SF1684 Algebra and geometry
Solutions to the final exam

Part A

1. Let P be the plane that is parallel to the vectors $(0, 3, 2)$ and $(1, 0, -2)$, and passes through the point $A = (0, -3, 1)$. Find the distance between the plane P and the point $B = (-1, 4, 3)$.

   Solution. The normal vector $\vec{n}$ to the plane P is given by

   $$\vec{n} := (0, 3, 2) \times (1, 0, -2) = (-6, 2, -3).$$

   For any point $A \in P$, the distance d between the plane P and a given ponint B can be computed as

   $$d = \|proj_{\vec{n}} \vec{AB}\| = \frac{|\vec{n} \cdot (\vec{B} - \vec{A})|}{\|\vec{n}\|} = \frac{(-6, 2, -3) \cdot (-1, 7, 2)}{\|(-6, 2, 3)\|} = 2.$$

   □

2. Let

$$A = \begin{bmatrix} 3 & -1 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & -1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Show that the vector $\vec{v}$ is an eigenvector for A, and find the corresponding eigenvalue. **(1 p)**

(b) The numbers 1 and 2 are eigenvalues of the matrix A. Find all the eigenvectors of A that correspond to these eigenvalues. **(3 p)**

(c) Find, if possible, a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. (You do not have to find P^{-1} explicitly). **(2 p)**

Solution. (a) Since $A\vec{v} = 3\vec{v}$, vector $\vec{v}$ is an eigenvector for A (with eigenvalue 3).

(b) In order to find all the non-zero eigenvectors with the eigenvalue 1, we look for non-zero solutions to the equation $(A - 1 \cdot I)\vec{x} = \vec{0}$. The solutions are given by

$$\vec{x} = s \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \text{ where } (s,t) \neq (0,0).$$

Note that the vectors multiplying the parameters s and t above are linearly independent.

To find all the eigenvectors with the eigenvalue two, we look for non-zero solutions to the equation $(A - 2 \cdot I)\vec{x} = \vec{0}$. The solutions are given by

$$\vec{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ where } s \neq 0.$$

(c) Since the eigenvectors corresponding to different eigenvalues are linearly independent, and since we have found two linearly independent eigenvectors with eigenvalue 1, we can define a basis for $\mathbb{R}^4$ consisting of the eigenvectors of A combining the results of (a) and (b) above. Matrix P is constructed by putting these eigenvectors as columns in a matrix. Matrix D is a diagonal matrix, having the corresponding eigenvalues (in the same order as the eigenvectors) as diagonal elements. Then we have:

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\square$

Part B

3. A linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $F(\vec{x}) = \vec{v} \times \vec{x}$, where $\vec{v} = (2, -2, 1)^T$
(here T denotes the transpose).
   (a) Find the standard matrix A for the transformation F; **(2 p)**
   (b) Find a basis for the null space $null(A)$ and a basis for the column space $Col(A)$ for A; **(3 p)**
   (c) Find all the solutions to the equation $F(\vec{x}) = (0, 2, 2)^T$. **(1 p)**

Solution. (a) For every $\vec{x} = (x, y, z) \in \mathbb{R}^3$ we have:

$$F(\vec{x}) = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ x & y & z \end{bmatrix} = \begin{bmatrix} -2z - y \\ -2z + x \\ 2y + 2x \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Hence, the standard matrix A for the transformation F is $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}$.

(b) With the help of Gauss method we can bring A to the row-echelon form:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} := R.$$

For the matrix R, columns nr 1 och 2 are linearly independent. The same has to hold for the matrix A since the elementary transformations do not change linear dependence between the columns. Hence, vectors $(0, 1, 2)$ and $(-1, 0, 2)$ form a basis for $Col(A)$.
All solutions to the equation $A\vec{x} = \vec{0}$ are given by $\vec{x} = s(2, -2, 1)$. Therefore, a basis for the null-space of A is given by vector $(2, -2, 1)$.

(c) Note that the system $A\vec{x} = \vec{v} \times \vec{x} = \vec{b}$ has a solution $\vec{x}$ if and only if $\vec{b}$ is orthogonal to the vector $\vec{v}$. We investigate if $\vec{b} = (0, 2, 2)$ is orthogonal to $\vec{v}$:

$$\vec{v} \cdot \vec{b} = (2, -2, 1) \cdot (0, 2, 2) = -2 \neq 0.$$

Hence, these vectors are not orthogonal, and equation $A\vec{x} = \vec{b}$ has no solution.

□

4. Let H be the plane passing through the origin $\mathbb{R}^3$ that is orthogonal to the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find an ON-basis for H. Add vector(s) to this basis in order to get an ON-basis, $\mathcal{B}$, for $\mathbb{R}^3$. **(2 p)**

(b) Let R be the linear operator acting as the rotation by the angle $\pi/4$ about $\vec{v}$ (in other words, the vectors are rotated be the angle $\pi/4$ about the line that is generated by the vector $\vec{v}$, and the rotation appears counterclockwise if one looks at H from the top of vector $\vec{v}$). Find the matrix for R with respect to the ON-basis $\mathcal{B}$ from question (a). **(2 p)**

(c) Find the matrix for R with respect to the standard basis for $\mathbb{R}^3$. **(2 p)**

Solution. (a) We start by normalizing vector $\vec{v}$: let

$$\vec{v_1} = \frac{1}{|\vec{v}|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

Now, let

$$\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v_3} = \vec{v_1} \times \vec{v_2} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}.$$

Then both vectors $\vec{v_2}$ and $\vec{v_3}$ are orthogonal to $\vec{v_1}$, hence, they lie in H. Moreover, these vectors have norm 1 and are orthogonal with each other. Since $\dim H = 2$, the pair of vectors $\{\vec{v_2}, \vec{v_3}\}$ is an ON-basis to H. At the same time, $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a positively oriented ON-basis for $\mathbb{R}^3$.

(b) By the geometric meaning of R we have: $R(\vec{v_1}) = \vec{v_1}$, $R(\vec{v_2}) = 1/\sqrt{2}\vec{v_2} + 1/\sqrt{2}\vec{v_2}$ and $R(\vec{v_3}) = -1/\sqrt{2}\vec{v_2} + 1/\sqrt{2}\vec{v_3}$. This implies that the matrix R with respect to the basis $\mathcal{B}$ is given by

$$[R]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

(c) The change of basis from $\mathcal{B}$ to the standard basis $\mathcal{E}$ is given by the matrix

$$T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix},$$

and since $\mathcal{B}$ is an ON-basis, we have $T^{-1} = T^t$. Therefore, the matrix for R with respect to the standard basis is

$$[R]_{\mathcal{E}} = T[R]_{\mathcal{B}}T^t = \begin{bmatrix} 1/2 + \sqrt{2}/4 & 1/2 - \sqrt{2}/4 & 1/2 \\ 1/2 - \sqrt{2}/4 & 1/2 + \sqrt{2}/4 & -1/2 \\ -1/2 & 1/2 & \sqrt{2}/2 \end{bmatrix}.$$

$\square$

Part C

5. Let V be the space of all real 2×2 matrices, and define the linear transformation $S : V \to V$ by $S(A) = A - A^T$.
   (a) Find a basis for the range of S. **(2 p)**
   (b) Find a basis for the kernel of S. **(4 p)**

   Solution. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$S(A) = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix} = (b-c) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   (a) This observation shows that the range of S spans by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, hence a basis is formed by a single matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (one element is always linearly independent).
   (b) The same observation above implies that the null-space of S consists of all matrices of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}$. Since

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

   we see that the null-space spans by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Since

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   implies $a = b = c = 0$, we conclude that these three matrices are linearly independent, hence form a basis for their span.

$\square$

6. The characterstic polynomial of a quadratic matrix A is the polynomial
$$p_A(\lambda) = \det(A - \lambda I),$$
where λ is a variable. If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial, and A is a quadratic matrix, we define $p(A)$ by
$$p(A) = a_n A^n + \cdots + a_1 A + a_0\, I.$$
Cayley-Hamilton's theorem says that $p_A(A) = 0$ for every quadratic matrix A (where the right-hand side is the zero matrix of the same size as A).
   (a) Prove Cayley-Hamilton's theorem in the special case when A is diagonalisable. Hint: Start by investigating $p_A(A)\,\vec{v}$ for suitable vectors $\vec{v}$. **(4 p)**
   (b) Let A be an invertible $n \times n$ matrix. Use Cayley-Hamilton's theorem to show that A^{-1} can be expressed as a linear combination of A^{n-1}, $A^{n-2}, \ldots, A$ and I. **(2 p)**

Solution. (a) Let A be a diagonalisable $n \times n$-matrix. Write the characteristic polynomial of A as $p_A(\lambda) = a_n \lambda^n + \cdots + a_1 \lambda + a_0$. Since A is diagonalisable, there exists a basis $\vec{v_1}, \ldots, \vec{v_n}$ till $\mathbb{R}^n$ consisting of eigenvectors of A. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. They are exactly the roots of the characteristic polynomial, i.e., we have $p_A(\lambda_i) = 0$, för $i = 1, \ldots, n$.
Hence, for an eigenvector v_i we can write
$$\begin{aligned}
p_A(A)\vec{v_i} &= (a_n A^n + \cdots a_1 A + a_0 I)\,\vec{v_i} \\
&= a_n A^n \vec{v_i} + \cdots a_1 A \vec{v_i} + a_0 \vec{v_i} \\
&= a_n A^{n-1} \lambda_i \vec{v_i} + \cdots a_1 \lambda_i \vec{v_i} + a_0 \vec{v_i} \\
&\vdots \\
&= a_n \lambda_i^n \vec{v_i} + \cdots a_1 \lambda_i \vec{v_i} + a_0 \vec{v_i} \\
&= (a_n \lambda_i^n + \cdots a_1 \lambda_i + a_0)\,\vec{v_i} \\
&= p_A(\lambda_i)\vec{v_i} = 0\vec{v_i} = \vec{0}.
\end{aligned}$$
It follows that $\dim p_A(A) = n$, dvs $p_A(A) = 0$.

(b) Again, let $p_A(\lambda) = a_n \lambda^n + \cdots + a_1 \lambda + a_0$. Then, by Cayley-Hamilton's theorem, we have
$$a_n A^n + \cdots + a_1 A + a_0 I = 0.$$
This can be rewritten as
$$(a_n A^{n-1} + \cdots + a_1 I)A = -a_0 I.$$
Multiplication by A^{-1} gives
$$a_n A^{n-1} + \cdots + a_1 I = -a_0 A^{-1},$$
i.e., $A^{-1} = -\frac{1}{a_0}(a_n A^{n-1} + \cdots + a_1 I)$, assuming that $a_0 \neq 0$.
Now suppose that $a_0 = 0$. Then $\lambda = 0$ is a solution to $\det(A - \lambda I) = 0$. But in this case $\det(A) = 0$, which is impossible since A är invertible.

□