KTH Teknikvetenskap

SF1684 Algebra and geometry
Exam
Monday, October 19, 2020

Time: 08:00-11:00
Help in any form (such as books/notes/computers/person etc.) is NOT allowed

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This exam consists of six problems, each worth 6 points.
Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of **problem 1**, which however cannot exceed 6 points.
Problems 3 and 4 constitute part B, and problems 5 and 6 part C. The latter is mostly for achieving a high grade. The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

Instruktions

- **To obtain points on a problem, it is required that:** the solution is well presented and easy to follow, neatly written with a handwriting which is easy to read, the notations introduced are clearly defined and the logical structure is clearly described in words or symbols.
- The reasoning should be well motivated and clearly explained. All steps in all calculations are to be presented clearly and should be easy to follow.
- Solutions and answers without correct, detailed and clear justifications will receive no points.

Good luck!

PART A

1. Let A be an invertible (3×3)-matrix, and let $B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 1 & 5 \end{bmatrix}$.

(a) Determine the reduced row echelon form of A. **(2 p)**
(b) Compute $\det(AB^3A^{-1}B^{-1})$. **(4 p)**

2. The lines L_1 and L_2 in $\mathbf{R}^3$ are given by

$$L_1 : \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{real numbers} \quad t,$$

$$L_2 : \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \text{real numbers} \quad s.$$

(a) Determine the intersection of L_1 and L_2. **(2 p)**
(b) Determine the angle between L_1 and L_2. **(2 p)**
(c) A plane passes through the origin and is orthogonal to the line L_1. Determine scalar equation for this plane.
(2 p)

PART B

3. Let S in $\mathbb{R}^5$ be the span

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

and let

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Determine the dimension of the orthogonal complement $S^\perp$. **(2 p)**
(b) Determine if $\vec{x}$ is in $S^\perp$. **(2 p)**
(c) Determine if $\vec{x}$ is in S. **(2 p)**

4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Show that the matrix A is not diagonalizable. **(3 p)**
(b) Let $\vec{v} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$. Compute (and simplify) $A^{83}\vec{v}$. (*Tips: write $\vec{v}$ as a linear combination of eigenvectors of A*). **(3 p)**

PART C

5. Let A be a symmetric (2×2)-matrix. Without using the Spetral Theorem, show

(a) that the characteristic polynomial of A have real roots only, **(3 p)**
(b) that there exist eigenvectors of A, which form a basis of $\mathbf{R}^2$. **(3 p)**

6. Let $P\colon V \to V$ be a linear transformation such that $P \circ P = P$. Such a transformation is called a projection.

(a) Show that for each vector $\vec{v}$ in V it holds that $\vec{v} - P(\vec{v})$ is in the kernel $\ker(P)$ of P. **(2 p)**
(b) Show that each vector $\vec{v}$ in V can be written as $\vec{v} = \vec{u} + \vec{w}$, where $\vec{u}$ is in $\ker(P)$ and $\vec{w}$ is in the image $\mathrm{Im(P)}$ of P. **(2 p)**
(c) Show that the sum in b) is unique, that is show that if $\vec{v} = \vec{u_1} + \vec{w_1} = \vec{u_2} + \vec{w_2}$ where $\vec{u_1},\ \vec{u_2}$ are in $\ker(P)$ and $\vec{w_1},\ \vec{w_2}$ are in $\mathrm{Im(P)}$, then $\vec{u_1} = \vec{u_2}$ and $\vec{w_1} = \vec{w_2}$. **(2 p)**