SF1625 (SF1685) Calculus in one variable
Solutions to exam 15.10.2020

DEL A

1. (a) Find all functions $y(t)$ that satisfy $y'' + y' - 2y = 0$. **(2 p)**
   (b) Find all functions $y(t)$ that satisfy **(4 p)**

$$\begin{cases} y'' + y' - 2y = 4e^{-t} \\ \lim\limits_{t\to\infty} y(t) = 0 \\ y(0) = 5. \end{cases}$$

Solution. (a) The given equation has the auxiliary equation $r^2 + r - 2 = 0$ which have the roots $r = 1$ and $r = -2$. Hence the general solution to the equation is given by

$$y = Ae^t + Be^{-2t}$$

where A and B are constants. We know from the theory that this is all solutions, that is, every solution is of this form.

(b) We look for a particular solution y_p of the inhomogeneous equation $y'' + y' - 2y = 4e^{-t}$. Since we have $4e^{-t}$ in the right hand side we make the anzats $y_p = ae^{-t}$, where a is a constant. Then we have $y_p' = -ae^{-t}$ and $y_p' = ae^{-t}$. Substitution into the equation yields

$$ae^{-t} - ae^{-t} - 2ae^{-t} = 4e^{-t}.$$

Dividing by e^{-t} (which never is zero) gives the equation $-2a = 4$, which has the solution $a = -2$. Hence $y_p = -2e^{-t}$ is a particular solution.

In part (a) we saw that $y_h = Ae^t + Be^{-2t}$ is the general solution to the corresponding homogeneous solution. Thus the general solution to the equation $y'' + y' - 2y = 4e^{-t}$ is given by

$$y = y_h + y_p = Ae^t + Be^{-2t} - 2e^{-t},$$

where A and B are constants.

Now we look for constants A and B so that the given conditions are satisfied. Seince $\lim_{t\to\infty} e^t = \infty$ (and $\lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} e^{-2t} = 0$) we see that we have to have $A = 0$ in order for the conition $\lim_{t\to\infty} y(t) = 0$ to be satisfied. Hence we must have

$$y = Be^{-2t} - 2e^{-t}.$$

For each choice of B we then have $\lim_{t\to\infty} y(t) = 0$.

The condition $y(0) = 5$ gives the equation $5 = y(0) = B - 2$, which implies that $B = 7$. Thus

$$y = 7e^{-2t} - 2e^{-t}$$

is the solution we are looking for.

Answer: a) $y = Ae^t + Be^{-2t}$ where A and B are constants, b) $y = 7e^{-2t} - 2e^{-t}$. $\square$

2. Evaluate the following integrals: **(3+3 p)**

$$\int_0^{\ln 2} \frac{e^x}{\sqrt{1+e^x}}\,dx \quad \text{och} \quad \int \frac{x+4}{x^2+2x}\,dx$$

Solution. (a) We make the substitution $u = 1 + e^x$, and get $du = e^x dx$ and the new limits of integration $u(0) = 2$ and $u(\ln 2) = 3$, which gives

$$\int_0^{\ln 2} \frac{e^x}{\sqrt{1+e^x}}\,dx = \int_2^3 \frac{du}{\sqrt{u}} = \left[2\sqrt{u}\right]_2^3 = 2(\sqrt{3}-\sqrt{2}).$$

(b) We begin by making a partial fraction decomposition. Since $\frac{x+4}{x^2+2x} = \frac{x+4}{x(x+2)}$ we try to find constants A and B such that

$$\frac{x+4}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}.$$

By multiplying with the denominator $x(x+2)$ we get $x+4 = A(x+2) + Bx$, which can be written

$$x + 4 = (A+B)x + 2A.$$

By identifying the corresponding coefficients on each side we get the system of equations

$$\begin{cases} A + B = 1 \\ 2A = 4, \end{cases}$$

which has the unique solution $A = 2, B = -1$. Hence we have

$$\frac{x+4}{x(x+2)} = \frac{2}{x} - \frac{1}{x+2}.$$

By integrating this we get

$$\int \frac{x+4}{x^2+2x}\,dx = \int \left(\frac{2}{x} - \frac{1}{x+2}\right) dx = 2\ln|x| - \ln|x+2| + C$$

Answer: a) $2(\sqrt{3}-\sqrt{2})$, b) $2\ln|x| - \ln|x+2| + C$.

□

DEL B

3. Let $f(x) = x \ln x$.
   (a) Find the Taylor polynomial of degree 2 for f about $x = 1$ and use it to find an approximate value of $3 \ln 3$. **(2 p)**
   (b) Is it true that the approximate value found in (a) differs by at most $1/10$ from the true value of $3 \ln 3$? **(3 p)**
   (c) Is the approximate value larger or smaller than $3 \ln 3$? **(1 p)**

Solution. We first notice that

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1, \; f''(x) = \frac{1}{x}, \; f'''(x) = -\frac{1}{x^2}.$$

(a) The Taylor polynomial that we are looking for is given by

$$P(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) + \frac{1}{2}(x-1)^2.$$

If we use this polynomial to get an approximate value of $3 \ln 3$ we get

$$3 \ln 3 = f(3) \approx P(3) = 2 + 2 = 4.$$

(b) By Taylor's formula we have

$$f(x) = P(x) + \frac{f'''(s)}{3!}(x-1)^3$$

where s is a number between 1 and x. Applying this formula with $x = 3$ yields

$$|3 \ln 3 - 4| = |f(3) - P(3)| = \left| \frac{f'''(s)}{3!}(3-1)^3 \right| = \left| \frac{1}{s^2} \cdot \frac{1}{3!} \cdot 2^3 \right| = \frac{4}{3s^2}.$$

where s is a number such that $1 < s < 3$. The question now is whether $|3 \ln 3 - 4|$ is larger or smaller than $1/10$. Since the function $g(x) = \frac{1}{x^2}$ is decreasing on $(0, \infty)$, and since $1 < s < 3$, we have $1/s^2 > 1/3^2$, which implies that

$$\frac{4}{3s^2} > \frac{4}{3 \cdot 3^2} = \frac{4}{27} > \frac{4}{40} = \frac{1}{10}.$$

Thus the approximate value 4 differs from $3 \ln 3$ by more than $1/10$.

(c) As we saw above (by applying Taylor's formula)

$$f(3) - P(3) = \frac{f'''(s)}{3!}(3-1)^3 = -\frac{8}{s^2 3!}$$

where $1 < s < 3$. Since the right hand side is negative it therefore follows that $f(3) - P(3) < 0$, that is, $3 \ln 3 - 4 < 0$. Thus the approximate values is larger than $3 \ln 3$.

Answer: a) $P(x) = (x-1) + \frac{1}{2}(x-1)^2$, and $3 \ln 3 \approx P(3) = 4$; b) The approximate value differs by more than $1/10$; c) the approximate is larger than $3 \ln 3$. $\square$

4. How many solutions does the equation $\dfrac{1}{x} + 2\arctan x = 3$ have?

Solution. Let
$$f(x) = \frac{1}{x} + 2\arctan x.$$
We note that f is defined for all $x \neq 0$. We also note that the given equation may be written $f(x) = 3$.

Since $1/x < 0$ ad $\arctan x < 0$ for all $x < 0$ we have $f(x) < 0$ for all $x < 0$. Thus the equation $f(x) = 3$ cannot have any solution in the interval $(-\infty, 0)$. It is thus enough to study the function $f(x)$ on the interval $(0, \infty)$.

We have
$$\lim_{x \to 0^+} f(x) = \infty \tag{1}$$
and
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{1}{x} + 2\arctan x \right) = 0 + 2 \cdot \frac{\pi}{2} = \pi > 3. \tag{2}$$
Moreover,
$$f'(x) = -\frac{1}{x^2} + \frac{2}{1+x^2} = \frac{2x^2 - (1+x^2)}{x^2(1+x^2)} = \frac{x^2-1}{x^2(1+x^2)}.$$
Hence the critical points are given by $x^2 - 1 = 0$, that is, f has the critical points $x = \pm 1$. As we noted above we only need to investigate f on the interval $(0, \infty)$. We get the following sign chart for the derivative: $f'(x) < 0$ for $0 < x < 1$, and $f'(x) > 0$ for $x > 1$. Thus f is strictly decreasing on $(0, 1]$ and strictly increasing on $[1, \infty)$.

Since f is strictly decreasing on the interval $(0, 1]$ the equation $f(x) = 3$ can have at most one solution in $(0, 1]$; and since f is strictly increasing on $[1, \infty)$ the equation $f(x) = 3$ can have at most one solution in $[1, \infty)$.

We now note that $f(1) = -1 + 2\arctan(1) = 2\frac{\pi}{4} - 1 = \frac{\pi}{2} - 1 < 3$. Since f is continuous on the interval $(0, 1]$, and since $f(1) < 3$ and (1) above holds (for example, we have $f(1/10) > 3$), the intermediate value theorem implies that there must be an x_1 in the interval $(0, 1)$ such that $f(x_1) = 3$. Since f is continuous on the interval $[1, \infty)$, and since $f(1) < 3$ and (2) above holds, the intermediate value theorem implies that there must be an x_2 in the interval $(1, \infty)$ such that $f(x_2) = 3$.

Conclusion: the equation $f(x) = 3$ has exactly two solutions.

Answer: the equation has exactly two solutions.

□

DEL C

5. Show that the following holds for all positive integers m och n:

$$\sum_{k=n+1}^{2n} \frac{1}{k} \leq \ln 2 \leq \sum_{k=m}^{2m-1} \frac{1}{k}.$$

Solution. Let $f(x) = 1/x$ an let l be a positive integer. Since f is decreasing on the interval $(0, \infty)$ we have that for each integer $k \geq 1$ there holds $1/(k+1) \leq 1/x \leq 1/k$ for all x such that $k \leq x \leq (k+1)$. This implies that

$$\frac{1}{k+1} \leq \int_k^{k+1} \frac{dx}{x} \leq \frac{1}{k}$$

(note that the length of the interval $[k, k+1]$ us 1; draw a figure). By using the first inequality we get

$$\sum_{k=l+1}^{2l} \frac{1}{k} = \frac{1}{l+1} + \frac{1}{l+2} + \ldots + \frac{1}{2l} \leq \int_l^{l+1} \frac{dx}{x} + \int_{l+1}^{l+2} \frac{dx}{x} + \ldots + \int_{2l-1}^{2l} \frac{dx}{x} =$$
$$= \int_l^{2l} \frac{dx}{x} = [\ln|x|]_l^{2l} = \ln(2l) - \ln(l) = \ln(2l/l) = \ln 2.$$

And if we use the second inequality we get

$$\sum_{k=l}^{2l-1} \frac{1}{k} = \frac{1}{l} + \frac{1}{l+1} + \ldots + \frac{1}{2l-1} \geq \int_l^{l+1} \frac{dx}{x} + \int_{l+1}^{l+2} \frac{dx}{x} + \ldots + \int_{2l-1}^{2l} \frac{dx}{x} =$$
$$= \int_l^{2l} \frac{dx}{x} = [\ln|x|]_l^{2l} = \ln(2l) - \ln(l) = \ln(2l/l) = \ln 2.$$

(Hint: draw a figure.)

By applying this with $l = n$ and $l = m$, respectively, we now get

$$\sum_{k=n+1}^{2n} \frac{1}{k} \leq \ln 2 \leq \sum_{k=m}^{2m-1} \frac{1}{k}.$$

□

6. (a) Assume that $f(0) = 0$ and that $|f(x)| > \sqrt{|x|}$ for all $x \neq 0$. Show that the function f is not differentiable at $x = 0$. **(3 p)**

   (b) Assume that the function g is defined on the whole real line and satisfies the following conditions: $g'(0) = k,\ g(0) \neq 0$ and $g(x + y) = g(x)g(y)$ for all x and y. Show that $g(0) = 1$ and that $g'(x) = kg(x)$ for all x. **(3 p)**

Solution. For the function f to be differentiable at $x = 0$ the limit

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

has to exist. And if this limit exists, the limit

$$\lim_{h\to 0} \left| \frac{f(h) - f(0)}{h} \right| = |f'(0)| \tag{3}$$

must also exist (since the function $g(x) = |x|$ is continuous).

From the assumptions on f we have

$$\left| \frac{f(h) - f(0)}{h} \right| = \frac{|f(h)|}{|h|} \geq \frac{\sqrt{|h|}}{|h|} = \frac{1}{\sqrt{|h|}}.$$

Since $\lim_{h\to 0} \frac{1}{\sqrt{|h|}} = \infty$ (note that this is an infinite limit; the limit does not exist) it follows from the above inequality that

$$\lim_{h\to 0} \left| \frac{f(h) - f(0)}{h} \right| = \infty,$$

(that is, the limit does not exist). Thus the limit (3) above does not exist, which implies that f is not differentiable at $x = 0$.

(b) By applying the relation $g(x + y) = g(x)g(y)$ with $x = y = 0$ we get

$$g(0) = g(0)g(0).$$

Since we have assumed that $g(0) \neq 0$ we can divide by $g(0)$, which gives $g(0) = 1$.

From the assumption $g'(0) = k$ we thus know that

$$k = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h) - 1}{h}.$$

Here we use that $g(0) = 1$.

Fix a real number x. We then get, by using the assumption on g, that

$$\frac{g(x + h) - g(x)}{h} = \frac{g(x)g(h) - g(x)}{h} = \frac{g(x)(g(h) - 1)}{h}.$$

This gives, if we use the limit above, that

$$g'(x) = \lim_{h\to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h\to 0} \frac{g(x)(g(h) - 1)}{h} = g(x) \left(\lim_{h\to 0} \frac{g(h) - 1}{h} \right) = g(x)k.$$

$\square$