KTH Teknikvetenskap

SF1684 Algebra and geometri
Tentamen med lösningsförslag
Friday April 22 april, 2022

1. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

(a) Determine a 2×2 matrix P such that $P^{-1}AP$ is a diagonal matrix. **(4 p)**

(b) Determine $A^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. **(2 p)**

Lösningsförslag.

(a) The characteristic polynomial of A is given by $\det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = (\lambda - 4)\cdot(\lambda - 2)$. Hence the eigenvalues of A are $\lambda_1 = 2$ och $\lambda_2 = 4$. In order to find the corresponding eigenvectors, we solve the augmented systems

$$\left[\begin{array}{cc|c} 3-2 & 1 & 0 \\ 1 & 3-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{cc|c} 3-4 & 1 & 0 \\ 1 & 3-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

with solutions given by $\vec{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively, where t is a real parameter. In order for the transition matrix P to give a diagonal matrix, the columns need to be eigenvectors. Hence we can choose

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

to get $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) Since $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2, we get $A^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2^{10} \\ 2^{10} \end{bmatrix} = \begin{bmatrix} -1024 \\ 1024 \end{bmatrix}$.

2. Let

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Determine the angle between $\vec{u}$ and $\vec{v}$. **(3 p)**

(b) Determine the intersection point between the line $(x, y, z) = (-1+t, 2+2t, 2t)$ and the plane through the origine that is spanned by the vectors $\vec{u}$ and $\vec{v}$. **(3 p)**

Lösningsförslag.

(a) Let α be the angle between $\vec{u}$ and $\vec{v}$. We have that

$$1 = \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \alpha.$$

Since $\|\vec{u}\| = \|\vec{v}\| = \sqrt{2}$ we get that $\cos \alpha = 1/2$ and hence $\alpha = \pi/3$.

(b) The plane spanned by $\vec{u}$ och $\vec{v}$ has a normal vector given by the cross product $\vec{u} \times \vec{v}$. We get

$$\vec{u} \times \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-0 \\ 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Hence the equation for the plane is given by $x - y + z = 0$ and we can find the intersection point by substituting the parameter form of the line in the equation.

$$(-1+t) - (2+2t) + 2t = 0 \quad \iff \quad t = 3.$$

Hence, the intersection point is $(-1+3, 2+2\cdot 3, 2\cdot 3) = (2, 8, 6)$.

3. Let Π be the plane given by the equation $2x + 3y - 6z = 0$.

(a) Determine the orthogonal projection of the vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ on the plane Π. **(3 p)**

(b) Determine the standard matrix A for the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that corresponds to the orthogonal projection on the plane Π. **(2 p)**

(c) Determine if the matrix A is invertible. **(1 p)**

Lösningsförslag.

(a) The projection is given by

$$\mathrm{Proj}_\Pi \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T \cdot \begin{bmatrix}2 & 3 & -6\end{bmatrix}^T}{\begin{bmatrix}2 & 3 & -6\end{bmatrix}^T \cdot \begin{bmatrix}2 & 3 & -6\end{bmatrix}^T} \begin{bmatrix}2\\3\\-6\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{2}{49}\begin{bmatrix}2\\3\\-6\end{bmatrix} = \frac{1}{49}\begin{bmatrix}45\\-6\\12\end{bmatrix}$$

(b) For the standard matrix of the projection, we need to compute also the images of the other two standard basis vectors, $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$ och $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$.

$$\mathrm{Proj}_\Pi \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \cdot \begin{bmatrix}2 & 3 & -6\end{bmatrix}^T}{\begin{bmatrix}2 & 3 & -6\end{bmatrix}^T \cdot \begin{bmatrix}2 & 3 & -6\end{bmatrix}^T} \begin{bmatrix}2\\3\\-6\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{3}{49}\begin{bmatrix}2\\3\\-6\end{bmatrix} = \frac{1}{49}\begin{bmatrix}-6\\40\\18\end{bmatrix}$$

$$\mathrm{Proj}_\Pi \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \cdot \begin{bmatrix}2 & 3 & -6\end{bmatrix}^T}{\begin{bmatrix}2 & 3 & -6\end{bmatrix}^T \cdot \begin{bmatrix}2 & 3 & -6\end{bmatrix}^T} \begin{bmatrix}2\\3\\-6\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{-6}{49}\begin{bmatrix}2\\3\\-6\end{bmatrix} = \frac{1}{49}\begin{bmatrix}12\\18\\13\end{bmatrix}$$

The standard matrix has the images of the standar basis vectors as its columns and is hence given by

$$A = \frac{1}{49}\begin{bmatrix}45 & -6 & 12\\ -6 & 40 & 18\\ 12 & 18 & 13\end{bmatrix}$$

(c) The matrix A is not invertible since the normal vector to the plane is mapped to zero.

4. Let $P = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ be the transition matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ where $\mathcal{B}$ and $\mathcal{C}$ are bases for the same subspace V of $\mathbb{R}^3$.

(a) Determine the dimension of the subspace V. **(1 p)**
(b) Determine the transition matrix from the basis $\mathcal{C}$ to the basis $\mathcal{B}$. **(2 p)**
(c) Give an example of a subspace V and bases $\mathcal{B}$ and $\mathcal{C}$ such that P is the transition matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$. **(3 p)**

Lösningsförslag.

(a) Since the transition matrix is of size 2×2 the subspace must have dimension 2.
(b) The opposite change of bases is given by the inverse of the transition matrix. Hence by $P^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$
(c) If we choose the subspace to by given by $z = 0$ and

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

we get the basis $\mathcal{B}$ by the transition matrix from that $\mathcal{B}$ to the basis$\mathcal{C}$ should be given by the coordinates of the basis vectors in $\mathcal{B}$ relative to the basis $\mathcal{C}$. Hence we have

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

5. Let V be a k-dimensional subspace of $\mathbb{R}^n$, let M_{nm} denote the set of $n \times m$-matrices, and let W be the set of $n \times m$-matrices A that satisfies that $\operatorname{Range}(A)$ is a subspace of V.

(a) Show that W is a subspace of M_{nm}. **(3 p)**
(b) Determine the dimension of W. **(3 p)**

Lösningsförslag.

(a) If B is a matrix with row space $V^\perp$ we have that a matrix A is in W precisely if $BA = 0$. Henc W is given as the set of solutions to a homogenous linear system of equation, which shows that it is a subspace of M_{nm}.
(b) For each column of A the soluges lösningsmängden av k parametrar eftersom B har rang $n-k$. Sammanlagt behövs därmed $k \cdot m$ parametrar och dimensionen för W är km.

6. An $n \times n$-matrix A is said to be *expansive* if $\|A\vec{x}\| > \|\vec{x}\|$ for all non-zero $\vec{x}$ i $\mathbb{R}^n$. We say that $\vec{x_0}$ in $\mathbb{R}^n$ is a *fixed point* to a mapping $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ if $f(\vec{x_0}) = \vec{x_0}$.

(a) Show that if A is an $n \times n$-matrix such that all eigenvalues of A have absolute value greater that 1, then the matrix $A - I_n$ is invertible. **(2 p)**
(b) Show that if A is an expansive $n \times n$-matrix, then the matrix $A - I_n$ is invertible. **(2 p)**
(c) If $\vec{b}$ is a vector in $\mathbb{R}^n$ and A is an $n \times n$-matrix, we define a mapping $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$f(\vec{x}) = A\vec{x} + \vec{b}, \qquad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

Show that if A is expansive, then f must have a unique fixed point. **(2 p)**

Lösningsförslag.

(a) The matrix $A - I_n$ is invertible if and only if 1 is not an eigenvalue of A. If all eigenvalues of A have absolute value greater than 1, then 1 cannot be an eigenvalue of A and hence the matrix $A - I_n$ is invertible.
(b) If 1 is an eigenvalue of A, there is a non-zero vector $\vec{x_o}$ with $A\vec{x_0} = \vec{x_0}$ and hance $\|A\vec{x_0}\| = \|\vec{x_0}\|$ which cannot happen if A is expansive.
(c) We have that

$$f(\vec{x}) = \vec{x} \quad \Longleftrightarrow \quad A\vec{x} + \vec{b} = \vec{x} \quad \Longleftrightarrow \quad (A - I_n)\vec{x} = -\vec{b}.$$

Since A is expansive, the matrix $A - I_n$ is invertible according to part (b) and hence there is a unique solution to $(A - I_n)\vec{x} = -\vec{b}$, which gives a unique fixed point of f.