

Skript Pattern Analysis Sommersemester 2017

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1 Density Estimation

Let $p(\vec{x})$ denote a probability density function pdf then:

1. $p(\vec{x}) \geq 0$
2. $\int_{-\infty}^{\infty} p(\vec{x}) d\vec{x} = 1$
3. $p(\vec{a} \leq \vec{x} \leq \vec{b}) = \int_{\vec{a}}^{\vec{b}} p(\vec{x}) d\vec{x}$

The task of density estimation is to obtain a continuous representation of the underlying pdf from a set of discrete samples (massumants). Note: that if we have the pdf we can do statistical analysis.

Parametric density estimation (mostly Pattern Recognition)

Make an assumption about the underlying distribution (e.g. Gaussian, GMM) and determine the best fitting distribution parameters from the data. (ML estimation, MAP estimation)

Non-parametric density estimation We make no assumption of the underlying Model.

1.1 Parzen-Rosenblatt estimator

???Idea: Quantify the number of samples with a window

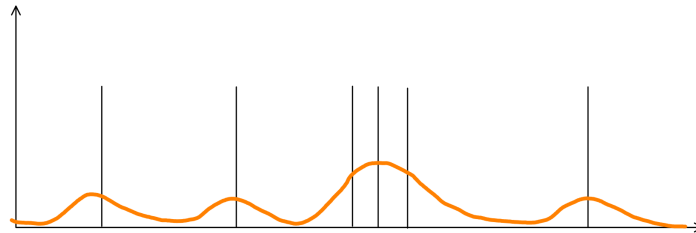


Figure 1: A PDF describing the distribution of measurements

The Parzen window estimator interpolates the pdf from the observations in the neighbourhood of a position x , using an appropriate kernel/window function.

Short derivition: Let p_R denote the probability that \vec{x} lies within region R :

$$p_R = \int_R p(\vec{x}) d\vec{x}$$

Now assume that $p(\vec{x})$ is approximately constant in R .

$$p_R \approx p(\vec{x}) \int_R d\vec{x}$$

For example, let R be a d -dimensional hypercube with side length h , then its volume¹² is h^d

$$p_R \approx p(\vec{x}) V_R$$

Let $p_R = \frac{k_R}{N}$, we determine the probability of making observations in region R by counting the samples in R ($= k_R$) and dividing by the total number of samples. Note: p_R is also called the “relative frequency”

$$p(\vec{x}) = \frac{p_R}{V_R} = \frac{k_R}{V_R N}$$

Let's write the parzen window estimator as a function of a kernel³ $k(\vec{x}; \vec{x}_i)$, then

$$p(\vec{x}) = \frac{1}{h^d N} \sum_{i=1}^N k(\vec{x}; \vec{x}_i)$$

where⁴

$$k(\vec{x}; \vec{x}_i) = \begin{cases} 1 & \text{when } \frac{|\vec{x}_i - \vec{x}|}{h} \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

equivalently, if we use a (multivariate) gaussian kernel:

$$k(\vec{x}; \vec{x}_i) = \frac{1}{(2\pi)^d |\Sigma|} e^{-(\vec{x} - \vec{x}_i)^T \Sigma^{-1} (\vec{x} - \vec{x}_i)}$$

A note on applications

- General remark: We obtain a continuous pdf, i.e. density estimation converts a list of measurements to a statistical model
- Specific example: We can sample from a pdf. This means that we have a principled way of generating new / more / ... data that behaves / looks / ... similarly to the observations.

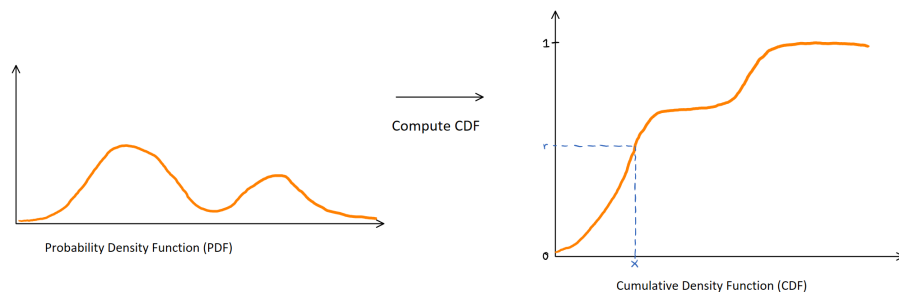
¹ $\int_R d\vec{x}$ is just the volume of R

² We also write V_R for the volume

³ Omit h^d if the kernel is gaussian

⁴ \vec{x}_i and \vec{x} are not farther apart than $0.5h$ in any dimension k

Q: How can we (practically) sample from a pdf?



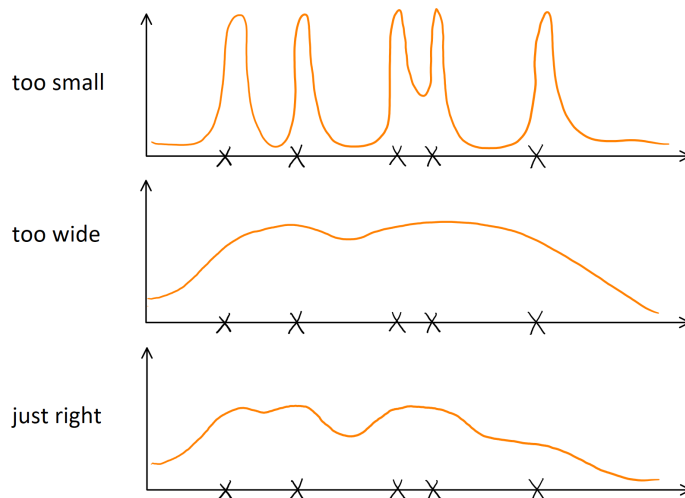
Compute through discretisation of the pdf $cdf[i] = cdf[i - 1] + pdf[i]$. Then draw a uniformly distributed number (r) between 0 and 1. The sampled value is x where $cdf[x] = r$

Q: How can we determine a good window / kernel width h ?
Let's do ML est. with a cross-validation (cv) (e.g. leave-one-sample-out cv)

$$p_{h,N-1}^j(\vec{x}) = \frac{1}{h^d N} \sum_{i=1 (i \neq j)}^N k(\vec{x}; \vec{x}_i)$$

We estimate the pdf from all samples except \vec{x}_j . \vec{x}_i will be used to evaluate the quality of the pdf using window size h .

Q: How do the results change with varying window size?



$$\hat{h} = \arg \max_h L(h) = \arg \max_h \prod_{j=1}^N p_{h,N-1}^j(\vec{x}_j) = \arg \max_h \sum_{j=1}^N \log p_{h,N-1}^j(\vec{x}_j)$$

The position of the maximum does not change, because the logarithm is a strictly monotonic function.

2 Mean Shift Algorithm

Purpose: Find maximum in pdf without actually performing a full density estimation.⁵

Potential applications: Clustering, segmentation, ...

Idea: Maxima can be found, where the gradient of the pdf is zero. (Assume that we have a full density estimator.)

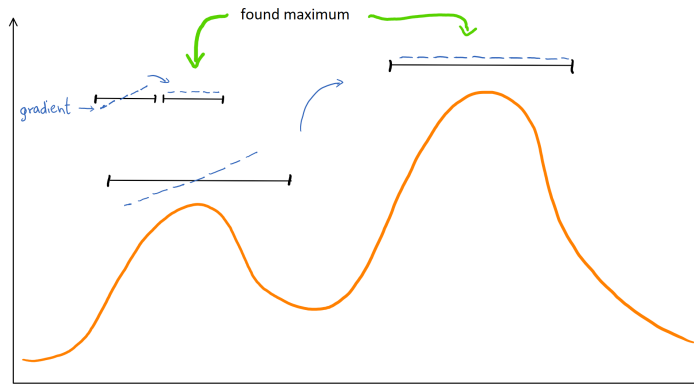


Figure 2: The kernel size indirectly controls the number of identified maxima

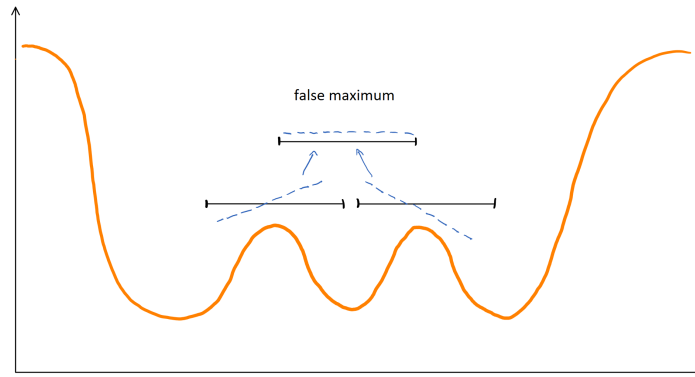


Figure 3: One of the issues is, the case when a zero gradient is just between two finer maxima

⁵This applies only in some cases, e.g. quickly finding clusters through particle tracing or with a downsampled PDF (see section 2)

Let

$$p(\vec{x}) = \frac{1}{N} \sum_{i=1}^N k_h(\vec{x}; \vec{x}_i)$$

denote the multivariate kernel density estimation. A local maximum of the pdf can be assumed where the gradient vanishes $\nabla p(\vec{x}) = 0$.

$$\nabla p(\vec{x}) = \nabla \left(\frac{1}{N} \sum_{i=1}^N k(\vec{x}; \vec{x}_i) \right) = \frac{1}{N} \sum_{i=1}^N \nabla k(\vec{x}; \vec{x}_i)$$

Let's assume that k_h is a radially symmetric kernel, i.e.

$$k(\vec{x}; \vec{x}_i) = c_d k_h(\|\vec{x}_i - \vec{x}\|^2)$$

$$\frac{\partial k_h(S)}{\partial S} = k'_h(S)$$

$$\frac{\partial S}{\partial \vec{x}} = \frac{\partial(\vec{x}_i - \vec{x})^T(\vec{x}_i - \vec{x})}{\partial \vec{x}} = -2(\vec{x}_i - \vec{x})$$

$$\nabla p(\vec{x}) = \frac{1}{N} \sum_{i=1}^N c_d k_h(\|\vec{x}_i - \vec{x}\|^2) (-2(\vec{x}_i - \vec{x})) \doteq \vec{0}$$

$\frac{1}{N}$ and c_d can be dropped then multiply out

$$\sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2) \vec{x}_i - \sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2) \vec{x} = \vec{0}$$

Then we get the mean shift vector

$$\frac{\sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2) \vec{x}_i}{\sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2)} - \vec{x} = \vec{0} \quad (1)$$

To perform a gradient ascent, compute the gradient, walk one step, re-compute the gradient, walk a step, ...

Mean shift algorithm (formalized)

1. Compute the mean shift vector $m(\vec{x}^{(t)})$ (see 1)
2. Update $\vec{x} : \vec{x}^{(t+1)} = \vec{x}^{(t)} + m(\vec{x}^{(t)}) = \vec{x}^{(t)} + \frac{\sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2) \vec{x}_i}{\sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2)} - \vec{x}^{(t)} =$

$$\frac{\sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2) \vec{x}_i}{\sum_{i=1}^N k'_h(\|\vec{x}_i - \vec{x}\|^2)}$$

Q: Why is it called “mean shift”?

If we plug in for k_h the Epanechnikov kernel. Then the computation breaks down to the mean of the samples in a circular (hyperspherical) around $\vec{x}^{(t)}$

Epanechnikov kernel

$$k_E(\vec{x}) = \begin{cases} c(1 - \vec{x}^T \vec{x}) & \text{when } \vec{x}^T \vec{x} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Abstract example: Assume we have a 2-D feature space

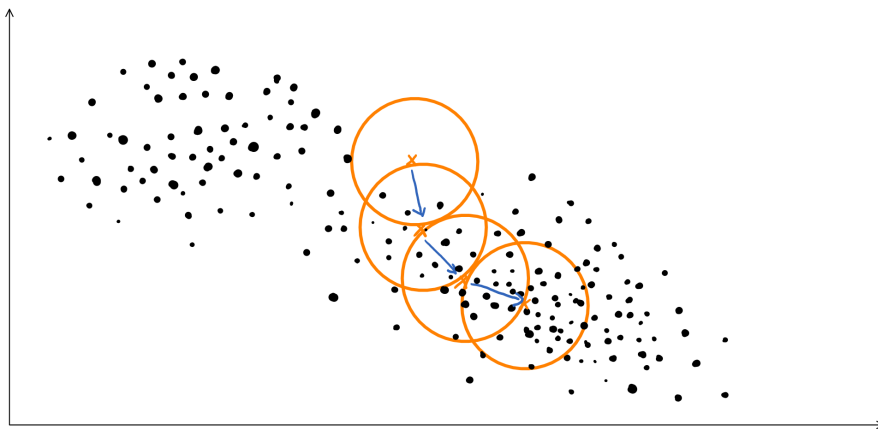


Figure 4: Mean Shift iterations with an Epanechnikov kernel

Specific example:

- (Color) quantization
 - Note: the RGB colorspace is not perceptually uniform (Lab or Luv are used in practice)
- (Color) segmentation
 - Similar in result to a super pixel segmentation: operate locally in the image
 - Incorporate the position of each pixel

Remarks on the found maxima:

- Different trajectories typically coverage only to **almost** the same peak, thus, we will have to post process the peaks and somehow reduce them.

- We don't have a guarantee to sit on top of a maximum when reaching a 0-gradient. This is due to the finite window size and the discrete representation of our density (see figure 2).
- If the amount of data is large, then it may become extremely costly to iteratively evaluate the “neighbourhood finder”. In that case we have to help ourselves, either with a smart data structure (oct-tree or a generalisation for many dimensions) or locality sensitive hashing (LSH).

3 Clustering

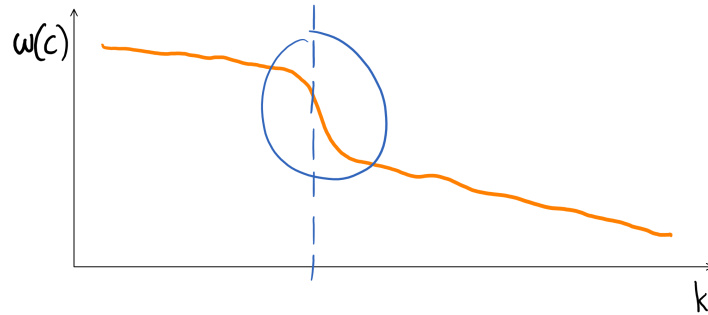
Variants of k-means: For clustering see section 14 of “The Elements of Statistical Learning” (Hastie, Tibshirani, Friedman - 2009). Section 14.3 “Cluster Analysis” introduces different flavours of k-means. Points of interest are:

- Proximity Matrices
- K-means
- Gaussian Mixtures as Soft K-means Clustering
- Vector Quantization

Q: How can we determine k?

Let $w(c)$ be distance from samples to cluster centres within clusters. We use this as a metric of quality in cluster analysis.

Approach 1: Track the rate of change of a quality metric (like $w(c)$). Proposed in “Pattern Classification” (Duda, Hart, Stork).



Approach 2: Let $w'(c)$ be a metric on a uniform distribution of samples. Relate change of $w(c)$ to change of $w'(c)$. Proposed in TEoSL.

