Basics of Monte Carlo Methods

Before discussing Sequential Monte Carlo methods and Particle Filters in more detail, we introduce the basics of Monte Carlo methods that will be needed later.

In this section we consider approximating for a fixed $n \in \mathbb{N}$ the probability density $\pi_n(x_{1:n})$ where $x_{1:n} := (x_1, \ldots, x_n)$. A Monte Carlo method approximates $\pi_n(x_{1:n})$ by the empirical measure

$$\hat{\pi}_n(x_{1:n}) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_{1:n}}(x_{1:n}) \,,$$

where the $X^i_{1:n}$, $i = 1, \ldots, N$ are N independent samples of $\pi_n(x_{1:n})$ and $\delta_{x_0}(x)$ denotes the Dirac delta mass located at x_0. The expectation of a test function $\phi_n : \mathcal{X}^n \to \mathbb{R}$ given by

$$\mathbb{E}_n(\phi_n) := \int \phi_n(x_{1:n}) \hat{\pi}_n(x_{1:n}) dx_{1:n}$$

can then be estimated by

$$\mathbb{E}^{MC}_n(\phi_n) := \int \phi_n(x_{1:n}) \hat{\pi}_n(x_{1:n}) dx_{1:n} = \frac{1}{N} \sum_{i=1}^{N} \phi_n(X^i_{1:n}) \,.$$

It is straightforward to check that this estimator is unbiased and that the variance of the approximation error decreases **independent of the dimension** of the space $\mathcal{X}^n$ at a rate of $\mathcal{O}(1/N)$. This is usually not the case with traditional numerical integration methods where an increase in the dimension of the integral makes its approximation considerably harder.

Note, however, that this Monte Carlo approach requires sampling from $\pi_n(\cdot)$ which might not be possible for complex high-dimensional distributions. To this end, we introduce a **proposal density** $q_n(x_{1:n})$ that is only required to be defined on the same support as $\pi_n(x_{1:n})$. In a Bayesian framwork writing

$$\pi_n(x_{1:n}) = \frac{\gamma_n(x_{1:n})}{Z_n} = \frac{\gamma_n(x_{1:n})}{\int \gamma_n(x_{1:n}) dx_{1:n}}$$

we require only that $\gamma_n : \mathcal{X}^n \to (0, \infty)$ is known pointwise, whereas the *normalising* *constant* Z_n might be unknown. With the porposal density this can be rewritten as

$$\pi_n(x_{1:n}) = \frac{w_n(x_{1:n})q_n(x_{1:n})}{Z_n} = \frac{w_n(x_{1:n})q_n(x_{1:n})}{\int w_n(x_{1:n})q_n(x_{1:n})dx_{1:n}} \tag{1}$$ {eq:is:density}

where $w_n(x_{1:n})$ is the **unnormalised weight** function

$$w_n(x_{1:n}) = \frac{\gamma_n(x_{1:n})}{q_n(x_{1:n})} .$$