
COMPLEX MANIFOLDS



NOTES PRODUCED BY

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COMMENTS & ACKNOWLEDGEMENTS

The primary goal of these notes is to aid in/test my understanding of the work I am doing for my PhD at Durham university. For this reason they are highly tailored to my own thought process, which will obviously have an effect on the level of detail both present and missing on the content that follows. That said, I do like to write my notes with a "imagine teaching this to someone" mentality, as it forces me to really think about the material rather than blindly copy it. For this reason (and because I'm producing the notes anyway) I am making these notes available on my website. For updated versions of these notes + all other stuff I've done, click the following link:

www.richiedadhley.com

Below I list the main sources used in producing these notes, and a more complete list of references can be found at the end of the document.

- *Complex Geometry: An Introduction*, D. Huybrechts:
This is a brilliant reference book. Basically any result you could want can be found in this book somewhere.
- *Lectures On Complex Geometry, Calabi–Yau Manifolds & Toric Geometry*, V. Bouchard:
These are a nice introductory set of notes to complex geometry and Calabi-Yau manifolds. They give lots of examples to help understand the somewhat condensed material.
- *Calabi-Yau Manifolds for Dummies*, Dennis:¹
Again these are a nice set of introductory notes, which discuss a lot of the same stuff as the Bouchard notes, although the latter is probably more extensive. A nice element to these notes is they continually work around the example of S^2 , showing how it is a complex manifold, and indeed a Kähler manifold, but how it fails to be Calabi-Yau.

Of course I would also like to extend a huge thank you to my supervisor Dr. Andreas Braun for suggesting and discussing the work with me.

Richie Dadhley

¹No second name given...

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0 | Introduction

Geometry has found immense use in the study of mathematical physics, and often provides a much more intuitive explanation to difficult physical problems. Perhaps the most obvious/prominent example of this is general relativity. This is built on the mathematical construction of *real* manifolds and their associated structures. However, of course the natural extension of such tools would be to consider the complex counterpart, *complex* manifolds. These notes aim to do just that, giving a somewhat smooth transition from a real manifold to a complex manifold, in a hopefully pedagogical manner.

The content is laid out as follows.

- Chapter 1 reviews real manifolds, and then introduces some perhaps new notions, such as (co)homology and Hodge theory.
- Chapter 2 then takes a "middle ground" approach to our problem, showing how we can make *almost* complex manifolds. These are real manifold on which we define complex tensor fields. We generalise the structures on our real manifolds to these cases and introduce some new ones, the most important of which are the *Chern classes*.
- Chapter 3 introduces so called *projective spaces*, and we study their properties (that are relevant to us). In particular we look at how to construct hypersurfaces in these spaces and derive their Chern classes.
- Chapter 4 finally introduces complex manifolds. We soon specialise to a particular class of complex manifolds known as *Kähler* manifolds. We then further constrain our attention to a subclass of these known as *Calabi-Yau* manifolds. It is these spaces we are fundamentally interested in.¹
- Chapter 5 then shows how to construct Calabi-Yau manifolds as submanifolds of our projective spaces. We give a general procedure of how to do this, and then consider specific examples to help ground the material.
- Chapter 6 briefly discusses orbifold singularities and shows that they arise in so-called *weighted* projective spaces.
- Chapter 7 then quickly summaries the work done.

¹We don't discuss this too much here, but the main reason for this is the fact that Calabi-Yau manifolds possess a lot of nice properties when it comes to compactifying string theories. In particular we can construct Calabi-Yau manifolds with real dimension $10 - 4 = 6$, which also manage to remove a significant proportion of the SUSY string theory needs, but that physical spacetime doesn't seem to possess.

1 | Real Manifolds

We start with a discussion of *real* manifolds. This serves two main purposes: 1) it allows us to set notation in a familiar setting; 2) a lot of the ideas that we will discuss for complex manifolds have some real counterpart. With the second purpose in mind, we point out now (and will try to mention as we go) the word "complex" in what follows does *not* always mean we are considering a complex manifold. For example it is possible to define complex-valued tensor fields even on a real manifold (although this is not often done in practice¹). I therefore think it is important that we keep track of what is specific to complex manifolds themselves and what holds in a more general idea.

1.1 Quick Review

A lot of the material in here is assumed to already be understood, at least to some degree. It can therefore be read in a more "skim read" fashion, but is included for pedagogical reasons. However some of the more detailed definitions are omitted² for brevity reasons. However any readers should expect a pretty steep climb in content as we progress through this chapter.

1.1.1 Topological Manifolds

First let's have a quick review on what a real manifold is and how we define it. A lot more details on these constructions can be found via any differential geometry textbook, or by the amazing online lecture course "Geometrical Anatomy of Theoretic Physics" by Dr. Frederic Schuller.³

Definition. [Topological Manifold] A *real topological manifold of dimension d* is a double⁴ $(\mathcal{M}, \mathcal{O})$, where \mathcal{M} is a set and \mathcal{O} is a (paracompact, Hausdorff) *topology*, where for every $p \in \mathcal{M}$ there is a neighbourhood U_p such that $x : U_p \rightarrow \mathbb{R}^d$ is a homeomorphism. We call the pair (U, x) a *coordinate chart* and the set of all coordinate charts defines an *atlas*. If we have two charts (U, x) and (V, y) where $U \cap V \neq \emptyset$, then we can consider the (*chart*) *transition maps* $y \circ x^{-1} : \mathbb{R}^d \supseteq x(U_p) \rightarrow y(V_p) \subseteq \mathbb{R}^d$, where obviously $p \in U \cap V$. We only require that these transition maps be continuous.

¹Or at least I don't suppose it is.

²e.g. the definition of a closed manifold is rammed into a footnote later in a very quick fashion

³See end of notes for some links.

⁴Well this whole construction is the topological manifold, but we get the idea.

The main point we want to stress here is that the definition of a topological manifold only requires that the transition maps be continuous. Of course we can impose further restrictions on these maps, therefore giving the manifold more *structure*. This will prove vital when it comes to defining a complex manifold later. Throughout these notes, unless specified otherwise, we shall actually assume that our transition maps are *smooth*, that is infinitely differentiable with continuous result, we call such a manifold a *smooth* (or differential) manifold.

We will also make extensive use of the tangent spaces. There are lots of ways one can define a tangent space, but here we just give a quick definition as we assume any readers are familiar with more complete definitions.

Definition. [Tangent Space] Let \mathcal{M} be a smooth manifold of dimension d . Consider a point $p \in \mathcal{M}$ and a chart (U, x) around p . Then if we denote the components of $x : U \rightarrow \mathbb{R}^n$ by $\{x^1, \dots, x^d\}$, i.e. $x^i : U \rightarrow \mathbb{R}$, then we define the *tangent space to p* , denoted $T_p\mathcal{M}$, to be the vector space spanned by

$$\left\{ \left(\frac{\partial}{\partial x^1} \right)_p, \dots, \left(\frac{\partial}{\partial x^d} \right)_p \right\}.$$

Proposition 1.1.1. *The following holds:*

$$\dim T_p\mathcal{M} = \dim \mathcal{M}$$

for all $p \in \mathcal{M}$.

Definition. [Cotangent Space] Given a manifold \mathcal{M} and its tangent spaces $T_p\mathcal{M}$, we define the *cotangent space to $p \in \mathcal{M}$* as the dual vector space $T_p^*\mathcal{M} \equiv (T_p\mathcal{M})^* := \{f : T_p\mathcal{M} \rightarrow \mathbb{R} \mid f \text{ is linear}\}$.

1.1.2 Bundles

We now need the notion of bundles. Again we are assuming that any readers are already familiar with the idea of a bundle and so we simply quickly state what a bundle is, mainly to fix notation. Again much more information can be found via Dr. Schuller's Geometrical Anatomy course.

Definition. [Bundle] A (smooth) *bundle* is a triple (E, π, \mathcal{M}) , where E and \mathcal{M} are (smooth) manifolds and $\pi : E \rightarrow \mathcal{M}$ is a surjective (smooth) mapping. E is known as the *total space*, \mathcal{M} the *base space* and π the *projection*. There are multiple ways to denote a bundle, the common ones being (E, π, \mathcal{M}) , $\pi : E \rightarrow \mathcal{M}$, $E \xrightarrow{\pi} \mathcal{M}$ and simply $E \rightarrow \mathcal{M}$. We will likely use all three interchangeably in these notes.

Definition. [Fibre] Given a bundle (E, π, \mathcal{M}) , we can define what is known as a *fibre at $p \in \mathcal{M}$* as the preimage under $\pi : E \rightarrow \mathcal{M}$, i.e.

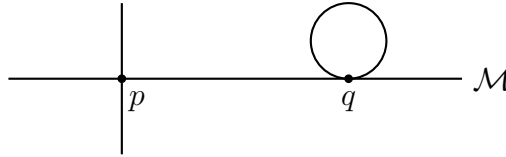
$$F_p := \text{preim}_\pi(p) := \{e \in E \mid \pi(e) = p\}.$$

Next we need the notion of a section.

Definition. [Section] Let (E, π, \mathcal{M}) be a bundle. Then a *section* is a map $\sigma : \mathcal{M} \supseteq U \rightarrow E$ such that $\pi \circ \sigma = \mathbb{1}_{\mathcal{M}}$, i.e. $\pi(\sigma(p)) = p$ for all $p \in \mathcal{M}$. In other words, a section maps a point $p \in \mathcal{M}$ into some point in the fibre F_p . Sections are not unique, and we denote the set of all sections on (E, π, \mathcal{M}) over region U by $\Gamma(E, U)$.

Note that we were careful to say the a section only maps a subset of \mathcal{M} to E . In this sense sections are a *local* quantity. Some bundles do admit a *global* section, but such bundles are necessarily *trivial* (to be defined in a moment).

It is important to note that for a given bundle, the fibres to each point need not be the same (i.e. isomorphic to each other). For example, we might have $F_p \cong \mathbb{R}$ but $F_q \cong S^1$, which we have tried to indicate in the following diagram where we depict $\mathcal{M} \cong \mathbb{R}$. Of course we do require that $\dim F_p = \dim F_q$ for all $p, q \in \mathcal{M}$, otherwise the dimension of E wouldn't be well defined.



However of course we can have the case when all the fibres are isomorphic, which leads us to the next definition.

Definition. [Fibre Bundle] Let (E, π, \mathcal{M}) be a bundle. Then if $F_p \cong F$, where F is some manifold, for all $p \in \mathcal{M}$ then we call the bundle a *fibre bundle* and call F the *typical fibre*. We will generally denote a fibre bundle by (E, π, \mathcal{M}, F) .

Given two fibre bundles over a common base space $(E_1, \pi_1, \mathcal{M})$ and $(E_2, \pi_2, \mathcal{M})$, we can form a new bundle by taking their tensor product

Definition. [Trivial Fibre Bundle] Let (E, π, \mathcal{M}, F) be a fibre bundle with typical fibre F . We call the fibre bundle *trivial* iff it is isomorphic as a bundle⁵ to the bundle $(\mathcal{M} \times F, \pi, \mathcal{M})$, which is clearly a fibre bundle with typical fibre F . For a trivial bundle the projection can really be thought of as a projection into the first slot, i.e. $\pi : (e, f) \mapsto e$.

Proposition 1.1.2. *A fibre bundle is trivial iff it admits a global section. When a bundle is trivial, we denote the space of sections simply as $\Gamma(E)$.*

Exercise

Prove the above proposition. *Hint: If stuck, see the proof of Proposition 1.1.3 below.*

So we can think of a trivial fibre bundle as one whose total space looks like $\mathcal{M} \times F$ everywhere. Clearly not all fibre bundles are trivial, however they are *all* necessarily *locally trivial*. This means that if we consider some open $U \subset \mathcal{M}$ then there exists a homeomorphism

$$\varphi_U : U \times F \rightarrow F_U := \text{preim}_\pi(U).$$

⁵This is defined in the expected way in terms of commuting diagrams.

We call such a map a *local trivialisation*, and as the name suggests it basically tells us that that any local patch of E is trivial. The classic examples of trivial vs non-trivial fibre bundles is a cylinder (which is given by $E \cong \mathbb{R} \times S^1$) and a Möbius strip, which is only locally trivial to $\mathbb{R} \times S^1$.

Vector Bundles

It is hopefully clear that we can given structure to a fibre bundle by giving the typical fibre structure. The most important case for us is when F is given the structure of a vector space.

Definition. [Real Vector Bundle] Let (E, π, \mathcal{M}, F) be a fibre bundle. Then we get a *vector bundle or rank k* if we equip the fibres with a dimension k real vector space structure. That is a vector bundle is a fibre bundle who's local trivialisation is

$$\varphi_U : U \times \mathbb{R}^k \rightarrow F_U.$$

We will typically denote a vector bundle by $F \rightarrow V$, where V represents the full vector space structure).

Definition. [Transition Functions] Given a vector bundle of rank k , (E, π, \mathcal{M}, V) , we can consider two local trivialisations on overlapping regions, i.e.

$$\varphi_U : U \times \mathbb{R}^k \rightarrow F_U$$

$$\varphi_V : V \times \mathbb{R}^k \rightarrow F_V,$$

where $U \cap V \neq \emptyset$. We can then consider the composite map

$$\begin{aligned} \varphi_U^{-1} \circ \varphi_V : (U \cap V) \times \mathbb{R}^k &\rightarrow (U \cap V) \times \mathbb{R}^k \\ (p, v) &\mapsto (p, g_{UV}(p)v) \end{aligned}$$

where $g_{UV} : (U \cap V) \rightarrow GL(k, \mathbb{R})$ are the *transition functions*. The transition functions can easily be shown to obey⁶

$$\begin{aligned} g_{UU}(p) &= \mathbb{1} & \forall p \in U \\ g_{UV}(p)g_{VU}(p) &= \mathbb{1} & \forall p \in U \cap V \\ g_{UV}(p)g_{VW}(p)g_{WU}(p) &= \mathbb{1} & \forall p \in U \cap V \cap W, \end{aligned} \tag{1.1}$$

known as the (Čech) cocycle conditions.

The transition functions are incredible useful as we can actually define a vector bundle using them. More specifically, we see that they encode information about the projection map: this is because they are defined in relation to the local trivialisations which contain information about the projection map. Putting this together with the fact that we can encode the information about \mathcal{M} via an open cover $\{U_i\}$, we see that we can determine a vector bundle (up to isomorphism) by the fibres, an open cover and the transition functions,

⁶Its worth checking these as an exercise!

which we now denote by $g_{ij} : (U_i \cap U_j) \rightarrow GL(k, \mathbb{R})$. Intuitively this makes sense, as we can think about the transition functions as a way of "patching together" the different parts of the bundle.

Proposition 1.1.3. *Let (E, π, \mathcal{M}, V) be a vector bundle of rank k . Then the bundle is trivial iff there exists k linearly independent, nowhere vanishing, global sections $\{\sigma_1, \dots, \sigma_k\} \subset \Gamma(E)$.*

Proof. First assume that (E, π, \mathcal{M}) is trivial. Then we have a global isomorphism $\varphi : \mathcal{M} \times V \rightarrow E$. Then if we pick a basis $\{e_1, \dots, e_k\}$ for V we can consider the maps

$$\sigma_i : p \rightarrow \varphi(p, e_i),$$

these are globally defined (as $\varphi(p, e_i)$ is⁷), and are sections as $\pi \circ \sigma_i = \mathbb{1}_{\mathcal{M}}$. Therefore $\{\sigma_i\} \subset \Gamma(E)$, are linearly independent and nowhere vanishing (as φ is an isomorphism, and so only vanishes when we consider the zero vector $0 \in V$).

Now assume that there exists k linearly independent, nowhere vanishing, global sections $\{\sigma_1, \dots, \sigma_k\} \subset \Gamma(E)$. Then *any* section $\tau \in \Gamma(E, U)$ can be expanded as

$$\tau = \sum_{i=1}^k f_i \sigma_i,$$

where $f_i \in C^\infty(\mathcal{M})$.⁸ However, any point $v_p \in \text{preim}_\pi(p)$ can be obtained by *some* section $\tau_p : p \rightarrow \text{preim}_\pi(p)$, and so from the above it follows that any point in the fibre at p can be expanded as

$$v_p = \sum_{i=1}^k A_i \sigma_i(p), \quad A_i \in \mathbb{R},$$

in other words the $\{\sigma_i(p)\}$ define a basis for the vector space $\text{preim}_\pi(p)$. As the $\{\sigma_i\}$ are globally defined, we can express any point in E via the following decomposition, which gives us the isomorphism

$$\begin{aligned} \varphi : E &\rightarrow \mathcal{M} \times V \\ e = \sum_{i=1}^k A_i \sigma_i(\pi(e)) &\mapsto \left(p = \pi(e), \sum_{i=1}^k A_i \sigma_i(p) \right). \end{aligned}$$

□

There is a specific category of vector bundles that will come up frequently for us later.

Definition. [Line Bundle] A vector bundle of rank $k = 1$ (i.e. the fibres are 1-dimensional vector spaces) is known as a *line bundle*. It is common to denote the total space of a line bundle by L .

⁷Also the smooth structure etc inherited from φ

⁸That is, it is a smooth function on \mathcal{M} .

Making New Bundles Out Of Old Ones

Finally, before defining tensor fields, we need to discuss how we can make new bundles out of existing ones.

We proceed by recalling that a vector bundle is determined (up to isomorphism) by an open cover, the fibres and the transition functions.

Definition. [Dual Bundle] Let (E, π, \mathcal{M}, V) be a vector bundle of rank k with transition functions $g_{ij} : (U_i \cap U_j) \rightarrow GL(k, \mathbb{R})$. Then we define the *dual bundle* as the bundle over the same base space whose fibres are given by the dual vector space V^* , and whose transition functions are given by the pointwise matrix inverse of g_{ij} .⁹

With some thought, it can be seen that the dual bundle maps sections of (E, π, \mathcal{M}, V) to smooth functions over \mathcal{M} , hence the name. This will become more clear in the next section when we construct the cotangent bundle.

Definition. [Whitney Sum Bundle] Consider two vector bundles of ranks k_1 and k_2 over the same base space: $(E_1, \pi_1, \mathcal{M}, V_1)$ and $(E_2, \pi_2, \mathcal{M}, V_2)$. Consider a common open cover $\{U_i\}$ ¹⁰ and denote the transition functions by $(g_1)_{ij} : (U_i \cap U_j) \rightarrow GL(k_1, \mathbb{R})$ and $(g_2)_{ij} : (U_i \cap U_j) \rightarrow GL(k_2, \mathbb{R})$.

We can construct a new bundle, known as the *Whitney sum bundle*, by taking the direct sum, i.e. we have a vector bundle over \mathcal{M} with total space $E_1 \oplus E_2$, fibres $V_1 \oplus V_2$ and whose transition functions are given by

$$(g_1)_{ij} \oplus (g_2)_{ij} : (U_i \cap U_j) \rightarrow GL(k_1 + k_2, \mathbb{R})$$

$$p \mapsto (g_1)_{ij}(p) \oplus (g_2)_{ij}(p) := \begin{pmatrix} (g_1)_{ij}(p) & 0 \\ 0 & (g_2)_{ij}(p) \end{pmatrix}.$$

Definition. [Tensor Product Bundle] Again consider two vector bundles with some common open cover with transition functions as above. We then define the *tensor product bundle* as the vector bundle with total space $E_1 \otimes E_2$, fibres $V_1 \otimes V_2$ and transition maps

$$(g_1)_{ij} \otimes (g_2)_{ij} : (U_i \cap U_j) \rightarrow GL(k_1 + k_2, \mathbb{R})$$

$$p \mapsto (g_1)_{ij}(p) \otimes (g_2)_{ij}(p) := \begin{pmatrix} (g_1)_{11}(p) \mathbf{g}_2(p) & \dots & (g_1)_{1k_1} \mathbf{g}_2(p) \\ \vdots & \ddots & \vdots \\ (g_1)_{k_1 1}(p) \mathbf{g}_2(p) & \dots & (g_1)_{k_1 k_1} \mathbf{g}_2(p) \end{pmatrix},$$

where $\mathbf{g}_2(p)$ represents the whole matrix of g_2 .

We want to introduce one more type of bundle, but first we need to remind ourselves/introduce of some notation.

⁹Note this is not the same as the inverse *function*: as a function $g_{ij} : (U_i \cap U_j) \rightarrow GL(k, \mathbb{R})$, so firstly the inverse would map from $GL(k, \mathbb{R})$ to $(U_i \cap U_j)$ which is the wrong behaviour, and besides not every such function is even invertible in the first place.

¹⁰This is always possible to form by taking what is known as the *refinement* of the local trivialisations of the two bundles.

Definition. [Antisymmetric Tensor Product] Consider some vector space V , and consider the n^{th} tensor product

$$\otimes^n V := \underbrace{V \otimes \dots \otimes V}_{n\text{-times}}.$$

We can consider the subspace of $\otimes^n V$ given by taking only the p^{th} order antisymmetric tensor product (also known as the *exterior product* or *wedge product*), which we denote by $\Lambda^p V$. It's clear that $1 \leq p \leq n$.

As a concrete example of a wedge product, given two vectors $v, w \in V$, we have

$$v \wedge w := v \otimes w - w \otimes v \in \Lambda^2 V.$$

Note that $\dim \Lambda^{\dim V} V = 1$, as it is formed by taking an antisymmetric product of *all* the basis elements, and so any element in $\Lambda^{\dim V}$ is related to another by a smooth function.

Definition. [Determinant Bundle] Let (E, π, \mathcal{M}, V) be a vector bundle of rank k . We can construct the *determinant bundle* as the k^{th} exterior product bundle. The fibres are then given by $\Lambda^k V$, and the transition functions as in the tensor product definition above, but now restricted to the antisymmetric product. However this is just the definition of the determinant of the matrix $\mathbf{g}(p)$, hence the name determinant bundle. Note also that, as explained above, the fibres are 1 dimensional, and so the determinant bundle is an example of a line bundle.

The determinant bundle is actually very important, and it gives rise to notions of tensor *densities*, which play a huge role in topics such as GR, where the familiar term $\sqrt{-g}$ used in GR integrals is a tensor density.

1.1.3 Tensor Fields

We are now in a good place to give a nice definition of tensor fields. We do this by recalling the tangent space to a point $p \in \mathcal{M}$ is a vector space with dimension $\dim T_p \mathcal{M} = \dim \mathcal{M}$. We have a tangent space for all $p \in \mathcal{M}$, and clearly $T_p \mathcal{M} \cong T_q \mathcal{M}$ for all $p, q \in \mathcal{M}$, and so we can form the trivial vector bundle with total space $E \cong \mathcal{M} \times T_p \mathcal{M}$. We denote the total space – in fact we will often just call the whole tangent bundle $T\mathcal{M}$ rather than $(T\mathcal{M}, \pi, \mathcal{M}, T_p \mathcal{M})$, for obvious reasons – by $T\mathcal{M}$ and the bundle is known as the *tangent bundle*. We similarly have the *cotangent bundle*, $T^* \mathcal{M}$, which is the dual bundle to the tangent bundle (as the fibres are the dual to the fibres in $T\mathcal{M}$).

Definition. [(Co)Vector Field] Consider the tangent bundle $T\mathcal{M}$. A smooth section $\sigma : \mathcal{M} \rightarrow T\mathcal{M}$ is known as a *vector field*. Likewise a *covector field* is a smooth section of the cotangent bundle $\omega : \mathcal{M} \rightarrow T^* \mathcal{M}$.

We can get a nice intuitive idea to why we define a vector field as above: we like to think of a vector field as a "vector at each point, that nicely follow on from each other". Well the fibres of the tangent bundle are exactly vectors at a point, and a section gives us one per point. The vague condition of "nicely follow on from each other" is taken care of by the more concrete requirement that the section be smooth.

Definition. [Tensor Field] Let $T\mathcal{M}$ and $T^*\mathcal{M}$ be the tangent and cotangent bundle to some manifold \mathcal{M} . A (r, s) -tensor field is a section of the product bundle

$$T : \mathcal{M} \rightarrow \otimes^r T\mathcal{M} \otimes^s T^*\mathcal{M},$$

i.e. it is a map

$$T : \underbrace{T^*\mathcal{M} \times \dots \times T^*\mathcal{M}}_{r\text{-times}} \times \underbrace{T\mathcal{M} \times \dots \times T\mathcal{M}}_{s\text{-times}} \rightarrow C^\infty(\mathcal{M}).$$

We denote the space of (r, s) -tensor fields as $\Gamma(\otimes^r T\mathcal{M} \otimes^s T^*\mathcal{M})$.

Note that vector fields are $(1, 0)$ -tensor fields and covector fields are $(0, 1)$ -tensor fields. There is a further subset of tensor fields which is very important, which we now define.

Differential Forms

Definition. [Differential Forms] A differential p -form on some manifold \mathcal{M} is a smooth section $\omega : \mathcal{M} \rightarrow \Lambda^p T^*\mathcal{M}$, i.e. it is a totally antisymmetric $(0, p)$ -tensor field. We denote the set of p -forms¹¹ on \mathcal{M} as $\Omega^p(\mathcal{M})$.

There are two special types of p -forms worth mentioning:

1. 0-forms: It follows from $\Lambda^0 T^*\mathcal{M} \cong C^\infty(\mathcal{M})$, that 0-forms are just smooth functions on \mathcal{M} .
2. $\dim \mathcal{M}$ -forms: As explained above, this is a one-dimensional space, and an element is known as a *top* form. A nowhere vanishing top form is called a *volume* form.

We conclude this "quick" review by noting/defining two operations on forms. Firstly note that the wedge product allows us to produce forms of higher degree, i.e. if $\omega \in \Omega^p \mathcal{M}$ and $\eta \in \Omega^q \mathcal{M}$, then $\omega \wedge \eta \in \Omega^{p+q} \mathcal{M}$. Next we define the exterior derivative.

Definition. [Exterior Derivative] The exterior derivative is a map

$$d : \Omega^p \mathcal{M} \rightarrow \Omega^{p+1} \mathcal{M}$$

defined to obey

- (i) For any smooth function $f \in C^\infty(\mathcal{M})$, we have $df(X) = X(f)$.
- (ii) It "squares to zero", i.e. $d \circ d\omega = 0$ for $\omega \in \Omega^p \mathcal{M}$ for all $1 \leq p \leq \dim \mathcal{M}$.
- (iii) It is graded Leibniz on the wedge product, easiest just written: if $\omega \in \Omega^p \mathcal{M}$ and $\eta \in \Omega^q \mathcal{M}$ then

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta.$$

¹¹We often drop the "differential" and just call them forms.

The Metric

We assume the reader is at least familiar with the importance of a metric (say, from a GR course), but for completeness we give a definition now.

Definition. [Metric] Let \mathcal{M} be a smooth manifold. Then the *metric*, g , is a $(0,2)$ -tensor field that is symmetric, $g(X, Y) = g(Y, X)$, and non-degenerate, $g(X, Y) = 0$ for all X implies $Y = 0$. A manifold equipped with a metric is called a metric manifold. We split these into two further categories:

- (i) If we require that is positive definite, i.e. $g(X, X) > 0$ for all $X \neq 0$, then we call the manifold *Riemannian*.
- (ii) If we allow $g(X, X) < 0$ then we call the manifold *pseudo-Riemannian*.

We can relate the two definitions to the *signature* of g , Riemannian metrics having all +s while pseudo-Riemannian having some +s and some -s. If we have only 1 minus we call the manifold Lorentzian.

As the metric is non-degenerate it can be inverted, i.e. we can define an inverse metric which is an element of the symmetric $(2,0)$ -tensor fields. What we are essentially doing is defining an (pseudo-)inner product on the tangent and cotangent spaces. We can extend this to define a (pseudo-)inner product for a general tensor space by the familiar "lowering"/"raising" of all the indices.

1.2 Poincaré Duality

We now present a very hand wavey/brief discussion of an important result in algebraic topology; the *Poincaré duality*. This section by no means claims to be a rigorous discussion, and definitely glosses over lots of subtleties. We do this for three reasons: 1) I am not a mathematician and so don't actually know these subtleties myself; 2) in these notes we will only really be concerned with the general concept of Poincaré duality; and 3) to save space — anyone interested in a much (much much) more detailed discussion is directed to the 250 or so pages in Hatcher's book.

1.2.1 Homology

Simplices

In order to understand homology, we first need to know what a simplex is. We give the following, geometrical, definition.

Definition. [n -Simplex (Triangle)] An n -simplex is the n -dimensional version of a triangle.

To help clarify the above definition, and to give some standard names, we list the first few simplices:

- (i) 0-simplex: a *point*.
- (ii) 1-simplex: a *line segment*.

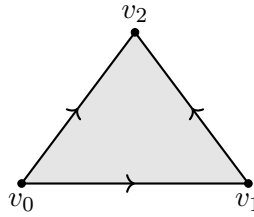
- (iii) 2-simplex: a *triangle*.
- (iv) 3-simplex: a *tetrahedron*.

A more formal definition of an n -simplex is as follows.

Definition. [n -Simplex (Convex Hull)] An n -simplex is n -dimensional polytope¹² which is the convex hull¹³ of $n + 1$ vertices.

We see, therefore, that we can think of an n -simplex as a collection of $(n + 1)$ -vertices, $\{v_0, \dots, v_n\}$, connected by straight lines, and then filling in the surface area. In fact we are also going to *order* the vertices, so that our lines now have an orientation given by increasing order. We then denote our simplex by $[v_0, \dots, v_n]$ where the ordering is read left to right. We will assume this ordering is understood whenever we say " n -simplex", i.e. we what we really mean is "an n -simplex with this vertices ordered".

Example 1.2.1. As an example, let's consider a 2-simplex. This is given by 3 vertices $[v_0, v_1, v_2]$, we depict this graphically as



Definition. [Standard n -Simplex] We define the *standard n -simplex* to be the n -simplex given by

$$\Delta^n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \quad \text{and} \quad t_i \geq 0 \forall i \right\}.$$

The standard simplex basically just corresponds to taking the vertices to lie unit distance along the axes of \mathbb{R}^{n+1} , and the ordering is given by the labelling of the axes. Now with a bit of thought, its easy to convince ourselves that *any* n -simplex is homeomorphic to the standard n -simplex Δ^n . We shall therefore just call our simplices $\Delta^n = [v_0, \dots, v_n]$.

Definition. [Face Of A Simplex] Consider an n -simplex $\Delta^n = [v_0, \dots, v_n]$. Now if we remove one of the vertices, say the i -th vertex, we are left with $[v_0, \dots, \hat{v}_i, \dots, v_n]$, where the hat indicates its removal. But this is simply a $(n - 1)$ -simplex. We call this resulting simplex a *face* of Δ^n .

Definition. [Boundary Of A Simplex] Let Δ^n be an n -simplex. The union of all the faces of Δ^n is called the *boundary* of Δ^n , and we denote it $\partial\Delta^n$. The name is hopefully clear

¹²Basically a shape with flat sides.

¹³Basically the surface.

from the geometrical point of view. For reasons that shall become clear soon, we define our boundary map by the formal sum

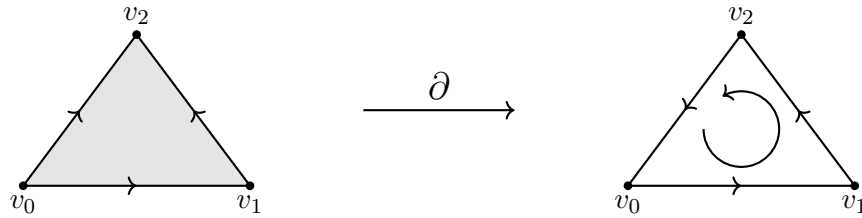
$$\partial : [v_0, \dots, v_n] \mapsto \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n], \quad (1.2)$$

where the sum should be thought of as "composing the arrows", which a minus sign indicating going against the ordering.

Example 1.2.2. To clarify the definition of boundary map given above, let's take the boundary of our 2-simplex from before. We have

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1],$$

which we can indicate pictorially as



where the arrow on the inside shows us the flow of the arrows.

(Simplicial) Homology

The idea of (simplicial) homology is to take the Δ^n s and to embed them into some topological space X , "gluing" them along the faces as we go. In this way we can build the space X using the Δ^n s. We denote the embedding maps by $\sigma_\alpha : \Delta^n \rightarrow X$, where the α index tells us which n we consider. If we place certain reasonable restrictions on these maps,¹⁴ we get what is known as a Δ -complex. We denote by $\Delta_n(X)$ the set of n -dimensional subspaces that are given by taking the disjoint union of (the image of) one or more Δ^n and the Δ^{n-1} simplices that "glue" them together.¹⁵ Essentially all we're saying is glue together one or more n -simplex in X along a common face (which is a $(n-1)$ -simplex), but keep track (i.e. disjoint union) of the simplices we have.

We now look at Example 1.2.2 again. We can think of the 2-simplex as a trivial element of $\Delta_n(X)$ given just by the embedding of the 2-simplex. Now we note that we can think of the triangle on the right of the diagram as an element of $\Delta_{n-1}(X)$, where we have glued three 1-simplices (the lines $[v_0, v_1]$, $[v_1, v_2]$ and $[v_2, v_0]$) together using three 0-simplices (the vertices v_0 , v_1 and v_2). By a (hopefully clear) extension of this example, we see that we can view our boundary maps instead as maps from $\Delta_n(X)$ to $\Delta_{n-1}(X)$. Explicitly, using Equation (1.2) for guidance, we define $\partial_n : \Delta_n(X) \rightarrow \Delta_{n-1}(X)$ by its action on the embedding maps:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]},$$

¹⁴See page 103 of Hatcher.

¹⁵For those interested, we are essentially talking about CW complexes.

where, by the linearity described above, we need only specify the map on the basis $(n-1)$ -simplices.

Lemma 1.2.3. *The boundary map is nilpotent, that is the composition $\partial_{n-1} \circ \partial_n : \Delta_n(X) \rightarrow \Delta_{n-2}(X)$ vanishes.*

Proof. By definition we have

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \partial_{n-1}\left(\sum_i (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\right) \\ &= \sum_{j \neq i} (-1)^i (-1)^j \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} \\ &= \sum_{j < i} (-1)^i (-1)^j \sigma_\alpha|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} \end{aligned}$$

where the second line follows from the fact that when we take out v_i , we should really relabel $v_j \rightarrow v_{j-1}$ for all $j > i$. Now simply relabel $i \leftrightarrow j$ in the last sum and then the last line cancels. \square

It follows from the above Lemma, that we have the following exact sequence of maps

$$\dots \longrightarrow \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \longrightarrow \dots \longrightarrow \Delta_1 \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0,$$

with $\partial_n \partial_{n+1} = 0$, i.e. $\text{Im } \partial_{n+1} \subseteq \ker \partial_n$, where Im / \ker stand for image and kernel, respectively. Such an sequence is known as *chain complex*. We call elements of $\text{Im } \partial_{n+1}$ *n-boundaries* and elements of $\ker \partial_n$ *n-cycles*.¹⁶

There is another sequence, which looks a lot like a chain complex, known as an exact sequence. An exact sequence is basically a chain complex but where we satisfy the equality, $\text{Im } \partial_{n+1} = \ker \partial_n$. The homology group is essentially a measure of how far off the chain complex is from being exact, as the next definition makes formal.

Definition. [Homology Group] Consider some chain complex $(C_\bullet, \partial_\bullet)$,¹⁷ we define the *n-th homology* to be the quotient

$$H_n^C := \frac{\ker \partial_n}{\text{Im } \partial_{n+1}}. \quad (1.3)$$

Elements of H_n^C are called *homology classes*, and are equivalence classes, i.e. for $\alpha, \beta \in C_n$

$$\alpha \sim \beta \quad \Longleftrightarrow \quad \alpha = \beta + \partial_{n+1}\gamma,$$

where $\gamma \in C_{n+1}$.

¹⁶These names are hopefully somewhat intuitive given the geometrical pictures: boundaries should be obvious, and a cycle we can think of as a collection of n -simplices which are glued together in "cyclic" fashion, i.e. they flow around the shape. With a bit of thought, it's clear such a cyclic shape will have vanishing boundary. It is important to note that such a cyclic shape need not be the boundary, though, which is the entire point of homology, so hang on!

¹⁷This notation is standard, the \bullet basically represents all the different n values. Note also as the wording indicates, there are homology groups are defined for general chain complexes, not just the $\Delta_n(X)$ stuff we're considering.

The homology group for the specific case we have constructed, i.e. $C_n = \Delta_n(X)$, is known as the n -th *simplicial homology group* of X , denoted $H_n^\Delta(X)$.

Ok so we have $H_n^\Delta(X)$, but what exactly does this tell us? Well, by definition, we get a non-vanishing n -th simplicial homology iff there exists a n -cycle which is *not* the boundary of some $(n+1)$ -simplex. With a bit of thought, it should be clear this is equivalent to saying that it's non-vanishing iff there is a n -dimensional subspace in X which is not the boundary of some $(n+1)$ -dimensional subspace which has no singularities (by which we mean you can't "poke a hole" in the latter space).¹⁸

It is actually the point in brackets above that is of interest to us: the n -th simplicial homology is non-vanishing *only* when there exists a n -dimensional subspace which does not contain any "holes", in other words it can be contracted away without leaving X . We try clarify this with the following examples and exercise but first we introduce the Betti numbers.

Definition. [Betti Numbers] We define the n -th *Betti number* as the dimension of the n -th homology group:

$$b^n := \dim H_n^\Delta(X). \quad (1.4)$$

Example 1.2.4. Let $X = \mathbb{R}^n$. It is hopefully clear that *any* k -cycle, which is a k -dimensional closed surface in \mathbb{R}^n , is given by the boundary of a $(k+1)$ -simplex, namely the $(k+1)$ -ball B^{k+1} .¹⁹ For example, if we take $n = 2$ then we have the 1-cycle given by a circle, but this is just the boundary of a disk. This is just the statement that $S^k = \partial B^{k+1}$. The only exception here is $k = 0$ as S^0 is a single point but B^1 is a line and the boundary of a line is two points. We therefore conclude²⁰

$$H_k^\Delta(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

where it is power of \mathbb{R} ²¹ that tells us "how many k -dimensional non-contractible loops we have" (from now on this is what we mean by k -dimensional hole).

Example 1.2.5. Let $X = S^2$, the 2-sphere. From the example above, it's hopefully clear that $H_1^\Delta(S^2) = 0$. This is just the statement that any loop on the surface of S^2 can be contracted away without leaving S^2 . However we see that $H_2^\Delta(S^2) \neq 0$ as we cannot contract the sphere itself away. In simplex language, we would need to embed a tetrahedron homeomorphically into S^2 , but this can't be done *precisely* because we don't include the interior of the ball. Said another way, we can think of S^2 as ∂B^3 , as above. The extension of these arguments show that

$$H_k^\Delta(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

¹⁸This is basically given by the fact that we require that $\sigma_\alpha : \Delta^n \rightarrow X$ need to map the interior of Δ^n to an open subspace of X .

¹⁹Or at least something that is homeomorphic to a B^{k+1} .

²⁰We argue the $k = n$ case by invoking the contractible away argument, as we obviously can't homeomorphically put a $(n+1)$ -dimensional space into \mathbb{R}^n .

²¹Really we should have a \mathbb{Z} here given the way we have defined homology. However for easy comparison to deRham cohomology shortly, we assume we have \mathbb{R} coefficients not \mathbb{Z} .

Exercise

Convince yourself (using geometric arguments) that the simplicial homology of the torus $T^2 \cong S^1 \times S^1$ is

$$H_n^\Delta(T) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

This result generalises to

$$H_n^\Delta(T^m) = \begin{cases} \mathbb{R} & k = 0, m \\ \mathbb{R}^m & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $T^m := \underbrace{S^1 \times \dots \times S^1}_{m\text{-times}}$.

Brief Comment on Singular Homology

As we have been careful to say, what we have defined is known as simplicial homology. There is another, closely related, type of homology known as *singular homology*. The main difference between the two is that in singular homology we only require that our maps $\sigma_\alpha : \Delta^n \rightarrow X$ be *immersions*, rather than embeddings. This basically means our map needs only to be continuous (and not necessarily injective) that we can "fold" the Δ^n into X , giving us some singular embedding. It is a fair question to ask "Why on EARTH would we consider such a wacky idea!?" The answer is that it actually allows us to see some really nice properties, e.g. two spaces X and Y that are homeomorphic must have isomorphic singular homology groups, in this sense singular homology is a property of the topology of X itself, rather than of some triangulation, which simplicial homology seems to be. We denote the n -th singular homology group of X simply by $H_n(X)$. We then claim, without proof (see Hatcher for details), the following Lemma holds.

Lemma 1.2.6. *The singular and simplicial homologies are isomorphic for all Δ -complexes.*

1.2.2 Cohomology

Just as a covector was the dual of a vector, *cohomology* is the dual of homology. We define it using the specific example of singular cohomology.

Singular Cohomology

Definition. [Singular Cohomology] Let $(C_\bullet(X), \partial_\bullet)$ be the singular homology of some topological space X . We then define the singular cochain complex (C_\bullet^*, d_\bullet) where

$$C^n \equiv C_n^* := \text{Hom}(C_n, \mathbb{R}), \quad \text{and} \quad d_n : C_n \rightarrow C_{n+1}$$

where Hom stands for homomorphisms. We then define the n -th singular cohomology equivalently to before, namely

$$H^n(X) = \frac{\ker d_n}{\text{Im } d_{n-1}},$$

where we note we have "raised" the n index, compared to singular homology. We call elements of $\ker d_n$ n -cocycles and elements of $\text{Im } d_{n-1}$ n -coboundaries.

Essentially what we have done is reverse the ordering of the sequence, so that our coboundary map d_n increases the index rather than decreases it. Note that the cohomology is well defined as we still have $d_n \circ d_{n-1} = 0$.

de Rham Cohomology

Before finally stating the Poincaré duality, we want to present the type of cohomology, that will be of most interest to us, *de Rham Cohomology*.

Recall that we defined the exterior derivative to be a map

$$d : \Omega^p \mathcal{M} \rightarrow \Omega^{p+1} \mathcal{M}$$

and condition (ii) told us that it is nilpotent. Well, if we are more correct and note that we actually have an exterior derivative for each $1 \leq p \leq \dim \mathcal{M}$, we see that we actually have a cochain complex $(\Omega^\bullet \mathcal{M}, d^\bullet)$. We can therefore define its cohomology, which we call the *de Rham cohomology of \mathcal{M}*

$$H_{dR}^p(\mathcal{M}) := \frac{\ker(d : \Omega^p(\mathcal{M}) \rightarrow \Omega^{p+1}(\mathcal{M}))}{\text{Im}(d : \Omega^{p-1}(\mathcal{M}) \rightarrow \Omega^p(\mathcal{M}))}.$$

It is standard to replace the names "cocycle" and "coboundary" with "closed" and "exact" when dealing with de Rham cohomology. That is, $\alpha \in \Omega^p(\mathcal{M})$ is closed if $d\alpha = 0$ and is exact if $\alpha = d\beta$ for some $\beta \in \Omega^{p-1}(\mathcal{M})$. So the de Rham cohomology is a measure on how a closed form fails to be exact.

Theorem 1.2.7 (de Rham). *Let \mathcal{M} be a smooth manifold, then we have an isomorphism*

$$H_{dR}^p(\mathcal{M}) \cong H^p(\mathcal{M}; \mathbb{R}), \quad (1.5)$$

where the right-hand side is the singular cohomology of \mathcal{M} , with coefficients in \mathbb{R} .²²

Proof. The proof of this theorem follows from the fact that we can only integrate a p -form on a p -dimensional manifold.²³ We then recall that the p -th singular cohomology is defined to be the dual of p -th singular homology, which in itself represents p -dimensional submanifolds in \mathcal{M} . Well we then just define the map

$$I : H_{dR}^p(\mathcal{M}) \rightarrow H^p(\mathcal{M}; \mathbb{R})$$

$$\omega \mapsto I_\omega,$$

²²This is why we took our coefficients above to be in \mathbb{R} .

²³For the sake of brevity, we do not show why this is the case here, but a discussion can be found in Lecture 12 of Dr. Schuller's Winter School On Gravity and Light course, notes for which are available on my website.
 note to self, maybe do put a short section in here as would make these notes more self contained.

which acts on a $[c] \in H_p(\mathcal{M}, \mathbb{R})$ as

$$I_\omega([c]) := \int_c \omega,$$

which we claim is well defined (i.e. independent of the representative c). \square

1.2.3 Poincaré Duality (Finally!)

We are now ready to finally state the Poincaré duality and why it is of interest to us. We state the duality in the form of a Lemma without proof (again see Hatcher for more details).

Lemma 1.2.8 (Poincaré Duality). *Let \mathcal{M} be an n -dimensional, oriented, closed²⁴ manifold, then we have an isomorphism between singular cohomology and singular homology:*

$$H^p(\mathcal{M}; \mathbb{R}) \cong H_{n-p}(\mathcal{M}; \mathbb{R}). \quad (1.6)$$

The reason we care about Poincaré duality is that, putting it together with Equation (1.5), we conclude

$$H_{dR}^p(\mathcal{M}) \cong H_{\dim \mathcal{M} - p}(\mathcal{M}; \mathbb{R}), \quad (1.7)$$

and so the p -th de Rham cohomology tells us how many $(\dim \mathcal{M} - p)$ -dimensional holes there are in \mathcal{M} . This is a highly non-trivial result and is incredibly powerful. Why? Well because it is practically relatively simple to calculate the de Rham cohomology of a space, whereas the singular homology is a bit of a pain. However the interesting topological information (i.e. the number of holes) is contained within the singular homology. So Equation (1.7) gives us a practical way to answer the question of "how many holes does our manifold have?" This motivates the redefinition of our Betti numbers, Equation (1.4):²⁵

Definition. [Betti Numbers (de Rham)] We define the n -th *Betti number* of a smooth manifold to be

$$b_n := \dim H_{dR}^n(\mathcal{M}). \quad (1.8)$$

We can then introduce the *Euler characteristic*²⁶

²⁴This means compact without boundary. A manifold is compact, if its underlying topological space is compact. A topological space is compact if every open cover (basically the union of the open sets that contains the full space) contains a finite subcover. These definitions are assumed understood, but if not any decent differential geometry textbook will explain this.

²⁵Note this is actually a new definition, which is indicated by the raising of the index.

²⁶The Euler characteristic is actually defined in terms of homology as the number of vertices minus the number of edges plus the number of faces of a given polyhedron. It turns out the following is equal to this.

Definition. [Euler Characteristic] We define the Euler characteristic of \mathcal{M} to be the alternating sum

$$\chi := \sum_{n=0}^{\dim \mathcal{M}} (-1)^n b^n. \quad (1.9)$$

1.2.4 Some Geometry Of Forms

Before moving on to discuss Hodge theory, let's just make some comments on the geometrical interpretations of forms. This short subsection is based off [1], and the interested reader is directed there for lots of nice diagrams to go along with the words.

1-Forms

The Poincaré duality actually allows us to give a nice geometrical interpretation to exact 1-forms. A 1-form α is exact if it is given by the exterior derivative of a smooth function, i.e. $\alpha = df$ where $f \in C^\infty(\mathcal{M})$. Well we have just seen that the Poincaré duality tells us that an exact p -form is isomorphic to a $(\dim \mathcal{M} - 1)$ -dimensional submanifold, which is the boundary of a $(\dim \mathcal{M} - 1 + 1)$ -dimensional submanifold. So what is it? Well with a bit of thought we can convince ourselves that df corresponds to the contour lines of f , that is the field f is our $\dim \mathcal{M}$ submanifold and its boundary is given by the contour lines.

This also allows us to understand, geometrically, why $df(X) := X(f)$ is a smooth function. If we think of df as the contour lines of f and X as a set of vectors at each point $p \in \mathcal{M}$, then the number we get out from $df(X)|_p$ is just given by the number of contour lines X "goes through". This is obviously a loose analogy (as how do you "count" the contours) but it is still a nice geometrical picture, I think.

Volume Forms

Recall that a volume form is a nowhere vanishing top form, i.e. $\omega \in \Omega^{\dim \mathcal{M}}(\mathcal{M})$ with $\omega(p) \neq 0$ for all $p \in \mathcal{M}$. Well, by Poincaré duality, this corresponds to a 0-dimensional submanifold, which is a scalar field (i.e. a number at every point). This number, is exactly the weight of the measure when we integrate over this manifold, hence the name *volume* form.

Wedge Products

Finally let's comment on the wedge product of two forms. Recall that if $\alpha \in \Omega^p(\mathcal{M})$ and $\beta \in \Omega^q(\mathcal{M})$ then $\alpha \wedge \beta \in \Omega^{p+q}(\mathcal{M})$. How does this translate under Poincaré duality? Well we're taking a $(\dim \mathcal{M} - p)$ -dimensional and a $(\dim \mathcal{M} - q)$ -dimensional submanifold and somehow putting them together to give a $(\dim \mathcal{M} - p - q)$ -dimensional submanifold. The geometrical picture of this is that the resulting submanifold is given by the intersection of the two original ones, which you should be able to convince yourself has the correct dimension. We can also account for the orientation of these manifolds under wedge products, but we won't discuss that further here (whenever needed later, we shall just claim results on how the resulting orientation comes out). For more details see section 4.3 of [1].

1.3 Hodge Theory

Right, we now want to introduce Hodge theory. This is a very powerful way of studying the differential forms of a manifold, and, as we will see, leads to a nice decomposition of the de Rham cohomology. The complex version of Hodge theory will prove immensely helpful to us later.

1.3.1 The Hodge Dual Operator

First we need to define the Hodge dual operator. This Hodge dual can be defined in a more general sense, however we focus on its definition in terms of its action on differential forms on a Riemannian manifold.

Definition. [Hodge Dual] Let $\alpha \in \Omega^p(\mathcal{M})$ be some generic p -form on a (pseudo-)Riemannian manifold \mathcal{M} . We define the *Hodge dual* as the map

$$\star : \Omega^p(\mathcal{M}) \rightarrow \Omega^{\dim \mathcal{M} - p}(\mathcal{M}).$$

We can define it completely via the following relation: let $\alpha, \beta \in \Omega^p(\mathcal{M})$, and let $\langle \alpha, \beta \rangle$ denote the inner product induced by the metric. Further let $\omega \in \Omega^{\dim \mathcal{M}}(\mathcal{M})$ be the unit volume form,²⁷ then the Hodge dual satisfies

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \omega. \quad (1.10)$$

It follows from the non-degeneracy of the inner product that Equation (1.10) tells us that the Hodge dual is unique. We can integrate both sides of Equation (1.10) on all of \mathcal{M} to obtain the square integrable inner product on p -forms, i.e.

$$\langle \alpha, \beta \rangle := \int_{\mathcal{M}} \alpha \wedge \star \beta, \quad (1.11)$$

which we note makes sense as $\alpha \wedge \star \beta$ is necessarily a top form and so can be integrated over \mathcal{M} .

Remark 1.3.1. We can give a slightly more "useful" definition of the Hodge star by using coordinates, namely, if $\dim \mathcal{M} = n$

$$(\star \omega)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \sqrt{|g|} \epsilon_{\mu_1 \dots \mu_{n-p} \nu_1 \dots \nu_p} \omega^{\nu_1 \dots \nu_p}.$$

We will rarely use indices in these notes, however, and so this is just included more to help "ground" the more abstract definitions above.

Remark 1.3.2. It is important to note that our definition of the Hodge dual depends on the metric. This is seen in the abstract definition by the introduction of the inner product $\langle \alpha, \beta \rangle$, and in the index version by the fact that we have raised the indices of the form $\omega_{\nu_1 \dots \nu_p}$ on the right-hand side. This is important in things like QFT where we contrast terms like $F \wedge F$ with $F \wedge \star F$; the former is purely topological, whereas the latter is metric dependent. We won't discuss this further here, but just include this as an important side remark.

²⁷I.e. the volume form with unit inner product with itself.

Self Dual & Anti-Self Dual Forms

One can relatively easily show that for $\alpha \in \Omega^p(\mathcal{M})$

$$\star(\star\alpha) = \pm(-1)^{p(n-p)}\alpha,$$

where we take the $+$ sign for Riemannian manifolds and the $-$ sign for Lorentzian ones. This is probably easiest shown using the component expansion above along with

$$\epsilon^{\mu_1 \dots \mu_{n-p} \rho_1 \dots \rho_p} \epsilon_{\nu_1 \dots \nu_{n-p} \rho_1 \dots \rho_p} = \pm p!(n-p)! \delta_{[\nu_1}^{\mu_1} \dots \delta_{\nu_{n-p}] }^{\mu_{n-p}} = \pm p! \delta_{\nu_1}^{\mu_1} \dots \delta_{\nu_{n-p}}^{\mu_{n-p}},$$

where again the $+$ sign is for Riemannian and $-$ for Lorentzian.

Now also note that, if $\dim \mathcal{M} = 2m$, i.e. is even, then $\star : \Omega^m(\mathcal{M}) \rightarrow \Omega^m(\mathcal{M})$ is a convolution. Putting this together with the Lorentzian result, for $\alpha \in \Omega^m(\mathcal{M})$,

$$\star(\star\alpha) = -(-1)^{m(2m-m)}\alpha = -(-1)^{m^2}\alpha = -\alpha$$

we see that our m -forms are eigenvectors of \star with eigenvalues $\pm i$. In this sense we can decompose as follows

$$\Omega^m(\mathcal{M}) = \Omega_+^m(\mathcal{M}) \oplus \Omega_-^m(\mathcal{M}),$$

where the subscripts indicate the eigenvalue, i.e. $\alpha \in \Omega_+^m(\mathcal{M})$ obeys $\star\alpha = +i\alpha$. We call elements of $\Omega_+^m(\mathcal{M})$ *self dual* forms and elements of $\Omega_-^m(\mathcal{M})$ *anti-self dual* forms.

Self dual and anti-self dual forms crop up all over the place in physics. We won't discuss these too much further in these notes, but we just note that this duality clearly reduces the number of degrees of freedom (i.e. if our m -form represents a collection of fields then the (anti-)self duality equates some of these fields).

1.3.2 Codifferential

Let's now go back to Equation (1.11). This should (hopefully) remind us of square integrable functions in topics such as quantum mechanics. Now also recall that we have an operator $d : \Omega^p(\mathcal{M}) \rightarrow \Omega^{p+1}(\mathcal{M})$. We now want to ask the question of is there some formal adjoint to this operator that we can insert into Equation (1.11). That is, is there an equivalent to the QM relation²⁸

$$\langle \psi, P\chi \rangle = \langle P^\dagger \psi, \chi \rangle?$$

The answer is yes, and we call it the *codifferential*. We define it in the following claim.

Claim 1.3.3. Let $\alpha \in \Omega^p(\mathcal{M})$ and $\beta \in \Omega^{p-1}(\mathcal{M})$, then, if \mathcal{M} is a closed manifold,²⁹ the following is true

$$\langle \alpha, d\beta \rangle = \langle d^\dagger \alpha, \beta \rangle, \quad (1.12)$$

where $d^\dagger : \Omega^p(\mathcal{M}) \rightarrow \Omega^{p-1}(\mathcal{M})$ is defined as

$$d^\dagger := \pm(-1)^{np+n-1} \star d \star, \quad (1.13)$$

where $\dim \mathcal{M} = n$, and again \pm corresponds to Riemannian vs Lorentzian. We call d^\dagger the codifferential operator.

²⁸If this doesn't look familiar, see Dr. Schuller's QM course. REF NEEDED.

²⁹This basically takes care of boundary terms.

Exercise

Given that, on a closed manifold, Stoke's theorem tells us

$$\int_{\mathcal{M}} d\omega = 0,$$

where $\omega \in \Omega^{\dim \mathcal{M}-1}(\mathcal{M})$ (i.e. the integral of an exact form vanishes), prove the above claim.

Hint: If you get very stuck, Prof. Tong's GR notes might not be a bad place to look...

We note that it follows from $d^2 = 0$ and $\star^2 \omega \propto \omega$ that $(d^\dagger)^2 = 0$, explicitly,

$$(d^\dagger)^2 \omega = \star d \star (\star d \star \omega) \propto \star d^2 \star \omega = 0.$$

In analogy to the d case, we call a form *coclosed* if $d^\dagger \alpha = 0$ and *coexact* if $\alpha = d^\dagger \beta$, where $\alpha \in \Omega^p(\mathcal{M})$ and $\beta \in \Omega^{p-1}(\mathcal{M})$.

1.3.3 Hodge Decomposition & Harmonic Forms

Now comes the important bit, we start by introducing the following definition.

Definition. [Laplacian] We define the Laplacian for differential forms via

$$\Delta_d := (d + d^\dagger)^2 = dd^\dagger + d^\dagger d, \quad (1.14)$$

where the second line follows from $d^2 = (d^\dagger)^2 = 0$.

Definition. [Harmonic Forms] We call a p -form α *harmonic* iff $\Delta_d \alpha = 0$. We denote the space of harmonic p forms on \mathcal{M} by $\mathcal{H}^p(\mathcal{M})$.

Proposition 1.3.4. *A p -form on a closed manifold is harmonic iff it is both closed and coclosed.*

Proof. Firstly it is clear that if $d\alpha = d^\dagger \alpha = 0$ then $\Delta_d \alpha = 0$. Now we need to show the reverse: recalling Equation (1.12), consider

$$\begin{aligned} \langle \Delta_d \alpha, \alpha \rangle &= \langle dd^\dagger \alpha, \alpha \rangle + \langle d^\dagger d \alpha, \alpha \rangle \\ &= \langle d^\dagger \alpha, d^\dagger \alpha \rangle + \langle d\alpha, d\alpha \rangle \\ &= \|d^\dagger \alpha\|^2 + \|d\alpha\|^2 \end{aligned}$$

where the last line defines what we mean by $\|\alpha\|^2$. Each of these terms are positive (they're norm-squares), and so if $\Delta_d \alpha = 0$ it follows that $d^\dagger \alpha = d\alpha = 0$. \square

Note that $\Delta_d : \Omega^p(\mathcal{M}) \rightarrow \Omega^p(\mathcal{M})$. This motivates the following theorem (which we do not prove).

Theorem 1.3.5 (Hodge Decomposition). *We can decompose the space of p -forms as follows*

$$\Omega^p(\mathcal{M}) = \mathcal{H}^p(\mathcal{M}) \oplus \text{Im } d_{p-1} \oplus \text{Im } d_{p+1}^\dagger. \quad (1.15)$$

That is, any p -form ω can be uniquely written as

$$\omega = d\alpha + d^\dagger\beta + \gamma,$$

where $\alpha \in \Omega^{p-1}(\mathcal{M})$, $\beta \in \Omega^{p+1}(\mathcal{M})$ and $\gamma \in \mathcal{H}^p(\mathcal{M})$.

The Hodge decomposition theorem allows us to prove the following, important, Lemma.

Lemma 1.3.6. *For some smooth manifold \mathcal{M} we have the following isomorphism*

$$\mathcal{H}^p(\mathcal{M}) \cong H_{dR}^p(\mathcal{M}). \quad (1.16)$$

Proof. First let's show that a harmonic form gives an element in $H_{dR}^p(\mathcal{M})$. Firstly note that if $\gamma \in \mathcal{H}^p(\mathcal{M})$ then, as we have shown, $d\gamma = 0$. We now just need to show that it is not exact. This follows immediately from the fact that the Hodge decomposition is unique and so, since $\mathcal{H}^p(\mathcal{M}) \subseteq \Omega^p(\mathcal{M})$, we have $\gamma \in \Omega^p(\mathcal{M})$ and so it must *not* lie in $\text{Im } d_{p-1}$, that is $\gamma \neq d\beta$ for some $\beta \in \Omega^{p-1}(\mathcal{M})$.

Now let's show that every equivalence class in $H_{dR}^p(\mathcal{M})$ contains a harmonic form. That is we want to show that if $[\omega] \in H_{dR}^p(\mathcal{M})$ that $\omega = d\alpha + \gamma$ for some harmonic form γ . Well again comparing this to the Hodge decomposition, basically all we need to show is that $d^\dagger\beta = 0$. Well note that $d\omega = 0$, and then consider

$$0 = \langle d\omega, \beta \rangle = \langle dd^\dagger\beta, \beta \rangle = \langle d^\dagger\beta, d^\dagger\beta \rangle = \|d^\dagger\beta\|^2 \iff d^\dagger\beta = 0,$$

where we have used that $d^2\alpha = d\gamma = 0$ in the decomposition of ω . This proves surjectivity, and then injectivity follows from the fact that the decomposition is unique. We can equally show it by proving an exact form $d\alpha$ is mapped to $0 \in \mathcal{H}^p(\mathcal{M})$.³⁰ \square

1.4 Holonomy

We now take a step back from discussing differential forms and cohomology to introduce what is known as *holonomy*. There is a lot that can be said about holonomy³¹ but here we shall just give a brief definition and description of holonomy.

Definition. [Holonomy] Let \mathcal{M} be a smooth manifold equipped with some connection³² ∇ , and consider a $X \in T_p\mathcal{M}$. Now consider a closed smooth loop $\gamma : [0, 1] \rightarrow \mathcal{M}$ with $\gamma(0) = \gamma(1) = p$. Now consider parallel transporting X around γ , the result will be, in general, some other element $X' \in T_p\mathcal{M}$. As $T_p\mathcal{M}$ is an $(\dim \mathcal{M})$ -dimensional vector space, we know we can relate X and X' via some $GL(\dim \mathcal{M}, \mathbb{R})$ action, i.e. $P_\gamma \in GL(\dim \mathcal{M}, \mathbb{R})$ where P_γ denotes the parallel transport along γ . We then define the *holonomy group at $p \in \mathcal{M}$* to be

$$\text{Hol}_p(\nabla) := \{P_\gamma \in GL(\dim \mathcal{M}, \mathbb{R}) \mid \gamma \text{ is a loop based at } p \in \mathcal{M}\}. \quad (1.17)$$

This is a Lie group, where multiplication is given by composition and the inverse is given by running around the path in the opposite direction.

³⁰This is a nice additional exercise.

³¹And I might later end up producing some more detailed notes on the topic.

As the notation suggests, the holonomy is a property of the connection ∇ , and so changing the connection changes the holonomy. This is not surprising, as it is the connection that defines what we mean by parallel transport. If we consider a (pseudo-)Riemannian manifold, then we know that there exists a unique connection which is *metric compatible*, which we call the Levi-Civita connection. The condition of being metric compatible basically means that lengths are preserved under parallel transport and so our holonomy group clearly restricts to $\text{Hol}_p(\nabla^{LC}) \subseteq O(\dim \mathcal{M})$. If we further require our manifold to be orientable, then we get $SO(\dim \mathcal{M})$.

Now, as we have been careful to indicate, the holonomy seems to depend on the choice of base point $p \in \mathcal{M}$. Of course in general this is true, however if we have a connected manifold then any two points $p, q \in \mathcal{M}$ can be connected by some smooth path $\tau : [0, 1] \rightarrow \mathcal{M}$ with $\tau(0) = p$ and $\tau(1) = q$, and so we can relate the holonomies at these two points, simply by

$$\text{Hol}_q(\nabla) = P_\tau \text{Hol}_p(\nabla) P_\tau^{-1}.$$

This provides an isomorphism between $\text{Hol}_p(\nabla)$ and $\text{Hol}_q(\nabla)$ and so it allows us to really speak about the holonomy of the manifold \mathcal{M} itself. We denote this by $\text{Hol}(\nabla; \mathcal{M})$, although in what follows we will basically always drop the ∇ , but again it is important to remember that the holonomy depends crucially on the connection.

It is hopefully clear that the holonomy group of a manifold is related to the curvature of the manifold and that $\text{Hol}(\mathcal{M}) = 0$ iff the Riemann tensor vanishes (i.e. the manifold is flat). In fact we can make an even nicer statement than this. The holonomy group is in fact a Lie group, and so we can consider its Lie algebra. This is going to be a measure of the *local* curvature (as the Lie algebra can be thought of as the local action of a group). Putting this together with the fact that parallel transporting a vector around an infinitesimal parallelogram is given by the Riemann tensor, i.e. that locally

$$[\nabla_X, \nabla_Y](V) = R(X, Y)V,$$

we see that the Lie algebra of $\text{Hol}_p(\mathcal{M})$ is generated by the matrices $R(X, Y) \in \text{End}(T_p \mathcal{M})$ where X, Y run over the elements of $T_p \mathcal{M}$. This is basically the content of the Ambrose-Singer theorem, however we will not discuss it in more detail in these notes.

³²We assume the reader is familiar with the idea of a connection and parallel transport from a GR course, e.g.

2 | Middle Ground

We will now start to make our first contact with something that looks "complex" in our geometry. However, as we noted at the start of the notes, it is important to note that everything we discuss in this chapter equally applies to real manifolds as well as complex ones. As we will make clear shortly, the idea is that essentially all that follows are properties of complex *vector bundles*, and it is possible to make a complex vector bundle over a real manifold: simply make the fibres complex vector spaces.

2.1 Almost Complex Structures

As we just explained, the aim is make a complex vector bundle over our real¹ manifold. We do that as by introducing the following definitions.

Definition. [Linear Complex Structure] Let V be some vector space. We call a linear map $J : V \rightarrow V$ a *linear complex structure* if it squares to -1 , i.e. $J^2 = -\mathbb{1}_V$.

Definition. [Almost Complex Structure] Let \mathcal{M} be a smooth manifold, then an *almost complex structure* is a linear complex structure on each tangent space $T_p\mathcal{M}$. That is, an almost complex structure is a $(1,1)$ -tensor field which we view as a map $J : T\mathcal{M} \rightarrow T\mathcal{M}$ such that $J^2 = -1$. We call the pair (\mathcal{M}, J) an *almost complex manifold*.

Proposition 2.1.1. *Let \mathcal{M} be a real smooth manifold, then \mathcal{M} admits an almost complex structure only if the dimension of \mathcal{M} is even.*

Proof. It follows from $J^2 = -\mathbb{1}$ that $(\det J)^2 = (-1)^{\dim \mathcal{M}}$, but if \mathcal{M} is real then $(\det J)^2 > 0$, and so we must have $\dim \mathcal{M} = 2m$ for $m \in \mathbb{Z}$. \square

Essentially an almost complex structure turns our tangent spaces into complex vector spaces. In fact, if we define the complexified tangent bundle by

$$T_{\mathbb{C}}\mathcal{M} := T\mathcal{M} \otimes \mathbb{C} \cong \mathbb{C}^{2m} \quad \text{where} \quad \dim_{\mathbb{R}}(\mathcal{M}) = 2m,$$

then we can naturally extend the definition of J to $J : T_{\mathbb{C}}\mathcal{M} \rightarrow T_{\mathbb{C}}\mathcal{M}$. We then see that elements $X \in T_{\mathbb{C}}\mathcal{M}$ are eigenvectors of J with eigenvalues $\pm i$ (as $J^2 X = -X$), which allows us to decompose as follows

$$T_{\mathbb{C}}\mathcal{M} = T\mathcal{M}^{(1,0)} \oplus T\mathcal{M}^{(0,1)},$$

¹As we might expect, a complex manifold is a particular kind of real manifold of twice the dimension, and so these definitions also apply to complex manifolds. We will make this more concrete in the next chapter.

where $X \in T\mathcal{M}^{(1,0)}$ obeys $JX = +iX$ and similarly $X \in T\mathcal{M}^{(0,1)}$ obeys $JX = -iX$. We call $T\mathcal{M}^{(1,0)}$ the *holomorphic tangent bundle* and $T\mathcal{M}^{(0,1)}$ the *antiholomorphic tangent bundle*. Of course we can equally define the (anti)holomorphic cotangent bundles in the obvious way. From here we can construct complex tensor fields of any form.

Remark 2.1.2. As we mentioned right at the start of the notes, and as indicated by the chapter title, it *is* possible to define complex tensor fields on a real manifold, as we have just done. This is *not* the same thing as having a complex manifold. This is one of the reasons we have set the notes out like this. With this noted, we continue to discuss complex tensor fields and their related objects within this "middle ground" chapter. As we have already hinted at, it will turn out that all complex manifolds are almost complex (but clearly the reverse is not true), and so everything that follows is applicable to complex manifolds and indeed is where we will use it.

2.2 Complex Differential Forms & Dolbeault Cohomology

Now we said above that we can define complex tensors of higher degree once we have our (anti)holomorphic (co)tangent bundles. This is true, but we need to be a bit more careful when looking at the decompositions. The result is exactly what we would expect, and we clarify what we mean by considering the complexified version of a $(0, p)$ -tensor field: the decomposition is given by

$$\otimes^p T_{\mathbb{C}}^* \mathcal{M} = \bigoplus_{p=r+s} T^* \mathcal{M}^{(r,s)} = \bigoplus_{j=0}^p T^* \mathcal{M}^{(p,p-j)},$$

where

$$T^* \mathcal{M}^{(r,s)} := \otimes^r T^* \mathcal{M}^{(1,0)} \otimes^s T^* \mathcal{M}^{(0,1)}.$$

As an explicit example, we have that

$$T_{\mathbb{C}}^* \mathcal{M} \otimes T_{\mathbb{C}}^* \mathcal{M} = T^* \mathcal{M}^{(2,0)} \oplus T^* \mathcal{M}^{(1,1)} \oplus T^* \mathcal{M}^{(0,2)}.$$

Note that

$$\overline{T^* \mathcal{M}^{(p,q)}} \cong T^* \mathcal{M}^{(q,p)}, \quad (2.1)$$

where the bar indicates complex conjugation. We will use this relation in a moment when discussing Hodge numbers.

The reason we used the example of complex $(0, p)$ -fields is indicated in the title of this section: we want to now study complex differential forms and define the equivalent of de Rham cohomology for them.

2.2.1 Complex Differential Forms

Definition. $[(p, q)\text{-Form}]$ Let (\mathcal{M}, J) be an almost complex smooth manifold. We then define a (p, q) -form to be an element of

$$\Omega^{p,q}\mathcal{M} := \Gamma(\Lambda^{p,q}\mathcal{M}), \quad \text{where} \quad \Lambda^{p,q}\mathcal{M} = \Lambda^p T^*\mathcal{M}^{(1,0)} \otimes \Lambda^q T^*\mathcal{M}^{(0,1)}.$$

It follows from above that we have the decomposition of complex p -forms via

$$\Lambda^p T_{\mathbb{C}}^*\mathcal{M} = \bigoplus_{j=0}^p \Lambda^{j,p-j}\mathcal{M}. \quad (2.2)$$

Next up we need the equivalent of our exterior derivative d . This also follows from a decomposition $d = \partial + \bar{\partial}$ where we have defined

$$\partial : \Omega^{p,q}\mathcal{M} \rightarrow \Omega^{p+1,q}\mathcal{M} \quad \text{and} \quad \bar{\partial} : \Omega^{p,q}\mathcal{M} \rightarrow \Omega^{p,q+1}\mathcal{M}.$$

Exercise

Using $d^2 = 0$ show that

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0. \quad (2.3)$$

Hint: Expand out d^2 and then look at what maps to where.

As it will be useful later (i.e. when constructing the so-called Kähler potential), we introduce another real operator $d^c : \Omega_{\mathbb{C}}^k(\mathcal{M}) \rightarrow \Omega_{\mathbb{C}}^{k+1}(\mathcal{M})$ defined by

$$d^c := i(\bar{\partial} - \partial),$$

which obeys

$$(d^c)^2 = 0, \quad dd^c + d^c d = 0, \quad \partial = \frac{1}{2}(d + id^c), \quad \bar{\partial} = \frac{1}{2}(d - id^c), \quad \text{and} \quad dd^c = 2i\partial\bar{\partial},$$

which are left as an exercise for the reader to prove.

2.2.2 Dolbeault Cohomology

Now we note that Equation (2.3) tells us, in particular, that $\bar{\partial}^2 = 0$, and so we can form the following chain complex

$$0 \xrightarrow{\bar{\partial}} \Omega^{p,0}\mathcal{M} \xrightarrow{\bar{\partial}} \Omega^{p,1}\mathcal{M} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{p,m}\mathcal{M} \xrightarrow{\bar{\partial}} 0,$$

which in turn lets us define a cohomology, known as the *Dolbeault cohomology*

$$H_{\bar{\partial}}^{p,q}(\mathcal{M}) = \frac{\ker(\bar{\partial} : \Omega^{p,q}(\mathcal{M}) \rightarrow \Omega^{p,q+1}(\mathcal{M}))}{\text{Im}(\bar{\partial} : \Omega^{p,q-1}(\mathcal{M}) \rightarrow \Omega^{p,q}(\mathcal{M}))}. \quad (2.4)$$

We make two immediate comments

- (i) We could just as easily have defined $H_{\partial}^{p,q}(\mathcal{M})$, as $\partial^2 = 0$.
- (ii) The Dolbeault cohomology depends on the almost complex structure J . This follows because it is J that allows us to decompose out complex tangent space, and so define what we mean by (p, q) .

2.2.3 Hodge Numbers

Definition. [Hodge Numbers] Let (M, J) be an almost complex smooth manifold with Dolbeault cohomology $H_{\partial}^{p,q}(\mathcal{M})$. We define the *Hodge numbers* to be

$$h^{p,q} := \dim_{\mathbb{C}} H_{\partial}^{p,q}(\mathcal{M}). \quad (2.5)$$

We often display Hodge numbers in a *Hodge Diamond* (where $\dim_{\mathbb{R}}(\mathcal{M}) = 2m$)

$$\begin{array}{ccccc}
 & & h^{m,m} & & \\
 & & \vdots & & \\
 & h^{m,m-1} & & h^{m-1,m} & \\
 h^{m,0} & \dots & & \dots & h^{0,m} \\
 & h^{1,0} & \vdots & h^{0,1} & \\
 & & h^{0,0} & &
 \end{array}$$

This seems like a lot, however the $(m+1)^2$ Hodge numbers are not independent. The relations depend on the type of manifold we are considering and what structures it has, but we notice already that Equation (2.1) tells us that complex conjugation of the tangent spaces gives us $h^{p,q} = h^{q,p}$. The Hodge star operator (which acts as we might imagine, namely $\star : \Omega^{p,q} \rightarrow \Omega^{m-p,m-q}$) also tells us that $h^{p,q} = h^{m-p,m-q}$.

2.2.4 de Rham Dolbeault Relation

Hodge numbers are clearly just the complex version of Betti numbers, but how are the two related? We can seek to an answer to this by recalling Equation (1.16), i.e. that there is an isomorphism between harmonic k -forms and the k -th de Rham cohomology. Why is this helpful? Well, if we complexify our harmonic k -forms, we can relate them, via Equation (2.2), to (p, q) -forms. If we can then obtain the (p, q) -form equivalent of Equation (1.16), we should be able to relate the de Rham and Dolbeault cohomologies. That is, we want to define harmonic (p, q) -forms, show they are isomorphic to $H_{\partial}^{p,q}(\mathcal{M})$. We do just this now.

We start by defining the formal adjoints of our $\partial/\bar{\partial}$ operators. This is done exactly as we might expect, namely

$$\partial^{\dagger} := \mp \star \partial \star \quad \text{and} \quad \bar{\partial}^{\dagger} := \mp \star \bar{\partial} \star, \quad (2.6)$$

where we note that the \mp factor comes from the fact that complex manifolds are always even dimensional, and so we have $(-1)^{pn+n-1} = -1$ for any p .

Exercise

Derive Equation (2.6) using $d^\dagger = \mp \star d \star$ and $d = \partial + \bar{\partial}$.

Next recall that a harmonic forms was defined via $\Delta_d \omega = 0$, where $\Delta_d = dd^\dagger + d^\dagger d$. With this in mind, we define the two Laplacians

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial \quad \text{and} \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial},$$

which we can easily check satisfy $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$. It is clear that we have two ways to define harmonic (p, q) -forms: namely w.r.t. Δ_∂ and $\Delta_{\bar{\partial}}$. As we defined our Dolbeault cohomology in relation to $\bar{\partial}$, we proceed with using the $\Delta_{\bar{\partial}}$ Laplacian, which we simply relabel Δ to lighten notation.

Definition. [Harmonic (p, q) -Form] A (p, q) -form $\omega \in \Omega^{p,q}(\mathcal{M})$ is called a *harmonic* (p, q) -form if $\Delta\omega = 0$. We denote the space of harmonic (p, q) -forms on \mathcal{M} by $\mathcal{H}^{p,q}(\mathcal{M})$.

Everything now follows through analogously to Hodge theory: i.e. we get the decomposition

$$\Omega^{p,q}(\mathcal{M}) = \mathcal{H}^{p,q}(\mathcal{M}) \oplus \text{Im } \bar{\partial}_{q-1} \oplus \text{Im } \bar{\partial}_{q+1},$$

and the isomorphism

$$\mathcal{H}^{p,q}(\mathcal{M}) \cong H_{\bar{\partial}}^{p,q}(\mathcal{M}).$$

So if we finally define $\mathcal{H}_{\mathbb{C}}^k(\mathcal{M})$ to be the space of complex harmonic k -forms, i.e. $\mathcal{H}_{\mathbb{C}}^k(\mathcal{M}) := \ker(\Delta_d : \Omega_{\mathbb{C}}^k(\mathcal{M}) \rightarrow \Omega_{\mathbb{C}}^k(\mathcal{M}))$, and recalling Equation (2.2), we finally obtain

$$\mathcal{H}_{\mathbb{C}}^k = \bigoplus_{j=0}^k \mathcal{H}^{j,k-j},$$

which in turn (using the related isomorphisms) gives

$$H_{dR}^k(\mathcal{M}; \mathbb{C}) = \bigoplus_{j=0}^k H_{\bar{\partial}}^{j,k-j}(\mathcal{M}),$$

where $H_{dR}^k(\mathcal{M}; \mathbb{C})$ is just the k -th de Rham cohomology w.r.t. complex forms.

It then follows from this decompositions that

$$b^k = \sum_{j=0}^k h^{j,k-j}. \quad (2.7)$$

Therefore, we can use the Hodge numbers to ask questions about the number of holes in our manifold. We can also, therefore, relate the Euler characteristic, Equation (1.9), to the Hodge numbers

$$\chi = \sum_{n=0}^{\dim \mathcal{M}} (-1)^n \sum_{j=0}^n h^{j,n-j}.$$

We can therefore calculate the Euler characteristic by adding up the rows of the Hodge diamond with alternating sign.

2.3 Chen Classes

So far we have just introduced the complex versions of the structures/operators on our real vector bundles to our complex vector bundles. We now want to introduce something very important that doesn't have a real vector bundle equivalent.

Definition. [Chern Class] Let (E, π, \mathcal{M}) be a complex vector bundle, and let A be the connection on E with associated curvature 2-form $F = dA + A \wedge A$ be the curvature 2-form.² Then we define the *total Chern class* of E as

$$c(E) := \det \left(1 + \frac{i}{2\pi} F \right) \quad (2.8)$$

Now, we note that F is a 2-form, and remembering that top forms are a thing, we see that if we expand Equation (2.8), the sequence will terminate at the top form contribution. That is, if the complex rank of E is k , then we can only have k powers of F before we get a vanishing result.³ We therefore get the *Chern classes*, defined via

$$c(E) = c_0(E) + c_1(E) + c_2(E) + \dots + c_k(E).$$

Remark 2.3.1. Actually, we should be a little more technically correct: the $c_i(E)$ we defined above are the *Chern forms*, the Chern classes are given by the cohomology classes of the Chern forms. This is why we put $[\cdot]$ around out expressions below: Chern classes are equivalence classes. However, in these notes we will use Chern form and Chern classes interchangeably.

Considering the standard results for the Taylor expansion of a determinant near the identity, we see from Equation (2.8) that

$$\begin{aligned} c_0(E) &= [1], \\ c_1(E) &= \left[\frac{1}{2\pi i} \text{Tr } F \right], \\ c_2(E) &= \left[\frac{1}{2} \left(\frac{i}{2\pi} \right)^2 (\text{Tr } F \wedge \text{Tr } F - \text{Tr}(F \wedge F)) \right] \\ &\vdots \\ c_k(E) &= \left[\left(\frac{i}{2\pi} \right)^k \det F \right]. \end{aligned} \quad (2.9)$$

We state the following claim without proof⁴

²More details on what this is if not familiar can be found in Dr. Schuller's Geometrical Anatomy course. **Note to self, if get time, would be good revision to type up stuff about this.**

³At first this might seem wrong, as F is a *two*-form and then surely we should only be able to have $k/2$ of them. However we must remember that E is a *complex* vector bundle and so F is a complex 2-form, and so we can have k powers of it, namely a (k, k) -form.

⁴This can be shown by considering the properties of the determinants of a short exact sequence.

Claim 2.3.2. Given a short exact sequence⁵ of complex vector bundles $0 \rightarrow E_1 \rightarrow E \rightarrow E_2$, the total Chern class obeys

$$c(E) = c(E_1)c(E_2). \quad (2.10)$$

2.3.1 First & Top Chern Classes

For reason that will become clearer later, the most important Chern classes for us will be the first Chern class $c_1(E)$ and the top Chern class $c_k(E)$. Let's discuss briefly why this is the case in turn.

First Chern Class

The first Chern class is a 2-form on E . Now, it turns out that if E is the complex tangent bundle of \mathcal{M} then the curvature 2-form F is actually (proportional to) the *Ricci curvature* R , more specifically, if we're dealing with the holomorphic tangent bundle, $F = -iR$. We therefore see that the first Chern class of the holomorphic tangent bundle is determined by the Ricci curvature⁶

$$c_1(T^{(1,0)}\mathcal{M}) \equiv c_1(\mathcal{M}) = \left[\frac{1}{2\pi} R \right]. \quad (2.11)$$

We see immediately from this that if the manifold is Ricci flat, i.e. $R = 0$, then the first Chern class vanishes. The natural question to ask is "does the reverse hold?" i.e. does the vanishing of the first Chern class tell us the manifold is Ricci flat. The answer is, in general, no however if \mathcal{M} is a certain type of complex manifold, namely a Kähler manifold (which we will discuss shortly), then it turns out to be true. This is the content of the Calabi conjecture, which was proved by Yau twenty years later. We will discuss this more later.

Top Chern Class

The top Chern class is a top form on E . If we again consider $E = T^{(1,0)}(\mathcal{M})$, then we note that $\dim_{\mathbb{R}} T^{(1,0)}\mathcal{M} = \dim_{\mathbb{R}} \mathcal{M}$, we see that we can integrate the top Chern class over \mathcal{M} itself.⁷ It turns out this top form in \mathcal{M} is actually what is known as the *Euler form*, and integrating it over \mathcal{M} gives you the Euler characteristic. That is (if $\dim_{\mathbb{R}} \mathcal{M} = 2m$)

$$\chi = \int_{\mathcal{M}} c_m(\mathcal{M}) \quad (2.12)$$

⁵It is assumed the reader knows what a short exact sequence is. If not see any decent differential geometry textbook.

⁶Here we define what we mean by $c_i(\mathcal{M})$, namely if we just write $c_i(\mathcal{M})$ we mean the i -th Chern class of the holomorphic tangent bundle.

⁷More technically we pullback the top Chern form on E to a top form on \mathcal{M} .

2.3.2 Chern Character

Before moving on to discuss projective spaces (in order to set up a nice discussion of complex projective spaces later), we introduce a useful quantity associated to Chern classes.

Definition. [Chern Character] Let E be a complex vector bundle of rank r , and express the total Chern class via $c(E) = \prod_{i=1}^r (1 + x_i)$. We define the *Chern character* to be

$$ch(E) := \sum_{i=1}^r e^{x_i}.$$

Now the Chern character seems like a strange thing to define, however we now note that it has the two nice properties that

$$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2) \quad \text{and} \quad ch(E_1 \otimes E_2) = ch(E_1)ch(E_2).$$

Next we note that if we have a complex line bundle L , then L has rank 1 and so our Chern class, as defined above, is simply $c(L) = (1 + x_1)$, but we can compare this to $c(L) = 1 + c_1(L)$ and conclude that $x_1 = c_1(L)$. We therefore have that

$$ch(L) = e^{c_1(L)} = \sum_{\ell=0}^{\infty} \frac{c_1(L)^\ell}{\ell!}$$

Now, it follows from the expressions above that if E is given by the direct sum of r line bundles $\{L_1, \dots, L_r\}$ then we have

$$ch(E) = ch(L_1 \oplus \dots \oplus L_r) = ch(L_1) + \dots + ch(L_r) = e^{c_1(L_1)} + \dots + e^{c_1(L_r)}.$$

If we now compare this to the fact that $ch(E) = \sum_i e^{x_i}$ when $c(E) = \prod_i (1 + x_i)$ we see that

$$c(L_1 \oplus \dots \oplus L_r) = (1 + c_1(L_1)) \dots (1 + c_1(L_r)),$$

and in particular

$$c(L^{\oplus r}) = (1 + c_1(L))^r \tag{2.13}$$

This is the nice result from the Whitney sum of line bundles, but we also have a nice result from the tensor product of line bundles as follows. If $E = L_1 \otimes \dots \otimes L_r$, then

$$ch(E) = ch(L_1 \otimes \dots \otimes L_r) = ch(L_1) \dots ch(L_r) = e^{x_1} \dots e^{x_r} = e^{x_1 + \dots + x_r},$$

where $x_i = c_1(L_i)$. Now comes the interesting bit: this is still a line bundle, as $\dim(V \otimes W) = \dim V \times \dim W$, so we can compare it to $ch(L) = e^{c_1(L)}$ and conclude that

$$c_1(L_1 \otimes \dots \otimes L_r) = c_1(L_1) + \dots + c_1(L_r).$$

What will be of particular use to us when trying to construct so-called Calabi-Yau manifolds later will be the specific case of this result

$$c_1(L^{\otimes d}) = 1 + dc_1(L), \tag{2.14}$$

where L is some line bundle.

3 | Projective Spaces

We now wish to talk about projective spaces. The reason we do this shall become much clearer when we talk about trying to construct our mysterious Calabi-Yau manifolds,¹ but we introduce them now in order to be able to talk about how all the above stuff applies to them, in particular their Chern classes.

3.1 Definition

Definition. [Projective Space] Let V be some vector space over a field K . We define the *projective space of V* , denoted $\mathbb{P}(V)$, as the equivalence classes of $V \setminus \{0_V\}$ where the equivalence relation is

$$v \sim w \iff v = \lambda w,$$

where $\lambda \in K^*$, i.e. a non-zero element of K .

Exercise

Convince yourself that $\dim \mathbb{P}(V) = (\dim V) - 1$

Claim 3.1.1. If V is a topological space, then we can make $\mathbb{P}(V)$ a topological space, using the so-called quotient topology.²

In the cases when $V = K^{n+1}$, we denote the projective space simply as $K\mathbb{P}^n$. In this case, we often denote an element of $K\mathbb{P}^n$ by $[k_0 : \dots : k_n]$, where of course $k_i \in K$. Of particular interest to us is going to

$$\mathbb{CP}^n := \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid (z_0, \dots, z_n) = (\lambda z_0, \dots, \lambda z_n), \text{ for } \lambda \in \mathbb{C}^*\}.$$

We are going to use \mathbb{CP}^n as a proxy for all that follows, as it allows us to be a bit more concrete. It should be reasonably clear how the definitions etc. that follow generalise to a generic projective space.

Although we have not formally introduced complex manifolds yet, it is hopefully obvious that \mathbb{C}^{n+1} is a complex manifold of complex dimension $(n+1)$, and so we conclude that \mathbb{CP}^n is a complex manifold of dimension n . If this is uncomfortable for any reason, we can think

¹As a semi-spoiler, we will construct them as hypersurfaces in complex projective spaces.

²Details of what this means can be found in Dr. Schuller's Geometrical Anatomy course.

of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)}$ and consider a real manifold instead. The charts in \mathbb{CP}^n are given by the open sets

$$U_i := \{[z_0, \dots, z_n] \mid z_i \neq 0\} \subset \mathbb{CP}^n.$$

It is clear that the set $\mathcal{U} := \{U_i \mid i = 0, \dots, n\}$ forms an open cover of \mathbb{CP}^n . Finally, we call the coordinates (z_0, \dots, z_n) the *homogeneous coordinates* of \mathbb{CP}^n . We will often use z to denote the full set (z_0, \dots, z_n) just to lighten notation.

3.2 Tautological & Hyperplane Line Bundles

We now want to construct two very important types of line bundle, defined on projective spaces. We give the definitions in a wordy manor (to avoid being too abstract) but of course they can be written down very concretely.

Definition. [Tautological Line Bundle] Consider the complex projective space \mathbb{CP}^n . There is a "natural" line bundle we can construct over this: namely attach to each point $[z_1 : \dots : z_{n+1}] \in \mathbb{CP}^n$ the line "projected away", i.e. the line given by $\pi^{-1}([z_0 : \dots : z_n]) = \{(\lambda z_0, \dots, \lambda z_n) \mid \lambda \in \mathbb{C}^*\} \subset \mathbb{C}^{n+1}$. This is known as the *tautological* (or *universal*) *line bundle*, and we denote it by $\mathcal{O}_{\mathbb{CP}^n}(-1)$.

Recalling that we can define a bundle via its open cover, fibres and transition functions, we give the following definition.

Definition. [Hyperplane Line Bundle] Consider the complex projective space \mathbb{CP}^n with tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$. Then we define the *hyperplane line bundle* to be its dual, i.e. is a bundle over \mathbb{CP}^n where the fibres are the dual space³ $(\mathbb{C} \setminus \{0\})^*$. The transition functions are given by $g_{ij} : U_i \cap U_j \rightarrow z_i/z_j$, where $U_i, U_j \in \mathcal{U}$. That is $g_{ij}([z]) = \frac{z_i}{z_j}[z]$. We denote the hyperplane line bundle by $\mathcal{O}_{\mathbb{CP}^n}(1)$.

Remark 3.2.1. If it's not clear why the hyperplane line bundle is in fact a line bundle, note that the fibres are the duals of the complex numbers. That is an element of the fibres of $\mathcal{O}_{\mathbb{CP}^n}(1)$ is a linear functional

$$\varphi : \mathbb{C}^* \rightarrow \mathbb{C}$$

which clearly just acts as $\varphi(z) = az$ for some $a \in \mathbb{C}$, but the basis of such a space is clearly just $\varphi = 1$, and so the fibres are one-dimensional.

Remark 3.2.2. As a technical aside, we have been a little sloppy with notation above. We denoted the tautological/hyperplane line bundles themselves using the $\mathcal{O}(\pm 1)$ notation. Really we should just use L/L^{-1} , and then $\mathcal{O}(\pm 1)$ denotes the sheaf of holomorphic sections⁴ $\Gamma(L)/\Gamma(L^{-1})$. However this is standard notation, and we shall use $\mathcal{O}(\pm 1)$ to denote both the bundle itself and sections of the bundle, with the understanding following from context.

³The asterisk here means dual, whereas the asterisk in the previous definition meant $\setminus \{0\}$. Annoying, I know, but we get the idea.

⁴As we're being more technical here, a holomorphic section obviously only makes sense if the base space is a complex manifold. This is simply because a section is a map $\sigma : \mathcal{M} \rightarrow E$, and so if we want to ask the question of "is it holomorphic", clearly \mathcal{M} and E must both be complex.

Recalling that the product of line bundles is again a line bundle, we introduce the notation

$$\mathcal{O}_{\mathbb{CP}^n}(d) := \otimes^d \mathcal{O}_{\mathbb{CP}^n}(1) \quad \text{and} \quad \mathcal{O}_{\mathbb{CP}^n}(-d) := \otimes^d \mathcal{O}_{\mathbb{CP}^n}(-1).$$

Now comes an important proposition that we will use later.

Proposition 3.2.3. *Any homogeneous polynomial of degree k in \mathbb{CP}^n can be canonically identified with the holomorphic sections $\mathcal{O}_{\mathbb{CP}^n}(k)$.*

Proof. Consider a polynomial of degree k in the homogeneous coordinates (z_0, \dots, z_n)

$$P_k(z) = \sum_{|\nu|=k} a_\nu z_0^{\nu_0} \dots z_n^{\nu_n},$$

where $a_\nu \in \mathbb{C}$, and the notation is hopefully understood (i.e. $\nu_0 + \dots + \nu_n = k$, but we can have a sum of different individual ν_i values). Now this is *not* a polynomial in \mathbb{CP}^n as it isn't scale invariant, i.e. $P_k(\lambda z) = \lambda^k P_k(z)$ but we want $P_k(\lambda z) = P_k(z)$. This is easily fixed by considering one of the charts $U_i \in \mathcal{U}$: we then simply divide by z_i^k , which we now write in a suggestive manner

$$s_i \equiv \frac{P_k(z)}{z_i^k} = \sum_{|\nu|=k} a_\nu \left(\frac{z_0}{z_i} \right)^{\nu_0} \dots \left(\frac{z_n}{z_i} \right)^{\nu_n}.$$

Now this is only defined on U_i (as this is where we are guaranteed $z_i \neq 0$), but it is hopefully clear that we can get a globally defined polynomial by patching together the different s_i by multiplying by $(z_i/z_j)^k$ on the overlap $U_i \cap U_j$. However we now notice that this is simply k times the hyperplane line bundle's transition functions $g_{ij} : U_i \cap U_j \rightarrow z_i/z_j$, so

$$s_j = g_{ij}^{-k} s_i.$$

We can therefore think of the global polynomial as a section of $\mathcal{O}_{\mathbb{CP}^n}(k)$. This map is clearly bijective, as an element of $\mathcal{O}_{\mathbb{CP}^n}(k)$ is a linear functional from⁵ $\mathbb{C}^k \rightarrow \mathbb{C}$, but this is basically the definition of a polynomial of degree k in \mathbb{CP}^n , which proves the proposition. \square

There is now an important Lemma associated to the proposition above.

Lemma 3.2.4. *The homogeneous coordinates of \mathbb{CP}^n can be identified as sections of the hyperplane line bundle.*

3.3 Chern Classes

We now want to find the Chern classes of \mathbb{CP}^n , the question is how do we do this? We start by clarifying what a vector field in $T^{(1,0)}\mathbb{CP}^n$ is, and in particular what a zero vector is here.

Recall that \mathbb{CP}^n is defined to be the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\lambda \in \mathbb{C}$. We can define this in terms of a projection $\pi(z) = [z]$, i.e. the fibres are given by the lines we project away. Now we can define a vector field in $T^{(1,0)}\mathbb{CP}^n$ by *pushing down*⁶ a vector \tilde{X} in $T(\mathbb{C}^{n+1} \setminus \{0\})$. That

⁵Note \mathbb{C}^k is fine as our polynomials s_i only have d variables: the term with z_i in $P_k(z)$ becomes 1 in s_i .

⁶To those unfamiliar, a push forward is basically a map that allows us to take a vector field from one space to another space given a map between the two. That is if $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ is our map then we get a map $\varphi_* : T_p\mathcal{M} \rightarrow T_{\varphi(p)}\mathcal{N}$ by $(\varphi_*X)(f) = X(f \circ \varphi)$.

is, consider some open subset $U \in \mathbb{CP}^n$, then we have an open subset in $\mathbb{C}^{n+1} \setminus \{0\}$ given by $\pi^{-1}(U)$. We define our \tilde{X} vector field over $\pi^{-1}(U)$, and then get a vector field over U as $X([z]) := \pi_{*z}\tilde{X}(z) = \pi_{*\lambda z}\tilde{X}(\lambda z)$, where the second equality is our projective condition.

Now we want to ask the question of "what is a zero vector in \mathbb{CP}^n ?" Well, it follows from above that $X([z]) = 0$ when $\pi_{*z}\tilde{X}(z) = 0$, but what is such a vector? Well, thinking geometrically, if \tilde{X} "points up the fibre" then the projection will shrink it down to a vanishing vector. In other words, if we split the tangent space at $z \in \mathbb{C}^{n+1} \setminus \{0\}$ into a vertical and horizontal subspace, defined precisely as

$$V_z(\mathbb{C}^{n+1} \setminus \{0\}) := \ker \pi_{*z},$$

and then $H_z(\mathbb{C}^{n+1} \setminus \{0\})$ defined as the rest, then if $\tilde{X}(z) \in V_z(\mathbb{C}^{n+1} \setminus \{0\})$, we get our zero vector $X([z]) = 0$. Finally, recalling that the fibres are given by scaling the point $[z] \in \mathbb{CP}^n$, we see that our zero vectors are given by the push downs of

$$\tilde{X} = \lambda \left(z_0 \frac{\partial}{\partial z_0} + \dots + z_n \frac{\partial}{\partial z_n} \right) = \lambda V_E,$$

where we have defined the *Euler vector field* $V_E := z_i \partial_{z_i}$, and where $\lambda \in \mathbb{C}$.

Ok why is this useful to us? Well we note that we can span our holomorphic tangent bundle $T^{(1,0)}\mathbb{CP}^n$ by the push downs of the vectors $\{s_i(z) \frac{\partial}{\partial z_i}\}$, where $s_i(z)$ is a section in $\mathcal{O}_{\mathbb{CP}^n}(1)$. In this way, we can define a surjective mapping

$$\varphi : \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \rightarrow T^{(1,0)}\mathbb{CP}^n,$$

where surjectivity is understood as we can produce a basis of $T^{(1,0)}\mathbb{CP}^n$. However we have just seen that the kernel of this map is the trivial line bundle \mathbb{C} (i.e. the λ appearing in front of V_E), and we can embed this into $\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}$, giving us the short exact sequence⁷

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \xrightarrow{\varphi} T^{(1,0)}\mathbb{CP}^n \longrightarrow 0,$$

and then recalling Equation (2.10) we have

$$c(\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}) = c(\mathbb{C}) \cdot c(T^{(1,0)}\mathbb{CP}^n).$$

Finally, using that trivially $c(\mathbb{C}) = 1$ and Equation (2.13) with $x = c_1(\mathcal{O}_{\mathbb{CP}^n}(1))$, we conclude (using $c(\mathbb{CP}^n) \equiv c(T^{(1,0)}\mathbb{CP}^n)$)

$$c(\mathbb{CP}^n) = (1 + x)^{n+1}. \quad (3.1)$$

3.3.1 Sum Of \mathbb{CP}^n s

We can slightly generalise the result above, by now considering the whole thing again but now over a sum of complex projective spaces. In other words our base space becomes

$$\mathbb{CP}^{n_1} \oplus \dots \oplus \mathbb{CP}^{n_\ell}.$$

⁷This is an example of an *Euler sequence*.

Basically the whole thing is completely analogous, however now our middle term in the sequence is

$$\mathcal{O}_{\mathbb{CP}^{n_1}}(1)^{\oplus(n_1+1)} \oplus \dots \oplus \mathcal{O}_{\mathbb{CP}^{n_\ell}}(1)^{\oplus(n_\ell+1)},$$

and similarly the holomorphic tangent space term changes. However clearly the expression above is just a Whitney sum of line bundles and so we have

$$c\left(\mathcal{O}_{\mathbb{CP}^{n_1}}(1)^{\oplus(n_1+1)} \oplus \dots \oplus \mathcal{O}_{\mathbb{CP}^{n_\ell}}(1)^{\oplus(n_\ell+1)}\right) = \prod_{i=1}^{\ell} (1 + x_i)^{n_i+1},$$

where $x_i = c_1(\mathcal{O}_{\mathbb{CP}^{n_i}}(1))$, and so we conclude

$$c(\mathbb{CP}^{n_1} \oplus \dots \oplus \mathbb{CP}^{n_\ell}) = \prod_{i=1}^{\ell} (1 + x_i)^{n_i+1}. \quad (3.2)$$

3.4 Weighted Projective Spaces

We conclude this chapter by discussing *weighted* projective spaces. These are basically exactly the same as "regular" projective spaces, but now each homogeneous coordinate has its own weight under scaling. That is, the weighted projective space $\mathbb{WCP}^{(k_0, \dots, k_n)}$ is defined the same as a projective space but now with equivalence relation

$$[z_0 : \dots : z_n] = [\lambda^{k_0} z_0 : \dots : \lambda^{k_n} z_n].$$

It is common to write a weighted projective space as \mathbb{WCP}^n and then stating the weights as an $(n+1)$ -tuple, i.e. we write " \mathbb{WCP}^n with weights (k_0, \dots, k_n) ". We sometimes also use the notation $\mathbb{WCP}_{k_0, \dots, k_n}^n$. We will likely use a combination of all of these.

As we might expect, $\mathbb{WCP}_{k_0, \dots, k_n}^n$ and \mathbb{CP}^n have a lot in common, however stuff is more subtle in the former. For example, let's consider trying to define a polynomial of degree d in $\mathbb{WCP}_{k_0, \dots, k_n}^n$. Let's illustrate why stuff is more subtle with an example.

Example 3.4.1. Consider $\mathbb{WCP}_{1,2}^2$. Let's define the polynomial

$$P(z_0, z_1) = z_0^2 z_1 + z_0^3,$$

this would be a polynomial of degree 3 in \mathbb{CP}^2 , but for $\mathbb{WCP}_{1,2}^2$ we have

$$P(\lambda z_0, \lambda z_1) = (\lambda z_0)^2 (\lambda^2 z_1) + (\lambda z_0)^3 = \lambda^4 z_0^2 z_1 + \lambda^3 z_0^3 \neq \lambda^d P(z_0, z_1).$$

Definition. [Quasihomogeneous Polynomial] We call a polynomial in $\mathbb{WCP}_{k_0, \dots, k_n}^n$ *quasi-homogeneous of degree d* if

$$P(\lambda z_0, \dots, \lambda z_n) = \lambda^d P(z_0, \dots, z_n)$$

for some $d \in \mathbb{N}$.

Now, we can still define the tautological line bundle over $\mathbb{WCP}_{k_0, \dots, k_n}^n$ as the line bundle with fibres

$$\pi^{-1}[z_0 : \dots : z_n] = (\lambda^{k_0} z_0, \dots, \lambda^{k_n} z_n).$$

We denote the space of holomorphic sections of this space by $\mathcal{O}_{\mathbb{WCP}_{k_0, \dots, k_n}^n}(-1)$. We then similarly have the hyperplane line bundle $\mathcal{O}_{\mathbb{WCP}_{k_0, \dots, k_n}^n}(1)$, given by the dual of the above. Note that the transition functions themselves are the same as before, i.e. $g_{ij} : U_i \cap U_j \rightarrow z_i/z_j$. The change comes by adapting Proposition 3.2.3:

Proposition 3.4.2. *Any quasihomogeneous polynomial of degree d in $\mathbb{WCP}_{k_0, \dots, k_n}^n$ can be canonically identifies with the holomorphic sections $\mathcal{O}_{\mathbb{WCP}_{k_0, \dots, k_n}^n}(d)$.*

The proof follows completely analogously to that of Proposition 3.2.3, however now Lemma 3.2.4 changes to

Lemma 3.4.3. *The homogeneous coordinate z_i of $\mathbb{WCP}_{k_0, \dots, k_n}^n$ can be identified as sections of $\mathcal{O}_{\mathbb{WCP}_{k_0, \dots, k_n}^n}(z_i)$.*

This is easily understood as $P_i([z]) = z_i$ is a quasihomogeneous polynomial of degree k_i . Indeed we can understand Proposition 3.2.3 and Lemma 3.2.4 simply as specialisations of the above with $k_1 = \dots = k_n = 1$.

We now proceed as before, and we arrive at an Euler sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{\mathbb{WCP}_{k_0, \dots, k_n}^n}(k_i)^{\oplus(\sum i)} \longrightarrow T^{(1,0)}\mathbb{WCP}_{k_0, \dots, k_n}^n \longrightarrow 0,$$

from which, recalling $c(\mathcal{O}_{\mathbb{WCP}_{k_0, \dots, k_n}^n}(k_i)) = (1 + k_i x_{k_0, \dots, k_n})$, we conclude that

$$c(\mathbb{WCP}_{k_0, \dots, k_n}^n) = \prod_i (1 + k_i x_{k_0, \dots, k_n}) \quad (3.3)$$

where, of course, $x_{k_0, \dots, k_n} = c_1(\mathcal{O}_{\mathbb{WCP}_{k_0, \dots, k_n}^n}(1))$.

As we will see later, the weightings in $\mathbb{WCP}_{k_0, \dots, k_n}^n$ give rise to some rather interesting behaviour, in particular we will get singularities!

4 | Complex Manifolds

We now finally want to actually define a complex manifold. The initial definition will, likely, not be surprising, however we will then redefine it in terms of stuff we introduced above.

4.1 Definition

Definition. [Complex Manifold] A *complex manifold* is a manifold \mathcal{M} of real dimension $2m$, but where our charts are now homeomorphic to \mathbb{C}^m , i.e. we have chart maps $\psi_i : U_i \rightarrow \mathbb{C}^m$, with $\{U_i\}$ being an open cover of \mathcal{M} . To get a smooth complex manifold, we further require that our chart transition maps $\psi_{ij} := \psi_i \circ \psi_j^{-1} : \psi_j(U_i \cap U_j) \rightarrow \psi_i(U_i \cap U_j)$ are holomorphic maps from \mathbb{C}^m to \mathbb{C}^m . We call \mathcal{M} a complex manifold of dimension m .

We can obviously view a complex manifold of dimension m as a particular kind of real manifold of dimension $2m$. Specifically, if \mathcal{M} is a m -dimensional complex manifold, and we express the chart transition functions as $\psi_{ij} = u_{ij} + iv_{ij}$ with u_{ij} and v_{ij} being real smooth functions, then we can view \mathcal{M} as a $(2m)$ -dimensional real manifold with chart transition maps $\{u_{ij}, v_{ij}\}$ subject the *Cauchy-Riemann* equations

$$\frac{\partial u_{ij}}{\partial x^\mu} = \frac{\partial v_{ij}}{\partial y^\mu} \quad \text{and} \quad \frac{\partial u_{ij}}{\partial y^\mu} = -\frac{\partial v_{ij}}{\partial x^\mu},$$

where we define x^μ and y^μ by $z^\mu = x^\mu + iy^\mu$.

Proposition 4.1.1. *Every orientable, 2-dimensional Riemannian manifold (\mathcal{M}, g) is a complex manifold.*

Proof. We do this by considering two overlapping charts $U, V \subset \mathcal{M}$ and show that the transition functions between these charts obey the Cauchy-Riemann equations. As we have a Riemannian manifold, we know that the metric can be written in any local chart as

$$g_U = \lambda_U^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly g_V with $\lambda_U^2 \rightarrow \lambda_V^2$. Now suppose we coordinates our two charts using (u_1, u_2) and (v_1, v_2) . Now consider a change of basis on the overlap $U \cap V$, we have

$$(g_U)_{ij} = \frac{\partial v_m}{\partial u_i} \frac{\partial v_n}{\partial u_j} (g_V)_{mn},$$

so pluggin in values for $i, j \in \{1, 2\}$, we have

$$\begin{aligned}\lambda_U^2 &= \lambda_V^2 \left[\left(\frac{\partial v_1}{\partial u_1} \right)^2 + \left(\frac{\partial v_2}{\partial u_1} \right)^2 \right] \\ \lambda_U^2 &= \lambda_V^2 \left[\left(\frac{\partial v_1}{\partial u_2} \right)^2 + \left(\frac{\partial v_2}{\partial u_2} \right)^2 \right] \\ 0 &= \lambda_V^2 \left(\frac{\partial v_1}{\partial u_1} \frac{\partial v_1}{\partial u_2} + \frac{\partial v_2}{\partial u_1} \frac{\partial v_2}{\partial u_2} \right),\end{aligned}$$

from which we conclude

$$\frac{\partial v_1}{\partial u_1} \frac{\partial v_1}{\partial u_2} + \frac{\partial v_2}{\partial u_1} \frac{\partial v_2}{\partial u_2} = 0, \quad \text{and} \quad \left(\frac{\partial v_1}{\partial u_1} \right)^2 + \left(\frac{\partial v_2}{\partial u_1} \right)^2 = \left(\frac{\partial v_1}{\partial u_2} \right)^2 + \left(\frac{\partial v_2}{\partial u_2} \right)^2.$$

Now consider the complex coordinates $z = u_1 + iu_2$ and $w = v_1 + iv_2$, then we have

$$\frac{\partial w}{\partial z} = \frac{\partial(v_1 + iv_2)}{\partial u_1} - i \frac{\partial(v_1 + iv_2)}{\partial u_2},$$

and so

$$\begin{aligned}\frac{\partial w}{\partial z} \cdot \frac{\partial \bar{w}}{\partial \bar{z}} &= \left[\frac{\partial(v_1 + iv_2)}{\partial u_1} - i \frac{\partial(v_1 + iv_2)}{\partial u_2} \right] \left[\frac{\partial(v_1 - iv_2)}{\partial u_1} - i \frac{\partial(v_1 - iv_2)}{\partial u_2} \right] \\ &= \left(\frac{\partial v_1}{\partial u_1} \right)^2 + \left(\frac{\partial v_2}{\partial u_1} \right)^2 - \left(\frac{\partial v_1}{\partial u_2} \right)^2 + \left(\frac{\partial v_2}{\partial u_2} \right)^2 - 2i \left(\frac{\partial v_1}{\partial u_1} \frac{\partial v_1}{\partial u_2} + \frac{\partial v_1}{\partial u_1} \frac{\partial v_1}{\partial u_2} \right)\end{aligned}$$

where the last line we have only written the terms that don't cancel in the expansion. But these are our two chart transition equations and so we conclude

$$\frac{\partial w}{\partial z} \cdot \frac{\partial \bar{w}}{\partial \bar{z}} = 0,$$

which clearly has the two solutions

$$\frac{\partial w}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \bar{w}}{\partial \bar{z}} = 0.$$

The second condition is what we want, as this is the Cauchy-Riemann equations, so let's see what happens if we take the first solution.

Consider the determinant of the Jacobian of the coordinate transformation

$$\det \begin{pmatrix} \frac{\partial w}{\partial z} & \frac{\partial w}{\partial \bar{z}} \\ \frac{\partial \bar{w}}{\partial z} & \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix} - \begin{pmatrix} \frac{\partial \bar{w}}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial \bar{z}} \end{pmatrix},$$

and then use our condition that $\frac{\partial w}{\partial z} = 0$ along with $\frac{\partial w}{\partial \bar{z}} = \overline{\frac{\partial \bar{w}}{\partial z}}$ to get

$$\det(\dots) = 0 - \left| \frac{\partial w}{\partial \bar{z}} \right|^2 < 0.$$

However, if (\mathcal{M}, g) is orientable then we require that the determinant of the Jacobian is always positive,¹ and so we must rule this solution out. We therefore conclude that our chart transition maps obey the Cauchy-Riemann equations, and so our manifold is complex. \square

¹For those not familiar, this is basically the definition of orientation.

Now recall that we can define an almost complex structure J on a manifold as a $(1,1)$ -tensor field that squares to -1 . Further recall that we can complexify the tangent space (which for a complex manifold is trivial: it is already complex) and define holomorphic and antiholomorphic tensor fields as sections of the appropriate vector bundles. Well we now introduce a property that an almost complex structure can have.

Definition. [Integrable Almost Complex Structure] Let \mathcal{M} be an almost complex manifold. If the Lie bracket of two holomorphic vector fields is again a holomorphic vector field, we say that the almost complex structure is *integrable*.

It is clear that if J is integrable then equally the Lie bracket of two antiholomorphic vector fields is again an antiholomorphic vector field.

Remark 4.1.2. Note that the integrability condition is really a property of the almost complex structure. This is because we define what we mean by "a holomorphic vector field" explicitly using J .

We can also define the following tensor field.

Definition. [Nijenhuis Tensor] Let \mathcal{M} be an almost complex manifold with almost complex structure J . Then we define the Nijenhuis tensor by

$$N_J(X, Y) = [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY]$$

where $X, Y \in T_{\mathbb{C}}\mathcal{M}$.

Proposition 4.1.3. *An almost complex structure is integrable iff the Nijenhuis tensor vanishes, i.e. $N_J(X, Y) = 0$ for smooth vector fields X, Y .*

Proof. First let's assume $N_J(X, Y) = 0$. Now, recall that we can decompose any vector field into a sum of a holomorphic and an antiholomorphic piece. Let's denote these by $X = X^+ + X^-$, i.e. $JX^{\pm} = \pm X^{\pm}$. Now, let's consider the specific case that $X = X^+$ and $Y = Y^+$, then we have

$$\begin{aligned} N_J(X^+, Y^+) &= [X^+, Y^+] + J[X^+, JY^+] + J[JX^+, Y] - [JX^+, JY^+] \\ &= 2[X^+, Y^+] + 2iJ[X^+, iY^+], \end{aligned}$$

but this must vanish as $N_J(X, Y) = 0$ for all X, Y , which includes purely holomorphic vector fields. We are therefore forced to conclude $J[X^+, Y^+] = i[X^+, Y^+]$, i.e. the Lie bracket of two holomorphic vector fields is again a holomorphic vector field, and so J is integrable.

Now suppose that J is integrable. Considering our decomposition above again, we have (using that $N_J(X^+, X^+) = N_J(Y^+, Y^+) = 0$, when J is integrable, as proved above)

$$\begin{aligned} N_J(X, Y) &= N_J(X^+, Y^+) + N_J(X^+, Y^-) + N_J(X^-, Y^+) + N_J(X^-, Y^-) \\ &= N_J(X^+, Y^-) + N_J(X^-, Y^+) \\ &= [X^+, Y^-] - iJ[X^+, Y^-] + iJ[X^-, Y^+] - [X^-, Y^+] + (+ \longleftrightarrow -) \\ &= 0 + 0, \end{aligned}$$

which completes the proof. \square

Note, as we wrote in the definition, the property of an integrable almost complex structure is valid for almost complex manifolds where the underlying manifold is purely real (i.e. it need not be complex). Equally the Nijenhuis tensor is well defined in this case. However, the reason we didn't introduce these in the "real manifolds" chapter because of the following theorem.

Theorem 4.1.4 (Nirnberg-Newlander). *Let \mathcal{M} be an almost complex manifold with an almost complex structure J . Then \mathcal{M} is a complex manifold iff the Nijenhuis tensor vanishes, i.e. if J is integrable. For this reason, we call an integrable almost complex structure simply a complex structure.*

Proof. Omitted, for now. [Read through this later and do proof.](#) \square

4.2 \mathbb{CP}^n

Claim 4.2.1. The manifold \mathbb{CP}^n is a complex manifold of dimension n .

Proof. We have basically already shown this. Recall that we can define charts for \mathbb{CP}^n by

$$U_i := \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\},$$

so the coordinates for each U_i are given by $\zeta_\mu^i = z_\mu/z_i$, and we go from one set of coordinates to another simply by multiplication of $\zeta_i^j = z_j/z_i$. In other words, on the overlap $U_i \cap U_j$ we have

$$\zeta_\mu^i = \frac{z_\mu}{z_i} = \frac{z_\mu}{z_j} \frac{z_j}{z_i} = \zeta_\mu^j \zeta_j^i,$$

which is well defined as both $z_i, z_j \neq 0$ on the intersection. This is clearly a holomorphic transition map and so \mathbb{CP}^n is a complex manifold. The dimension follows from the fact that each U_i has n coordinates, i.e. $\mu = 1, \dots, n$ (note it's n and not $n+1$ as $\zeta_i^i = 1$ and so is not a coordinate). \square

4.2.1 Submanifolds Of \mathbb{CP}^n

We have just shown that \mathbb{CP}^n is a complex manifold, we now claim (without proof) that it is also a compact manifold. This is great, but, for reasons that will become clearer later, we now want to ask the question of how we get submanifolds of \mathbb{CP}^n that are also compact, complex manifolds. This question is answered in the next theorem.

Theorem 4.2.2 (Chow). *Any compact complex manifold that is a submanifold of \mathbb{CP}^n can be constructed by considering the zero locus of a finite number of homogeneous polynomial equations.*

We do not prove this theorem, but just clarify that it seems reasonable: a homogeneous polynomial is a polynomial of the homogeneous coordinates (z_0, \dots, z_n) , and if we construct a polynomial out of them, and consider the zero locus (i.e. the points at which $P(z) = 0$) then we can use this condition to relate one of the coordinates to some of the others. In this way we reduce the dimension of the manifold we are considering by one. If we take two such polynomials and consider their mutual zero locus (i.e. the points when both $P_1(z)$ and $P_2(z)$

vanish), then we reduce the dimension by 2. This idea clearly generalises to saying that for every polynomial we introduce, we reduce the dimension by one. We call a manifold produced by the common zero locus of a finite collection of polynomials a *complete intersection*. Of course this does not prove that the resulting space is a compact, complex manifold, but we accept that as true and move on.

Given Chow's theorem, we can ask the question "what are the Chern classes of the resulting complex submanifolds?" The answer to this question will prove immensely useful to us later, but we shall answer it now.

Let $X \subset \mathbb{CP}^n$ be a smooth hypersurface given by the zero locus of a homogeneous polynomial of degree d , $P(z)$, which we recall can be identified with a section of $\mathcal{O}_{\mathbb{CP}^n}(d)$. We now define the *normal bundle* of X to be

$$N_X := \frac{T^{(1,0)}\mathbb{CP}^n|_X}{T^{(1,0)}X},$$

that is, the normal bundle is we construct the normal bundle by considering the holomorphic tangent space to \mathbb{CP}^n , restricting it to $X \subset \mathbb{CP}^n$ and then quotienting by the holomorphic tangent space of X itself. Now this name is suggestive: we want to think of the normal bundle as being the tangent bundle to X , but where we only consider the vectors that are tangent to X , as seen in \mathbb{CP}^n . Indeed this is exactly what it is, as we now try outline.

Take any vector field in $V = T^{(1,0)}\mathbb{CP}^n|_X$. For simplicity, we restrict ourselves to a single point $x \in X$ so that we are dealing with a single vector space. Now we decompose this vector at x into a piece that is tangential to X , which we define as an element of $T^{(1,0)}X$ and a piece that is orthogonal (i.e. normal) to this. Our quotienting map is then

$$v \sim w \iff v = w + u_X,$$

where $u_X \in T^{(1,0)}\mathbb{CP}^n$ only has a tangential element. Well clearly this is going to leave us just with the piece that is normal to X , hence giving us the normal bundle. This idea is depicted in the following figure.

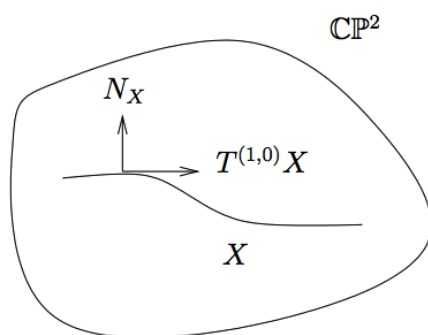


Figure 4.1: A depiction of the normal bundle N_X of a 1-dimensional hypersurface $X \subset \mathbb{CP}^2$. Image from [2].

Now comes the crucial point: as we mentioned already, we can view X as the zero locus of our polynomial $P(z)$. However recall that Proposition 3.2.3 told us that a polynomial of degree d can be identified with a section of $\mathcal{O}_{\mathbb{CP}^n}(d)$, from which we conclude that X should

be identified with the zeros in the fibres of $\mathcal{O}_{\mathbb{CP}^n}(d)$. In fact the normal bundle N_X of X is actually just given by $\mathcal{O}_{\mathbb{CP}^n}(d)|_X$.² Finally, noting that essentially what we said above about the splitting of $T^{(1,0)}\mathbb{CP}^n|_X$ into the normal bundle and $T^{(1,0)}X$ is just the statement that

$$T^{(1,0)}\mathbb{CP}^n|_X = T^{(1,0)}X \oplus N_X = T^{(1,0)}X \oplus \mathcal{O}_{\mathbb{CP}^n}(d)|_X,$$

we have the (split) short exact sequence

$$0 \longrightarrow T^{(1,0)}X \longrightarrow T^{(1,0)}\mathbb{CP}^n|_X \longrightarrow \mathcal{O}_{\mathbb{CP}^n}(d)|_X \longrightarrow 0,$$

from which we can compute the total Chern class of X , again using Equation (2.10), as

$$c(X) = \frac{c(T^{(1,0)}\mathbb{CP}^n|_X)}{c(\mathcal{O}_{\mathbb{CP}^n}(d)|_X)} = \frac{c(\mathbb{CP}^n)}{c(\mathcal{O}_{\mathbb{CP}^n}(d))},$$

where we have used that the total Chern class doesn't depend on whether we restrict to X or not. So finally recalling Equation (3.1) and Equation (2.14) (which tells us that $c(\mathcal{O}_{\mathbb{CP}^n}(d)) = 1 + dx$) we finally conclude

$$c(X) = \frac{(1+x)^{n+1}}{1+dx}. \quad (4.1)$$

where as always $x = c_1(\mathcal{O}_{\mathbb{CP}^n}(d))$.

4.2.2 Generalising

We can generalise this result to the cases when we consider a complete intersection manifold, i.e. our submanifold is now given by the common zero locus of multiple homogeneous polynomials. Let's say there are k polynomials of degrees d_i , $i \in \{1, \dots, k\}$. Then it is hopefully intuitively clear that in this case we have that the result $N_X = \mathcal{O}_{\mathbb{CP}^n}(d)|_X$ generalises to

$$N_X = \mathcal{O}_{\mathbb{CP}^n}(d_1)|_X \oplus \dots \oplus \mathcal{O}_{\mathbb{CP}^n}(d_k)|_X,$$

i.e. each $\mathcal{O}_{\mathbb{CP}^n}(d_i)$ term represents the polynomial of degree d_i , and the direct sum the fact that we must satisfy all of them. Now recalling that each $\mathcal{O}_{\mathbb{CP}^n}(d_i)$ is itself a line bundle, we can use Equation (2.13) to obtain

$$c(N_X) = \prod_{i=1}^k (1 + d_i x),$$

which gives us

$$c(X) = \frac{(1+x)^{n+1}}{\prod_{i=1}^k (1 + d_i x)}.$$

We can generalise this result further by allowing our base space to be a sum of complex projective spaces. However we need to be a bit more careful than simply plugging Equation (3.2) into the numerator of the above expression. The reason is that our polynomials

²Note to self: Go over explanation in Griffiths and Harris on why this is true (page 146).

could have different degrees in the different \mathbb{CP}^{n_i} s. For example, if we had $\mathbb{CP}^2 \oplus \mathbb{CP}^3$, which is a complex 5-dimensional manifold, we can produce a complex 2-dimensional manifold by introducing 3 polynomials. These polynomials can be of different degrees to each other, but we also have to take into account how the degree of each polynomial is distributed across the \mathbb{CP}^2 and \mathbb{CP}^3 . We summarise this information in a *configuration matrix*. Say, for example, our polynomials had degrees $(1, 3)$, $(4, 2)$ and $(5, 0)$, where (i, j) means degree i in the homogeneous coordinates of \mathbb{CP}^2 and degree j in the homogeneous coordinates of \mathbb{CP}^3 , then our configuration matrix would be

$$\begin{array}{c|ccc} \mathbb{CP}^2 & 1 & 4 & 5 \\ \mathbb{CP}^3 & 3 & 2 & 0 \end{array} \Big|_{\chi},$$

where we have also indicated that we normally include the Euler characteristic in the bottom right. To be completely clear on what the polynomials above are, if we denote the homogeneous coordinates of \mathbb{CP}^2 by (z_0, z_1, z_3) and those of \mathbb{CP}^3 by (w_0, w_1, w_2, w_3) , then a *particular* example would be

$$\begin{aligned} P_1(z, w) &= z_0 w_0^2 w_1 + z_2 w_0 w_2 w_3, \\ P_2(z, w) &= z_1^3 z_2 w_1 w_3, \\ P_3(z, w) &= z_0^2 z_1^2 z_3 + z_0^4 z_3. \end{aligned}$$

Luckily, the result for the total Chern class is relatively simple, given what we already know: the polynomials above are simply sections in $\mathcal{O}_{\mathbb{CP}^2}(1) \otimes \mathcal{O}_{\mathbb{CP}^3}(3)$, $\mathcal{O}_{\mathbb{CP}^2}(4) \otimes \mathcal{O}_{\mathbb{CP}^3}(2)$ and $\mathcal{O}_{\mathbb{CP}^2}(5)$, respectively. To write down the final result we want, we now consider the completely general configuration matrix

$$\begin{array}{c|ccc} \mathbb{CP}^{n_1} & d_1^1 & \dots & d_k^1 \\ \vdots & \vdots & & \vdots \\ \mathbb{CP}^{n_\ell} & d_1^\ell & \dots & d_k^\ell \end{array} \Big|_{\chi},$$

we get that the total Chern class of X is given by

$$c(X) = \frac{\prod_{i=1}^{\ell} (1 + x_i)^{n_i+1}}{\prod_{r=1}^k (1 + \sum_{s=1}^{\ell} d_r^s x_s)},$$

which is hopefully not too hard to see.

4.3 Kähler Geometry

So far, a great deal of effort has gone into discussing properties of complex manifolds, but we are yet to really talk about specific types of complex manifolds and why we care about them. We will now do just that, first discussing Kähler manifolds and then going on to study Calabi-Yau manifolds.

Remark 4.3.1. The way these notes are laid out is a bit different to how (at least I've seen) other notes on complex geometry are done. Specifically, other notes tend to introduce Kähler and Calabi-Yau manifolds earlier on, in order to give a motivation from the start as to why

we are doing what we are doing. They then sprinkle all (and more) of the stuff we have discussed throughout these discussions. The reason I have decided to lay the notes out this way is because it allows us to make the introduction of Kähler/Calabi-Yau manifolds very concise and we can immediately discuss how the above structures apply to these cases. This is done essentially to allow myself to make sure I understand where the definitions apply in a general sense. This remark is just to point this out.

4.3.1 Kähler Manifold

Definition. [Hermitian Metric/Manifold] Let (\mathcal{M}, J, g) be a complex manifold equipped with a Riemannian metric g . We call the metric *Hermitian* if the following holds: $g(X, Y) = g(JX, JY)$ for all $X, Y \in T\mathcal{M}$. We call the resulting manifold a *Hermitian manifold*.

If preferred, we can express the above in component notation as $g_{ab} = J_a^c J_b^d g_{cd}$. Now we recall that J actually induces a decomposition $T\mathcal{M} = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}$, so the natural question is "how does the Hermitian metric decompose here?" Well, if we label the holomorphic components by Greek letters α, β etc, and antiholomorphic components by barred Greek letters $\bar{\alpha}, \bar{\beta}$ etc, and recalling the $JX^\pm = \pm iX^\pm$, it is hopefully clear that in this notation we have

$$J_b^a = i\delta_\beta^\alpha - i\delta_{\bar{\beta}}^{\bar{\alpha}},$$

where it is understood that $\alpha/\bar{\alpha}$ match with a and $\beta/\bar{\beta}$ match with b and correspond to projecting onto the holomorphic/antiholomorphic parts. If we then plug this into our component expression for g , we get

$$\begin{aligned} g_{\alpha\beta} + g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\bar{\beta}} &= -(\delta_\alpha^\zeta - \delta_{\bar{\alpha}}^{\bar{\zeta}})(\delta_\beta^\gamma - \delta_{\bar{\beta}}^{\bar{\gamma}})(g_{\zeta\gamma} + g_{\bar{\zeta}\gamma} + g_{\zeta\bar{\gamma}} + g_{\bar{\zeta}\bar{\gamma}}) \\ &= -(\delta_\alpha^\zeta - \delta_{\bar{\alpha}}^{\bar{\zeta}})(g_{\zeta\beta} + g_{\bar{\zeta}\beta} - g_{\zeta\bar{\beta}} - g_{\bar{\zeta}\bar{\beta}}) \\ &= -(g_{\alpha\beta} - g_{\bar{\alpha}\beta} - g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\bar{\beta}}), \end{aligned}$$

and so comparing components on each side we see we are forced to conclude that $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$, so in other words

$$g_{ab} = g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}}.$$

This tells us that a Hermitian metric is a section $g \in \Gamma(T^{(1,0)}\mathcal{M} \otimes T^{(0,1)}\mathcal{M})$.

Claim 4.3.2. A complex manifold always admits a Hermitian metric.

We do not prove this here, but simply state it to be clear that the Hermiticity of the metric is really a property of the metric itself, not the manifold \mathcal{M} .

Definition. [Hermitian Form] Let (\mathcal{M}, J, g) be a Hermitian manifold. We can use the Hermitian metric to define a $(1, 1)$ -form via

$$\omega(X, Y) = g(JX, Y) \tag{4.2}$$

for all vector fields X, Y , which we call the *Hermitian form*.

Note the Hermitian form really is a form, i.e. it is antisymmetric. This follows from the fact that g is Hermitian and so

$$\omega(Y, X) = g(JY, X) = g(X, JY) = g(JX, J^2Y) = -g(JX, Y) = -\omega(X, Y),$$

where we have used the symmetry of the metric along with $J^2 = -1$. In Greek letter component form we have

$$\omega_{ab} = ig_{a\bar{b}} - ig_{\bar{a}b},$$

which again shows explicitly that $\omega \in \Omega^{(1,1)}\mathcal{M}$.

Proposition 4.3.3. *Let (\mathcal{M}, J, g) be a Hermitian manifold with associated Hermitian form ω . Then the Levi-Civita connection associated to g satisfies*

$$d\omega(X, Y, Z) - d\omega(Z, JY, JZ) + 2g((\nabla_X J)Y, Z) = 0.$$

Proof. Omitted. Essentially its just a "plug-in-and-go" job, details of which can be found on page 35 of the Calabi-Yau for Dummies notes. \square

Lemma 4.3.4. *On a Hermitian manifold (\mathcal{M}, J, g) with Hermitian form ω and Levi-Civita connection ∇ , the following three are equivalent:*

- (i) $\nabla\omega = 0$ (i.e. $\nabla_X\omega = 0$ for arbitrary vector field X),
- (ii) $\nabla J = 0$, and
- (iii) $d\omega = 0$.

Proof. Omitted. Again see the Calabi-Yau for Dummies notes. \square

Definition. [Kähler Form/Metric/Manifold] Let (\mathcal{M}, J, g) be a Hermitian manifold with associated Hermitian form ω . If ω is closed, i.e. $d\omega = 0$, then we call it the *Kähler form*, we call g a *Kähler metric* and the whole thing a *Kähler manifold*.

Remark 4.3.5. For those familiar, note that the Kähler form is a symplectic form: that is it is a closed, non-degenerate 2-form. We will not discuss symplectic geometry any further in these notes, we just point this out here.

Proposition 4.3.6. *Any submanifold of a Kähler manifold is itself a Kähler manifold.*

Proof. We do not prove this in detail, but simply point out that it is reasonable: the Kähler form is globally defined and closed, so if we restrict it to some submanifold, we will again get a closed $(1, 1)$ -form defined over all of our submanifold. A bit more technically, this is seen by the fact that the exterior derivative commutes with the pullback, and we can pull the Kähler form back from \mathcal{M} onto the submanifold, and so $d(\varphi^*\omega) = \varphi^*(d\omega) = 0$, and so the induced form is closed. \square

Kähler Class & Kähler Cone

Definition. [Kähler Class] Let (\mathcal{M}, J, g) be a Kähler manifold with associated Kähler form ω . By definition, ω is a closed $(1, 1)$ -form and so its equivalence class is an element of $H_{\bar{\partial}}^{1,1}(\mathcal{M})$. If we view the manifold as a real manifold of dimension $2m$, we equally get that $[\omega] \in H_{dR}^2(\mathcal{M}; \mathbb{R})$, and we call this latter class the *Kähler class*.

Proposition 4.3.7. *The Kähler class of a compact Kähler manifold is non-trivial.*

Proof. First we claim (without proof) that

$$\text{Vol}(Y) = \frac{1}{r!} \int_Y \omega^r,$$

where $Y \subset \mathcal{M}$ is a closed, complex r -dimensional submanifold, and where ω^r means r wedge products of ω . We note that this integral at least makes sense as $\omega^r \in \Omega^{r,r}(\mathcal{M})$, which in terms of the real manifold picture is an element of $\Omega^{2r}(\mathcal{M})$ and here $\dim_{\mathbb{R}}(Y) = 2r$. Now we know from Stoke's theorem that this integral only depends on the cohomology class (as the integral of an exact form vanishes on a closed manifold) and so the result is only dependent on $[\omega] \in H_{dR}^2(\mathcal{M}; \mathbb{R})$,³ i.e. on the Kähler class. Now it follows from $\text{Vol}(Y) > 0$ that $\omega > 0$ which in particular tells us that ω is not exact, and so $[\omega] \neq [0]$. \square

We now note that we never said that the Kähler form was unique. This follows from the fact that a complex manifold may admit more than one, distinctly different, Hermitian metric. Indeed, in general, one can produce any different (i.e. not even in the same Kähler class) Kähler forms for a given complex manifold. This then motivates the following definition.

Definition. [Kähler Cone] Let (\mathcal{M}, J) be a complex manifold admitting Kähler metrics. Each Kähler metric g gives rise to a Kähler class $[\omega_g] \in H_{\bar{\partial}}^{1,1}(\mathcal{M})$. We define the *Kähler cone*, \mathcal{K} , to be the set of Kähler classes, i.e.

$$\mathcal{K} := \{[\omega_g] \in H_{\bar{\partial}}^{1,1}(\mathcal{M}) \mid g \text{ is a Kähler metric on } \mathcal{M}\}.$$

The Kähler cone is important, as it seems to suggest that there is some link between the Hodge number $h^{1,1}$ and Kähler structure of moduli space of \mathcal{M} . That is, $h^{1,1}$ seems to indicate to us how many different Kähler manifolds we can obtain from a given complex manifold.

Kähler Potential

Now there is an important structure that exists on Kähler manifolds, which we now outline.

Let (\mathcal{M}, J, g) be a Kähler manifold. Then we can construct a real, closed 2-form from a smooth function ϕ simply by $dd^c\phi$. Now we recall that $dd^c = 2i\partial\bar{\partial}$, and so we see that $dd^c\phi$ is actually a closed $(1, 1)$ -form. This is true even outside Kähler manifolds, of course, but the important point comes when we realise that the Kähler form is a non-degenerate, closed $(1, 1)$ -form, and so *locally* we can always relate it to $dd^c\phi$. The natural question to ask is "when can this

³At first we might think it should depend on $[\omega^r] \in H_{dR}^{2r}(\mathcal{M}; \mathbb{R})$. However we note that the wedge product descends to cohomology, which follows from the fact that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$: additional exercise, check that this does indeed imply that if α, β are closed then $\alpha \wedge \beta$ is closed and also that if *either* α or β are exact then $\alpha \wedge \beta$ is exact.

be done globally?" The answer is "never", and this follows simply from the fact that $dd^c\phi$ is exact but we just showed that the Kähler class is non-trivial and so ω cannot be globally exact. We therefore conclude

On a Kähler manifold we can always express the Kähler form *locally* as $\omega = dd^c\phi$ for some real smooth function ϕ , but we can never do this globally. We call ϕ the *Kähler potential*.

A related result is the following.

Lemma 4.3.8. *We can parameterise the space of Kähler metrics for a given Kähler class by non-constant smooth functions on \mathcal{M} .*

Proof. Let's suppose ω_1 and ω_2 are two different Kähler forms on \mathcal{M} but that $[\omega_1] = [\omega_2]$, i.e. they are in the same Kähler class. Well, then, by definition, the difference $\omega_1 - \omega_2$ is a globally defined exact $(1,1)$ -form, but any globally defined exact $(1,1)$ -form can be written as $dd^c\psi$ for some real smooth function ψ , and so our two Kähler forms are related by $\omega_1 = \omega_2 + dd^c\psi$.

We now have to show that the choice of $dd^c\psi$ is unique, up to a constant. Well suppose that $dd^c\psi_1 = \omega_1 - \omega_2 = dd^c\psi_2$, for two different smooth functions ψ_1, ψ_2 . Then, using the linearity of dd^c , we have $dd^c(\psi_1 - \psi_2) = 0$, which tells us that $\psi_1 - \psi_2$ is a constant. We therefore conclude that the space of non-constant smooth functions on \mathcal{M} parameterise the representations of the Kähler class, which once we put together with the fact that the Kähler forms are explicitly linked to the Kähler metric, we prove our Lemma. \square

4.3.2 \mathbb{CP}^n Is Kähler

We now want to prove that \mathbb{CP}^n is a Kähler manifold. We already know that it is a complex manifold, so we just need to show that it admits a Kähler metric. We now construct this metric and it's associated Kähler form.

In order to construct a Kähler metric, we first need a Hermitian metric. Similarly to our construction of the Euler vector in Section 3.3, we do this by considering a metric on $\mathbb{C}^{n+1} \setminus \{0\}$ and then pushing it down onto \mathbb{CP}^n .

Recalling that a Hermitian metric is a section in $\Gamma(T^{(1,0)}\mathcal{M} \otimes T^{(0,1)}\mathcal{M})$, we consider the standard Hermitian metric (which we express in terms of the line element) on \mathbb{C}^{n+1}

$$ds^2 = |dz|^2 = dz_0 \otimes d\bar{z}_0 + \dots + dz_n \otimes d\bar{z}_n.$$

However this won't work as it doesn't project down nicely: that is it is *not* invariant under $z \rightarrow \lambda z$. With this in mind we suggest the following metric

$$ds^2 = \frac{|z|^2|dz|^2 - (z \cdot d\bar{z})(\bar{z} \cdot dz)}{|z|^4}$$

where $z \cdot d\bar{z} = z_0 d\bar{z}_0 + \dots + z_n d\bar{z}_n$ and similarly for $\bar{z} \cdot dz$. It is easily seen that this is indeed Hermitian, and it also respects our scaling, and so gives rise to an Hermitian metric on \mathbb{CP}^n . This is known as the *Fubini-Study metric*, and we write it in slightly more useful form, namely⁴

$$ds^2 = \frac{z_\alpha \bar{z}^\alpha dz_\beta d\bar{z}^\beta - \bar{z}_\alpha z_\beta dz^\alpha d\bar{z}^\beta}{(z_\tau \bar{z}^\tau)^2} = \frac{1}{z_\tau \bar{z}^\tau} \left(\delta_{\alpha\beta} - \frac{\bar{z}_\alpha z_\beta}{z_\gamma \bar{z}^\gamma} \right) dz^\alpha d\bar{z}^\beta.$$

⁴Note this notation makes sense as everything is summed over and so it's fine to write \bar{z}^α etc. (i.e. not using $\bar{\alpha}$ index) as both α and $\bar{\alpha}$ have the same range.

Now recalling that $\omega_{ab} = ig_{\alpha\bar{\beta}} - ig_{\bar{\alpha}\beta}$, we can use the above result to write down a potential Kähler form. We *cannot* conclude that this is the Kähler form just yet, though. However if we show that the local version of this result can be reproduced via $\partial\bar{\partial}\phi$ for some scalar field ϕ , then we can conclude that ω is the Kähler form and so we have a Kähler manifold.

Recall that our charts for \mathbb{CP}^n are given by $U_\alpha = \{[z_0 : \dots : z_n] \mid z_\alpha \neq 0\}$. We can use the scale invariance to use the representative given by dividing through by z_i everywhere. For concreteness we pick U_0 , but of course this choice is arbitrary and so the following result holds in all charts. We then define $Z_i := z_i/z_1$ where $i \in \{1, \dots, n\}$. Our contraction then becomes

$$|z|^2|_{U_0} = 1 + Z_i \bar{Z}^i,$$

and so the local expression for our line element is

$$ds^2|_{U_0} = \frac{1}{1 + Z_k \bar{Z}^k} \left(\delta_{ij} - \frac{\bar{Z}_i Z_j}{1 + Z_\ell \bar{Z}^\ell} \right) dZ^i d\bar{Z}^j$$

Now consider the global smooth function $\phi = \log(|z|^2)$. On U_0 this becomes $\phi|_{U_0} = \log(1 + Z_i \bar{Z}^i)$. Then take the derivatives

$$\begin{aligned} \partial_i \bar{\partial}_j \phi|_{U_0} &= \partial_i \left(\frac{Z_k \delta_j^k}{1 + Z_\ell \bar{Z}^\ell} \right) \\ &= \frac{1}{1 + Z_\ell \bar{Z}^\ell} \left(\delta_{ik} \delta_j^k - \frac{Z_j \bar{Z}^r \delta_{ri}}{1 + Z_s \bar{Z}^s} \right) \\ &= \frac{1}{1 + Z_k \bar{Z}^k} \left(\delta_{ij} - \frac{Z_j \bar{Z}_i}{1 + Z_\ell \bar{Z}^\ell} \right), \end{aligned}$$

where to get to the last line we have done some relabelling. This is just the expression appearing in $ds^2|_{U_0}$, and so we conclude that

$$g_{i\bar{j}} = \partial_i \bar{\partial}_j \phi|_{U_0}$$

We can then see from here that ϕ is in fact our Kähler potential and so we have proved that \mathbb{CP}^n is a Kähler manifold.

We conclude this subsection by recalling that Proposition 4.3.6 told us that any submanifold of a Kähler manifold is itself Kähler. Well we spent quite a bit of time earlier finding submanifolds of \mathbb{CP}^n in terms of the zero-locus of polynomials. We now see that these submanifolds are Kähler, a fact that will prove incredibly useful for us when trying to construct Calabi-Yau manifolds shortly.

4.3.3 Holonomy

We now note that Lemma 4.3.4 tells us that we could equally define a Kähler manifold by the conditions $\nabla\omega = 0$ or $\nabla J = 0$. The latter of these two is very important and interesting. Why? Well recall that our decomposition into holomorphic and antiholomorphic tensors depends on J . So if J is covariantly constant, i.e. not effected by parallel transport so $\nabla J = 0$, then our decomposition is unaffected by parallel transport. This tells us that if we equip a Kähler manifold with the Levi-Civita connection then the holomorphicity of a tensor field is preserved under parallel transport. The reason this is important, is that we recall that

holonomy is defined by parallel transporting a vector around a closed loop, and so it follows that the holonomy of a Kähler manifold w.r.t. the Levi-Civita connection is restricted. In particular we see that the holonomy group must preserve both the length of the vector as well as its complex properties. In other words, because $P_\gamma X \in T^{(1,0)}\mathcal{M}$ if $X \in T^{(1,0)}\mathcal{M}$ and recalling that $\overline{T^{(1,0)}\mathcal{M}} = T^{(0,1)}\mathcal{M}$, the holonomy group must preserve the Hermiticity, and so we have

$$\text{Hol}(\mathcal{M}_{\text{Kähler}}) \subseteq U(\dim_{\mathbb{C}} \mathcal{M}).$$

4.4 Calabi-Yau Geometry

We are finally ready to begin discussing Calabi-Yau manifolds. These are a particular kind of Kähler manifold and play a huge role in studying compactifications of string theories. Recall that we showed all the way back in Section 2.3.1 that if a manifold has vanishing Ricci curvature that the first Chern class vanishes. We said that the reverse was not true in general, but that Calabi and Yau showed that if the manifold was in fact a Kähler manifold that it was. That is, if \mathcal{M} is a Kähler manifold then $c_1(\mathcal{M}) = 0$ if and only if \mathcal{M} is Ricci flat. We state this more formally now.

Theorem 4.4.1 (Yau). *Let (\mathcal{M}, J, g) be a compact Kähler manifold with associated Kähler form ω . Further let R be $(1, 1)$ -form which represents the first Chern class of \mathcal{M} , i.e. $[R] \propto c_1(\mathcal{M})$. Then there exists a **unique** Kähler metric \tilde{g} on \mathcal{M} with associated Kähler form $\tilde{\omega}$ such that $[\tilde{\omega}] = [\omega]$ and the Ricci form associated to \tilde{g} is R .*

This theorem is not easy to prove⁵ however its importance lies in the fact that we get the following immediate corollary.

Corollary 4.4.2. *Let (\mathcal{M}, J, g) be compact Kähler manifold with Kähler form ω . Then, if $c_1(\mathcal{M}) = 0$ there exists a unique equivalent Kähler form, $[\tilde{\omega}] = [\omega]$, such that \tilde{g} is Ricci flat.*

This gives us a definition of a Calabi-Yau manifold, however there are many equivalent ways of defining one. We group the common ones in the following definition.

Definition. [Calabi-Yau Manifold] Let (\mathcal{M}, J, g) be a Kähler manifold of real dimension $2m$. Then we call it a *Calabi-Yau* manifold if any of the following hold:

- (i) \mathcal{M} is Ricci flat, $R = 0$;
- (ii) The first Chen class vanishes, $c_1(\mathcal{M}) = 0$;
- (iii) The holonomy group is restricted to $\text{Hol}(\mathcal{M}) \subseteq SU(m)$;
- (iv) The canonical bundle is trivial (i.e. admits a global, non-vanishing section);
- (v) \mathcal{M} admits a globally defined, nowhere vanishing holomorphic m -form.

⁵As all the notes I've read say...

A comparison between these different definition can be found in Bouchard's notes, but here we just point out the thing that is of most interest to us: recall that $h^{1,1}$ tells us the number of inequivalent Kähler forms for a given complex manifold, so putting this together with Yau's theorem we see that $h^{1,1}$ counts the number of possible Ricci flat Kähler forms for our manifold. In other words:

$h^{1,1}$ tells us how many different Calabi-Yau manifolds we can define for a given complex manifold.

4.4.1 Hodge Numbers

We now want to study the Hodge numbers for a Calabi-Yau manifold. We recall that we have already shown that complex conjugation gives $h^{p,q} = h^{q,p}$ and that Hodge star gives $h^{p,q} = h^{m-p,m-q}$. Well for Calbi-Yau manifolds we have further restrictions, which we now outline:

- Condition (v) in the definition tells us the $h^{m,0} = 1$. Why? Well a nowhere vanishing holomorphic m -form is clearly a holomorphic volume form, Ω and therefore any element in $\Omega^{m,0}$ can be expressed in terms of this holomorphic m -form, i.e. $\alpha = f\Omega$ for some holomorphic function f .
- Next, given a class $[\alpha] \in H^{0,q}(\mathcal{M})$, there is a unique class $[\beta] \in H^{0,m-q}(\mathcal{M})$ such that

$$\int_{\mathcal{M}} \alpha \wedge \beta \wedge \Omega = 1,$$

where Ω is our unique holomorphic volume form. This follows simply from the fact that the above integrand is clearly the (m,m) volume form, which is unique. We therefore have $h^{0,q} = h^{0,m-q}$. We can put this together with our complex conjugation condition to get that $h^{p,0} = h^{m-p,0}$.

- Finally we claim (without proof [Look up.](#)) that $h^{1,0} = 0$.

In physics we are mostly interested in Calabi-Yau 3-folds, that is Calabi-Yau manifolds of complex dimension 3. The reason for this is that M-theory is 10-dimensional and so if we compactify M-theory on some real 6-dimensional space then we stand a chance of getting something related to our observed reality, i.e. we get a 4-dimensional spacetime. For this reason, we write down the Hodge diamond explicitly for a Calabi-Yau three fold.

$$\begin{array}{ccccccc}
& & & & 1 & & \\
& & & & 0 & & 0 \\
& & 0 & & h^{1,1} & & 0 \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
& & 0 & & h^{1,1} & & 0 \\
& & 0 & & 0 & & \\
& & & & 1 & &
\end{array} \tag{4.3}$$

The Euler characteristic also nicely simplifies for Calabi-Yau manifolds, and in particular for a Calabi-Yau 3-fold we have

$$\chi = 2(h^{1,1} - h^{2,1}) \tag{4.4}$$

It turns out for a Calabi-Yau 3-fold that $h^{2,1}$ classifies the infinitesimal deformations of the complex structure, in the same way that $h^{1,1}$ classifies the infinitesimal deformations of the Kähler structure. More details on why this is the case will come later.⁶

In practice, for a Calabi-Yau 3-fold we compute the Euler characteristic by recalling Equation (2.12), i.e. that χ is given by the integral over our top Chern class, which here is $c_3(\mathcal{M})$:

$$\chi = \int_{\mathcal{M}} c_3(\mathcal{M}).$$

We then employ techniques to calculate $h^{2,1}$ and then we can use Equation (4.4) to find $h^{1,1}$. We will do all of this for explicit examples in a moment.

Mirror Symmetry

Before moving on to discuss explicit examples, we first make a brief comment on mirror symmetry. Note, if swapped $h^{1,1} \iff h^{2,1}$ in our Hodge diamond Equation (4.3), we would get back a Calabi-Yau 3-fold but now with the complex moduli and Kähler structure moduli swapped. Clearly these two Calabi-Yau manifolds are geometrically different, however it turns out that if we use them as the compactifying dimensions in string theory, they give symmetric results. We will not discuss this further⁷ but simply show a pretty diagram to highlight how elegant this symmetry is.

⁶I am still learning about this!

⁷Again because I am still learning about this.

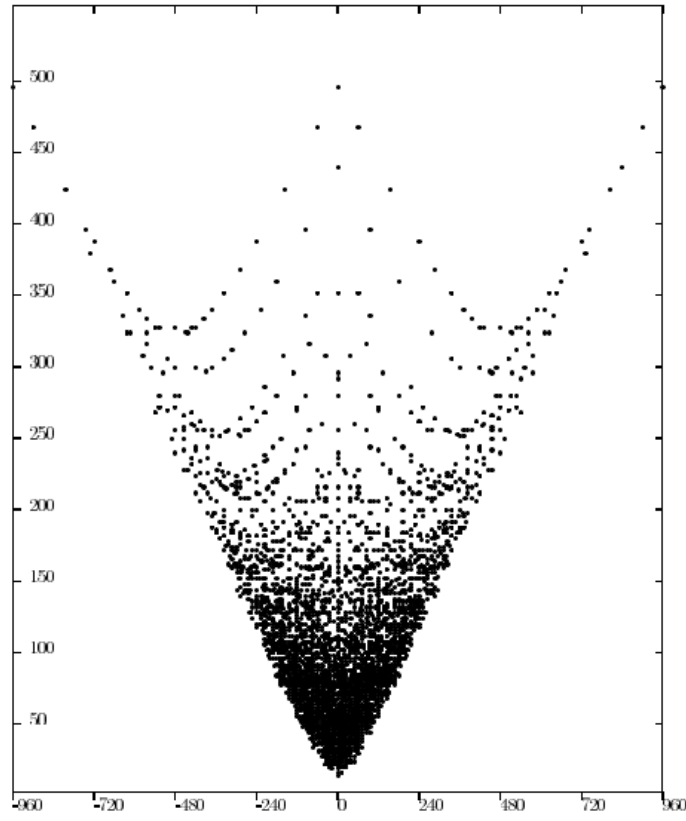


Figure 4.2: A plot of $h^{1,1} + h^{2,1}$ against $\chi = 2(h^{1,1} - h^{2,1})$. Figure taken from [3].

5 | Constructing Calabi-Yaus In \mathbb{CP}^n

We now want to actually try explicitly find out where we can construct Calabi-Yau manifolds. Of course the first thing we require is that the manifold is Kähler, we then need to check if we can impose some condition on this manifold such that the first Chern class vanishes.

Well we have already shown that \mathbb{CP}^n is Kähler, and in fact we have claimed that submanifolds of \mathbb{CP}^n defined by the zero-locus of degree polynomials in \mathbb{CP}^n are Kähler. Now we see why we invested so much effort in finding expressions for the total Chern class of these manifolds: we can expand these expressions and obtain a formula for $c_1(X)$ in terms of the dimension of the polynomials, which in turn might allow us to fix $c_1(X) = 0$. We use the most general case of an arbitrary configuration matrix and then drastically simplify (e.g. consider a single polynomial in \mathbb{CP}^n):

5.1 General Result

Recall that for the general configuration matrix

$$\begin{array}{c} \mathbb{CP}^{n_1} \\ \vdots \\ \mathbb{CP}^{n_\ell} \end{array} \left| \begin{array}{ccc} d_1^1 & \dots & d_k^1 \\ \vdots & & \vdots \\ d_1^\ell & \dots & d_k^\ell \end{array} \right|_\chi,$$

we get that the total Chern class of X is given by

$$c(X) = \frac{\prod_{i=1}^{\ell} (1 + x_i)^{n_i+1}}{\prod_{r=1}^k (1 + \sum_{s=1}^{\ell} d_r^s x_s)}.$$

Well we can expand the numerator and denominator in powers of x and read the first Chern class off as the term linear in x . We obtain

$$c_1(X) = \sum_{i=1}^{\ell} \left(n_i + 1 - \sum_{r=1}^k d_r^i \right) x_i \quad (5.1)$$

and so

We get a Calabi-Yau manifold of dimension $(\sum_{i=1}^{\ell} n_i - k)$ when

$$\sum_{r=1}^k d_r^i = n_i + 1 \quad \forall i \in \{1, \dots, \ell\}. \quad (5.2)$$

If we continue the expansion of our total Chern class, we can find the top Chern class, which if we then integrate over X gives us our Euler characteristic. This is, of course, technically correct although it is often quite hard to compute in practise. However integrating on \mathbb{CP}^n is *much* simpler, so we ask the question "is there any way we can get the result of $\int_X c_{\text{top}}(X)$ as an integral over \mathbb{CP}^n ?" The answer is yes and it the content of the next theorem.

Theorem 5.1.1. *Let \mathcal{M} be an n -dimensional manifold, and let $X \subset \mathcal{M}$ be some closed, k -dimensional submanifold. Then for any closed k -form $[\tau] \in H_{dR}^k(\mathcal{M}; \mathbb{R})$ there exists a closed $(n - k)$ -form $[\eta_X] \in H_{dR}^{n-k}(\mathcal{M}; \mathbb{R})$ such that*

$$\int_X \tau = \int_{\mathcal{M}} \tau \wedge \eta_X. \quad (5.3)$$

We call η_X the Poincaré dual class to X .¹

Proof. We give a somewhat intuitive proof here (although it is perhaps not the most rigorous). For a moment let's forget that X is some submanifold in \mathcal{M} . We then see that τ must be related to the volume form on X . Locally we can write the volume form on X as

$$\Omega_X = dx^1 \wedge \dots \wedge dx^k,$$

where (dx^1, \dots, dx^k) are the coordinate basis of the cotangent bundle. Now, we imagine embedding X into \mathcal{M} , but bringing our basis along for the ride. We then complete this basis to give a full basis for \mathcal{M} , (or we can just imagine doing the embedding in such a way that the (dx^1, \dots, dx^k) align with the first k basis elements in \mathcal{M}). Let's denote the completed basis by $(dx^1, \dots, dx^k, dy^1, \dots, dy^{n-k})$. Now \mathcal{M} carries its own volume form, which we can write locally as

$$\Omega_{\mathcal{M}} = dx^1 \wedge \dots \wedge dx^k \wedge dy^1 \wedge \dots \wedge dy^{n-k} = \Omega_X \wedge dy^1 \wedge \dots \wedge dy^{n-k}.$$

So we simply define our $\eta_X = dy^1 \wedge \dots \wedge dy^{n-k}$, and then we get

$$\int_{\mathcal{M}} \tau \wedge \eta_X = \int_{\mathcal{M}} \tau \wedge dy^1 \wedge \dots \wedge dy^{n-k} = \int_X \tau,$$

where the last equality follows from the rules of integration. In this sense we can think of η_X as a delta function which restricts the integral over \mathcal{M} to the integral over X . \square

This is great, apart from we now have to ask ourselves "what is η_X , explicitly?" Well, it is hopefully clear that our η_X is a top form in the (co)normal bundle N_X , i.e. it only contains

¹This name comes preciey from Poincaré duality as we defined it before. That is X is a k -dimensional closed manifold (and so essentially an element of the k -th homology group), so by Poincaré duality there should exist some closed $(n - k)$ -form, which is exactly our η_X .

dy^i terms, which are normal to the cotangent bundle of T^*X . Now if we are considering the special cases when $N_X = E|_X$, which we are doing for \mathbb{CP}^n — recall that $N_X = \mathcal{O}_{\mathbb{CP}^n}(d)|_X$ for $E = \mathcal{O}_{\mathbb{CP}^n}(d)$ — then the top form is just given by the top Chern class $c_r(E)$, where r is the rank of E . So finally using that for us $\dim X = \sum_{i=1}^{\ell} n_i$ and $\dim N_x = k$ (that is each polynomial increases the dimension of the normal bundle by 1) we have

$$\chi = \int_{\mathbb{CP}^{n_1} \oplus \dots \oplus \mathbb{CP}^{n_\ell}} c_{\sum_{i=1}^{\ell} n_i}(X) \wedge c_k(E), \quad (5.4)$$

where E is given by the slightly complicated looking expression in terms of $\mathcal{O}_{\mathbb{CP}^n}(d)$ terms.

5.2 Calabi-Yau 3-folds

We now want to simplify the above formula a bit by considering the specific cases when we want to get a Calabi-Yau 3-fold.

5.2.1 The Quintic In \mathbb{CP}^4

Let's start with the simplest case: a single polynomial in \mathbb{CP}^n . Here we have $\ell = 1$ and $k = 1$ and so Equations (5.1) and (5.2) become

$$c_1(X) = (n + 1 - d)x \quad \implies \quad (n - 1)\text{-dimensional Calabi-Yau if } d = n + 1.$$

So if we want to construct a Calabi-Yau 3-fold we have to consider a *quintic* in \mathbb{CP}^4 . This is a very important example of a Calabi-Yau manifold and we now explore it in a bit more detail.

Denoting the space by Q , we have that the total Chern class is

$$c(Q) = \frac{(1+x)^5}{1+5x} = 1 + 10x^2 - 40x^3,$$

where the second line follows from expanding and truncating at x^3 as $\dim_{\mathbb{C}}(Q) = 3$. So we see that $c_3(Q) = -40x^3$. Next we have that our normal bundle $N_Q = \mathcal{O}_{\mathbb{CP}^4}(5)$ is a line bundle and so

$$c_k(E) = c_1(\mathcal{O}_{\mathbb{CP}^4}(5)) = 5x,$$

where we have used $c(\mathcal{O}_{\mathbb{CP}^n}(d)) = 1 + dx$. So we can compute our Euler characteristic via Equation (5.4)

$$\chi(Q) = \int_{\mathbb{CP}^4} (-40x^3) \wedge (5x) = -200.$$

We can therefore summarise the Calabi-Yau manifold coming from the quintic in \mathbb{CP}^4 via the following configuration matrix

$$Q = \mathbb{CP}^4 | 5 |_{-200}.$$

Ok, great so we have constructed a collection of Calabi-Yau manifolds, we now want to ask the question of "how many are there?" We recall this question is answered exactly by the value of $h^{1,1}$, so if we can just work out $h^{2,1}$ then we can use Equation (4.4) to find $h^{1,1}$ and be done!

So how do we find $h^{2,1}$? Well we recall that it classifies the infinitesimal deformations of the complex structure on Q . This complex structure is inherited from the complex structure in \mathbb{CP}^4 , and how it is inherited clearly depends on how Q is embedded into \mathbb{CP}^4 . That is, if we deform how Q sits in \mathbb{CP}^4 , then we change how the tangent spaces of Q align with the tangent spaces of \mathbb{CP}^4 , therefore altering the what we call a holomorphic vs. antiholomorphic vector in Q , which is exactly an alteration of the complex structure. Well the different embeddings of Q into \mathbb{CP}^4 are given precisely by the number of free parameters in our defining polynomial. So we just need to compute this.

Now it is a fact that the number of independent degree d polynomials in $(n+1)$ variables is given by the binomial coefficient $\binom{d+n}{n}$, and so our quintic polynomial starts off with $\binom{9}{4} = 126$ parameters. However we need to account for coordinate transformations (i.e. homogeneous linear change of variables) as well as the scaling. These collectively add up to² $(n+1)^2$ which for us is $5^2 = 25$, which finally leaves us with $h^{2,1} = 126 - 25 = 101$. So plugging this into Equation (4.4) we conclude

$$-200 = 2(h^{1,1} - 101) \implies h^{1,1} = 1.$$

That is

There is a *single* Calabi-Yau manifold given by the zero-locus of a quintic polynomial in \mathbb{CP}^4 .

We can summarise this using the Hodge diamond Equation (4.3)

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & 0 & & 0 \\
 & & 0 & & 1 & & 0 \\
 1 & & 101 & & 101 & & 1 \\
 & & 0 & & 1 & & 0 \\
 & & 0 & & 0 & & \\
 & & & & 1 & &
 \end{array}$$

5.2.2 Complete Intersection Manifolds

We now go to the slightly more complicated case where we still only have a single \mathbb{CP}^n but now we can use multiple polynomials. Here we have $\ell = 1$ and $k = n - 3$, i.e. we need to reduce down to an 3-dimensional manifold. Our Calabi-Yau condition is simply

$$\sum_{r=1}^{\ell-3} d_r = n + 1.$$

²Basically the homogeneous linear transformations of $(n+1)$ variables are given by the group $PGL(n+1, \mathbb{C})$, which is defined to be $GL(n+1, \mathbb{C})$ modded out by our scaling, so it has dimension $(n+1)^2 - 1$, with the -1 corresponding exactly to our scaling, so when we add this back in we're just left with $(n+1)^2$.

We now note that we actually require that $d_r \geq 2$ for all r . Why? Well imagine we have some \mathbb{CP}^n and one of our polynomials has degree 1. Well, we can always use a coordinate transformation such that this polynomial simply sets one of the homogeneous coordinates to zero, but then this just leaves us with $(k-1)$ polynomials in \mathbb{CP}^{n-1} .

Given this, we can see that there are actually only five solutions. We display them via their configuration matrices below (without their Euler characteristics)

$$\mathbb{CP}^4|5|, \quad \mathbb{CP}^5|33|, \quad \mathbb{CP}^5|24|, \quad \mathbb{CP}^6|223| \quad \text{and} \quad \mathbb{CP}^7|2222|.$$

For clarity on why we can't have anymore, let's imagine we considered \mathbb{CP}^8 . To get a 3-fold, we would need to consider 5 polynomials whose degrees sum up to 9. However we cannot do this if we also require that $d_r \geq 2$ for all r . The same idea applies to higher n .

5.2.3 Tian-Yau Manifold

Of course we can generate other Calabi-Yau 3-folds by allowing our base spaces to be given by a direct sum of \mathbb{CP}^n s. For example,

$$\begin{array}{c} \mathbb{CP}^3 \\ \mathbb{CP}^3 \end{array} \left| \begin{array}{ccc} 1 & 3 & 0 \\ 1 & 0 & 3 \end{array} \right|_{-18},$$

is a Calabi-Yau 3-fold. This particular example is known as the *Tian-Yau manifold*. For completeness, the Hodge diamond for the Tian-Yau manifold is

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 0 & & 0 \\ & & 0 & & 14 & & 0 \\ 1 & & 23 & & 23 & & 1 \\ & & 0 & & 14 & & 0 \\ & & 0 & & 0 & & \\ & & & & 1 & & \end{array}$$

Hopefully the construction of Calabi-Yau 3-folds is now clear and so we don't discuss generating Calabi-Yaus in this way any further.

Exercise

Show that the Tian-Yau manifold is indeed a Calabi-Yau 3-fold, and verify its Euler characteristic is -18 .

Hint: If stuck on finding χ , see the section below on K3 manifolds, then come back. Equally see Bouchard's notes for more details.

5.2.4 Holomorphic $(3,0)$ -Form

Recall that we said one of the equivalent conditions for a Calabi-Yau 3-fold was that there exists a globally defined, nowhere vanishing holomorphic $(3,0)$ -form. We now want to outline

how we calculate this form, and we shall use the quintic in \mathbb{CP}^4 as an example, and then we will state how this generalises after.

We start by defining a $(4, 0)$ form on \mathbb{C}^5 by

$$\tau = \sum_{\mu=0}^4 dz_0 \wedge \dots \wedge z_\mu \wedge \dots \wedge dz_4,$$

where we note that the z_μ is *not* dz_μ (as then we would have a $(5, 0)$ -form). It is hopefully clear that τ is holomorphic, however we note that it is not well defined in \mathbb{CP}^4 : this is simply because it is not invariant under the scaling $z_\mu \rightarrow \lambda z_\mu$. In fact it scales as $\tau \rightarrow \lambda^5 \tau$ under our scaling. So what do we do? Well we note that this scaling is *exactly* the scaling of our defining polynomial P_Q ,³ so we know that the form $\tilde{\tau} = \tau/P_Q$ is projectively well defined.

This is great, however we have the big problem: $\tilde{\tau}$ is singular exactly on our Calabi-Yau, i.e. when $P_Q = 0$. So what do we do? Well, this problem hopefully "smells like" residue problems in complex analysis, and so we consider a small curve γ_{P_Q} in \mathbb{CP}^4 which circles the point $P_Q = 0$. We then define

$$\Omega := \int_{\gamma_{P_Q}} \tilde{\tau} = \int_{\gamma_{P_Q}} \frac{\tau}{P_Q}.$$

The claim is that this is a nowhere vanishing holomorphic $(3, 0)$ -form on Q , i.e. when $P_Q = 0$. To see this, consider a coordinate patch and use $dz_0 = \frac{\partial z_0}{\partial P_Q} dP_Q$, then we get

$$\Omega = \int_{\gamma_{P_Q}} \frac{\sum_{\mu=1}^4 dP_Q \wedge dz_1 \wedge \dots \wedge z_\mu \wedge \dots \wedge dz_4}{(\partial P_Q / \partial z_0) P_Q} = (2\pi i) \left(\frac{\sum_{\mu=1}^4 dz_1 \wedge \dots \wedge z_\mu \wedge \dots \wedge dz_4}{(\partial P_Q / \partial z_0)} \right)_{P_Q=0},$$

where the second line is our residue theorem, i.e. we did the integral around γ_{P_Q} . This is a holomorphic $(3, 0)$ -form, is projectively well defined and lives on Q , which is exactly what we wanted.

So how do we adapt this result to the case when we have a complete intersection, but still only one \mathbb{CP}^n ? We outline the idea here, and leave it for the reader to make sure they understand why this works.

We define an $(n, 0)$ -form on \mathbb{C}^{n+1} by

$$\tau = \sum_{\mu=0}^n dz_0 \wedge \dots \wedge z_\mu \wedge \dots \wedge dz_n.$$

We then make this projectively well defined by dividing by the the product of the defining polynomials,

$$\tilde{\tau} = \frac{\tau}{\prod_{r=1}^{n-3} P_r},$$

which we note is projectively well defined precisely because $(n+1) = \sum_{r=1}^{n-3} d_r$. Again this is singular, but now at all the points $P_r = 0$, so we consider the $(n-3)$ -dimensional contour given by

$$\Gamma_{n-3} = \gamma_1 \times \dots \times \gamma_{n-3},$$

³A lot of authors use Q to denote both the Calabi-Yau space as the polynomial. In these notes I am going to try to be careful to differentiate between the two, at least for now.

where γ_r means the contour circling $P_r = 0$. We then get the holomorphic $(3, 0)$ -form as

$$\Omega = \int_{\Gamma} \tilde{\tau} = \int_{\gamma_1 \times \dots \times \gamma_{n-3}} \frac{\tau}{\prod_{r=1}^{n-3} P_r}.$$

which is well defined on our Calabi-Yau 3-fold.

We now just want to generalise this to the case when our base space is given by multiple \mathbb{CP}^n s. Well each of these \mathbb{CP}^n s carries its own set of homogeneous coordinates, and so we need to find a τ on each one. So if we consider our usual general case (i.e. ℓ \mathbb{CP}^n s) we define

$$\tau_i = \sum_{\mu=0}^{n_i+1} dz_0^i \wedge \dots \wedge z_{\mu}^i \wedge \dots \wedge dz_{n_i}^i,$$

and then define

$$\tau = \prod_{i=1}^{\ell} \tau_i,$$

which is a form on $\prod_{i=1}^{\ell} \mathbb{CP}^{n_i+1}$. We now need to divide by our defining polynomials

$$\tilde{\tau} = \frac{\tau}{\prod_{r=1}^N P_r},$$

where $N = (\sum_{i=1}^{\ell} n_i) - 3$, i.e. the number of polynomials needed to give us a 3-fold. Next we define our contour

$$\Gamma_N = \gamma_1 \times \dots \times \gamma_N$$

and finally obtain our holomorphic $(3, 0)$ -form

$$\Omega = \int_{\Gamma_N} \tilde{\tau} = \int_{\gamma_1 \times \dots \times \gamma_N} \frac{\tau}{\prod_{r=1}^N P_r}.$$

5.3 K3 Surfaces

We have just constructed a Calabi-Yau 3-fold as the quintic in \mathbb{CP}^4 . We now want to look at constructing another class of interesting Calabi-Yau manifolds, namely 2-folds. As we will see, interestingly, the Euler characteristics of *all* 2-folds will be the same, in which sense we can view all 2D Calabi-Yau manifolds as deformations of each other. The deformation family are known as *K3 surfaces*.

5.3.1 In \mathbb{CP}^3

We start with the easy case of a polynomial in \mathbb{CP}^3 . From above, we know that this polynomial must be of degree $3+1 = 4$, i.e. this is how we get vanishing first Chern class in Equation (5.1). The total Chern class is then given by

$$c(K) = \frac{(1+x)^4}{(1+4x)}, \quad \text{with} \quad x = c[\mathcal{O}_{\mathbb{CP}^3}(1)],$$

we can expand this out, and truncate at the x^2 term (as K is a 2-fold so we can have at max a $(2, 2)$ -form), and then read off the top Chern class. We get

$$c_2(K) = 6x^2.$$

Next we again use that our normal bundle $N_K = \mathcal{O}_{\mathbb{CP}^3}(4)$ is a line bundle and so the top Chern class is simply $4x$. We can then compute the Euler characteristic:

$$\chi(K) = \int_{\mathbb{CP}^3} 6x^2 \wedge 4x = 24.$$

The Hodge diamond for a Calabi-Yau 2-fold is simply

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ & & h^{1,1} & & \\ 1 & & & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

which comes from $h^{0,0} = 1$ and $h^{1,0} = 0$ along with the relations

$$h^{0,0} = h^{2,2} = h^{0,2} = h^{2,0}, \quad \text{and} \quad h^{1,0} = h^{0,1} = h^{1,2} = h^{2,1}.$$

So we just need to compute $h^{1,1}$, which we relate to the Euler characteristic:

$$\chi(K) = \sum_{k=0}^4 (-1)^k b^k = 2b^0 - 2b^1 + b^2 = 2 - 0 + 2 + h^{1,1},$$

where we have used $b^0 = b^4$ and $b^1 = b^3$, so we conclude

$$h^{1,1} = 20.$$

5.3.2 In $\mathbb{CP}^1 \oplus \mathbb{CP}^2$

We can now repeat the construction for the complete intersection of two hypersurfaces in $\mathbb{CP}^1 \oplus \mathbb{CP}^2$. From the work above, we have the configuration matrix

$$\begin{array}{c|c} \mathbb{CP}^1 & 2 \\ \mathbb{CP}^2 & 3 \end{array} \Big|_{\chi},$$

where the degrees of the polynomials are hopefully clear. The total Chern class is given by

$$c(K_{1,2}) = \frac{(1+x_1)^2(1+x_2)^3}{(1+2x_1+3x_2)} \quad \text{where} \quad x_i = c[\mathcal{O}_{\mathbb{CP}^i}(1)].$$

Again we now expand and truncate at the second order term, however first we note something. Really what we are after is the Euler characteristic, which is given by integrating over $\mathbb{CP}^1 \oplus$

\mathbb{CP}^2 , so our final integrand must be of the form $x_1x_2^2$, as these are the top forms on the two spaces. We therefore drop anything containing a x_1^2 term, leaving us with

$$c_2(K_{1,2}) = 6x_1x_2 + 3x_2^2.$$

Again our normal bundle is simply a line bundle, and we have

$$c_1(N_{K_{1,2}}) = 2x_1 + 3x_2$$

Our Euler characteristic is then given by

$$\begin{aligned} \chi(K_{1,2}) &= \int_{K_{1,2}} (6x_1x_2 + 3x_2^2) \\ &= \int_{\mathbb{CP}^1 \oplus \mathbb{CP}^2} (6x_1x_2 + 3x_2^2) \wedge (2x_1 + 3x_2) \\ &= \int_{\mathbb{CP}^1 \oplus \mathbb{CP}^2} (18 + (-1)^2 6)x_1x_2^2 \\ &= 24, \end{aligned}$$

where the $(-1)^2$ factor is coming from having to commute x_1 through x_2 , which are *forms*, remember. We therefore see that the Euler characteristic of $K_{1,2}$ is equal to that of K , as promised at the start of this section.

6 | Quick Comment On \mathbb{WCP}^n & Orbifold Singularities

We have seen how to construct Calabi-Yau manifolds in projective spaces, the next step is obviously to consider *weighted* projective spaces. However, as we will see, things here are more subtle and singularities arise! These singularities can be fixed using the method of *toric geometry*, but we will discuss this construction in a separate set of notes.

Let's consider the specific case of a single polynomial in $\mathbb{WCP}_{1,1,1,1,4}^4$, which we coordinatise using $[x_1, x_2, x_3, x_4, y]$. We have from Equation (3.3) that (using A to denote our ambient space, i.e. the \mathbb{WCP}_{11114}^4)

$$c_1(A) = 8H,$$

where we have changed to the notation of $H = c_1(\mathcal{O}_{\mathbb{WCP}_{11114}^4})$, so as to not confuse this with the homogeneous coordinates. So if we want to have some subspace X that is Calabi-Yau, we are going to need a quasihomogenous polynomial of degree 8, which is hopefully clear from the previous calculations (i.e. the denominator in $c(X)$ is still just $1 + dH$).

Let's consider the case $y^2 = P_8(x_i)$, where $P_8(x_i)$ is some degree 8 polynomial in the x_i s. Now say we pick the patch $y = 1$, that is consider the chart where $y \neq 0$ and use the scaling to set $y = 1$. We now note that this scaling is *not* unique. That is, we can further multiply by any of the solutions to $\lambda^4 = 1$, but of course which solution we pick will effect the x_i values. In other words, we conclude that the x_i s are not free complex numbers but in fact

$$x_i \in \frac{\mathbb{C}}{\mathbb{Z}_4}.$$

We see that this problem goes away when $x_i = 0$, and we call this point a *fixed point*. This is an example of what is known as an *orbifold singularity*, which we do not discuss too much further here.¹ Another way to see this result is to note that A has a *fractal hyperplane class*. That is, if we consider the smooth point $p = [0 : 0 : 0 : 0 : 1]$, which is defined by the 4 polynomials $y = x_1 = x_2 = x_3 = 0$, then we have

$$\int_p 4H \cdot H \cdot H \cdot H = 1 \quad \Longleftrightarrow \quad H^4 = \frac{1}{4},$$

where the $4H$ and H factors are just the first Chern classes of the polynomials, and the 1 comes from simply integrating over p .

¹They will be discussed in the Toric Geometry notes.

Remark 6.0.1. It is important to note that this singularity is a property of the ambient space $A = \mathbb{WCP}_{11114}^4$ *not* the Calabi-Yau space X . This is seen simply from the fact that $p = [0 : 0 : 0 : 0 : 1]$ is not a solution to our defining polynomial.

As we mentioned at the start of this chapter, these singularities can actually be easily fixed using the techniques of toric geometry. For example, the resolution of the orbifold singularity in $\mathbb{WCP}_{1,1,n}^2$ is given by the *Hirzebruch surface* F_n . Again this will be discuss more in the toric geometry notes.

7 | Quick Summary

Let's just quickly remind ourselves what we have discussed in these notes.

- In Chapter 1 we discussed real manifolds in some detail, introducing perhaps previously unfamiliar structures such as de Rham cohomology and Hodge theory. We also briefly discussed holonomy at the end. The main results from this chapter were the procedure for constructing (co)homology groups, Poincaré duality, Betti numbers, Hodge decomposition and harmonic forms.
- Next, in Chapter 2 we took a "middle ground" approach to going from real to complex manifolds, by introducing an almost complex structure. We showed how to complexify the tangent bundle and so define complex tensor fields on a real manifold. We then adapted the above constructions to these complexified cases, also introducing Chern classes. The main results are the definitions of (p, q) -forms, the Dolbeault cohomology, Hodge numbers/diamond, and of course Chern classes.
- Chapter 3 then introduced complex projective spaces, and discussed how to construct hypersurfaces in these spaces by considering the zero locus of homogeneous polynomials. We then studied the total Chern classes of these spaces, as well as direct sums of different \mathbb{CP}^n s. We then did the same thing for weighted projective spaces. The main results are the total Chern class formulae, Equations (3.1) to (3.3).
- In Chapter 4 we finally introduced complex manifolds. We then discussed certain classes of complex manifolds, in particular Kähler manifolds. This was done by introducing the Hermitian form associated to the Hermitian metric and saying such a form is Kähler when it is closed. We then showed that \mathbb{CP}^n is a Kähler manifold, constructing the Fubini-Study metric. We could then introduce Calabi-Yau manifolds as a subset of Kähler manifolds in which the first Chern class vanished, while also introducing several other, obviously equivalent, definitions. We then discussed how the Hodge numbers on a Calabi-Yau manifold are related to each other and wrote down the Hodge diamond explicitly for a Calabi-Yau 3-fold, Equation (4.3). We then made a brief comment on Mirror symmetry at the end. The main results of this chapter is the definition of a Calabi-Yau manifold and the relation between the Hodge numbers, Euler characteristic and Chern classes.
- Then in Chapter 5 we discussed in some detail how to construct Calabi-Yau manifolds in (products of) \mathbb{CP}^n . We saw that projective spaces themselves are not Calabi-Yau, but that we could form Calabi-Yau manifolds by considering hypersurfaces in such spaces.

We showed that the first Chern class vanishes only when the polynomial degree(s) equal (the sum of) $n + 1$. We then worked through a few explicit examples of Calabi-Yau 3-folds, discussing the quintic in \mathbb{CP}^4 and the Tian-Yau manifold. We also worked through how to construct the holomorphic $(3, 0)$ -form for our Calabi-Yau 3-folds. We then concluded this chapter with a brief discussion of K3 surfaces. The main results were the Calabi-Yau condition stated above, and the details of the calculations.

- Finally in Chapter 6 we touched on orbifold singularities, showing how weighted projective spaces contain such singularities. As mentioned in the chapter, we will discuss these more in some follow up notes on toric geometry.

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