Weakly calibrated currents

Kurt Pagani
pagani@scios.ch

June 9, 2013

Abstract

We show

1 Introduction

Denote by $\mathcal{D}_p(\mathbb{R}^n)$ the space of infinitely differentiable p-forms on $\mathbb{R}^n$ with compact support, and by $\mathcal{D}'_p(\mathbb{R}^n)$ the corresponding space of p-currents (DeRham currents to be precise). One may think of Distributions with values in the finite dimensional Grassmann space $\wedge_p\mathbb{R}^n$, whose elements are the p-vectors

$$v = \tfrac{1}{p!}\sum_{\alpha} v^{\alpha} e_{\alpha} = \sum_{1\leq\alpha_1<\ldots<\alpha_p\leq n} v^{\alpha_1\cdots\alpha_p} e_1\wedge\ldots\wedge e_p,$$

where $e_1,\ldots,e_n$ denotes the standard basis of $\mathbb{R}^n$. For any $C\subset\mathcal{D}_p(\mathbb{R}^n)$ we define

$$\Phi_C(T) = \mathtt{sup}\{T(\omega) : \omega\in C\}, \tag{1}$$

for all $T\in\mathcal{D}'_p(\mathbb{R}^n)$. The set $\{T\in\mathcal{D}'_p(\mathbb{R}^n) : |\Phi_C(T)|<\infty\}$ is called the domain of Φ_C, as it is usual in convexity theory. It is easily seen that Φ_C is a positively homogeneous, lower semicontinuous and convex function. Moreover, it is the support function of the set C, thus we could assume that C is a closed convex subset since the the support functions of C and $\overline{\mathtt{conv}(C)}$ coincide. Recall that the boundary of a p-current is defined as the $(p-1)$-current

$$\partial T(\phi) = T(d\phi),$$

so that for smooth objects the Stokes theorem results:

$$\int_{\partial T}\omega = \int_T d\omega.$$

For such smooth objects that we can integrate over, we use the notation $[\![.]\!]$. For instance, let γ be a smooth curve joining the points $a,b\in\mathbb{R}^n$, then

$$[\![\gamma]\!](\omega) = \int_I \langle\gamma'(t),(\omega\circ\gamma)(t)\rangle\,dt$$

and

$$\partial[\![\gamma]\!](\phi) = [\![\gamma]\!](d\phi) = [\![b]\!](\phi) - [\![a]\!](\phi) = \phi(b)-\phi(a).$$

With that definition we have $\partial[\![\gamma]\!] = [\![\partial\gamma]\!]$ in this special case, what is not true generally, of course. Furthermore, the integrand $\langle\gamma'(t),(\omega\circ\gamma)(t)\rangle\,dt$ is just the pull-back of the one form ω by the mapping $\gamma : I\to\mathbb{R}^n$, where $I=[a,b]$. Therefore we may also write

$$[\![\gamma]\!](\omega) = (\gamma_{\#}[\![I]\!])(\omega) = [\![I]\!](\gamma^{\#}\omega),$$

where $\gamma_{\#}$ and $\gamma^{\#}$ denotes the push-forward and pull-back respectively. We adopt this notation for any mapping, so that, for example, $G_{u,\#}[\![\Omega]\!]$ means integration over the graph of the mapping $u:\Omega\to\mathbb{R}^n$.

Definition 1. *Let $1 \leq p \leq n$, $C \subset \mathcal{D}_p(\mathbb{R}^n)$, $S \in \mathcal{D}'_{p-1}(\mathbb{R}^n)$, then set*

$$\alpha(C,S) = \mathtt{inf}\{\Phi_C(T) : T \in \mathcal{D}'_p(\mathbb{R}^n), \partial T = S\}, \tag{2}$$

where Φ_C as in (1)*.*

By the well known saddle point property we have

$$\alpha(C,S) = \inf_{\partial T = S} \sup_C T(\omega) \geq \sup_C \inf_{\partial T = S} T(\omega),$$

and if there is a pair[1] (T_0, ω_0) such that

$$T_0(\omega) \leq T_0(\omega_0) \leq T(\omega_0)$$

for all $(T, \omega) \in \{\partial T = S\} \times C$, then

$$\alpha(C,S) = \inf_{\partial T = S} \sup_C T(\omega) = \sup_C \inf_{\partial T = S} T(\omega) = T_0(\omega_0).$$

Definition 2. *Let $1 \leq p \leq n$, $C \subset \mathcal{D}_p(\mathbb{R}^n)$, $S \in \mathcal{D}'_{p-1}(\mathbb{R}^n)$, then set*

$$\beta(C,S) = \mathtt{sup}\{S(\phi) : \phi \in \mathcal{D}_{p-1}(\mathbb{R}^n), d\phi \in C\}. \tag{3}$$

It may be the case that C does not contain any exact forms, therefore, and because we use the extended real number system, let us agree to set $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Note that β is the support function of the set $d^{-1}C$, thus $\beta(C,S) = \Phi_{d^{-1}C}(S)$.

Definition 3. *A p-form ω_0 is called a calibration for $C \subset \mathcal{D}_p(\mathbb{R}^n)$ if*

1. *$\omega_0 \in C$,*
2. *$d\omega_0 = 0$, and*
3. *$\int_{\mathbb{R}^n} \omega_0 = 0$ if $p = n$.*

A p-current T_0 is said to be calibrated by ω_0 if $T_0(\omega_0) = \Phi_C(T_0)$.

Now, we can come to the main point: suppose T_0 is calibrated by ω_0, then

$$\Phi_C(T) - \Phi_C(T_0) = \Phi_C(T) - T_0(\omega_0) \geq T(\omega_0) - T_0(\omega_0) = \partial B(\omega_0) = B(d\omega_0) = 0,$$

thus

$$\Phi_C(T) \geq \Phi_C(T_0)$$

for all T such that $\partial T = \partial T_0$. Recall that the compactly supported de Rham cohomology groups for $\mathbb{R}^n$ are given by $H^p_c(\mathbb{R}^n) = \delta_{p,n}\mathbb{R}$, therefore we had to add item 3 in the definition of a calibration for $p = n$ in order to guarantee that the difference $T - T_0$ is a boundary ∂B when $p = n$. It can be omitted if one uses $\mathcal{E}_p(\mathbb{R}^n)$ instead of $\mathcal{D}_p(\mathbb{R}^n)$.

Definition 4. *We call a p-current T_0 weakly calibrated on $C \subset \mathcal{D}_p(\mathbb{R}^n)$ if there is a sequence $\{d\phi_j\}_{j \in \mathbb{N}} \subset C$ of exact forms such that*

$$\lim_{j \to \infty} T_0(d\phi_j) = \Phi_C(T_0).$$

Obviously, every calibrated current is also weakly calibrated (choose $d\phi_j = \omega_0$ for all j, since ω_0 has to be exact on $\mathbb{R}^n$).

[1] a saddle point actually

2 Results

Proposition 1. *Suppose $T_0 \in \mathcal{D}'_p(\mathbb{R}^n)$ with $\partial T_0 = S$ is weakly calibrated on $C \subset \mathcal{D}_p(\mathbb{R}^n)$, then*

$$\Phi_C(T_0) = \alpha(C,S) = \beta(C,S), \tag{4}$$

that is

$$\Phi_C(T_0) \leq \Phi_C(T)$$

for all T with the same boundary S.

Proof. For all T with boundary S holds $T(d\phi_j) = S(\phi_j)$ and we always have $\beta(C,S) \leq \alpha(C,S)$ because of

$$S(\phi) = \partial T(\phi) = T(d\phi) \leq \Phi_C(T),$$

therefore

$$\Phi_C(T) - \Phi_C(T_0) = \Phi_C(T) - \lim_{j\to\infty} T_0(d\phi_j) \geq \liminf_{j\to\infty}(T(d\phi_j) - T_0(d\phi_j)) = 0.$$

Now, since $\Phi_C(T_0) = \alpha(C,S)$ and $\lim_{j\to\infty} T_0(d\phi_j) = \lim_{j\to\infty} S(\phi_j) = \alpha(C,S) \leq \beta(C,S)$ it follows necessarily that $\alpha = \beta$. □

The proof above shows that a necessary condition for a calibrated current (weakly or not) is given by $\alpha(C,S) = \beta(C,S)$. Recalling the saddle point property mentioned in the introduction, it holds

$$\alpha(C,S) = \inf_{\partial T = S} \sup_C T(\omega) \geq \sup_C \inf_{\partial T = S} T(\omega) \geq \beta(C,S),$$

where all inequalities may be strict.

References

[1] L.E. Payne, G.A. Philippin *On Maximum Principles for a Class of Nonlinear Second-Order Elliptic Equations.* Journal of Differential Equations,37,39-48,1980.

[2] L.E. Payne *"Best Possible" Maximum Principles.* Banach Center Publications,Vol. 11, PWN Polish Scientific Publishers, Warsaw 1985.

[3] R.P. Sperb *Maximum Principles and their Applications.* Academic Press, 1981.

[4] D. Gilbarg, N.S. Trudinger *Elliptic Partial Differential Equations of Second Order.* Classics in Mathematics, Springer Verlag.

[5] F. Rellich *Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral.* Math. Zeitschrift, 46, 635-636, 1940.