8

Statistical Intervals for a Single Sample

© jkullander / iStockphoto

Chapter Outline

Introduction

Engineers are often involved in estimating parameters. For example, there is an ASTM Standard E23 that defines a technique called the *Charpy V-notch method for notched bar impact testing* of metallic materials. The impact energy is often used to determine whether the material experiences a ductile-to-brittle transition as the temperature decreases. Suppose that we have tested a sample of 10 specimens of a particular material with this procedure. We know that we can use the sample average $\overline{X}$ to estimate the true mean impact energy μ. However, we also know that the true mean impact energy is unlikely to be exactly equal to your estimate. Reporting the results of your test as a single number is unappealing because nothing inherent in $\overline{X}$ provides any information about how close it is to μ. Our estimate could be very close, or it could be considerably far from the true mean. A way to avoid this is to report the estimate in terms of a range of plausible values called a **confidence interval**. A confidence interval always specifies a confidence level, usually 90%, 95%, or 99%, which is a measure of the reliability of the procedure. So if a 95% confidence interval on the impact energy based on the data from our 10 specimens has a lower limit of 63.84J and an upper limit of 65.08J, then we can say that at the 95% level of confidence any value of **mean** impact energy between 63.84J and 65.08J is a plausible value. By *reliability*, we mean that if we repeated this experiment over and over again, 95% of all samples would produce a confidence interval that contains the true mean impact energy, and only 5% of the time would the interval be in error. In this chapter, you will learn how to construct confidence intervals and other useful types of statistical intervals for many important types of problem situations.

Learning Objectives

After careful study of this chapter, you should be able to do the following:

1. Construct confidence intervals on the mean of a normal distribution, using either the normal distribution or the *t* distribution method
2. Construct confidence intervals on the variance and standard deviation of a normal distribution
3. Construct confidence intervals on a population proportion
4. Use a general method for constructing an approximate confidence interval on a parameter
5. Construct prediction intervals for a future observation
6. Construct a tolerance interval for a normal population
7. Explain the three types of interval estimates: confidence intervals, prediction intervals, and tolerance intervals

---

In the previous chapter, we illustrated how a point estimate of a parameter can be estimated from sample data. However, it is important to understand how good the estimate obtained is. For example, suppose that we estimate the mean viscosity of a chemical product to be $\hat{\mu} = \overline{x} = 1000$. Now because of sampling variability, it is almost never the case that the true mean μ is exactly equal to the estimate $\overline{x}$. The point estimate says nothing about how close $\hat{\mu}$ is to μ. Is the process mean likely to be between 900 and 1100? Or is it likely to be between 990 and 1010? The answer to these questions affects our decisions regarding this process. Bounds that represent an interval of plausible values for a parameter are examples of an interval estimate. Surprisingly, it is easy to determine such intervals in many cases, and the same data that provided the point estimate are typically used.

An interval estimate for a population parameter is called a **confidence interval**. Information about the precision of estimation is conveyed by the length of the interval. A short interval implies precise estimation. We cannot be certain that the interval contains the true, unknown population parameter—we use only a sample from the full population to compute the point estimate and the interval. However, the confidence interval is constructed so that we have high confidence that it does contain the unknown population parameter. Confidence intervals are widely used in engineering and the sciences.

A **tolerance interval** is another important type of interval estimate. For example, the chemical product viscosity data might be assumed to be normally distributed. We might like to calculate limits that bound 95% of the viscosity values. For a normal distribution, we know that 95% of the distribution is in the interval

$$\mu - 1.96\sigma, \mu - 19.6\sigma$$

However, this is not a useful tolerance interval because the parameters μ and σ are unknown. Point estimates such as $\overline{x}$ and s can be used in the preceding equation for μ and σ. However, we need to account for the potential error in each point estimate to form a tolerance interval for the distribution. The result is an interval of the form

$$\overline{x} - ks,\ \overline{x} + ks$$

where k is an appropriate constant (that is larger than 1.96 to account for the estimation error). As in the case of a confidence interval, it is not certain that the tolerance interval bounds 95% of the distribution, but the interval is constructed so that we have high confidence that it does.

Tolerance intervals are widely used and, as we will subsequently see, they are easy to calculate for normal distributions.

Confidence and tolerance intervals bound unknown elements of a distribution. In this chapter, you will learn to appreciate the value of these intervals. A **prediction interval** provides bounds on one (or more) **future observations** from the population. For example, a prediction interval could be used to bound a single, new measurement of viscosity—another useful interval. With a large sample size, the prediction interval for normally distributed data tends to the tolerance interval, but for more modest sample sizes, the prediction and tolerance intervals are different.

Keep the purpose of the three types of interval estimates clear:

- A confidence interval bounds population or distribution parameters (such as the mean viscosity).
- A tolerance interval bounds a selected proportion of a distribution.
- A prediction interval bounds future observations from the population or distribution.

Our experience has been that it is easy to confuse the three types of intervals. For example, a confidence interval is often reported when the problem situation calls for a prediction interval.

8-1 Confidence Interval on the Mean of a Normal Distribution, Variance Known

The basic ideas of a confidence interval (CI) are most easily understood by initially considering a simple situation. Suppose that we have a normal population with unknown mean μ and known variance σ^2. This is a somewhat unrealistic scenario because typically both the mean and variance are unknown. However, in subsequent sections, we will present confidence intervals for more general situations.

8-1.1 DEVELOPMENT OF THE CONFIDENCE INTERVAL AND ITS BASIC PROPERTIES

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and known variance σ^2. From the results of Chapter 5, we know that the sample mean $\overline{X}$ is normally distributed with mean μ and variance σ^2/n. We may **standardize** $\overline{X}$ by subtracting the mean and dividing by the standard deviation, which results in the variable

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{8-1}$$

The random variable Z has a standard normal distribution.

A **confidence interval** estimate for μ is an interval of the form $l \le \mu \le u$, where the end-points l and u are computed from the sample data. Because different samples will produce different values of l and u, these end-points are values of random variables L and U, respectively. Suppose that we can determine values of L and U such that the following probability statement is true:

$$P\{L \le \mu \le U\} = 1 - \alpha \tag{8-2}$$

where $0 \le \alpha \le 1$. There is a probability of $1 - \alpha$ of selecting a sample for which the CI will contain the true value of μ. Once we have selected the sample, so that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and computed l and u, the resulting **confidence interval** for μ is

$$l \le \mu \le u \tag{8-3}$$

The end-points or bounds l and u are called the **lower-** and **upper-confidence limits (bounds)**, respectively, and $1 - \alpha$ is called the **confidence coefficient**.

In our problem situation, because $Z = (\overline{X} - \mu)/(\sigma/\sqrt{n})$ has a standard normal distribution, we may write

$$P\left\{-z_{\alpha/2} \le \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right\} = 1 - \alpha$$

Now manipulate the quantities inside the brackets by (1) multiplying through by $\sigma/\sqrt{n}$, (2) subtracting $\overline{X}$ from each term, and (3) multiplying through by −1. This results in

$$P\left\{\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha \tag{8-4}$$

This is a **random interval** because the end-points $\overline{X} \pm Z_{\alpha/2}\sigma/\sqrt{n}$ involve the random variable $\overline{X}$. From consideration of Equation 8-4, the lower and upper end-points or limits of the inequalities in Equation 8-4 are the lower- and upper-confidence limits L and U, respectively. This leads to the following definition.

Confidence Interval on the Mean, Variance Known

If $\overline{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ CI on μ is given by

$$\overline{x} - z_{\alpha/2}\sigma/\sqrt{n} \le \mu \le \overline{x} + z_{\alpha/2}\sigma/\sqrt{n} \tag{8-5}$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.

The development of this CI assumed that we are sampling from a normal population. The CI is quite robust to this assumption. That is, moderate departures from normality are of no serious concern. From a practical viewpoint, this implies that an **advertised** 95% CI might have actual confidence of 93% or 94%.

Example 8-1 **Metallic Material Transition** ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at 60°C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. Assume that impact energy is normally distributed with $\sigma = 1J$. We want to find a 95% CI for μ, the mean impact energy. The required quantities are $z_{\alpha/2} = z_{0.025} = 1.96$, $n = 10$, $\sigma = 1$, and $\overline{x} = 64.46$. The resulting 95% CI is found from Equation 8-5 as follows:

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$64.46 - 1.96\frac{1}{\sqrt{10}} \le \mu \le 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \le \mu \le 65.08$$

Practical Interpretation: Based on the sample data, a range of highly plausible values for mean impact energy for A238 steel at 60°C is $63.84J \le \mu \le 65.08J$.

Interpreting a Confidence Interval

How does one interpret a confidence interval? In the impact energy estimation problem in Example 8-1, the 95% CI is $63.84 \le \mu \le 65.08$, so it is tempting to conclude that μ is within

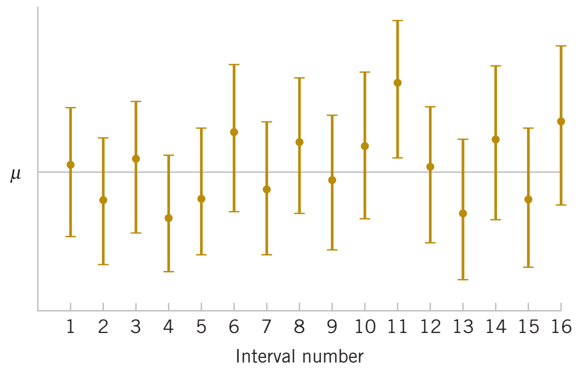

FIGURE 8-1 Repeated construction of a confidence interval for μ.

this interval with probability 0.95. However, with a little reflection, it is easy to see that this cannot be correct; the true value of μ is unknown, and the statement $63.84 \leq \mu \leq 65.08$ is either correct (true with probability 1) or incorrect (false with probability 1). The correct interpretation lies in the realization that a CI is a *random interval* because in the probability statement defining the end-points of the interval (Equation 8-2), L and U are random variables. Consequently, the correct interpretation of a $100(1-\alpha)\%$ CI depends on the relative frequency view of probability. Specifically, if an infinite number of random samples are collected and a $100(1-\alpha)\%$ confidence interval for μ is computed from each sample, $100(1-\alpha)\%$ of these intervals will contain the true value of μ.

The situation is illustrated in Fig. 8-1, which shows several $100(1-\alpha)\%$ confidence intervals for the mean μ of a normal distribution. The dots at the center of the intervals indicate the point estimate of μ (that is, $\bar{x}$). Notice that one of the intervals fails to contain the true value of μ. If this were a 95% confidence interval, in the long run only 5% of the intervals would fail to contain μ.

Now in practice, we obtain only one random sample and calculate one confidence interval. Because this interval either will or will not contain the true value of μ, it is not reasonable to attach a probability level to this specific event. The appropriate statement is that the observed interval $[l, u]$ brackets the true value of μ with **confidence** $100(1-\alpha)$. This statement has a frequency interpretation; that is, we do not know whether the statement is true for this specific sample, but the *method* used to obtain the interval $[l, u]$ yields correct statements $100(1-\alpha)\%$ of the time.

Confidence Level and Precision of Estimation

Notice that in Example 8-1, our choice of the 95% level of confidence was essentially arbitrary. What would have happened if we had chosen a higher level of confidence, say, 99%? In fact, is it not reasonable that we would want the higher level of confidence? At $\alpha = 0.01$, we find $z_{\alpha/2} = z_{0.01/2} = z_{0.005} = 2.58$, while for $\alpha = 0.05$, $z_{0.025} = 1.96$. Thus, the **length** of the 95% confidence interval is

$$2\left(1.96\sigma/\sqrt{n}\right) = 3.92\sigma/\sqrt{n}$$

whereas the length of the 99% CI is

$$2\left(2.58\sigma/\sqrt{n}\right) = 5.16\sigma/\sqrt{n}$$

Thus, the 99% CI is longer than the 95% CI. This is why we have a higher level of confidence in the 99% confidence interval. Generally, for a fixed sample size n and standard deviation σ, the higher the confidence level, the longer the resulting CI.

The length of a confidence interval is a measure of the **precision** of estimation. Many authors define the half-length of the CI (in our case $z_{\alpha/2}\sigma/\sqrt{n}$) as the bound on the error in estimation of the parameter. From the preceeding discussion, we see that precision is inversely

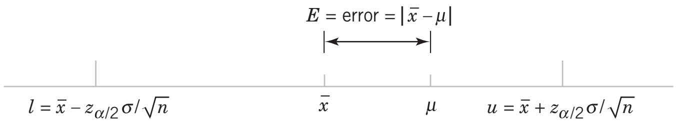

FIGURE 8-2 Error in estimating μ with $\bar{x}$.

related to the confidence level. It is desirable to obtain a confidence interval that is short enough for decision-making purposes and that also has adequate confidence. One way to achieve this is by choosing the sample size n to be large enough to give a CI of specified length or precision with prescribed confidence.

8-1.2 CHOICE OF SAMPLE SIZE

The precision of the confidence interval in Equation 8-5 is $2z_{\alpha/2}\sigma/\sqrt{n}$. This means that in using $\bar{x}$ to estimate μ, the error $E = |\bar{x} - \mu|$ is less than or equal to $z_{\alpha/2}\sigma/\sqrt{n}$ with confidence $100(1-\alpha)$. This is shown graphically in Fig. 8-2. In situations whose sample size can be controlled, we can choose n so that we are $100(1-\alpha)\%$ confident that the error in estimating μ is less than a specified bound on the error E. The appropriate sample size is found by choosing n such that $z_{\alpha/2}\sigma/\sqrt{n} = E$. Solving this equation gives the following formula for n.

Sample Size for Specified Error on the Mean, Variance Known

If $\bar{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error $|\bar{x} - \mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \tag{8-6}$$

If the right-hand side of Equation 8-6 is not an integer, it must be rounded up. This will ensure that the level of confidence does not fall below $100(1-\alpha)\%$. Notice that $2E$ is the length of the resulting confidence interval.

Example 8-2 **Metallic Material Transition** To illustrate the use of this procedure, consider the CVN test described in Example 8-1 and suppose that we want to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at 60°C has a length of at most $1.0J$. Because the bound on error in estimation E is one-half of the length of the CI, to determine n, we use Equation 8-6 with $E = 0.5$, $\sigma = 1$, and $z_{\alpha/2} = 1.96$. The required sample size is,

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left[\frac{(1.96)1}{0.5}\right]^2 = 15.37$$

and because n must be an integer, the required sample size is $n = 16$.

Notice the general relationship between sample size, desired length of the confidence interval $2E$, confidence level $100(1-\alpha)$, and standard deviation σ:

- As the desired length of the interval $2E$ decreases, the required sample size n increases for a fixed value of σ and specified confidence.
- As σ increases, the required sample size n increases for a fixed desired length $2E$ and specified confidence.
- As the level of confidence increases, the required sample size n increases for fixed desired length $2E$ and standard deviation σ.

8-1.3 ONE-SIDED CONFIDENCE BOUNDS

The confidence interval in Equation 8-5 gives both a lower confidence bound and an upper confidence bound for μ. Thus, it provides a two-sided CI. It is also possible to obtain one-sided confidence bounds for m by setting either the lower bound $l = -\infty$ or the upper bound $u = \infty$ and replacing $z_{\alpha/2}$ by z_α.

One-Sided Confidence Bounds on the Mean, Variance Known

A $100(1-\alpha)\%$ **upper-confidence bound** for μ is

$$\mu \le \bar{x} + z_\alpha \sigma / \sqrt{n} \tag{8-7}$$

and a $100(1-\alpha)\%$ **lower-confidence bound** for μ is

$$\bar{x} - z_\alpha \sigma / \sqrt{n} = l \le \mu \tag{8-8}$$

Example 8-3 **One-Sided Confidence Bound** The same data for impact testing from Example 8-1 are used to construct a lower, one-sided 95% confidence interval for the mean impact energy. Recall that $\bar{x} = 64.46$, $\sigma = 1J$, and $n = 10$. The interval is

$$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} \le \mu$$

$$64.46 - 1.64 \frac{1}{\sqrt{10}} \le \mu$$

$$63.94 \le \mu$$

Practical Interpretation: The lower limit for the two-sided interval in Example 8-1 was 63.84. Because $z_\alpha < z_{\alpha/2}$, the lower limit of a one-sided interval is always greater than the lower limit of a two-sided interval of equal confidence. The one-sided interval does not bound μ from above so that it still achieves 95% confidence with a slightly larger lower limit. If our interest is only in the lower limit for μ, then the one-sided interval is preferred because it provides equal confidence with a greater limit. Similarly, a one-sided upper limit is always less than a two-sided upper limit of equal confidence.

8-1.4 GENERAL METHOD TO DERIVE A CONFIDENCE INTERVAL

It is easy to give a general method for finding a confidence interval for an unknown parameter θ. Let $X_1, X_2, \ldots, X_n$ be a random sample of n observations. Suppose that we can find a statistic $g(X_1, X_2, \ldots, X_n; \theta)$ with the following properties:

1. $g(X_1, X_2, \ldots, X_n; \theta)$ depends on both the sample and θ.
2. The probability distribution of $g(X_1, X_2, \ldots, X_n; \theta)$ does not depend on θ or any other unknown parameter.

In the case considered in this section, the parameter $\theta = \mu$. The random variable $g(X_1, X_2, \ldots, X_n; \mu) = (\bar{X} - \mu) / (\sigma / \sqrt{n})$ satisfies both conditions; the random variable depends on the sample and on μ, and it has a standard normal distribution because σ is known. Now we must find constants C_L and C_U so that

$$P\left[C_L \le g\left(X_1, X_2, \ldots, X_n; \theta\right) \le C_U\right] = 1 - \alpha \tag{8-9}$$

Because of property 2, C_L and C_U do not depend on θ. In our example, $C_L = -z_{\alpha/2}$ and $C_U = z_{\alpha/2}$. Finally, we must manipulate the inequalities in the probability statement so that

$$P\left[L\left(X_1, X_2, \ldots, X_n\right) \le \theta \le U\left(X_1, X_2, \ldots, X_n\right)\right] = 1 - \alpha \tag{8-10}$$

This gives $L(X_1, X_2, \ldots, X_n)$ and $U(X_1, X_2, \ldots, X_n)$ as the lower and upper confidence limits defining the $100(1-\alpha)$ confidence interval for θ. The quantity $g(X_1, X_2, \ldots, X_n; \theta)$ is often called a *pivotal quantity* because we pivot on this quantity in Equation 8-9 to produce Equation 8-10. In our example, we manipulated the pivotal quantity $(\bar{X}-\mu)/(\sigma/\sqrt{n})$ to obtain $L(X_1, X_2, \ldots, X_n) = \bar{X} - z_{\alpha/2}\sigma/\sqrt{n}$ and $U(X_1, X_2, \ldots, X_n) = \bar{X} + z_{\alpha/2}\sigma/\sqrt{n}$.

Example 8-4 **The Exponential Distribution** The exponential distribution is used extensively in the fields of reliability engineering and communications technology because it has been shown to be an excellent model for many of the kinds of problems encountered. For example, the call-handling (processing) time in telephone networks often follows an exponential distribution. A sample of $n = 10$ calls had the following durations (in minutes):

$$x_1 = 2.84,\ x_2 = 2.37,\ x_3 = 7.52,\ x_4 = 2.76,\ x_5 = 3.83,\ x_6 = 1.32,\ x_7 = 8.43,\ x_8 = 2.25,\ x_9 = 1.63 \text{ and } x_{10} = 0.27.$$

Assume that call-handling time is exponentially distributed. Find a 95% two-sided CI on both the parameter λ of the exponential distribution and the mean call-handling time.

If X is an exponential random variable, it can be shown that $2\lambda\sum_{i=1}^{n} X_i$ is a chi-square distributed random variable with $2n$ degrees of freedom (the chi-square distribution will be formally introduced in Section 8.3). So we can let $g(x_1, x_2, \ldots x_n; \theta)$ in Equation (8-9) equal $2\lambda\sum_{i=1}^{n} X_i$ and let C_L and C_U in that equation be the lower-tailed and upper-tailed 2½ percentage points of the chi-square distribution, which are given in Appendix Table IV. For $2n = 2(10) = 20$ degrees of freedom, these percentage points are $C_L = 9.59$ and $C_U = 34.17$, respectively. Therefore, Equation (8-9) becomes

$$P\left(9.59 \le 2\lambda\sum_{i=1}^{n} X_i \le 34.17\right) = 0.95$$

Rearranging the quantities inside the probability statement by dividing through by $2\sum_{i=1}^{n} X_i$ gives

$$P\left(\frac{9.59}{2\sum_{i=1}^{n} X_i} \le \lambda \le \frac{34.17}{2\sum_{i=1}^{n} X_i}\right) = 0.95$$

From the sample data, we find that $\sum_{i=1}^{n} x_i = 33.22$, so the lower confidence bound on λ is

$$\frac{9.59}{2\sum_{i=1}^{n} x_i} = \frac{9.59}{2(33.22)} = 0.1443$$

and the upper confidence bound is

$$\frac{34.17}{2\sum_{i=1}^{n} x_i} = \frac{34.17}{2(33.22)} = 0.5143$$

The 95% two-sided CI on λ is

$$0.1443 \le \lambda \le 0.5143$$

The 95% confidence interval on the mean call-handling time is found using the relationship between the mean μ of the exponential distribution and the parameter λ; that is, $\mu = 1/\lambda$. The resulting 95% CI on μ is $1/0.5143 \le \mu = 1/\lambda \le 1/0.1443$, or

$$1.9444 \le \mu \le 6.9300$$

Therefore, we are 95% confident that the mean call-handling time in this telephone network is between 1.9444 and 6.9300 minutes.

8-1.5 LARGE-SAMPLE CONFIDENCE INTERVAL FOR μ

We have assumed that the population distribution is normal with unknown mean and known standard deviation σ. We now present a **large-sample CI** for μ that does not require these assumptions. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with unknown mean μ and variance σ^2. Now if the sample size n is large, the central limit theorem implies that $\bar{X}$ has approximately a normal distribution with mean μ and variance σ^2/n. Therefore, $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$ has approximately a standard normal distribution. This ratio could be used as a pivotal quantity and manipulated as in Section 8-1.1 to produce an approximate CI for μ. However, the standard deviation σ is unknown. It turns out that when n is large, replacing σ by the sample standard deviation S has little effect on the distribution of Z. This leads to the following useful result.

Large-Sample Confidence Interval on the Mean

When n is large, the quantity

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has an approximate standard normal distribution. Consequently,

$$\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}} \tag{8-11}$$

is a **large-sample confidence interval** for μ, with confidence level of approximately $100(1 - \alpha)\%$.

Equation 8-11 holds regardless of the shape of the population distribution. Generally, n should be at least 40 to use this result reliably. The central limit theorem generally holds for $n \ge 30$, but the larger sample size is recommended here because replacing s with S in Z results in additional variability.

Example 8-5 **Mercury Contamination** An article in the 1993 volume of the *Transactions of the American Fisheries Society* reports the results of a study to investigate the mercury contamination in large-mouth bass. A sample of fish was selected from 53 Florida lakes, and mercury concentration in the muscle tissue was measured (ppm). The mercury concentration values were

1.230	1.330	0.040	0.044	1.200	0.270
0.490	0.190	0.830	0.810	0.710	0.500
0.490	1.160	0.050	0.150	0.190	0.770
1.080	0.980	0.630	0.560	0.410	0.730
0.590	0.340	0.340	0.840	0.500	0.340
0.280	0.340	0.750	0.870	0.560	0.170
0.180	0.190	0.040	0.490	1.100	0.160
0.100	0.210	0.860	0.520	0.650	0.270
0.940	0.400	0.430	0.250	0.270	

The summary statistics for these data are as follows:

Variable	***N***	**Mean**	**Median**	**StDev**	**Minimum**	**Maximum**	***Q*1**	***Q*3**
Concentration	53	0.5250	0.4900	0.3486	0.0400	1.3300	0.2300	0.7900

Figure 8-3 presents the histogram and normal probability plot of the mercury concentration data. Both plots indicate that the distribution of mercury concentration is not normal and is positively skewed. We want to find an approximate 95% CI on μ. Because $n > 40$, the assumption of normality is not necessary to use in Equation 8-11. The required quantities are $n = 53$, $\bar{x} = 0.5250$, $s = 0.3486$, and $z_{0.025} = 1.96$. The approximate 95% CI on μ is

$$\bar{x} - z_{0.025}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

$$0.5250 - 1.96\frac{0.3486}{\sqrt{53}} \le \mu \le 0.5250 + 1.96\frac{0.3486}{\sqrt{53}}$$

$$0.4311 \le \mu \le 0.6189$$

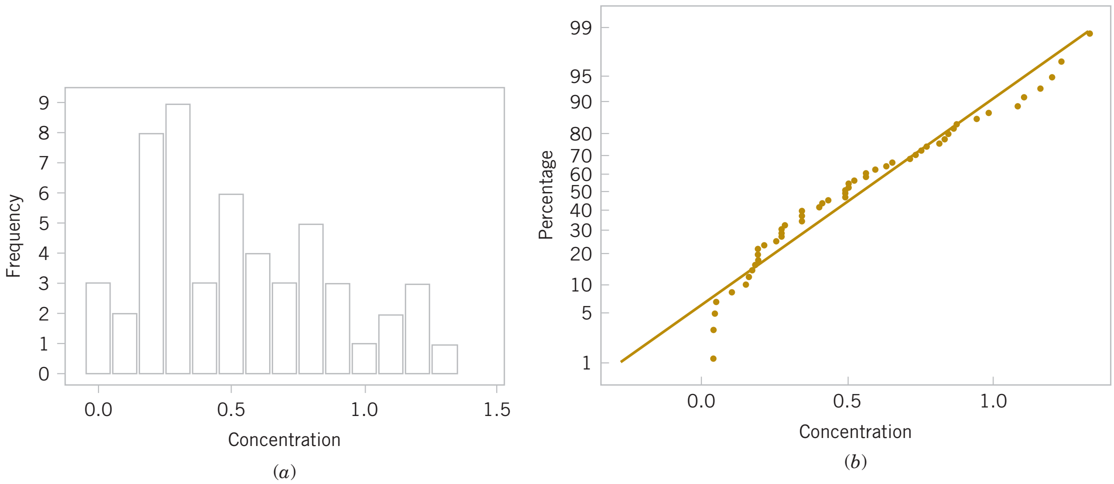

FIGURE 8-3 Mercury concentration in largemouth bass. (a) Histogram. (b) Normal probability plot.

Practical Interpretation: This interval is fairly wide because there is substantial variability in the mercury concentration measurements. A larger sample size would have produced a shorter interval.

Large-Sample Confidence Interval for a Parameter

The large-sample confidence interval for μ in Equation 8-11 is a special case of a more general result. Suppose that θ is a parameter of a probability distribution, and let $\hat{\Theta}$ be an estimator of θ. If $\hat{\Theta}$ (1) has an approximate normal distribution, (2) is approximately unbiased for θ, and (3) has standard deviation $\sigma_{\hat{\Theta}}$ that can be estimated from the sample data, the quantity $(\hat{\Theta} - 0)/\sigma_{\hat{\Theta}}$ has an approximate standard normal distribution. Then a large-sample approximate CI for θ is given by

Large-Sample Approximate Confidence Interval

$$\hat{\theta} - z_{\alpha/2}\sigma_{\hat{\Theta}} \le \theta \le \hat{\theta} + z_{\alpha/2}\sigma_{\hat{\Theta}} \qquad (8\text{-}12)$$

Maximum likelihood estimators usually satisfy the three conditions just listed, so Equation 8-12 is often used when $\hat{\Theta}$ is the maximum likelihood estimator of θ. Finally, note that Equation 8-12 can be used even when $\sigma_{\hat{\Theta}}$ is a function of other unknown parameters (or of θ). Essentially, we simply use the sample data to compute estimates of the unknown parameters and substitute those estimates into the expression for $\sigma_{\hat{\Theta}}$.