KTH Teknikvetenskap

SF1625 Calculus in one variable
Solutions to the exam 25.10.2016

PART A

1. (a) Determine the Taylor polynomial of degree 2 of $\ln(1+x)$ around $x=0$. **(2 p)**
   (b) Approximate the value of $\ln(6/5)$ with an error less than $3/1000$. **(2 p)**

Solution. If $f(x) = \ln(1+x)$ we have that $f'(x) = 1/(1+x)$, $f''(x) = -1/(1+x)^2$ and $f'''(x) = 2/(1+x)^3$. Therefore, according to the Taylor formula, for x near 0:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{\frac{2}{(1+c)^3}}{3!}x^3$$

for some c between 0 and x. We then have that the Taylor polynomial of $\ln(1+x)$ of degree 2, around $x=0$ is $x - \frac{x^2}{2}$. With $x = 1/5$ we get the approximation

$$\ln 6/5 \approx \frac{1}{5} - \frac{1}{2 \cdot 25} = \frac{10}{50} - \frac{1}{50} = \frac{9}{50}$$

with an error of

$$\frac{\frac{2}{(1+c)^3}}{3!}0.2^3 \leq \frac{0.2^3}{3} \leq 3 \cdot 10^{-3}.$$

$\square$

2. (a) Determine all primitive functions to $\dfrac{1}{\sqrt{x}+1}$. (Hint: Substitute $u = \sqrt{x}$.) **(2 p)**

   (b) Compute the limit **(2 p)**

$$\lim_{x\to 0} \frac{2\cos(x) - 2 + x^2}{x^4}.$$

Solution. (a) We use $u = \sqrt{x}$, with $du = dx/2\sqrt{x}$, and obtain:

$$\begin{aligned}\int \frac{1}{\sqrt{x}+1}\,dx &= \int \frac{2u}{u+1}\,du \\ &= 2\int 1 - \frac{1}{u+1} \\ &= 2u - 2\ln(u+1) + C \\ &= 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C\end{aligned}$$

where C is any constant.

(b) The Taylor polynomial of $\cos(x)$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{24} + E(x),$$

where the rest term $E(x)$ is of degree ≥ 6. We then get that

$$2\cos(x) - 2 + x^2 = \frac{x^4}{12} + 2E(x),$$

and the sought limit is

$$\lim_{x\to 0} \frac{2\cos(x) - 2 + x^2}{x^4} = \lim_{x\to 0}\left(\frac{1}{12} + 2\frac{E(x)}{x^4}\right) = \frac{1}{12}.$$

□

3. Let L be the tangent line at $(1, e)$ to the curve $y = xe^{x^2}$. Find the point of intersection of L with the x-axis. **(4 p)**

Solution. Let $f(x) = xe^{x^2}$. Then $f'(x) = e^{x^2} + xe^{x^2}2x = (1+2x^2)e^{x^2}$. We get $f'(1) = 3e$ and so the equation of the tangent line is

$$y - e = 3e(x - 1).$$

The intersection with the x-axis is obtained when $y = 0$, i.e. whten $-e = 3e(x - 1)$ i.e. when $x = 2/3$.

$\square$

PART B

4. We study an electrical circuit containing a voltage source, en coil with inductance 1 henry, a resistor with resistance 15 ohm and a condenser with capacitance $1/50$ farad. The electrical current i through the circuit satisfies the differential equation

$$i''(t) + 15i'(t) + 50i(t) = 0.$$

Solve the differential equation and determine the electrical current at time t if $i(0) = 0$ ampere and $i'(0) = 1$ ampere/second. **(4 p)**

Solution. The characteristic equation $r^2+15r+50 = 0$ has solutions $r = -5$ and $r = -10$, and so the solution to the differential equation is

$$i(t) = Ae^{-5t} + Be^{-10t}$$

where A and B are arbitrary constants. The condition $i(0) = 0$ gives $A + B = 0$ and $i'(0) = 1$ gives $-5A-10B = 1$ and using this we can determine $A = 1/5$ and $B = -1/5$. The current at time t is therefore

$$i(t) = \frac{1}{5}e^{-5t} - \frac{1}{5}e^{-10t} \quad \text{ampere.}$$

$\square$

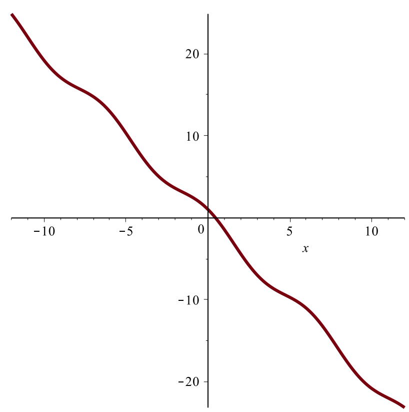
5. (a) Sketch the curve $y = \cos x - 2x$. **(2 p)**
   (b) Using your solution of (a) above, show that the equation $\cos x = 2x$ has exactly one solution. **(1 p)**
   (c) Determine an interval of length at most $\frac{1}{2}$ that contains the solution of the equation $\cos x = 2x$. (The length of an interval $[a, b]$ is $|b - a|$.) **(1 p)**

Solution. (a) Put $f(x) = \cos x - 2x$. We see f is defined and continuous for all x. We differentiate and obtain $f'(x) = -\sin x - 2$. Since the values of the sine function is in the interval $[-1, 1]$ we conclude that $f'(x) < 0$ for all x. It follows that f is strictly decreasing on the entire x axis. Since further $\lim_{x\to\infty}(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$ and $f(0) = 1$ we can now sketch the curve (see below).

(b) With $f(x) = \cos x - 2x$ as above, we have $\cos x = 2x \Leftrightarrow f(x) = 0$. Since f is strictly decreasing (see above) there can be at most one x ssuch that $f(x) = 0$. Since $f(0) = 1$ and $f(1) = \cos 1 - 2 < 0$ and f is continuous on $[0, 1]$ it follows from the intermediate value theorem that there is at least one x such that $f(x) = 0$. This x must lie between 0 and 1. We have thus proven that there is exactly one x such that $\cos x = 2x$.

(c) We have that $f(0) = 1$, a positve number, and that $f(1/2) = \cos(1/2) - 1 \leq 0$, a negative number. The solution to the equation $\cos(x) = 2x$ is then in the interval $[0, \frac{1}{2}]$, which is of length $\frac{1}{2}$.

$\square$



6. Consider the function
$$f(x) = \frac{e^x + e^{-x}}{2},$$
defined on the interval $-1 \leq x \leq 1$.
   (a) Write down an integral giving the arc length of the curve $y = f(x)$. **(2 p)**
   (b) Compute the length of the curve. **(2 p)**

Solution. The length L of the curve $y = f(x)$ pon the interval $a \leq x \leq b$ is given by
$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$
In our case
$$f'(x) = \frac{e^x - e^{-x}}{2}$$
and
$$1 + (f'(x))^2 = 1 + \frac{(e^x)^2 - 2 + (e^{-x})^2}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2$$
The length of our curve is
$$\begin{aligned}\int_{-1}^1 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx &= \int_{-1}^1 \frac{e^x + e^{-x}}{2}\, dx \\ &= \left[\frac{e^x - e^{-x}}{2}\right]_{-1}^1 \\ &= e - \frac{1}{e}\end{aligned}$$
□

PART C

7. (a) Define what it means for a function f to be continuous at point a. **(1 p)**
   (b) Define what it means for a function f to be differentiable at point a. **(1 p)**
   (c) Determine the numbers a and b so that the function f given by

$$f(x) = \begin{cases} x^2 & x \leq 1 \\ ax + b & x > 1 \end{cases}$$

   is both continuous and differentiable at $x = 1$. **(2 p)**

Solution. (a) f is continuous at a if f is defined at a and f has a limit when $x \to a$ and

$$\lim_{x \to a} f(x) = f(a)$$

(b) f is differentiable at a if f is defined in a neighborhood of a and the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists as a finite number.

(c) For f to be continuous at $x = 1$ it is required that the function value and the limit are equal. We see that $f(1) = 1$ which is also the limit as x tends to 1 from the left. The limit from the right is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax + b) = a + b$$

For continuity to hold, it therefore is required that $a + b = 1$. For differentiability we also need that

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

exists and is finite. If we let $h \to 0^-$ we get

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = 2.$$

If we let $h \to 0^+$ we get

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{a(1+h) + b - a \cdot 1 - b}{h} = a$$

It follows that f is both continuous and differentiable if $a = 2$ and $b = -1$

□

8. Let the function f be defined by

$$f(t) = \begin{cases} \cos^2 t, & 0 \leq t \leq 1 \\ t^2 + 1 & t > 1 \end{cases}$$

Compute for all numbers $x \geq 0$ the integral **(4 p)**

$$\int_0^x f(t)\,dt.$$

Solution. If $x \in [0,1]$ we get

$$\int_0^x f(t)\,dt = \int_0^x \cos^2 t\,dt = \int_0^x \frac{1+\cos 2t}{2}\,dt = \frac{x}{2} + \frac{\sin 2x}{4}.$$

If $x > 1$ we get

$$\begin{aligned} \int_0^x f(t)\,dt &= \int_0^1 \cos^2 t\,dt + \int_1^x (t^2+1)dt \\ &= \frac{1}{2} + \frac{\sin 2}{4} + \frac{x^3}{3} + x - \frac{4}{3} \\ &= \frac{\sin 2}{4} + \frac{x^3}{3} + x - \frac{5}{6}. \end{aligned}$$

□

9. Compute the limit **(4 p)**

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{n+k}{n^4}\right)^{1/3}$$

Solution. We observe that

$$\sum_{k=1}^{n}\left(\frac{n+k}{n^4}\right)^{1/3}=\sum_{k=1}^{n}\left(\frac{n+k}{n}\right)^{1/3}\cdot\frac{1}{n}$$

is a Riemann sum with n equal subintervals to the integral

$$\int_0^1 (1+x)^{1/3}\,dx.$$

Since the integrand is continuous on the interval of integration (including the endpoints), the sequence of Riemann sums converge to the integral when $n\to\infty$. Therefore,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{n+k}{n^4}\right)^{1/3}=\int_0^1(1+x)^{1/3}\,dx=\left[\frac{3}{4}(1+x)^{4/3}\right]_0^1=\frac{3}{4}(2^{4/3}-1)$$

$\square$