Example 1. Consider the following real symmetric positive definite 4×4 matrix

$$\mathbf{A} = 25 \cdot \mathrm{diag}(-1, 2, -1) = \begin{pmatrix} 50 & -25 & 0 & 0 \\ -25 & 50 & -25 & 0 \\ 0 & -25 & 50 & -25 \\ 0 & 0 & -25 & 50 \end{pmatrix}$$

that arises, for example, when discretizing a 1D Laplacian by finte differences.

Suppose we want to numerically compute a specific eigenpair of $\mathbf{A}$ using the Rayleigh-Quotient iteration.

The eigenpairs $(\lambda_i, \boldsymbol{v}_i)$, $i = 1, \ldots, 4$, of $\mathbf{A}$ can be computed analytically. The eigenvalues in increasing order of magnitude are

$$\lambda_1 = 9.549, \quad \lambda_2 = 34.5492, \quad \lambda_3 = 65.4508, \quad \lambda_4 = 90.4508 \tag{1}$$

with associated normalised eigenvectors such that $\|v_i\|_2 = 1$

$$\boldsymbol{v}_1 = \begin{pmatrix} 0.3717 \\ 0.6015 \\ 0.6015 \\ 0.3717 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0.6015 \\ 0.3717 \\ -0.3717 \\ -0.6015 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0.6015 \\ -0.3717 \\ -0.3717 \\ 0.6015 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 0.3717 \\ -0.6015 \\ 0.6015 \\ -0.3717 \end{pmatrix}. \tag{2}$$

Now consider the Rayleigh quotient iteration. Suppose we want to compute the second eigenpair but we only have an approximation of the eigenvalue but not of the eigenvector. If we chose $\sigma = 35$ as the initial shift and $x = (0.5\ 0.5\ 0.5\ 0.5)^{\mathsf{T}}$ we get the following results after the first four iterations:

Iteration	Eigenvalue approx.	Eigenvector approx.
0	35	$(0.5, 0.5, 0.5, 0.5)^{\mathsf{T}}$
1	11.6438	$(-0.2483, -0.6621, -0.6621, -0.2483)^{\mathsf{T}}$
2	9.5524	$(0.3764, 0.5986, 0.5986, 0.3764)^{\mathsf{T}}$
3	9.5492	$(-0.3717, -0.6015, -0.6015, -0.3717)^{\mathsf{T}}$
4	9.5492	$(0.3717, 0.6015, 0.6015, 0.3717)^{\mathsf{T}}$

Even tough the initial eigenvalue aproximation was very good the algorithm converges to a different eigenvector. This is due to the fact, that the initial eigenvalue approximation is very good already.