

# Degenerate crossing number and signed reversal distance\*

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## Abstract

The degenerate crossing number of a graph is the minimum number of crossings among all its drawings, where edges are represented as simple arcs and multiple edges passing through the same point are counted as a single crossing. Interpreting each crossing as a cross-cap induces an embedding into a non-orientable surface. In 2007, Mohar showed that the degenerate crossing number of a graph is at most its non-orientable genus and he conjectured that these quantities are equal for every graph. He also made the stronger conjecture that this also holds for any loopless pseudotriangulation with a fixed embedding scheme.

In this paper, we prove a structure theorem that almost completely classifies the loopless 2-vertex embedding schemes for which the degenerate crossing number equals the non-orientable genus. In particular, we provide a counterexample to Mohar’s stronger conjecture, but show that in the vast majority of the 2-vertex cases, the conjecture does hold.

The reversal distance between two signed permutations is the minimum number of reversals that transform one permutation to the other one. If we represent the trajectory of each element of a signed permutation under successive reversals by a simple arc, we obtain a drawing of a 2-vertex embedding scheme with degenerate crossings. Our main result is proved by leveraging this connection and a classical result in genome rearrangement (the Hannenhalli-Pevzner algorithm) and can also be understood as an extension of this algorithm when the reversals do not necessarily happen in a monotone order.

## 1 Introduction

A *cross-cap drawing* of a graph  $G$  is a drawing of  $G$  on the sphere with  $g$  distinct points, called *cross-caps*, such that the drawing is an embedding except at the cross-caps, where multiple edges are allowed to cross transversely, as pictured in Figure 1. In [13], Pach and Tóth introduced the *degenerate crossing number*, denoted by  $Cr_{deg}(G)$  which in this language is the minimum number of cross-caps for a cross-cap drawing of  $G$  to exist, where edges are required to be drawn as simple arcs. In [11], Mohar removed the constraint that edges be simple arcs, leading to the *genus crossing number*, which he proved to be equal to the non-orientable genus of the graph, denoted by  $g(G)$ . He then made an enticing conjecture claiming that these two crossing numbers are equal. We say that a cross-cap drawing of graph  $G$  is *perfect* if there are  $g(G)$  cross-caps and every edge intersects each cross-cap at most once. Then this conjecture can be restated as follows:

**Conjecture 1.1.** [11, Conjecture 3.1, Proposition 3.3] *Every simple graph has a perfect cross-cap drawing.*

Mohar went further and conjectured that for any loopless graph embedding, there exists a perfect cross-cap drawing that is compatible with this embedding, in a sense that we now

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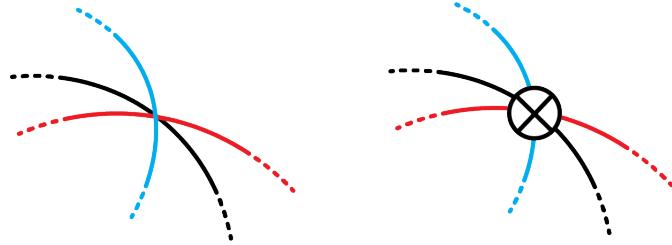


Figure 1: A degenerate crossing and a cross-cap placed at this crossing.

describe. The terminology that we introduce is equivalent to the PD1S in [11, Section 3]. A *pseudo-triangulation* is a cellularly embedded multi-graph  $\phi: G \rightarrow S$  in which each face has degree three. Denote by  $S^2 \setminus g\otimes$ , the sphere minus  $g$  tiny disks, and by  $(S^2 \setminus g\otimes)/\sim$  the space obtained by quotienting the boundary of each disk with the antipodal map. Topologically, this amounts gluing a Möbius band on each missing disk, thus yielding the non-orientable surface of genus  $g$ , denoted by  $N_g$ . We say that an embedded multi-graph  $\phi: G \rightarrow N_g$  *admits a cross-cap drawing*  $\phi': G \rightarrow S^2$ , if there is a homeomorphism  $f: N_g \rightarrow (S^2 \setminus g\otimes)/\sim$  such that  $f(\phi(G)) = \phi'(G)$ .

**Conjecture 1.2.** [11, Conjecture 3.4] *For any positive integer  $g$ , every loopless pseudo-triangulation of  $N_g$  admits a perfect cross-cap drawing.*

An even stronger conjecture was hinted at in [11, Paragraph following Conjecture 3.4], suggesting that one could possibly remove the loopless assumption if one forbids separating loops. This strengthening was disproved by Schaefer and Štefankovič [14, Theorem 7].

In addition to their motivation from crossing number theory, these conjectures would also shed light on the difficult task of visualizing high genus embedded graphs, providing an alternate approach to that of Duncan, Goodrich and Kobourov [5], who rely on canonical polygonal schemes [10].

**Our results.** A big step towards both these conjectures was achieved by Schaefer and Štefankovič, who proved [14, Theorem 10] that any multi-graph embedded on a non-orientable surface of genus  $g$  admits a cross-cap drawing with  $g$  cross-caps, in which each edge enters each cross-cap at most *twice*. This theorem applies in particular to one-vertex embedding schemes, and thus suggests a natural approach towards proving Conjectures 1.1 and 1.2. First contract a spanning tree to obtain a one-vertex graph and apply this theorem. Then, edges might enter cross-caps twice, but since the initial graph is loopless, one could hope to uncontract some edges so as to spread these two cross-caps on two edges, thus obtaining a perfect cross-cap drawing. Our first result shows that this approach cannot work, as some loopless 2-vertex schemes do not admit perfect cross-cap drawings.

**Theorem 1.3.** *A loopless 2-vertex embedding scheme that consists of exactly one non-trivial positive block and one non-trivial negative block admits no perfect cross-cap drawing.*

We refer to Figure 2 for an example that should provide an intuitive idea of the notion of blocks, and to Section 2 for the precise definition. As a corollary, we obtain a counter-example to Conjecture 1.2:

**Corollary 1.4.** *There exist a loopless pseudo-triangulation  $G$  that admits no perfect cross-cap drawing.*

Our second contribution and main theorem is a converse to Theorem 1.3.

**Theorem 1.5.** *For any embedding  $G$  of a loopless 2-vertex graph on  $N_g$ , at least one of the following is true.*

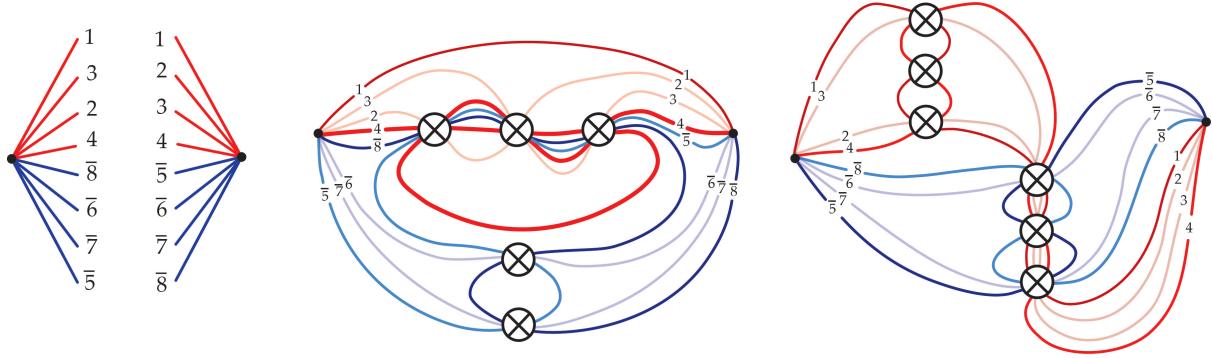


Figure 2: Left: A loopless 2-vertex scheme made of a positive block, in red, consisting of only positive edges, and a negative block, in blue, consisting of negative edges. Middle: a cross-cap drawing showing that it has non-orientable genus 5. The bolded edge enters a cross-cap twice. Right: A cross-cap drawing where each edge enters each cross-cap at most once requires 6 cross-caps.

1.  $G$  admits a perfect cross-cap drawing with  $g$  cross-caps,
2. or the reduced graph of  $G$  is one of the two schemes pictured in Figure 3,



Figure 3: The only two exceptions to perfect cross-cap drawings.

We refer to Section 2 for the definition of blocks and reduced graphs. Essentially, Theorem 1.5 shows that apart from two narrow families of exceptional cases, all the loopless 2-vertex embeddings do satisfy Conjecture 1.2. As an illustration, Figure 4 shows that while the example in Figure 2 does not admit a perfect cross-cap drawing, surprisingly, it does after *adding* two edges to it. It directly follows from Theorem 1.5 that under standard random models, any loopless 2-vertex embedding scheme admits a perfect cross-cap drawing asymptotically almost surely.

**Techniques and connections to signed reversal distance.** Our focus on the 2-vertex case in Theorem 1.5 is further motivated by a connection (introduced in [7]) to computational genomics. An important problem in computational biology is to compute various notions of distance between two genomes (see [4, 6]). Remarkably, one of the most biologically relevant distances is also one of the few that can be calculated efficiently: a one chromosome genome is encoded by a signed permutation (i.e., permutations of integers in which each element has a sign)  $\pi = (\pi_1, \dots, \pi_n)$ . The *reversal* of the interval  $(i, j)$  acts on  $\pi$  by reversing the order of the elements  $\pi_i, \dots, \pi_j$  as well as their signs, it maps

$$(\pi_1, \pi_2, \dots, \pi_{i-1}, \pi_i, \pi_{i+1} \dots, \pi_{j-1}, \pi_j, \pi_{j+1} \dots \pi_n),$$

to

$$(\pi_1, \pi_2, \dots, \pi_{i-1}, \overline{\pi_j}, \overline{\pi_{j-1}} \dots, \overline{\pi_{i+1}}, \overline{\pi_i}, \pi_{j+1} \dots \pi_n).$$

The *reversal distance* between signed permutations  $\pi, \pi'$ , denoted by  $d(\pi, \pi')$  is the minimum number of reversals needed to transform  $\pi$  to  $\pi'$ . A celebrated algorithm of Hannenhalli and Pevzner [8] (see also [2, 3]) computes in polynomial time the reversal distance between two signed permutations.

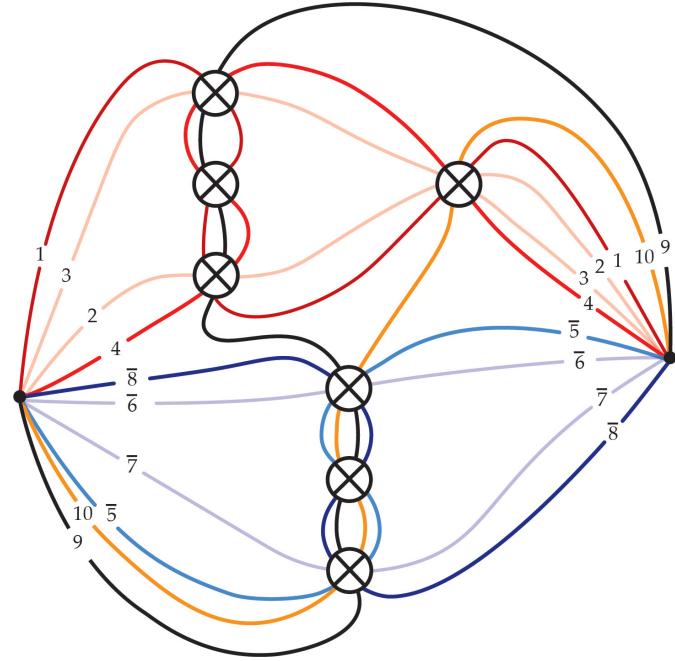


Figure 4: A perfect cross-cap drawing of Figure 2 with two additional edges.

Now, a signed permutation is exactly the combinatorial data in the embedding scheme of a 2-vertex loopless graph, and when tracing the action of each element of a signed permutation under the action of reversals, one obtains a perfect cross-cap drawing of that embedding scheme, where each reversal corresponds to a cross-cap and each edge is an x-monotone curve, and in particular no edge enters twice the same cross-cap (see Figure 5 for an illustration).

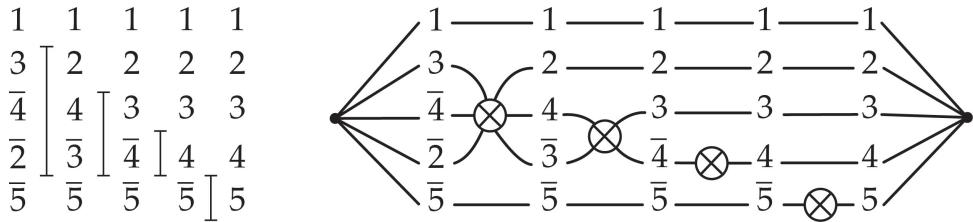


Figure 5: From sorting permutations with reversals to perfect cross-cap drawings.

This easily implies the inequalities:  $g(\pi) \leq Cr_{deg}(\pi) \leq d(\pi, id)$ . It turns out that the inequality  $g(\pi) \leq d(\pi, id)$  is central to the reversal distance theory, and the cases of equality are well understood. As we explain in Section 4 these arguments are natural from the point of view of embedding schemes. Conversely, Theorem 1.2 can be reinterpreted in the setting of signed permutations as providing an extension of the Hannenhalli-Pevzner theory.

The proof of Theorem 1.2 consists of two steps which can readily be made algorithmic: we first *reduce* a signed permutation  $\pi$  to a simpler one  $\pi|$  for which we can prove that  $g(\pi|) = d(\pi|, id)$  (Lemma 5.1), then we devise a technique to *blow up* (Lemma 5.3) the cross-cap drawing of the reduced signed permutation  $\pi|$ , yielding a perfect cross-cap drawing of the original signed permutation.

## 2 Preliminaries

### Embedding schemes

In this article, we work with multi-graphs, possibly with loops and multiple edges. An *embedding* of a graph  $G$  on a surface  $S$  is an injective map  $\phi : G \rightarrow S$ . We consider two embeddings equivalent if they are homeomorphic. The *faces* of an embedded graph are the connected components of  $S \setminus \phi(G)$ . An embedding is *cellular* if its faces are homeomorphic to topological disks. The Euler genus,  $eg(G)$ , of a cellular embedding of a graph  $G$  is the quantity  $2 - v + e - f$ , where  $v$ ,  $e$  and  $f$  denote respectively the number of vertices, edges and faces of the embedding of  $G$ . If  $S$  is orientable, an embedding can be described combinatorially by a *rotation system*, that is, the set of cyclic permutations encoding the order of the edges around each vertex. When  $S$  is non-orientable, which will always be the case in this paper, some additional data is required to encode an embedding, which is encompassed in the concept of an *embedding scheme*. We introduce the main definitions and refer to Mohar and Thomassen [12, Section 3.3] for extensive background.

**Definition 2.1** (Embedding scheme). *An embedding scheme consists of a triple  $(G, \rho, \lambda)$  where*

- $G$  is a graph,
- $\rho = \{\rho_v, v \in V(G)\}$ , where each  $\rho_v$  is a cyclic permutation of the edges incident to  $v$ , and
- $\lambda$  is a function that assigns a signature  $\{+1, -1\}$  to each edge of  $G$ .

From an embedding scheme  $(G, \rho, \lambda)$ , one can naturally recover the set of *facial cycles* by following the edges and switching sides according to their signatures. Then, pasting a topological disk on each facial cycle yields a cellular embedding of  $G$ . Given an embedding scheme  $(G, \rho, \lambda)$ , a *flip* at a vertex  $v$  is another embedding scheme of the same graph, in which we reverse the order of the edges incident to  $v$  and invert the signature of those edges incident to  $v$  that are not loops. We say that two embedding schemes  $(G, \rho, \lambda)$  and  $(G, \rho', \lambda')$  are *equivalent* if one can go from one to the other one by a sequence of flips. Two embedding schemes are equivalent if and only if they induce equivalent cellular embeddings [12, Theorem 3.3.1]. This justifies that equivalence classes of embedding schemes and embedded graphs can be considered as being two representations of the same objects, and we switch freely between the two points of views in this article, sometimes using the shorthand  $G$  to denote  $(G, \rho, \lambda)$ .

A closed curve in an embedding scheme is one-sided (resp. two-sided) if and only if the signatures of its edges multiply to  $-1$  (resp.  $+1$ ). An embedding scheme is called *orientable* if it only contains two-sided closed curves; otherwise it is called *non-orientable*. A closed curve in a non-orientable embedding scheme is *separating* if cutting along  $\gamma$  yields two connected components. A closed curve in a non-orientable embedding scheme is *orienting* [7, 14] if by cutting along  $\gamma$ , we obtain a connected orientable surface.

For two edges  $a$  and  $b$  in a 2-vertex embedding scheme, we denote by  $a \cdot b$  the concatenation of  $a$  and  $b$  which we will interpret as a cycle. Denoting the vertices of the scheme by  $v_1$  and  $v_2$ , we define a *wedge* between  $a$  and  $b$ ,  $\omega_{a,b}$ :

- If both  $a$  and  $b$  are negative, then  $\omega_{a,b}$  contains all the half-edges in the interval  $(a, b)$  in both  $\rho_{v_1}$  and  $\rho_{v_2}$ .
- If at least one of them is positive, then  $\omega_{a,b}$  contains all the half-edges in the interval  $(a, b)$  in  $\rho_{v_1}$  and  $(b, a)$  in  $\rho_{v_2}$ .

We say that a wedge *encloses* an edge if it contains both its half-edges or none of them. For example  $w_{1,4}$  in Figure 6 encloses all the edges of the graph. We can recognize orienting and separating curves in an embedding scheme with the following lemmas.

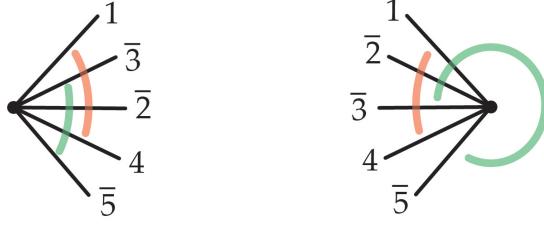


Figure 6: The wedges  $\omega_{1,4}$  and  $\omega_{3,5}$  depicted in pink and green respectively.

**Lemma 2.2.** *For two edges  $a$  and  $b$  in a non-orientable loopless 2-vertex embedding scheme, the cycle  $a.b$  is orienting,*

- if at least one of  $a$  and  $b$  is positive and  $\omega_{a,b}$  contains exactly one end of all negative edges and encloses all the positive edges, or
- if both  $a$  and  $b$  are negative and their wedge contains exactly one end of all positive edges and encloses all the negative edges.

*Proof.* We prove the lemma by contracting the edge  $a$  in order to obtain a one-vertex embedding scheme  $G'$ . Note that the topological type of the cycle formed by the edges  $a$  and  $b$  is the same as the loop  $b$  in  $G'$ . Denote the vertices of  $G$  by  $v_1$  and  $v_2$ . When we contract a positive edge  $e$  in  $G$ , we obtain an embedding scheme  $G'$  with a single vertex  $w$  such that the cyclic permutation of the edges around  $w$  after contraction is  $\rho_w = \rho_{v_1} \rho_{v_2}$  where the edge  $e$  has been removed from both cyclic permutations and the notation means that have been concatenated at  $e$ . On the other hand, when we contract a negative edge  $e$  in  $G$ ,  $\rho_w = \rho_{v_1} \overline{\rho_{v_2}}$ , where the edge  $e$  has been removed from both permutations, the signature of all the edges are reversed and the notation means that they have been concatenated at  $e$ . The ends of the loop  $b$  subdivide the half-edges around  $w$  into two sets. We can see that the half-edges in  $\omega_{a,b}$  in  $G$  correspond to one of the sets of half-edges divided by  $b$  in  $\rho_w$ .

By [7, Lemma 2.3], a loop  $o$  in a 1-vertex non-orientable embedding scheme is orienting if and only if its ends *alternate* with the ends of all negative loops in the cyclic permutation around the vertex and *enclose* the ends of any positive loop; i.e. the ends of  $o$  does not alternate with the ends of any positive loop.

To prove the first case, without loss of generality we can assume that  $a$  is positive. We contract the edge  $a$  in  $G$ . Since the wedge  $\omega_{a,b}$  in  $G$  contains exactly one end of each negative edge, the ends of the loop  $b$  alternate with the ends of negative loops in  $\rho_w$ . Similarly, since  $\omega_{a,b}$  encloses all the positive edges, the ends of  $b$  enclose the ends of any positive loop in  $\rho_w$ . Therefore  $b$  is orienting in  $G'$  and this implies that the cycle formed by  $a$  and  $b$  is orienting in  $G$ . For the proof of the second case in 1 we proceed identically by contracting the negative edge  $a$ . This finishes the proof.  $\square$

**Lemma 2.3.** *For two edges  $a$  and  $b$  in a loopless 2-vertex embedding scheme, the cycle  $a.b$  is separating if  $a$  and  $b$  have the same signature and  $\omega_{a,b}$  encloses all the edges.*

*Proof.* We contract  $a$ . The loop  $b$  has positive signature in  $G'$  and since  $\omega_{a,b}$  contains both ends of any edge inside it then the ends of  $b$  separate the ends of the other loops in  $G'$ , i.e. the ends of no loop alternates with those of  $b$ . Such a loop is separating the surface, and therefore  $a$  and  $b$  form a separating cycle in  $G$ .  $\square$

See Figure 6 for an example of separating and orienting cycle in a 2-vertex scheme.

## 2-vertex embedding schemes and signed permutations.

This paper almost exclusively deals with loopless graphs with two vertices, and in that setting embedding schemes take a particularly simple form. Without loss of generality, one can number the edges so that the cyclic permutation around one of the vertices is the identity. Then the data of the embedding scheme just consists of the cyclic permutation around the other vertex, and the signature of the edges, and thus this amounts to a *signed cyclic permutation*: a cyclic permutation where each number is additionally endowed with a + or - sign. We also use an overline notation  $\bar{i}$  to depict negative signs. Therefore, in what follows we freely identify a signed permutation and a 2-vertex embedding scheme. Two edges in a 2-vertex embedding scheme are *homotopic* if either they are both positive and appear consecutively in increasing order (e.g.,  $(i, i+1)$ ) or they are both negative and they appear in the reverse order (e.g.,  $(\bar{i+1}, \bar{i})$ ).

A *positive block* in a signed permutation is an interval  $I = (\pi_i, \dots, \pi_j)$  where all the elements are positive,  $\pi_i < \pi_j$ , all the integers in  $[\pi_i, \pi_j]$  are contained in  $I$ . A *negative block* in a signed permutation is an interval  $I = (\bar{\pi_i}, \dots, \bar{\pi_j})$  where all the elements are negative,  $\pi_i > \pi_j$ , all the integers in  $[\pi_j, \pi_i]$  are contained in  $I$ . A block is *non-trivial* if it is not already sorted, i.e., it is not equal to  $(\pi_i, \pi_i+1, \dots, \pi_j-1, \pi_j)$  or to  $(\bar{\pi_i}, \bar{\pi_i-1}, \dots, \bar{\pi_j+1}, \bar{\pi_j})$ . Our concept of blocks is similar to the notion of hurdles of [8], or more accurately, to the notion of unoriented components in [3]. In both cases, we call  $\pi_i$  and  $\pi_j$  the frames of the block. A block is called *minimal* if it does not contain any block except itself. We say that a signed permutation is *reduced* if it has no blocks. Given a signed permutation  $\pi$ , its reduced permutation  $\pi|$  is a permutation in which we replace every minimal block with a single element of the same sign, and we iterate this process until we arrive at a reduced permutation.

## Cross-cap drawings

A cross-cap drawing of an embedding scheme  $(G, \rho, \lambda)$  is a cross-cap drawing of  $(G, \rho, \lambda)$  or of an equivalent scheme (under flips), that respects the cyclic permutations and signatures on the edges, i.e., if an edge has signature +1 (resp. -1) then it enters an even (resp. odd) number of cross-caps. This definition is equivalent to the cross-cap drawings of embeddings defined in the introduction. For non-orientable embedding schemes, by Euler's formula, the minimum number of cross-caps for a drawing coincides with the Euler genus. For orientable embedding schemes of non-zero Euler genus, one needs exactly one additional cross-cap:

**Lemma 2.4** ([14, Lemma 6]). *Let  $(G, \rho, \lambda)$  be an orientable embedding scheme with non-zero Euler genus. Then any cross-cap drawing of  $G$  requires  $eg(G) + 1$  cross-caps, in particular  $eg(G) + 1$  is odd.*

The next lemma allows us to recognize types of curves in a cross-cap drawing.

**Lemma 2.5.** *For any cross-cap drawing of a non-orientable embedding scheme:*

1. *A closed curve is one-sided (two-sided) if and only if it enters an odd (even) number of cross-caps.*
2. *A closed curve is orienting (separating) if and only if it enters each cross-cap an odd (even) number of times.*

We refer to [14, Lemma 3] and [14, Lemma 4] for proofs.

## Reversal distance and monotone cross-cap drawings.

Signed permutations model genomes with a single chromosome in computational biology where they come endowed with the reversal distance. The reversal distance  $d(\pi, id)$  is the smallest

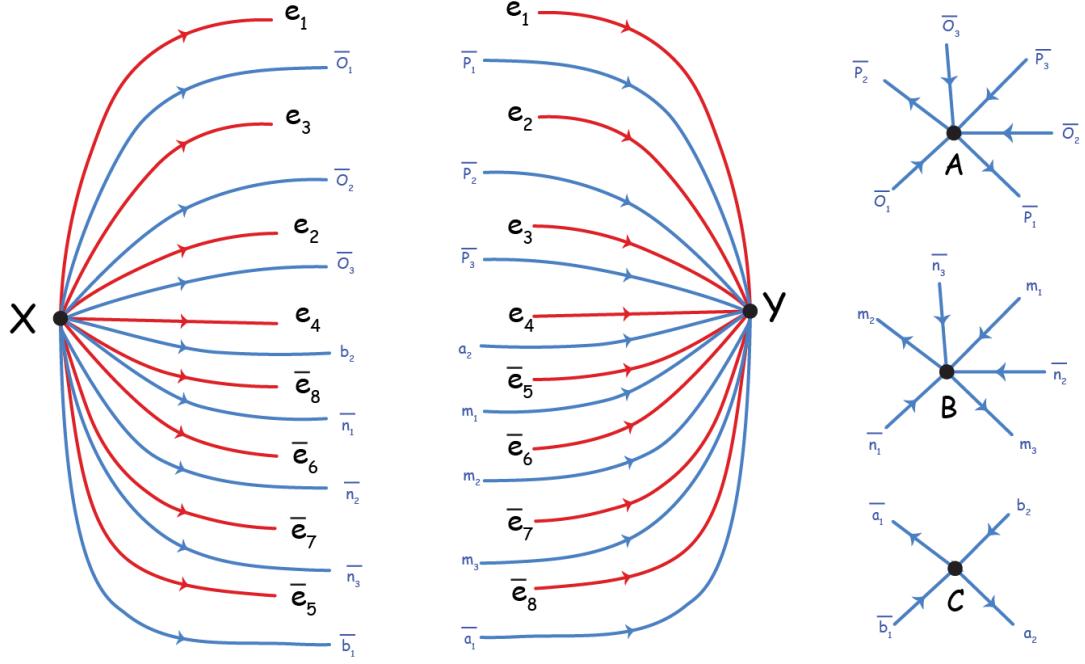


Figure 7: A pseudo-triangulation of  $N_5$  admitting no perfect cross-cap drawing.

number  $d$  such that there exists a sequence  $\{\pi = \pi^1, \pi^2, \dots, \pi^d = id\}$  such that  $(\pi^i)$  and  $(\pi^{i+1})$  differ by a signed reversal. We call such a (not necessarily minimizing) sequence, a *path* of signed permutations.

To any path of signed permutations  $\{\pi^1, \pi^2, \dots, \pi^d\}$  we can associate a cross-cap drawing. We place a source vertex at  $(-1, n/2)$  and a terminal vertex at  $(d+1, n/2)$ . Edges will be  $x$ -monotone piece-wise linear curves between these two vertices, and at each crossing between two such curves we introduce a cross-cap. The edge  $j$  emanates from  $(-1, n/2)$  to  $(0, \pi_j^1)$ , for each  $k \leq d$  it passes through  $(k, \pi_k^j) \in \mathbb{R}^2$ , and finally it connects  $(d, \pi_j^d)$  to the terminal vertex at  $(d+1, n/2)$ . In the remaining of this article, we often forget about the vertices  $(-1, n/2)$  and  $(d+1, n/2)$  in our illustrations as they play no role, and we always assume that  $\pi^d$  is the identity (see Figure 5).

### 3 The counterexample

In this section, we provide a family of 2-vertex embedding schemes that do not admit a perfect cross-cap drawing. Then we provide an explicit pseudo-triangulation of  $N_5$  (depicted in Figure 7), disproving Conjecture 1.2.

**Remark 3.1.** *If an embedding scheme  $C$  has one positive and one negative block, then so does its flipped version, therefore we do not need to account for the possible flip in the proof of Theorem 1.3.*

In order to prove Theorem 1.3, we rely on Lemmas 3.1 and 3.2.

**Lemma 3.2.** *Let  $G$  be an embedding scheme with a non-trivial positive block  $A$  and a non-trivial negative block  $B$ , then  $g(G) = g(A) + g(B) - 1$ .*

*Proof.* Assume that the positive block has edges labelled  $A = \{e_1, \dots, e_k\}$  and the edges of the negative block are  $B = \{\bar{e}_{k+1}, \bar{e}_{k+2}, \dots, \bar{e}_{k+l}\}$ . Notice that  $f(C) = f(A) + f(B) - 1$ , indeed the face  $e_1, e_k$  and the face  $\bar{e}_{k+1}, \bar{e}_{k+l}$  are the outer faces of  $A$  and  $B$ , and they merge to become

the face  $e_1, \bar{e}_{k+1}, e_k, \bar{e}_{k+l}$ . Hence by Euler's formula  $eg(C) = eg(A) + eg(B) + 1$ . We know that  $g_A = eg(A) + 1$  and  $g_B = eg(B) + 1$ . On the other hand, a cycle in  $C$  that contains one edge from  $A$  and one edge from  $B$  is one-sided, therefore  $C$  is a non-orientable, hence  $eg(C) = g_C$ . All in all we can conclude

$$g_C = eg(C) = eg(A) + eg(B) + 1 = g_A - 1 + g_B - 1 + 1 = g_A + g_B - 1$$

as claimed.  $\square$

We now have all the tools to prove Theorem 1.3, and refer to Figure 2 for an example to help follow the proof.

*Proof of Theorem 1.3.* Let  $C$  be a concatenation of a positive block  $A$  with frames  $a_1$  and  $a_2$  and a negative block  $B$  with frames  $b_1$  and  $b_2$ . Let us assume that  $\phi$  is a perfect cross-cap drawing of  $C$ . From Lemma 2.2 we derive that  $a_1 \cdot b_1$  and  $a_1 \cdot b_2$  are orienting curves, hence by Lemma 2.5 each of them enters each cross-cap once. Lemma 2.3 implies that  $b_1 \cdot b_2$  is separating. Therefore by Lemma 2.5,  $b_1$  and  $b_2$  enter the same cross-caps and do not enter any cross-cap that  $a_1$  enters. Similarly,  $a_1 \cdot a_2$  is separating and hence they enter the same cross-caps and no cross-cap that  $b_1$  and  $b_2$  enter. Then  $A$  is drawn with  $g_A$  cross-caps and  $B$  is drawn with  $g_B$  cross-caps that are disjoint from the cross-caps that  $A$  entered. But by Lemma 3.2 the non-orientable genus of  $C$  is  $g_A + g_B - 1$ . Therefore, there are not enough cross-caps available to draw both  $A$  and  $B$ . This concludes.  $\square$

Corollary 1.4 follows at once as we can always add edges and vertices to a scheme to triangulate it without adding loops nor changing its genus, and any perfect cross-cap drawing of the triangulation restricts to a cross-cap drawing of the scheme. We provide in Figure 7 an example of such a pseudo-triangulation.

## 4 Topology of the reversal distance

In this section, we recall some well-known results from the genomics rearrangements literature<sup>1</sup>, which we interpret in the language of embedding schemes (see also [9] for an alternate topological interpretation of these arguments).

### 4.1 The Bafna-Pevzner inequality from Euler's formula.

Let  $\pi$  be a signed cyclic permutation, which, as explained in Section 2, we think of as a 2-vertex embedding scheme, which requires  $g(\pi)$  cross-caps to be drawn. We can compute its number of faces, which we denote by  $f(\pi)$  and the number of elements in the permutation corresponds to the number of edges in the scheme, which we denote by  $e(\pi)$ . Then Euler's formula reads  $2 - eg(\pi) = 2 - e(\pi) + f(\pi)$  which simplifies to  $eg(\pi) = e(\pi) - f(\pi)$ . By Lemma 2.4, we thus have  $d(\pi, id) \geq e(\pi) - f(\pi)$ .

A very similar inequality was first discovered by Bafna and Pevzner [1, Theorem 2] without reference to embeddings. Figure 8 shows an example where the inequality is strict: the embedding scheme has Euler genus two, non-orientable genus three and one can show that the signed permutation requires four reversals to be sorted. However, as pictured on the right, it does admit a perfect cross-cap drawing with three cross-caps. Necessarily, in that example, the cross-caps can not be interpreted as reversals: this is apparent here as they do not occur in a monotone order.

The starting idea of the Hannenhalli-Pevzner (HP) algorithm is to identify intervals in a signed permutation where applying a reversal is clearly making progress. Given a signed cyclic

<sup>1</sup>The literature primarily deals with sorting standard permutations, while here we are sorting cyclic permutations. In our description, we directly translate their techniques to this cyclic setting.

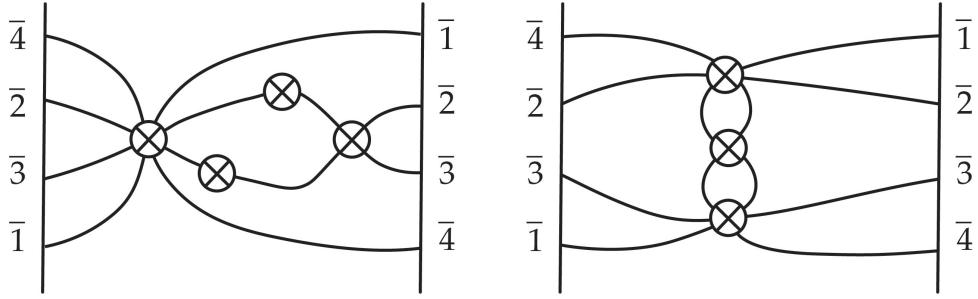


Figure 8: The embedding scheme depicted in this picture has non-orientable genus 3 but requires four reversals (left). However, it admits a perfect cross-cap drawing with three cross-caps (right).

permutation  $\pi$ , we call a pair of consecutive integers  $i$  and  $i+1$  *reversible* if they have opposite signs in  $\pi$  (this is called an *oriented pair* in [3]). For a given reversible pair there exist two reversals  $\sigma, \sigma'$  such that  $i$  and  $i+1$  are homotopic in  $\pi \cdot \sigma$  and in  $\pi \cdot \sigma'$ . These two reversals are equivalent in the sense that the two permutations that they yield are flipped versions of each other. The following lemma follows from an Euler characteristic argument.

**Lemma 4.1.** *If  $i, i+1$  are a reversible pair in  $\pi$  and  $\sigma$  is a reversal that turns them into homotopic curves in  $\pi \cdot \sigma$ , then  $eg(\pi \cdot \sigma) = eg(\pi) - 1$ .*

*Proof.* Since  $e(\pi) = e(\pi \cdot \sigma)$ , we need to show that  $f(\pi \cdot \sigma) = f(\pi) + 1$ . The edges  $i$  and  $i+1$  appear in some face  $a$  in  $\pi$ . We claim that the faces of  $\pi \cdot \sigma$  are the same as the faces of  $\pi$  except that  $a$  is subdivided into the bigon  $(i, i+1)$  and another face which contains the edges of  $a$  minus  $i$  and  $i+1$ . Indeed every other face is not disrupted by the reversal. This finishes the proof.  $\square$

## 4.2 The HP algorithm for reduced signed permutations

It is immediate to see that blocks do not contain reversible pairs of edges, and thus non-trivial blocks form a natural obstruction to applying the reversals described before. Given a reversible pair  $(i, i+1)$  in the signed permutation  $\pi$ , let  $\sigma$  be a reversal that turns  $i$  and  $i+1$  into homotopic edges. The *score* of  $(i, i+1)$  is the number of reversible pairs in  $\pi \cdot \sigma$ .

**HP algorithm:** While there is a reversible pair, reverse a pair of maximal score [2].

**Theorem 4.2.** *If a signed permutation  $\pi$  is non-orientable and has no non-trivial blocks then  $d(\pi, id) = eg(\pi)$ , and the HP algorithm gives a sequence of reversals of this optimal length.*

This theorem follows at once from the following lemma.

**Lemma 4.3.** *If  $\pi$  is a non-orientable signed permutation without non-trivial blocks,  $(i, i+1)$  is a reversible pair of maximal score, and  $\sigma$  is a reversal that makes  $i$  and  $i+1$  homotopic, then if  $\pi \cdot \sigma$  is not the identity, it is non-orientable and has no non-trivial blocks.*

The proof is almost identical to that of [2, Theorem 10], see Appendix A.

*Proof of Theorem 4.2.* By the previous lemma, if  $\pi$  is non-orientable has no non-trivial block and  $\sigma$  is a reversal of maximum score, then  $\pi \cdot \sigma$  is also non-orientable and also has no non-trivial block. By induction,  $d(\pi \cdot \sigma, id) = e(\pi \cdot \sigma) - f(\pi \cdot \sigma) = eg(\pi \cdot \sigma)$ . Therefore, by Lemma 4.1,  $eg(\pi) \leq d(\pi, id)$ , and  $d(\pi, id) \leq d(\pi \cdot \sigma, id) + 1 = eg(\pi \cdot \sigma) + 1 = e(\pi) - f(\pi) = eg(\pi)$ .  $\square$

## 5 Perfect drawings for most 2-vertex graph embeddings

We say that a cross-cap drawing is *fantastic* if it is perfect and every edge enters at least one cross-cap.

**Lemma 5.1.** *Every reduced loopless 2-vertex graph embedding scheme that is different from  $(1)$ ,  $(1, \bar{2})$  or  $(1, \bar{3}, \bar{4}, 2)$  admits a fantastic cross-cap drawing.*

The proof is based on an induction and an exhaustive analysis of all the loopless 2-vertex embedding schemes of genus 2 and 3, as pictured in Figure 9. We first prove the following lemma.

**Lemma 5.2.** *Let  $G$  be an orientable scheme of non-orientable genus  $g(G)$  that has no non-trivial block. We can add an edge to this embedding scheme such that we obtain a non-orientable scheme of genus  $g(G)$  without any block.*

*Proof.* Let us assume that the signatures of the edges are positive. Choose an edge  $f$ . Add a negative edge  $e$  consecutive to  $f$ , such that in the cyclic permutation of one vertex we see  $ef$  and in the other vertex we see  $fe$ . Let us call the new embedding scheme by  $G'$ . The scheme  $G'$  is non-orientable. Since  $G$  is orientable, we know that  $g(G) = eg(G) + 1$ . Since  $e(G') = e(G) + 1$ , it is enough to show that  $f(G') = f(G)$ . By looking at the face cycles of both schemes, we can see that all face cycles are intact except a face  $(i, f, j, \dots)$  in  $G$  that is turned to a face cycle  $(i, e, f, e, j, \dots)$  ( $i$  and  $j$  are two adjacent edges to  $f$  in the cyclic permutation of the vertices). Since the edge  $e$  has an opposite signature compared to the edges of  $G$ ,  $e$  cannot create a block in  $G'$ . The proof for the case where all the edges of  $G$  are negative can be obtained by flipping. This finishes the proof.  $\square$

*Proof of Lemma 5.1.* Let  $\pi$  be the signed permutation associated to the embedding scheme. If the embedding scheme is orientable, we first add one edge without changing its genus to make it non-orientable while keeping it reduced (this is possible by Lemma 5.2). Now, since it is reduced and non-orientable, by Theorem 4.2, the HP algorithm provides a path in the reversal graph  $\{\pi = \pi^1, \pi^2, \dots, \pi^g = id\}$ , where  $g$  is the non-orientable genus of  $\pi$ . We distinguish cases depending on the value of  $g$ .

If  $g \geq 3$ , we consider the sub-path  $\{\pi^1, \pi^2, \dots, \pi^k\}$ , with  $g - k = 3$  and realize this sub-path as a cross-cap drawing  $\phi$  as described in Section 2. Notice that if there exists a fantastic cross-cap drawing  $\phi'$  for  $\pi^k$ , we can concatenate  $\phi$  with  $\phi'$  to obtain a fantastic cross-cap drawing for  $\pi$ . Now  $\pi^k$  is a reduced signed permutation of non-orientable genus 3. The proof then proceeds via an exhaustive case analysis. Without loss of generality, we can assume that

- There is no non-trivial block in  $\pi^k$  since that is preserved by the HP algorithm.
- There are no homotopic edges since, for any collection of homotopic edges, one can remove all but one and add them in the end identical to the remaining one.
- There is at least one edge of signature  $-1$  and one edge of signature  $+1$  in  $\pi^k$ . Otherwise,  $\pi^k$  is orientable, which is impossible since by Theorem 4.2, the HP algorithm preserves non-orientability.
- $\pi^k$  is maximal while preserving these three properties. Indeed, otherwise, we can add edges, draw the resulting scheme and remove these superfluous edges at the end.

With these simplifying assumptions at our disposal, we can exhaustively enumerate all the genus 3 embedding schemes matching these assumptions. The numbers are small enough that this can be done by hand, we ran a computer search for safety. One obtains that all the maximal schemes have their all faces of degree 4, and thus have six edges: for all reduced schemes

with some faces of degree higher than four, one can always add an edge within a face while keeping the fact that it is reduced. Then, there are exactly eight loopless 2-vertex embedding schemes matching our assumptions, their signed permutations are:  $(1, \bar{6}, 5, \bar{4}, 3, \bar{2})$ ,  $(1, \bar{6}, \bar{3}, 5, 4, \bar{2})$ ,  $(1, \bar{3}, \bar{6}, 5, \bar{4}, 2)$ ,  $(1, 5, \bar{3}, \bar{6}, 4, 2)$ ,  $(1, \bar{4}, 6, 3, \bar{5}, 2)$ ,  $(1, \bar{6}, 3, \bar{4}, 5, \bar{2})$ ,  $(1, \bar{4}, 6, \bar{2}, 5, \bar{3})$ , and  $(1, \bar{4}, \bar{6}, \bar{2}, 5, 3)$ . Fantastic drawings of each of them are provided in Figure 9.

If  $g = 2$ , there are exactly three reduced schemes:  $(1, \bar{3}, \bar{4}, 2)$ ,  $(1, \bar{2}, 3, \bar{4})$  and  $(1, 3, \bar{2})$ . Fantastic drawings of the second and third case (or rather its flipped version) are provided in Figure 9.

If  $g = 1$ , there is a single reduced scheme:  $(1, \bar{2})$ .

If  $g = 0$ , there is a single reduced scheme:  $(1)$ .  $\square$

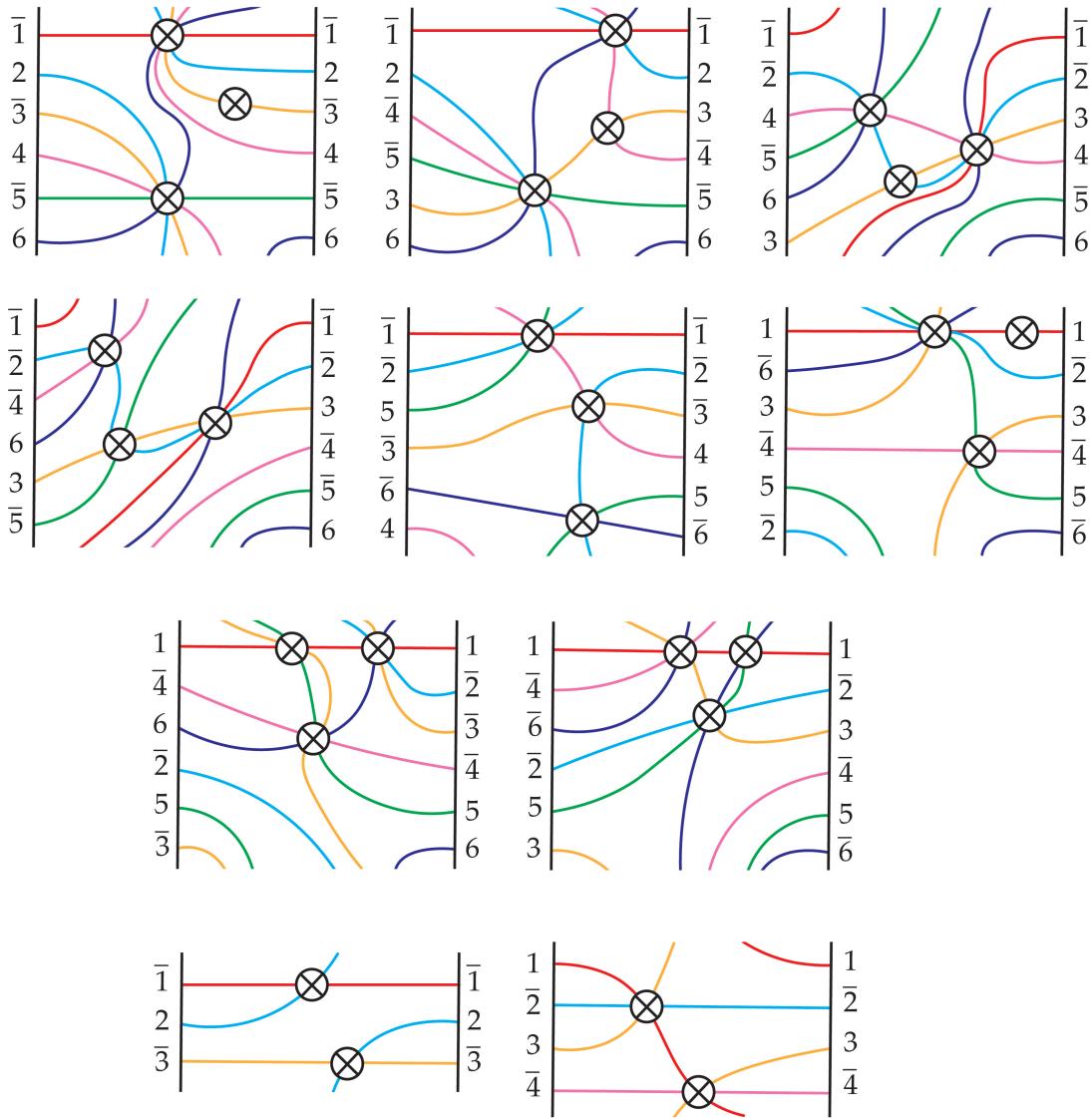


Figure 9: Fantastic drawings of genus-2 and genus-3 embedding schemes (these are cylindrical drawings: the top is identified to the bottom).

In order to prove Theorem 1.5, our strategy is to first look for a fantastic cross-cap drawing for the reduced graph of an embedding scheme. Then, to obtain a drawing for the initial graph, we need to bring back the blocks that we replaced and extend the drawing to the edges of the block. This is achieved via the following blowing-up operation.

**Blowing up a cross-cap.** Let  $\phi$  be a fantastic drawing for  $G$ . By the definition of fantastic drawing, any reduced edge enters at least one cross-cap. Let  $e$  be a reduced edge that corresponded to a block  $X$  with frames  $a$  and  $b$  and let  $\mathbf{c}$  be a cross-cap in  $\phi$  that  $e$  enters. By Lemma 2.4,  $X$  needs an odd number of cross-caps to be drawn, exactly  $eg(X) + 1$ . We replace  $\mathbf{c}$  by  $eg(X) + 1$  cross-caps and we draw the frames of  $X$ ,  $a$  and  $b$  as follows. We draw  $a$  following  $e$  thoroughly. To draw  $b$  we follow  $e$  except that we make  $b$  enter the new  $eg(X) + 1$  cross-caps in the reversed order that  $e$  enters. All the other edges outside of  $X$  that were entering  $e$  are now drawn in the same way as  $a$  or  $b$  depending on how they were crossing  $e$  at  $\mathbf{c}$ . Finally we remove  $e$  (see Figure 10).

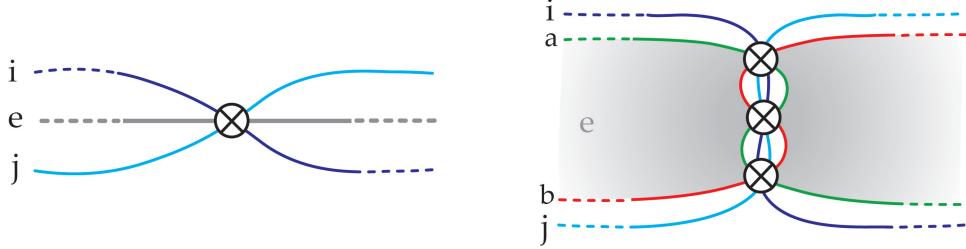


Figure 10: Blowing up a cross-cap to draw the frames of a block

Repeatedly blowing up a drawing of a reduced permutation  $\pi|$  yields a drawing for all edges of  $\pi$  except for the edges inside the blocks. The following lemma shows that these edges can be added in this cross-cap drawing.

**Lemma 5.3.** *Let  $\pi$  be a signed permutation on  $n$  elements such that all the elements form a non-trivial minimal negative block. The associated embedding scheme admits a perfect cross-cap drawing in which the frames of  $\pi$ , i.e. elements 1 and  $n$ , enter all the cross-caps but in opposite order.*

*Proof.* Let us denote by  $g(\pi) = eg(\pi) + 1$  the non-orientable genus of the associated embedding scheme. We know that  $\pi_1 = \bar{n}$  and  $\pi_n = \bar{1}$ . Let us define  $\pi'$  from  $\pi$  by replacing  $\bar{1}$  by  $n + 1$ . The following lemma is proved using the HP algorithm and Theorem 4.2.  $\square$

**Lemma 5.4.** *The optimal number of reversals to go from  $\pi'$  to the permutation  $(2, 3, \dots, n + 1)$  is  $g(\pi') = g(\pi)$ . There exists a sequence of such reversals such that no reversal is applied on the element  $n + 1$ .*

*Proof.* Note that the associated embedding scheme to  $\pi'$  is a non-orientable scheme and therefore  $g(\pi') = eg(\pi')$ . The number of edges in  $\pi$  and  $\pi'$  are equal therefore to show that  $eg(\pi') = g(\pi) = eg(\pi) + 1$ , it is enough to show that  $f(\pi') = f(\pi) - 1$ . Let  $\pi = (\bar{n}, \dots, \bar{2}, \dots, \bar{j}, \bar{1})$ . Then  $\pi' = (\bar{n}, \dots, \bar{2}, \dots, \bar{j}, n + 1)$ .  $\pi$  has a face  $f_1 = (0, n)$  and  $f_2 = (2, n, j, \dots)$ . Replacing  $-1$  with  $n + 1$ , the faces  $f_1$  and  $f_2$  merge to a single face  $f = (2, n + 1, n, n + 1, j, \dots)$  in  $\pi'$  (see Figure 11). The other faces in  $\pi'$  are the same as the faces in  $\pi$  other than  $f_1$  and  $f_2$ . This implies that  $\pi'$  has one face less than  $\pi$ .

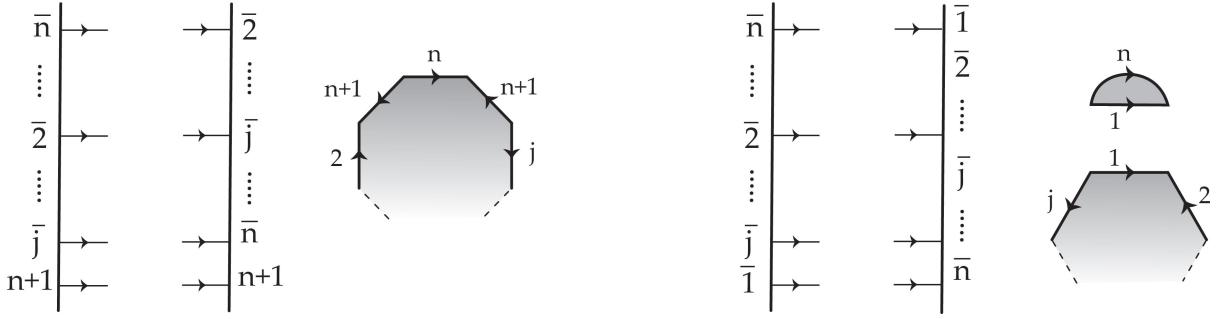


Figure 11: Faces of  $\pi'$  (left) and  $\pi$  (right).

Furthermore,  $\pi'$  does not contain any block and therefore it is reduced. It is non-orientable by construction. Having a reversible pair  $(i, i+1)$ , there are two reversals that make  $i$  and  $i+1$  homotopic. By running the HP-algorithm on  $\pi'$  and choosing the reversal at each step that does not reverse  $n+1$  we obtain a desired sequence of reversals. This finishes the proof of the lemma.  $\square$

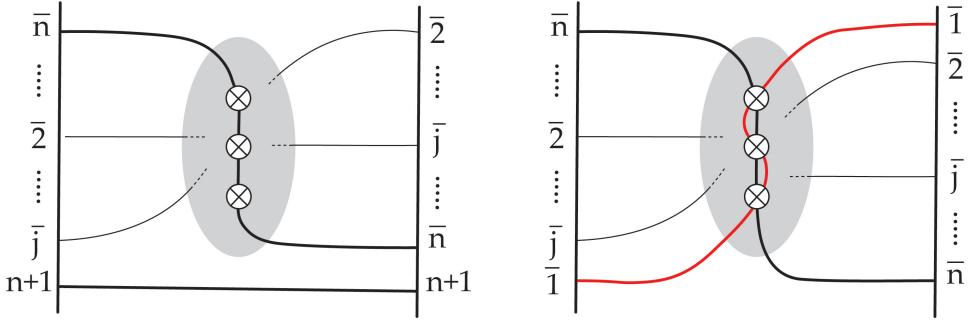


Figure 12: From a cross-cap drawing of  $\pi'$  to a cross-cap drawing of  $\pi$ .

*Proof of Theorem 1.5.* Let  $G$  denote an embedding scheme for a loopless 2-vertex graph of non-orientable genus  $g$ . We denote by  $G'$  the reduced scheme, i.e., the scheme obtained after recursively replacing minimal blocks with a curve of the corresponding sidedness. If  $G'$  is exactly one of the two graphs depicted in Figure 3, the conclusion of the theorem holds. If  $G'$  consists of a single edge, up to flipping  $G$  we can assume that this edge has negative signature. Then this edge can be drawn in the natural way with a single cross-cap. In all the other cases, by Lemma 5.1,  $G'$  admits a fantastic cross-cap drawing.

In order to finish the proof, we explain how to inductively put back minimal blocks of  $G$  in the place of the corresponding reduced edge in  $G'$ . If  $A$  is such a minimal block and is negative and we denote by  $e$  the corresponding reduced edge,  $e$  is one-sided and thus goes through at least one cross-cap. We blow up this cross-cap, replacing it by exactly the odd number of cross-caps required to draw  $A$ . By Lemma 5.3, since  $A$  is minimal, it is reduced and thus it can be drawn using these blown-up cross-caps, in such a way that the frames of the block enter the blown-up cross-caps in opposite orders. This process is pictured in Figure 12. If  $A$  is a positive minimal block and  $e$  is the corresponding reduced edge, since the drawing of  $G'$  is fantastic, we know that  $e$  enters at least two cross-caps. We first make the entirety of  $A$  enter the first of these cross-caps. Thus, there remains to draw the flipped version of  $A$ , which is now a negative block. This is achieved as before by blowing-up the second cross-cap, and appealing to Lemma 5.3 to draw the negative block within the space bordered by the two frames.

This process shows how to obtain a perfect cross-cap drawing of  $G$  from a fantastic cross-cap drawing of the reduced graph  $G'$ , concluding the proof.  $\square$

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## A Appendix

**Lemma 4.3.** *If  $\pi$  is a non-orientable signed permutation without non-trivial blocks,  $(i, i + 1)$  is a reversible pair of maximal score, and  $\sigma$  is a reversal that makes  $i$  and  $i + 1$  homotopic, then if  $\pi \cdot \sigma$  is not the identity, it is non-orientable and has no non-trivial blocks.*

The following is similar to the definition of overlapping graph and the proof of Theorem 1 in [2].

Two intervals  $(i, j)$  and  $(k, l)$  are called *interleaving* in a cyclic permutation, if we either have  $\pi^{-1}(i) < \pi^{-1}(k) < \pi^{-1}(i + 1)$  or  $\pi^{-1}(i) < \pi^{-1}(l) < \pi^{-1}(i + 1)$ .

**Interleaving graph.** Given a permutation  $\pi$  on  $\{1, \dots, n\}$ , Let  $\pi^*$  be the permutation of size  $2n$  on elements  $\{i^l, i^r \text{ for } 1 \leq i \leq n\}$  in which  $\pi_{2k-1}^* = \pi_i^l$  and  $\pi_{2k}^* = \pi_i^r$  when  $\pi_i$  is positive and  $\pi_{2k-1}^* = \pi_i^r$  and  $\pi_{2k}^* = \pi_i^l$  when  $\pi_i$  is negative. We build the *interleaving graph*  $I_\pi$  as follows:

- The graph has  $n$  vertices labelled by  $(i, i+1)$  for  $i$  modulo  $n$ . A vertex  $(i, i+1)$  is called *reversible* if  $(i, i+1)$  is a reversible pair, otherwise it is called *non-reversible*. We denote a reversible pair by a white vertex and a non-reversible pair by a black one.
- Two vertices  $(i, i+1)$  and  $(j, j+1)$  are connected if the intervals  $(i^r, i+1^l)$  and  $(j^r, j+1^l)$  are interleaving in  $\pi^*$ .

See Figure 13 for an example.

A connected component in  $I_\pi$  is *non-trivial* if it has more than one vertex and it is called *orientable* if it only contains non-reversible vertices.

**Remark A.1.** *If  $(i, i+1)$  are a homotopic pair in  $\pi$  then the vertex  $(i, i+1)$  is an isolated vertex. A non-trivial block and the vertices in  $I_\pi$  associated to the pair of edges that belong to the block correspond to a non-trivial orientable connected component in  $I_\pi$ . Note that the reverse is not true and an orientable connected component does not always correspond to a non-trivial block in the permutation.*

**Lemma A.2.** *Let  $\pi$  be a signed permutation for which  $I_\pi$  has a non-trivial orientable connected component  $U$ . Then either  $U$  corresponds to a non-trivial block or  $\pi$  is orientable.*

*Proof.* We say that an element  $i$  belongs to  $U$  if  $(i, i+1) \in U$  or  $(i-1, i) \in U$ . By orientability, all the elements belonging to  $U$  have the same signature, let us first assume that it is positive. Let us furthermore assume, for the sake of contradiction, that  $\pi$  is not orientable and that  $U$  does not correspond to a block. Since  $\pi$  is not orientable, not every element has positive signature, so there is at least one element that does not belong to  $U$ . Let  $a$  be an element belonging to  $U$  such that  $a-1$  does not belong to  $U$ . Without loss of generality, we can fix an origin to the signed permutation  $\pi$  at  $a-1$ . Now let  $b$  be an element that belongs to  $U$  and such that for any element  $j$  that belongs to  $U$ ,  $a, j$  and  $b$  appear in this cyclic order in  $\pi$ . We claim that either  $a-2 = b$ , or  $a-2$  does not appear between  $a$  and  $b$  in  $\pi$ . Indeed, if  $a-2$  appears between  $a$  and  $b$ , then  $(a-2^r, a-1^l)$  would interleave with one pair on a path between  $(a, a+1)$  and  $(b, b \pm 1)$  in  $U$ , and thus  $a-1$  would belong to  $U$ . Therefore, either  $a-2 = b$  or  $a-2$  does not belong to  $U$ . Inductively, none of the elements that are not in  $[a, b]$  lie between  $[a, b]$  in  $\pi$ . Similarly, all the elements within  $[a, b]$  lie between  $a$  and  $b$  in  $\pi$ , as otherwise the smallest one that does not, call it  $k$ , does not belong to  $U$ , yet  $(k-1^r, k^l)$  interleaves with a pair on a path between  $(a, a+1)$  and  $(b, b \pm 1)$  in  $U$ , contradicting the fact that  $k$  is not in  $U$ . We conclude that  $a$  and  $b$  are the frames of a block, which is non-trivial since there is at least one pair in  $U$ . This is a contradiction. The case where all the signatures in  $U$  are negative follows from the fact that flipping  $\pi$  does not change its interleaving graph nor the orientability of its components.  $\square$

When we apply a reversal on a reversible pair  $(i, i+1)$ , the effect on  $I_\pi$  is to complement the subgraph induced by the vertex  $(i, i+1)$  and its neighbors in  $I_\pi$ . Also if a vertex in this subgraph was reversible, it gets non-reversible and vice versa. Indeed, let  $(j, j+1)$  be a reversible vertex connected to  $(i, i+1)$ . This means that the intervals  $(j^r, j+1^l)$  and  $(i^r, i+1^l)$  are interleaving. Without loss of generality let us assume that  $j^r$  is the element that belongs to the interval  $(i^r, i+1^l)$  and it is positive. Reversing the elements between  $(i, \pi_{\pi^{-1}(i+1)-1})$  makes  $i$  and  $i+1$  homotopic and isolates  $(i, i+1)$ . Also it makes  $j$  negative and therefore the pair  $(j, j+1)$  is not reversible anymore. Similarly it can be seen that if  $(j, j+1)$  and  $(k, k+1)$  are two neighbors of  $(i, i+1)$  in  $I_\pi$  and the intervals  $(j^r, j+1^l)$  and  $(k^r, k+1^l)$  interleave, after applying the reversal they stop being interleaved and therefore vertices  $(j, j+1)$  and  $(k, k+1)$  are not connected anymore in the interleaving graph. This explains the complementing of the induced subgraph. Figure 13 depicts the effect of applying the reversal on the pair  $(4, 5)$ .

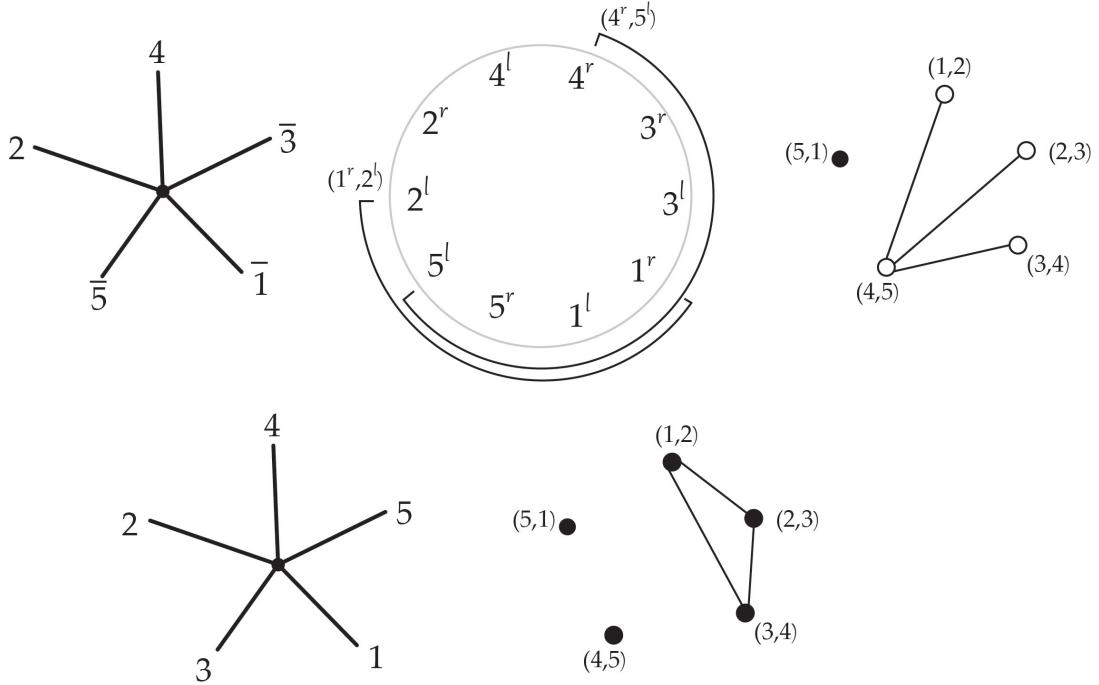


Figure 13: Figure depicts a permutation  $\pi$ , its associated doubled permutation  $\pi^*$  and its interleaving graph  $I_\pi$ . At the bottom we can see the effect of reversing elements 1, 3, 5 on the interleaving graph; this reversal makes 4 and 5 homotopic.

*Proof of Lemma 4.3.* To prove the lemma, we first show that the number of non-trivial orientable connected components in  $I_{\pi \cdot \sigma}$  cannot be more than  $I_\pi$ . Let us assume that this is the case and that by applying  $\sigma$  we create such a component  $C$ . In this case, we claim that the score of any pair  $(j, j+1)$  for which the corresponding vertex is in  $C$ , is higher than the score of the pair  $(i, i+1)$ .

Denote by  $\#^+(i, i+1)$  (resp.  $\#^-(i, i+1)$ ) the number of reversible (resp. non-reversible) pairs of edges adjacent edges to  $(i, i+1)$  in  $I_\pi$ . If  $\pi$  has  $k$  reversible pairs of edges, then the score( $i, i+1$ ) =  $k - \#^+(i, i+1) + \#^-(i, i+1)$ .

Before applying the reversal, every non-reversible vertex that is connected to  $(i, i+1)$  has to be connected to  $(j, j+1)$ . This is because otherwise after applying the reversal this vertex will be a reversible vertex connected to  $(j, j+1)$  and therefore belonging to  $C$  which is not possible. This implies that  $\#^-(i, i+1) \leq \#^-(j, j+1)$ .

Before applying the reversal, every reversible vertex  $(t, t+1)$  that is connected to  $(j, j+1)$  has to be connected to  $(i, i+1)$ . This is because if  $(t, t+1)$  is not connected to  $(i, i+1)$ , after applying the reversal this vertex remains connected to  $(j, j+1)$  without changing its reversibility. This means that  $(t, t+1)$  is a reversible pair that belongs to  $C$  which is not possible. This implies that  $\#^+(j, j+1) \leq \#^+(i, i+1)$ . The equality does not happen since the component  $C$  has more than one vertex and a vertex in  $C$  to which  $(j, j+1)$  is connected after the reversal, is a reversible vertex that used to be connected to  $(i, i+1)$  but not to  $(j, j+1)$ . Therefore  $\#^+(j, j+1) < \#^+(i, i+1)$ .

We have that score( $i, i+1$ ) =  $k - \#^+(i, i+1) + \#^-(i, i+1) < k - \#^+(j, j+1) + \#^-(j, j+1) = \text{score}(j, j+1)$  which contradicts our assumption. This finishes the proof of the claim.

Now, the assumptions of Lemma 4.3 imply that there are no non-trivial orientable connected components in  $I_\pi$ . Thus there are also none in  $I_{\pi \cdot \sigma}$ . A non-trivial block or the entire scheme being orientable but not the identity would induce such a non-trivial orientable connected component. This concludes the proof.  $\square$