Teacher's Copy

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where a and b are positive parameters. If we further assume decreasing returns to scale, then a + b < 1. For simplicity, let's consider the symmetric case where $a = b = \frac{1}{4}$

$$Q = L^{\frac{1}{4}}K^{\frac{1}{4}} \tag{3}$$

Substituting Equation 3 into Equation 1 gives us

$$\pi(K, L) = PL^{\frac{1}{4}}K^{\frac{1}{4}} - wL - rK \tag{4}$$

The first order conditions are

$$\begin{aligned} \frac{\partial\pi}{\partial L} &= P\left(\frac{1}{4}\right)L^{-\frac{3}{4}}K^{\frac{1}{4}} - w = 0 \\ \frac{\partial\pi}{\partial K} &= P\left(\frac{1}{4}\right)L^{\frac{1}{4}}K^{-\frac{3}{4}} - r = 0 \end{aligned} \tag{5}$$

This system of equations define the optimal L and K for profit maximization. Rewriting the first equation in Equation 5 to isolate K

$$\begin{aligned} P\left(\tfrac{1}{4}\right)L^{-\frac{3}{4}}K^{\frac{1}{4}} &= w \\ K &= \left(\tfrac{4w}{p}L^{\frac{3}{4}}\right)^4 \end{aligned}$$

Substituting into the second equation of Equation 5

$$\begin{aligned} \tfrac{P}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}} &= \left(\tfrac{P}{4}\right)L^{\frac{1}{4}}\left[\left(\tfrac{4w}{p}L^{\frac{3}{4}}\right)^4\right]^{-\frac{3}{4}} = r \\ &= P^4\left(\tfrac{1}{4}\right)^4 w^{-3}L^{-2} = r \end{aligned}$$

Re-arranging to get L by itself gives us

$$L^* = \left(\frac{P}{4}w^{-\frac{3}{4}}r^{-\frac{1}{4}}\right)^2$$

Taking advantage of the symmetry of the model, we can quickly find the optimal K

$$K^* = \left(\frac{P}{4}r^{-\frac{3}{4}}w^{-\frac{1}{4}}\right)^2$$

L* and K^* are the firm's factor demand equations.

Optimization with Constraints

The Lagrange Multiplier Method

Sometimes we need to to maximize (minimize) a function that is subject to some sort of constraint. For example

$$\textit{Maximize} \quad z = f(x, y)$$

$$\textit{subject to the constraint} \quad x + y \leq 100$$

For this kind of problem there is a technique, or *trick*, developed for this kind of problem known as the *Lagrange Multiplier method*. This method involves adding an extra variable to the problem called the lagrange multiplier, or λ.

We then set up the problem as follows:

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1. Create a new equation form the original information

$$L = f(x,y) + \lambda(100 - x - y)$$
$$or$$
$$L = f(x,y) + \lambda\,[Zero]$$

2. Then follow the same steps as used in a regular maximization problem

$$\frac{\partial L}{\partial x} = f_x - \lambda = 0$$
$$\frac{\partial L}{\partial y} = f_y - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

3. In most cases the λ will drop out with substitution. Solving these 3 equations will give you the constrained maximum solution

Example 1:

Suppose $z = f(x,y) = xy$. and the constraint is the one from above. The problem then becomes

$$L = xy + \lambda(100 - x - y)$$

Now take partial derivatives, one for each unknown, including λ

$$\frac{\partial L}{\partial x} = y - \lambda = 0$$
$$\frac{\partial L}{\partial y} = x - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

Starting with the first two equations, we see that $x = y$ and λ drops out. From the third equation we can easily find that $x = y = 50$ and the constrained maximum value for z is $z = xy = 2500$.

Example 2:

Maximize

$$u = 4x^2 + 3xy + 6y^2$$

subject to

$$x + y = 56$$

Set up the Lagrangian Equation:

$$L = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned} L_x &= 8x + 3y - \lambda = 0 \\ L_y &= 3x + 12y - \lambda = 0 \\ L_\lambda &= 56 - x - y = 0 \end{aligned}$$

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From the first two equations we get

$$\begin{aligned} 8x + 3y &= 3x + 12y \\ x &= 1.8y \end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned} 56 - 1.8y - y &= 0 \\ y &= 20 \end{aligned}$$

therefore

$$x = 36 \qquad \lambda = 348$$

Example 3: Cost minimization

A firm produces two goods, x and y. Due to a government quota, the firm must produce subject to the constraint $x + y = 42$. The firm's cost functions is

$$c(x, y) = 8x^2 - xy + 12y^2$$

The Lagrangian is

$$L = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

The first order conditions are

$$\begin{aligned} L_x &= 16x - y - \lambda = 0 \\ L_y &= -x + 24y - \lambda = 0 \\ L_\lambda &= 42 - x - y = 0 \end{aligned} \qquad (6)$$

Solving these three equations simultaneously yields

$$x = 25 \qquad y = 17 \qquad \lambda = 383$$

Example of duality for the consumer choice problem

Example 4: Utility Maximization

Consider a consumer with the utility function $U = xy$, who faces a budget constraint of $B = P_x x + P_y y$, where B, P_x and P_y are the budget and prices, which are given.

The choice problem is

Maximize

$$U = xy \qquad (7)$$

Subject to

$$B = P_x x + P_y y \qquad (8)$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x - P_y y) \qquad (9)$$

The first order conditions are

$$\begin{aligned} Z_x &= y - \lambda P_x = 0 \\ Z_y &= x - \lambda P_y = 0 \\ Z_\lambda &= B - P_x x - P_y y = 0 \end{aligned} \tag{10}$$

Solving the first order conditions yield the following solutions

$$x^M = \frac{B}{2P_x} \quad y^M = \frac{B}{2P_y} \quad \lambda = \frac{B}{2P_x P_y} \tag{11}$$

where x^M and y^M are the consumer's Marshallian demand functions.

Example 5: Minimization Problem

Minimize

$$P_x x + P_y y \tag{12}$$

Subject to

$$U_0 = xy \tag{13}$$

The Lagrangian for the problem is

$$Z = P_x x + P_y y + \lambda(U_0 - xy) \tag{14}$$

The first order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda y = 0 \\ Z_y &= P_y - \lambda x = 0 \\ Z_\lambda &= U_0 - xy = 0 \end{aligned} \tag{15}$$

Solving the system of equations for x, y and λ

$$\begin{aligned} x^h &= \left(\frac{P_y U_0}{P_x}\right)^{\frac{1}{2}} \\ y^h &= \left(\frac{P_x U_0}{P_y}\right)^{\frac{1}{2}} \\ \lambda^h &= \left(\frac{P_x P_y}{U_0}\right)^{\frac{1}{2}} \end{aligned} \tag{16}$$

Application: Intertemporal Utility Maximization

Consider a simple two period model where a consumer's utility is a function of consumption in both periods. Let the consumer's utility function be

$$U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

where c_1 is consumption in period one and c_2 is consumption in period two. The consumer is also endowments of y_1 in period one and y_2 in period two.

Let r denote a market interest rate with the consumer can choose to borrow or lend across the two periods. The consumer's intertemporal budget constraint is

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

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Why Is this Method Applied?

The Lagrange method is frequently used in economics, mainly because the Lagrange multiplicator(s) has an interesting interpretation. The Lagrange multiplicator represents the shadow price of the constraint that it is multiplied with; it measures how much the optimal value of the objective function $f(x_1^*, x_2^*)$ would change if the constraint would be relaxed marginally (i.e. if the constant c would increase marginally).

Example: Utility Maximization

We want to maximize $u(x_1, x_2) = x_1 x_2$ subject to the budget constraint $p_1 x_1 + p_2 x_2 = m$:

$$\max_{x_1, x_2} \; x_1 x_2 \qquad \text{s.t.} \qquad p_1 x_1 + p_2 x_2 = m.$$

The Lagrangian is thus given by

$$L(x_1, x_2, \lambda) = x_1 x_2 - \lambda[p_1 x_1 + p_2 x_2 - m].$$

The optimal solutions are given by

$$\begin{aligned} x_1^* &= \frac{m}{2p_1}, \\ x_2^* &= \frac{m}{2p_2}, \\ \lambda^* &= \frac{m}{2p_1 p_2}. \end{aligned}$$

In this case λ^* measures the marginal utility of income, i.e. λ^* measures how much utility would increase at the optimal values x_1^* and x_2^* if the individual's income were increased marginally:

$$\begin{aligned} u(x_1^*, x_2^*) &= x_1^* x_2^* = \frac{m^2}{4p_1 p_2} \equiv u^*(p_1, p_2, m) \\ \Rightarrow \frac{du^*}{dm} &= \frac{m}{2p_1 p_2} = \lambda^*. \end{aligned}$$

Example: Cost Minimization

The utility function is given by $u(x_1, x_2) = x_1 x_2$. We want to minimize the expenditures, given by $E(x_1, x_2) = p_1 x_1 + p_2 x_2$, for attaining utility level $\overline{u}$:

$$\min_{x_1, x_2} \; p_1 x_1 + p_2 x_2 \qquad \text{s.t.} \qquad x_1 x_2 = \overline{u}.$$

The Lagrangian is thus given by

$$M(x_1, x_2, \mu) = p_1 x_1 + p_2 x_2 - \mu[x_1 x_2 - \overline{u}].$$

The optimal solutions are given by

$$\begin{aligned} x_1^h &= \sqrt{\frac{p_2 \overline{u}}{p_1}}, \\ x_2^h &= \sqrt{\frac{p_1 \overline{u}}{p_2}}, \\ \mu^h &= \sqrt{\frac{p_1 p_2}{\overline{u}}}. \end{aligned}$$

In this case μ^h measures the marginal cost of $\overline{u}$, i.e. μ^h measures how much expenditures would increase at the optimal values x_1^h and x_2^h if the individual's utility level $\overline{u}$ were increased marginally:

$$\begin{aligned} E(x_1^h, x_2^h) &= p_1 x_1^h + p_2 x_2^h = 2\sqrt{p_1 p_2 \overline{u}} \equiv E^h(p_1, p_2, \overline{u}) \\ \Rightarrow \frac{dE^h}{d\overline{u}} &= \sqrt{\frac{p_1 p_2}{\overline{u}}} = \mu^h. \end{aligned}$$

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- The relative factor price $\frac{w_1}{w_2}$ gives the "rate at which factors can be exchanged in the market".

11.3 Some Examples

11.3.1 Fixed Proportions

$$\begin{aligned} C(w_1, w_2, y) &= \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\ \text{s.t } y &= \min\{x_1, x_2\} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} x_1(w_1, w_2, y) &= y \\ x_2(w_1, w_2, y) &= y \end{aligned}$$

$\Rightarrow$

$$C(w_1, w_2, y) = (w_1 + w_2)\, y$$

11.3.2 Cobb Douglas

$$\begin{aligned} C(w_1, w_2, y) &= \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\ \text{s.t } y &= x_1^a x_2^b \end{aligned}$$

Solving the constraint we get

$$x_2 = y^{\frac{1}{b}} x_1^{-\frac{a}{b}}$$

Plugging into the objective we get

$$C(w_1, w_2, y) = \min_{x_1} w_1 x_1 + w_2 y^{\frac{1}{b}} x_1^{-\frac{a}{b}}$$

Or, you may write this as

$$\max_{x_1} -w_1 x_1 - w_2 y^{\frac{1}{b}} x_1^{-\frac{a}{b}}$$

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The first order condition is

$$-w_1 - w_2 y^{\frac{1}{b}} \left(-\frac{a}{b}\right) x_1^{-\frac{a}{b}-1} = 0$$

or

$$\begin{aligned} w_1 &= w_2 y^{\frac{1}{b}} \frac{a}{b} x_1^{-\left(\frac{a+b}{b}\right)} \Leftrightarrow \text{multiply with } x_1^{\frac{a+b}{b}} \\ w_1 x_1^{\frac{a+b}{b}} &= w_2 y^{\frac{1}{b}} \frac{a}{b} x_1^{-\left(\frac{a+b}{b}\right)} x_1^{\frac{a+b}{b}} = w_2 y^{\frac{1}{b}} \frac{a}{b} \end{aligned}$$

or

$$\begin{aligned} x_1^{\frac{a+b}{b}} &= \frac{w_2}{w_1} \frac{a}{b} y^{\frac{1}{b}} \Leftrightarrow \\ x_1 (w_1, w_2, y) &= \left(\frac{w_2}{w_1} \frac{a}{b}\right)^{\frac{b}{a+b}} y^{\frac{1}{a+b}} \end{aligned}$$

Symmetrically we get (either by observing the symmetry or by plugging back in constraint) that

$$x_2 (w_1, w_2, y) = \left(\frac{w_1}{w_2} \frac{b}{a}\right)^{\frac{a}{a+b}} y^{\frac{1}{a+b}}$$

and plugging this into the objective we get the cost function

$$C (w_1, w_2, y) = w_1 \left(\frac{a w_1}{b w_2}\right)^{\frac{b}{a+b}} y^{\frac{1}{a+b}} + w_2 \left(\frac{b w_2}{a w_1}\right)^{\frac{a}{a+b}} y^{\frac{1}{a+b}},$$

which looks really ugly. However, competitive analysis assumes that prices and factor prices are exogenous for the firm. Hence, from the perspective of analyzing the supply problem for the firm *the really important property of the cost function is to say how costs change with output.* This exercise is for fixed factor prices and we note that we then may write the resulting cost function for a Cobb Douglas technology as

$$C (y) = K y^{\frac{1}{a+b}},$$

where

$$K = w_1 \left(\frac{a w_1}{b w_2}\right)^{\frac{b}{a+b}} + w_2 \left(\frac{b w_2}{a w_1}\right)^{\frac{a}{a+b}}$$

We then see that

$$C' (y) = K \frac{1}{a+b} y^{\frac{1}{a+b}-1}$$

is:

1. Increasing in y if $a + b < 1$. That is, the *marginal cost* is increasing when there is *decreasing returns to scale.*

2. Decreasing in y if $a + b > 1$. That is, the marginal cost is decreasing with increasing returns to scale.

3. Constant in y if $a + b = y$. That is the marginal cost is constant with constant returns to scale.

11.3.3 A Remark about Notation

Once again, note that:

- w_1, w_2, y are *parameters* of the cost minimization problem

- x_1, x_2 are the *choice variables.*

The solution to the cost minimization problem will then in general give the choice variables as functions of the parameters. We write these as

$$x_1 (w_1, w_2, y)$$
$$x_2 (w_1, w_2, y)$$

and call them conditional factor demands. Now, plugging in these in the objective we get the *cost function*

$$C (w_1, w_2, y) = w_1 x_1 (w_1, w_2, y) + w_2 x_2 (w_1, w_2, y) .$$

One of the more confusing aspects of economics is that sometimes we write something as a function of a long list of parameters and sometimes we write the same thing as a function of a shorter list, maybe just a single parameter. This practice simply reflects that for some purposes we want to keep a bunch of parameters constant and for other purposes we want to see what happens when we change these parameters. For that reason, the list of parameters that is explicitly introduced in the notation depends on the question we want to ask.

$$w_2 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_2}$$

Divide them through (both sides) to get:

$$\frac{w_1}{w_2} = \frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}}$$

But what are the partial derivatives of the production - remember they are simply the marginal products of each input, $MP_1(x_1, x_2)$ and $MP_2(x_1, x_2)$. Thus we get:

$$\frac{w_1}{w_2} = \frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)}$$

which is what we had before! Then this equation, together with the constraint $f(x_1, x_2) = y$ can be used to solve for the conditional input demands $\hat{x}_1(w_1, w_2, y)$ and $\hat{x}_2(w_1, w_2, y)$.

An Example With A Specific Production Function:

Take $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$ and let input prices be $w_1 = 1$, $w_2 = 2$. We want to solve the firm's cost minimization problem of producing y units of output. We will use all three methods discussed above and obtain the same results:

– substitution from the constraint
– using the optimality condition $\frac{MP_1(x_1,x_2)}{MP_2(x_1,x_2)} = \frac{w_1}{w_2}$
– using the Lagrange method (optional)

Let us first write down the problem:

$$\min_{x_1, x_2} \; 1x_1 + 2x_2$$
$$\text{s.t. } x_1^{1/3} x_2^{1/3} = y$$

Method 1 (Direct Substitution)

1. We first have to express x_2 in terms of x_1 from the technology constraint $f(x_1, x_2) = y$. The easiest way to to it is get rid of the exponents first by raising the constraint to power 3 - we get:

$$x_1 x_2 = y^3$$

thus $x_2 = \frac{y^3}{x_1}$.

2. Now let's plug the expression for x_2 obtained above in our cost function to get the following minimization problem:

$$\min_{x_1} \; x_1 + 2\frac{y^3}{x_1}$$

3. We solve the above problem of just one unknown (x_1) by taking the first derivative of the function being minimized and setting it to zero:

$$1 + 2y^3\left(-\frac{1}{x_1^2}\right) = 0$$

(we use that the derivative of $\frac{1}{x_1}$ is $-\frac{1}{x_1^2}$). Remember y is given, i.e. treated as if it were a number!. From the above equation we can solve for the optimal x_1 as function of y (in general it is also a function of w_1 and w_2 but remember we plugged numbers for these already):

$$\frac{1}{x_1^2} = \frac{1}{2y^3} \text{ or } x_1^2 = 2y^3, \text{ or } \hat{x}_1(y) = \sqrt{2}y^{3/2}$$

which is the **conditional input demand for input 1**.

4. Now how do we find the optimal quantity demanded of input 2 (i.e. its conditional input demand, $\hat{x}_2$)? Remember we know that $\hat{x}_2 = \frac{y^3}{\hat{x}_1}$ so we have:

$$\hat{x}_2(y) = \frac{y^3}{\sqrt{2}y^{3/2}} = \frac{y^{3/2}}{\sqrt{2}}$$

which is the **conditional input demand for input 2**. Notice that both conditional input demands depend on the given level of output, y that the firm wants to produce.

5. The **minimum cost of producing y units of output** then can be found by simply substituting the conditional input demands obtained above into the cost function:

$$\hat{c}(y) = 1\hat{x}_1(y) + 2\hat{x}_2(y) = \sqrt{2}y^{3/2} + \frac{2}{\sqrt{2}}y^{3/2} = 2\sqrt{2}y^{3/2}$$

Method 2 (using the optimality condition)

This method works even if it is not obvious how to express x_2 in terms of x_1 from the constraint. However, it relies on tangency and interiority of the solution.

1. To be able to write down the optimality condition for cost minimization we need to compute the marginal products of each input, i.e. the partial derivatives of the production function with respect to x_1 and x_2. We have:

$$\begin{aligned} MP_1(x_1, x_2) &= \frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{1}{3}x_1^{-2/3}x_2^{1/3} \\ MP_2(x_1, x_2) &= \frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{1}{3}x_1^{1/3}x_2^{-2/3} \end{aligned}$$

(remember when taking partial derivatives you hold all variables that you are not taking the derivative with respect to as constant).

2. Thus the optimality condition for our particular production function and input prices is (cancelling the 1/3s):

$$\frac{x_1^{-2/3}x_2^{1/3}}{x_1^{1/3}x_2^{-2/3}} = \frac{1}{2} \text{ or collecting the exponents, } \frac{x_2}{x_1} = \frac{1}{2}$$

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(please remember how to manipulate fractions of numbers raised to different powers!). From above we get: $x_2 = \frac{1}{2}x_1$.

3. We can use the above relationship between x_2 and x_1 and plug it in the second equation that they must satisfy, namely that output should be y :

$$x_1^{1/3}(\frac{1}{2}x_1)^{1/3} = y$$

Again, raising both sides to power 3 helps:

$$\frac{1}{2}x_1^2 = y^3 \text{ or, } x_1^2 = 2y^3, \text{ so } \hat{x}_1(y) = \sqrt{2}y^{3/2}$$

or the conditional input demand for input one is the same thing as before once again. Then

$$\hat{x}_2(y) = \frac{1}{2}\hat{x}_1(y) = \frac{y^{3/2}}{\sqrt{2}}$$

again same as before. Clearly then the minimized costs will be also the same.

Method 3 (using Lagrange, OPTIONAL)

1. We write the Lagrangean for our specific problem:

$$\Lambda(x_1, x_2, \lambda) = 1x_1 + 2x_2 - \lambda(x_1^{1/3}x_2^{1/3} - y)$$

2. Taking the partial derivatives with respect to the arguments of the Lagrangean we get:

$$\begin{aligned} 1 - \lambda\frac{1}{3}x_1^{-2/3}x_2^{1/3} &= 0 \\ 2 - \lambda\frac{1}{3}x_1^{1/3}x_2^{-2/3} &= 0 \\ x_1^{1/3}x_2^{1/3} - y &= 0 \end{aligned}$$

3. Re-arrange the first two equations as described in the general example above and divide them through to get:

$$\frac{1}{2} = \frac{x_1^{-2/3}x_2^{1/3}}{x_1^{1/3}x_2^{-2/3}}$$

which is the same what we had in Method 2. Thus you can use the rest of the steps in Method 2 to finish the solution.

Constrained Optimization: The Method of Lagrange Multipliers:

Suppose the equation $p(x,y) = -2x^2 + 60x - 3y^2 + 72y + 100$ models profit when x represents the number of handmade chairs and y is the number of handmade rockers produced per week. The optimal (maximum) situation occurs when $x = 15$ and $y = 12$. However due to an insufficient labor force they can only make a total of 20 chairs and rockers per week ($x + y = 20$). So how many chairs and how many rockers will give the realistic maximum profit? We will come back to this questions shortly but first we will look at the following example.

Using Level Curves and the constraint function to determine optimal points:

A Company has a determined that its production function is the Cobb-Douglas function $f(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$ where x is the number of labor hours and y is the number of capitol units. The budget constraint for the company is given by $100x + 100y = 400000$.

a) If the company decides to spend $300, 000 on x then how much can be spent on y under this budget constraint?

solution:
If $300000 is spent on x then $100x = 300000$ and $x = 3000$. There is $100000 left to spend on y therefore $100y = 100000$ and $y = 1000$.

In this case total production will be $f(3000, 1000) = (3000)^{\frac{2}{3}}(1000)^{\frac{1}{3}} = (208)(10) = 2080$ units.

Suppose the amount spent on x is changed to $350,000. How will that change production? Following the same procedure as above $x = 3500$ and $y = 500$. The production f(3500, 500) then be 1829 units.

Is there a way to determine values for x and y that will give the optimal production possible given this budget constraint?

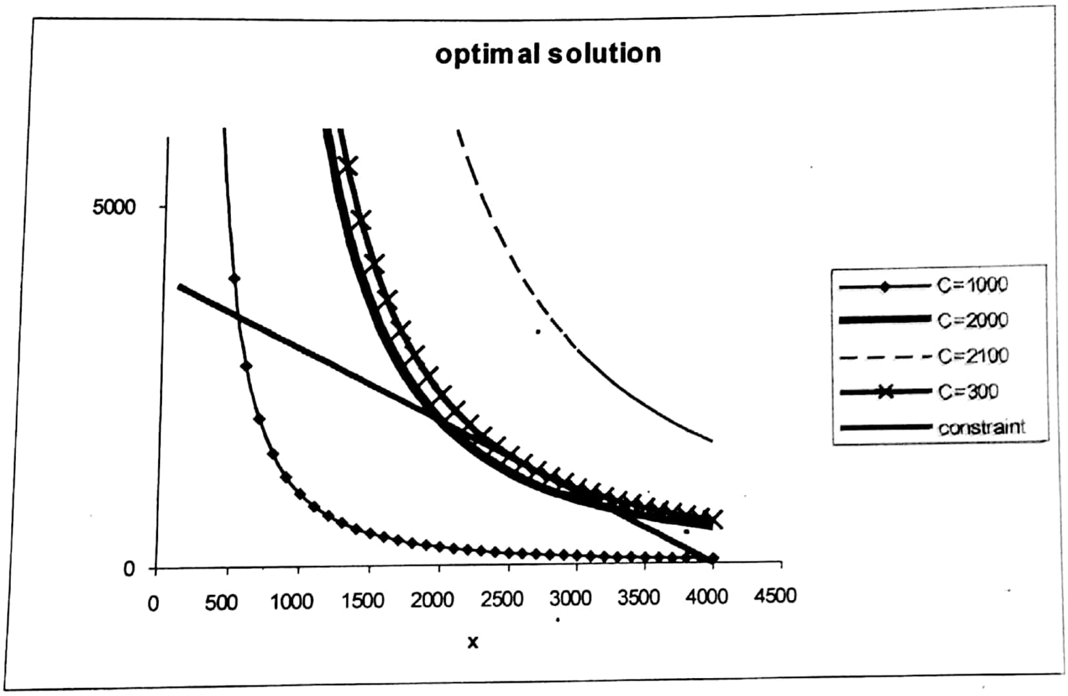
we will first try to find the optimal situation using level curves with the constraint function.

It turns out that the global maximum or global minimum occurs where the graph of the constraint equation is tangent to one of the level curves of the original function.

In order to plot the level curves we must solve for y to get $\frac{f(x,y)}{x^{\frac{2}{3}}} = y^{\frac{1}{3}}$ -> $\frac{C^3}{x^2} = y$

where C now represents values of f(x,y). Now we will plot y for various values of C. (The choice of values for C is determined mostly by common sense.) In this case we will let C = 1000, C = 2000 and C = 3000. After looking at the resulting level curves and constraint , I decide to add another value C = 2100. Now I can make a good guess

After considering the graph below, we can guess the optimal value will occur for x = 2600 and y = 1400.



There is also an algebraic approach to finding the optimal solution given a certain constraint. We will use the method of Lagrange Multipliers to find the maximum situation in the problem above. In the proceeding sections you have learned how to use partial derivatives to find the optimal situation for a multivariable equation . Now we will expand on this process to find the optimal situation with a constraint. Below is an outline of the process we will use followed examples.

Procedure for Applying the Method of Lagrange Multipliers:

In order to maximize or minimize the function f(x,y) which is subject to the constraint g(x, y) = k we will follow the following procedure.

Step #1 First create the LaGrange Function. This function is composed of the function to be optimized combined with the constraint function in the following way:

$L(x, y) = f(x, y) - \lambda[g(x, y) - k]$

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Step #2 Now find the partial derivative with respect to each variable x, y and the Lagrange multiplier λ of the function shown: $L(x, y) = f(x, y) - \lambda[g(x, y) - k]$

Step #3 Set each of the partial derivatives equal to zero to get $L_x = 0$, $L_y = 0$ and $L_\lambda = 0$ Using $L_x = 0$, $L_y = 0$, proceed to solve for x and solve for y in terms of λ. Now substitute the solutions for x and y so that $L_\lambda = 0$ is in terms of λ only. Now solve for λ and use this value to find the optimal values x and y.

Note: If M is the max or min value of f(x, y) subject to the constraint g(x, y) = k, then the Lagrange multiplier λ is the rate of change in M with respect to k. (i.e. λ = dM/dk and therefore λ approximates the change in M resulting in a one unit increase in k.)

Example 1

We will revisit the Cobb-Douglas function $f(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$ and the budget constraint function $100x + 100y = 400000$. We will use the method of Lagrange multiplier described above to find the optimal solutions.

First we will create the Lagrange equation

$$\boxed{L(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}} - \lambda(100x + 100y - 400000}$$

Now find $L_x = 0$, $L_y = 0$ and $L_\lambda = 0$.

$$L_x = \frac{2}{3}x^{\frac{-1}{3}}y^{\frac{1}{3}} - 100\lambda$$
$$Ly = \frac{1}{3}x^{\frac{2}{3}}y^{\frac{-2}{3}} - 100\lambda$$
$$L_\lambda = -100x - 100y + 400000$$

Using $L_x = 0$, $L_y = 0$ to solve for x and y

Set the first derivative $L_x = 0$ to get and solve for $\lambda = \dfrac{\frac{2}{3}x^{\frac{-1}{3}}y^{\frac{1}{3}}}{100}$ and then set $L_y = 0$ and solve for λ again to get $\lambda = \dfrac{\frac{1}{3}x^{\frac{2}{3}}y^{\frac{-2}{3}}}{100}$

We now need to use the results to turn the equation $L_\lambda = 0$ into an equation of one variable only. To do this we will set the λ equations equal to each other and solve for y as shown below.

$\frac{\frac{2}{3}x^{\frac{-1}{3}}y^{\frac{1}{3}}}{100} = \frac{\frac{1}{3}x^{\frac{2}{3}}y^{\frac{-2}{3}}}{100}$ -> $\frac{\frac{2}{3}x^{\frac{-1}{3}}y^{\frac{1}{3}}}{100} = \frac{\frac{1}{3}x^{\frac{2}{3}}y^{\frac{-2}{3}}}{100}$ -> multiply both sides by 100 and then

by 3 to get $2x^{\frac{-1}{3}}y^{\frac{1}{3}} = x^{\frac{2}{3}}y^{\frac{-2}{3}} \rightarrow \frac{2y^{\frac{1}{3}}}{y^{\frac{-2}{3}}} = \frac{x^{\frac{2}{3}}}{x^{\frac{-1}{3}}} \rightarrow 2y = x \rightarrow y = \frac{1}{2}x$

Now substitute the values of y into $L_\lambda = 0$ to get $-100x - 100(\frac{1}{2}x) + 400000 = 0$ and solve for x = 2666.66 and then find y to be y = 1333.33.

The value of x and y that will maximize production is x = 2666 and y = 1333 and the maximum production is 2116 units.

Example 2:

We will now revisit the profit equation introduced at the beginning of this section.
$p(x,y) = -2x^2 + 60x - 3y^2 + 72y + 100$

where x represents the number of handmade chairs and y is the number of handmade rockers produced per week. The constraint on this profit is g(x, y) = $x + y = 20$.

In order to find the optimal situation given this constraint we will follow the method given above. First we will create the LaGrange equation

$$\boxed{L(x, y) = p(x, y) - \lambda[g(x, y) - k] = (-2x^2 + 60x - 3y^2 + 72y + 100) - \lambda(x + y - 20)}$$

Now find $L_x = 0$, $L_y = 0$ and $L_\lambda = 0$.

$$L_x = -4x + 60 - \lambda = 0$$

$$L_y = -6y + 72 - \lambda = 0$$

$$L_\lambda = -x - y + 20 = 0$$

Using $L_x = 0$, $L_y = 0$ to solve for x and y to get $x = 15 - \frac{1}{4}\lambda$ and $y = 12 - \frac{1}{6}\lambda$. Now substitute the values of x and y into $L_\lambda = 0$ to get
$L_\lambda = -x - y + 20 = -(15 - \frac{1}{4}\lambda) - (12 - \frac{1}{6}\lambda) + 20 = (-27 + \frac{10}{24}\lambda) + 20 = -7 + \frac{5}{12}\lambda = 0$ and $\lambda = 16.8$ so $x = 10.8 \approx 10$ and $y = 9.2 \approx 9$. Therefore the maximum profit will be achieved when 10 chairs and 9 rockers are produced. (Remember that the maximum profit is x = 15 and y = 12 when there is not constraint.)

Notice that the value of $\lambda = 16.8$ and if the constraint was increased from 20 to 21 then the maximum profit would increase by \$16.80

Example 3:

Amanda is getting a new dog. She wants to build a pen for her dog in the back yard. The pen will be rectangular using 200 feet of fence. Amanda plans to build the pen up against the wall of her house so that she will only need three sides of fence. She wants to build a pen with the maximum amount of area. Therefore we need to maximize the area equation A = xy.

Since Amanda has a constraint in the amount of fencing she can use, we will use the LaGrange method and create the following equation.

$L(x,y) = xy - \lambda(x + y - 200)$

Now find $L_x = 0$, $L_y = 0$ and solve for x and y to get that $x = \lambda$ and $y = 2\lambda$. Find $L_\lambda = 0$ and substitute in the values for x and y to put the equation in terms of λ. Now solve to get that $\lambda = 50$. Therefore the optimal values are x = 50 and y = 100 feet of fencing. (Notice that if the constant 200 feet of fencing is increased by on foot to 201 feet then by marginal analysis the maximum value for the area will increase by 50 square feet. Determine what the maximum area would be for 201 feet of fencing) Answer is ____

Example 4 The function U = f(x,y) represent the utility or customer satisfaction derived by a consumer from the consumption of a certain amount of product x and a certain amount of product y. In the example below the maximum utility is determined when the given budget constraint affects the amount of each product produced.

The function $f(x,y) = x^2y^2$ is a utility function with a budget constraint given to be g(x,y) = 2x + 4y = 40.

The first step is to create the LaGrange equation $L(x,y) = x^2y^2 - \lambda(2x + 4y - 40)$

$$L_x = \frac{\partial L}{\partial x} = 2xy^2 - 2\lambda = 0 \quad \text{and therefore } \lambda = xy^2$$

$$L_y = \frac{\partial L}{\partial y} = 2x^2y - 4\lambda = 0 \quad \text{and therefore } \lambda = \frac{1}{2}x^2y$$

The next step is to find $L_\lambda = -2x - 4y + 40 = 0$ and then use the partials above to help transform L_λ into an equation with just one variable. It does not matter if it is terms of x only, y only or λ only (as in the two examples above)

In this case it is easier to set the λ equations equal to each other and solve for either x or y as seen below.

set $\quad xy^2 = \frac{1}{2}x^2y \rightarrow \frac{y^2}{y} = \frac{1}{2}\frac{x^2}{x} \rightarrow y = \frac{x}{2}$

Going back and substitute into $L_\lambda = -2x - 4y + 40 = -2x - 4(\frac{x}{2}) + 40 = 0$ and solve to get x = 10 and then y = 5. (You will also get $\lambda = xy^2 = (10)(25) = 250$. What is the meaning of this information ? It means that if the constant is increased to 41 the maximum utility will increase by 250)

Practice Problems

Problem 1

1) Find the minimum value of the function $f(x, y) = x^2 + 2y^2 - xy$ subject to the constraint $2x + y = 22$.

Problem 2

The highway Department is planning to build a picnic area for motorist along a major highway. It is rectangular with area of 5,000 square yards. It is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing needed to complete the job?

Problem 3

Using the information given in problem 2 graph the level curves and the constraint function and indicate where the minimum point will occur.

Problem 4

Two women have decided to start a cottage industry that will be run out of their homes. They are going to make two types of specialty shirts, rhinestone and embroidery. The total gross profit is given by $P(x, y) = 16x + 20y - x^2 - y^2$ where x is the number of rhinestone shirts made and y is the number of embroidery shirts made per month. Together, they are able to make at most a total of six shirts per month. What is the number of each type of shirt that they should try to make and sell in order to maximize their profit each month.?

Problem 5

If the equation $f(x, y, z) = x^2 + y^2 + z^2$ is subject to the constraint $x - y + 2z = 6$, find the optimal point under these conditions.