The Complexity of Finding Tarski Fixed Points

Master Thesis

April 19, 2024

Nils Jensen

Advised by

Prof. Dr. Bernd Gärtner

Sebastian Hasselbacher

Abstract

Insert the abstract here.

Contents

List of Figures

List of Tables

1 Introduction

Write the introduction here. This is a test.

Preliminaries 2

The aim of this chapter is to introduce the complexity class TNFP, and some of its subclasses, in particular PPAD, PLS and EOPL. We will also introduce the TARSKI problem.

2.1 Total search problems

The study of complexity classes originally works with so-called *decision-problems*, which are the question of deciding on the membership in a set — also called a *language*. Now while these problems are interesting, real world questions or problem often ask for an explicit anwser. For instance while deciding if a function has a global minimum is a decision problem, we are interrested in actually finding this minimum, which is not a decision problem.

Notable such problems include deciding on whether a boolean formula can be satified or if a k-Clique exist in a given graph.

Even though as we will see it can be transformed into one

This is where so called *search problems* come into play:

Definition 2.1 — Search Problem.
A *search problem* is given by a relation $R \subset \{0,1\}^* \times \{0,1\}^*$. For a given *instance* $I \in \{0,1\}^*$ the computational problem, to find a *solution* $s \in \{0,1\}^*$, that satisfies: $(I,s) \in R$ or output “No” if no such s exists.

The “No” case can be encoded as some special binary string.

Now of course we can view these search problems as decision problems by looking at the corresponding decision problem given by the language:

$$\mathscr{L}_R = \{I \in \{0,1\}^* | \ \exists s \in \{0,1\}^* \ : \ (I,s) \in R\}$$

We can then ask the classical complexity questions about these search problems, i.e. whether these search problems are in P? NP? whether they are NP-Hard? One easily observes that search problems are always at least as hard as just deciding whether a solution exist. This is because solving a search problem also solves the underlying decision problem. This leads to the natural question: what if we remove the underlying decision problem? This can be done by garanteeing that “No” is never a solution. We call these problems where every instance admits a solution *total search problems*.

Definition 2.2 — Total search problems.
A *total search problem* is a search problem given by a relation $R \subset \{0,1\}^* \times \{0,1\}^*$, such that for every given instance $I \in \{0,1\}^*$ there is a solution $s \in \{0,1\}^*$, that satisfies: $(I, s) \in R$.

The complexity class **TNFP** as introduced in [1] is simply the class of all total search problems that lie in **NP**. Examples of **TNFP** problems are:

[1]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

This means that **TNFP** can be seen as an intermediate class between **P** and **NP**, containing all search problems where a solution is guaranteed to exist, and where one can efficiently check the feasibility of a candidate solution.

- Factoring, the problem of finding the prime factors of a number,
- Nash, the problem of finding a nash equilibrium in a bimatrix game,
- Minimize, the problem of finding the global minimum of a convex function.

Similarly to decision problem we can also define reduction inside **TNFP**.

Definition 2.3 — Reduction.
For two problem $R, S \in$ **TNFP**, we say that R *reduces* (many to one) to S if there exist polynomial time computable functions $f : \{0,1\}^* \to \{0,1\}^*$ and $g : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ such that for $I, s \in \{0,1\}^*$: if $(f(I), s) \in S$ then $(I, g(I, s)) \in R$. This means that if s is a solution to an instance $f(I)$ in S, we can compute $g(I, s)$ a solution to an instance I in R

Saying *one can reduce R onto S* can be understood as saying *if one can solve S efficiently then I can solve R efficiently.*

2.2 An excursion into Binary Circuits

TODO

2.3 Subclasses of TNFP

Because the existence of complete **FNP**-Problems in **TNFP** would imply **NP** = **coNP**, as described in [2]. Because this is a very unexpected outcome we cannot expect to find complete problems in **TNFP**. This means that we should use other tools to study the structure of **TNFP**.

[2]: Megiddo et al. (1991), *On total functions, existence theorems and computational complexity*

One of the challenges is that **TNFP** is a so-called *semantic* class. By semantic class we mean a class for which it is difficult to check if that Turing Machine defines a language in this class. A *syntactic* class is a class for which it is easy to check that the accepted language of a Turing Machine indeed belongs to the class. These

Examples of syntactic classes include **P** and **NP**.

terms are defined in more detail in [3]. Hence we want to study syntactic subclasses of **TNFP**. One of the proposed methods [1] is to categorize total search problems with respect to the existence results which allow them to be *total*. This is what leads to the complexity classes we will discuss next.

[3]: Papadimitriou (1994), *Computational complexity*

[1]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

Polynomial Local Search (PLS)

The existence results which gives rise to **PLS** is "*every directed acyclic graph has a sink*". We can then construct the class **PLS** by defining it as all problems which reduce to finding the sink of a directed acyclic graph (DAG).

Formally we first define the problem LOCALOPT as in [4]:

[4]: Johnson et al. (1988), *How easy is local search?*

LOCALOPT
Input: Two binary circuits $P, S : [2^n] \to [2^n]$.
Output: A vertex $v \in [2^n]$ such that $P(S(v)) \geq P(v)$.

S can be seen as a proposed successor, and P as a potential. The goal is to find a local minima v of the potential.

One might ask why this is equivalent to finding the sink of a DAG? The circuit S defines a directed graph, which might contain cycles. Only keeping the edge on which the potential decreases (strictly) leads to a DAG, with as sinks exactly the v such that $P(S(v)) \geq P(v)$. Now we can define **PLS**:

Definition 2.4 — Polynomial Local Search (PLS).
The class **PLS** is the set of all **TNFP** problems that reduce to LOCALOPT.

One of the reasons we think that studying very "easy" problems such as **PLS** is that we strongly believe that there is no clever way of solving these problems without actually walking through the graph. Hence if we have a graph of exponentially large size it seems very unlikely that one can find an efficient way of solving the problem. Hence all problems in **PLS** can be though of as including the fundamental difficulty of no beeing able to do better than to walk along some graph.

By "easy" we mean that the problem can be solved by simply walking through the graph, and checking whether every vertex is a local minima.

Polynomial Parity Argument on Directed Graphs (PPAD)

Now we want to discuss the complexity class **PPAD**, introduced by Papadimitriou as one of the first syntatic subclasses of **TNFP** in [1]. The existence result giving rise to this class is that "*If a directed graph has an unbalanced vertex, then it has at least one other unbalanced vertex*". **PPAD** can be defined using the problem END-OF-LINE as introduced in [5].

[1]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

[5]: Daskalakis et al. (2009), *The Complexity of Computing a Nash Equilibrium*

End-of-Line
Input: Boolean circuit $S, P : \{0,1\}^n \to \{0,1\}^n$ such that $P(0^n) = 0^n \neq S(0^n)$ (0^n is a source.)
Output: An $x \in \{0,1\}^n$ such that either:
- $P(S(x)) \neq x$ (x is a sink) or
- $S(P(x)) \neq x \neq 0^n$ (x is a non non-standard source)

Here S can be thought of giving the successor of a vertex, and P as giving the predecessor of a vertex.

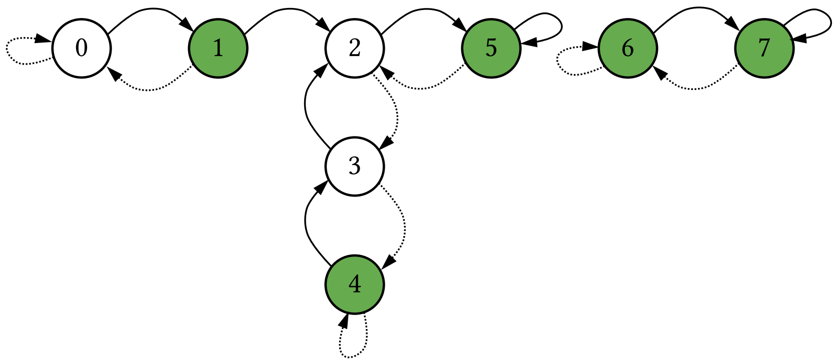

Figure 2.1: Example of an End-of-Line Problem with $n = 3$ (8 vertices). The circuit S is represented by solid lines and the circuit P by dashed lines. The solutions are the sinks $x = 5$, $x = 7$ and $x = 1$, aswell as the sources $x = 4$ and $x = 6$.

These boolean circuits represent a directed graph with maximal in and out degree 1, by having an edge from x to y if and only if $S(x) = y$ and $P(y) = x$. The goal is to find a sink of the graph, or another source. It can be shown that the general case of finding a second imbalanced vertex in a directed graph (a problem called Imbalance) can be reduced to End-of-Line [6]. Now we can define the complexity class **PPAD** as follows:

Notice that End-of-Line allows cycles, and that these do not induce solutions.

[6]: Goldberg et al. (2021), *The Hairy Ball problem is PPAD-complete*

Definition 2.5 — PPAD.
The class **PPAD** is the set of all **TNFP** problems that reduce to End-of-Line.

End of Potential Line (EOPL)

Next we want to discuss the complexity class **EOPL** as introduced in [7]. The existence results which gives rise to **EOPL** is that *"in a directed acyclic graph, there must be at least two unbalanced vertices"*. Similarly to **PLS** acyclicity will be enforced using a potential.

[7]: Fearnley et al. (2018), *End of Potential Line*

End of Potential Line

Input: Two boolean circuits $S, P : \{0,1\}^n \to \{0,1\}^n$, and a boolean circuit $V : \{0,1\}^n \to [2^n - 1]$, such that 0^n is a source, (i.e. $P(0^n) = 0^n \neq S(0^n)$).

Output: An $x \in \{0,1\}^n$ such that either:

- $P(S(x)) \neq x$ (x is a sink)
- $S(P(x)) \neq x \neq 0^n$ (x is a *non-standard source*)
- $S(x) \neq x$, $P(S(x)) = x$ and $V(S(x)) \leq V(x)$ (violation of the monoticity of the potential)

Here S can be thought of giving the successor of a vertex, and P as giving the predecessor of a vertex. V can be though of as a potential which is suppose to be monotonously increasing along the line.

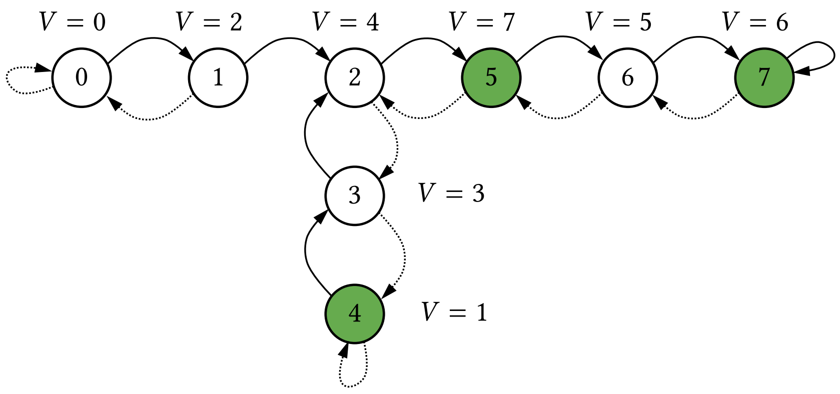

Figure 2.2: Example of an **EOPL** Problem with $n = 3$ (8 vertices). The circuit S is represented by solid lines and the circuit P by dashed lines. The solutions are the sink $x = 7$, the violation of potential at $x = 5$ and the non-standard source $x = 4$.

S and P can be though of as representing a directed line. Finding another source (a non-standard source), is a violation, as a directed line only has one source. The potential serves a garantee of acyclicity. Now we can define the complexity class **EOPL**.

Definition 2.6 — EOPL.
The class **EOPL** is the set of all **TNFP** problems that reduce to End of Potential Line.

2.4 The Tarski Problem

Next we want to introduce the Tarski Problem. Before we do this we recall that there is a partial order on the d dimensional latice $[N]^d$, given by $x \leq y$ iff $x_i \leq y_i$ for all $i \in \{1, \dots, d\}$. The name originates from Tarski's fixed point Theorem as introduced in [8] which we remind the reader of below:

[8]: Tarski (1955), *A lattice-theoretical fixpoint theorem and its applications.*

Theorem 2.1 — Tarski's fixed point Theorem.
Let $f : [N]^d \to [N]^d$ a function on the d-dimentional lattice. If f is monotonous (with respect to the previously discussed partial order), then f has a fixed point, i.e. there is an $x \in [N]^d$ such that $f(x) = x$.

This theorem is also known as the Knaster-Tarski Theorem in the litterature.

A proof of this theorem can be found in the previously mentionned work [8]. Without surprise the TARSKI problem as defined in [9], is now to find such a fixed-point. Formally we define the problem as follows:

[9]: Etessami et al. (2020), *Tarski's Theorem, Supermodular Games, and the Complexity of Equilibria*

TARSKI
Input: A boolean circuit $f : [N]^d \to [N]^d$.
Output: Either:

- An $x \in [N]^d$ such that $f(x) = x$ (fixed point) or
- $x, y \in [N]^d$ such that $x \leq y$ and $f(x) \not\leq f(y)$ (violation of monoticity).

This is of course a total search problem, as there will always either be a fixed point, or a point violating monoticity. We now want to summarize where TARSKI lies inside of TNFP. It has been shown in [9] that TARSKI lies in both PLS and $\mathsf{P}^{\mathsf{PPAD}}$. Previous work [10], showed that many-to-one reductions and Turing-reduction onto PPAD are equivalent. In particular this means that $\mathsf{P}^{\mathsf{PPAD}} = \mathsf{PPAD}$, and that TARSKI also lies in PPAD.

[10]: Buss et al. (2012), *Propositional proofs and reductions between NP search problems*

2.5 Structure of PLS ∩ PPAD

Now that we have established that TARSKI lies inside PLS ∩ PPAD, we want to discuss the structure of PLS ∩ PPAD and describe recent advances in the study of this class.

Reducing TARSKI onto EOPL | 3

In this chapter we will discuss how one can reduce TARSKI onto **EOPL**. We will do this by introducing and using SPERNER's Lemma, an indroducing a new problem: TARSKI*. We will show also that an instance of the original TARSKI problem can be solved using a polynomial number of calls to TARSKI*. Then we will give a proof that TARSKI* is in **PPAD**, an argue that the same construction also shows that TARSKI* is **EOPL**. We will conclude by arguing that $P^{\mathsf{EOPL}} = \mathsf{EOPL}$.

3.1 Introducing TARSKI*

We want to start by introducing a new problem, TARSKI*. This problem can be thought of, as a subproblem in order to solve TARSKI, as we will argue. A standard strategy to solve TARSKI is to use a *divide and conquer* strategy, as for instance in [9]. We want to construct a problem, which allows us to divide the TARSKI problem into two smaller problems, where solving the smaller of the two leads to a solution. We propose the following problem:

[9]: Etessami et al. (2020), *Tarski's Theorem, Supermodular Games, and the Complexity of Equilibria*

For the sake of generality and for the proofs in the following we introduce the problem on a general latice $L = N_1 \times \cdots \times N_d$, such that $N_i \leq 2^n$.

TARSKI*
Input: A boolean circuit $f : L \to L$.
Output: Either:
(T*1) Two points $x, y \in L$ such that $\|x - y\|_\infty \leq 1$, $x \leq f(x)$ and $y \geq f(y)$, or
(T*2) A violation of mononicity: Two points $x, y \in L$ such that $x \leq y$ and $f(x) \nleq f(y)$.

We now want to show that TARSKI* can be seen as a subproblem of TARSKI.

Claim 3.1
An instance of TARSKI can be solved using $\mathcal{O}(d \cdot n)$ calls of TARSKI* and up to $\mathcal{O}(d)$ additional steps.

Proof. We will show that we can use a single call of TARSKI* to either find a violation of monoticity, a fixpoint, or an instance of

TARSKI which has at most half as many points, and must contain a solution. We proceed by case distingtion:

Case 1: If $x = y$, then x is a fixpoint, and we are done.

Case 2: If either $f(x) = x$ or $f(y) = y$, then we are done, because we have found a fixpoint.

Case 3.1: If $x < y$ and $f(x) \not\leq f(y)$, we have a violation of monoticity, which solves the given TARSKI instance.

Case 3.2: If $x < y$ and $f(x) \leq f(y)$, we claim that we can solve the TARSKI instance in $\mathcal{O}\left(\|x-y\|_1\right)$ additional function calls. Notice that x and y can be thought of as being vertices on the same hypercube of length 1, because $\|x-y\|_\infty \leq 1$. Now notice that because $f(x) > x$ (if not see case 2), there is at least one dimension $i \in \{1,\dots,d\}$ such that $f(x)[i] > x[i]$. Also notice that in this dimension i if $f(y)[i] < y[i]$, then because $|x[i]-y[i]| \leq \|x[i]-y[i]\|_\infty \leq 1$, we would have a violation of the monoticity of f in this dimension. Therefore we must have $f(y)[i] = y[i]$. The same argument shows that if in any dimension j $f(y)[j] < y[j]$, then $f(x)[j] = x[j]$. Therefore we know that because there must be at least one such dimension i and j we have:

$$\|f(x)-f(y)\|_\infty \leq \|x-y\|_\infty \leq 1 \quad \text{and} \quad \|f(x)-f(y)\|_1 \leq \|x-y\|_1 - 2$$

Hence we can now repeat the same argumentation with $f(x)$ and $f(y)$, and we can do this at most $\mathcal{O}\left(\|x-y\|_1\right)$ times, until we find a violation of monoticity or a fixpoint. Because $\|x-y\|_1 \leq d$, this will take at most $\mathcal{O}(d)$ additional steps.

Case 4: If $x \not\leq y$, then we can partition the set of lattice points into two sets S_x and S_y, as follows:

$$S_x = \{z \in L : z \geq x\} \quad \text{and} \quad S_y = \{z \in L : z \leq y\}.$$

These two sets are disjoint: if there was a $z \in S_x \cap S_y$, then $x \leq z \leq y$, which would imply $x \leq y$, which is a contradiction. We will show that S_x must contain a solution to the TARSKI instance. If for some $z \in S_x$ we have $f(z) \notin S_x$, then we have $f(z) \not\leq f(x)$, because or else we have $f(z) \leq f(x) \leq x$, which contradicts the assumption, hence x, z are two points withnessing a violation of monoticity of f. This means that S_x froms a new valid instance of TARSKI. By the same argumentation S_y also forms a valid instance of TARSKI and hence it suffices to solve the smaller of the two instances. In particular because they are disjoint, one of the instances S_x or S_y contains less than half of the lattice points of L, and hence we can solve the instance in $\mathcal{O}\left(\log 2^{dn}\right) = \mathcal{O}(d \cdot n)$ calls of TARSKI*. □

We do not actually need to check these points, it suffice to have the algorithm stop if at any point it notices that f leaves S_x.

3.2 SPERNER'S Lemma

Of course the previous discussions assume, that TARSKI* is a total problem, that is, that every instance has a solution, which we will prove in this section, in order to conclude that that TARSKI* is in **TNFP**. In order to do this we introduce SPERNER'S Lemma, as introduced and proven in [11]. A more modern presentation and proof can be found in [12].

[11]: Sperner (1928), *Neuer beweis für die invarianz der dimensionszahl und des gebietes*

[12]: Aigner et al. (2018), *Proofs from THE BOOK*

Theorem 3.1 — Sperner's Lemma.
TODO.

For us to be able to use SPERNER'S Lemma, on our TARSKI* instances, we need to define a coloring of the vertices of L. We propose the following coloring $l : L \to \{0, \dots, d\}$:

A vertex colored 0 indicates that the function points "forwards" in all dimensions, a vertex colored i for $i \geq 1$ indicates that the function points "backwards" in at least the i-th dimension.

$$l(x) = \begin{cases} 0 & \text{if } x \leq f(x) \\ 1 & \text{else if } x[1] > f(x)[1] \\ & \vdots \\ d & \text{else if } x[d] > f(x)[d] \end{cases}$$

We are now ready to use SPERNER'S Lemma to show that TARSKI* is a total search problem.

Claim 3.2
Finding a cell with all colors, yields a solution to TARSKI*, in $\mathcal{O}(d)$ steps.

Proof. Assume we have found a simplex, with vertices colored $0, \cdot, d$. Let us denote x_i the vertex colored i, for $i \in \{0, \dots, d\}$. Notice that all of these vertices are by construction contained in some cell (hypercube of length 1), let $\mathbf{0}$ be the smallest vertex of this hypercube and $\mathbf{1}$ the largest. In particular this means that for all i we have:

$$\mathbf{0} \leq x_i \leq \mathbf{1} \quad \text{and} \quad f(x_i)[i] < x_i[i] \quad \text{for } i > 0$$

We now proceed by case distinction:

Case 1: If x_0 is a fixed point, then $x = y = x_0$ is a solution to TARSKI*.

Case 2: If $x_0 \neq f(x_0)$ and $x_0 = \mathbf{0}$. Then there is an i such that $f(x_0)[i] > x_0[i]$, which means that $f(x_0[i]) - x_0[i] \geq 1$. At the same time we must have $f(x_i)[i] < x_i[i]$ and $x_0[i] - x_i[i]$ because $x_0 = \mathbf{0}$, and hence $x_i[i] - f(x_i)[i] \geq 1$. Now we get:

$$f(x_0)[i] - f(x_i)[i] = \underbrace{f(x_0)[i] - x_0[i]}_{\geq 1} + \underbrace{x_0[i] - x_i[i]}_{\geq 0} + \underbrace{x_i[i] - f(x_i)[i]}_{\geq 1}$$
$$f(x_0)[i] - f(x_i)[i] \geq 2$$

This implies that $f(x_0) \not\leq f(x_i)$, and hence x_0, x_i are two points witnessing a violation of monoticity of f, which form a solution to TARSKI*.

Case 3: If $x_0 \neq f(x_0)$ and $x_0 \neq \mathbf{0}$. We claim that either $f(\mathbf{0}) \leq \mathbf{0}$, or we have a violation of monoticity. Assume for the sake of contradiction that there is an i such that $f(\mathbf{0})[i] > \mathbf{0}[i]$. Then we must have $f(x_i)[i] < x_i[i]$ hence we get: $f(\mathbf{0})[i] \not\leq f(x_i)[i]$, which is a violation of monoticity. This means that either we can return $y = x_0$ and $x = \mathbf{0}$ as a solution to TARSKI*, or x_i and $\mathbf{0}$ as a violation of mononicity. □

Now to show that we have a total search problem we only need to show that such a cell colored $0, \dots, d$ always exists. In order to do this we need to use a variation of SPERNER's Lemma, which was introduced by Papadimitriou in [13].

[13]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

Theorem 3.2 — SPERNER's Lemma on Hypercubes.
TODO.

This will allow us to show that TARSKI* is a total search problem, and hence in **TNFP**:

Theorem 3.3
TARSKI* is in **TNFP**.

3.3 Reducing TARSKI* onto PPAD

We now want to show that TARSKI* is in **PPAD**. In order to do this we will use the problem SPERNER, as introduced in [13].

[13]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

SPERNER
Input: A coloring $c : L \to \{0, \dots, d\}$ of the vertices of L.
Output: A cell $C \subset L$ such that for all $i \in \{0, \dots, d\}$ there is a vertex $x \in C$ such that $c(x) = i$.

Papadimitriou showed that SPERNER is **PPAD**-complete, in [13], which means that it suffices to reduce TARSKI* onto SPERNER, in order to show that TARSKI* is in **PPAD**.

Theorem 3.4
TARSKI* is in PPAD.

Proof. TODO □

Appendix

Bibliography

[1] Christos H. Papadimitriou. 'On the complexity of the parity argument and other inefficient proofs of existence'. In: *Journal of Computer and System Sciences* 48.3 (June 1994), pp. 498–532. DOI: 10.1016/S0022-0000(05)80063-7. (Visited on 03/05/2024) (cited on pages 3, 4).

[2] Nimrod Megiddo and Christos H. Papadimitriou. 'On total functions, existence theorems and computational complexity'. In: *Theoretical Computer Science* 81.2 (Apr. 1991), pp. 317–324. DOI: 10.1016/0304-3975(91)90200-L. (Visited on 03/05/2024) (cited on page 3).

[3] Christos H. Papadimitriou. *Computational complexity*. Reading (Mass): Addison-Wesley, 1994 (cited on page 4).

[4] David S. Johnson, Christos H. Papadimitriou, and Mihalis Yannakakis. 'How easy is local search?' In: *Journal of Computer and System Sciences* 37.1 (Aug. 1988), pp. 79–100. DOI: 10.1016/0022-0000(88)90046-3. (Visited on 03/06/2024) (cited on page 4).

[5] Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitriou. 'The Complexity of Computing a Nash Equilibrium'. In: *SIAM Journal on Computing* 39.1 (Jan. 2009), pp. 195–259. DOI: 10.1137/070699652. (Visited on 03/11/2024) (cited on page 4).

[6] Paul W. Goldberg and Alexandros Hollender. 'The Hairy Ball problem is PPAD-complete'. In: *Journal of Computer and System Sciences* 122 (Dec. 2021), pp. 34–62. DOI: 10.1016/j.jcss.2021.05.004. (Visited on 03/10/2024) (cited on page 5).

[7] John Fearnley et al. *End of Potential Line*. Apr. 18, 2018. URL: http://arxiv.org/abs/1804.03450 (visited on 03/02/2024) (cited on page 5).

[8] Alfred Tarski. 'A lattice-theoretical fixpoint theorem and its applications.' In: *Pacific Journal of Mathematics* 5.2 (Jan. 1, 1955), pp. 285–309 (cited on pages 6, 7).

[9] Kousha Etessami et al. 'Tarski's Theorem, Supermodular Games, and the Complexity of Equilibria'. In: (2020). In collab. with Thomas Vidick. Artwork Size: 19 pages, 651206 bytes ISBN: 9783959771344 Medium: application/pdf Publisher: Schloss Dagstuhl – Leibniz-Zentrum für Informatik Version Number: 1.0, 19 pages, 651206 bytes. DOI: 10.4230/LIPICS.ITCS.2020.18. (Visited on 02/24/2024) (cited on pages 7, 8).

[10] Samuel R. Buss and Alan S. Johnson. 'Propositional proofs and reductions between NP search problems'. In: *Annals of Pure and Applied Logic* 163.9 (Sept. 2012), pp. 1163–1182. DOI: 10.1016/j.apal.2012.01.015. (Visited on 02/24/2024) (cited on page 7).

[11] E. Sperner. 'Neuer beweis für die invarianz der dimensionszahl und des gebietes'. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 6.1 (Dec. 1928), pp. 265–272. DOI: 10.1007/BF02940617. (Visited on 04/18/2024) (cited on page 10).

[12] Martin Aigner and Günter M. Ziegler. *Proofs from THE BOOK*. 6th ed. 2018. Berlin, Heidelberg: Springer Berlin Heidelberg : Imprint: Springer, 2018. 1 p. (cited on page 10).

[13] Christos H. Papadimitriou. 'On the complexity of the parity argument and other inefficient proofs of existence'. In: *Journal of Computer and System Sciences* 48.3 (June 1994), pp. 498–532. DOI: 10.1016/S0022-0000(05)80063-7. (Visited on 03/22/2024) (cited on page 11).

Alphabetical Index