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## **Bachelor Thesis**

# **Simulation of Proof Systems**

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Acht Jahre nachdem die P-NP-Frage von Cook gestellt wurde [Coo71], war es auch Cook zusammen mit Reckhow, der den engen Zusammenhang dieser Frage mit Beweissystemen feststellte [CR79].

Die Bedeutung der bis heute ungeklärten P-NP-Frage ist immens. Die Gleichheit dieser Mengen würde bedeuten, dass man zu jedem Problem, dessen Lösung sich leicht überprüfen lässt, auch leicht eine Lösung finden kann. Es wäre beispielsweise genau so schwer, einen Beweis zu verifizieren, wie einen Beweis zu finden. Es würde kaum einen Unterschied machen, ob man die PIN zu einer EC-Karte herausfinden möchte, oder überprüfen möchte, ob die eingegebene Nummer die korrekte ist. Aufgrund dieser unwirklich erscheinenden Implikationen glauben die meisten Komplexitätstheoretiker, dass  $P \neq NP$ .

Der Zusammenhang der P-NP-Frage mit Beweissystemen lässt sich wie folgt formulieren. Existiert kein polynomiell beschränktes Beweissystem für TAUT, dann folgt  $P \neq NP$ . Diesen Satz werden wir im Laufe dieser Arbeit beweisen, und anschließend noch etwas tiefer in die Theorie der Beweissysteme einsteigen.

Zunächst wird der Leser in die wichtigsten Definitionen und Begriffe aus dem Gebiet der Beweissysteme eingeführt. Anschließend werden in Kapitel 3 einige wichtige Ergebnisse zusammengefasst. In Kapitel 4 wird bewiesen, dass in bestimmten superpolynomiellen Komplexitätsklassen Sprachen existieren, die keine optimalen Beweissysteme besitzen. Abschließend werden die Ergebnisse der Arbeit zusammengefasst und ein Ausblick auf weitere interessante und offene Fragen gegeben.

# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
<b>3</b>	<b>A brief Overview of Proof Systems</b>	<b>7</b>
3.1	Languages possessing optimal proof systems . . . . .	7
3.2	Proof Systems and the P versus NP question . . . . .	8
<b>4</b>	<b>A set in <math>\text{co-NEXP} \setminus \text{OPT}</math></b>	<b>11</b>
4.1	Basic Construction . . . . .	11
4.2	Making the Language Tally . . . . .	12
4.3	Super-Sparse Sets without an Optimal Proof System . . . . .	13
4.4	Adding Redundancy . . . . .	14
4.5	Putting the Results Together . . . . .	14
<b>5</b>	<b>Conclusion and future work</b>	<b>15</b>
	<b>Bibliography</b>	<b>16</b>

# 1 Introduction

After Cook stated 1971 the P versus NP question [Coo71], he gave the main motivation to study proof systems in 1979. Cook and Reckhow showed in their article the close relation between the separation of complexity classes and the existence of polynomially bounded proof systems [CR79].

Despite its importance, the P versus NP question remains still open. Informally speaking,  $P = NP$  means that for every problem that has an efficiently verifiable solution, we can find that solution efficiently as well. While most theoreticians assume that  $P \neq NP$ , their equality would have heavy implications. One consequence affects the security of communication in the Internet, as public-key cryptography depends on the existence of certain problems in NP that are not efficiently to decide. If there are no problems with solutions fast to verify but hard to find, as  $P = NP$  would imply, the small lock beside the URL in one's browser could not indicate a secure communication anymore [For09].

$P = NP$  would also have fundamental implications on mathematics: As mathematical proofs must by definition be efficiently verifiable,  $P = NP$  would imply that they are efficiently to find [CR79]. This argument to believe that  $P \neq NP$  is further illustrated in the following quotation, taken from Aaronson's Blog<sup>1</sup>.

“If  $P = NP$ , then the world would be a profoundly different place than we usually assume it to be. There would be no special value in ‘creative leaps’, no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss; everyone who could recognize a good investment strategy would be Warren Buffett.”

Closely related to the P versus NP problem is the question if  $NP = co-NP$ . If  $P = NP$ , then  $NP = co-NP$ , since P is closed under complement. In return, if one can separate NP from co-NP, then  $P \neq NP$ . To connect the field of proof systems with the P versus NP questions, we state

**Proposition 1** ([KMT03], [CR79]).  *$NP = co-NP$  if and only if a polynomially bounded proof system for TAUT exists.*

In order to introduce the reader into the field of proof systems, we will first define the important notions used related to it. Subsequently, we will prove proposition 1 and give an overview of important results about proof systems in chapter 3. These results will connect proof systems with the P versus NP question as mentioned above and will give a basis for the proofs in the subsequent chapter. In chapter 4 we will proof the main theorem of this thesis, showing that there are languages without optimal proof systems in all super-polynomial complexity classes. This chapter will also analyze the inner structure of these languages that do not possess an optimal proof system.

Finally, we will give a conclusion and look forward to currently unresolved problems and further questions.

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<sup>1</sup><http://www.scottaaronson.com/blog/>

## 2 Preliminaries

As mentioned before, we will first introduce important symbols and definitions. Although some familiarity with standard notions of complexity theory is assumed, we will here define most of the notions used in this thesis. For the most important ones, we will give a short discussion.

Let  $\Sigma = \{0, 1\}$  denote the alphabet. The output of a Turing transducer  $M$  on input  $x \in \Sigma^*$  is denoted by  $M(x)$ . If the transducer  $M$  does not accept or runs forever on input  $x$ , we define  $M(x) = \perp$ . We say a Turing transducer *calculates* a partial function  $f$ , if  $M(x) = f(x)$  for all  $x \in \Sigma^*$ . We further define  $\text{time}_M(x)$  as the number of steps the transducer  $M$  runs on input  $x \in \Sigma^*$ . With  $\mathcal{FP}$  we denote the set of all partial functions  $f$  for which a Turing transducer  $M$  calculating  $f$  exists such that  $\text{time}_M(x) \leq p(|x|)$  for a polynomial  $p$ .

**Definition 1.** A function  $h \in \mathcal{FP}$  is called *proof system* for a language  $L$  if the range of  $h$  is  $L$ . A string  $w$  with  $h(w) = x$  is called an *h-proof* for  $x$ .  $h$  is called a *polynomially bounded proof system*, if there is a polynomial  $p$  such that for every  $x \in L$ , there is a *h-proof*  $w$  with  $|w| \leq p(|x|)$ .

With this definition, a proof system for  $L$  is basically a polynomial-time bounded function that enumerates  $L$ . Notice, although its time bound against the input, the shortest proof of a string  $w \in L$  can be very long. To give an example, let  $h$  be defined by

$$\text{sat}(x) = \begin{cases} \varphi & (x = \langle a, \varphi \rangle \text{ and } \alpha \text{ is an satisfying assignment for } \varphi), \\ \perp & (\text{otherwise}). \end{cases}$$

Then  $h$  is a proof system for SAT. It is an open question, whether  $\text{sat}$  is p-optimal. Köbler and Messner showed, that this question is equivalent to a variety of well studied complexity theoretic assumptions [KM00]. We will show some of the connections in the next chapter.

There may be various proof systems for a language  $L$ . In order to make them comparable, we define the notion of *simulation* of proof systems. It turns out that the notion of simulation corresponds in a certain way with the notion of many-one reducibility [KM00]. In the following definitions, we will keep this correspondence.

**Definition 2.** Let  $h$  and  $h'$  be proof systems for a language  $L$ . If there is a polynomial  $p$  and a function  $f$  such that for all  $w \in \Sigma^*$

$$h(f(w)) = h'(w)$$

and  $|f(w)| \leq p(|w|)$ , then  $h$  *simulates*  $h'$ .

Informally speaking,  $f$  translates  $h$ -proofs into  $h'$ -proofs and keeps them short. In the given definition,  $f$  could be hard or even impossible to calculate. Hence we define a stronger version of this notion, demanding  $f \in \mathcal{FP}$ .

**Definition 3.** Again, let  $h$  and  $h'$  be proof systems for a language  $L$ . If  $h$  *simulates*  $h'$  with a function  $f$  and additionally  $f \in \mathcal{FP}$ ,  $h$  *p-simulates*  $h'$ .

Notice, if  $f$  is a function that can be calculated in polynomial time  $p$ , then we obtain  $|f(w)| \leq p(|w|)$  as required in the definition of simulation. With a proof system p-simulating another, we can

translate proofs as above in polynomial short time. With the notion of simulation of proof systems, we can compare different proof systems for a language  $L$ . With respect to these notions, we will define a notion of the best proof system as follows.

**Definition 4.** *A proof system  $h$  for  $L$  is called optimal, if it simulates every proof system for  $L$ . It is called  $p$ -optimal, if it  $p$ -simulates every proof system for  $L$ .*

Like noted above, this definition corresponds with the definition of complete problems in respect of many-one-reducibility [KMT03]. We will investigate further on the connection between complete problems and optimal proof systems later in lemma 2.

The existence of optimal proof systems for a arbitrary language  $L$  is an important complexity theoretic question. For languages in  $P$  and  $NP$ , there is always a optimal proof system, as we will see in 4. For super-polynomial time complexity classes, there are languages without an optimal proof system, as we will show in chapter 4. For that reason, we will define a complexity class containing all languages possessing an optimal proof system.

**Definition 5.** *Let  $OPT$  be the complexity class of all languages that have a optimal proof system.*

Observe that for  $OPT$  we use the weaker notion of simulation. As noted above, we can easily state a proof system for languages in  $P$  or  $NP$ , therefore we obtain  $NP \subseteq OPT$ . It is an open questions whether  $OPT \subseteq NP$ .

With these notions, we will take a look at important results in the field of optimal proof systems in the next chapter. For notions not defined in this thesis, refer to a standard work of computational complexity like the one from Papadimitriou [Pap94].

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## 3 A brief Overview of Proof Systems

After defining the important notions for this thesis, we will give a brief overview of some important results in the field of optimal proof systems.

### 3.1 Languages possessing optimal proof systems

One basic lemma that is widely used formalizes a part of the connection between optimal proof systems and polynomial many-one-reducibility. Later in this thesis, we will use it to proof corollary 11. The following proof is mainly taken from Köbler et al. We will omit the proof for optimal proof systems, as it easily follows from the proof for p-optimal ones.

**Lemma 2** ([KMT03]). *If  $A$  has a (p-)optimal proof system and if  $B \leq_m^p A$ , then  $B$  has a (p-)optimal proof system, too.*

*Proof.* Let  $h$  be a p-optimal proof system for  $A$  and let  $B$  many-one reduce to  $A$  via  $f \in \mathcal{FP}$ , that is  $x \in B \Leftrightarrow f(x) \in A$ . Then  $h'$  defined by

$$h'(\langle x, w \rangle) = \begin{cases} x & (h(w) = f(x)), \\ \perp & (\text{otherwise}), \end{cases}$$

is a proof system for  $B$ , as  $h(w) = f(x) \in A$  is equivalent to  $x \in B$ . To show  $h'$  is optimal, let  $g'$  be a proof system for  $B$ . In order to obtain a proof system for  $A$ , let  $g$  be

$$g(bw) = \begin{cases} h(w) & (b = 0), \\ f(g'(w)) & (b = 1). \end{cases}$$

Since both  $h(w)$  and  $f(g'(w))$  are in  $A$  and  $h$  is a proof system for  $A$ ,  $g$  is also a proof system for  $A$ . As  $h$  is p-optimal, there is a function  $t \in \mathcal{FP}$  translating  $g$ -proofs to  $h$ -proofs implying that

$$h(t(1w)) = g(1w) = f(g'(w)).$$

This implies  $h'(\langle g'(w), t(1w) \rangle) = g'(w)$ . Hence,  $h'$  p-simulates  $g'$ . □

In contraposition to this, we can state that for  $B \leq_m^p A$ , if  $B$  has no (p-)optimal proof system, then  $A$  has not either.

Using this lemma, for a language  $L \in \text{OPT}$ , we can construct further languages possessing optimal proof systems. We call two languages  $A$  and  $B$  *polynomial time equivalent*, if  $A \leq_m^p B$  and  $B \leq_m^p A$ . With this notion, we can easily state the following.

**Corollary 3.** *For polynomial time equivalent languages  $A$  and  $B$ ,  $A \in \text{OPT}$  if and only if  $B \in \text{OPT}$ . Especially, if  $L \notin \text{OPT}$ , then for a  $b \in \Sigma$ ,  $bL := \{bw : w \in L\} \notin \text{OPT}$  as well as  $0L \cup 1L \notin \text{OPT}$ .*

*Proof.* From  $A \leq_m^p B$  we obtain  $B \in \text{OPT} \Rightarrow A \in \text{OPT}$ , using lemma 2. Similar, we obtain  $A \in \text{OPT} \Rightarrow B \in \text{OPT}$  from  $B \leq_m^p A$ .

We now construct polynomial time functions in order to show that  $L$ ,  $bL$  and  $0L \cup 1L$  are polynomial time equivalent. With the function discarding the first bit of any word,  $bw \mapsto w$ , we obtain  $bL \leq_m^p L$ . With the function adding  $b$  to the start of any word,  $w \mapsto bw$ , we obtain  $L \leq_m^p bL$ . With  $x \mapsto 0x$ , we obtain  $L \leq_m^p 0L \cup 1L$ , as  $x \in L \Leftrightarrow 0x \in 0L \cup 1L$ . Using  $bw \mapsto w$ , we further obtain  $0L \cup 1L \leq_m^p L$ .  $\square$

The following lemma gives a partial answer to the basic question what languages do have optimal proof systems. Although Koebler et al. omitted the proof of the following theorem due to its straightforwardness, we will give a proof for the purpose of this thesis.

**Lemma 4** ([KMT03]). (i) *Every language in P has a p-optimal proof system.*

(ii) *Every language in NP has an optimal proof system.*

*Proof.* (i) Let  $L \in \text{P}$ . Then there is a function  $f \in \mathcal{FP}$  with  $f(w) = 1 \Leftrightarrow w \in L$ . To show there is a proof system, let  $h$  be defined by

$$h(w) = \begin{cases} w & (f(w) = 1), \\ \perp & (\text{otherwise}). \end{cases}$$

Then  $h$  is a proof system for  $L$ . To show  $h$  is optimal, let  $h'$  be an arbitrary proof system for  $L$ . Then  $h' \in \mathcal{FP}$  by definition and we can translate  $h'$ -proofs with  $h'$  in polynomial time into  $h$ -proofs, in formulas

$$h(h'(w)) = h'(w).$$

Therefore,  $h$  p-simulates every proof system  $h'$ .

(ii) Let  $L \in \text{NP}$ . Then there is a nondeterministic Turing transducer  $M$  deciding  $L$  in polynomial time. Let  $f_i(x) \in \mathcal{FP}$  be the function calculating the  $i$ -th path of the nondeterministic calculation of  $M$ . Finally, let  $h$  be defined by

$$h(\langle i, w \rangle) = \begin{cases} w & (f_i(w) \text{ accepts}), \\ \perp & (\text{otherwise}). \end{cases}$$

Then  $h \in \mathcal{FP}$  is a proof system for  $L$ . To show  $h$  is optimal, let  $h'$  be an arbitrary proof system for  $L$ . Let  $g$  be a function that maps an  $w \in L$  to an  $i$  such that  $f_i(w)$  accepts in polynomial time. Notice that  $g$  may be not in  $\mathcal{FP}$ . With these definitions, we obtain

$$h(\langle g(h'(w)), h'(w) \rangle) = h'(w).$$

Therefore,  $h$  simulates every proof system  $h'$  via the translating function

$$w \mapsto \langle g(h'(w)), h'(w) \rangle.$$

$\square$

## 3.2 Proof Systems and the P versus NP question

As we have seen,  $\text{NP} \subseteq \text{OPT}$ . It is an open question whether  $\text{NP} \supseteq \text{OPT}$ . Lemma 4 connects to the P-versus-NP-question by stating different properties for P and NP. If one would find a set in NP without an p-optimal proof system, one would have separated P from NP.



**Corollary 5.** *If there is no  $p$ -optimal proof system for SAT, then  $P \neq NP$ .*

In order to prove proposition 1, we will show two lemmata of which the first is taken from the work of Cook and Reckhow [CR79]. It gives an equivalent formulation of the question whether  $NP = co-NP$ .

**Lemma 6.**  *$NP = co-NP$  if and only if  $TAUT \in NP$ .*

*Proof.* Without further assumptions, we can show by using a nondeterministic Turing transducer, that an arbitrary formula is not a tautology in polynomial time by guessing and verifying an assignment for which the formula is falsified. Thus,  $\overline{TAUT} \in NP$ .

For every  $L \in NP$ , there is a function  $f \in \mathcal{FP}$  such that  $x \in L \Leftrightarrow f(x) \in \overline{TAUT}$ . We will show this as follows. As SAT is  $\leq_m^p$ -complete for NP, we can reduce every  $L \in NP$  and obtain an  $f' \in \mathcal{FP}$  with  $x \in L \Leftrightarrow f'(x) \in SAT$ . Hence, there is a truth assignment for  $f'(x)$  such that  $f'(x)$  yields true. By negating, we obtain the existence of a truth assignment such that  $\neg f'(x)$  is false. Consequently,  $\neg f'(x) \in \overline{TAUT}$ . Notice that the negation of  $f'(x)$  can be done efficiently. By choosing  $f(x) = \neg f'(x)$ , we obtain  $f \in \mathcal{FP}$  with the stated properties.

Assume  $TAUT \in NP$ , and let  $L$  be an arbitrary language with  $L \in NP$ . As shown above, there is a function  $f \in \mathcal{FP}$  such that  $x \in L \Leftrightarrow f(x) \in \overline{TAUT}$  respectively  $x \in \overline{L} \Leftrightarrow f(x) \in TAUT$ . Since  $TAUT \in NP$ , for any given  $x \in \Sigma^*$  we can in nondeterministic polynomial time decide whether  $f(x)$  is in TAUT. Thus we also can decide in nondeterministic polynomial time whether  $x \in \overline{L}$ , as  $f \in \mathcal{FP}$ . It follows that  $\overline{L} \in NP$ . As this proves that NP is closed under complement, we obtain  $NP \subseteq co-NP$ . To see that  $co-NP \subseteq NP$ , let an arbitrary language  $\overline{L}$  be in co-NP. By definition we obtain  $L \in NP$ . As NP is closed under complement,  $\overline{L} \in NP$ . Thus,  $co-NP \subseteq NP$ .

Assume  $TAUT \notin NP$ , then  $\overline{TAUT} \notin co-NP$ . As we have seen above,  $\overline{TAUT} \in NP$ . Hence  $NP \neq co-NP$ .  $\square$

The second lemma connects with the theory of proof systems by formulating a necessary and sufficient condition for a language being in NP [CR79].

**Lemma 7.** *A set  $L \neq \emptyset$  is in NP if and only if  $L$  has a polynomially bounded proof system.*

*Proof.* Assume  $L \in NP$ , then some nondeterministic Turing transducer  $M$  accepts  $L$  in polynomial time. Let  $f_i(x) \in \mathcal{FP}$  be the function calculating the  $i$ -th path of the nondeterministic calculation of  $M$ . We define  $f$  by

$$f(\langle i, x \rangle) = \begin{cases} x & (f_i(x) \text{ accepts}), \\ \perp & (\text{otherwise}). \end{cases}$$

Then  $f$  is a polynomially bounded proof system for  $L$ .

Conversely, assume  $f$  is a polynomially bounded proof system for  $L$ . Then a nondeterministic Turing transducer on input  $y$  can guess a proof  $x$  and verify  $f(x) = y$ .  $\square$

Putting lemma 6 and 7 together, we obtain proposition 1,

$$NP = co-NP \Leftrightarrow \text{there is a polynomially bounded proof system for TAUT.}$$

Using this theorem, one tried to separate NP from co-NP by studying more and more powerful proof systems, showing that they are not polynomially bounded [KMT03]. As mentioned before, there was no success on answering this questions until now. To take the notion of optimal proof systems

into account, one could ask if there is an optimal or even p-optimal proof system for TAUT. If that were the case, then the existence of one specific proof system that is not polynomially bounded would suffice to prove that  $NP \neq co-NP$  and hence  $P \neq NP$  [KMT03].

Krajíček and Pudlák proved a sufficient condition for the existence of optimal proof systems for TAUT [KP89].

**Theorem 8.** *If  $NE = co-NE$  then optimal proof systems for TAUT exist. If  $E = NE$  then p-optimal proof systems for TAUT exists.*

We will omit their proof in this thesis, since it uses many notions of formal logics and a huge equivalence theorem not introduced here.

**Corollary 9.** *As a conclusion, we can state the following implication diagram.*

$$\begin{array}{ccccccc}
 & & & & & E = NE & \\
 & & & & & \Downarrow & \\
 P = NP & \Rightarrow & NP = co-NP & \Leftrightarrow & TAUT \in NP & \Leftrightarrow & TAUT \in OPT \\
 & & & & & \Uparrow & \\
 & & & & & NE = co-NE & 
 \end{array}$$

## 4 A set in $\text{co-NEXP} \setminus \text{OPT}$

In the main theorem of this thesis, we will show that in certain complexity classes above of NP, there are always languages that possess no optimal proof system. To formalize this, we say a function is *time-constructible*, if there is a Turing transducer that stops on input  $n$  after  $t(n)$  steps.

### 4.1 Basic Construction

**Theorem 10.** *Let  $t : \mathbb{N} \rightarrow \mathbb{N}$  be a time-constructible function such that for every polynomial  $p$  there is a number  $n$  with  $p(n) \leq t(n)$ . Then there is a language  $L \in \text{co-NTIME}(t(n))$  that has no optimal proof system.*

Notice, if for function  $f$  there is an  $n \in \mathbb{N}$  with  $f(n) \geq p(n)$  for every polynomial  $p$ , then there are for every polynomial  $p$  infinitely many  $n \in \mathbb{N}$  for which  $f(n) \geq p(n)$  is satisfied. Otherwise, we could construct a polynomial  $q$  that is greater than  $f$  for every  $n \in \mathbb{N}$ . To see this, let  $p$  be a polynomial for which  $f(n) \geq p(n)$  for only finitely many  $n \in \mathbb{N}$ . Then we can determine the greatest difference between  $f$  and  $p$  by taking the maximum of the set  $\eta = \{f(n) - p(n), f(n) \geq p(n)\}$ . Then for the polynomial  $p'(n) = \eta + p(n) + 1$  there is no  $n$  with  $f(n) \geq p'(n)$ , which contradicts the assumption that for every polynomial  $p$  there is a natural number such that  $f(n) \geq p(n)$ .

Messner showed that under the same presumptions as in our theorem, there is a language  $L \in \text{co-NTIME}(t(n))$  without an optimal acceptor [Mes99]. He also proved that the existence of an optimal acceptor is equivalent to the existence of an optimal proof system for every p-cylinder  $L$ . We will here give a proof that is based on the work of Messner, but stays in the notion of proof systems.

Before formally proving theorem 10, let us briefly discuss the main idea of the proof. As a first observation, we notice that we can enumerate all  $\mathcal{FP}$ -functions  $f_i$  such that  $\text{time}(f_i) \leq n^i + i$ . We then construct a language  $L$  by uniting languages  $L_i$ . We will construct them in a way such that there are only long  $f_i$ -proofs for all strings in  $L_i$ . Afterwards we assume there is an optimal proof system  $g$  for  $L$ . As for every  $\mathcal{FP}$ -function there is a subset that has only long proofs, there is such a subset  $L_g$  for  $g$ . But as it turns out we can state a proof system for  $L_g$  that has short proofs, we can combine both proof systems to a system that runs polynomial equally fast on nearly every string in  $L$ , but way faster on strings in  $L_g$ . Hence,  $g$  cannot be an optimal proof system.

*Formal proof of theorem 10.* Let  $f_1, f_2, \dots$  be an enumeration of all  $\mathcal{FP}$ -functions with  $\text{time}(f_i) \leq n^i + i$ . We obtain this enumeration by enumerating all Turing transducers (using Gödel's enumeration) and combine each transducer with a clock that ensures a maximum runtime of  $n^i + i$  steps for a transducer  $M_i$ . For any  $i > 0$ , let  $L_i$  be the regular language described by the expression  $0^i 10^*$ . Define

$$L'_i = \{x \in L_i \mid \forall y \in \Sigma^* |y|^{2i} \leq t(|x|) \implies f_i(y) \neq x\}.$$

That is, as long as you put strings of length  $|y|^{2i} \leq t(|x|)$  into  $f_i$ , you will not obtain  $x$ . Let  $L = \bigcup_{i>0} L'_i$ .

First, we obtain  $L \in \text{co-NTIME}(t(n))$ . To show this, one considers

$$L \in \text{co-NTIME}(t(n)) \Leftrightarrow \bar{L} = \overline{\bigcup_{i>0} L'_i} = \bigcap_{i>0} \bar{L}'_i \in \text{NTIME}(t(n)).$$

By negating the condition for  $L'_i$ , we obtain

$$\bar{L}'_i = \{x \in \Sigma^* \mid x \notin L_i \vee (\exists y \in \Sigma^* |y|^{2i} \leq t(|x|) \wedge f_i(y) = x)\}.$$

For any given  $x$ , we can decide in polynomial time whether it is in any  $L_i$  or not. If it is not, then  $x$  is in  $\bar{L}'_i$  for all  $i > 0$  and therefore  $x \in \bar{L}$ , so we are done. If it is in any  $L_i$ , it is in exactly one  $L_i$ . Let  $i^*$  be the set with  $x \in L_{i^*}$ . We can simulate a polynomial-time machine calculating  $f_{i^*}(y)$  on every input  $y \in \Sigma^*$  with  $|y|^{2i^*} \leq t(|x|)$ . If, and only if, there is a path with  $f_{i^*}(y) = x$ , then  $x \in \bar{L}$ . In both cases,  $\bar{L} \in \text{NTIME}(t(n))$ .

have a look at simulation runtime

For a proof system  $f_i$  with  $f_i(\Sigma^*) = L$ , we observe that  $L'_i = L_i$ . Assume there is an  $x = 0^i 1 z \in L_i$  that is not in  $L'_i$ . Then there is an  $y$  with  $|y|^{2i} \leq t(|x|)$  and  $f_i(y) = x$ . Since  $f_i$  is a proof system for  $L$ , this yields  $x = 0^i 1 z \in L$  and so  $x \in L'_i$ , which contradicts the assumption. Therefore, for any  $y$  with  $f_i(y) = x \in L_i$  we know that  $|y|^{2i} > t(|x|)$ . Speaking informally, every proof system  $f_i$  for  $L$  has long proofs on  $L'_i \subset L$ .

Assume now, for contradiction, that  $f_i$  is an optimal proof system for  $L$ . Let  $g$  be a function defined as

$$g(bx) = \begin{cases} f_i(x) & (b = 0), \\ x & (b = 1 \text{ and } x = 0^i 10^* \in L_i = L'_i). \end{cases}$$

$g$  is a proof system for  $L$  with polynomial length-bounded proofs for all  $x \in L_i$ . As  $f_i$  is optimal, there is a function  $f^*$  such that for all  $x \in L'_i$ ,  $f_i(f^*(x)) = g(x)$  and  $|f^*(x)| \leq p(|x|)$  for a polynomial  $p$ . Let  $q$  be the polynomial  $q(n) = p(n)^{2i}$ . Then  $p(|x|) \leq q(|x|)^{2i}$ . As there is an  $n$  with  $q(n) \leq t(n)$ , there is an  $x$  in  $L_i$  such that  $|f^*(x)| \leq p(|x|) \leq q(|x|) = p(|x|)^{2i} \leq t(|x|)$ . According to the definition of  $L'_i$ , this yields  $f_i(f^*(x)) \neq x$ . Therefore,  $f_i$  is not optimal on  $L'_i$ , which contradicts the assumption that  $f_i$  is optimal on  $L$ .  $\square$

Using the relation to many-one-hard reducible sets proven in lemma 2, we obtain

**Corollary 11.** *No set  $\leq_m^p$ -hard for co-NE has an optimal proof system.*

*Proof.* With  $t(n) = 2^n$ , we can get an  $L \in \text{co-NE}$  that has no optimal proof system. Any  $\leq_m^p$ -hard set  $A$  for co-NE is  $L \leq_m^p A$ . Together with lemma 2 we obtain, that  $A$  cannot have optimal proof system.  $\square$

## 4.2 Making the Language Tally

Now, let us take a closer look at this set  $L$  that has no optimal proof system. One first observation is that  $L$  is sparse. As every  $L'_i$  only contains strings that are of the form  $0^i 10^*$ ,  $L$  is a subset of the regular language  $L_R = 0^* 10^*$ . Therefore, the density of  $L_R$  is an upper bound for the density of  $L$ . In  $L_R$ , there are exactly  $n$  words of length  $n$ . As a consequence,  $\text{dens}_{L_R}(n) = \sum_{i=1}^n i = \frac{n^2+n}{2}$ . Hence,  $L_R$  and  $L$  are both sparse. But we can construct even less dense languages by constructing tally sets without optimal proof systems.

**Theorem 12.** *If  $L \subseteq \{x \in \Sigma^*, x = 0^* 10^*\}$  has no optimal proof system, then there is a  $T \in \text{TALLY}$ , that has no optimal proof system and is polynomial time equivalent to  $L$ .*

*Proof.* Using the Cantor pairing function, we can construct a bijective function  $t \in \mathcal{FP}$  that maps each element of  $L$  one-to-one to an element of  $T \subseteq \text{TALLY}$ . Since all strings in  $L$  are almost tally—namely every string in  $L$  contains only one 1— $t$  is a polynomial time function. Let  $g$  be an optimal proof system for  $T = \{t(x), x \in L\}$ . Then  $t^{-1} \circ g$  is a proof system for  $L$ . Let  $h$  be an arbitrary proof system for  $L$ . Then  $t \circ h$  is a proof system for  $T$ . As  $g$  is optimal, there is a polynomial bounded  $f$  with  $g(f(x)) = t(h(x))$ . It follows that  $t^{-1}(g(f(x))) = h(x)$  for all proof systems  $h$  for  $L$ . Hence,  $t^{-1} \circ g$  is an optimal proof system for  $L$ , which contradicts the assumption that  $L$  has no optimal proof system. Using  $t$  we obtain  $L \leq_m^p T$  and  $T \leq_m^p L$ .  $\square$

Using this theorem, we can further improve our results on the density of languages not possessing an optimal proof system. For any tally language  $p(n) = n$  is an upper bound of its density.

**Corollary 13.** *There is a language  $L$  with  $\text{dens}_L(n) \leq n$  that possesses no optimal proof system.*

### 4.3 Super-Sparse Sets without an Optimal Proof System

By adapting the proof of our main theorem 10, we can show that there are even arbitrary sparse sets in  $\text{co-NTIME}(t(n))$  that do not possess an optimal proof system. In order to do so, we have to slightly tighten the requirements of theorem 10. To formalize the notion of arbitrary sparse, we introduce a polynomial-time computable, strictly increasing function  $u : \mathbb{N} \rightarrow \mathbb{N}$ . Notice, as  $u$  is strictly increasing, we have  $u(n) \geq n$  for all  $n \in \mathbb{N}$ .

We will use  $u$  to constrain the length of any string in  $L$ . Therefore, we define

$$\tilde{L}'_i = \{x \in L'_i : \exists k \in \mathbb{N} |x| = u(k)\}.$$

By definition we obtain  $\tilde{L}'_i \subseteq L'_i$ . We further define  $\tilde{L} = \bigcup_{i \geq 0} \tilde{L}'_i$ . By doing so, in  $\tilde{L}$  will only be strings whose length is in the range of  $u$ . Informally speaking, the faster  $u$  is growing, the more sparse becomes  $\tilde{L}$ . More formally, we can state an upper bound for the density of  $\tilde{L}$  as follows.

$$\begin{aligned} \text{dens}_{\tilde{L}}(n) &= \sum_{k=0}^n \#\{x \in \tilde{L} : |x| = k\} \\ &\leq \sum_{k=0}^n \#\{x \in \Sigma^* : x = 0^a 10^b \text{ and } |x| = k\} \end{aligned}$$

The faster  $u$  is growing, the more summands in this sum are zero, as there are many  $k$  not in the range of  $u$ . The exact count of summands is  $u^{-1}(n)$ , which we will define as the the greatest number  $x$  for which  $u(x) \leq n$ . For each summand,  $k$  is an upper bound of its value, and as  $k \leq n$  we obtain

$$\text{dens}_{\tilde{L}}(n) \leq u^{-1}(n) \cdot n.$$

give an example

In order to adapt the proof of theorem 10 to this new circumstances, we have to only one issue. It occurs in the proof that  $\tilde{L}$  is in  $\text{co-NTIME}(t(n))$ . In order to decide whether an given string  $x$  is in the complement  $\tilde{L}$ , we have to additionally check in polynomial time whether there is an  $k \in \mathbb{N}$  such that  $|x| = u(k)$ . Since  $u \in \mathcal{FP}$  and is strictly increasing, this is possible. As for any  $k > |x|$ , it follows  $u(k) \geq |x|$ , we have to calculate at most  $|x|$  times the value of  $u(k)$  in order to check if there is an  $k$  with  $u(k) = |x|$ . If there is no such  $k$ , then the given  $x$  is for sure in the complement of  $\tilde{L}$  and we are done. If there is such a  $k$ , we proceed with the algorithm as described before.

Adapting the proof as described yields

**Corollary 14.** *Let  $t : \mathbb{N} \rightarrow \mathbb{N}$  be a time-constructible function such that for every polynomial  $p$  there is a number  $n$  with  $p(n) \leq t(n)$ . Let  $u$  be a polynomial-time computable, strictly increasing function  $u : \mathbb{N} \rightarrow \mathbb{N}$ . Then there is a language  $L \in \text{co-NTIME}(t(n))$  that has no optimal proof system; the density of  $L$  is bounded by*

$$\text{dens}_L(n) \leq u^{-1}(n) \cdot n,$$

where  $u^{-1}$  denotes the smallest number  $m$  for which  $u(m) \geq n$ .

## 4.4 Adding Redundancy

Given a language  $L$  not possessing an optimal proof system, we can show that there is even a mitotic language that has no optimal proof system. To show this, we mainly rely on results proven in chapter 3.

**Theorem 15.** *If  $L$  possesses no optimal proof system, there is an polynomial time equivalent, many-one-mitotic language  $L' = 0L \cup 1L$  not possessing an optimal proof system, too.*

*Proof.* As  $L$  has no optimal proof system, lemma ?? states that  $L' = 0L \cup 1L$  is polynomial time equivalent and does not possess an optimal proof system.

Let  $A$  be the language  $0\Sigma^*$ , then  $\overline{A} = 1\Sigma^*$ . We obtain  $L' \cap A = 0L$  and  $L' \cap \overline{A} = 1L$ . As shown in lemma ??,  $L$  and  $bL$  are polynomial time equivalent. Using  $0x \mapsto 1x$ ,  $0L$  and  $1L$  are also polynomial time equivalent. Hence,  $L'$  is many-one-mitotic.  $\square$

## 4.5 Putting the Results Together

In this section, we will show that we can combine all results to one language, that is arbitrary sparse, tally, has no optimal proof system and possesses redundancy.

**Corollary 16.** *Let  $t : \mathbb{N} \rightarrow \mathbb{N}$  be a time-constructible function such that for every polynomial  $p$  there is a number  $n$  with  $p(n) \leq t(n)$ . Let  $u$  be a polynomial-time computable, strictly increasing function  $u : \mathbb{N} \rightarrow \mathbb{N}$ . Then there is a language  $L \in \text{co-NTIME}(t(n))$  that possesses no optimal proof system and*

- (i) *has its density bound by  $\text{dens}_L(n) \leq u^{-1}(n) \cdot n$ , where  $u^{-1}$  denotes the smallest number  $m$  for which  $u(m) \geq n$ ,*
- (ii) *is many-one-mitotic,*
- (iii) *is tally.*

*Proof.* Corollary 14 yields a language  $L \in \text{co-NTIME}(t(n))$  that has no optimal proof system and fulfills our demands on its sparseness. By using theorem 15, we obtain the language  $0L \cup 1L$  that is polynomial time-equivalent to  $L$  and possesses no optimal proof system. Finally, we can use theorem 12 to construct a tally set  $T \notin \text{OPT}$  and is polynomial time-equivalent to  $L$ .  $\square$

$T \in \text{co-NTIME}(t(n))?$

## 5 Conclusion and future work

What a great work!

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Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen Hilfsmittel und Quellen als die angegebenen benutzt habe. Weiterhin versichere ich, die Arbeit weder bisher noch gleichzeitig einer anderen Prüfungsbehörde vorgelegt zu haben.

Würzburg, den \_\_\_\_\_, \_\_\_\_\_  
(Nils Wisiol)