SF1625 Envariabelanalys
Solutions to exam 2021.01.07

DEL A

1. (a) Evaluate the integral $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\, dx.$ **(3 p)**

   (b) Find a primitive function of $f(x) = \arctan x$. **(3 p)**

Solution. (a) We make the substitution $u = \cos x$ and get $du = -\sin x dx$ and the new limits of integration $u(0) = 1$ och $u(\pi/4) = 1/\sqrt{2}$, which gives

$$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\, dx = -\int_1^{1/\sqrt{2}} \frac{du}{u^2}\, du = \int_{1/\sqrt{2}}^{1} \frac{du}{u^2} = \left[-\frac{1}{u}\right]_{1/\sqrt{2}}^{1} = \sqrt{2} - 1.$$

(b) Integration by parts gives

$$\int x \cos 2x\, dx = x\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}\, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C.$$

Hence, for example $F(x) = \dfrac{x \sin 2x}{2} + \dfrac{\cos 2x}{4}$ is a primitive function of f. $\square$

2. Let $f(x) = \ln(1+x^2)$.
   (a) Find the domain of definition of f and calculate $f'(x)$. **(2 p)**
   (b) Find the Taylor polynomial of order 2 for f about $x = 0$. **(2 p)**
   (c) Evaluate the limit $\lim_{x\to 0} \dfrac{f(x)}{x^2}$. **(2 p)**

Solution. (a) We know that $\ln t$ is defined for all $t > 0$. Since $1 + x^2 \geq 1 > 0$ for all real numbers x we thus have that $f(x) = \ln(1+x^2)$ is defined for all real numbers x. Furthermore, differentiation gives

$$f'(x) = \frac{1}{1+x^2} \cdot 2x = \frac{2x}{1+x^2}.$$

(b) The Taylor polynomial we are looking for is given by $P(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$. Since

$$f''(x) = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

we now get

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = \ln 1 + 0 \cdot x + \frac{2}{2}x^2 = x^2.$$

(c) From (b) above it follows that $\ln(1+x^2) = x^2 + O(x^3)$ when $x \to 0$. This gives

$$\lim_{x\to 0} \frac{\ln(1+x^2)}{x^2} \lim_{x\to 0} \frac{x^2 + O(x^3)}{x^2} = \lim_{x\to 0} \frac{1+O(x)}{1} = 1.$$

One can also apply l'Hôpitals rule to evaluate the limit.

□

DEL B

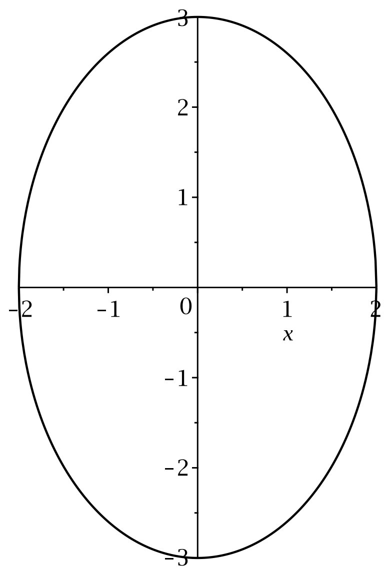
3. (a) Parametrize the curve $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. Also sketch the curve. **(2 p)**

   (b) What is the maximal area of a rectangle if its corners are on the curve in (a) and its sides are parallel to the coordinate axes? **(4 p)**

Solution. (a) One parametrization is

$$\begin{cases} x(t) = 2\cos t \\ y(t) = 3\sin t \end{cases}$$

where t runs from 0 to 2π. The curve (ellipse) looks like this:



(b) The rectangle becomes symmetric so it is enough to find the area of the part that lies in the first quadrant, and multiply this area by 4.

Then we try to optimize the area of a rectangle with corners at $(0,0)$, $(x,0)$, $(y,0)$ and (x,y), where also $x^2/4 + y^2/9 = 1$. The last condition means (note that we are in the first quadrant)

$$y = 3\sqrt{1 - \frac{x^2}{4}}.$$

The area in the first quadrant is thus given by

$$xy = 3x\sqrt{1 - \frac{x^2}{4}}$$

where x can be between 0 and 2. Thus, we need to maximize the function

$$f(x) = 3x\sqrt{1 - \frac{x^2}{4}} \text{ when } x \in [0, 2].$$

The function is continuous on the closed and bounded interval so we know that maximum exists. It can be attained at a critical point, a singular point or an endpoint of the interval. At the endpoints 0 and 2 the function has the value 0. We differentiate and get

$$f'(x) = 3\sqrt{1 - \frac{x^2}{4}} + 3x\frac{-2x/4}{2\sqrt{1 - \frac{x^2}{4}}} = 3\frac{1 - \frac{x^2}{2}}{\sqrt{1 - \frac{x^2}{4}}}.$$

We see that f is differentiable on the interval $(0, 2)$ and the derivative has one single zero in this interval, namely at $x = \sqrt{2}$. It follows that this must be our maximum point and that the largest value of f is $f(\sqrt{2}) = 3$. Since this value is a quarter of the maximal area that we are looking for, it thus follows that this area is 12.

Answer: 12

$\square$

4. Let $f(x) = \dfrac{\ln x}{x^2}$.
   (a) Make a sign chart for the derivative and find all local extreme points. Use the sign chart, together with appropriate limits, to sketch the curve $y = f(x)$. **(3 p)**
   (b) Determine which of the following improper integrals converge:

$$\int_0^1 f(x)\,dx, \qquad \int_1^\infty f(x)\,dx.$$

   Evaluate the integrals in case they converge. **(3 p)**

Solution. (a) We that f is defined for all $x > 0$ and that

$$f'(x) = \frac{(1/x)x^2 - (\ln x)(2x)}{x^4} = \frac{x - 2x(\ln x)}{x^4} = \frac{1 - 2\ln x}{x^3} \text{ for } x > 0.$$

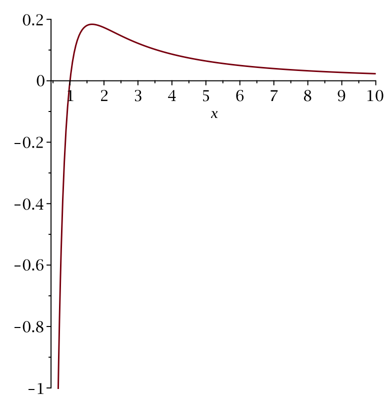
From this is follows that $f'(x) = 0$ exactly when $2\ln x = 1$, that is, when $x = e^{1/2}$. We get the following sign chart for the derivative: $f'(x) > 0$ for $0 < x < e^{1/2}$, and $f'(x) < 0$ for $x > e^{1/2}$. Hence $x = e^{1/2}$ is a local maximum point (in fact the function f also has an absolute maximum at this point). To sketch the curve we evaluate the following limits:

$$\lim_{x\to 0^+} \frac{\ln x}{x^2} = -\infty$$

and

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = 0.$$

We now get the following sketch:



(b) We first look for at primitive function of $f(x)$. Integration by parts gives

$$\int \frac{\ln x}{x^2}\,dx = -\left(\frac{1}{x}\right)\ln x - \int \frac{1}{x}\left(-\frac{1}{x}\right)\,dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\left(\frac{\ln x}{x} + \frac{1}{x}\right) + C.$$

Here we used that the derivative of $\ln x$ is $1/x$ and that $-1/x$ is a primitive function of $1/x^2$.

This now gives

$$\int_0^1 f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 f(x)\,dx = \lim_{\varepsilon\to 0^+}\left[-\left(\frac{\ln x}{x}+\frac{1}{x}\right)\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+}\left(1+\frac{1+\ln\varepsilon}{\varepsilon}\right) = -\infty.$$

Thus this integral is divergent.

Furthermore,

$$\int_1^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx = \lim_{R\to\infty}\left[-\left(\frac{\ln x}{x}+\frac{1}{x}\right)\right]_1^R = \lim_{R\to\infty}\left(-\frac{\ln R}{R}-\frac{1}{R}+1\right) = 1.$$

The integral is thus convergent, and the value is 1.

$\square$

DEL C

5. Let the function f be defined for all real numbers x by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

(a) At which points is f continuous? **(2 p)**
(b) At which points is f differentiable? **(2 p)**
(c) Find the range of f. **(2 p)**

Solution. (a) Since f is given by an elementary expression in the interval $(0, \infty)$ if follows that f is continuous at all points in this interval. Since f is given by an elementary expression in the interval $(-\infty, 0)$ if follows that f is continuous at all points in this interval. Since $\lim_{x \to 0} f(x) = 0 = f(0)$ it follows that f is continuous at 0. Hence f is continuous on the whole real line.

(b) Let $x \neq 0$. Then $f'(x) = e^{-1/x^2} \cdot \frac{2}{x^3}$, so we see that f at least is differentiable for all $x \neq 0$. At the point 0 we have to use the definition of the derivative. We see that

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h} = 0$$

so f is differentiable also at 0. Hence f is differentiable on the whole real line. Note that

$$\lim_{h \to 0^+} \frac{e^{-1/h^2}}{h} = \lim_{t \to \infty} \frac{e^{-t^2}}{1/t} = \lim_{t \to \infty} t e^{-t^2} = 0$$

and

$$\lim_{h \to 0^-} \frac{e^{-1/h^2}}{h} = \lim_{t \to -\infty} \frac{e^{-t^2}}{1/t} = \lim_{t \to -\infty} t e^{-t^2} = 0.$$

Thus we indeed have $\lim_{h \to 0} \dfrac{e^{-1/h^2}}{h} = 0$.

(c) We see that $f'(x) > 0$ on the interval $x > 0$ so f is strictly increasing here. We see that $f'(x) < 0$ on the interval $x < 0$ so f is strictly decreasing here. Moreover, $f'(0) = 0$ as we saw above. Hence f must attain its smallest value at the point $x = 0$ and this smallest value is 0. From the above it also follows that f cannot have a largest value, but since $\lim_{x \to \pm\infty} f(x) = 1$ and f is continuous and decreasing and increasing, respectively, as we noted above, if follow that the domain is $[0, 1)$

Svar: (a) All x
(b) All x
(c) $[0, 1)$

□

6. Show that for every constant $c > 0$ the following holds: **(6 p)**

$$\frac{\pi}{2\sqrt{c}} \leq \sum_{n=0}^{\infty} \frac{1}{n^2+c} \leq \frac{\pi}{2\sqrt{c}} + \frac{1}{c}$$

Solution. Take $c > 0$ and let $f(x) = \dfrac{1}{x^2+c}$. We note that f is positive and continuous and that f is decreasing on the interval $[0, \infty)$. We have

$$\int_0^\infty \frac{dx}{x^2+c} = \lim_{R\to\infty} \int_0^R \frac{dx}{x^2+c} = \lim_{R\to\infty} \int_0^R \frac{dx}{c(1+(x/\sqrt{c})^2)} = \lim_{R\to\infty} \left[\frac{1}{\sqrt{c}} \arctan(x/\sqrt{c})\right]_0^R =$$
$$= \lim_{R\to\infty} \frac{1}{\sqrt{c}} \arctan(R/\sqrt{c}) = \frac{\pi}{2\sqrt{c}}.$$

Since this improper integral is convergent it follows from Cauchys integral test that the series

$$\sum_{n=0}^{\infty} \frac{1}{n^2+c}$$

is convergent.

Since f is positive and decreasing on $[0, \infty)$ it follows that we for each $n \geq 0$ have $f(n+1) \leq f(x) \leq f(n)$ for all $n \leq x \leq n+1$. This gives (draw a figure)

$$f(n+1) \leq \int_n^{n+1} f(x)\,dx \leq f(n) \tag{1}$$

for all $n \geq 0$. If we now use this inequality we get

$$\sum_{n=1}^{N} \frac{1}{n^2+c} \leq \int_0^N \frac{dx}{x^2+c} < \int_0^\infty \frac{dx}{x^2+c} = \frac{\pi}{2\sqrt{c}}$$

for all $N \geq 1$. Since this holds for all $N \geq 1$ we must have

$$\sum_{n=1}^{\infty} \frac{1}{n^2+c} \leq \frac{\pi}{2\sqrt{c}}$$

and thus

$$\sum_{n=0}^{\infty} \frac{1}{n^2+c} = \frac{1}{c} + \sum_{n=1}^{\infty} \frac{1}{n^2+c} \leq \frac{1}{c} + \frac{\pi}{2\sqrt{c}}.$$

The inequality (1) also gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2+c} > \sum_{n=0}^{N} \frac{1}{n^2+c} \geq \int_0^{N+1} \frac{dx}{x^2+c}$$

for all $N \geq 1$. Since this holds for all $N \geq 1$ we must have

$$\sum_{n=1}^{\infty} \frac{1}{n^2+c} \geq \int_0^\infty \frac{dx}{x^2+c} = \frac{\pi}{2\sqrt{c}}.$$

Thus we have shown that

$$\frac{\pi}{2\sqrt{c}} \leq \sum_{n=0}^{\infty} \frac{1}{n^2+c} \leq \frac{\pi}{2\sqrt{c}} + \frac{1}{c}.$$

□