

# The Complexity of Finding Tarski Fixed Points

Master Thesis

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# Abstract

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## 1.1 Total Search Problems

The study of computational complexity is central to computer science, its primary goal is to establish lower bounds on the complexity of various problems. Specifically, complexity theory attempts to prove that certain problems cannot be solved faster than a given time, as a function of the input size. This endeavor has proven to be particularly challenging for many problems where there is a significant gap between the best-known lower bounds and the upper bounds determined by existing algorithms.

A fundamental tool in complexity theory is the concept of reduction, which makes it possible to compare the difficulty of two problems. We say that a problem  $P_1$  is reducible to another problem  $P_2$  if  $P_1$  can be solved efficiently by solving  $P_2$ . This concept underlies the classification of problems into complexity classes-groups of problems that are mutually reducible.

Traditionally, complexity theory has focused on decision problems, which involve determining whether a given object has a given property. Examples include determining whether a graph contains a  $k$  clique, or whether a number is prime. These problems typically require a decision about whether an object belongs to a set of objects—a language—defined by a particular property.

However, real-world challenges often extend beyond simple decision making into the realm of search problems. In practical scenarios, the existence of a solution is typically assumed, and the task is not just to verify its existence, but to compute the solution itself. For example, instead of just detecting the presence of a  $k$  clique in a graph, one may need to explicitly find this clique or confirm its absence. Similarly, instead of just recognizing a number as prime, one might need to determine its prime factors.

Within this broader category of search problems lies a special subclass known as *total search problems*. These are characterized by the guaranteed existence of a solution, often proved by mathematical theorems. A notable example within this subclass is the problem of identifying a sink in a directed acyclic graph, where the existence of such a sink is guaranteed.

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Here, *efficiently* generally means in polynomial time. We will define this and related concepts more strictly later.

## 1.2 The TFNP landscape

The class of **TNFP**, is the pendant of **NP**, in the sense that it is the class of all total search problems, where a solution can be checked for validity in polynomial time. The study of this complexity class has been a active research subject in the last years and has given rise to many interesting results.

TODO: Write some intuition about the subclasses.

## 1.3 The TARSKI problem

The main problem we study in this thesis is the TARSKI problem. The name sake of the TARSKI problem is TARSKI'S FIXED POINT THEOREM, which states that every monotone function on a complete lattice has a fixed point[1]. The TARSKI problem is the problem of finding such a fixed point for a given function  $f$  on a complete lattice  $L$ , or to find a violation of monotonicity of this function. By Tarski's theorem, this problem is guaranteed to have a solution, and hence is a total search problem.

[1]: Tarski (1955), *A lattice-theoretical fixpoint theorem and its applications*.

## 1.4 Current algorithms for solving TARSKI

TODO: Write this section.

## 1.5 Location of TARSKI in TNFP

It is known that the TARSKI problem lies in **PPAD** and in **PLS**. A very recent breakthrough has shown that the class  $\mathbf{PPAD} \cap \mathbf{PLS} = \mathbf{EOPL}$  [2]. This result immediately implies that the TARSKI problem is in **EOPL**, which in turn means that there must be a reduction from TARSKI to **EOPL**-complete problems, in particular to the **ENDOFPOTENTIALLINE** problem.

[2]: Goos et al. (2022), *Further Collapses in TFNP*

## 1.6 Thesis Outline

TODO: Write this section.



The aim of this chapter is to introduce the complexity class **TNFP**, and some of its subclasses, in particular **PPAD**, **PLS** and **EOPL**. We will also introduce the **TARSKI** problem.

## 2.1 Total search problems

The study of complexity classes originally works with so-called *decision-problems*, which are the question of deciding on the membership in a set — also called a *language*. Now while these problems are interesting, real world questions or problem often ask for an explicit answer. For instance while deciding if a function has a global minimum is a decision problem, we are interested in actually finding this minimum, which is not a decision problem.

This is where so called *search problems* come into play:

### Definition 2.1 — Search Problem.

A *search problem* is given by a relation  $R \subset \{0,1\}^* \times \{0,1\}^*$ . For a given *instance*  $I \in \{0,1\}^*$  the computational problem, to find a *solution*  $s \in \{0,1\}^*$ , that satisfies:  $(I, s) \in R$  or output “No” if no such  $s$  exists.

Now of course we can view these search problems as decision problems by looking at the corresponding decision problem given by the language:

$$\mathcal{L}_R = \{I \in \{0,1\}^* \mid \exists s \in \{0,1\}^* : (I, s) \in R\}$$

We can then ask the classical complexity questions about these search problems, i.e. whether these search problems are in **P**? **NP**? whether they are **NP**-Hard? One easily observes that search problems are always at least as hard as just deciding whether a solution exist. This is because solving a search problem also solves the underlying decision problem. This leads to the natural question: what if we remove the underlying decision problem? This can be done by guaranteeing that “No” is never a solution. We call these problems where every instance admits a solution *total search problems*.

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Notable such problems include deciding on whether a boolean formula can be satisfied or if a  $k$ -Clique exist in a given graph.

Even though as we will see it can be transformed into one

The “No” case can be encoded as some special binary string.

**Definition 2.2 — Total search problems.**

A *total search problem* is a search problem given by a relation  $R \subset \{0,1\}^* \times \{0,1\}^*$ , such that for every given instance  $I \in \{0,1\}^*$  there is a solution  $s \in \{0,1\}^*$ , that satisfies:  $(I, s) \in R$ .

The complexity class **TNFP** as introduced in [3] is simply the class of all total search problems that lie in **NP**. Examples of **TNFP** problems are:

- **FACTORING**, the problem of finding the prime factors of a number,
- **NASH**, the problem of finding a nash equilibrium in a bimatrix game,
- **MINIMIZE**, the problem of finding the global minimum of a convex function.

Similarly to decision problem we can also define reduction inside **TNFP**.

**Definition 2.3 — Reduction.**

For two problem  $R, S \in \mathbf{TNFP}$ , we say that  $R$  *reduces* (many to one) to  $S$  if there exist polynomial time computable functions  $f : \{0,1\}^* \rightarrow \{0,1\}^*$  and  $g : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$  such that for  $I, s \in \{0,1\}^*$ : if  $(f(I), s) \in S$  then  $(I, g(I, s)) \in R$ . This means that if  $s$  is a solution to an instance  $f(I)$  in  $S$ , we can compute  $g(I, s)$  a solution to an instance  $I$  in  $R$ .

[3]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

This means that **TNFP** can be seen as an intermediate class between **P** and **NP**, containing all search problems where a solution is guaranteed to exist, and where one can efficiently check the feasibility of a candidate solution.

Saying *one can reduce  $R$  onto  $S$*  can be understood as saying *if one can solve  $S$  efficiently then I can solve  $R$  efficiently*.

## 2.2 An excursion into Binary Circuits

TODO

## 2.3 Subclasses of TNFP

Because the existence of complete **FNP**-Problems in **TNFP** would imply  $\mathbf{NP} = \mathbf{coNP}$ , as described in [4]. Because this is a very unexpected outcome we cannot expect to find complete problems in **TNFP**. This means that we should use other tools to study the structure of **TNFP**.

One of the challenges is that **TNFP** is a so-called *semantic* class. By semantic class we mean a class for which it is difficult to check if that Turing Machine defines a language in this class. A *syntactic* class is a class for which it is easy to check that the accepted language of a Turing Machine indeed belongs to the class. These

[4]: Megiddo et al. (1991), *On total functions, existence theorems and computational complexity*

Examples of syntactic classes include **P** and **NP**.

terms are defined in more detail in [5]. Hence we want to study syntactic subclasses of **TNFP**. One of the proposed methods [3] is to categorize total search problems with respect to the existence results which allow them to be *total*. This is what leads to the complexity classes we will discuss next.

[5]: Papadimitriou (1994), *Computational complexity*

[3]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

### 2.3.1 Polynomial Local Search (PLS)

The existence results which gives rise to **PLS** is “every directed acyclic graph has a sink”. We can then construct the class **PLS** by defining it as all problems which reduce to finding the sink of a directed acyclic graph (DAG).

Formally we first define the problem **LOCALOPT** as in [6]:

#### **LOCALOPT**

**Input:** Two binary circuits  $P, S : [2^n] \rightarrow [2^n]$ .

**Output:** A vertex  $v \in [2^n]$  such that  $P(S(v)) \geq P(v)$ .

[6]: Johnson et al. (1988), *How easy is local search?*

$S$  can be seen as a proposed successor, and  $P$  as a potential. The goal is to find a local minima  $v$  of the potential.

One might ask why this is equivalent to finding the sink of a DAG? The circuit  $S$  defines a directed graph, which might contain cycles. Only keeping the edge on which the potential decreases (strictly) leads to a DAG, with as sinks exactly the  $v$  such that  $P(S(v)) \geq P(v)$ . Now we can define **PLS**:

#### **Definition 2.4 — Polynomial Local Search (PLS).**

The class **PLS** is the set of all **TNFP** problems that reduce to **LOCALOPT**.

One of the reasons we think that studying very “easy” problems such as **PLS** is that we strongly believe that there is no clever way of solving these problems without actually walking through the graph. Hence if we have a graph of exponentially large size it seems very unlikely that one can find an efficient way of solving the problem. Hence all problems in **PLS** can be thought of as including the fundamental difficulty of not being able to do better than to walk along some graph.

By “easy” we mean that the problem can be solved by simply walking through the graph, and checking whether every vertex is a local minima.

### 2.3.2 Polynomial Parity Argument on Directed Graphs (PPAD)

Now we want to discuss the complexity class **PPAD**, introduced by Papadimitriou as one of the first syntactic subclasses of **TNFP** in [3]. The existence result giving rise to this class is that “If a directed graph has an unbalanced vertex, then it has at least one

[3]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

other unbalanced vertex". **PPAD** can be defined using the problem **END-OF-LINE** as introduced in [7].

#### END-OF-LINE

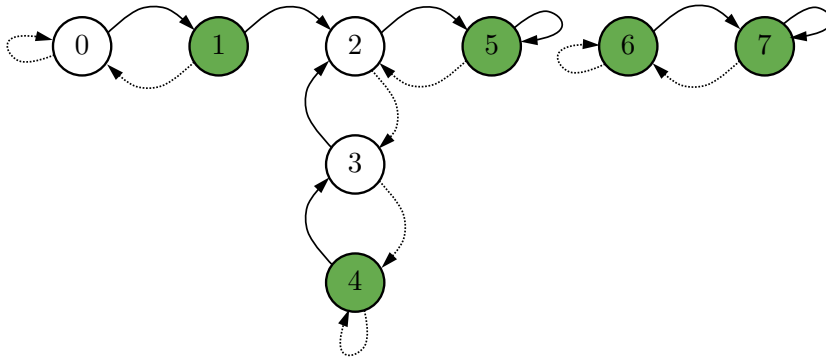
**Input:** Boolean circuit  $S, P : \{0,1\}^n \rightarrow \{0,1\}^n$  such that  $P(0^n) = 0^n \neq S(0^n)$  ( $0^n$  is a source.)

**Output:** An  $x \in \{0,1\}^n$  such that either:

- $P(S(x)) \neq x$  ( $x$  is a sink) or
- $S(P(x)) \neq x \neq 0^n$  ( $x$  is a non non-standard source)

[7]: Daskalakis et al. (2009), *The Complexity of Computing a Nash Equilibrium*

Here  $S$  can be thought of giving the successor of a vertex, and  $P$  as giving the predecessor of a vertex.



**Figure 2.1:** Example of an **END-OF-LINE** Problem with  $n = 3$  (8 vertices). The circuit  $S$  is represented by solid lines and the circuit  $P$  by dashed lines. The solutions are the sinks  $x = 5$ ,  $x = 7$  and  $x = 1$ , as well as the sources  $x = 4$  and  $x = 6$ .

These boolean circuits represent a directed graph with maximal in and out degree 1, by having an edge from  $x$  to  $y$  if and only if  $S(x) = y$  and  $P(y) = x$ . The goal is to find a sink of the graph, or another source. It can be shown that the general case of finding a second imbalanced vertex in a directed graph (a problem called **IMBALANCE**) can be reduced to **END-OF-LINE** [8]. Now we can define the complexity class **PPAD** as follows:

#### Definition 2.5 — PPAD.

The class **PPAD** is the set of all **TNFP** problems that reduce to **END-OF-LINE**.

Notice that **END-OF-LINE** allows cycles, and that these do not induce solutions.

[8]: Goldberg et al. (2021), *The Hairy Ball problem is PPAD-complete*

### 2.3.3 End of Potential Line (EOPL)

Next we want to discuss the complexity class **EOPL** as introduced in [9]. The existence results which gives rise to **EOPL** is that “in a directed acyclic graph, there must be at least two unbalanced vertices”. Similarly to **PLS** acyclicity will be enforced using a potential.

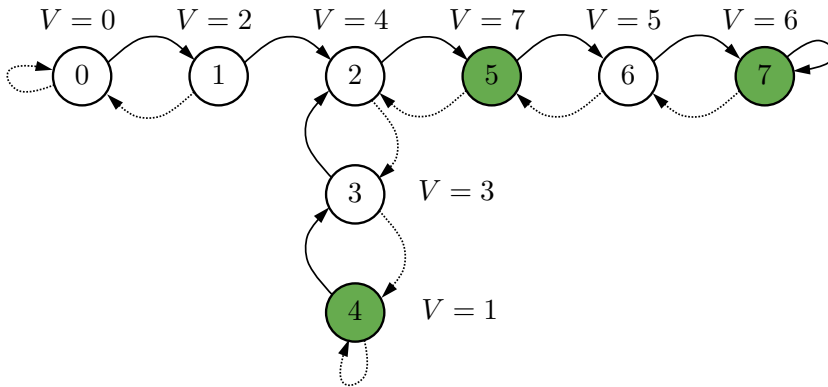
[9]: Fearnley et al. (2018), *End of Potential Line*

**END OF POTENTIAL LINE**

**Input:** Two boolean circuits  $S, P : \{0, 1\}^n \rightarrow \{0, 1\}^n$ , and a boolean circuit  $V : \{0, 1\}^n \rightarrow [2^n - 1]$ , such that  $0^n$  is a source, (i.e.  $P(0^n) = 0^n \neq S(0^n)$ ).

**Output:** An  $x \in \{0, 1\}^n$  such that either:

- $P(S(x)) \neq x$  ( $x$  is a sink)
- $S(P(x)) \neq x \neq 0^n$  ( $x$  is a *non-standard source*)
- $S(x) \neq x$ ,  $P(S(x)) = x$  and  $V(S(x)) \leq V(x)$  (violation of the monotonicity of the potential)



Here  $S$  can be thought of giving the successor of a vertex, and  $P$  as giving the predecessor of a vertex.  $V$  can be thought of as a potential which is supposed to be monotonously increasing along the line.

**Figure 2.2:** Example of an EOPL Problem with  $n = 3$  (8 vertices). The circuit  $S$  is represented by solid lines and the circuit  $P$  by dashed lines. The solutions are the sink  $x = 7$ , the violation of potential at  $x = 5$  and the non-standard source  $x = 4$ .

$S$  and  $P$  can be thought of as representing a directed line. Finding another source (a non-standard source), is a violation, as a directed line only has one source. The potential serves a guarantee of acyclicity. Now we can define the complexity class **EOPL**.

**Definition 2.6 — EOPL.**

The class **EOPL** is the set of all **TNFP** problems that reduce to **END OF POTENTIAL LINE**.

## 2.4 The TARSKI Problem

Next we want to introduce the TARSKI Problem. Before we do this we recall that there is a partial order on the  $d$  dimensional lattice  $[N]^d$ , given by  $x \leq y$  iff  $x_i \leq y_i$  for all  $i \in \{1, \dots, d\}$ . The name originates from TARSKI's fixed point Theorem as introduced in [1] which we remind the reader of below:

**Theorem 2.7 — Tarski's fixed point Theorem.**

Let  $f : [N]^d \rightarrow [N]^d$  a function on the  $d$ -dimensional lattice. If  $f$  is monotonous (with respect to the previously discussed partial order), then  $f$  has a fixed point, i.e. there is an  $x \in [N]^d$  such that  $f(x) = x$ .

[1]: Tarski (1955), *A lattice-theoretical fixpoint theorem and its applications*.

This theorem is also known as the Knaster–Tarski Theorem in the literature.

A proof of this theorem can be found in the previously mentioned work [1]. Without surprise the TARSKI problem as defined in [10], is now to find such a fixed-point. Formally we define the problem as follows:

**TARSKI**

**Input:** A boolean circuit  $f : [N]^d \rightarrow [N]^d$ .

**Output:** Either:

- An  $x \in [N]^d$  such that  $f(x) = x$  (fixed point) or
- $x, y \in [N]^d$  such that  $x \leq y$  and  $f(x) \not\leq f(y)$  (violation of monotonicity).

[10]: Etessami et al. (2020), *Tarski's Theorem, Supermodular Games, and the Complexity of Equilibria*

This is of course a total search problem, as there will always either be a fixed point, or a point violating monotonicity. We now want to summarize where TARSKI lies inside of **TNFP**. It has been shown in [10] that TARSKI lies in both **PLS** and  $\mathbf{P}^{\mathbf{PPAD}}$ . Previous work [11], showed that many-to-one reductions and Turing-reduction onto **PPAD** are equivalent. In particular this means that  $\mathbf{P}^{\mathbf{PPAD}} = \mathbf{PPAD}$ , and that TARSKI also lies in **PPAD**.

[11]: Buss et al. (2012), *Propositional proofs and reductions between NP search problems*

## 2.5 Structure of $\mathbf{PLS} \cap \mathbf{PPAD}$

Now that we have established that TARSKI lies inside  $\mathbf{PLS} \cap \mathbf{PPAD}$ , we want to discuss the structure of  $\mathbf{PLS} \cap \mathbf{PPAD}$  and describe recent advances in the study of this class.

# Reducing TARSKI to PPAD

In this chapter, we explore the membership of TARSKI to the complexity class **PPAD**. We begin by presenting an established proof of the reduction of this problem to BROUWER [10], focusing on a high-level overview. Subsequently, we introduce a novel problem, TARSKI\*, which facilitates a divide and conquer approach to solving TARSKI by leveraging the structure of the function  $f$ . This new formulation allows us to provide an alternative proof of TARSKI's membership in PPAD using *Sperner's Lemma* instead of the traditional *Brouwer's Fixed Point Theorem*. This approach not only simplifies the proof but also sets the stage for further reduction of TARSKI\* to EOPL in the subsequent chapter.

## 3.1 Presentation of the known reduction of TARSKI to PPAD

We want to give a high level presentation of the proof of TARSKI membership in **PPAD** from [10], which will help us motivate the introduction of TARSKI\* and the subsequent use of *Sperner's Lemma*. The proof given by Etessami et al. relies on *Brouwer's fixed point theorem*, which we introduce below.

### Theorem 3.1 — Brouwer's fixed point theorem.

Let  $K \subset \mathbb{R}^d$  be a compact, convex set. Then every continuous function  $f : K \rightarrow K$  has a fixed point  $x^* \in K$ , i.e.  $f(x^*) = x^*$ .

The original proof can be found in [12], a simpler proof relying on SPERNER'S LEMMA can be found in [13]. This theorem gives rise to a total search problem which we call BROUWER:

#### BROUWER

**Input:** A continuous function  $f : K \rightarrow K$ .

**Output:** A fixed point  $x^* \in K$  such that  $f(x^*) = x^*$ .

The problem BROUWER was first introduced and shown to be **PPAD**-complete in [14]. This means that it suffices to reduce TARSKI to BROUWER in order to show that TARSKI is in **PPAD**. We will actually reduce TARSKI to at most polynomially many instances of BROUWER, which will allow us to show that TARSKI is in  $\mathbf{P}^{\mathbf{PPAD}}$ . This means that we will show a Turing reduction of TARSKI to BROUWER, which suffice as **PPAD** is closed under Turing reductions [11].

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[10]: Etessami et al. (2020), *Tarski's Theorem, Supermodular Games, and the Complexity of Equilibria*

[10]: Etessami et al. (2020), *Tarski's Theorem, Supermodular Games, and the Complexity of Equilibria*

[12]: Brouwer (1911), *Über Abbildung von Mannigfaltigkeiten*

[13]: Aigner et al. (2018), *Proofs from THE BOOK*

We leave out the technical detail of how this function is given using boolean circuits, and how precise the output needs to be, as it is not relevant for this high level presentation.

[14]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

[11]: Buss et al. (2012), *Propositional proofs and reductions between NP search problems*

The idea of the the reduction is to extend the discrete function  $f$ , to a function  $\tilde{f} : [0, 2^n - 1]^d \rightarrow [0, 2^n - 1]^d$ , such that  $\tilde{f}$  interpolates the lattice function  $f$ , is continuous and piecewise linear between lattice points, and hence continuous. This can be achieved using a simplicial decomposition of each cell of the lattice. Now we have an instance of BROUWER, and hence we can find a fixed point  $x^*$  of  $\tilde{f}$ . Of course, this fixed point does not need to be *integral*. The key insight is that we can use this fixed point to reduce the search area for a integral fixed point by at least half, or find a violation of monotonicity. In particular, either there is a fixed point in both  $\{x \in [2^n - 1]^d : x \geq x^*\}$  and  $\{x \in [2^n - 1]^d : x \leq x^*\}$ , or there is a violation of monotonicity in the cell containing  $x^*$ . We can repeat this procedure always halving the search area, which allows us to solve a TARSKI instance using at most  $\mathcal{O}(d \cdot n)$  calls to BROUWER.

We call a point *integral* if it belongs to the original lattice.

### 3.2 Introducing TARSKI\*

In the previous section, we have seen that TARSKI can be reduced to a polynomial number of BROUWER instances. We would like to study a single such reduction, in order to give an alternative proof that TARSKI is in PPAD. In order to do this, we introduce a new problem, TARSKI\*. This problem can be thought of as a subproblem towards solving TARSKI. A standard strategy to solve TARSKI is to use a *divide and conquer* strategy, as for instance used in [10]. We want to construct a problem, which allows us to divide the TARSKI problem into two smaller problems, where solving the smaller of the two leads to a solution.

[10]: Etessami et al. (2020), *Tarski's Theorem, Supermodular Games, and the Complexity of Equilibria*

For the sake of generality and for the proofs in the following we introduce the problem on the integer lattice  $L = N_1 \times \dots \times N_d$ , such that  $N_i \leq 2^n$  for all  $i \in \{1, \dots, d\}$ . We propose the following problem:

#### TARSKI\*

**Input:** A boolean circuit  $f : L \rightarrow L$ .

**Output:** Either:

- (T\*1) Two points  $x, y \in L$  such that  $\|x - y\|_\infty \leq 1$ ,  $x \leq f(x)$  and  $y \geq f(y)$ , or
- (T\*2) A violation of monotonicity: Two points  $x, y \in L$  such that  $x \leq y$  and  $f(x) \not\leq f(y)$ .

We now want to show that TARSKI\* can be seen as a subproblem of TARSKI.



**Claim 3.2**

An instance of TARSKI can be solved using  $\mathcal{O}(d \cdot n)$  calls to TARSKI\* and up to  $\mathcal{O}(d)$  additional function evaluations.

*Proof.* We will show that we can use a single call of TARSKI\* to either find a violation of monotonicity, a fixpoint, or an instance of TARSKI which has at most half as many points, and must contain a solution. Let  $x, y$  be the two points outputted by a Turing machine solving TARSKI\* on a function  $f$ . We proceed by case distinction:

**Case 1:** If either  $f(x) = x$  or  $f(y) = y$ , then we are done, because we have found a fixpoint.

**Case 2.1:** If  $x < y$  and  $f(x) \not\leq f(y)$ , we have a violation of monotonicity, which solves the given TARSKI instance.

**Case 2.2:** If  $x < y$  and  $f(x) \leq f(y)$ , we claim that we can solve the TARSKI instance in  $\mathcal{O}(\|x - y\|_1)$  additional function calls. Notice that we have  $\|x - y\|_\infty \leq 1$ . Now notice that because  $f(x) > x$  (if not see case 1), there is at least one dimension  $i \in \{1, \dots, d\}$  such that  $f(x)[i] > x[i]$ . Also notice that in this dimension  $i$  if  $f(y)[i] < y[i]$ , then because  $|x[i] - y[i]| \leq \|x - y\|_\infty \leq 1$ , we would have a violation of the monotonicity of  $f$  in this dimension. Therefore we must have  $f(y)[i] = y[i]$ . The same argument shows that if in any dimension  $j$  we have  $f(y)[j] < y[j]$ , then  $f(x)[j] = x[j]$ . Therefore we know that because there must be at least one such dimension  $i$  and  $j$  we have:

$$\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty \leq 1 \text{ and } \|f(x) - f(y)\|_1 \leq \|x - y\|_1 - 2$$

Hence we can now repeat the same argumentation with  $f(x)$  and  $f(y)$ , and we can do this at most  $\mathcal{O}(\|x - y\|_1)$  times, until we find a violation of monotonicity or a fixpoint. Because  $\|x - y\|_1 \leq d$ , this will take at most  $\mathcal{O}(d)$  additional steps.

**Case 3:** If  $x \not\leq y$ , then we can partition the set of lattice points into two sets  $S_x$  and  $S_y$ , as follows:

$$S_x = \{z \in L : z \geq x\} \quad \text{and} \quad S_y = \{z \in L : z \leq y\}.$$

These two sets are disjoint: if there was a  $z \in S_x \cap S_y$ , then  $x \leq z \leq y$ , which would imply  $x \leq y$ , which is a contradiction. We will show that  $S_x$  must contain a solution to the TARSKI instance. If for some  $z \in S_x$  we have  $f(z) \notin S_x$ , then we have  $f(z) \not\leq f(x)$ , which means that  $z$  and  $x$  form a violation of monotonicity. This means that  $S_x$  forms a new valid instance of TARSKI. By the same argumentation  $S_y$  also forms a valid instance of TARSKI and hence it suffices to recursively solve the smaller of the two instances. In particular because they are disjoint, one of the instances  $S_x$  or  $S_y$  contains less than half of the lattice points of  $L$ , and hence we can solve the instance in  $\mathcal{O}(\log 2^{dn}) = \mathcal{O}(d \cdot n)$  calls of TARSKI\*.

□

Now that we know that TARSKI\* is a good stepping stone towards solving TARSKI, we want to investigate why TARSKI\* lies in PPAD.

### 3.3 Sperner's Lemma

The preceding discussion hinges on the assumption that TARSKI\* is a total problem, implying that every instance of the problem is guaranteed a solution. In this section, we will substantiate this claim, establishing TARSKI\*'s classification within **TNFP**. Rather than employing *Brouwer's fixed point Theorem* — a cornerstone of continuous topology — we pivot to its discrete analogue, *Sperner's Lemma*, a foundational result in combinatorial topology. This approach is particularly apt for two main reasons:

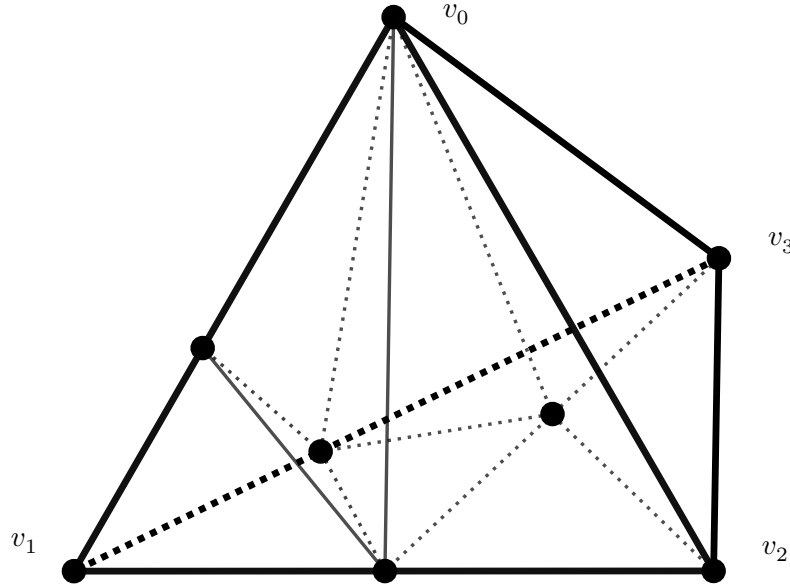
- We are working on a discrete lattice, and hence it seems more natural to use a discrete tool.
- Papadimitriou proved that BROUWER is PPAD-complete by reducing BROUWER to SPERNER [14]. Hence by reducing to BROUWER, we introduce continuity into the problem, which is not necessary, as it gets removed again behind the scenes.

[14]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

Our goal is to apply *Sperner's Lemma* on the integer lattice. This is not directly possible, as *Sperner's Lemma* is defined on a simplicial decomposition of a simplex. Hence we will first introduce *Sperner's Lemma* for simplices, and then show how it can be adapted to work on an integer lattice.

### 3.3.1 Sperner's Lemma for Simplices

Before we introduce the Lemma itself, we want to define the setting of the result. We consider a  $d$ -dimensional simplex<sup>1</sup> with vertices  $v_0, v_1, \dots, v_d$ . We now consider a *simplicial subdivision* of this simplex. This means that we partition the simplex into smaller simplices. We give an example of such a partition in Figure 3.1 in the 3-dimensional case.



1: By  $d$  dimensional simplex we mean the convex Hull of these  $d+1$  points in  $\mathbb{R}^d$

**Figure 3.1:** Setup for SPERNER'S LEMMA in the 3-dimensional case. The large simplex spanned by  $v_0, v_1, v_2, v_3$  is subdivided into smaller simplices.

Now we introduce a coloring  $c$  of the vertices of this subdivision with colors  $\{0, 1, \dots, d\}$ . We want to enforce that the vertices  $v_i$  of the large simplex are colored with color  $i$ , and that the vertices on a subsimplex  $\{v_{i_0}, \dots, v_{i_k}\}$  are colored with colors  $i_0, \dots, i_k$ . We give an example of such a coloring in 2 dimensions in Figure 3.2.

We now introduce Sperner's Lemma, which was first proven in [15], and for which a more modern proof can be found in [13].

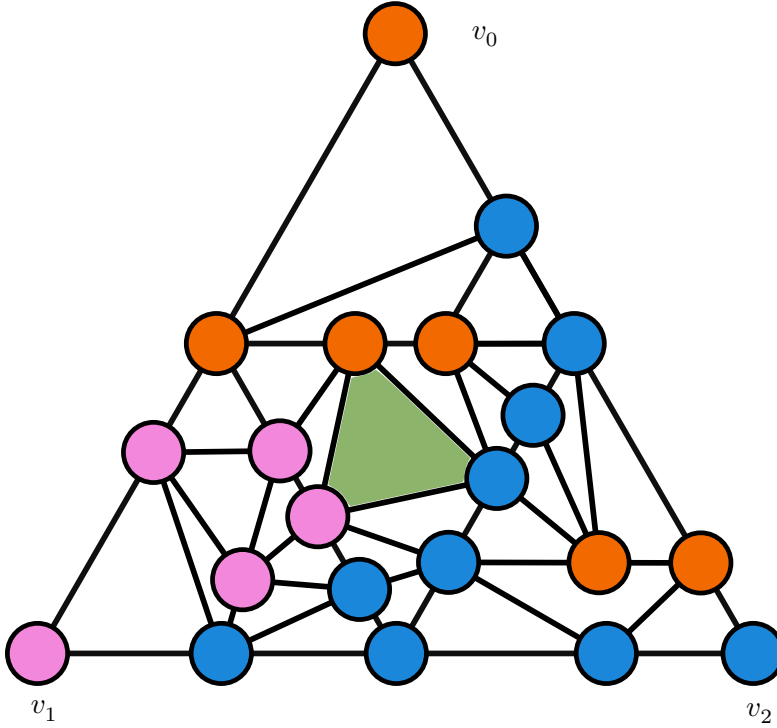
#### Theorem 3.3 — Sperner's Lemma.

Suppose that a  $d$ -dimensional simplex with vertices  $v_0, \dots, v_d$  is subdivided into smaller simplices. Now color every vertex with a color  $\{0, \dots, d\}$  such that  $v_i$  is colored  $i$ , and the vertices on a subsimplex  $\{v_{i_0}, \dots, v_{i_k}\}$  are colored with colors  $i_0, \dots, i_k$ . Then there is a subsimplex, with vertices of every color.

[15]: Sperner (1928), *Neuer beweis für die invarianz der dimension-zahl und des gebietes*

[13]: Aigner et al. (2018), *Proofs from THE BOOK*

We give an example of a 2-dimensional simplex, which is subdivided into smaller simplices, and colored according to *Sperner's Lemma* in Figure 3.2.



**Figure 3.2:** Example of SPERNER'S LEMMA in the two dimensional case, with 3 colors: orange (0), purple (1) and blue (2). The subsimplex spanned by  $v_0$  and  $v_1$  only contains blue and purple vertices, the subsimplex spanned by  $v_1$  and  $v_2$  contains only purple and blue vertices and the subsimplex spanned by  $v_0$  and  $v_2$  contains only orange and blue vertices. *Sperner's Lemma* implies that there must be a subsimplex (colored in green), which contains all colors.

### 3.3.2 Sperner's Lemma for an integer lattice

Now that we have introduced *Sperner's Lemma* for a integer lattice. The motivation is to be able to find a region of a colored lattice which contains all colors under certain conditions. Instead of looking for a subsimplex, we will look for a *cell*<sup>2</sup> of the lattice, which contains all colors.

In order to do this we proceed as follows. We take the  $d$ -dimensional lattice  $L = [N_1] \times \dots \times [N_d]$ , we subdivide each cell into simplices<sup>3</sup>. We set  $v_0 = (0, \dots, 0)$ ,  $v_1 = (N_1 - 1, 0, \dots, 0)$ ,  $\dots$ ,  $v_d = (0, \dots, 0, N_d - 1)$ . We give an example of such a subdivision in the 3-dimensional case in Figure 3.3. Notice that we can deform the lattice and we obtain an equivalent simplex, and a simplicial decomposition of this simplex.

This means that under the appropriate conditions — which we will detail next — we can apply *Sperner's Lemma* to the lattice. Assume that we color all vertices of the lattice with colors  $\{0, \dots, d\}$ , such that  $v_i$  is colored  $i$ , and every vertex  $x$  with  $x[i] = 0$ , is *not* colored  $i$  for  $i \in \{1, \dots, d\}$ . Then we can apply *Sperner's Lemma* to this simplicial decomposition of the lattice, and we will find a simplex which contains all colors. Of course because every subsimplex is included in exactly one cell by construction, there must be a cell which contains all colors. This motivates the definition of the total problem SPERNER which was introduced and shown to be PPAD-complete in [14]. We introduce the problem for

2: By cell we mean a unit hypercube of the integer lattice

3: How this is done is not relevant in this chapter but will be discussed in the next chapter.

[14]: Papadimitriou (1994), *On the complexity of the parity argument and other inefficient proofs of existence*

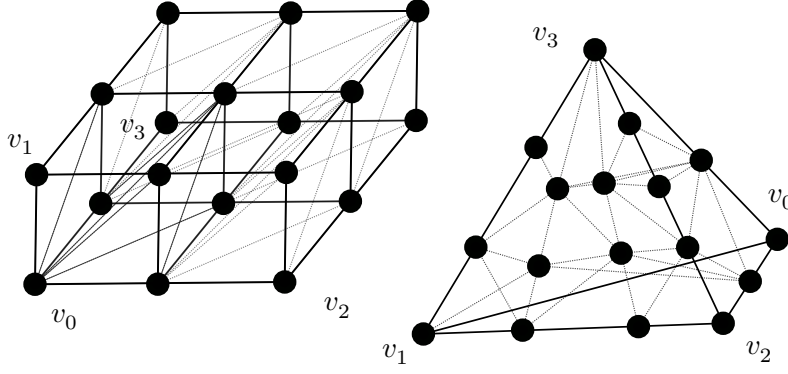


Figure 3.3: Example of the simplicial decomposition of a lattice in the 3 dimensional case on the left, and the equivalent simplicial decomposition on the right of a simplex  $v_0, v_1, v_2, v_3$ .

a general lattice  $L = N_1 \times \dots \times N_d$ , such that  $N_i \leq 2^n$ .

#### SPERNER

**Input:** A coloring  $c : L \rightarrow \{0, \dots, d\}$  of the vertices of  $L$ , such that for every  $i \in \{0, \dots, d\}$  the the vertices  $\{x \in L : x[i] = 0\}$  are not colored  $i$ .

**Output:** A cell  $C$  such that for all  $i \in \{0, \dots, d\}$  there is a vertex  $x \in C$  such that  $c(x) = i$ .

Next we will use this problem to show that TARSKI\* is a total search problem, and hence lies in PPAD.

### 3.4 Reducing TARSKI\* to SPERNER

For us to be able to use SPERNER's Lemma on our TARSKI\* instances, we need to define a coloring of the vertices of  $L$ . We propose the following coloring  $c : L \rightarrow \{0, \dots, d\}$ :

$$c(x) = \begin{cases} 0 & \text{if } x \leq f(x) \\ 1 & \text{else if } x[1] > f(x)[1] \\ \vdots & \\ d & \text{else if } x[d] > f(x)[d] \end{cases}$$

A vertex colored 0 indicates that the function points *weakly forwards* in all dimensions, a vertex colored  $i$  for  $i \geq 1$  indicates that the function points *backwards* in at least the  $i$ -th dimension.

We now need two results. First we need to show that a cell with all colors always exists, which will allow us to show that TARSKI\* is a total search problem. Second we need to show that finding a cell with all colors, yields a solution to TARSKI\*, in polynomial time.

#### Claim 3.4

For any TARSKI\* instance, with vertices colored as above, there is always a cell with all colors.

*Proof.* This claim follows directly from SPERNER'S Lemma, and the coloring we have defined. There can never be a vertex colored  $i$  with  $x[i] = 0$ , because this would imply that  $f(x)[i] < x[i]$ , which is a contradiction to the construction of the function. Hence by dividing each cell of the lattice into simplices, we can apply SPERNER'S Lemma to show that a cell with all colors always exists. The vertices we use as the vertices of the large simplex are  $\{(0, \dots, 0), (2^n - 1, 0, \dots, 0), \dots, (0, \dots, 2^n - 1)\}$ .  $\square$

### Claim 3.5

Finding a cell with all colors yields a solution to TARSKI\*, in  $\mathcal{O}(d)$  additional steps.

*Proof.* Assume we have found a simplex, with vertices colored  $0, \dots, d$ . Let us denote  $x_i$  the vertex colored  $i$ , for  $i \in \{0, \dots, d\}$ . Notice that all of these vertices are by construction contained in some cell (hypercube of length 1), let  $\mathbf{0}$  be the smallest vertex of this hypercube and  $\mathbf{1}$  the largest. In particular this means that for all  $i$  we have:

$$\mathbf{0} \leq x_i \leq \mathbf{1} \quad \text{and} \quad f(x_i)[i] < x_i[i] \quad \text{for } i > 0$$

We now proceed by case distinction:

**Case 1:** If  $x_0$  is a fixed point, then  $x = y = x_0$  is a solution to TARSKI\*.

**Case 2:** If  $x_0 \neq f(x_0)$  and  $x_0 = \mathbf{0}$ . Then there is an  $i$  such that  $f(x_0)[i] > x_0[i]$ , which means that  $f(x_0)[i] - x_0[i] \geq 1$ . At the same time we must have  $f(x_i)[i] < x_i[i]$  and  $x_0[i] - x_i[i] \leq 0$  because  $x_0 = \mathbf{0}$ , and hence  $x_i[i] - f(x_i)[i] \geq 1$ . Now we get:

$$\begin{aligned} f(x_0)[i] - f(x_i)[i] &= \underbrace{f(x_0)[i] - x_0[i]}_{\geq 1} + \underbrace{x_0[i] - x_i[i]}_{\geq 0} + \underbrace{x_i[i] - f(x_i)[i]}_{\geq 1} \\ f(x_0)[i] - f(x_i)[i] &\geq 2 \end{aligned}$$

This implies that  $f(x_0) \not\leq f(x_i)$ , and hence  $x_0, x_i$  are two points witnessing a violation of monotonicity of  $f$ , which form a solution to TARSKI\*.

**Case 3:** If  $x_0 \neq f(x_0)$  and  $x_0 \neq \mathbf{0}$ . We claim that either  $f(\mathbf{0}) \leq \mathbf{0}$ , or we have a violation of monotonicity. Assume for the sake of contradiction that there is an  $i$  such that  $f(\mathbf{0})[i] > \mathbf{0}[i]$ . Then we must have  $f(x_i)[i] < x_i[i]$  hence we get:  $f(\mathbf{0})[i] \not\leq f(x_i)[i]$ , which is a violation of monotonicity. This means that either we can return  $y = x_0$  and  $x = \mathbf{0}$  as a solution to TARSKI\*, or  $x_i$  and  $\mathbf{0}$  as a violation of monotonicity.

This shows that we can solve a TARSKI\* instance in  $\mathcal{O}(d)$  additional steps.  $\square$

This shows that TARSKI\* is a total search problem, and can be reduced to SPERNER. Hence TARSKI\* lies in **PPAD**, and by using that  $\mathbf{P}^{\mathbf{PPAD}} = \mathbf{PPAD}$  we have shown that TARSKI lies in **PPAD**, without relying on BROUWER.

In the previous chapter, we exhibited how one can demonstrate the membership of Tarski in **PPAD** through a reduction to Sperner. We now demonstrate that the same approach yields a reduction to **ENDOFPOTENTIALLINE**, which lies within **EOPL**. This will necessitate a more meticulous examination of the structure of a Tarski instance and the induced colouring of the lattice points. In order to achieve our objective, we must first construct a specific simplicial decomposition of the lattice. This will be done with the intention of obtaining certain useful properties. Ultimately, our goal is to demonstrate that for a monotone Tarski instance, the associated **EndOfLine** instance does not contain any cycles. This will prove sufficient to establish a reduction to **ENDOFPOTENTIALLINE**.

## 4.1 Choosing a simplicial decomposition of the lattice — Freudenthal’s Simplicial Decomposition

In the previous chapter, we left the choice of a specific simplicial decomposition of the lattice open, as it did not contribute to our reduction. In this chapter, we aim to be more precise in our approach by selecting a specific simplicial decomposition that will enable us to derive structural results. We begin by outlining the desired properties of our simplicial decomposition. The most fundamental property is that every simplex of the decomposition must be contained within a single cell of the lattice. This implies that we can limit our inquiry to the identification of a simplicial decomposition of a single  $d$ -dimensional hypercube of side-length 1. Additionally, it is important to note that our objective does not entail the introduction of any new vertices; instead, we seek a decomposition of the hypercube that can be expressed as a set of subsets of the hypercube’s vertices. Finally, we wish for the vertices of a given simplex be totally ordered with respect to the partial order defined in Section 2.4. This will allow us to argue that two vertices, inside a given simplex, are always comparable, and thus their images through  $f$  must also be comparable, which will be useful.

Such a decomposition exists, and is known in the literature as *Freudenthal’s simplicial decomposition* [16]. We will introduce it

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[16]: Freudenthal (1942), *Simplizialzerlegungen von Beschränkter Flachheit*



in a combinatorial way here, and refer the reader to the original paper for a geometric construction of the same decomposition.

**Definition 4.1 — Freudenthal's Simplicial Decomposition.**

Consider a unit hypercube  $[0, 1]^d$  in  $\mathbb{R}^d$  and consider  $S_d$  the group of all permutations of the dimensions of the hypercube  $\{1, \dots, d\}$ . For every permutation  $\pi \in S_d$ , define the simplex  $S_\pi$  as the convex hull of the vertices:

$$\begin{aligned} v_0 &= (0, 0, \dots, 0) \\ v_1 &= v_0 + e_{\pi(1)} \\ v_2 &= v_1 + e_{\pi(2)} \\ &\vdots \\ v_d &= v_{d-1} + e_{\pi(d)} = (1, 1, \dots, 1) \end{aligned}$$

Here we will use the notation  $e_i$  to denote the  $i$ -th unit vector in  $\mathbb{R}^d$ .

The set of such simplexes  $\mathcal{S} = \{S_\pi : \pi \in S_d\}$  is Freudenthal's simplicial decomposition of the hypercube  $[0, 1]^d$ .

We want to begin by arguing why this decomposition is well-defined. We begin by showing that every point of the hypercube is contained in at least one simplex of  $\mathcal{S}$ .

**Lemma 4.2**

Let  $x = (x[1], \dots, x[d]) \in [0, 1]^d$ , let  $\pi \in S^d$  be the permutation such that  $x[\pi(1)] \leq x[\pi(2)] \leq \dots \leq x[\pi(d)]$ . Then  $x \in S_\pi$ .

*Proof.* We want to show that  $x$  is a convex combination of the vertices of  $S_\pi$ . We define the following sequence of real numbers:

$$\begin{aligned} \lambda_0 &= x[\pi(1)] \\ \lambda_1 &= x[\pi(2)] - x[\pi(1)] \\ \lambda_2 &= x[\pi(3)] - x[\pi(2)] \\ &\vdots \\ \lambda_{d-1} &= x[\pi(d)] - x[\pi(d-1)] \\ \lambda_d &= 1 - x[\pi(d)] \end{aligned}$$

Notice that we have  $\lambda_i \geq 0$  for all  $i$  and  $\sum_{i=0}^d \lambda_i = 1$ , by telescoping the sum. We can now write  $x$  as a convex combination

of the vertices of  $S_\pi$  as follows by noticing that  $v_i = \sum_{j=0}^i e_{\pi(j)}$ :

$$\begin{aligned} \sum_{i=0}^d \lambda_i v_i &= \sum_{i=0}^d \lambda_i \left( \sum_{j=0}^i e_{\pi(j)} \right) = \sum_{i=0}^d \sum_{j=1}^i \lambda_i e_{\pi(j)} \\ &= \sum_{j=1}^d \sum_{i=0}^j \lambda_i e_{\pi(j)} = \sum_{j=1}^d e_{\pi(j)} \sum_{i=0}^j \lambda_i = \sum_{j=1}^d e_{\pi(j)} x[\pi(j)] = x \end{aligned}$$

This shows that  $x$  is a convex combination of the vertices of  $S_\pi$ , and thus  $x \in S_\pi$ .  $\square$

Next we want to discuss why this really forms a partition of the hypercube. Of course a given point  $x$  can be contained in multiple simplexes, but we want to show that this does not happen apart from on the boundary of the simplices.

#### Lemma 4.3

Let  $S_\pi \in \mathcal{S}$  be a simplex. Then the *interior* of  $S_\pi$  is:

$$\text{int}(S_\pi) = \{x \in [0, 1]^d : 0 < x[\pi(1)] < x[\pi(2)] < \dots < x[\pi(d)] < 1\}$$

*Proof.* The same proof as for lemma 4.2, holds with the added constraint that all  $\lambda_i > 0$ , this then shows that these points are in the interior of the simplex.  $\square$

These two lemma's together show that we have a well-defined simplicial decomposition of the hypercube. We can now use this decomposition to prove some structural results about the lattice points of a TARSKI instance. We start by showing that this simplicial decomposition has the desired properties.

#### Lemma 4.4

Let  $S_\pi \in \mathcal{S}$  be a simplex. Then the vertices of  $S_\pi$  are totally ordered with respect to the partial order defined in Section 2.4. In particular we claim that:

$$v_0 < v_1 < v_2 < \dots < v_d$$

*Proof.* Because this relation is transitive it suffice to show that  $v_i < v_{i+1}$  for all  $i \in \{0, \dots, d-1\}$ . This follows immediately from the construction of the  $v_i$  as we have  $v_i[j] = v_{i+1}[j]$  for all  $j \neq \pi(i+1)$  and  $v_i[\pi(i+1)] = v_{i+1}[\pi(i+1)] - 1$ .  $\square$

This directly implies the following corollary.

**Corollary 4.5**

For two vertices  $x, y$  of any simplex  $S \in \mathcal{S}$ , if for any  $i \in \{1, \dots, d\}$  we have  $x[i] < y[i]$ , then  $x < y$ . In particular  $x \not< y$  is equivalent to  $x > y$ .

Notice that this is not the case for any two points in the hypercube, as the partial order is not a total order. This is why choosing a simplicial decomposition with this property will be crucial in the following sections. Next we want to introduce a new notation which will allow us describe these simplices more succinctly. Assume that a permutation  $\pi$  of the dimensions, induces a simplex  $S_\pi$ , with vertices  $v_0, \dots, v_d$ , as defined in Definition 4.1. Then we will denote the  $d$ -dimensional simplex  $S_\pi$  as:

$$v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} v_2 \xrightarrow{\pi(3)} \dots \xrightarrow{\pi(d)} v_d$$

This notation means that we obtain  $v_i$  by moving by one unit-length in the direction  $\pi(i)$  from  $v_{i-1}$ . We already briefly discussed how the faces of a given simplex are given. We will also describe how to describe these faces in our notation. We will denote the face of  $S_\pi$  obtained by removing the vertex  $v_i$  as:

$$v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \dots v_{i-1} \xrightarrow{\pi(i), \pi(i+1)} v_{i+1} \xrightarrow{\pi(i+2)} \dots \xrightarrow{\pi(d)} v_d$$

We can remark the following about the faces of a simplex.

**Remark 4.6**

For a given  $d - 1$  dimensional simplex  $F$  in  $\mathcal{S}$  we have that:

(1) If  $F$  is of the form:

$$F : v_0 \xrightarrow{\pi(1)} \cdots v_{i-1} \xrightarrow{\pi(i), \pi(i+1)} v_{i+1} \xrightarrow{\pi(i+2)} \cdots \xrightarrow{\pi(d)} v_d$$

Then  $F$  is a face of exactly two simplices  $S_1$  and  $S_2$ :

$$S_1 : v_0 \xrightarrow{\pi(1)} \cdots v_{i-1} \xrightarrow{\pi(i+1)} w_i \xrightarrow{\pi(i)} v_i \xrightarrow{\pi(i+2)} \cdots \xrightarrow{\pi(d)} v_d$$

$$S_2 : v_0 \xrightarrow{\pi(1)} \cdots v_{i-1} \xrightarrow{\pi(i)} w'_i \xrightarrow{\pi(i+1)} v_i \xrightarrow{\pi(i+2)} \cdots \xrightarrow{\pi(d)} v_d$$

(2) If  $F$  is of the form:

$$F : v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \cdots \xrightarrow{\pi(d-1)} v_{d-1}$$

Then  $F$  is a face of exactly two simplices  $S_1$  and  $S_2$ :

$$S_1 : v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \cdots \xrightarrow{\pi(d-1)} v_{d-1} \xrightarrow{\pi(d)} w_d$$

$$S_2 : w_0 \xrightarrow{\pi(d)} v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \cdots \xrightarrow{\pi(d-1)} v_{d-1}$$

Notice that the case (1) is the case where the face is inside the cell, and the case (2) is the case where the face is on the border of the cell.

We want to discuss what simplices of the decomposition neighbor each other. We claim that a given simplex has  $d - 1$  neighboring simplices inside a given cell, and two neighboring simplices in neighboring cells. More precisely we have the following lemma.

**Lemma 4.7 — Neighboring Simplices.**

Let  $S_\pi \in \mathcal{S}$  be a simplex:

$$v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} v_2 \xrightarrow{\pi(3)} \dots \xrightarrow{\pi(d)} v_d$$

Then the following simplices are neighbors of  $S_\pi$ :

- ▶  $v_0 \xrightarrow{\pi(2)} v_1 \xrightarrow{\pi(1)} \dots v_{i-1} \xrightarrow{\pi(i+1)} w_i \xrightarrow{\pi(i+1)} v_i \xrightarrow{\pi(i+2)} \dots \xrightarrow{\pi(d)} v_d$ ,  
for all  $i \in \{1, \dots, d-1\}$ , where  $w_i$  is the vertex obtained by moving one unit in the direction  $\pi(i+1)$  from  $v_{i-1}$ .
- ▶  $w_d \xrightarrow{\pi(d)} v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \dots \xrightarrow{\pi(d-1)} v_{d-1}$ , where  $w_d$  is the vertex obtained by moving one unit in the direction  $-\pi(d)$  from  $v_0$ .
- ▶  $v_2 \xrightarrow{\pi(2)} \dots \xrightarrow{\pi(d-1)} v_{d-1} \xrightarrow{\pi(d)} v_d \xrightarrow{\pi(1)} w_1$ , where  $w_1$  is the vertex obtained by moving one unit in the direction  $\pi(1)$  from  $v_d$ .

*Proof.* The proof follows directly by enumerating the faces of  $S_\pi$  and using remark 4.6.  $\square$

## 4.2 Orientation of a the simplicial decomposition

In this section we want to discuss how to orient the simplicial decomposition of the lattice, we defined in the previous section. This will be important as we will argue in the next section, that the existence of a cycle would contradict the orientation of the simplicial decomposition. We will start by defining what we mean by an orientation of a simplex and then discuss how to extend this to a general simplicial complex.

### 4.2.1 Orientation of a simplex

**Definition 4.8 — Orientation of a simplex.**

An *orientation* of a simplex  $S$  spanned by the vertices  $v_0, \dots, v_d$  is a choice of a permutation of the vertices  $[v_{\pi(0)}, \dots, v_{\pi(d)}]$ .

Notice that this leaves us with  $d!$  possible orientations of a simplex. Our notion of orientability should only lead to two possible classes of orientations, as an orientation of a 1-simplex is simply a choice of direction, and an orientation of a 2-simplex is a choice of a cyclic order of the vertices. Hence we want to define when two orientations are equivalent.

**Definition 4.9 — Equivalent orientations.**

Two orientations  $\pi$  and  $\sigma$  of a simplex  $S$  are *equivalent* if they differ by an even permutation. That is if  $\sigma = \pi \circ \tau$  for some permutation  $\tau$  with an even number of inversions.

In particular we give a more explicit definition of the equivalence of orientations of a 2-simplex, by relying on a total order  $\preceq$  of the vertices. We then get the following useful lemma:

**Lemma 4.10**

Two orientations  $\sigma, \tau$  of a simplex  $S$  are equivalent if and only if  $\text{sgn}(\sigma) = \text{sgn}(\tau)$ , with respect to the total order  $\preceq$ .

For a lattice this can be achieved by defining  $\preceq$  to be the lexicographic order of the vertices.

We would like to define the *opposite orientation* of a simplex, which should be an orientation which has the opposite sign with respect to the total order  $\preceq$ . This can be achieved by setting:

$$-[v_0, v_1, v_2, \dots, v_d] = [v_1, v_0, v_2, \dots, v_d]$$

We then have that the opposite orientation is not equivalent to the original orientation. This way we have a representative of both equivalence classes.

This means that we now have two equivalence classes of orientations for any simplex. We want to discuss how an orientation of a simplex extends to the faces of this simplex next. Notice that the faces of a simplex are themselves simplices, and thus have an orientation. Let  $[v_0, \dots, v_d]$  be an orientation of a simplex  $S$ . Now notice that every face can be obtained by removing one of the vertices  $v_j$  of  $S$ . Hence for every face  $F$ , the permutation  $[v_0, \dots, \hat{v}_j, \dots, v_d]$  is an orientation of  $F$ . But the orientation  $-[v_0, \dots, \hat{v}_j, \dots, v_d]$  is also a valid orientation of  $F$ . For reasons which will become apparent later we define the induced orientation of a face as follows:

We use the notation  $\hat{v}_j$  to denote that  $v_j$  is missing.

**Definition 4.11 — Induced orientation of a face.**

Let  $\sigma = [v_0, \dots, v_d]$  be an orientation of a simplex  $S$ . The *induced orientation* of a face  $F$  of  $S$ , which is obtained by removing the vertex  $v_j$  from the vertex, is the orientation:

$$\sigma_j = (-1)^j \cdot [v_0, \dots, \hat{v}_j, \dots, v_d]$$

We claim that the induced orientations of faces, yields a consistent orientation of the simplex, that is that for every  $d-2$ -simplex  $E$  which is a face of two  $d-1$ -simplices  $S_1$  and  $S_2$ , the induced orientations of  $E$  in  $S_1$  and  $S_2$  are opposite.

**Claim 4.12**

Let  $F_1$  and  $F_2$  be two  $d - 1$ -simplices in  $S$  which share a common face  $E$ . Then the induced orientations of  $E$  in  $S_1$  and  $S_2$  are opposite.

*Proof.* Let  $[v_0, \dots, v_d]$  be an orientation of  $S$ . The face  $E$  is obtained by removing two vertices  $v_i, v_j$  from  $S$ . Without loss of generality assume that  $F_1$  is obtained by removing  $v_i$  from  $S$  and  $F_2$  is obtained by removing  $v_j$  from  $S$ . Then the induced orientations  $S_1$  and  $S_2$  are:

$$S_1 : (-1)^i \cdot [v_0, \dots, \hat{v}_i, \dots, v_d]$$

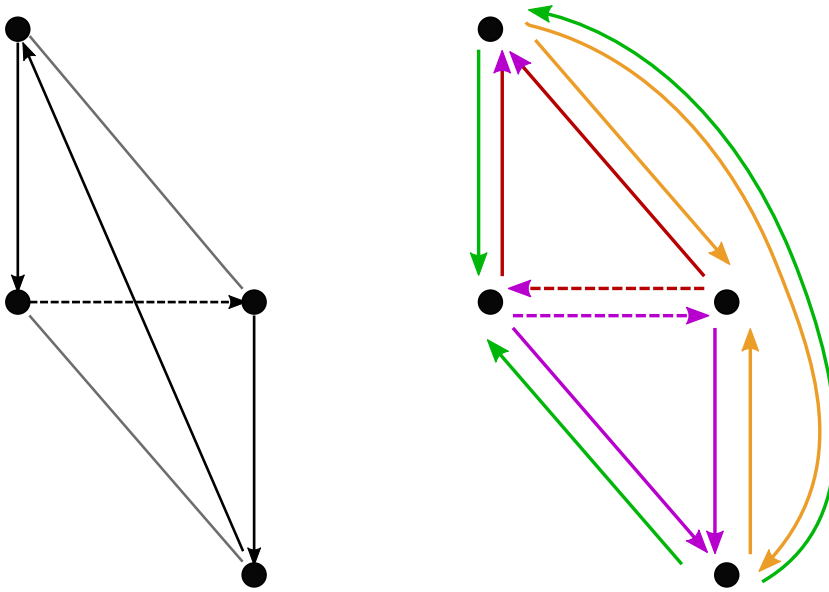
$$S_2 : (-1)^j \cdot [v_0, \dots, \hat{v}_j, \dots, v_d]$$

Now without loss of generality assume that  $i < j$ , then we have that the induced orientations of  $E$  in  $S_1$  and  $S_2$  are:

$$E \text{ in } S_1 : (-1)^i \cdot (-1)^{j-1} \cdot [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{d-1}]$$

$$E \text{ in } S_2 : (-1)^j \cdot (-1)^i \cdot [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{d-1}]$$

This shows that the induced orientations of  $E$  in  $S_1$  and  $S_2$  are opposite.  $\square$



**Figure 4.1:** Example of the orientation of a 3-simplex on the left, and the induced orientation of the faces on the right.

We give an example of the orientation of a 3-simplex and its faces in Figure 4.1. We can now discuss how we can extend this notion to a general simplicial complex.

### 4.2.2 Orientation of a simplicial complex

A simplicial complex can be thought of as a collection of simplices which are be glued together on their face. Our goal is now to extend this notion of orientation to these simplicial complexes. Formally we define a simplicial complex as follows [17]:

[17]: Munkres (2018), *Elements of algebraic topology*

**Definition 4.13 — Simplicial complex.**

A *simplicial complex*  $\mathcal{K}$  in  $\mathbb{R}^d$  is a collection of simplices such that:

- (1) Every face of a simplex in  $\mathcal{K}$  is also in  $\mathcal{K}$ .
- (2) The intersection of any two simplices in  $\mathcal{K}$  is a face of both simplices.

Notice that the lattice points which we are interested in, together with Freudenthal's simplicial decomposition of each cell, form a simplicial complex. We now want to define an orientation of a simplicial complex. Of course such an orientation relies on an orientation of each simplex, and we want to make sure that these orientations are in some sence "compatible" on the faces of the simplicial complex. We will define this notion in the following definition.

**Definition 4.14 — Orientation of a simplicial complex.**

An *orientation* of a simplicial complex  $\mathcal{K}$  is a choice of an orientation of every  $d$ -simplex in  $\mathcal{K}$ , such that for every intersection of two simplices  $S_1, S_2 \in \mathcal{K}$ , the induced orientation of the face  $F = S_1 \cap S_2$  in  $S_1$  and  $S_2$  are opposite.

If such an orientation exists, we say that the simplicial complex is *orientable*.

We now claim that the simplicial complex formed by the lattice points and Freudenthal's simplicial decomposition is orientable. This will be crucial in the next section, where we will argue that the existence of a cycle in the ENDOFLINE instance would contradict the orientation of the simplicial complex. In particular this shows that a Mobius Strip or the higher dimensional equivalents do not exist in our simplicial complex.

**Claim 4.15**

The simplicial complex formed by the lattice points and Freudenthal's simplicial decomposition is orientable.

*Proof.* We will give an orientation of every  $d$  simplex, and then show that the induced orientation of the faces of the simplicial complex are opposite. Let  $\pi \in S^d$  be a permutation of the dimensions, and  $v_0 \in L$  a vertex of the lattice. We then obtain



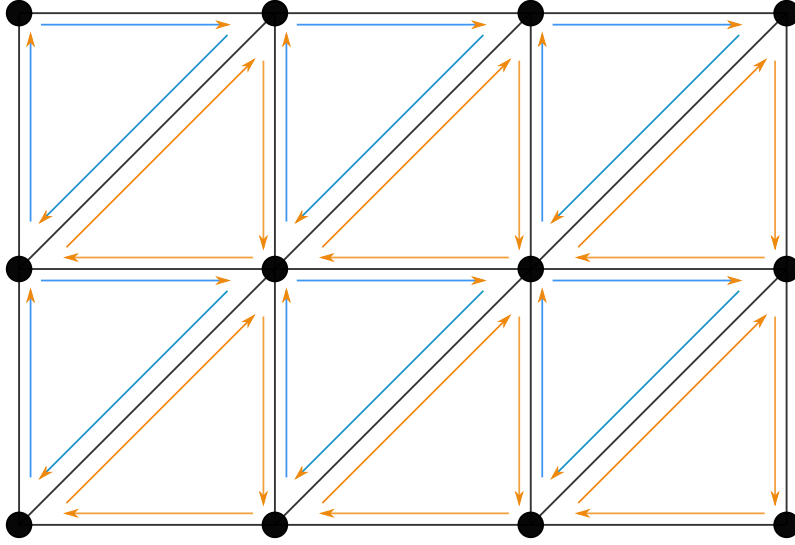


Figure 4.2: Example of the orientation of a Freundenthal's simplicial complex in 2 dimensions.

a simplex  $S_\pi \in \mathcal{S}$  as previously described:

$$v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} v_2 \xrightarrow{\pi(3)} \dots \xrightarrow{\pi(d)} v_d$$

We now orient  $S_\pi$  using the permutation:

$$\sigma = \text{sgn}(\pi) \cdot [v_0, \dots, v_d]$$

First we notice that for all  $d - 2$  simplices, two neighboring  $d - 1$ -simplices are contained in exactly one  $d$  simplex of the decomposition, and hence the orientation is consistent, as discussed in claim 4.12.

Now let us look at a common face  $F$  of two  $d$ -simplices  $S_1$  and  $S_2$ . We proceed by case distinction:

**Case 1:** Assume that  $S_1$  and  $S_2$  are in the same cell, then  $F$  is of the form:

$$F : v_0 \xrightarrow{\pi(1)} \dots v_{i-1} \xrightarrow{\pi(i), \pi(i+1)} v_{i+1} \xrightarrow{\pi(i+1)} v_i \xrightarrow{\pi(i+2)} \dots \xrightarrow{\pi(d)} v_d$$

And we have that  $S_1$  and  $S_2$  are of the form:

$$S_1 : v_0 \xrightarrow{\pi(1)} \dots v_{i-1} \xrightarrow{\pi(i+1)} w_i \xrightarrow{\pi(i)} v_i \xrightarrow{\pi(i+2)} \dots \xrightarrow{\pi(d)} v_d$$

$$S_2 : v_0 \xrightarrow{\pi(1)} \dots v_{i-1} \xrightarrow{\pi(i)} w'_i \xrightarrow{\pi(i+1)} v_i \xrightarrow{\pi(i+2)} \dots \xrightarrow{\pi(d)} v_d$$

We immediately notice that  $\text{sgn}(S_1) = -\text{sgn}(S_2)$ . We remove a vertex  $w_i, w'_i$  of the same rank in  $S_1$  and  $S_2$  in order to obtain  $F$ . Hence the induced orientation of  $F$  in  $S_1$  and  $S_2$  are opposite.

By abuse of notation we will denote by  $\text{sgn}(S_1)$  the sign of the permutation inducing  $S_1$ .

**Case 2:** Next assume that  $S_1$  and  $S_2$  are in neighboring cells, then as dicussed in Remark 4.6  $F$  is of the form:

$$F : v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \dots \xrightarrow{\pi(d-1)} v_{d-1}$$

And we have that  $S_1$  and  $S_2$  are of the form:

$$\begin{aligned} S_1 : v_0 &\xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \dots \xrightarrow{\pi(d-1)} v_{d-1} \xrightarrow{\pi(d)} w_d \\ S_2 : w_0 &\xrightarrow{\pi(d)} v_0 \xrightarrow{\pi(1)} v_1 \xrightarrow{\pi(2)} \dots \xrightarrow{\pi(d-1)} v_{d-1} \end{aligned}$$

We once again must proceed by case distinction.

**Case 2.1:** If  $d$  is even, then:  $\text{sgn}(S_1) = -\text{sgn}(S_2)$ , and we remove a vertex of rank  $d$  in  $S_1$  and of rank 0 in  $S_2$  to obtain  $F$ . We have  $(-1)^d = (-1)^0 = 1$  and hence the induced orientation of  $F$  in  $S_1$  and  $S_2$  are opposite.

**Case 2.2:** If  $d$  is odd, then  $\text{sgn}(S_1) = \text{sgn}(S_2)$ , and we remove a vertex of rank  $d$  in  $S_1$  and of rank 0 in  $S_2$  to obtain  $F$ . We have  $(-1)^d = -1$  and  $(-1)^0 = 1$  and hence the induced orientation of  $F$  in  $S_1$  and  $S_2$  are opposite.

This shows that the simplicial complex formed by the lattice points and Freudenthal's simplicial decomposition is orientable.  $\square$

We give an example of such an orientation in Figure 4.2. We have now introduced the necessary tools to argue that a certain type of cycle cannot exist as we will argue in the following.

### 4.3 Sequences of simplices

In this section we want to introduce and study *sequences of simplices*. We will show that they have some nice properties in regard to the orientation of the simplicial decomposition which we previously discussed. This will be useful as we will argue that paths in the ENDOFLINE instance are sequences of simplices, and that the orientation of the simplicial decomposition will prevent these paths from forming cycles. We will start by defining what we mean by a sequence of simplices.

**Definition 4.16 — Sequence of simplices.**

A *sequence of simplices*, or *simplicial sequence* is a sequence  $(S_i)_{i=1}^k$  of  $d$ -dimensional simplices  $S_i \in \mathcal{S}$  such that:

- (1)  $S_{i+1} \not\subset \{S_1, \dots, S_i\}$  for all  $i \in \{1, \dots, k-1\}$ .
- (2)  $S_i$  and  $S_{i+1}$  share a  $d-1$ -dimensional face  $F_i$  for all  $i \in \{1, \dots, k-1\}$ .

Observe that because of the orientation of the simplicial complex, the orientation of the faces of the simplices in a sequence are consistent. We want to show that this consistent orientation also extends to faces of the simplices in the sequence. In order to do this we need to introduce what we mean by *sequence of faces*. We will then show that the orientation of the simplicial complex implies that the orientation of the faces of the simplices in a sequence are consistent.

**Definition 4.17 — Sequence of faces.**

A *sequence of faces* of a simplicial sequence  $(S_i)_{i=1}^k$  is a sequence  $(L_i)_{i=1}^k$  of simplices such that:

- (1)  $L_i$  is a subsimplex of the simplices  $S_i$  for all  $i \in \{1, \dots, k\}$ .
- (2)  $L_i$  and  $L_{i+1}$  share a  $d-2$ -dimensional face  $G_i$  for all  $i \in \{1, \dots, k-1\}$ .

Notice that the face  $L_i$  can be of dimension  $d-1$  or  $d-2$ , in this definition.

We now want to show that such a sequence of face is consistent with the orientation of the simplicial complex.

**Proposition 4.18 — Orientation of sequences of faces.**

Let  $(S_i)_{i=1}^k$  be a simplicial sequence, and  $(L_i)_{i=1}^k$  a sequence of faces of the simplices in the sequence. Then the orientation of the faces  $L_i$  is consistent.

*Proof.* It suffices to show that for any two faces  $L_i$  and  $L_j$ , the induced orientation of  $Q = L_i \cap L_j$  in  $L_i$  and  $L_j$  are opposite. We proceed by case distinction:

**Case 1:** Assume that  $L_i$  and  $L_j$  are faces of the same simplex  $S$ , then  $Q$  is a face of  $S$  and the induced orientation of  $Q$  in  $L_i$  and  $L_j$  are opposite by Claim 4.12.

**Case 2:** Assume that  $L_i$  and  $L_j$  are faces of two simplices  $S_1$  and  $S_2$  which share a common face  $F$ . Then  $Q$  is a face of  $F$  and the induced orientation of  $Q$  in  $L_i$  and  $L_j$  are opposite by Claim 4.12.

This shows that the orientation of the faces of the simplices in a sequence is consistent.  $\square$

## 4.4 Properties of the coloring of TARSKI instances

In this section we want to discuss different properties with the coloring of TARSKI instances have. This will be helpful in arguing that the resulting ENDOFLINE instance does not contain any cycles. We will start with general properties and then move on to properties of sequences of simplices.

### 4.4.1 General properties of the coloring

In this section we will assume that we are working on a integer lattice  $L$ , and that for a function  $f : L \rightarrow L$ , the points have been colored  $c : L \rightarrow \{0, \dots, d\}$  as in Section 3.4. Now we are ready to present a first observation, which will be a helpful stepping stone for more advanced results.

Recall that the coloring was given by:

$$c(x) = \begin{cases} 0 & \text{if } x \leq f(x) \\ 1 & \text{else if } x[1] > f(x)[1] \\ \vdots & \\ d & \text{else if } x[d] > f(x)[d] \end{cases}$$

#### Lemma 4.19

Assume that  $f$  is monotone and that we have  $x_i, x_j \in L$ ,  $c(x_i) = i$  and  $c(x_j) = j$  for  $i, j \in \{1, \dots, d\}$  and  $x_i[i] = x_j[i]$ , then either:

- (1)  $i \geq j$  or
- (2)  $i < j$  and  $x_i \not\leq x_j$

*Proof.* Assume that  $i < j$  and  $x_i \geq x_j$ . We must then have  $f(x_j)[i] \geq x_j[i] = x_i[i] > f(x_i)[i]$ . Now by monotonicity of  $f$  we must have  $f(x_i) \geq f(x_j)$ , which is not possible if  $f(x_j)[i] > f(x_i)[i]$ . Hence we must have  $x_i \not\leq x_j$ . This shows that the lemma holds.  $\square$

For vertices of a given simplex we get the following corollary.

#### Corollary 4.20

Assume that  $f$  is monotone and that we have  $x_i, x_j \in S$ , for some simplex  $S \in \mathcal{S}$ . Further assume that  $c(x_i) = i$  and  $c(x_j) = j$  for  $i, j \in \{1, \dots, d\}$  with  $i < j$  and that  $x_i[i] = x_j[i]$ , then  $x_i < x_j$ .

Notice that if we assume that  $x_i$  and  $x_j$  are in the same simplex of the simplicial decomposition, then the condition  $x_i \not\leq x_j$  is equivalent to  $x_i < x_j$ .

*Proof.*  $x_i \leq x_j$ , follows immediately. Because  $x_i$  and  $x_j$  are colored differently, they can not be equal which shows the strict inequality.  $\square$

#### 4.4.2 Properties of sequences of simplices

Now we want to work with sequences of simplices, and show that the coloring of the vertices of these simplices have some nice properties. We start by defining what we mean by a sequence of simplices. Let  $C \subset \{0, \dots, d\}$  be a subset of colors.

**Definition 4.21 — Valid sequence of simplices.**

A *valid sequence of simplices for colors  $C$*  is a sequence  $(S_i)_{i=1}^k$  of simplices as defined previously in Definition 4.16 such that the  $F_i$  are colored exactly colors of  $C$ .

Notice that this means that the first and last simplex of the sequence could be colored with any color. All other simplices must be colored with colors in  $C$ .

These sequences are the objects that latter get reduced to paths in the ENDOFLINE instance, which is why we want to study them in detail. We define the some more terminology to help us with this.

**Definition 4.22 — Cycle.**

A *cycle of simplices for colors  $C$*  is a valid sequence  $(S_i)_{i=1}^k$  of simplices  $S_i \in \mathcal{S}$  for colors  $C$  such that  $S_{k+1} = S_1$ .

Note that the empty sequence, and all sequences consisting of a single simplex are cycles.

**Definition 4.23 — Maximal sequence.**

A *maximal sequence of simplices for colors  $C$*  is a valid sequence  $(S_i)_{i=1}^k$  of simplices  $S_i \in \mathcal{S}$  for colors  $C$  such that:

- (1) There is no simplex  $S_{k+1} \in \mathcal{S}$  such that  $(S_i)_{i=1}^{k+1}$  is a valid sequence.
- (2) There is no simplex  $S_0 \in \mathcal{S}$  such that  $(S_i)_{i=0}^k$  is a valid sequence.

Intuitively we say that a sequence is maximal if we cannot make it longer by adding simplices at the beginning or end.

Finally we want to define the sequence of all transitions between simplices in a sequence.

**Definition 4.24 — Transition sequence.**

The *transition sequence* of a valid sequence  $(S_i)_{i=1}^k$  of simplices  $S_i \in \mathcal{S}$  is the sequence  $(F_i)_{i=1}^{k-1}$  of  $(d-1)$ -dimensional faces  $F_i = S_i \cap S_{i+1}$ .

We now are ready to study the properties of these sequence in more detail. We now restrict ourselves to the case where  $C \subset \{0, \dots, d\}$  contains exactly  $d$  colors (i.e. only one color is left out). Notice that for a valid sequence  $(S_i)_{i=1}^k$  we then have that the transition sequence  $(F_i)_{i=1}^{k-1}$  is a sequence of  $(d-1)$ -dimensional simplices  $S_i$  which are colored with all  $d$  colors of  $C$ . This means that for every  $j \in C$  we get a sequence of vertices  $(x_i^j)_{i=1}^k$  such that  $x_i^j \in F_i$  and  $c(x_i^j) = j$ . We will now study this special case in more detail.

**Lemma 4.25**

Let  $S_i, F_i$  and  $x_j$  be as above. For any  $i \in \{1, \dots, k-1\}$  there is exactly one  $j \in C$  such that we have  $x_i^j \neq x_{i+1}^j$ .

*Proof.*  $F_i$  and  $F_{i+1}$  are two faces of the same  $d$  dimensional simplex, and thus they share exactly  $d-1$  vertices. This means that there is exactly one vertex  $x$  which is in  $F_i$  but not in  $F_{i+1}$ , and exactly one vertex  $y$  which is in  $F_{i+1}$  but not in  $F_i$ . This means that there is exactly one  $j$  such that  $x_i^j = x$  and  $x_{i+1}^j = y$ .  $\square$

Now for a valid sequence  $(S_i)_{i=1}^k$  of simplices, inside the color set  $C = \{0, \dots, d-1\}$  we can consider all the vertices that are not colored with the color 0. These vertices have a nice structure as the following proposition shows.

**Proposition 4.26**

Let  $(S_i)_{i=1}^k$  be a valid sequence of simplices for colors  $C = \{0, \dots, d-1\}$ , then defining  $L_i$  to be the face of  $S_i$  which is spanned by the vertices colored with colors in  $C \setminus \{0\}$ . Then the sequence  $(L_i)_{i=1}^k$  is a sequence of faces as defined in Definition 4.17.

*Proof.* We need to show that the two conditions set by Definition 4.17 are satisfied. The first condition is immediate as  $L_i$  is a face of  $S_i$ . For the second condition we need to show that  $L_i$  and  $L_{i+1}$  share a  $d-2$ -dimensional face for every  $i \in \{1, \dots, k-1\}$ . In order to see this notice that  $S_i$  and  $S_{i+1}$ , share a common face  $F_i$ , which contains exactly one vertex colored with color 0. This means that  $L_i$  and  $L_{i+1}$  share a  $d-2$ -dimensional face of  $F_i$ , and hence a  $d-2$  dimensional face of  $S_i$  and  $S_{i+1}$ .  $\square$

As a direct consequence of this proposition we get the following corollary, which will be a key tool in the discussion on cycles in the ENDOFLINE instance.

**Corollary 4.27**

The  $(L_i)_{i=1}^k$  defined above can be oriented consistently.

*Proof.* This is a direct consequence of Proposition 4.18, which showed that the orientation of the faces of a sequence of simplices is consistent, when using the induced orientation of the faces by the individual simplices.  $\square$

## 4.5 No cycles in the ENDOFLINE instance

We are now ready to show that the ENDOFLINE instance does not contain any cycles. We will do this by showing that the existence of a cycle would contradict the orientation of the simplicial complex. A cycle is a valid sequence such that  $S_0$  and  $S_k$  are a  $(d - 1)$ -simplex as a face. We will show that certain situations cannot occur due to the orientation of the simplicial complex. And we will then argue that these situations must occur in a cycle. This will be enough to show that the ENDOFLINE instance does not contain any cycles.

Before we start we want to make an observation on how the dimension  $d$  plays together with the orientation of the simplicial complex.

### Lemma 4.28

Let  $l \in \{1, \dots, d - 1\}$  be a dimension. Let  $S$  be a  $d$ -simplex with colors  $C = \{0, \dots, d - 1\}$  in the colored simplicial complex, such that  $S$  is of the form:

$$(S) : \quad v_0 \xrightarrow{l} v_1 \rightarrow \dots \rightarrow v_d$$

and assume that the face  $F$  spanned by  $v_1, \dots, v_d$  is a rainbow face. Then we must have for the colors:

$$(F) : \quad c(v_1) \rightarrow \dots \xrightarrow{d} \dots 0 \rightarrow \dots \rightarrow c(v_d)$$

*Proof.* Every color  $c \in \{0, \dots, d - 1\}$  appears exactly once in the face  $F$ . If the color  $c \neq 0$  appear after 0, then by Corollary 4.20 we must have that we move in dimension  $c$  between 0 and  $c$ :

$$(F) : \quad c(v_1) \rightarrow \dots 0 \rightarrow \dots \xrightarrow{c} \dots \rightarrow c$$

Because we have this for every color  $c_i \neq 0$ , which appears after 0 in  $F$  we must have:

$$(F) : \quad c(v_1) \rightarrow \dots 0 \xrightarrow{c_1} c_1 \xrightarrow{c_2} c_2 \dots \xrightarrow{c_k} c_k$$

Now it is clear that because no vertex is colored with  $d$ , we must have that the change in dimension  $d$  occurs before the vertex colored 0 appears. This shows that we must have:

$$(F) : \quad c(v_1) \rightarrow \dots \xrightarrow{d} \dots 0 \rightarrow \dots \rightarrow c(v_d)$$

This shows the Lemma. □

We now want to show that a sequence of colored simplices as defined previously, can only cross a given hyperplane, given by

fixing a dimension  $l$ , at most once. This will be a key tool in the discussion on cycles in the ENDOFLINE instance. Formally we have the following proposition.

**Proposition 4.29**

Let  $l \in \{1, \dots, d-1\}$  be a dimension. Consider the hyperplane  $H$  given by fixing  $l = L$  for some  $L \in [N]$ . Then consider the sequence of simplices  $(S_i)_{i=1}^k$  such as defined previously. Then there can not be two  $i \neq j$  such that  $F_i \subset H$  and  $F_j \subset H$ .

In other words  $H$  is given by  $H = \{x \in \mathbb{R}^d \mid x[l] = L\}$

Before we prove this we want to detail why this is a very powerful result. This means that in all dimensions appart from  $d$  the sequences of simplices which induce the paths in the ENDOFLINE instance are monotone. Now let us prove this result.

*Proof.* For the sake of contradiction assume that there are two such  $i \neq j$ , such that  $F_i \subset H$  and  $F_j \subset H$ . Without loss of generality we can further assume that  $i < j$  and that for all  $k \in \{i+1, \dots, j-1\}$  we have  $F_k \not\subset H$ , if so then replace  $j$  with the smallest such  $k$ .

Now notice that  $S_i$  and  $S_j$  are on opposite sides of  $H$ , and that  $S_{i+1}$  and  $S_j$  are on the same side of  $H$ . Now let us consider the sequence of colored simplices  $S_{i+1}, \dots, S_j$ . Notice that Lemma 4.28, which we proved earlier, implies that both  $F_i$ , and  $F_j$  must be of the following form:

$$(F_i) : c(v_1) \rightarrow \dots \xrightarrow{d} \dots 0 \rightarrow \dots \rightarrow c(v_d)$$

$$(F_j) : c(w_1) \rightarrow \dots \xrightarrow{d} \dots 0 \rightarrow \dots \rightarrow c(w_d)$$

We will now proceed by case distinction:

**Case 1:** Assume that  $F_i$  and  $F_j$  are comparable. We want to show that this is not possible. Assume without loss of generality that  $F_i$  is smaller than  $F_j$ . Then the vertex colored 0 in  $F_i$  must be smaller than the vertex colored  $l$  in  $F_j$ . This leads to a violation of monotonicity according to Corollary 4.20.

By *comparable* we mean that the cell containing  $F_i$  is either larger or smaller than the cell containing  $F_j$ .

**Case 2:** Assume that  $F_i$  and  $F_j$  are not comparable. Our goal is to show that in this case we do not have a consistent orientation of the faces  $L_i$  and  $L_{j-1}$ .

□



# APPENDIX

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