KTH VETENSKAP OCH KONST

KTH Teknikvetenskap

SF1684 Algebra and geometry
Exam
January 12, 2022

Time: 08:00-11:00

No books/notes/calculators etc. allowed.
Examiner: Maria Saprykina

This exam consists of six assignments, each worth 6 points.

Part A comprises the first two assignments. The bonus points from the seminars will be automatically added to the total score of the assignment 1, which however cannot exceed 6 points.

The next two assignments constitute part B, and the last two assignments part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To obtain points on a problem, it is required that:

- The solution is well presented and easy to follow.
- The solutions are neatly written with a handwriting which is easy to read.
- The notations introduced are to be clearly defined, the logical structure clearly described in words or symbols. The reasoning should be well motivated and clearly explained. All steps in all calculations are to be presented clearly and should be easy to follow.

Solutions and answers without correct, detailed and clear justifications will receive no points.

Part A

1. Let P be the plane that is parallel to the vectors $(0,3,2)$ and $(1,0,-2)$, and passes through the point $A=(0,-3,1)$. Find the distance between the plane P and the point $B=(-1,4,3)$.

2. Let

$$A=\begin{bmatrix}3 & -1 & 2 & -1\\ 0 & 1 & 2 & 0\\ 0 & 0 & 3 & 0\\ 2 & -1 & 2 & 0\end{bmatrix} \quad \text{and} \quad \vec{v}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

(a) Show that the vector $\vec{v}$ is an eigenvector for A, and find the corresponding eigenvalue. **(1 p)**

(b) The numbers 1 and 2 are eigenvalues of the matrix A. Find all the eigenvectors of A that correspond to these eigenvalues. **(3 p)**

(c) Find, if possible, a diagonal matrix D and an invertible matrix P such that $A=PDP^{-1}$. (You do not have to find P^{-1} explicitly). **(2 p)**

Part B

3. A linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $F(\vec{x}) = \vec{v} \times \vec{x}$, where $\vec{v} = (2, -2, 1)^T$ (here T denotes the transpose).

(a) Find the standard matrix A for the transformation F; **(2 p)**

(b) Find a basis for the null space $null(A)$ and a basis for the column space $Col(A)$ for A; **(3 p)**

(c) Find all the solutions to the equation $F(\vec{x}) = (0, 2, 2)^T$. **(1 p)**

4. Let H be the plane passing through the origin $\mathbb{R}^3$ that is orthogonal to the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find an ON-basis for H. Add vector(s) to this basis in order to get an ON-basis, $\mathcal{B}$, for $\mathbb{R}^3$. **(2 p)**

(b) Let R be the linear operator acting as the rotation by the angle $\pi/4$ about $\vec{v}$ (in other words, the vectors are rotated be the angle $\pi/4$ about the line that is generated by the vector $\vec{v}$, and the rotation appears counterclockwise if one looks at H from the top of vector $\vec{v}$). Find the matrix for R with respect to the ON-basis $\mathcal{B}$ from question (a). **(2 p)**

(c) Find the matrix for R with respect to the standard basis for $\mathbb{R}^3$. **(2 p)**

Part C

5. Let V be the space of all real 2×2 matrices, and define the linear transformation $S : V \to V$ by $S(A) = A - A^T$.

(a) Find a basis for the range of S. **(2 p)**

(b) Find a basis for the kernel of S. **(4 p)**

6. The characterstic polynomial of a quadratic matrix A is the polynomial

$$p_A(\lambda) = \det(A - \lambda I),$$

where λ is a variable. If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial, and A is a quadratic matrix, we define $p(A)$ by

$$p(A) = a_n A^n + \cdots + a_1 A + a_0 I.$$

Cayley-Hamilton's theorem says that $p_A(A) = 0$ for every quadratic matrix A (where the right-hand side is the zero matrix of the same size as A).

(a) Prove Cayley-Hamilton's theorem in the special case when A is diagonalisable. Hint: Start by investigating $p_A(A)\,\vec{v}$ for suitable vectors $\vec{v}$. **(4 p)**

(b) Let A be an invertible $n \times n$ matrix. Use Cayley-Hamilton's theorem to show that A^{-1} can be expressed as a linear combination of A^{n-1}, $A^{n-2}, \ldots, A$ and I. **(2 p)**