

## INTRODUCTION

We consider the symmetric eigenvalue problem, i. e. given a symmetric matrix  $A \in \mathbb{R}^{n \times n}$  we want to find  $0 \neq x \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$  such that

$$Ax = \lambda x. \quad (1)$$

We assume that  $A$  is positive definite which implies, since  $A$  is also symmetric, that all eigenvalues of  $A$  are positive. In the following we are especially interested in finding *interior* eigenvalues of  $A$ . By this we mean that if the eigenvalues of  $A$  are labeled in decreasing order of magnitude  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$  we are looking for  $\lambda_k$  where  $1 < k < n$ .

**Example 1** (Structural mechanics, discretization of PDEs). This example describes an eigenvalue problem that naturally arises in the investigation of resonance frequencies. Consider the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2)$$

posed on a domain  $\Omega \subset \mathbb{R}^n$ . We will now consider the one-dimensional case, i. e.  $n = 1$ , with  $\Omega = (0, 1)$ . Equation (2) then simplifies to

$$\begin{aligned} -u''(x) &= f(x) && \text{for all } x \in (0, 1), \\ u(0) &= u(1) = 0. \end{aligned} \quad (3)$$

Taylor expansion of  $u$  around some  $x \in (0, 1)$  yields the *central finite difference quotient*

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2).$$

If we only consider (3) in the discrete mesh points  $x_i = ih$ ,  $i = 1, \dots, m-1$ , for  $h = m^{-1}$  and neglect the  $\mathcal{O}(h^2)$  term in the approximation of  $u''$  we can approximate (3) by

$$\frac{1}{h^2}(-u_{i-1} + 2u_i - u_{i+1}) = f(x_i), \quad (4)$$

where  $u_i := u(x_i)$  and  $u_0 = u_m = 0$ . This leads to the following system of linear equations

$$Au := \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-2} \\ b_{m-1} \end{pmatrix} := \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{m-2}) \\ f(x_{m-1}) \end{pmatrix}. \quad (5)$$

Obviously  $A$  is symmetric. Since it is also strictly diagonally dominant with positive entries on the diagonal it follows that it is positive definite.