Exercise Sheet 01 - Nils Friess, Leonardo Valle

April 28, 2022

[1]:
```
from functools import partial
from fractions import Fraction

import numpy as np
from scipy.special import p_roots

import matplotlib.pyplot as plt
plt.rcParams["figure.dpi"] = 120
```

Exercise 4

We begin by defining the function f we want to integrate over.

[2]:
```
def f(y):
    if not isinstance(y,list):
        y = [y]

    s = len(y)
    res = np.power(2 + 1/(2.*s), s)

    for yj in y:
        res *= np.power(yj, 1 + 1/(2.*s))

    return res
```

Next, we implement the one-dimensional trapezoidal rule.

[3]:
```
def trapezoidal1d(f, a, b, N):

    ys = np.linspace(a,b,N)
    ys = ys[1:-1]
    h = ys[1] - ys[0]

    res = (f(a) + f(b))/2.
    for y in ys:
        res += f(y)

    return h*res
```

Using this rule, we can implement the product trapezoidal rule for arbitrary dimensions using a recursive approach.

[4]:
```
class MultiDQuadrature:
    def __init__(self,rule1d, f, a, b, N):
        self.rule1d = rule1d
        self.f = f
        self.a = a
        self.b = b
        self.N = N
        self.maxdepth = len(a)

    def integrate(self,*args, **kwargs):
        depth = kwargs.pop('depth', 0)

        if depth == self.maxdepth - 1:
            func = self.f
        else:
            func = partial(self.integrate, depth=depth+1)

        res = self.rule1d(func, self.a[depth], self.b[depth], self.N)

        return res
```

Below, we plot the quadrature error against the total number of evaluation points for different dimensions.

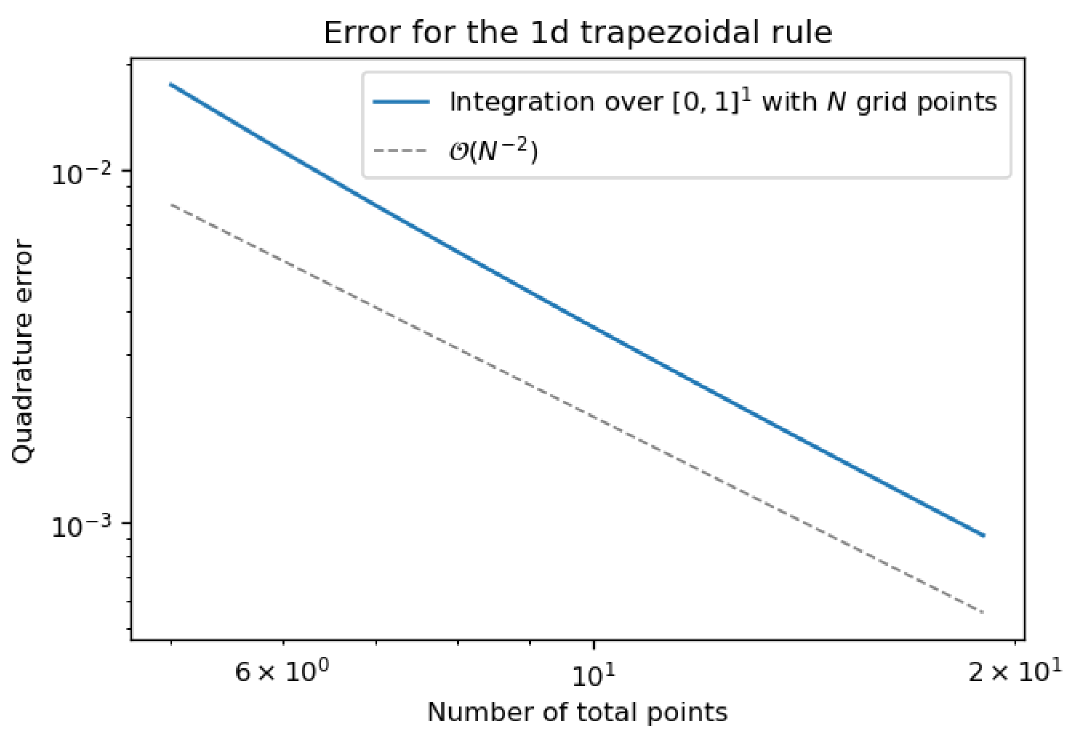
We can clearly observe the curse of dimensionality since the quadrature error is of the order

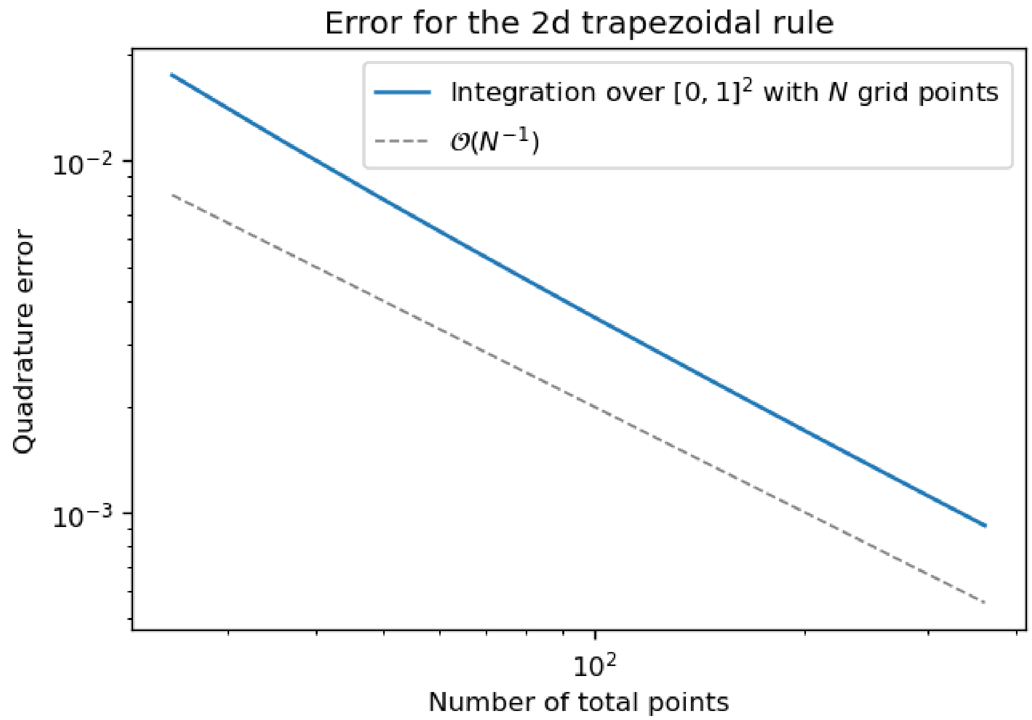
$$\mathcal{O}(N^{-2/d}),$$

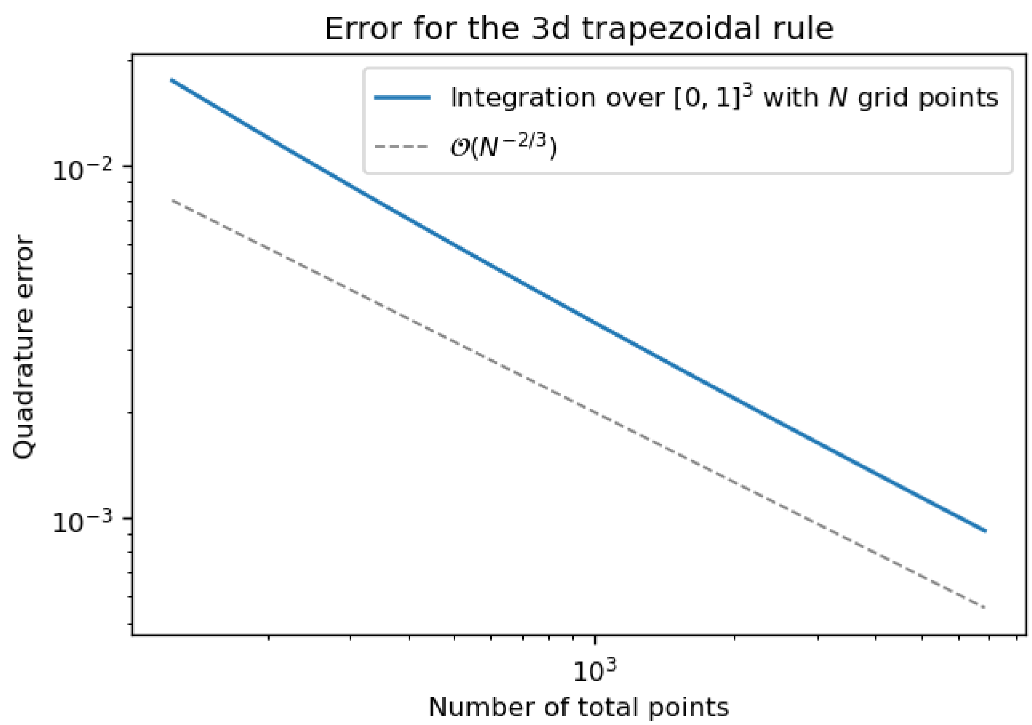
where d is the dimension and N is the number of total evaluations (the factor 2 comes from the fact that the trapezoidal rule is of order 2 in 1D).

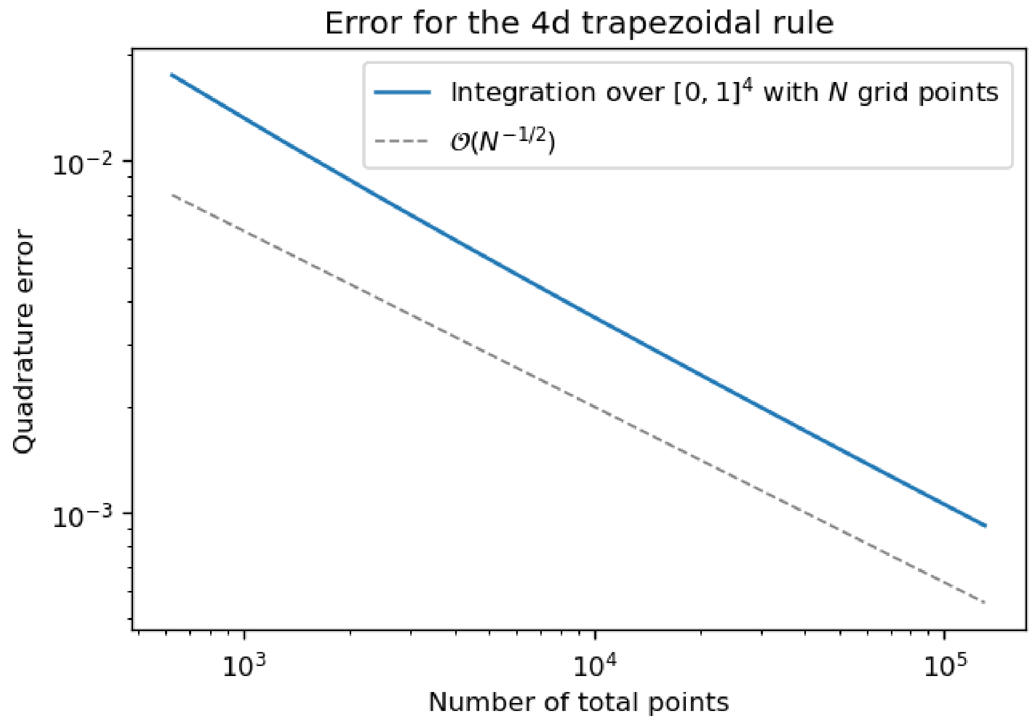
[5]:
```
dimensions = [1,2,3,4,5]

for dim in dimensions:
    Ns = np.arange(5,20)

    a = np.zeros((dim,))
    b = np.ones((dim,))

    res = np.array([MultiDQuadrature(trapezoidal1d,f,a,b,N).integrate() for N␣
 ↪in Ns])

    error = np.abs(res - 1.)
```

```
    plt.loglog(np.array(Ns)**dim, error, label=f"Integration over $[0,1]^{dim}$␣
↪with $N$ grid points")

    # Plot convergence order
    x0 = Ns[0]**dim
    x1 = Ns[-1]**dim
    y = lambda x : 0.2/x**(2/dim)
    y0 = y(x0)
    y1 = y(x1)

    order = Fraction(2,dim)

    plt.loglog([x0,x1], [y0,y1], '--', color='gray',␣
↪label=f"$\mathcal{{O}}(N^{{-{order}}})$", linewidth=1)

    plt.ylabel('Quadrature error')
    plt.xlabel('Number of total points')
    plt.title(f"Error for the {dim}d trapezoidal rule")
    plt.legend()
    plt.show();
```

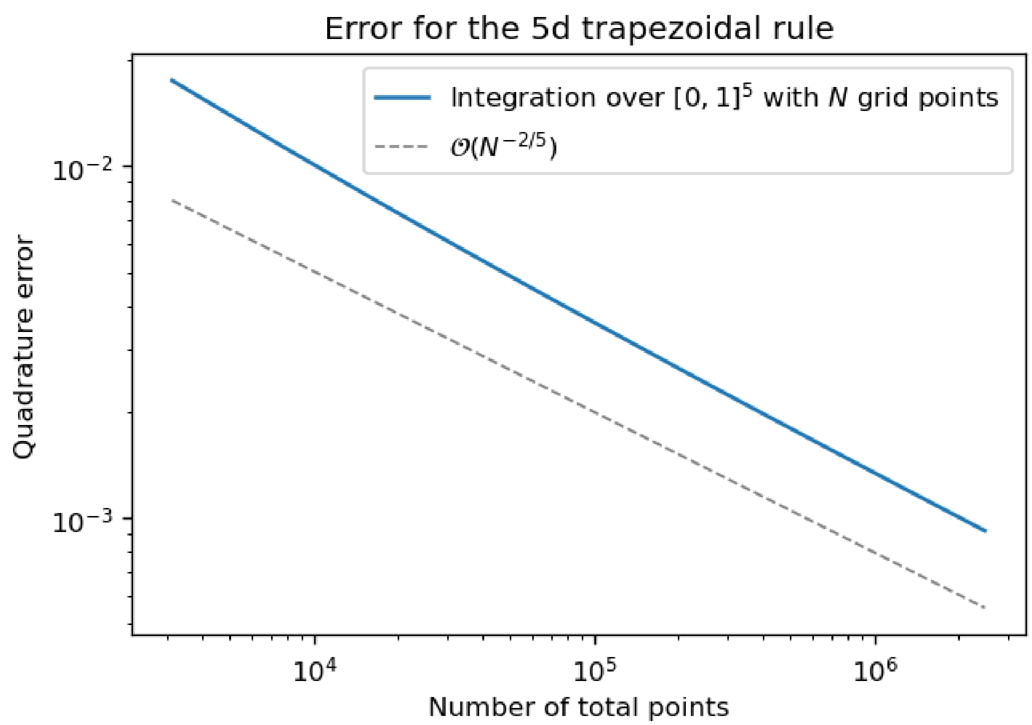



Error for the 2d trapezoidal rule
Integration over $[0, 1]^2$ with N grid points
$\mathcal{O}(N^{-1})$
10^{-2}
10^{-3}
Quadrature error
10^2
Number of total points


Error for the 3d trapezoidal rule
Integration over $[0, 1]^3$ with N grid points
$\mathcal{O}(N^{-2/3})$
10^{-2}
10^{-3}
Quadrature error
10^3
Number of total points



Error for the 4d trapezoidal rule
Integration over $[0, 1]^4$ with N grid points
$\mathcal{O}(N^{-1/2})$
10^{-2}
10^{-3}
Quadrature error
10^3
10^4
10^5
Number of total points

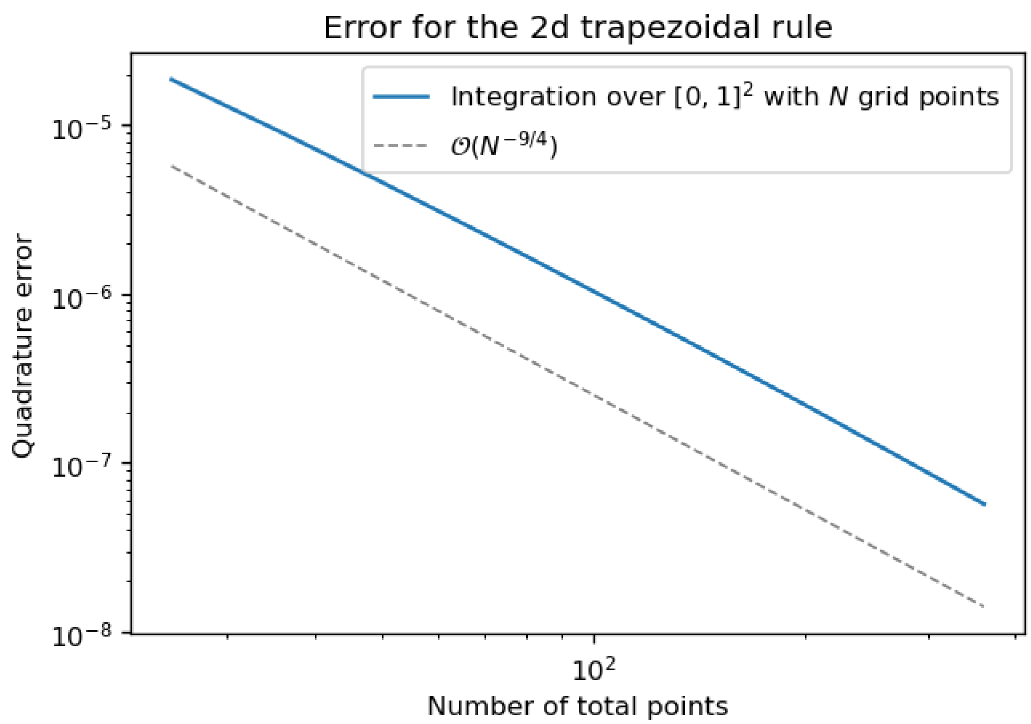


Error for the 5d trapezoidal rule
Integration over $[0, 1]^5$ with N grid points
$\mathcal{O}(N^{-2/5})$
10^{-2}
10^{-3}
Quadrature error
10^4
10^5
10^6
Number of total points

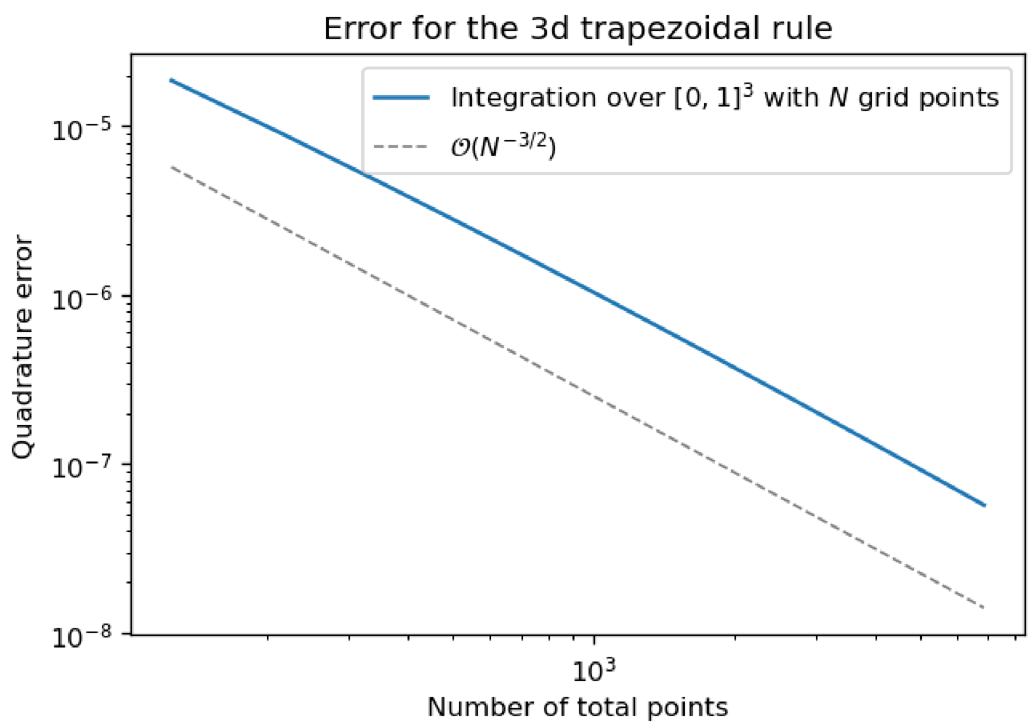
We repeat the experiment for a Gauss quadrature rule. We observe essentially the same behaviour with slightly different rates (for the function used in this exercise, we observe a rate of about $\mathcal{O}(N^{-9/(2d)})$).

[6]:
```
def gauss(f,a,b,N):
    [x,w] = p_roots(N+1)
    res = 0.5*(b-a) * sum(w*f(0.5*(b-a)*x+0.5*(b+a)))
    return res
```

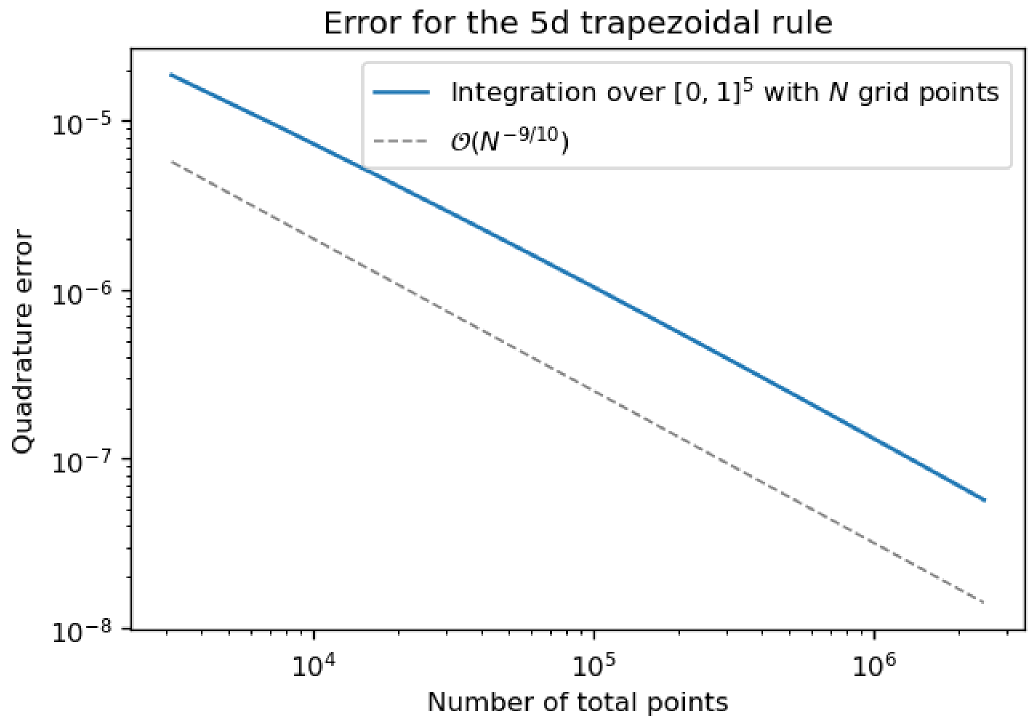
[8]:
```
dimensions = [2,3,5,6,7]

for dim in dimensions:
    Ns = np.arange(5,20)

    a = np.zeros((dim,))
    b = np.ones((dim,))

    res = np.array([MultiDQuadrature(gauss,f,a,b,N).integrate() for N in Ns])

    error = np.abs(res - 1.)

    plt.loglog(np.array(Ns)**dim, error, label=f"Integration over $[0,1]^{dim}$␣
 ↪with $N$ grid points")

    order = Fraction(9,2*dim)

    # Plot convergence order
    x0 = Ns[0]**dim
    x1 = Ns[-1]**dim
    y = lambda x : 0.008/x**(order)
    y0 = y(x0)
    y1 = y(x1)

    plt.loglog([x0,x1], [y0,y1], '--', color='gray',␣
 ↪label=f"$\mathcal{{O}}(N^{{-{order}}})$", linewidth=1)

    plt.ylabel('Quadrature error')
    plt.xlabel('Number of total points')
    plt.title(f"Error for the {dim}d trapezoidal rule")
    plt.legend()
    plt.show();
```



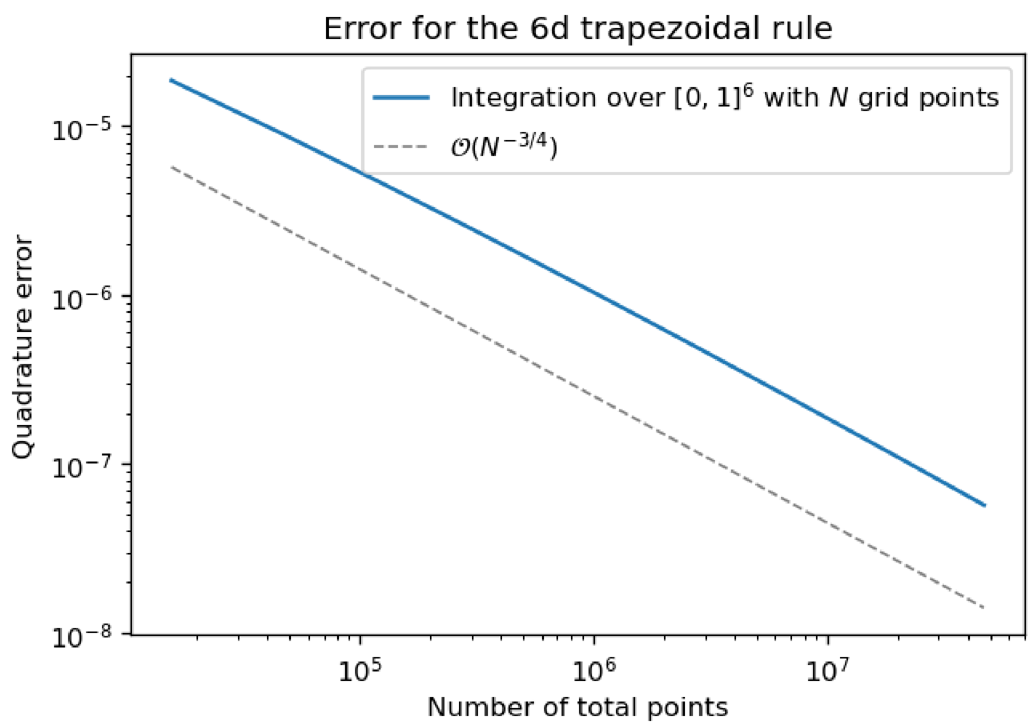
Error for the 2d trapezoidal rule
Integration over $[0, 1]^2$ with N grid points
$\mathcal{O}(N^{-9/4})$
Quadrature error
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{2}
Number of total points



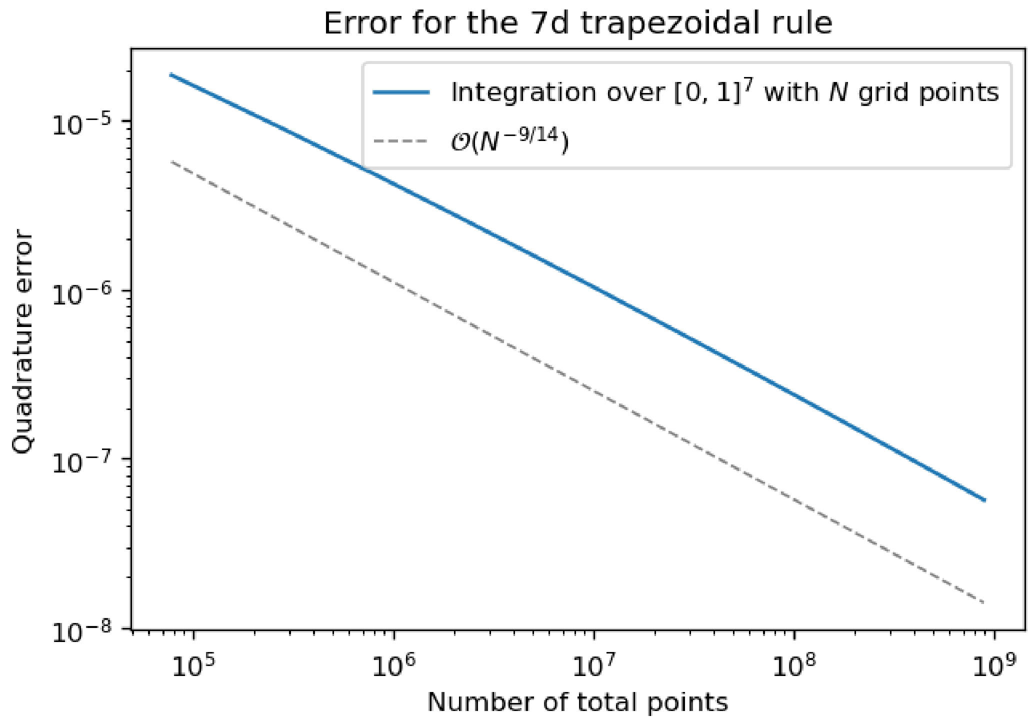
Error for the 3d trapezoidal rule
Integration over $[0, 1]^3$ with N grid points
$\mathcal{O}(N^{-3/2})$
Quadrature error
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{3}
Number of total points



Error for the 5d trapezoidal rule
Integration over [0, 1]^5 with N grid points
$\mathcal{O}(N^{-9/10})$
Quadrature error
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{4}
10^{5}
10^{6}
Number of total points



Error for the 6d trapezoidal rule
Integration over [0, 1]^6 with N grid points
$\mathcal{O}(N^{-3/4})$
Quadrature error
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{5}
10^{6}
10^{7}
Number of total points



Error for the 7d trapezoidal rule
Integration over $[0, 1]^7$ with N grid points
$\mathcal{O}(N^{-9/14})$
Quadrature error
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^5
10^6
10^7
10^8
10^9
Number of total points