

INTRODUCTION

We consider the symmetric eigenvalue problem, i. e. given a symmetric matrix $A \in \mathbb{R}^{n \times n}$ we want to find $0 \neq x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$Ax = \lambda x. \quad (1)$$

We assume that A is positive definite which implies, since A is also symmetric, that all eigenvalues of A are positive. In the following we are especially interested in finding *interior* eigenvalues of A . By this we mean that if the eigenvalues of A are labeled in decreasing order of magnitude $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ we are looking for λ_k where $1 < k < n$.

Example 1 (Structural mechanics, discretization of PDEs). This example describes an eigenvalue problem that naturally arises in the investigation of resonance frequencies. Consider the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2)$$

posed on a domain $\Omega \subset \mathbb{R}^n$. We will now consider the one-dimensional case, i. e. $n = 1$, with $\Omega = (0, 1)$. Equation (2) then simplifies to

$$\begin{aligned} -u''(x) &= f(x) & \text{for all } x \in (0, 1), \\ u(0) &= u(1) = 0. \end{aligned} \quad (3)$$

Taylor expansion of u around some $x \in (0, 1)$ yields the *central finite difference quotient*

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \mathcal{O}(h^2).$$

If we only consider (3) in the discrete mesh points $x_i = ih$, $i = 1, \dots, m-1$, for $h = m^{-1}$ and neglect the $\mathcal{O}(h^2)$ term in the approximation of u'' we can approximate (3) by

$$\frac{1}{h^2}(-u_{i-1} + 2u_i - u_{i+1}) = f(x_i), \quad (4)$$

where $u_i := u(x_i)$ and $u_0 = u_m = 0$. This leads to the following system of linear equations

$$Au := \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-2} \\ b_{m-1} \end{pmatrix} := \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{m-2}) \\ f(x_{m-1}) \end{pmatrix}. \quad (5)$$

Obviously A is symmetric. Since it is also strictly diagonally dominant with positive entries on the diagonal it follows that it is positive definite.