SF1625 Calculus in one variable
Solutions to the exam 2017-01-09

PART A

1. (a) Show that $F(x) = \ln(x + \sqrt{x^2 + a^2}) + C$ is an anti-derivative of $f(x) = 1/\sqrt{x^2 + a^2}$ for each choice of the constant C. **(2 p)**

   (b) Compute the integral
$$\int_0^3 \frac{dx}{\sqrt{x^2 + 16}}$$
   and simplify the answer. **(2 p)**

Lösning. A. We need to show that $F'(x) = f(x)$. We differentiate and obtain
$$F'(x) = \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + a^2}}\right) \cdot 2x = \frac{1}{\sqrt{x^2 + a^2}} = f(x).$$
We compute the integral using the anti-derivative (with $a = 4$ and $C = 0$) :
$$\int_0^3 \frac{dx}{\sqrt{x^2 + 16}} = \left[\ln(x + \sqrt{x^2 + 16})\right]_0^3 = \ln 8 - \ln 4 = \ln 2$$

$\square$

2. (a) Compute the integral $\displaystyle\int_0^{1/2} \arcsin x \, dx$. **(2 p)**

   (b) Compute the limit $\lim_{n\to\infty}\{a_n\}$ where $a_n = \displaystyle\sum_{i=2}^{n} \frac{2}{3^i}$. **(2 p)**

Lösning. (a) We integrate by parts:

$$\begin{aligned}\int_0^{1/2} \arcsin x \, dx &= [x \arcsin x]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} \, dx \\ &= \frac{\pi}{12} + [\sqrt{1-x^2}]_0^{1/2} \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.\end{aligned}$$

(b) The geometric series $\sum_{i=0}^{\infty} \frac{1}{3^i}$ converges to $\frac{1}{1-\frac{1}{3}} = \frac{3}{2}$. This implies that $\sum_{i=0}^{\infty} \frac{2}{3^i} = 3$, and the sum we are seeking is

$$3 - \frac{2}{1} - \frac{2}{3} = \frac{1}{3}.$$

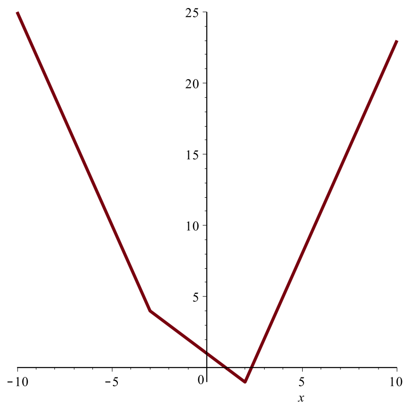
$\square$

3. The function f is given by $f(x) = -6 + |x+3| + |4-2x|$. Determine all real numbers x that solves the equation $f(x) < 0$. (Hint: Sketch the graph). **(4 p)**

Lösning. From the definition we have that $|x+3| = x+3$ if $x+3 \geq 0$, and that $|x+3| = -(x+3)$ if $x+3 < 0$, and similarily for $|4-2x|$. The interesting breaking points are the values of x such that $x+3=0$ and $4-2x=0$. In other words $x=-3$ and $x=2$. Thus our function is

$$f(x) = \begin{cases} -3x-5, & x < -3 \\ -x+1, & -3 \leq x < 2 \\ 3x-7, & x \geq 2 \end{cases}$$

Using this we can sketch the graph $y = f(x)$,



We get that the graph of the function intersects the x-axis in two points. These points are given as $-x+1=0$, that is $x=1$, and $3x-7=0$, that is $x=\frac{7}{3}$. This means that $f(x) < 0$ exactly when $1 < x < \frac{7}{3}$.

$\square$

PART B

4. The position $y(t)$ of a particle in a force field at time t satisfies the differential equation
$$y''(t) + ay'(t) + by(t) = 0.$$
   (a) If you know $y(t) = 3e^{-3t} + 6e^{2t}$. What are the values of a and b? **(2 p)**
   (b) Solve the differential equation when $a = -5$ and $b = 6$, with the initial value conditions $y(0) = 1$ and $y'(0) = 4$. **(2 p)**

Lösning. (a) With $y(t) = 3e^{-2t} + 6e^{t}$ as a solution to the differential equation, $r = -3$ and $r = 2$ must satisfy $r^2 + ar + b = 0$. In that case
$$r^2 + ar + b = (r+3)(r-2) = r^2 + r - 6$$
and so $a = 1$ and $b = -6$.

(b) With $a = -5$ and $b = 6$ the characteristic equation is
$$r^2 - 5r + 6 = (r-2)(r-3) = 0.$$
The solutions are $r = 2$ and $r = 3$. The solutions to the differential equation are then of the form $y(t) = Ce^{2t} + De^{3t}$. Initial conditions give that $y(0) = 1 = C + D$, and $y'(0) = 4$ gives $2C + 3D = 4$. The solution to the linear system of equations is $D = 2$ and $C = -1$. Thus $y(t) = -e^{2t} + 2e^{3t}$.

□

5. Let $f(x) = \arctan x^2$.
   (a) Compute the Taylor polynomial of degree 1 of $f(x)$ around $x = 1$. **(2 p)**
   (b) Show that $|\frac{\pi}{4} + \frac{1}{10} - f(1,1)| < 1/50$. **(2 p)**

Lösning. With $f(x) = \arctan x^2$ it holds that $f'(x) = \dfrac{2x}{1+x^4}$ and $f''(x) = \dfrac{-6x^4+2}{(1+x^4)^2}$. At $x = 1$ we have

$$f(1) = \frac{\pi}{4}, \quad f'(1) = 1$$

and the Taylor polynomial of degree one of f around $x = 1$ is therefore

$$p(x) = \frac{\pi}{4} + 1 \cdot (x-1).$$

The approximation we get of $f(1.1)$ using this is

$$f(1.1) \approx \frac{\pi}{4} + \frac{1}{10} \ (\approx 0.885)\,.$$

The error is, for some c between 1 and 1.1, given by

$$\left|\frac{f''(c)}{2} \cdot 10^{-2}\right| = \left|\frac{-6c^4+2}{2(1+c^4)^2}\right| \cdot 10^{-2} < \frac{5}{4} \cdot 10^{-2} < \frac{1}{50}.$$

At the first inequality we used in the numerator that $c^4 < 2$ and in the denominator that $1 + c^4 \geq 2$, when c is between 1 and 1.1.

$\square$

6. The ellipse E with half axes $a > 0$ and $b > 0$ is given by the equation $x^2/a^2 + y^2/b^2 = 1$. Find the largest possible area of a rectangle with all corners on the ellipse E and where the sides of the rectangle are parallell to the principal axes. **(4 p)**

Lösning. In a standard coordinate system the ellips is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and if we choose a point on it in the firsta quadrant this determines a rectangle, and all admissible rectangles are found this way. The point has coordinates

$$(x, y) = \left(x, b\sqrt{1 - \frac{x^2}{a^2}}\right).$$

The area of the rectangle is

$$A(x) = 4xb\sqrt{1 - \frac{x^2}{a^2}}, \quad 0 \le x \le a.$$

We want to maximize A. Since A is continuous and the domain of definition is closed and bounded we know a maximum is attained, either at a boundary point or at singular point or at a critical point. We can rule out the boundary points, because ty $A(0) = A(a) = 0$ and this cannot be a max because A assumes positive values. We differentiate:

$$A'(x) = 4b\sqrt{1 - \frac{x^2}{a^2}} - \frac{4x^2 b}{a^2\sqrt{1 - \frac{x^2}{a^2}}} = \frac{4b(a^2 - 2x^2)}{a^2\sqrt{1 - \frac{x^2}{a^2}}}.$$

. We see A is differentiable for all x between 0 and a and so there are no singular points in the open interval. The maximum must therefore be assumed at a critical point. Let us find those:

$$A'(x) = 0 \Longleftrightarrow x = \frac{a}{\sqrt{2}}.$$

Since this is the only critical point and we know that max is assumed at a critical point, this must be our max point. The maximum value of the area is therefore

$$A(a/\sqrt{2}) = 2ab.$$

□

PART C

7. Let f be a function defined on the interval $I = [a, b]$. Assume that f is continuous, increasing and positive on I. Let $A(x)$ denote the area between the graph of f and above $[a, x]$, for any given x in I. Show the Fundamental Theorem of Calculus saying that A is differentiable, and that $A'(x) = f(x)$. **(4 p)**

   Lösning. The definition of the derivative gives that
   $$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}.$$
   We consider the case with $h > 0$. We have that $A(x+h) - A(x)$ is the area under the function graph of f, over the interval $[x, x+h]$. The function f is assumed to be increasing, so
   $$f(x)h \leq A(x+h) - A(x) \leq f(x+h)h,$$
   and we get that
   $$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h).$$
   The limit then squeezes the middle term
   $$f(x) \leq \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \leq_{h\to 0} f(x+h) = f(x).$$
   In the right hand side we use that the function f is continuous. Similar arguement is applied for $h < 0$, and the conclusion is that $A'(x)$ exists and that $A'(x) = f(x)$.

   $\square$

8. The function f is given by
$$f(x) = \frac{x}{1+x^2} + \frac{4}{5}\arctan x.$$
Determine the biggest open interval containing $x = 1$, where f is invertible. **(4 p)**

Lösning. We differentiate and get
$$f'(x) = \frac{1+x^2-2x^2}{(1+x^2)^2} + \frac{4}{5}\cdot\frac{1}{1+x^2} = \frac{1}{5}\cdot\frac{9-x^2}{(1+x^2)^2}$$
We see the derivative exists for all x and $f'(x) = 0 \Leftrightarrow x = \pm 3$. We study the sign of the derivative:

If $x < -3$ then$f'(x) < 0$ and f is strictly decreasing.
If $-3 < x < 3$ then $f'(x) > 0$ and f strictly increasing.
If $x > 3$ then $f'(x) < 0$ and f is strictly decreasing.

The point $x = 1$ is between -3 and 3 where f is strictly increasing. It follows that $-3 < x < 3$ is the largest open interval where f is invertible.

□

Svar: $-3 < x < 3$

9. (a) Determine whether there exists a number $R > 0$ such that **(2 p)**

$$\int_1^R \frac{\sin^2 x}{x^2}\,dx > 100.$$

(b) Determine whether there exists a number $R > 0$ such that **(2 p)**

$$\sum_{n=1}^{R} \frac{1}{\sqrt{n^2 + \frac{1}{2}}} > 100.$$

Lösning. (a) For all $x \geq 1$ it holds that $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$. Since

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty}\left[-\frac{1}{x}\right]_1^R = 1$$

fit follows that the integral

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{\sin^2 x}{x^2}\,dx$$

is convergent and that its value is nonnegative and less than 1. Since

$$\int_1^R \frac{\sin^2 x}{x^2}\,dx$$

is an increasing function of R (the integrand is positive) sit follows that there can not be a number $R > 0$ that makes this integral larger than 100. Answer no.

(b) For all $n \geq 1$ it holds that

$$\frac{1}{\sqrt{n^2+\frac{1}{2}}} \geq \frac{1}{\sqrt{n^2+n^2}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{n}.$$

It follows that

$$\sum_{n=1}^{R} \frac{1}{\sqrt{n^2+\frac{1}{2}}} \geq \frac{1}{\sqrt{2}}\sum_{n=1}^{R}\frac{1}{n}.$$

Since the altter sum tends to ∞ when $R \to \infty$ it now follows that there has to exist a number $R > 0$ making

$$\sum_{n=1}^{R} \frac{1}{\sqrt{n^2 + \frac{1}{2}}} > 100.$$

$\square$