Transport

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1 Introduction and Notation

In the sequel we are going to use a lot of results and terminology of *mass transportation theory*, whereby [6] serves as the main reference.

Let $\mathcal{P}(\mathbb{R}^n)$ denote the space of probability measures on $\mathbb{R}^n$ and for $\varphi \in L^2(\mathbb{R}^n)$ let $\hat{\varphi}$ denote its (unitary) Fourier transform. Each normalized $\varphi \in L^2(\mathbb{R}^n)$ gives rise to a measure

$$\nu_\varphi(f) = \int_{\mathbb{R}^n} f(x)|\varphi(x)|^2\, dx,$$

where $f \in C_0(\mathbb{R}^n)$, the continuous functions with compact support. Then we define the mapping

$$\mu : L^2(\mathbb{R}^n) \cap \{||\varphi|| = 1\} \longrightarrow \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n) \tag{1}$$

$$\varphi \longmapsto \nu_\varphi \otimes \nu_{\hat{\varphi}},$$

that means μ_φ is the (unique) product measure with marginals ν_φ and $\nu_{\hat{\varphi}}$. Furthermore we denote by $\Gamma(\varphi)$ the subset of $\mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ whose elements have the aforementioned marginals.

Let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be lower semi-continuous and bounded below then we call

$$K_H(\varphi) = \inf_{\gamma \in \Gamma(\varphi)} \int_{\mathbb{R}^n \times \mathbb{R}^n} H(q,p)\, d\gamma(q,p) \tag{2}$$

the *Kantorovich* energy of φ. Similarly we call

$$E_H(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} H(q,p)\, d\mu_\varphi(q,p) \tag{3}$$

the *Schrödinger* energy, for reasons that will be enlightened further below. Monge's formulation of the *optimal transport problem* reads in our case as

$$M_H(\varphi) = \inf_T \left\{ \int_{\mathbb{R}^n} H(q, T(q) d\nu_\varphi : T_\# \nu_\varphi = \nu_{\hat{\varphi}} \right\}, \tag{4}$$

which means to find a minimizing map $T : \mathbb{R}^n \to \mathbb{R}^n$ that transports the measure ν_φ to $\nu_{\hat{\varphi}}$ by pushing forward:

$$T_\# \nu_\varphi(f) := \nu_\varphi(f \circ T) = \int_{\mathbb{R}^n} f(T(q)) |\varphi(q)|^2 \, dq = \int_{\mathbb{R}^n} f(p) |\hat{\varphi}(p)|^2 \, dp.$$

If all quantities involved were smooth enough and T one to one then we would get the condition

$$|\varphi(q)|^2 = |\hat{\varphi}(T(q))|^2 \, |\det DT(q)| \tag{5}$$

by a simple change of coordinates. So it is by no means for sure whether admissible maps exist at all.

1.1 Schrödinger Energy

Suppose H has the familiar form $H(x,k) = \frac{\hbar^2 |k|^2}{2m} + V(x)$, then we easily calculate that

$$K_H(\varphi) = E_H(\varphi) = \frac{\hbar^2}{2m} \int_{\mathbb{R}^n} |k|^2 |\hat{\varphi}(k)|^2 dk + \int_{\mathbb{R}^n} V(x) |\varphi(x)|^2 dx$$

holds. Furthermore, if $\partial_j \varphi \in L^2(\mathbb{R}^n)$ then the above expression reduces to

$$K_H(\varphi) = E_H(\varphi) = \int_{\mathbb{R}^n} \left(\frac{\hbar^2}{2m} |\nabla \varphi(x)|^2 + V(x) |\varphi(x)|^2 \right) dx. \tag{6}$$

Whether the energy is finite or not will also depend on the behavior of V of course. In a similar way the above deduction holds whenever the *cost function* H has the form $H(q,p) = T(p) + V(q)$ or even $H = T(p)V(q)$, that is the *Kantorovich energy* coincides with E_H which in turn means that the *transference plan* $\mu_\varphi = \nu_\varphi \otimes \nu_{\hat{\varphi}}$ is optimal. Villani notes with reference to the sand pile example [6]

> ...this corresponds to the most stupid transportation plan that one may imagine: any piece of sand, regardless of its location, is distributed over the entire hole, proportionally to the depth.

He certainly would not claim that quantum mechanics is stupid, however, we recognize that the procedure mentioned is just another formulation of the uncertainty principle (replacing sand pile/hole by position/momentum). This is in strong contrast to the corresponding *Monge* problem (omitting the factor $\hbar^2/2m$ from now on),

$$M_H(\varphi) = \inf_T \left\{ \int_{\mathbb{R}^n} \left(|T(x)|^2 + V(x)\right) d\nu_{\varphi(x)} : T_\# \nu_\varphi = \nu_{\hat{\varphi}} \right\},$$

where, since T is a map, there is no such distribution (*mass* cannot be split by Monge transport). By a theorem of Brenier-McCann [5], there is a convex function ϕ on $\mathbb{R}^n$ such that

$$(\nabla\phi)_\# \nu_\varphi = \nu_{\hat{\varphi}},$$

whence we have [1]

$$K_H(\varphi) \leq M_H(\varphi) \leq \int_{\mathbb{R}^n} \left(|\nabla\phi(x)|^2 + V(x)\right) |\varphi(x)|^2 dx.$$

Now, if we suppose for the moment the existence of a ground state $\varphi_0 > 0$ to E_H (more precisely to the self-adjoint operator corresponding to H), we would find the identities

$$E_H(\varphi_0) = K_H(\varphi_0) = \int_{\mathbb{R}^n} \left(|\nabla \log \varphi_0(x)|^2 + V(x)\right) |\varphi_0(x)|^2 dx \leq M_H(\varphi_0).$$

This leads to the question:

Can $\nabla\phi = -\nabla \log \varphi_0$ be a *Brenier* map?

In the first place $\phi = -\log \varphi_0$ had to be convex, or equivalently, the ground state $\varphi_0(x) = Ce^{-\phi(x)}$ should be `log`-concave, a property that is not uncommon for certain potentials V. A far more stringent condition, however, is the requirement $(-\nabla \log \varphi_0)_\# \nu_{\varphi_0} = \nu_{\hat{\varphi}_0}$, which, assuming some smoothness and recalling (5), reads as

$$|\varphi_0(q)|^2 = |\hat{\varphi}_0(\nabla\phi(q))|^2 \, |\det D^2\phi(q)|. \tag{7}$$

Actually, the ground state of the harmonic oscillator $\varphi_{ho}(x) = Ce^{-\frac{1}{2}|x|^2}$ satisfies the above equation and consequently in this particular case we have $K_H(\varphi_{ho}) = E_H(\varphi_{ho}) = M_H(\varphi_{ho})$ with transport map $T(x) = \nabla\phi(x) = -\nabla \log \varphi_{ho}(x) = x$, the identity map. Are there others? We do not know.

[1] notice that K_H usually is a relaxation of M_H since an admissible transport map T always gives raise to a transference plan $(id \times T)_\# \nu_\varphi \in \Gamma(\varphi)$

1.2 General H

If the Hamilton function H does not split up as above then we only have $K_H(\varphi) \leq E_H(\varphi)$ and since the infimum in (1) is always attained [2] , there is a $\gamma_\varphi \in \Gamma(\varphi)$ such that $\gamma_\varphi(H) \leq \mu_\varphi(H)$. In the following let us denote by $\Gamma_n = \mathbb{R}^n \oplus \mathbb{R}^n$ a $2n$-dimensional phase space, where there should be no confusion among the meanings of Γ, e.g. we have $\Gamma(\varphi) \subset \mathcal{P}(\Gamma_n)$. The minimization problem

$$\lambda_0 = \inf \left\{ \int_{\Gamma_n} H d\mu_\varphi : \varphi \in L^2(\mathbb{R}^n), ||\varphi||_2 = 1 \right\} \tag{8}$$

may have or may not have a solution. This depends on the function H under consideration and even if there is a solution it is by no means granted that it would be a ground state to a corresponding self-adjoint *Hamiltonian*. Existence questions will not be our concern at this point, therefore we will take the existence of a minimizer $\varphi_0 \in L^2(\mathbb{R}^n)$ for granted. Since we have assumed the function H to be bounded below (and l.s.c) it is obvious that

$$\lambda_0 \geq \inf_{\Gamma_n} H > -\infty$$

and moreover it holds that $\lambda_0 = E_H(\varphi_0) \geq K_H(\varphi_0)$. When we define (assuming H fixed)

$$F_\varphi(x) = \int_{\mathbb{R}^n} H(x,k)\, |\hat{\varphi}(k)|^2 dk$$

as well

$$G_\varphi(k) = \int_{\mathbb{R}^n} H(x,k)\, |\varphi(x)|^2 dx$$

and recall that $\mu_\varphi = \nu_\varphi \otimes \nu_{\hat{\varphi}}$ holds by definition, we obtain

$$E_H(\varphi) = \int_{\Gamma_n} H(x,k)\, d\mu_\varphi(x,k) = \int_{\mathbb{R}^n} F_\varphi(x) d\nu_\varphi(x) = \int_{\mathbb{R}^n} G_\varphi(k) d\nu_{\hat{\varphi}}(k).$$

Now we may state the *Euler equations* which a minimizer had to satisfy.

Proposition 1. *Let $\varphi_0 \in L^2(\mathbb{R}^n)$ be a critical point of $E_H(\varphi)$, then it satisfies the equation (in $\mathcal{D}'(\mathbb{R}^n)$)*

$$(2E_0 - F_{\varphi_0}(x))\, \varphi_0(x) = \int_{\mathbb{R}^n} G_{\varphi_0}(k)\, \hat{\varphi}_0(k)\, e^{i\langle k,x\rangle}\, dm_n(k), \tag{9}$$

where $E_0 = E_H(\varphi_0)$ and $dm_n(k) := (2\pi)^{-\frac{n}{2}}\, dk$.

[2] under the conditions given at the beginning

Whether (9) is valid almost everywhere w.r.t. Lebesgue measure depends (here again) on the function H. The inverse Fourier transform on the right hand side should be understood symbolically, unless $G_{\varphi_0}\hat{\varphi}_0 \in L^1(\mathbb{R}^n)$. In a compact notation the equation for a critical point of E_H is

$$F_\varphi\,\varphi + \widehat{G_\varphi} \star \varphi = 2\lambda\,\varphi$$

where the convolution is defined here as $(f\star g)(x) = \int_{\mathbb{R}^n} f(x-y)\,g(y)\,dm_n(y)$. It is easily checked that in case of $H(x,k) = |k|^2 + V(x)$, (9) reduces to $(-\Delta + V(x))\varphi_0 = E_0\varphi_0$. Our main interest, however, is H being the indicator function of an open subset of Γ_n which is obviously bounded, measurable and lower semi-continuous. This is, as will be outlined farther below, connected to the question:

> How big can we make
>
> $$\int_\Lambda |\varphi(x)|^2\,|\hat{\varphi}(k)|^2 dx\,dk, \tag{10}$$
>
> given a compact subset Λ of phase space Γ_n?

Actually the question may be posed for $\Lambda \subset \Gamma_n$ having finite Lebesgue measure.

1.3 Duality

One of the corner stones of mass transportation theory certainly is *Kantorovich's* duality formula ([6], Theorem 1.3) which, translated to our needs, says

$$K_H(\varphi) = \sup_{T(k)+V(x)\le H(x,k)} \left\{ \int_{\mathbb{R}^n} T(k)|\hat{\varphi}(k)|^2\,dk + \int_{\mathbb{R}^n} V(x)|\varphi(x)|^2\,dx \right\}, \tag{11}$$

where the functions T, V may either be any bounded continuous functions on $\mathbb{R}^n$ or by extension $(T,V) \in L^1(\nu_\varphi) \times L^1(\nu_{\hat{\varphi}})$, satisfying the inequality $T + V \le H$ point-wise in the first case and almost everywhere (with respect to the measures) in the second case. We cite one other result from [6] which will be needed later on (a precursor of Strassen's theorem, Theorem 1.27): Let U be a nonempty open subset of Γ_n, then

$$\inf_{\gamma\in\Gamma(\varphi)} \int_U d\gamma = \sup_{A\subset\mathbb{R}^n} \left\{ \int_A |\varphi(x)|^2\,dx - \int_{A_U} |\hat{\varphi}(k)|^2\,dk : A \textit{ closed} \right\}, \tag{12}$$

where $A_U := \{k \in \mathbb{R}^n : \exists x \in A, (x,k) \notin U\}$. This result implies, setting $H = \chi_U$,

$$E_{\chi_U}(\varphi) \geq K_{\chi_U}(\varphi) = \sup\{\nu_\varphi(A) - \nu_{\hat{\varphi}}(A_U) : A \subset \mathbb{R}^n, A \ \mathit{closed}\}.$$

Note that we use the notation $\nu(A)$ and $\nu(\chi_A)$ interchangeably when there is no danger of confusion (i.e. we identify a set with its indicator function).

1.4 Symplectic transformations

Let $M : \Gamma_n \to \Gamma_n$ be a symplectic transformation, represented by a matrix of the form (we use the same symbol)

$$M = M^{A,B,C,D} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the $n \times n$ block matrices A, B, C, D satisfy the equations:

$$\begin{gathered} A^T D - C^T B = I \\ A^T C = C^T A \\ D^T B = B^T D. \end{gathered}$$

Then we obtain for any $f \in C_0(\Gamma_n)$:

$$M_\# \mu_\varphi(f) = \mu_\varphi(f \circ M) = \int_{\Gamma_n} f(Ax + Bk, Cx + Dk) d\mu_\varphi(x,k).$$

The inverse M^{-1} of M is easily calculated using the symplectic condition $M^T J M = J$ to

$$M^{-1} = J^{-1} M^T J = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}$$

which implies:

$$M_\# \mu_\varphi(f) = \int_{\Gamma_n} f(\xi, \eta) \, |\varphi(D^T \xi - B^T \eta)|^2 \, |\hat{\varphi}(-C^T \xi + A^T \eta)|^2 d\xi d\eta.$$

Simple examples (e.g. $n = 1$ and $\phi(x) = C \exp(-\alpha|x|)$) show that we cannot expect the image measure $M_\# \mu_\varphi$ being an element of some $\Gamma(\psi)$. However, two special cases immediately spring to mind:

$$M_\#^{A,0,0,D} \mu_\varphi(f) = \int_{\Gamma_n} f(\xi, \eta) \, |\varphi(D^T \xi|^2 \, |\hat{\varphi}(A^T \eta)|^2 d\xi d\eta.$$

and

$$M_\#^{0,B,C,0} \mu_\varphi(f) = \int_{\Gamma_n} f(\xi, \eta) \, |\varphi(-B^T \eta)|^2 \, |\hat{\varphi}(-C^T \xi)|^2 d\xi d\eta.$$

In the first case we have $B = C = 0$, so that $A^T D = I$ by the symplectic conditions above. The second case requires $-C^T B = I$ by the same reasoning since $A = D = 0$. Hence there are two subgroups generated by matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & A^{-T} \end{bmatrix} \textit{and} \begin{bmatrix} 0 & B \\ -B^{-T} & 0 \end{bmatrix}.$$

For these, the image measures are indeed of the form $d\mu_\psi$. If we use the notation $\varphi_A(x) = \varphi(Ax)$ we may state:

$$M_{\#}^{A,0,0,A^{-T}} \mu_\varphi = \mu_{\varphi_{A^{-1}}} \tag{13}$$

and

$$M_{\#}^{0,B,-B^{-T},0} \mu_\varphi = \mu_{\widehat{\varphi_{B^{-1}}}}. \tag{14}$$

This follows by straightforward computation. Finally we want to mention the special case $B = I_n$ (the identity matrix in $\mathbb{R}^n$), giving M=J, thus

$$J_{\#}\mu_\varphi(H) = \mu_\varphi(H \circ J) = \mu_{\hat{\varphi}}(H),$$

which is equivalent to $E_{H\circ J}(\varphi) = E_H(\hat{\varphi})$. In other words, if H is invariant under the canonical transformation $x' = k, k' = -x$ and if φ_0 is a unique positive minimum of E_H, then $\varphi_0(x) = C\exp(-|x|^2/2)$.

1.5 Orthonormal Sequences in $L^2(\mathbb{R}^d)$

Let $\{\varphi_j\}_{j\in J}$ be an orthonormal sequence in $L^2(\mathbb{R})$, then a result by H. S. Shapiro, meanwhile known as *Shapiro's Umbrella Theorem*, states that if given two functions $f(x)$ and $g(k)$ in $L^2(\mathbb{R})$ such that

$$|\varphi_j(x)| \le |f(x)|, \ |\hat{\varphi}_j(k)| \le |g(k)|$$

for all $j \in J$ and for almost all x, k in $\mathbb{R}$ then J must be finite. We refer to [3] and the references therein for background information and more details. Recently. E. Mallinikova ([4],Th. 1.2) showed the following localization property of a orthonormal sequecne $\{\varphi_j\}_{j=1}^N$:

$$N - |A||B| \le \frac{3}{2}\sum_{j=1}^{N}\left(\sqrt{\nu_{\varphi_j}(\mathbb{R}^d\backslash A)} + \sqrt{\nu_{\hat{\varphi}_j}(\mathbb{R}^d\backslash B)}\right) \tag{15}$$

where $A, B \subset \mathbb{R}^d$ are arbitrary measurable sets with finite Lebesgue measure (i.e. $|A|, |B| < \infty$). Remembering the definition of the Radon measures ν_φ at the beginning, $\nu_\varphi(\mathbb{R}^d\backslash A)$ is just

$$\int_{\mathbb{R}^d\backslash A} |\varphi(x)|^2\, dx.$$

The inequality (15) immediatley leads to a quantitative version of the *Umbrella theorem* ([EM,Th. 4]) as well as to the general inequality

$$\sum_{j=1}^{N} \int_{\Gamma_d} (|x|^p + |k|^p)\; d\mu_{\varphi_j}(x,k) \geq C\, N^{1+\frac{p}{2d}}, \tag{16}$$

where C depends only on $p > 0$ and d. Moreover, it is also shown that the inequality is sharp up to a multiplicative constant.

1.6 The Nazarov-Jaming Inrequality

Another important result we shall need is the following inequality obtained by Nazarov for the case $d = 1$ and extended by Jaming [2] to $d \geq 1$.

Let $A, B \subset \mathbb{R}^d$, each having finite Lebesgue measure, then there are positive constants [3] α, β and $\eta(A,B)$ such that

$$\nu_\varphi(\mathbb{R}^d \backslash A) + \hat{\nu}_\varphi(\mathbb{R}^d \backslash B) \geq \alpha e^{-\beta\eta(A,B)} \tag{17}$$

holds for all $\varphi \in L^2(\mathbb{R}^d)$, $||\varphi|| = 1$. The constant η is given by

$$\eta(A,B) = \begin{cases} |A||B| & : d = 1 \\ \min(|A|\,|B|, |A|^{1/d}\, w(B), w(A)\, |B|^{1/d}) & : d \geq 1 \end{cases}$$

with $w(A)$ the *average width* of A (see [2] for the precise definition).

1.7 Scaling

For $\lambda > 0$ let $\varphi_\lambda(x)$ denote the scaled function $\lambda^n \varphi(\lambda x)$, then $||\varphi_\lambda|| = 1$ whenever $\varphi \in L^2(\mathbb{R}^n)$ and $||\varphi|| = 1$. The Fourier transform $\widehat{\varphi_\lambda}$ of φ_λ is easily calculated to be equal to $\hat{\varphi}_{1/\lambda}$, therefore

$$\mu_{\varphi_\lambda}(f) = \int_{\Gamma_n} f(x,k)\, |\varphi_\lambda(x)|^2\, |\hat{\varphi}_{1/\lambda}(k)|^2\, dx\, dk, \tag{18}$$

for all $f \in C_0(\Gamma_n)$. The coordinate change $\xi = \lambda x, \eta = k/\lambda$ yields

$$\mu_{\varphi_\lambda}(f) = \int_{\Gamma_n} f(\frac{\xi}{\lambda}, \lambda\,\eta)\, |\varphi(\xi)|^2\, |\hat{\varphi}(\eta)|^2\, d\xi\, d\eta = \int_{\Gamma_n} f(\frac{\xi}{\lambda}, \lambda\,\eta) d\mu_\varphi(\xi,\eta). \tag{19}$$

Recall that $\mathrm{supp}(\mu_\varphi) = \mathrm{supp}(\nu_\varphi) \times \mathrm{supp}(\nu_{\hat{\varphi}})$, so we get

$$\lim_{\lambda \to 0} \mu_{\varphi_\lambda}(f) = \int_{\mathbb{R}^n} f(x,0)\, dx \tag{20}$$

[3] Clearly, the constants may depend on the dimension d, although we do not explicitly outline this point by notation.

and correspondingly

$$\lim_{\lambda\to\infty} \mu_{\varphi_\lambda}(f) = \int_{\mathbb{R}^n} f(0,k)\,dk, \tag{21}$$

that is $\mu_{\varphi_{1/j}}$ converges *vaguely* to $(1\times\delta_0)\,dx\,dk$ while μ_{φ_j} tends to $(\delta_0\times 1)\,dx\,dk$ for $j\to\infty$. This behavior is best visualized when taking a function $f\in C_0(\Gamma_n)$ whose support is the ball $\{|x|^2+|k|^2\le 1\}$, then the scaled support degenerates like an *ellipse* with one half-axis equal to λ and the other equal to $1/\lambda$ (which indeed is the case for $n=1$).

Note: we say that a sequence of measures μ_j converges *vaguely to a measure μ if $\mu_j(f)\to\mu(f)$ for all continuous functions f with compact support, whereas we speak of weak convergence if the functions f are continuous and bounded.*

2 Maximal Probability of Closed Sets

Definition 1. *Let Λ be a closed subset of Γ_n, then we define*

$$e(\Lambda) = \sup\left\{\int_\Lambda d\mu_\varphi : \varphi\in L^2(\mathbb{R}^n),\ ||\varphi||=1\right\}. \tag{22}$$

Lemma 1. *Let A,B be subsets of $\mathbb{R}^n$ having finite Lebesgue measure, that is $|A|+|B|<\infty$, then exists a ψ such that*

$$\nu_\psi(A)+\nu_{\hat\psi}(B)=0. \tag{23}$$

Proof. This follows by Corollary 2.5.A in [1]. Actually it is shown that there always is a $\varphi\in L^2(\mathbb{R}^n)$ such that for any given pair g,h of functions in $L^2(\mathbb{R}^n)$ the restriction of φ to A and that of $\hat\varphi$ to B coincides with the restriction of g to A and h to B respectively. $\square$

Proposition 2. *Let each of A,B be the complement of a bounded open subset in $\mathbb{R}^n$, then*

$$e(A\times B)=1.$$

Proof. Set $U=\mathbb{R}^n\backslash A$, $V=\mathbb{R}^n\backslash B$, then for each φ we have $\mu_\varphi(A\times B) = \nu_\varphi(A)\,\nu_{\hat\varphi}(B) = (1-\nu_\varphi(U))\,(1-\nu_{\hat\varphi}(V))$. By the lemma above we may choose a ψ such that $\nu_\psi(A)=\nu_{\hat\psi}(B)=0$, thus $\mu_\psi(A\times B)=1$. $\square$

Proposition 3. *Let A,B be subsets of $\mathbb{R}^n$ such that $|A|+|B|<\infty$, then for every normalized $\varphi\in L^2(\mathbb{R}^n)$*

$$\mu_\varphi(\chi_{A\times B}) \le \left(1-\frac{\alpha}{2}e^{-\beta\,\eta(A,B)}\right)^2 \tag{24}$$

with constants α,β and η as in (17).

Proof. Using (17) we get $2 - (\nu_\varphi(A) + \nu_{\hat{\varphi}}(B)) \geq \alpha e^{-\beta\,\eta(A,B)}$. Dividing both sides by two and applying the arithmetic-geometric mean inequality yields $\sqrt{\nu_\varphi(A)\nu_{\hat{\varphi}}(B)} \leq 1 - \frac{\alpha}{2} e^{-\beta\,\eta(A,B)}$, what implies (24). □

Corollary 1. *Let $\Lambda \subset \Gamma_n$ be compact, then*

$$e(\Lambda) \leq \left(1 - \frac{\alpha}{2} e^{-\beta\,\eta(\pi_1(\Lambda),\pi_2(\Lambda))}\right)^2, \tag{25}$$

where $\pi_1, \pi_2 : \Gamma_n \to \mathbb{R}^n$ are the standard projections and the constants α, β, η are as in (17).

Proof. The images of the projections π_1, π_2 are again compact, thus measurable and of finite Lebesgue measure. □

If we replace compactness by finite Lebesgue measure or closed only then we have to deal with analytic sets. Since

$$e(\Lambda) = \sup_\varphi \mu_\varphi(\Lambda) = 1 - \inf_\varphi \mu_\varphi(\Gamma_n \backslash \Lambda)$$

we have the relation to E_H with $H = \chi_{\Gamma_n \backslash \Lambda}$. Since $U = \Gamma_n \backslash \Lambda$ is open we can also apply (12).

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