Analysis II

1 Ordinary Differential Equations

Differential Equation An equation where the unknown (or unknowns) is a function f, and the equation relates values of f at a point x with values of derivatives of the function at the *same point* x.

Ordinary Differential Equation A differential equation where the function f has one variable only.

Order of ODEs The order of an ODE is defined as the order of it's biggest derivative.

Linear Differential Equation Has the form

$$y^{(k)} + a_{k-1}(x)y^{k-1} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where $a_1, ..., a_{k-1}, b$ are continuous functions $I \to \mathbb{C}$. It is important that **all derivatives have power one**, and there is **no product** of two different derivatives. Also, neither y nor its derivatives are **inside another function**.

If $b = 0$, the ODE is homogeneous.

Initial Condition A set of equations:

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{k-1}(x_0) = y_{k-1}$$

Theorem Let $I \subset \mathbb{R}$ be an open interval, $k \geq 1$ an integer. Let

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_0(x)y = b(x)$$

be a linear ODE over I with continuous coefficients. Then

1. Let S_0 be the set of solutions when $b = 0$ (i.e. the associated homog. ODE). Then so is a vector space of dimension k. If $f_1, ..., f_k$ are the solutions then so is $\alpha_1 f_1 + ... + \alpha_k f_k$.
2. For any initial conditions (i.e. for any choice of $x_0 \in I$ and $(y_0, ..., y_{k-1}) \in \mathbb{C}^k$ there is a **unique** solution $f \in S_0$ such that $f(x_0) = y_0, f'(x_0) = y_1, ..., f^{(k-1)}(x_0) = y_{k-1}$.
3. For any arbitrary $b(x)$, the set of solutions of the ODE is

$$S_b = \{f + f_p | f \in S_0\}$$

   where f_p is a "particular" solution of the ODE.
4. For any initial condition there is a unique solution $f \in S_b$.

Solve linear differential equation of order 1

1. Solve homogeneous equation for f_0
2. Solve inhomogeneous equation for f_h
3. Add solutions $f_h + f_0$

Step 1 - Homogeneous solution In this step we first want $\frac{y'(x)}{y(x)} = ...$, then take the integral of both sides such that we get $\ln|y(x)| = \int ... + c$. We then solve for y.

Any solution of $y' + a(x)y = 0$ is of the form

$$f(x) = Ke^{-A(x)}$$

where $A(x) = \int a(x)\, dx$.

If the initial condition $f(x_0) = y_0$ is given then we can determine K. The unique solution is then

$$f(x) = y_0 \exp(A(x_0) - A(x))$$

Step 2 - Solution of inhomog. equation $y' + a(x)y = b(x)$. We need to find **one** particular function f_p such that $f_p' + a(x)f_p = b(x)$. Then all the solutions are of the form

$$f = f_p(x) + f_0 = f_p(x) + Ke^{-(A(x))}$$

Method 1: Educated Guess Assume that the solution will have a similar form to the function $b(x)$. If $b(x)$ is e.g. a polynomial of degree m we can try $f_p = c_m x^m + ... + c_0$. If $b(x) = e^{3x}$ we can try $f_p = e^{3x}$.

Example: $y' + 2y = x$. We assume f_p is also a polynomial of degree 1, i.e. $f_p = c_1 x + c_0$. We then get

$$\begin{aligned}
&(c_1 x + c_0)' + 2(c_1 x + c_0) = x \\
&c_1 + 2c_1 x + 2c_0 = x \\
&2c_1 = 1 \qquad\qquad 2c_0 + c_1 = 0 \\
&c_1 = \frac{1}{2} \qquad\qquad c_0 = \frac{-1}{4} \\
&f_p = \frac{1}{2}x - \frac{1}{4}
\end{aligned}$$

So the solution of $y' + 2y = x$ is of the form

$$\frac{1}{2}x - \frac{1}{4} + Ke^{-2x}$$

Method 2: Variation of parameters Assumes that a particular solution has a similar form to the homogeneous solution but the constant in the hom. solution will now be replaced by a function of x. If $f_p = K(x)e^{-A(x)}$, we'll put this f_p into the diff. equation and see what it forces $K(x)$ to satisfy.

Example: $y' + 2y = x$. $f_0 = Ke^{-2x}$ is the hom. solution. We guess for $f_p = K(x)e^{-2x}$ and put this in the ODE:

$$\begin{aligned}
&\left(K(x)e^{-2x}\right)' + 2\left(K(x)e^{-2x}\right) = x \\
&K'(x)e^{-2x} + K(x)e^{-2x}(-2) + 2K(x)e^{-2x} = x \\
&K'(x)e^{-2x} = x \\
&K(x) = \int xe^{2x}\, dx = ... = \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x} \\
&f_p = K(x)e^{-2x} = \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x}e^{-2x} \\
&\qquad = \frac{1}{2}x - \frac{1}{4}
\end{aligned}$$

Hence $\mathcal{L} = \left\{ Ke^{-2x} + \frac{1}{2}x - \frac{1}{4} \,\middle|\, K \in \mathbb{C} \right\}$

Method 3: Integration Factor If we have

$$\frac{dy}{dx} + a(x) \cdot y = b(x)$$

we can multiply both sides with $e^{A(x)} = e^{\int a(x)dx}$. We then arrive at

$$y = \left(\int b(x)e^{A(x)}\, dx\right) \cdot e^{-A(x)}$$

Linear differential Equations with constant coefficients

We are looking for a solution of $Dy = b(x)$ for an equation Dy of the following form:

$$\frac{d^k}{dx^k} + a_{k-1}\frac{d^{k-1}}{dx^{k-1}} + ... + a_0$$

1. Solve homogeneous equation (f_0)
2. Solve inhomogeneous equation (f_p)
3. Combine 1. and 2. $(f_0 + f_p)$

Step 1

1. Find characteristic polynomial
   $(y'' + 3y = b \implies P(\lambda) = \lambda^2 + 3)$
2. Find the roots of the charac. polynomial
3. Use the following theorem:
   λ is root of $P_D(x) \iff De^{\lambda x} = 0$
4. We then get $f_0 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + ...$

If we have an **imaginary root** $\alpha \in \mathbb{C} \setminus \mathbb{R}$: we use

$$c \cdot e^{(a+bi)x} = c_1 \cdot e^{ax}\cos(bx) + c_2 \cdot e^{ax}\sin(bx)$$

If a root has **multiplicity** $k > 1$ we use

$$c_1 e^{\lambda x} + c_2 x e^{\lambda x} + ... + c_k x^{k-1} e^{\lambda x}$$

Step 2 (Educated Guess) This method is called "method of undetermined coefficients".

1. Choose "Ansatz" which is similar to $b(x)$
2. Put it into the ODE to calculate the constants

$b(x)$	Ansatz for f_p
$ae^{\alpha x}$	$ce^{\alpha x}$
$a\sin(\beta x)$	$D\sin(\beta x) + E\cos(\beta x)$
$a\cos(\beta x)$	$D\sin(\beta x) + E\cos(\beta x)$
$ae^{\alpha x}\sin(\beta x)$	$De^{\alpha x}\sin(\beta x) + Ee^{\alpha x}\cos(\beta x)$
$ae^{\alpha x}\cos(\beta x)$	$De^{\alpha x}\sin(\beta x) + Ee^{\alpha x}\cos(\beta x)$
$P_n(x)$	$Q_n(x)$
$P_n(x)e^{\alpha x}$	$Q_n(x)e^{\alpha x}$
$P_n(x)e^{\alpha x}\cos(\beta x)$	$Q_n(x)e^{\alpha x}\sin(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)$
$P_n(x)e^{\alpha x}\sin(\beta x)$	$Q_n(x)e^{\alpha x}\sin(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)$

If $b(x)$ is a solution of the homog. equation, try $xb(x), x^2b(x), \dots$, depending on the multiplicity of the eigenvalue. E.g. $b(x) = 10e^{2x}$. Try $y_p = Kxe^{2x}$.

Example: $y'' + y' - 6y = 3e^{-4x}$.
Homogeneos solution $f_h = c_1e^{2x} + c_2e^{-3x}$
For part. sol. we try "Ansatz" $y_p = Ke^{-4x}$:
$(Ke^{-4x})'' + (Ke^{-4x})' - 6Ke^{-4x} = 3e^{-4x} \implies \dots \implies K = \frac{1}{2}$
Particular solution: $f_p = \frac{1}{2}e^{-4x}$
General solution: $f_h + f_p = c_1e^{2x} + c_2e^{-3x} + \frac{1}{2}e^{-4x}$
Solution Set: $\mathcal{L} = \left\{ c_1e^{2x} + c_2e^{-3x} + \frac{1}{2}e^{-4x} \;\middle|\; c_1, c_2 \in \mathbb{C} \right\}$

Separation of variables A linear equation of first order is called separable if it is of the form $y' = b(x)g(y)$. We can separate the variables:

$$\frac{dy}{g(y)} = b(x)dx$$

Example: $e^{2y}y' = x \implies \frac{dy}{e^{-2y}} = xdx$
$\implies \int \frac{dy}{e^{-2y}} = \int xdx \implies \frac{e^{2y}}{2} = \frac{x^2}{2} + c$
$\implies \dots \implies y = \frac{1}{2}\log(x^2 + c)$

2 Differential Calculus in $\mathbb{R}$

2.1 Multi-Variable Functions

Vector field A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Polynomial in n variables A poly. of degree $\leq d$ is a finite sum of monomials of degree $\leq d$.
$P(x, y, z) = x^3 + 2x^2y + xyz + 5z^4$

Monomial is a function of the form
$f : (x_1, x_2, \dots, x_n) \longrightarrow \alpha x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$
with degree $d = d_1 + d_2 + \dots + d_n$.

Cartesian product

$$\begin{aligned} f_1(x) &: \mathbb{R}^n \longrightarrow \mathbb{R}^s \\ x &\longrightarrow f_1(x) \\ f_2(x) &: \mathbb{R}^n \longrightarrow \mathbb{R}^t \\ x &\longrightarrow f_2(x) \\ f(x) &: \mathbb{R}^n \longrightarrow \mathbb{R}^{s+t} \\ x &\longrightarrow (f_1(x), f_2(x)) \end{aligned}$$

Composition $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad g(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^t$
. $\qquad g \circ f = g(f(x)) : \mathbb{R}^n \longrightarrow \mathbb{R}^t$

Functions of separated values These are functions that map a vector from $\mathbb{R}^n$ to a scalar:

$$\begin{aligned} f &: \mathbb{R}^n \longrightarrow \mathbb{R} \\ x &= (x_1, \dots, x_n) \to f_1(x_1)f_2(x_2)\cdots f_n(x_n) \end{aligned}$$

Convergence of sequences We have a sequence $(x_k)_k \subset \mathbb{R}^m$ in $\mathbb{R}^n$, $x_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$. Let $y = (y_1, \dots, y_n) \in \mathbb{R}$.
We say that the sequence (x_k) converges to y as $k \to \infty$ (i.e. $x_k \longrightarrow y$) if

$$\forall \epsilon > 0 \; \exists N \geq 1 \text{ s.t. } \forall k \geq N : ||x_k - y|| < \epsilon$$

Lemma For a sequence (x_k) in $\mathbb{R}$, $(y_1, \dots, y_n) = y \in \mathbb{R}^n$. Then the **following are equivalent**:

1. $\lim x_k = y$
2. For each i, $1 \leq i \leq n$ the sequence $(x_{k,i})_k \subset \mathbb{R}$ of real numbers converge to $y_i \in \mathbb{R}$
3. The sequence of real numbers $||x_k - y|| \longrightarrow 0$.

Definition Limit Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m, x_0 \in X, y \in \mathbb{R}^m$.
We say f has a limit as $x \to x_0$ if

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } \forall x \in X : ||x - x_0|| < \delta \implies ||f(x) - y|| < \epsilon$$

Limit proposition Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m, x_0 \in X, y \in \mathbb{R}^m$. Then we have:

$$\lim_{x \to x_0} f(x) = y \iff$$
$$\forall \text{ sequence } (x_k) \text{ with } \lim x_k = x_0 \text{ we have } \lim f(x_k) = y$$

Definition Continuous Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$.
We say f is continuous at x_0 if

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } ||x - x_0|| < \delta \implies ||f(x) - f(x_0)|| < \epsilon$$

Continuous proposition

f is continuous at $x_0 \iff$
$\forall$ sequence $(x_k) \subset X$ with $\lim x_k = x_0$ we have $\lim f(x_k) = f(x_0)$

This implies:
$$\lim f(x_k) = f(\lim x_k)$$

Proof of not continuous Assume the following:

$$\lim_{(x,y) \to (0,0)} f(x, y)$$

With two variables, choose a y in terms of x and show that for two different y the limit is not the same.

Continuous with polar coordinates For functions of two variables polar coordinates can be helpful. We choose $x = r\cos\theta$ and $y = r\sin\theta$. We then take the limit $r \to 0$ and observe whether the limit depends on r (i.e. continuous) or only θ (i.e. not continuous).

If the limit only depends on θ it matters from which direction we approach the point.

Lemma Sandwich If f, g, h are functions $\mathbb{R}^n \to \mathbb{R}$ with $f(x) < g(x) < h(x) \forall x \in \mathbb{R}^n$.
Let $a \in \mathbb{R}^n$. If $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$,
then $\lim_{x \to a} g(x) = L$ as well.

Bounded set A set $X \subseteq \mathbb{R}^n$ is bounded if the set $\{||x|| \mid x \in X\}$ is bounded in $\mathbb{R}$.

Closed set A set $X \subset \mathbb{R}^n$ is closed if every sequence $(x_k) \subset X$ that converges n $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$ we have that $y \in X$. This means that the limits of sequences in X must also be in X.

Compact set A set $x \subset \mathbb{R}$ is called compact if it is closed and bounded.

Theorem Inverse of closed functions Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ continuous. Then for every $Y \subseteq \mathbb{R}^m$ closed, the set $f^{-1}(Y)$ is closed.

Example: $f:\mathbb{R}^n \longrightarrow \mathbb{R}$
$X = \{x \in \mathbb{R}^n \mid a \leq f(x) \leq b\} = f^{-1}([a,b])$ is closed.

Theorem Min-Max Let $X \subseteq \mathbb{R}^n$ compact. Let $f: X \longrightarrow \mathbb{R}$ a continuous function. Then f is bounded and achieves it's min and max.

i.e. $\exists x_+, x_- \in X$ s.t. $f(x_+) = \mathrm{Sup}_{x\in X} f(x)$ resp.
$f(x_-) = \mathrm{Inf}_{x\in X} f(x)$

Open set $X \subseteq \mathbb{R}^n$ is called open if it's complement in $\mathbb{R}^n$, $\mathbb{R}^n \setminus X$ is closed.

This is equivalent to $\forall x \in X$, $\exists r > 0$
s.t. the set $\{y \in \mathbb{R}^n \mid ||y - x|| < r\} = B_r(x) \subset X$

2.2 Derivatives in $\mathbb{R}^n$

Partial derivative If $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and we want to study f around $x_0 = (x_{0,1}, x_{0,2}, ..., x_{0,n})$. For each j, we consider the 1-variable function $g_j(t) := f(x_{0,1}, x_{0,2}, ..., x_{0,j-1}, t, x_{0,j+1}, ..., x_{0,n})$ defined on the set $A = \{t \in \mathbb{R} \mid (x_{0,1}, ..., t, ..., x_{0,n}) \in X\}$ where we fixed all variables except the j-th one.

$$\frac{dg_j}{dt}(x_{0,j}) = \lim_{h\to 0} \frac{g(x_{0,j}+h) - g(x_{0,j})}{h}$$

If this limit exists we say that f has partial derivative with respect to x_j at the point x_0, and for the limit we write $\frac{\delta f}{\delta x_i}(x_0)$ where x_0 is a fixed point.

Example: $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x,y) \to (x^2 + xy)\sin y$
$\frac{\delta f}{\delta x}(x,y) = \sin y(2x + y)$
$\frac{\delta f}{\delta y}(x,y) = x \sin y + (x^2 + xy)\cos y$

Directional derivative We can approach also along another direction (with vector e):

$$\lim_{h\to 0} \frac{f(x_0 + he) - f(x_0)}{h}$$

Jacobi matrix Consider a linear map $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$, $x \to Ax$ with $A_{m,n}$ Matrix.

Then we have $\frac{\delta f}{\delta x_j}(x) = \begin{pmatrix} a_{1,j} \\ a_{2,j} \\ ... \\ a_{m,j} \end{pmatrix}$ (the j-th column of A).

If we consider $Ax = \begin{pmatrix} f_1(x) \\ f_2(x) \\ ... \\ f_m(x) \end{pmatrix}$, we then get

$$f_i(x) = a_{i,1}x_1 + a_{i,2}x_2 + ... + a_{i,n}x_n \ , \ 1 \leq i \leq m$$

With respect to the j-th variable we have $\frac{\delta f_i}{\delta x_j} = a_{i,j}$. If we put all these derivatives into a matrix we get the Jacobi matrix:

$$J_f := \left(\frac{\delta f_i}{\delta x_j}(x)\right) = A$$

Gradient of a function Consider $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$. If all partial derivatives exist then the column vector $\nabla f(x_0) = \begin{pmatrix} \frac{\delta f}{\delta x_1}(x) \\ \frac{\delta f}{\delta x_2}(x) \\ ... \\ \frac{\delta f}{\delta x_n}(x) \end{pmatrix}$ is called the gradient of f at x_0.

Note $\nabla f(x_0) = (J_f(x_0))^T$

Add / subtract partial derivatives If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ have partial derivatives wrt. to x_j then so does $f+g$, $f-g$.

$$\frac{\delta(f+g)}{\delta x_j} = \frac{\delta f}{\delta x_j} + \frac{\delta g}{\delta x_j}$$

Product of partial derivatives If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ have partial derivatives wrt. to x_j then so does fg.

$$\frac{\delta(fg)}{\delta x_j} = \left(\frac{\delta f}{\delta x_j}\right) g + f\left(\frac{\delta g}{\delta x_j}\right)$$

Quotient of partial derivatives Similar for $\frac{f}{g}$ if $g(x_0) \neq 0$:

$$\frac{\delta(f/g)}{\delta x_j}(x_0) = \frac{\frac{\delta f}{\delta x_j}(x_0)\cdot g(x_0) + f(x_0)\cdot \frac{\delta g}{\delta x_j}(x_0)}{g^2(x_0)}$$

Higher order differentiation If $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, then $\frac{\delta f}{\delta x_i}$ are themselves functions of n variables and can be partial differentiated:

$$\frac{\delta}{\delta x_j}\left(\frac{\delta f}{\delta x_i}\right) = \delta_{x_j, x_i}$$

Example: $f(x,y,z) = x^3yz^2 + \cos x + z^2$.
$\frac{\delta f}{\delta x} = 3x^2yz^2 - \sin x$ $\quad$ $\frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = 3x^2z^2$

Definition Total differential Let $X \subset \mathbb{R}^n$ open, $x_0 \in X$, $f : X \longrightarrow \mathbb{R}^m$ a function.

We say f is differentiable at x_0 with differential u if there exists a linear map $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that:

$$. \lim_{x\to x_0, x\neq x_0} = \frac{f(x) - (f(x_0) - u(x - x_0))}{||x - x_0||} = 0$$

This means that the affine linear map $g(x) = f(x_0) + u(x - x_0)$ approximates f well (which means that it goes faster to 0 as $||x - x_0||$).

The linear map $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called (total) differential of f at x_0. It is denoted by $df(x_0)$, $d_{x_0}f$

Theorem Differentiable $\implies$ continuous Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at x_0.

- f is continuous at x_0
- f has all partial derivatives at x_0
- The matrix that represents the differential in the standard basis is the Jacobi matrix:

$$J_f(x_0) = \left(\frac{\delta f_i}{\delta x_j}(x_0)\right)_{1\leq i\leq m, 1\leq j\leq n}$$

If $f : X \longrightarrow \mathbb{R}^m$ has all partial derivatives

$$\frac{\delta f_i}{\delta x_j} : X \longrightarrow \mathbb{R}^m$$

and if these functions are all continuous in X then f is differentiable on X (i.e. it is continuous).

Properties of the differential

- If $f, g : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ differentiable in x_0 then so is $f + g$ and $d_{x_0}(f+g) = d_{x_0}f + d_{x_0}g$
- $f, g : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$. If f and g are differentiable at x_0 then so is fg. If g is non-zero then also $\frac{f}{g}$.

$$d_{x_0}(fg) = (d_{x_0}f)g(x_0) + f(x_0)(d_{x_0}g)$$

Chain rule Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ open and $f : X \longrightarrow Y$, $g : Y \longrightarrow \mathbb{R}^p$ differentiable functions. Then $g \circ f : X \longrightarrow \mathbb{R}^p$ is differentiable. We have

$$d_{x_0}(g\circ f) = d(g\circ f)(x_0) = dg(f(x_0))\cdot df(x_0)$$

In particular for the Jacobi Matrix we have

$$J_{g\circ f}(x_0) = J_g(f(x_0))\cdot J_f(x_0)$$

Directional derivative In general given $v \in \mathbb{R}^n$ we define the directional derivative of f at x_0 in the direction of f as the derivative at $t = 0$ of $g(t) = f(x_0 + tv)$.

$$d_v f(x_0) = df(v; x_0) = \lim_{t\to 0}\frac{f(x_0 + tv) - f(x_0)}{t}$$

Theorem Directional Derivative $f: X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ differentiable at $x_0 \in X$ and $v \in \mathbb{R}^n$, $v \neq 0$. Then the directional derivative of f at x_0 in the direction of v exists and

$$\frac{d}{dt} f(x_0 + tv)\Big|_{t=0} = (d_{x_0} f)(v) = J_f(x_0) \cdot v$$

Gradient of a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

- Points in the direction of "greatest increase". For any $v \in \mathbb{R}^n$ the directional derivative $d_{x_0} f(v) = J_f(x_0) \cdot v$ is $< \nabla f, v >$, suppose $||v|| = 1$
- Consider the level sets of a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $L_c := \{x \in \mathbb{R}^n | f(x) = c\}$. $\nabla f(x_0)$ is perpendicular to the level set.

Polar coordinates

$$f: [0, \infty) \times [0, 2\pi) \longrightarrow \mathbb{R}^2$$

$$(r, \phi) \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} = \begin{pmatrix} f_1(r,\phi) \\ f_2(r,\phi) \end{pmatrix}$$

$$J_f(r,\phi) = \begin{pmatrix} \frac{\delta f_1}{\delta r} & \frac{\delta f_1}{\delta \phi} \\ \frac{\delta f_2}{\delta r} & \frac{\delta f_2}{\delta \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}$$

$\det J_f(r,\phi) = r(\cos^2\phi + \sin^2\phi) = r$

Change of variables Let $X \subset \mathbb{R}^n$ be open and $f: X \longrightarrow \mathbb{R}^n$ differentiable.

We say f is a change of variables around $x_0 \in X$ if there is a radius $r > 0$ such that the restriction of f to the Ball $B_r(x_0) := \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}$ has the property that the image $Y = f(B_r(x_0))$ is open in $\mathbb{R}^n$ and there exists a differential map $g: Y \longrightarrow B$ such that $f \circ g = id$ and $g \circ f = id$.

Inverse Function Theorem Let $x \subseteq \mathbb{R}^n$ open, $f: X \longrightarrow \mathbb{R}^n$ differentiable. If $x_0 \in X$ is such that $\det(J_f(x_0)) \neq 0$ i.e. $J_f(x_0)$ is invertible then f is a change of variables around x_0. Moreover the Jacobian of g is determined by $J_g(f(x_0)) = J_f(x_0)^{-1}$

Classes (Smoothness) Let $f: X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. We say f is differentiable of class C^1 if f is differentiable on X and all its partial derivatives are continuous. The set of all C^1 functions is denoted by $C^1(X: \mathbb{R}^m)$

Let $k \geq 2$. We say $f \in C^k(X: \mathbb{R}^m$ or f is of class C^k if it is differentiable and each $\delta_{x_i} f: X \longrightarrow \mathbb{R}^m$ $1 \leq i \leq n$ is of class C^{k-1}.

We say f is smooth or C^∞ if $f \in C^k$ $\forall k$.

Theorem Order of differentiation If $f \in C^k$, $k \geq 2$ then the partial derivatives of order $\leq k$ are independent of the order of differentiation.

Hessian Matrix Let $f: X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$. If $f \in C^2(X: \mathbb{R})$, $x_0 \in X$. Then the $n \times n$ Matrix

$$\left(\frac{\delta^2 f(x_0)}{\delta x_i \delta x_j} \right)_{1 \leq i,j \leq n}$$

is called the Hessian of f at x_0. It is denoted by $\nabla^2 f(x_0)$ or $\text{Hess}_f(x_0)$

Taylor polynomial The first order Taylor polynomial of f at point x_0 is defined as:

$$\begin{aligned} T_1 f(x_0; y) :=& f(x_0) + \nabla f(x_0) \cdot y \\ =& f(x_0) + \frac{\delta f}{\delta x_1}(x_0) y_1 + \frac{\delta f}{\delta x_2}(x_0) y_2 + ... + \frac{\delta f}{\delta x_n}(x_0) y_n \end{aligned}$$

The second order **Taylor polynomial** at x_0 is

$$T_2 f(x_0; y) := f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2} y \cdot \text{Hess}_f(x_0) \cdot y^T$$

For higher dimensions we define the Taylor polynomial of order k as

$$T_k f(x_0; y) = \sum_{|m| \leq k} \frac{1}{m!} \cdot \delta_x^{|m|} f(x_0) \cdot y^m$$

where

- $|m| = m_1 + m_2 + ... + m_n$
- $m! = m_1! m_2! ... m_n!$
- $y^m = y_1^{m_1} y_2^{m_2} ... y_n^{m_n}$
- $\delta_x^{|m|} f = \frac{\delta^{|m|} f}{\delta x_1^{m_1} \delta x_2^{m_2} ... \delta x_n^{m_n}}$

y is defined as $y = y_0 - x_0$ where y_0 is the point at which we want to evaluate.

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f \in C^2$, $a \in \mathbb{R}^2$, $y = (y_1, y_2)$.

$$\begin{aligned} T_2 f(a; y) &= f(a) + \frac{\delta f}{\delta x_1}(a) y_1 + \frac{\delta f}{\delta x_2}(a) y_2 \\ &+ \frac{1}{2!} \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \frac{\delta^2 f}{\delta x_1^2}(a) & \frac{\delta^2 f}{\delta x_1 \delta x_2}(a) \\ \frac{\delta^2 f}{\delta x_2 \delta x_1}(a) & \frac{\delta^2 f}{\delta x_2^2}(a) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= f(a) + \frac{\delta f}{\delta x_1}(a) y_1 + \frac{\delta f}{\delta x_2}(a) y_2 \\ &+ \frac{1}{2} \frac{\delta^2 f}{\delta x_1^2}(a) y_1^2 + \frac{\delta^2 f}{\delta x_1 \delta x_2}(a) y_1 y_2 + \frac{1}{2} \frac{\delta^2 f}{\delta x_2^2}(a) y_2^2 \end{aligned}$$

Extrema of functions Let $f: X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ open. A point $x_0 = (x_{0,1}, ..., x_{0,n})$ is a local max (resp. local min) if we can find a neighborhood $B_{x_0}(r) := \{x \in \mathbb{R}^n | \, |x - x_0| < r\}$ with $B_{x_0}(r) \subset X$ s.t. $\forall x \in B_{x_0}(r):$ $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$)

If $x_0 \in X$ is a local extrema then $\nabla f(x_0) = 0$.

Definition Critical point A point $x_0 \in X$ is called a critical point of f if $\nabla f(x_0) = 0$.

***f* attains max, min** If $f: X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable on the interior of X, and if X is closed and bounded then global extrema exist and is either at a point $x_0 \in$ Interor of X for which $\nabla f(x_0) = 0$ or $x_0 \in$ Boundary of X.

Non-degenerate critical point x_0 is a non-degenerate critical point of $f \in C^2(X, \mathbb{R})$ if $\det(\text{Hess}_f(x_0)) \neq 0$.

X-Definite matrices A symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, $\det A \neq 0$ is called

1. Positive definite $(A > 0)$
   $\iff$ $xAx^T > 0 \; \forall x \in \mathbb{R}^n$
   $\iff$ all eigenvalues of A are positive
   $\iff$ all principal minors of A are positive (each square sub-matrix containing $a_{0,0}$ has a positive determinant)
2. Negative definite $(A < 0)$
   $\iff$ $-A$ is positive definite
   $\iff$ all eigenvalues of A are negative
3. Indefinite if it has positive and negative eigenvalues

Determine local extrema Let $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $f \in C^2(X, \mathbb{R})$ and $x_0 \in X$ be a critical point of f, $\nabla f(x_0) = 0$. Then

1. If $\text{Hess}_f(x_0) > 0$ then x_0 is a local minimum
2. If $\text{Hess}_f(x_0) < 0$ then x_0 is a local maximum
3. If $\text{Hess}_f(x_0)$ is indefinite then x_0 is a saddle point

To find the local extrema do the following:

1. Calculate ∇f and determine the critical points for which $\nabla f = 0$
2. Check if the Hessian has det $\neq 0$. If yes, determine if the matrix is pos. or neg. definite or if it is indefinite.
3. Check the border lines for any local extrema

If the Hessian matrix is degenerate (i.e. $\det \text{Hess}_f = 0$, we cannot do the above. Instead we want to search two lines along which x_0 is a local minimum resp. a local maximum.

2.3 Integration in $\mathbb{R}^n$

Path Integral Let $\gamma : [a, b] \longrightarrow \mathbb{R}^n$ a curve in $\mathbb{R}^n$, $X \subset \mathbb{R}^n$ a subset of $\mathbb{R}^n$ which contains the image of γ, $v : X \longrightarrow \mathbb{R}^n$ a continuous function.

The following integral is called the line (path) integral of v along γ:

$$\int_a^b \langle\, v(\gamma(t))\, ,\, \gamma'(t)\,\rangle\, dt = \int_\gamma v(s) \cdot ds \in \mathbb{R}$$

Reparametrization The path integral is independent of orientation preserving representation of the curve:

If $\gamma : [a, b] \longrightarrow \mathbb{R}^n$, $\sigma : [c, d] \longrightarrow [a, b]$ s.t. $\sigma(c) = a$, $\sigma(d) = b$ and $\sigma'(t) > 0$ then $\widetilde{\gamma} := \gamma \circ \sigma : [c, d] \longrightarrow \mathbb{R}^n$ is a reparametrization of γ. We have:

$$\int_\gamma f \cdot ds = \int_{\widetilde{\gamma}} f \cdot ds$$

Concatenation Let $\gamma_1 : [a, b] \longrightarrow X \subset \mathbb{R}^n$, $\gamma_2 : [c, d] \longrightarrow X$ be two paths with $\gamma_1(b) = \gamma_2(c)$.

Define $\gamma_1 + \gamma_2$ as the path formed by concatenation of these two curves:

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t - b + c) & t \in [b, d + b - c] \end{cases}$$

Then

$$\int_{\gamma_1 + \gamma_2} f \cdot ds = \int_{\gamma_1} f \cdot ds + \int_{\gamma_2} f \cdot ds$$

Opposite direction If $\gamma : [a, b] \longrightarrow \mathbb{R}^n$ a path. Let $-\gamma : [a, b] \longrightarrow \mathbb{R}^n$ be the same path traced in the opposite direction s.t. $(-\gamma)(t) := \gamma(a + b - t)$.

Then

$$\int_{-\gamma} f \cdot ds = -\int_\gamma f \cdot ds$$

Potential for function A differentiable function $g : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t. $\nabla g = f$ is called a potential for f.

Then

$$\int_\gamma f \cdot ds = \int_\gamma \nabla g \cdot ds = \int_a^b \langle\, \nabla g(\gamma(t))\, ,\, \gamma'(t)\,\rangle\, dt$$

Attention: there is not always a g s.t. $\nabla g = f$!

Conservative Vector Field If $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is conservative, for any $x_1, x_2 \in X$ the line integral

$$\int_\gamma f \cdot ds = F(\gamma(b)) - F(\gamma(a))$$

of a curve in X from x_1 to x_2 is independent of the curve. f must be continuous to be considered conservative.

Path Connected Let $X \subset \mathbb{R}^n$ open. X is path connected if for every pair of points in $x, y \in X$, $\exists$ a C^1 path $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma([0, 1]) \subset X$.

When can we use potential? Let f be a continuous vector field on an open, path-connected set $X \subset \mathbb{R}^n$. Then the following are equivalent:

1. f is the gradient of a function $g : X \longrightarrow \mathbb{R}$, i.e. $f = \nabla g$
2. The line integral of f is independent of the path between the 2 points
3. The line integral of f along closed paths are zero

Curl of a function Let $f : X \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, C^1 a vector field. Then the curl of f is defined

$$\text{curl}(f) := \begin{pmatrix} \delta_y f_3 - \delta_z f_2 \\ \delta_z f_1 - \delta_x f_3 \\ \delta_x f_2 - \delta_y f_1 \end{pmatrix}$$

Condition for conservative vector field Let $X \subseteq \mathbb{R}^n$ open, $f : X \longrightarrow \mathbb{R}^n$ C^1 a vector field with $f(x) = (f_1(x), ..., f_n(x))$.

If f is conservative then

$$\frac{\delta f_j}{\delta x_i} = \frac{\delta f_i}{\delta x_j} \qquad 1 \leq i, j \leq n$$

This means that

$$f \text{ is conservative} \implies \text{curl}(f) = 0$$

Remark: In general $\text{curl}(f) = 0$ doesn't imply that f is conservative. But if f is defined on a star shaped region then these conditions are sufficient.

Riemann integral Is defined for $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and defines the volume below the hyperplane spanned by the function.

Riemann integral properties Let $f, g : Q \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ integrable, $\alpha, \beta \in \mathbb{R}$. Then

1. $\alpha f + \beta g$ is also integrable and
$$\int (\alpha f + \beta g) dx = \alpha \int f\, dx + \beta \int g\, dx$$
2. If $f(x) \leq g(x)\ \forall x \in Q$ then
$$\int_Q f\, dx \leq \int_Q g\, dx$$
3. If $f(x) \geq 0$ then
$$\int_Q f\, dx \geq 0$$
4. $$\left| \int_Q f\, dx \right| \leq \int_Q |f|\, dx \leq (\sup f)_Q \cdot (\text{vol} Q)$$

Fubini's theorem
If $Q = I_1 \times I_2 \times ... \times I_n = [a_1, b_1] \times ... \times [a_n, b_n]$ then

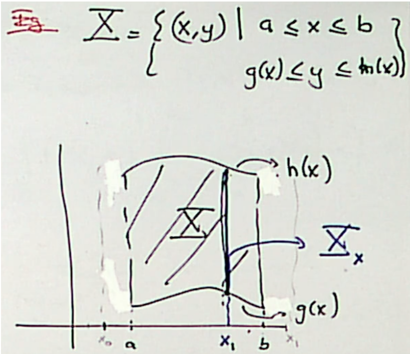
$$\int_Q f\, dx = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(... \left(\int_{a_n}^{b_n} f(x_1, ..., x_n)\, dx_n \right) ... \right) dx_2 \right) dx_1$$

Integrable If f is continuous and bounded on Q then f is integrable.

General Fubini Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $n = n_1 + n_2$ with $n_1, n_2 \geq 1$. For $x \in \mathbb{R}^n$ write $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$.

For $x_1 \in \mathbb{R}^{n_1}$, define $X_{x_1} := \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}$.
Define $X_1 := \{x_1 \in \mathbb{R}^{n_1} \mid X_{x_1} \neq \emptyset\} \subset \mathbb{R}^{n_1}$.
If $g(x_1) := \int\limits_{X_{x_1}} f(x_1, x_2)\, dx_2$ is continuous on X_1, then

$$\int\limits_X f(x)\, dx = \int\limits_{X_1} g(x_1)\, dx_1 = \int\limits_{X_1} \int\limits_{X_{x_1}} f(x_1, x_2)\, dx_2\, dx_1$$



m-parametrized set For $1 \leq m \leq n$ a m-parametrized set or parametrized m-set is a continuous function $\phi : [a_1, b_1] \times ... \times [a_m, b_m] \longrightarrow \mathbb{R}^n$ which is C^1 on $(a_1, b_1) \times ... \times (a_m, b_m)$.

Negligible subset A subset $Y \subset \mathbb{R}^n$ is called negligible if there exist finitely many ϕ_i, parametrized m_i-sets with $m_i < n$ s.t. $1 \leq i \leq k$:

$$Y \subset \bigcup_{i=1}^{k} \phi_i(x_i)$$

where $\phi_i : X_i \longrightarrow \mathbb{R}^n$. This means that

- $n = 1$ - union of finitely many points are negligible
- $n = 2$ - union of finitely many images of parametrized curves and finitely many points
- $n = 3$ - union of finitely many images of parametrized surfaces, curves and points

Integral of negligible subsets If $Y \subset \mathbb{R}^n$ is negligible, closed and bounded, then for all continuous functions $f : Y \longrightarrow \mathbb{R}$

$$\int\limits_Y f(x_1, ..., x_n)\, dx_1 ... dx_n = 0$$

Domain additivity If $X = A_1 \cup A_2$, A_1, A_2 bounded and closed then for $f : X \longrightarrow \mathbb{R}$:

$$\int\limits_X f\, dx = \int\limits_{A_1} f\, dx + \int\limits_{A_2} f\, dx - \int\limits_{A_1 \cap A_2} f\, dx$$

In case $A_1 \cap A_2$ is negligible we can omit the last integral.

Unbounded Integrals Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, X a non-compact set, s.t. $\int\limits_K f\, dx$ exists for every K, compact ($=$ closed, bounded) subset of X.
Suppose we have a sequence of regions X_n such that

1. $X_n \subset X$ bounded, closed
2. $X_n \subset X_{n+1}$
3. $\bigcup X_n = X$

If $\lim_{n \to \infty} \int\limits_{X_n} f$ exists then we say $\int\limits_X f\, dx$ converges.

Change of variables

$$\int\limits_X f(\phi(x)) \cdot |\det J_\phi(x)|\, dx = \int\limits_Y f(y)\, dy$$

This holds for $X, Y \subset \mathbb{R}^n$ compact subsets and $\phi : X \longrightarrow Y$, where when ϕ is restricted to the open set $\tilde{X}$ (i.e. X without it's border), ϕ is a C^1 bijective map.

Example Polar Coordinates Let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\phi(r, \theta) = (x, y) = (r\cos\theta, r\sin\theta)$.

$$|\det J_f| = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} \end{vmatrix} = r$$

Integrate $f(x) = x$ over the area of the pos-pos unit circle quarter, i.e. $x^2 + y^2 = 1$.

$$\int\limits_0^{\pi/2} \int\limits_0^1 (r\cos\theta)\, r\, dr\, d\theta = \int\limits_0^{\pi/2} \left(\int\limits_0^1 r^2\, dr \right) \cos\theta\, d\theta$$

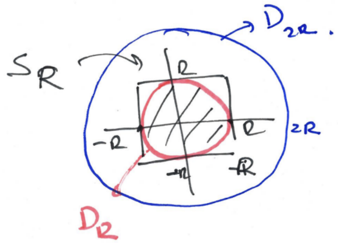
$$= \int\limits_0^{\pi/2} \left(\frac{r^3}{3} \bigg|_0^1 \right) \cos\theta\, d\theta$$

$$= \frac{1}{3} \int\limits_0^{\pi/2} \cos\theta\, d\theta = \frac{1}{3}$$

Example Gaussian Curve

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-x^2 - y^2}\, dx\, dy = ?$$

.
Consider $\lim_{R \to \infty} \int\limits_{S_R} \int e^{-x^2 - y^2}\, dx\, dy$ where



Then

$$\int\limits_{D_R} \int e^{-x^2 - y^2}\, dx\, dy = \int\limits_0^R \int\limits_0^{2\pi} e^{-r^2}\, r\, d\theta\, dr = 2\pi \int\limits_0^R e^{-r^2}\, r\, dr$$

$$= \frac{2\pi}{2}(1 - e^{-R}) = \pi(1 - e^{-R})$$

because $\int e^{-r^2} r\, dr = \int e^{-u} \frac{du}{2} = \frac{-1}{2} e^{-u}$ with $r^2 = u$, $2r\, dr = du$.
We know $D_R \leq S_R \leq D_{2R}$, and if we let $R \longrightarrow \infty$ we get

$$\pi \leq \lim_{R \to \infty} \int\limits_{S_R} \int e^{-x^2 - y^2}\, dx\, dy \leq \pi$$

$$\Longrightarrow \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-x^2 - y^2}\, dx\, dy = \pi$$

Hence we get the area below the gaussian curve:

$$\left(\int\limits_{-\infty}^{\infty} e^{-x^2}\, dx \right) \left(\int\limits_{-\infty}^{\infty} e^{-y^2}\, dy \right) = \pi$$

$$\Longrightarrow \int\limits_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Example Change of Variables

$$\int\limits_D \int e^{\frac{y-x}{y+x}}\, dx\, dy$$

With change of variables: $u = y - x$ and $v = y + x$. We calculate $x = \frac{v-u}{2}$ and $y = \frac{u+v}{2}$.
We have $\phi(u, v) = (x, y) = \left(\frac{v-u}{2}, \frac{u+v}{2}\right)$ and

$$|\det J_\phi| = \left| \det \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right| = \frac{1}{2}$$

and hence $dx\, dy = \frac{1}{2}\, du\, dv$.

$$\int\limits_D \int e^{\frac{y-x}{y+x}}\, dx\, dy = \int\limits_X \int e^{\frac{u}{v}} \frac{1}{2}\, du\, dv = ...$$

Cylindrical coordinates

$$\phi : (r, \theta, z) \longrightarrow (x, y, z) = (r\cos\theta, r\sin\theta, z)$$

$$\det|J_\phi(r,\theta,z)| = r \implies dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Spherical coordinates

$$\phi : \rho = [0,\infty) \times \theta = [0,\pi] \times \phi = [0,\pi] \longrightarrow \mathbb{R}^3$$

where

$x = \rho\sin\phi\cdot\cos\theta,$

$y = \rho\sin\phi\cdot\sin\theta,$

$z = \rho\cos\phi$

Green's Theorem Let $f : X \longrightarrow \mathbb{R}^2$, a C^1 vector field, X closed and bounded where $\delta X = \bigcup_{i=1}^n \gamma_i$ union of simple closed curves so that X is always to the left of the curve $\gamma = \bigcup_{i=1}^n \gamma_i$ then

$$\int_X\int \left(\frac{\delta f_2}{\delta x} - \frac{\delta f_1}{\delta_y}\right) dxdy = \int_\gamma f\,ds$$

Examples with curl = 1

1. $f = (0, x)$
2. $f = (-y, 0)$
3. $f = (\frac{-y}{2}, \frac{x}{2})$

Example area bounded by curves To calculate the area we look for a vector field $f = (f_1, f_2)$ s.t. $\text{curl} f = 1$.

Find the area enclosed by the curve $\gamma(t) = (t^2, \frac{t^3}{3} - t)$ for $-\sqrt{3} \le t \le \sqrt{3}$.

Choose $f = (0, x)$. Then

$$\int_\gamma f\,ds = \int_{-\sqrt{3}}^{\sqrt{3}} f(\gamma(t))\cdot\gamma'(t)\,dt$$
$$= \int_{-\sqrt{3}}^{\sqrt{3}} (0, t^2)\cdot(2t, t^2-1)\,dt = \ldots = \frac{8}{5}\sqrt{3}$$

3 Other

3.1 Integration

Trick 1: Linearität
Stellt meist einen ersten Schritt dar, um das Problem auf kleinere Probleme zu reduzieren. Wir nutzen dabei aus, dass

$$\int_a^b \alpha f(x) + \beta g(x)\;dx = \alpha\int_a^b f(x)\;dx + \beta\int_a^b g(x)\;dx$$

Beispiel:

$$\int_{-1}^1 \frac{(6x^2+3)\sin(x^3)}{2}\;dx = \int_{-1}^1 3x^2\sin(x^3)\;dx + \frac{3}{2}\int_{-1}^1 \sin(x^3)\;dx$$

Trick 2: Substitution

$$f(g(x)) + c = \int g'(x)f'(g(x))\;dx$$

$$\int f(y)\;dy = \int g'(x)f(g(x))\;dx$$

$$\int_{g(a)}^{g(b)} f(y)\;dy = \int_a^b g'(x)f(g(x))\;dx$$

Wichtig: Die Grenzen müssen ebenfalls substituiert werden!

Beispiel: berechne $\int_0^2 x\cdot e^{x^2}dx$. Wir substituieren $u = x^2$. es gilt $\frac{du}{dx} = 2x \iff dx = \frac{du}{2x}$.

$$\int_0^2 x\cdot e^{x^2}dx = \int_0^4 x\cdot e^u\frac{du}{2x} = \int_0^4 \frac{1}{2}e^u du$$
$$= \frac{1}{2}e^u\Big|_0^4 = \frac{1}{2}(e^4 - 1)$$

Beispiel: ($u = 1 + x^3$, $\frac{du}{dx} = 3x^2$, $dx = \frac{du}{3x^2}$)

$$\int_0^2 x^2\log(1+x^3)\;dx = \int_1^9 x^2\log u\;\frac{du}{3x^2} = \frac{1}{3}\int_1^9 \log u\;du$$
$$= \frac{1}{3}\left[x(\log x - 1)\right]_1^9 = \ldots = 3\log 9 - \frac{8}{3}$$

Beispiel: ($u = x^2$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$)

$$\int_0^1 2x^3x^{x^2}\;dx = \int_0^1 2xue^u\;\frac{du}{2x} = \int_0^1 ue^u\;du$$
$$\stackrel{\text{PI}}{=} [ue^u]_0^1 - \int_0^1 e^u\;du = [ue^u]_0^1 - [e^u]_0^1 = e - (e-1) = 1$$

Trick 3: Partielle Integration

$$\int f'(x)g(x)\;dx = f(x)g(x) - \int f(x)g'(x)\;dx$$

$$\int_a^b f'(x)g(x)\;dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\;dx$$

Beispiel:

$$\int \log x\;dx = \int 1\cdot\log x\;dx = x\log x - \int x\cdot\frac{1}{x}\;dx$$
$$= x\log x - x + c$$

Trick 4: ungerade Funktionen
Sei $f(x) : [-a, a] \to \mathbb{R}$ stetig und ungerade, d.h. $f(-x) = -f(x)$. Dann gilt:

$$\int_{-a}^a f(x)\;dx = 0$$

Trick 5: Partialbruchzerlegung

$$\frac{dx + f}{ax^2 + bx + c} = \frac{A}{Bx + C} + \frac{D}{Ex + F}$$

Bei doppelten Nullstellen (z.B. $(x+1)^2$) folgende Nenner wählen: $x + 1$ und $(x+1)^2$.

Hat der Nenner Grad $n > 1$, für den Zähler ein Polynom mit Grad $n-1$ wählen: $(x^2+1) \implies \frac{ax+b}{x^2+1}$

Wenn der Zähler grösseren Grad hat als der Nenner: Polynomdivision (Zähler durch Nenner):

$$\frac{x^3 + 2x^2 - 4x + 2}{(x-1)(x+3)} = \mathbf{x} + \frac{\mathbf{(2\text{-}x)}}{(x-1)(x+3)}$$

weil $x^3 + 2x^2 - 4x + 2 = \mathbf{x}\cdot(x-1)(x+3) + \mathbf{(2\text{-}x)}$

Komplexe Nullstelle: $x^2 + px + q \implies \frac{Ax+B}{x^2+px+q}$

Beispiel:

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{a}{x-1} + \frac{b}{x+1}$$
$$= \frac{a(x+1) + b(x-1)}{(x-1)(x+1)}$$
$$\implies a(x+1) + b(x-1) = 1$$
$$\implies a = \frac{1}{2}, b = -\frac{1}{2} \text{ (Nullstellen einsetzen)}$$

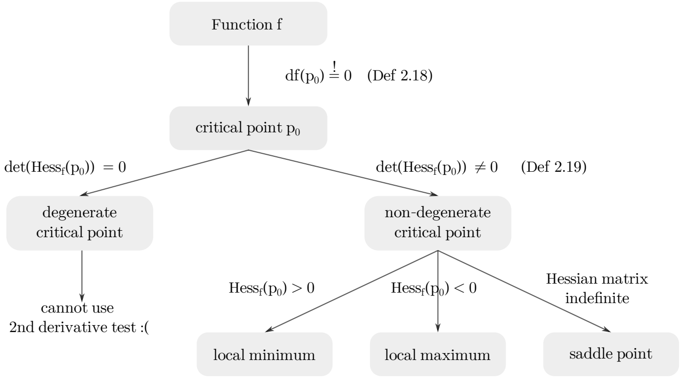
Beispiel:

$$\frac{9x^3 - 3x + 1}{x^3 - x^2} = 9 + \frac{9x^2 - 3x + 1}{x^2(x-1)} \text{ (Polynomdivision)}$$
$$\implies \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{9x^2 - 3x + 1}{x^2(x-1)} \text{ (Nullstellen)}$$
$$\implies A(x)(x-1) + B(x-1) + C(x^2) = 9x^2 - 3x + 1$$
$$\implies B = -1, C = 7 \implies A = 2$$

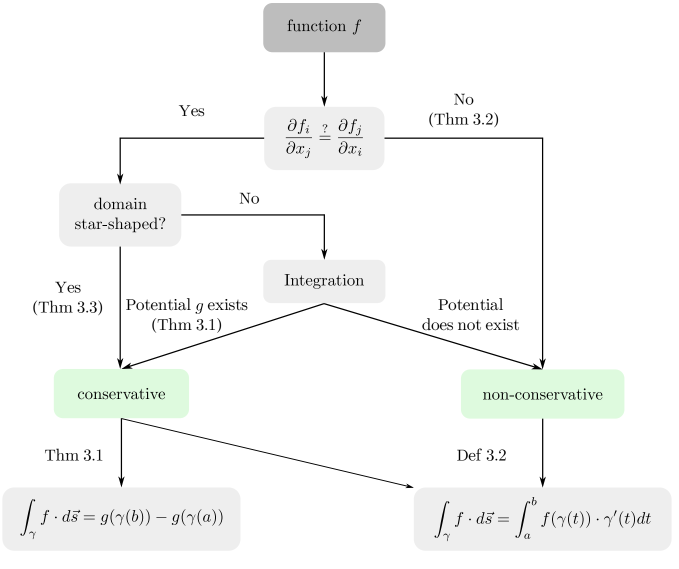
3.2 Differentials and Derivatives

f'(x)	**f**(x)	**F**(x)
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$ + C
$\cos(x)$	$\sin(x)$	-$\cos(x)$
$2\sin(x)\cos(x)$	$\sin^2$(x)	$\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right)$
-$\sin(x)$	$\cos(x)$	$\sin(x)$
-$2\sin(x)\cos(x)$	$\cos^2$(x)	$\frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right)$
$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\tan(x)$	-$\ln\lvert\cos(x)\rvert$

3.3 Determine Function Extrema



3.4 Conservative functions



3.5 Identities

p-q-Formel $x^2 + p \cdot x + q = 0$ $\left(p = \frac{b}{a},\ q = \frac{c}{a}\right)$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Mitternachtsformel: $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Normalform Kreisgleichung: $r^2 = (x - m_1)^2 + (y - m_2)^2$

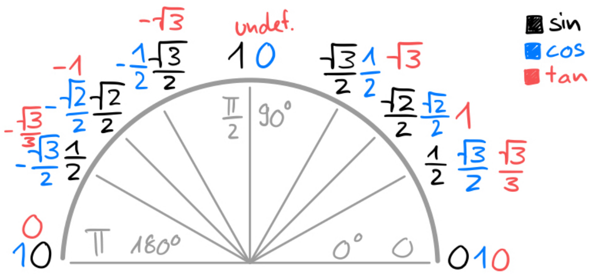
mit Mittelpunkt (m_1, m_2)

Wurzeln:

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$
$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$
$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$$

Logarithmus:

$$\log_a(u \cdot v) = \log_a(u) + \log_a(v)$$
$$\log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)$$
$$\log_a(u^v) = v \cdot \log_a(u)$$
$$\log_a\left(\sqrt[n]{u}\right) = \frac{1}{n} \cdot \log_a(u)$$
$$\log_b(u) = \frac{\log_a(u)}{\log_a(b)}$$



Binomialkoeffizient: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Stirling's formula: $n! \approx \frac{\sqrt{2\pi n} \cdot n^n}{e^n}$

Inverse $f^{-1}(y)$ berechnen:

$$f(x) = y = \ln(x - 17) + 2 \iff f^{-1}(y) = x = e^{y-2} + 17$$

- $e^{ix} = \cos(x) + i\sin(x)$
- $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$
- $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$
- $\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}$
- $\int\limits_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$
- $\sin^2(x) + \cos^2(x) = 1$
- $\tan(x) = \frac{\sin(x)}{\cos(x)}$
- $\sin(x \pm y) = \sin(x) \cdot \cos(y) \pm \cos(x) \cdot \sin(y)$
- $\cos(x \pm y) = \cos(x) \cdot \cos(y) \mp \sin(x) \cdot \sin(y)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$