Topological Spaces

October 26th, 2020

1 Topological Spaces

Definition. A **topology** on a set X is a collection $\mathcal{T}$of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$

(2) For any subcollection of $\mathcal{T}$, indexed by set I, we have: $\displaystyle\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}$

(3) For any finite subcollection of $\mathcal{T}$ with n elements, we have: $\displaystyle\bigcap_{i=1}^{n} U_i \in \mathcal{T}$

A set for which a topology $\mathcal{T}$ is specified is called a **topological space**. And the element of $\mathcal{T}$ is called **Open Set**

With the element of $\mathcal{T}$ is defined as open set, we could say a topology is a collection of subsets of X such that $\varnothing$ and X itself are open and satisfies that arbitrary unions and finite intersections of open sets are open. We often write set X and its topology $\mathcal{T}$ as the ordered pair: $(X, \mathcal{T})$. And when we say :"Let XXX be open sets", that means we defined a topology on X and $\mathcal{T}$ consists the subsets mentioned above.

EXAMPLE. If X is any set, the collection of all subsets of X is a topology on X, called **discrete topology**. The collection which has only $\varnothing$ and X itself is called **trivial topology**.

EXAMPLE. Let X be a set; let $\mathcal{T}_f$ be the collectino of all subsets U of X such that $X-U$ is either finite or all of X. Then $\mathcal{T}_f$ is a topology of X, called ***finite complement topology***. Note that $varnothing = U - U$ is finite and $U = U - \varnothing$, therefore we have $\varnothing$ and U belong to $\mathcal{T}_f$. Let $\{U_\alpha\}$ be a subcollection of $\mathcal{T}$ indexed by I. Then we have:

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Since each $X - U_\alpha$ is finite, we have $X - \bigcup U_\alpha$ is finite. If $U_1, ..., U_n \in \mathcal{T}_f$. Then:

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

Since each $X - U_i$ is finite, the finite union of sets with finite cardinal numbers are also finite. Thus $\displaystyle\bigcap_{i=1}^{n} U_i \in \mathcal{T}_f$

In conclusion, $\mathcal{T}_f$ is a topology on set X.

EXAMPLE. Let X be a set and $\mathcal{T}$ a topology on X. If Y is a subset of U. We define the following collection:

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathrm{T}\}$$

It is easy to see that $\mathcal{T}_Y$ is a topology on Y:

$$\varnothing = \varnothing \cap Y \quad Y = X \cap Y$$

If $\{V_\alpha\}$ is a subcollection of $\mathcal{T}_Y$, then each V_α could be written as $U_\alpha \cap Y$, we have:

$$\bigcup V_\alpha = \bigcup (U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y$$

Note that $\bigcup U_\alpha$ is in $\mathcal{T}$,hence we have $\bigcup V_\alpha \in \mathcal{T}_Y$.
If $V_i = U_i \bigcap Y, i = 1, 2, ..., n$ is a finite collection of $\mathcal{T}_Y$. Then:

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (U_i \cap Y) = (\bigcap_{i=1}^{n} U_i) \bigcap Y$$

Note that $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$, thus we have $\bigcap\limits_{i=1}^{n} V_i \in \mathcal{T}_Y$. The above new collection consists of the intersection of Y and open sets are called ***subspace topology***, and therefore, Y is a topological space.

REMARK. It is easy to see that a set could be assigned with different topology. A typical example is the discrete topology and trivial topology of the same set X. These two topology represents different topological structure. A trivial topology looks like a steel while discrete topology is fine enough to make generate any subset of X.

Definition. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T} \subset \mathcal{T}' (\mathcal{T} \subsetneqq \mathcal{T}')$,we say that $\mathcal{T}'$ is ***finer***(***strictly finner***) than $\mathcal{T}$, or $\mathcal{T}$ is ***coarser***(***stricly coarser***) than $\mathcal{T}'$. We say $\mathcal{T}$ is ***comparable*** with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$

Sometimes we also say that $\mathcal{T}'$ is larger than $\mathcal{T}$ or $\mathcal{T}$ is smaller than $\mathcal{T}'$, but not as vivid as finer.

2 Closed Sets and Limit Point

2.1 Closed Set

Definition. Let $(X, \mathcal{T})$ be a topological space. We say a subset A of X is ***closed*** if $X - A$ is open.

EXAMPLE. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{T}$ be the discrete topology, then any subset of X is a closed set. On the other hand, let $\mathcal{T}$ be trivial topology, then any subset that is neither $\varnothing$ nor X is neither open nor closed.

EXAMPLE. Let $(\mathbb{R}^2, \mathcal{T})$ be a topological space and $\mathcal{T}$ generated by all open ball. And consider the set:

$$\{(x, y) \mid x \geq 0, y \geq 0\}$$

The set is closed as its complement is:

$$(-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$$

And each of them are open.

EXAMPLE. Let $(\mathbb{R}, \mathcal{T})$ be a topological space with topology $\mathcal{T}$ consists of all open sets under the metric space $(\mathbb{R}, \mathrm{d})$. Consider $Y = [0,1] \cup (2,3)$ and the subspace topology.
We claim hat $[0,1]$ is an open set of Y, because $[0,1] = (-1, \frac{3}{2}) \cap Y$. Similarly, $(2,3)$ is also open in Y. And the complement of each of them is another interval, therefore $[0,1]$ and $(2,3)$ are both open and closed.

REMARK. By these three examples, we could see that a subset of X can be open, or closed, or both, or neither. In addtition, we see that a subset is open(closed) or not depends on the whole space you consider: $[0,1]$ in EXAMPLE3 is not open in $\mathbb{R}$ but open in Y.$(2,3)$ is not closed in $\mathbb{R}$ but closed in Y.

One might think that a single point is a closed set. This wrong intuition comes from the special topological space $\mathbb{R}$. But the following example shows exactly that a single point might not be a closed set:

$$X = \{0,1\}, \mathcal{T} = \{\{0\}, \{0,1\}, \varnothing\}$$

Then $\{0\}$ is not a closed set as $\{1\}$ is not open. However, $\{0\}$ is dense in X as its closure equals to X, which we will talk about later.

Theorem 1. *Let X be a topology space. Then the following conditions hold:*
(1) *$\varnothing$ and X are closed*
(2) *For any collection of closed set $\{V_\alpha \mid \alpha \in I\}$, we have $\bigcap\limits_{\alpha \in I} V_\alpha$ is closed*
(3) *The intersection of any finite many closed sets are closed.*

Proof. (1) is trivial with $\varnothing = X - X$ and $X = X - \varnothing$.
As for (2), notice that :

$$\bigcap_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} U_\alpha^c = (\bigcup_{\alpha \in I} U_\alpha)^c$$

where U_α is an open set. And we denote $X - U_\alpha$ with U_α^c. (3) follows the same way with the fact that:

$$\bigcup_{i=1}^{n} V_\alpha = \bigcup_{i=1}^{n} U_\alpha^c = (\bigcap_{i=1}^{n} U_\alpha)^c$$

Theorem 2. *Let Y be a subsapce of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Consider the subspace topology of Y and let V_Y is a closed set under such subspace topology. Then we have $V_Y = Y - U_Y$ for some open set U_Y in Y. With the definition of subspace topology, we have $U_Y = U \cap Y$ with U an open set in X. Then $V_Y = Y - U_Y = Y - U \cap Y = Y - U = Y \cap (X - U)$ where $(X - U)$ is closed in X. Therefore if V_Y is closed in Y, then V_Y is intersection of Y and a closed set in X.
On the other hand, if $V_Y = Y \cap V$ for some closed set V of X. We have $V_Y = Y \cap (X - U) = Y - U = Y - (Y \cap U)$, which is closed in Y

REMARK. General speaking, a set that is closed in a subspace may not be closed in the larger topological space. For example, let $X = \mathbb{R}$ and open set consists of conventional open set in $\mathbb{R}$. Consider the subspace Y generated by the intersection of $[0,1)$ and X. Then

notice that $[0, \frac{1}{2})$ is open in Y as $[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap [0, 1)$. Therefore $Y - [0, \frac{1}{2}) = [\frac{1}{2}, 1)$ is closed in Y, however, it's not closed in $\mathbb{R}$.
But we have the following theorem explained the so called "transitivity" of closed property:

Theorem 3. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. By theorem 2, $A = Y \cap V$ with some V closed in X. Therefore, A is closed in X by the fact that the intersection of two closed sets is closed.

2.2 Limit Point and Closure

Definition. Let X be a topological space and A a subset of X. An element x of X is said to be ***limitpoint*** of A if: for every open set U that contains x, $U \cap A \neq \varnothing$ or $\{x\}$.

Definition. Let A be a subset of the topologival space X; let $A^{'}$ be the set of all limit points of A, we define the closure of A as the union of A and $A^{'}$, denoted by $\bar{A}$. Which is:

$$\bar{A} = A \cup A^{'}$$

Theorem 4. *Let X be a topological space. Then A is closed in X if and only if:$\bar{A} = A$*

Proof. $\Leftarrow$: If $\bar{A} = A$, we need to show that A is closed, or to show that $X - A$ is open. For any element $x \in X - A$, x is neither an element of A nor the limit point of A. $x \notin A^{'}$ means there is some open set U that contains x but $U \cap A = \varnothing$ or $\{x\}$. Now that $x \notin A$, we have $U \cap A = \varnothing$. For any $x \in X - A$, we have such open set U_x. And thus:

$$X - A = \bigcup_{x \in (X-A)} U_x$$

is union of open set in X, therefore an open set. Hence we have A is closed.

$\Rightarrow$: If A is closed. To prove $A = \bar{A}$, we only need to show that $A^{'} \subset A$, which is: any limit point of A is in A. Suppose x is a limit point of A but $x \in X - A$. Then notice that $X - A$ is an open set that contains x but $(X - A) \cap A = \varnothing$, which contradicts the definition of limit point. Therefore any limit point of A is in A, and hence $A = \bar{A}$.

Theorem 5. *Let X be a topological space and A a subset of X, then $\bar{A}$ is the smallest closed set that contains A.*

Proof. The proof are divided into two parts:

(i) $\bar{A}$ is closed.

(ii) Every closed set that contains A must contain $\bar{A}$.

For (ii), we only need to show that every closed set that contains A must contain the limit point of A. This is easy to show: Let B a closed set that contains A and x a limit point of A, then x must be a limit point of B as $A \subset B$. By theorem 4 and the fact that B is closed, we have: $x \in \bar{B} = B$. Therefore, $\bar{A} \subset B$
For (i), we only need to show that $\bar{A} = \bar{\bar{A}}$ by theorem 4. which is concluded as the following lemma.

Lemma 6. Let X be a topological space and A a subset of X, then $\bar{\bar{A}} = \bar{A}$.

Proof. $\bar{A} \subset \bar{\bar{A}}$ according to the definition of closure. As for the other side ,we need to show that the limit point of $\bar{A}$ is in $\bar{A}$.
If x is a limit point of $\bar{A}$. If $x \in A$, we're done. Otherwise let U be any open set that contains x, we have:

$$U \cap \bar{A} \neq \varnothing, \{x\}$$

We claim that x is a limit point of A, by claiming that $U \cap A \neq \varnothing$(of course it can't be $\{x\}$ as $x \notin A$).

(i) If $U \cap A \neq \varnothing$, we're done.

(ii) Otherwise $U \cap A = \varnothing$ but $U \cap A' \neq \varnothing$. $U \cap A' \neq \varnothing$ shows that there is some point, say y, is a limit point of A, and $y \notin A$. Therefore $U \cap A \neq \varnothing$ as U is an open set containing y, this contradicts that assumption that $U \cap A = \varnothing$

In conclusion, $U \cap A \neq \varnothing$ and thus x is a limit point of A by definition.

Both sides contains the other side, therefore we have:$\bar{\bar{A}} = \bar{A}$.

By using the result of lemma 6, we may draw the conclusion of theorem 5 as explained in the proof.

REMARK. The name "closure" means that $\bar{A}$ remains constant under the map by mapping a set of topological space into the union of A and A'. Or, as explained int theorem 5, closure is the smallest closed set that contains A. Further more, we can easily prove the closure of A has an equvalent definition:

$$\bar{A} = \bigcap_{A \subset V, V\ closed} V$$

So far,we have actually given two ways of explaining what is a closed set is. One by clarifying the relationship between open set and closed set; and the other by using the definition of limit point. Theorem 4, 5 and lemma 6 has showed the equivalence of these two expression, and we conclude it as:

Theorem 7. *Let X be a topological space, a subset A of X is closed iff every limit point of A is in A.*

Proof. Omitted, see theorem 4.

The following theorem describes the closure of a subset in subspace.

Theorem 8. *Let X be a topological space and Y a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X, Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. By theorem 5 and its remark, we know that the closure of A in Y, denoted by $\bar{A_Y}$, equals to the insersection of all closed set in Y that contains A. Note that any closed set in Y equals to the intersection of Y and a closed set in X. Therefore we have:

$$\begin{aligned}
\bar{A_Y} &= \bigcap_{A \subset V_Y, V_Y\ closed\ in\ Y} V_Y \\
&= \bigcap_{A \subset V \cap Y, V\ closed\ in\ X} (V \cap Y) \\
&= (\bigcap_{A \subset V, V\ closed\ in\ X} V) \bigcap Y \\
&= \bar{A_X} \bigcap Y
\end{aligned}$$

REMARK. The equivalence between the second line and the third line is easy to prove with the following claim:

$$A \subset V, V \textit{ closed in } X \Leftrightarrow A \subset V \cap Y, V \cap Y \textit{ closed in } Y$$

A question is that whether the following proposition is true:

Proposition. Let X be a topological space and Y is a subspace of X; A is a subset of X(not Y), then:

$$\overline{A \cap Y} = \bar{A}_X \cap Y$$

Unfortunately, this proposition is false. But we have the left side subsets the right side.

We give a theorem about the closure of the union of two subsets and the closure of its intersection:

Theorem 9. *Let X be a topological space and A, B subsets of X. Then we have:*

$$\overline{A \cup B} = \bar{A} \cup \bar{B}, \overline{A \cap B} \subset \bar{A} \cap \bar{B}$$

Proof. We first prove that $\overline{A \cup B} = \bar{A} \cup \bar{B}$. For any $x \in \overline{A \cup B}$, if $x \in A \cup B$ then x belongs to the right side. Otherwise if x is the limit point of $A \cup B$ and $x \notin A \cup B$. We will show that $x \in A^{'}$ or $x \in B^{'}$. If $x \notin A^{'}$ and $x \notin B^{'}$, then there are some open sets W, V such that $x \in W, x \in V$ but $W \cap A = \varnothing, V \cap B = \varnothing$. Therefore: $(W \cap V) \cap (A \cup B) = \varnothing$, with $W \cap V$ an open set contains x, which contradicts the assumption that x is a limit point of $A \cup B$. Therefore, $x \in A^{'}$ or $x \in B^{'}$ and $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

For the other inclusion:$\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. For $x \in \bar{A} \cup \bar{B}$, assume $x \in \bar{A}$, then $x \in A$ or $x \in A^{'}$. If $x \in A$ then $x \in A \cup B \subset \overline{A \cup B}$; If $x \in A^{'}$ then x is obviously a limit point of $A \cup B$ as $A \subset A \cup B$, which means $x \in (A \cup B)^{'} \subset \overline{A \cup B}$ and thus $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Therefore we have the first part of this theorem proved.The second part follows the same way.

REMARK. $\overline{A \cap B} \subsetneqq \bar{A} \cap \bar{B}$ might be true. Consider $X = \mathbb{R}$ and its conventional topology. Let $A = [0, 1), B = (1, 2]$, then we have:

$$\overline{A \cap B} = \varnothing \subsetneqq \bar{A} \cap \bar{B} = \{1\}$$

2.3 Dense Set

Definition. Let X be a topological space and A a subset of X. A is said to be a ***dense*** set of X if $\bar{A} = X$.

In other words, a set A is called dense, if any point x in X belongs to A or x is a limit point of A. There are many examples of dense set, the most common one, which is frequently mentioned in Mathematical Analysis, is that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition. Let X be a topological space, the ***interior*** of A is defined as the union of all open sets that is contained in A, denoted as **IntA**. The element of Int$A$ is called ***interior point***

It's easy to see the following relation between Int$A$, $\bar{A}$ and A:

$$\text{Int}A \subset A \subset \bar{A}$$

And further more, A is open iff Int$A = A$, A is closed iff $\bar{A} = A$.

Theorem 10. *X is a topological space and A a subset of X. The following conditions are equalvalent:*

(i) x is an interior point of A.

(ii) There is an open set U, such that: $x \in U \subset A$

Proof. (i)$\Rightarrow$ (ii):

$$x \in \text{Int}A \Rightarrow x \in \bigcup_{U \subset A, U \text{ open}} U$$

Therefore, there is an open set in the right side that contains x and it's done.
(ii)$\Rightarrow$ (i):

$$x \in U \subset \bigcup_{U \subset A, U \text{ open}} U = \text{Int}A$$

Topology Basis And Continuous Functions

November 21st, 2020

1 Basis For Topology

Definition. **(Topology Basis)** Let $(X, \mathcal{T})$ be a topological space. A **basis** for topology $\mathcal{T}$ is a collection of subsets $\mathcal{B}$ such that for any open set $U \in \mathcal{T}$,

$$U = \bigcup_{\alpha \in I} B_\alpha, B_\alpha \in \mathcal{B}$$

In other words, any open set could be denoted as the union of a collection of subsets in $\mathcal{B}$

EXAMPLE. Let $X = \mathbb{R}$ and $\mathcal{T}$ be the conventional topology on $\mathbb{R}$. Let $\mathcal{B}$ consists of all open interval with rational endpoint, which is:

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q}, a < b\}$$

Then for any $U \underset{open}{\subset} \mathbb{R}$, and any $x \in U$, there is an interval I with: $x \in I_x \subset U$ and therefore: $U = \bigcup\limits_{x \in U} I_x$.

Note that we say $\mathcal{B}$ is a basis for some specific topology if $\mathcal{B}$ satisfies the above definition. Under such condition, we already pointed out what topology this basis is corresponding to. The next question is: what kind of collection of subsets could be a basis for some topology on X?(If we didn't specify the topology yet)

Suppose there is a collection of subsets of X, say $\mathcal{B}$. If B is a basis for some topology $\mathcal{T}$, then every open set in $\mathcal{T}$ could be denoted as union of elements in $\mathcal{B}$. Note that $X \in \mathcal{T}$ therefore $X = \bigcup\limits_{\alpha \in I} B_\alpha$.

Moreover, consider $U, V \in \mathcal{T}$, and denote them as union of basis elements of $\mathcal{B}$:

$$U = \bigcup_{\alpha \in I} U_\alpha, U_\alpha \in \mathcal{B}$$
$$V = \bigcup_{\beta \in J} V_\beta, V_\beta \in \mathcal{B}$$

Then $U \cap V$ is supposed to be denoted as union of basis elements in $\mathcal{B}$. However, notice that:

$$U \cap V = \bigcup_{\alpha \in I} \bigcup_{\beta \in J} (U_\alpha \cap V_\beta)$$

Therefore, we only need that $U_\alpha \cap V_\beta$ could be denoted as union of basis elements in $\mathcal{B}$.So far, we have got a sufficient condition that makes $\mathcal{B}$ be a basis of some topology on X:

Theorem 1. *Let X be a set and $\mathcal{B}$ be a collection of subsets in X. If $\mathcal{B}$ satifies the following two requirements, then $\mathcal{B}$ is a basis for some topology:*
(i) For any $x \in X$, there is some $B \in \mathcal{B}$ such that $x \in B \subset X$
(ii) For any $B_1, B_2 \in mathcalB$ with $U \cap V \neq \varnothing$, and any $x \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

Proof. Define $\mathcal{T} = \{\bigcup_{\alpha \in I} B_\alpha \mid B_\alpha \in \mathcal{B}\}$, then it's easy to verify that $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis of $\mathcal{T}$.

Definition. If $\mathcal{B}$ satisfies these two conditions in theorem 1, then we define **the topology generated by** $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

REMARK. The topology generated by $\mathcal{B}$ is apparently equivalent to the topology mentioned in the proof of theorem 1. And now we know the topology generated by $\mathcal{B}$ is actually those sets which can be denoted as union of elements in $\mathcal{B}$.

The word "basis" might be confusing. We know the open set of $\mathcal{B}$-generated topology equals to the union of some subsets in $\mathcal{B}$ but this expression is not unique. However, in other subjects, for example, linear algebra , a basis means element could be uniquely expressed as linear combination of basis elements.

However, there is one thing follows the same for the basis concepts in linear albegra and topology, that is there might be multiple basis for the same topology,or linear space.But the following theorem, gives a sufficient condition for equivalent topology.

Theorem 2. *We say two basis are **equalvalent** if they generates the same topology. If two basis $\mathcal{B}_1$, $\mathcal{B}_2$ satisfies the following two conditions, then they are equivalent.*
(i) For any $B_1 \in \mathcal{B}_1$, any $x \in B_1$, there is some $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$
(ii) For any $B_2 \in \mathcal{B}_2$, any $x \in B_2$, there is some $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subset B_2$

Proof. Trivial. The condition specifies that every element in $\mathcal{B}_1$ could be expressed as union of elements in $\mathcal{B}_2$ and vice versa.

Definition. (**subbasis**) A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals to X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersection of elements of $\mathcal{S}$

2 Continuous Function

2.1 Continuous Function

Definition. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open set V of Y, the set $f^{-1}(V)$ is an open set of X

It is easy to see that the continuity of a function f depends not only on the function itself, but also the specified topology of its domain and range. To emphasize this fact, we may say f is continuous ***relative*** to specific topologies on X and Y.

EXAMPLE. In analysis, a real-value function of real variable is said to be continuous if it's continuous at every point of its domain. And a function $f : \mathbb{R} \to \mathbb{R}$ is continous at x_0 is define as follows:

$\forall \epsilon > 0$, there is some δ such that $\mid f(x) - f(x_0) \mid$ if $\mid x - x_0 \mid < \delta$.
Then a continous real variable function is continous from $\mathbb{R}$ to $\mathbb{R}$.

We have seen many other theorems about continuity in mathmatical analysis, for exmaple, a continuous function would map a limit point to a limit point, which is $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$. Some of these theorems are generalized for more general space, the following theorem describes this.

Theorem 3. *Let X and Y be topological spaces; let $f : X \to Y$. Then the following conditions are equivalent:*

(1) f is continuous

(2) Let $\mathcal{B}$ be a basis for Y, then $f^{-1}(B_\beta)$ is open in X

(3) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$

(4) For every subset B of Y, one has $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$

(5) For every closed set B of Y, $f^{-1}(B)$ is closed in X.

Proof. (**1**) $\Rightarrow$ (**2**): Every $B_\beta \in \mathcal{B}$ is an open set in Y, therefore $f^{-1}(B_\beta)$ is open in X.

(**1**) $\Rightarrow$ (**3**): Let x is a limit point of A, it's sufficient to show that $f(x)$ is a limit point of $f(A)$ if $f(x) \notin f(A)$ (There are other situations where $x \in A$ or $f(x) \in f(A)$, but it's simple). We prove as proceed: For any open set U of Y that contains $f(x)$, consider $f^{-1}(U)$, note that $x \in f^{-1}(U)$ as $f(x) \in U$. And by definition of continuous function we have $f^{-1}(U)$ is an open set of X that contains x. Therefore $f^{-1}(U) \cap A \neq \varnothing$ and $U \cap f(A) \neq \varnothing$. By definition, $f(x)$ is a limit point of $f(A)$.

(**2**) $\Rightarrow$ (**1**) We will show that f is continous if it satisfies condition in (2). For any open set U in Y, we can denote U as union of the basis elements, such that:$U = \bigcup_{\beta\in I} B_\beta$, and by theorem about inverse image of union, we have:

$$f^{-1}(\bigcup_{\beta\in I} B_\beta) = \bigcup_{\beta\in I} f^{-1}(B_\beta)$$

, which is open in X. The proof is done.

(**3**) $\Rightarrow$ (**4**) By the definition of inverse image, it's sufficient to show that $f(\overline{f^{-1}(B)}) \subset \bar{B}$. By (3), we have:

$$f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \bar{B}$$

The last subsets holds as $f(f^{-1}(B) \subset B$.

(**4**) $\Rightarrow$ (**5**) If F is closed in Y, then $\bar{F} = F$. Then we have: $\overline{f^{-1}(F)} \subset f^{-1}(\bar{F}) = f^{-1}(F)$ by (3). It's obvious that $f^{-1}(F) \subset \overline{f^{-1}(F)}$ and hence $f^{-1}(F) = \overline{f^{-1}(F)}$, which shows $f^{-1}(F)$ is closed in X

(**5**) $\Rightarrow$ (**1**) For any open set $U \in Y$, consider $f^{-1}(U)$. Let $V = Y \setminus U$, then V is closed and $f^{-1}(U) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Notice that $f^{-1}(V)$ is closed, therefore $f^{-1}(U)$ is open and f is a continous function.

2.2 Limit and Hausdorff Space

We mention before the proof of theorem 3, that in $\mathbb{R}-\mathbb{R}$ function we have $\lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n)$ if f is continous. We may generalize this theorem if we define the **limit** of a sequence of point in topological space.

Definition. Let X be a topological space, $\{x_n\} \subset X$ is a sequence of point in X, we say x is the **limit** of $\{x_n\}$, denoted by $\lim\limits_{n\to\infty} x_n = x$(this denotion might be invalid), if :

$$U \underset{open}{\subset} X, x \in X \Rightarrow \exists N \in \mathbb{N}, st. \forall n > N, x_n \in U$$

The open set U of X that contains x is called a **neighborhood** of x

REMARK. The limit of $\{x_n\}$ might not be unique in a general topological space.

Consider $X = \{0, 1\}, \mathcal{T} = \{\varnothing, X, \{0\}\}$. And consider a point sequence :$\{x_n = 0\}, \forall n \in \mathbb{N}$. Then both 0 and 1 ar e limit of this sequence by definition.

As explained above, $\lim\limits_{n\to\infty} x_n = x$ is an invalid denotion if there are two limits. However, we may put a constraints on the space to make there exists only one limit(if there is any) for a sequence.

Definition. **(Hausdorff space)** Let X be a topological space. X is said to be a **Hausdorff space** if for any $x_1, x_2 \in X$ and $x_1 \neq x_2$, there exists two open sets $U_1, U_2 \subset X$ such that $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \varnothing$

Theorem 4. *Let X be a Hausdorff space, then any sequence $\{x_n\}$ has unique limit (if there is any)*

Proof. If there are two limits of $\{x_n\}$, say a and b. Then by definition of Hausdorff space, there are two open sets U_a, U_b contains a and b respectively but $U_a \cap U_b = \varnothing$. According to the definition of limit, there exists N_1, N_2 such that $x_n \in U_a, \forall n > N_1$; $x_n \in U_b, \forall n > N_2$. Let $N = \max(N_1, N_2)$, then $x_n \in U_a \cap U_b, n > N$, a contradiction.

There are multiple other interesting facts about Hausdorff space. Recall that we mentioned a fact in last section: a single point might not be closed in a general topological space. A typical example is as follows:

$$X = \{a, b, c\}, \mathcal{T} = \{\{a, b\}, \{b\}, \{b, c\}, \varnothing, X\}$$

In this topological space, $\{b\}$ is not closed. And further more, the sequence $\{x_n\}$ where $x_n = b, \forall n \in \mathbb{N}$ has two limits a and c.

We have proved in Hausdorff space, sequence has only one limit. Further more, a single point is closed in Hausdorff space.

Theorem 5. *Let X be a Hausdorff space, then every finite set of X is closed.*

Proof. It suffices to show that any single point is closed. Let $\{x_0\}$ be a one-point set. For any $x \in X, x \neq x_0$, there is some open set U_x contains x but does not intersect x_0, therefore x is not a limit point of x_0. We have this fact for arbitraty $x \neq x_0$. So the closure of $\{x_0\}$ is $\{x_0\}$ itself. Therefore $\{x_0\}$ is closed .

The condition that finite point sets be closed is called $\mathbf{T_1}$ **axiom**. And T_1 axiom is **weaker** than Hausdorff condition, which means: a topological space that satisfies T_1 axiom may not be a Hausdorff space. We will explain the reason in later chapters.

Theorem 6. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. $(\Leftarrow)$: If any neighborhood of x contains infinitely many points of A, the neighborhood must contain some points other than x iteself, so that x is a limit point of A.
$(\Rightarrow)$: Conversely, suppose that x is a limit point of A, and suppose there is some open set U containing x but intersects only finitely many points with A. Consider $A \setminus \{x\} \cap U$, denoted by V, then V also has finitely many points.

By T_1 axiom, V is closed in X and $X \setminus V$ is open. Notice that $U \cap (X \setminus V)$ is an open set that contains x. However, this open set intersects an empty set with A, which contradicts the assumption that x is a limit point of A.

Now let's back to the topic of continous function. The following theorem is a generalized property of its $\mathbb{R}$ version

Theorem 7. *Let X and Y be Hausdorff spaces. And f a continous function from X to Y. Let $\{x_n\}$ be a sequence in X and converged to x, then sequence$\{f(x_n)\}$ has a limit, and $\lim\limits_{n\to\infty} f(x_n) = f(\lim\limits_{n\to\infty} x_n)$*

Proof. Let $\lim\limits_{n\to\infty} x_n = x$, then we only need to show that any open set of Y that contains $f(x)$ must contain all elements after of $\{f(x_n)\}$ after some N.

Let $V \underset{open}{\subset} Y$ and $f(x) \in V$, then $x \in f^{-1}(V)$ and $f^{-1}(V) \underset{open}{\subset} X$. Therefore, according to the definition of limit, $\exists N \in \mathbb{N}, \forall n > N, x_n \in f^{-1}(V)$. Therefore, $\forall n > N, f(x_n) \in V$ and the proof is done.

2.3 Homeomorphisms

Definition. (**Homeomorphism**) Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both f and f^{-1} are continuous, then f is called **homeomorphism**

The condition that $f^{-1} : Y \to X$ is continuous says that for each open set of X, $(f^{-1})^{-1}(U)$ is open in Y. Notice that $(f^{-1})^{-1}(U) = f(U)$. So the condition actually says that $f(U)$ is open if and only if U is open.

The homeomorphism says that not only every element of X corresponds to some element of Y, there is also a corresponsence between the collection of open sets of X and Y. This is quite similar to the concept of ***Isomorphism*** which preserves the algebraic structure. ***Homeomorphism*** is a bijection that preserves the ***topological structure***

Let $f : X \to Y$ be an injective continuous function, where X and Y are topological spaces. Let $f(X)$ is subspace topology. Then $f' : X \to f(X)$ is a bijection . If f' happens to be a homeomorphism, then f is called an ***imbedding***

Munkres's book provides a very classical example where a function is bijective and continuous without being homeomorphism, which shows that the continuity of f^{-1} is neccessary.

Compactness of Topological Space

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1 Compact Spaces

Compact Spaces is a kind of special topological spaces. In such a topological space, a local property may be true in the whole space. In mathmatical analysis, we have already seen some compact spaces, for example, the closed interval. A basis but important theorem in analysis says that a continuous function must be bounded in a closed interval. The key point of the proof to this theorem is the concept of *compactness*.

Definition. Let X be topological space. A collection $\mathcal{A}$ of subsets of X is said to be a ***covering*** of X, if their union equals to X.If elements of this collection are all open sets, then $\mathcal{A}$ is said to be an ***open covering*** of X.

Definition. A topological space X is said to be ***compact***, if every open covering of X has finite subcollection that covers X.

Finite subcollection means we can pick up finite many open set to form a new collection. Here are some examples about compact topological spaces.

EXAMPLE. The following subspace of $\mathbb{R}$ is compact:

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{N}^+\}$$

Let $\mathcal{A}$ be an open covering of X, we will pick up finite of them to cover X. Since $0 \in X = \bigcup_{U \in \mathcal{A}} U$, there must be some open set that contains 0, we pick up this open set. Notice that 0 is a limit point of $X \setminus \{0\}$, there are only finite many points not included in this open set. So we can pick finite many open sets to cover them.

EXAMPLE. Consider $\mathbb{R}$ and general topology on $\mathbb{R}$, then $(0, 1]$ is not compact. The reason is that we have an open cover:

$$\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{N}^+\}$$

but $\mathcal{A}$ has no finite sub-cover.

Definition. Let X be a topological space and Y a subset of X. Y is said to be a ***compact set*** (of X), if any open covering of Y has finite subcover.

REMARK. We say a collection $\mathcal{A}$ of X is a cover of Y, if:

$$Y \subset \bigcup_{U \in \mathcal{A}} U$$

and $\mathcal{A}$ is said to be an open cover iff every elements of $\mathcal{A}$ is an open set.

Different from the definition of *compact space*, a *compact set* specifies the compactness of a subset,but there are no substantial difference between this two definitions. We will demonstrate you a theorem proof(very easy, just follow the definition):

Theorem 1. *Let X be a topological space. Y is a compact set if and only if Y is compact space under subspace topology.*

Now we may not distinguish *compact set* and *compact space* deliberately.

Theorem 2. *Every closed set of a compact space is a compact set, thus a compact space.*

Proof. Let X be a compact space, and Y a closed set of X. We shall see that every open cover of Y has a finite sub-cover.

Let $\mathcal{A} = \{U_\alpha \mid \alpha \in I\}$ is an open cover of Y, st. $Y \subset \bigcup_{\alpha \in I} U_\alpha$. Then $\bigcup_{\alpha \in I} U_\alpha \cup Y^c$ is an open covering of X as Y is closed in $X(Y^c = X \setminus Y)$. Thus there are finite sub-cover of X for X is compact. Let: $\bigcup_{i=1}^{n} U_i = X$, which is also a finite sub-cover of Y. If Y^c is one of these open sets, kick it out, and we get a finite sub-cover from $\mathcal{A}$ for Y, which concludes that Y is compact set in X.

In mathmatical analysis, we have proved so-called "finite-covering theorem" for closed interval, therefore every closed interval of $\mathbb{R}$ is compact. One may naively think that compact set must be closed set. This is not true: Consider $X = \{0, 1, 2\}, \mathcal{T} = \{\varnothing, X, 1, 2\}$. Then $\{1\}$ is compact as there are totally finite open set, however, $\{1\}$ is not closed. But we will see this assertion is true in some more particular space.

Theorem 3. *Every compact set of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and Y a compact set of X. We will show that Y is closed. We will prove that every elements not in Y is also not a limit point of Y.

Fix a point of Y^c, say x. For any $y \in Y$, there exists two open set U_y, V_y, such that:$x \in U_y, y \in V_y$ but $U_y \cap V_y = \varnothing$ for X is Hausdorff. $\{V_y \mid y \in Y\}$ is obviously an open covering of Y. Then there are finite sub-cover,say $\{V_{y_1}, V_{y_2}, ..., V_{y_n}\}$. Consider $U = \bigcup_{i=1} U_{y_i}$. Then U is an open set that contains x, and it's easy to prove that $U \cap V_{y_i} = \varnothing, i = 1, 2, ..., n$, thus $U \cap Y = \varnothing$. This indicates that x is not a limit point of Y. Hence, $\bar{Y} = Y$ and Y is closed. The proof is done.

The following theorem is a direct corollary of ***Theorem 2***(not ***Theorem 3***). It discusses what a compact set is in a more special space.

Theorem 4. ***(Heine–Borel theorem)*** *A subset of $\mathbb{R}^n$ is compact if and only if it's closed and bounded*

Proof. We will just give a sketch of proof for this theorem.
($\Rightarrow$): If Y is compact, it must be closed as $\mathbb{R}^n$ is Hausdorff. To see Y is bounded, we consider the distance between any point of Y and the original point $\mathbf{0}$. For any $y \in Y$, there is an open ball $B(y, r_y)$ that contains y but doesn't contain $\mathbf{0}$. Then $Y \subset \bigcup_{y \in Y} B_y$. We can pick finite of these open ball to cover Y, say $B(y_1, r_{y_1}), ..., B(y_n, r_{y_n})$. Then every point of Y must be one of these open balls. Assume $y \in B(y_1, r_{y_1})$ then $d(y, \mathbf{0}) \leq d(y, y_1) + d(y_1, \mathbf{0})$. Note that there are only finite many open ball, then we can let $M = \max r_{y_i}, N = \max d(y_i, \mathbf{0})$. Thus $d(y, \mathbf{0}) \leq M + N$ and the proof is done.
($\Leftarrow$) If Y is bounded, then $Y \subset [a_1, b_1] \times \cdots [a_n, b_n]$ for some a_i, b_i. We denote this cubic with U. Note that Y is also closed in U because $U \cap Y$ is closed in U. We assert that U is compact, and by *theorem 3*, Y is compact.
(There are many methods to proving U is compact. A direct way is use the same technique of proving compactness of closed interval.)

Theorem 5. *The image of a compact space under a continuous function is still compact.*

Proof. To simplify this question, we will prove that a continous function maps a compact set to a compact set.

Let X, Y be topological space and $f : X \to Y$ a continuous function. Let $A \subset X$ a compact set, we will prove that $f(A)$ is a compact set of Y. Consider an open covering of $f(A)$, say $\{U_\alpha \mid \alpha \in I\}$. And consider $\{f^{-1}(U_\alpha) \mid \alpha \in I\}$ in X. Since f is continuous, each $f^{-1}(U_\alpha)$ is open in X. And obviously $A \subset \bigcup\limits_{\alpha \in I} f^{-1}(U_\alpha)$. A has a finite sub-cover, say: $A \subset \bigcup\limits_{i=1}^{n} f^{-1}(U_i)$. Then $f(A) \subset \bigcup\limits_{i=1}^{n} U_i$ and the proof is done.

Theorem 6. *Let $f : X \to Y$ be a bijection continous function. If X is compact and Y is Hausdorff, then f is homeomorphism.*

Proof. For any open set $U \subset X$, we have U^c is closed in X, therefore, compact in X(*theorem 2*). Thus, $f(U^c)$ is compact in Y(*theorem 5*), and therefore closed (*theorem 3*) Note that $f(U^c) = f(U)^c$ as f is bijective, we have $f(U)$ is open. Therefore, $f(U)$ is open if and only if U is open, the proof is done.