 7.4,

Theorem 8.3. *The class $[\mathbb{O}]$ is a subgroup of $\Gamma(D)$, and each switching class $[g]$ is a coset of $[\mathbb{O}]$:*

$$[g] = g + [\mathbb{O}] = \{g + \mathbb{O}_{AB} \mid A \cup B = D \text{ a bipartition}\}.$$

8.2 Structure of switching classes

We shall now prove that one can determine locally whether or not a graph h belongs to a switching class $[g]$ by looking at the subgraphs induced by the 3-subsets (i.e., triangles) of the domain D.

Let $g\colon E_2(D) \to \mathbb{Z}_2$ and $F \subseteq E_2(D)$. We let

$$g(F) = \sum_{e \in F} g(e) \qquad (\text{in } \mathbb{Z}_2) \tag{8.2}$$

be the parity of the number of edges e of g in F (such that $g(e) = 1$).

Lemma 8.3. *Let $g\colon E_2(D) \to \mathbb{Z}_2$ and let $F \subseteq E_2(D)$ be a nonempty subset such that each $x \in D$ is incident with an even number of edges of g in F. Then $g(F) = g^{\sigma}(F)$ for all selectors $\sigma \in S(D)$.*

Proof. For $x \in D$, let F_x be the number of ordered pairs $(x, y) \in F$. Then

$$\begin{aligned} g^{\sigma}(F) &= \sum_{(x,y)\in F} g^{\sigma}(x,y) = \sum_{(x,y)\in F} \sigma(x) + g(x,y) + \sigma(y) \\ &= \sum_{(x,y)\in F} g(x,y) + \sum_{x\in D} F_x \cdot \sigma(x) \\ &= g(F) + \sum_{x\in D} F_x \cdot \sigma(x), \end{aligned}$$

where $F_x = 0$ in $\mathbb{Z}_2$. □

Consider **triangles**

$$T = \langle x, y, z\rangle = \{(x, y), (x, z), (y, z)\},$$

where x, y and z are different vertices.

Theorem 8.4. *Let $g\colon E_2(D) \to \mathbb{Z}_2$. Then $h \in [g]$ if and only if $g(T) = h(T)$ for every triangle T.*

Proof. We may assume that $|D| \geq 3$. Every vertex occurs an even number of times in each triangle T, and thus, by Lemma 8.3, $g^{\sigma}(T) = g(T)$ for all σ.

For the converse, suppose that $g(T) = h(T)$ for all triangles T. Fix a vertex $x \in D$, and let σ and τ be defined by

$$\begin{aligned} &\sigma(x) = 0 = \tau(x), \\ &\sigma(y) = g(x, y) \quad\text{and}\quad \tau(y) = h(x, y) \quad\text{for } y \neq x. \end{aligned}$$

We have $g^{\sigma}(x, y) = 0 = h^{\tau}(x, y)$ for all y ($\neq x$). We show that $g^{\sigma} = h^{\tau}$, from which it follows that $h = (g^{\sigma})^{\tau} = g^{\tau+\sigma}$, and thus that $h \in [g]$ as required. Let $y \neq z$ be in $D \setminus \{x\}$. By Lemma 8.3, for the triangle $T = \langle x, y, z\rangle$, $h(T) = h^{\tau}(T)$ and $g(T) = g^{\sigma}(T)$, where

$$h^{\tau}(y, z) = h^{\tau}(T) = h(T) = g(T) = g^{\sigma}(T) = g^{\sigma}(y, z). \tag{8.3}$$

□

Automorphisms

We say that $g, h\colon E_2(D) \to \mathbb{Z}_2$ are **strictly isomorphic** if there exists a permutation $\alpha\colon D \to D$ such that for all $(x, y) \in E_2(D)$,

$$g(x, y) = h(\alpha(x), \alpha(y)). \tag{8.4}$$

In this case, we write $h = \alpha(g)$. Strictly isomorphic graphs are naturally isomorphic, but in (8.4) we do not allow permuting the colours. (Well, there are now only two colours.)

A permutation $\alpha\colon D \to D$ is called an **automorphism** of a switching class $[g]$ if $\alpha(h) \in [g]$ for all $h \in [g]$. Therefore a permutation α of the domain is an automorphism of $[g]$ if $[g]$ is closed under α. This includes the requirement (8.4).

We observe first that if $\alpha\colon D \to D$ is a permutation and $g, h\colon E_2(D) \to \mathbb{Z}_2$ are any two graphs, then $\alpha(g+h) = \alpha(g) + \alpha(h)$.

Lemma 8.4. *Let α be a permutation on D, $\sigma\colon D \to \mathbb{Z}_2$ a selector, and $g\colon E_2(D) \to \mathbb{Z}_2$ a graph. Then*

$$\alpha(g^\sigma) = \alpha(g)^{\sigma\alpha^{-1}}.$$

Proof. Indeed,

$$\begin{aligned}
\alpha(g^\sigma)(x,y) &= g^\sigma(\alpha^{-1}(x), \alpha^{-1}(y)) \\
&= \sigma(\alpha^{-1}(x)) + g(\alpha^{-1}(x), \alpha^{-1}(y)) + \sigma(\alpha^{-1}(y)) \\
&= \sigma(\alpha^{-1}(x)) + \alpha(g)(x,y) + \sigma(\alpha^{-1}(y)) \\
&= (\alpha(g))^{\sigma\alpha^{-1}}(x,y).
\end{aligned}$$

□

According to the next exercise, in order for a permutation α to be an automorphism of a switching class $[g]$, it only needs to preserve one of the graphs in $[g]$.

Exercise 8.3. Let $\alpha\colon D \to D$ be a permutation and $g\colon E_2(D) \to \mathbb{Z}_2$ a graph. Show that $\alpha([g]) = [\alpha(g)]$. In particular, if $\alpha(h) \in [g]$ for some $h \in [g]$, then α is an automorphism of $[g]$.

We prove that every automorphism of a switching class has a fixed point.

Theorem 8.5. *Let $g\colon E_2(D) \to \mathbb{Z}_2$. A permutation $\alpha\colon D \to D$ is an automorphism of the switching class $[g]$ if and only if there exists a graph $h \in [g]$ such that $\alpha(h) = h$.*

Proof. In the converse implication the claim follows from Exercise 8.3.

Suppose now that α is an automorphism of $[g]$. Let τ be such that $\alpha(g) = g^\tau$. By Lemma 8.4,

$$\alpha(g^\sigma) = \alpha(g)^{\sigma\alpha^{-1}} = g^{\tau+\sigma\alpha^{-1}},$$

for all σ, and thus it is sufficient to find a selector σ that satisfies

$$\sigma = \tau + \sigma\alpha^{-1}. \tag{8.5}$$

Denote the orbit of the vertex $x \in D$ by

$$A_x = \{\alpha^i(x) \mid i \geq 0\} = \{x_0, x_1, \ldots, x_{r-1}\} \quad (\text{where } \alpha(x_i) = x_{i+1 (\mathrm{mod}\, r)}). \tag{8.6}$$

Claim ($\star$). *For all x, $|A_x \cap \tau^{-1}(1)|$ is even or, for all x, $|A_x \cap \tau^{-1}(0)|$ is even.*

To see this, let $X \subseteq D$ be any closed set with respect to α, i.e., $\alpha(X) = X$. Then $g[X]$ and $g^\tau[X]$ are strictly isomorphic, and so they have equally many edges. Let

$$B_0 = X \cap \tau^{-1}(0) \quad \text{and} \quad B_1 = X \cap \tau^{-1}(1).$$

Now $g[X]+g^{\tau}[X] = (g+g^{\tau})[X]$ is a complete bipartite graph with the bipartition $X = B_0 \cup B_1$, by Lemma 8.1. The subgraphs $g[X]$ and $g^{\tau}[X]$ agree on the sets B_0 and B_1, since τ is constant on them. Hence the edges of $g[X]$ and $g^{\tau}[X]$ from $B_0 \times B_1$ form a partition of $B_0 \times B_1$. Therefore they have the same number of edges (with label 1) in $B_0 \times B_1$. Consequently, $|B_0 \times B_1|$ is even, and hence $|B_0|$ or $|B_1|$ is even.

Consider then any orbit A_x such that $|A_x \cap \tau^{-1}(0)|$ is odd. By the above, $|A_x \cap \tau^{-1}(1)|$ is even. Let A_y be another orbit and suppose that $|A_y \cap \tau^{-1}(1)|$ is odd, and thus $|A_y \cap \tau^{-1}(0)|$ is even. The set $X = A_x \cup A_y$ is closed with respect to α, and

$$|X \cap \tau^{-1}(i)| = |A_x \cap \tau^{-1}(i)| + |A_y \cap \tau^{-1}(i)|.$$

In both cases, $|X \cap \tau^{-1}(i)|$ is odd; a contradiction. This proves Claim ($\star$).

Since $g^{\tau} = g^{\overline{\tau}}$ and $\overline{\tau}^{-1}(1) = \tau^{-1}(0)$, we may assume that for all orbits, $|A_x \cap \tau^{-1}(1)|$ is even for all $x \in D$. Thus for each orbit A_x as in (8.6),

$$\sum_{i=0}^{r-1} \tau(x_i) = 0. \tag{8.7}$$

Let $\sigma\colon D \to \mathbb{Z}_2$ be defined as follows. For each orbit A_x as in (8.6), let

$$\sigma(x_0) = 0 \quad \text{and} \quad \sigma(x_j) = \sum_{i=1}^{j} \tau(x_i) \quad \text{for } j \in [1, r-1].$$

We have $\sigma(x_{i+1}) - \sigma(x_i) = \tau(x_{i+1})$ for all $i \in [0, r-2]$, and, by (8.7), also $\sigma(x_0) - \sigma(x_{r-1}) = \tau(x_0)$. Hence $\sigma(y) - \sigma(\alpha^{-1}(y)) = \tau(y)$ for all $y \in A_x$, and for all orbits A_x. This proves (8.5) for σ. □

8.3 Special problems

Eulerian graphs

A graph $g\colon E_2(D) \to \mathbb{Z}_2$ is said to be **eulerian** if there exists a closed walk $(x_0, x_1, \dots, x_m)$ in g containing every edge exactly once.

Theorem 8.6. *A graph g is eulerian if and only if g is connected and it is **even**, i.e., every vertex of g has an even degree $d_g(x)$, where*

$$d_g(x) = |\{y \mid g(x,y) = 1\}|.$$

Exercise 8.4. If $g, h\colon E_2(D) \to \mathbb{Z}_2$ are even, so is their sum $g + h$.

Theorem 8.7. *Assume that $|D|$ is odd. Then every switching class $[g]$, with $g\colon E_2(D) \to \mathbb{Z}_2$, contains a unique eulerian graph.*

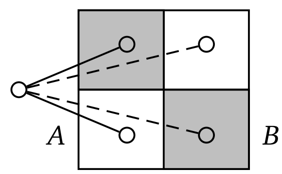
Proof. For $g\colon E_2(D)\to\mathbb{Z}_2$ let

$$\sigma(x)=\begin{cases}1 & \text{if } d_g(x) \text{ is odd,}\\ 0 & \text{if } d_g(x) \text{ is even.}\end{cases}$$

The Handshaking Lemma guarantees that in every graph the number of vertices of odd degree is even, and since $|D|$ is assumed to be odd, there is an odd number of vertices with an even degree. From this it follows that g^σ is even. Indeed, let, for a vertex x,

$$A=\{y \mid \sigma(y)=0 \text{ and } g(x,y)=1\} \text{ and } B=\{y \mid \sigma(y)=0 \text{ and } g(x,y)=0\}.$$

Then $|A|$ and $|B|$ are of different/same parity depending on if $\sigma(x)=1$ or 0.



For the uniqueness, note that the even graphs form a subgroup of $\Gamma(D)$. Indeed, the discrete graph $\mathbb{O}$ is even, and if g and h are even, so is their sum $g+h$ by the previous exercise. Consequently, if both g and g^σ (with $g\neq g^\sigma$) are even for a selector σ, so is the complete bipartite graph $\mathbb{O}_{OI}=g+g^\sigma$, where $O=\sigma^{-1}(0)$ and $I=\sigma^{-1}(1)$. Now, $O\neq D$ and $I\neq D$ (since $g^\sigma\neq g$), and so if $\mathbb{O}_{OI}$ is even, then both $|O|$ and $|I|$ are even, and thus so is $|D|=|O|+|I|$. This shows that $[g]$ contains a unique even graph, whenever $|D|$ is odd. □

A graph g is said to be **odd**, if the degrees of its vertices are odd.

Theorem 8.8. *Let $g\colon E_2(D)\to\mathbb{Z}_2$ be a graph where $|D|$ is even. Then either $[g]$ has no even and no odd graphs, or exactly half of its graphs are even while the other half are odd.*

Proof. Write $x\sim_g y$ if the degrees of x and y have the same parity. This relation is an equivalence relation on D. For an elementary selector σ at a vertex x, we obtain

$$d_{g^\sigma}(y)=\begin{cases}|D|-1-d_g(y) & \text{if } y=x,\\ d_g(y)+1 & \text{if } g(x,y)=0,\\ d_g(y)-1 & \text{if } g(x,y)=1,\end{cases}$$

and therefore $d_g(y)$ and $d_{g^\sigma}(y)$ have different parities for each $y\in D$. Consequently, the relations $\sim_g$ and $\sim_{g^\sigma}$ agree (since $|D|$ is even).

Now, by (8.1), every selector σ, which is not identically zero ($\sigma\neq\zeta$), is a sum of elementary selectors, $\sigma=\sum_{i=0}^k\sigma_i$, where $\sigma^{-1}(1)=\{x_0,\dots,x_k\}$ and σ_i is the elementary selector at x_i. Therefore $\sim_g$ and $\sim_{g^\sigma}$ agree for all selectors, and, by the above, $d_g(x)$ and $d_{g^\sigma}(x)$ have the same parity if and only if k is even. The claim follows from this. □

Pancyclic graphs

A graph $g\colon E_2(D) \to \mathbb{Z}_2$ is said to be **hamiltonian** if it has a **Hamilton cycle**:

$$D = \{x_0, x_1, \ldots, x_{n-1}\} \text{ with } g(x_i, x_{i+1(\mathrm{mod}\, n)}) = 1 \text{ for all } i \in [0, n-1],$$

where each vertex is visited once. Moreover, g is said to be **pancyclic** if it has a cycle C_i for all lengths i with $3 \le i \le n$. In particular, each pancyclic graph is hamiltonian.

The proof of Theorem 8.9 uses the following result, the proof of which we shall omit.

Lemma 8.5. *Let $g\colon E_2(D) \to \mathbb{Z}_2$ be a graph with $|D| = n$ that has a vertex x of degree $d_g(x) > (n-1)/2$ such that $g[D \setminus \{x\}]$ is hamiltonian. Then g is pancyclic.*

We show that each class $[g]$ has a pancyclic graph unless g is a complete bipartite graph.

Theorem 8.9. *For every $g\colon E_2(D) \to \mathbb{Z}_2$ with $|D| = n \ge 3$, the switching class $[g]$ contains a pancyclic graph if and only if $[g] \ne [\mathbb{O}]$.*

Proof. By Theorem 8.4, no graph from $[\mathbb{O}]$ has a cycle of length three, and thus the graphs in $[\mathbb{O}]$ are not pancyclic.

We prove the claim in the opposite direction by induction on n. For $n = 3$, the claim is clear. Suppose then that $n \ge 4$ and let $g \notin [\mathbb{O}]$. By Theorem 8.4, g contains a 3-subset inducing a subgraph with an odd number of edges, and hence there exists a vertex x such that $g[D \setminus \{x\}]$ contains such a 3-subset, from which we deduce, again by Theorem 8.4, that $g[D \setminus \{x\}] \notin [\mathbb{O}]$. By the induction hypothesis, there exists a selector $\sigma\colon (D \setminus \{x\}) \to \mathbb{Z}_2$ such that

$$h = g[D \setminus \{x\}]^\sigma \ \ (= g^\sigma[D \setminus \{x\}])$$

is pancyclic. Let $(x_1, x_2, \ldots, x_{n-1})$ be a Hamilton cycle of h. There are two extensions of σ to D depending on the value, 0 or 1, of x. We let these extensions be σ_0 and σ_1 such that $g^{\sigma_0}(x, x_1) = 0$ and $g^{\sigma_1}(x, x_1) = 1$. Also, let $g_i = g^{\sigma_i}$ for $i = 0, 1$. Now h is a substructure (an induced subgraph) of both g_0 and g_1, and hence if either of these graphs is hamiltonian, it is pancyclic.

It is easy to show that if $d_{g_i}(x) > (n-1)/2$ for $i = 0$ or 1, then g_i necessarily contains a cycle C_3 as its subgraph on one of the 3-subsets $\{x, x_j, x_{j+1}\}$, and in this case, g_i has a Hamilton cycle

$$(x, x_{j+1}, \ldots, x_{n-1}, x_1, \ldots, x_j),$$

and therefore it is pancyclic. We may thus assume that

$$d_{g_0}(x) = \frac{1}{2}(n-1) = d_{g_1}(x)$$

since $d_{g_0}(x) + d_{g_1}(x) = n-1$. Consequently, n is odd. We also conclude that the neighbourhoods of x in g_0 and in g_1 are

$$N_{g_0}(x) = \{x_i \mid i \text{ even}\} \text{ and } N_{g_1}(x) = \{x_i \mid i \text{ odd}\}. \tag{8.8}$$

Consider first the case $1 \le 2j \le n-1$. By (8.8), the sequence $(x, x_{2j+1}, \dots, x_{n-1}, x_1, \dots, x_{2j-1})$ is a Hamilton cycle of $g_1[D-\{x_{2j}\}]$. As in the above, we may assume that $d_{g_1}(x_{2j}) = (n-1)/2$, since, otherwise, Lemma 8.5 guarantees the existence of a pancyclic graph g^τ, where either $\tau = \sigma_1$ or τ is obtained from σ_1 by changing its value at x_{2j}. Since $g_1(x, x_{2j}) = 0$, we deduce that $d_h(x_{2j}) = (n-1)/2$.

A similar argument for $g[D-\{x_{2j+1}\}]$ with $1 \le 2j+1 \le n-1$ yields that $d_h(x_{2j+1}) = (n-1)/2$ (or we have a pancyclic graph in the switching class).

It follows that in h either

$$N_{g_1}(x) = \{y \mid h(x_{2j}, y) = 1\} \text{ for all } x_{2j} \text{ and}$$
$$N_{g_0}(x) = \{y \mid h(x_{2j+1}, y) = 1\} \text{ for all } x_{2j+1}\,, \tag{8.9}$$

or

$$\text{there exists a cycle } (x_t, x_k, x_{k+1}) \text{ of three vertices in } h\,. \tag{8.10}$$

The condition (8.9) implies that h is a complete bipartite graph with the bipartition $D = N_{g_0}(x) \cup N_{g_1}(x)$ contradicting the assumption $h \notin [\mathbb{O}]$. The condition (8.10) ensures a Hamilton cycle

$$(x_t, x_{k+1}, \dots, x_{n-1}, x_1, \dots, x_{t-1}, x, x_{t+1}, \dots, x_k)$$

either in g_1 (if t is even) or in g_0 (if t is odd). This proves the claim. □

Exercise 8.5. A switching class $[g]$ of a graph g contains a hamiltonian graph if and only if g is not a complete bipartite graph with an odd number of vertices.

Trees

A switching class two (or more) trees, see Example 8.2, but these trees are all isomorphic, and, in fact, a switching class can contain two trees only in some special cases.

Lemma 8.6. *Let $g, g^\sigma : E_2(D) \to \mathbb{Z}_2$ be trees where σ is a selector. Assume that both $O = \sigma^{-1}(0)$ and $I = \sigma^{-1}(1)$ are nonempty. Let C be a connected component of $g[I]$. Then*

(a) $|C| \le 2$.
(b) If $C = \{x, y\}$ with $x \ne y$, then for all $z \in O$,

$$\textit{either } \; g(z,x) = 1 \; \textit{ or } \; g(z,y) = 1 \textit{ but not both}\,. \tag{8.11}$$

(c) either $g[I]$ or $g[O]$ is discrete.

Proof. Since σ is constant on I, we have $g[I] = g^\sigma[I]$. Thus these two subgraphs have the same connected components. Let $C \subseteq I$ be such a connected component. By acyclicity, for each $z \in O$, there can be at most one $x \in C$ such that $g(z,x) = 1$. Similarly, there can be at most one $x \in C$ such that $g^\sigma(z,x) = 1$.

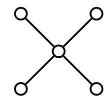
For each $(z,x) \in O \times I$, either $g(z,x) = 1$ or $g^\sigma(z,x) = 1$, and (a) and (b) follow.

For (c), suppose that, for some $x, y \in I$, $g(x,y) = 1$ and $u, v \in O$, $g(u,v) = 1$. By (b), we can assume that $g(u,x) = 1$. By acyclicity, we have to have that $g(v,x) = 0$ and $g(v,y) = 0$, contradicting case (b). □

Before the main theorem for trees, we descibe the exceptional trees whose isomorphic copies can occur in a switching class more than once.

A complete bipartite graph $\mathbb{O}_{AB}$ is said to be a $\mathbb{O}_{k,m}$-graph if $|A| = k$ and $|B| = m$.

A tree $g: E_2(D) \to \mathbb{Z}_2$ is called a **star at** x if it is the complete bipartite $\mathbb{O}_{1,n-1}$-graph $\mathbb{O}_{\{x\},D\setminus\{x\}}$. A $\mathbb{O}_{1,4}$-star is on the right:



The path P_t consists of t vertices, say $x_1, x_2, \ldots, x_t$, with $P_t(x_i, x_{i+1}) = 1 = P_t(x_{i+1}, x_i)$ for all $i \in [1, t-1]$, and otherwise $P_t(x, y) = 0$; for a P_7, see Fig. 8.2.
The special graph T_7 is (isomorphic to) the tree in Fig. 8.2.

Fig. 8.2. P_7 and T_7

The graph $P_t(m, r)$ is obtained from the path P_t by replacing the leaves x_1 and x_t by the stars $\mathbb{O}_{1,m}$ at x_1 and $\mathbb{O}_{1,r}$ at x_t, see Fig. 8.3 for $P_2(m, r)$ and $P_4(m, r)$ (where the black vertices have a special meaning in the proof of Theorem 8.10).

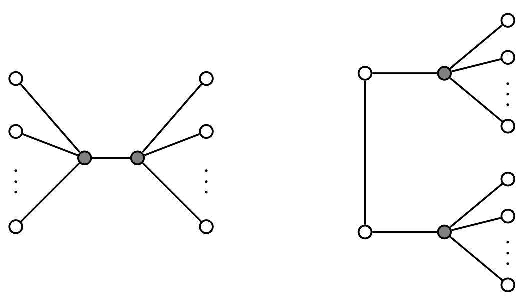


Fig. 8.3. $P_2(m, r)$ and $P_4(m, r)$

The graph $\mathbb{O}^*_{1,m}$ if it is obtained from the star $\mathbb{O}_{1,m}$ by replacing the leaves by edges P_2, see Fig. 8.4.
The graph $\mathbb{O}_{1,3}(m, r)$ is obtained from the star $\mathbb{O}_{1,3}$ by replacing two of its leaves x and y by stars $\mathbb{O}_{1,m}$ at x and $\mathbb{O}_{1,r}$ at y, see Fig. 8.4.

Theorem 8.10. *Each switching class contains at most one tree up to isomorphism.*

Proof. Assume that $g: E_2(D) \to \mathbb{Z}_2$ is a tree and that σ is a selector such that also g^σ is a tree. Let $O = \sigma^{-1}(0)$ and $I = \sigma^{-1}(1)$. We assume that these sets are nonempty; otherwise $g^\sigma = g$.

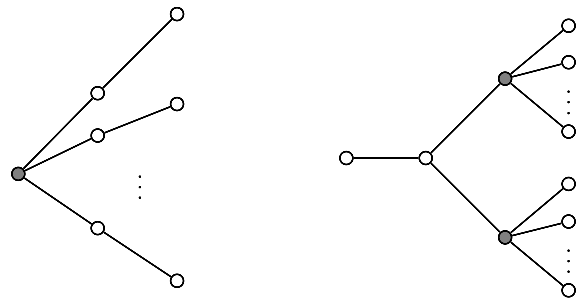


Fig. 8.4. $\mathbb{O}^*_{1,m}$ and $\mathbb{O}_{1,3}(m,r)$

By Lemma 8.6, we can assume that $g[O]$ is discrete; otherwise consider $\overline{\sigma}$, for which $g^\sigma = g^{\overline{\sigma}}$ by Lemma 8.1.

Let $n = |D|$, $p = |I|$. By Lemma 8.6, $g[I]$ contains edges $g(x_i, y_i) = 1$, for $i \in [1,r]$, such that the sets $\{x_i, y_i\}$ are the non-singleton connected components of $g[I]$.

Since g is a tree, it has exactly $n-1$ edges, and therefore there are $(n-1)-r$ edges in $O \times I$. Similarly g^σ has exactly $n-1$ edges, and of these $p(n-p)-(n-1-r)$ are in $O \times I$ (since $g^\sigma[I]$ has the same r edges as $g[I]$). Therefore, the number of edges of g^σ satisfies the equality $n-1 = p(n-p)-(n-1-r)+r$, from which we obtain

$$(p-2)n = (p-2)(p+1)+(p-2r). \tag{8.12}$$

If $p = 1$, then $r = 0$, and $n = 1$. This is a trivial case.

If $p = 2$, then $p = 2r$ and hence $r = 1$, and in this case, $I = \{x_1, y_1\}$ with $g(x_1, y_1) = 1$ (since $g[O]$ is discrete). By (8.11), $O = A \cup B$, where

$$A = \{z \in O \mid g(z, x_1) = 1\} \text{ and } B = \{z \in O \mid g(z, y_1) = 1\}$$

form a partition of O. Therefore g is a $P_2(m,t)$ of Fig. 8.3, where the black vertices are in I, and $m = |A|$, $t = |B|$. Here also g^σ is a $P_2(m,t)$, and thus isomorphic to g.

Assume then that $p > 2$. Now the equation (8.12) becomes

$$n = p+1+\frac{p-2r}{p-2}. \tag{8.13}$$

Clearly, either $2r = p$ (perfect matching of $g[I]$) or $r \in \{0,1\}$. These cases give the following solutions.

If $2r = p$, then $n = p+1$, and $g[I]$ has r edges with no isolated vertices, and $g[O]$ is a singleton graph. Thus g is a $K^*_{1,r}$ of Fig. 8.4, where the black vertex is in O. In this case, g^σ is isomorphic to g.

If $r = 1$, then $n = p+2$, and $g[I]$ has exactly one edge (x_1, y_1) and $p-2$ isolated vertices, and $g[O]$ is a discrete graph of two vertices, say z_1, z_2. By (8.11), there are two options: either z_1 and z_2 are connected to the same or to different vertices of $\{x_1, y_1\}$. It follows that g is either $\mathbb{O}_{1,3}(m,t)$ or $P_4(m,t)$ of Fig. 8.4 and Fig. 8.3, respectively. (Here $m+t = p-2$, and m and t are the numbers of the neighbours of z_1 and z_2 in $I \setminus \{x_1, y_1\}$.) As is easy to see, g^σ is isomorphic to g in both of these cases.

If $r = 0$, then $p = 3$ or $p = 4$, since now $p-2$ divides p. In this case, $n = 7$. By inspection (of an appendix of Harary's book), there are 11 non-isomorphic trees on seven vertices. Of these trees seven have a bipartition into a 3-vertex subset and a 4-vertex subset. Since both g and g^σ are trees, g contains no independent set with two vertices in I and two vertices in O. Moreover, no vertex in I (respectively in O) can be connected to all vertices in O (respectively in I), for otherwise that vertex would be isolated in g^σ. After these considerations there remains only two trees, P_7 and T_7, of seven vertices. For both of these trees, g^σ is isomorphic to g. $\square$

The proof of Theorem 8.10 reveals a more detailed result stating that a switching class can contain two or more trees only in the special cases of Fig. 8.3, 8.4 and 8.2.

Corollary 8.1. *If a switching class contains more than one tree g, then g is one of $P_2(m, t+1)$, $P_4(m, t)$, $\mathbb{O}_{1,3}(m, t)$, $disc1, t+1^*$, P_7, or T_7 for some $m, t \geq 0$.*

Two-graphs

The properties of the switching classes of undirected graphs are formulated in many research articles in terms of two-graphs. A **two-graph** Ω on a domain D is a family of 3-subsets of D such that each 4-subset of D contains an even number of 3-subsets from Ω.

Exercise 8.6. Let Ω be a two-graph and $C \subseteq D$ a subset of its domain. Then $\{A \in \Omega \mid A \subseteq C\}$ is a two-graph on C.

Let $g\colon E_2(D) \to \mathbb{Z}_2$, and let $\Omega(g)$ be the family of all 3-subsets A of D such that $g[A]$ has an odd number of edges of g, i.e., for a 3-subset $\{x, y, z\} \in \Omega(g)$,

$$\{x, y, z\} \in \Omega(g) \iff g(\langle x, y, z\rangle) = 1\,,$$

where $\langle x, y, z\rangle = \{(x, y), (x, z), (y, z)\}$ is a triangle on the vertices of A.

Exercise 8.7. Let $|D| \geq 3$ be a finite set.

(i) Show that for a family of 3-subsets Ω of D, Ω is a two-graph if and only if there exists $g\colon E_2(D) \to \mathbb{Z}_2$ such that $\Omega = \Omega(g)$.
(ii) Show that for $g, h\colon E_2(D) \to \mathbb{Z}_2$, $\Omega(g) = \Omega(h)$ if and only if $[g] = [h]$.

Notes on references

- Switching of graphs was defined by VAN LINT AND SEIDEL (1966) in connection with a problem of finding equilateral n-tuples of points in elliptic geometry. For further results on switching of (undirected) graphs and two-graphs, we refer to SEIDEL (1976) and SEIDEL AND TAYLOR (1981), where the authors give several connections to other parts of mathematics.

- Theorems 8.4 and 8.7 were proved by SEIDEL (1976).
- Theorem 8.5 was first proved by MALLOWS AND SLOANE (1975) and explicitly stated by CAMERON (1977a,1977b). For further results of automorphisms, check HARRIES AND LIEBECK (1978) and LIEBECK (1982).
- Hamiltonian graphs in switching classes have been considered by KRATOCHVÍL, NEŠETŘIL AND ZÝKA (1992). Theorem 8.9 for the pancyclic graphs is due to EHRENFEUCHT, HAGE, HARJU AND ROZENBERG (1998). For Lemma 8.5, see HÄGGKVIST, FAUDREE AND SCHELP (1981).
- The results on trees in switching classes, see Theorem 8.10, come from HAGE AND HARJU (2000), where one can find more details on acyclic graphs in switching classes. For a complete characterization of acyclicity, see HAGE AND HARJU (2004) where a characterization is given by means of forbidden induced subgraphs. Apart from switches of the cycles C_n for $n \geq 7$, there are only finitely many forbidden graphs. They live in 24 switching classes, all having at most 9 vertices. The total number of forbidden graphs is 905.

  Trees and their connection to switching classes have been considered also in CAMERON (1994).
- Two-graphs were introduced originally by HIGMAN (1964) in the context of sporadic groups.

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