KTH Teknikvetenskap

SF1684 Algebra and geometri
Exam
Friday April 22 april, 2022

Time: 08:00–11:00
No books/notes/calculators etc. allowed

Examinator: Maria Saprykina

This exam consists of six problems, each worth 6 points. Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points. The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 3 points.

DEL A

1. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

(a) Determine a 2×2 matrix P such that $P^{-1}AP$ is a diagonal matrix. **(4 p)**

(b) Determine $A^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. **(2 p)**

2. Let

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Determine the angle between $\vec{u}$ and $\vec{v}$. **(3 p)**

(b) Determine the intersection point between the line $(x, y, z) = (-1 + t, 2 + 2t, 2t)$ and the plane through the origine that is spanned by the vectors $\vec{u}$ and $\vec{v}$. **(3 p)**

DEL B

3. Let Π be the plane given by the equation $2x + 3y - 6z = 0$.

(a) Determine the orthogonal projection of the vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ on the plane Π. **(3 p)**

(b) Determine the standard matrix A for the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that corresponds to the orthogonal projection on the plane Π. **(2 p)**

(c) Determine if the matrix A is invertible. **(1 p)**

4. Let $P = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ be the transition matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ where $\mathcal{B}$ and $\mathcal{C}$ are bases for the same subspace V of $\mathbb{R}^3$.

(a) Determine the dimension of the subspace V. **(1 p)**

(b) Determine the transition matrix from the basis $\mathcal{C}$ to the basis $\mathcal{B}$. **(2 p)**

(c) Give an example of a subspace V and bases $\mathcal{B}$ and $\mathcal{C}$ such that P is the transition matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$. **(3 p)**

DEL C

5. Let V be a k-dimensional subspace of $\mathbb{R}^n$, let M_{nm} denote the set of $n \times m$-matrices, and let W be the set of $n \times m$-matrices A that satisfies that $\mathrm{Range}(A)$ is a subspace of V.

(a) Show that W is a subspace of M_{nm}. **(3 p)**

(b) Determine the dimension of W. **(3 p)**

6. An $n \times n$-matrix A is said to be *expansive* if $\|A\vec{x}\| > \|\vec{x}\|$ for all non-zero $\vec{x}$ i $\mathbb{R}^n$. We say that $\vec{x_0}$ in $\mathbb{R}^n$ is a *fixed point* to a mapping $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ if $f(\vec{x_0}) = \vec{x_0}$.

(a) Show that if A is an $n \times n$-matrix such that all eigenvalues of A have absolute value greater that 1, then the matrix $A - I_n$ is invertible. **(2 p)**

(b) Show that if A is an expansive $n \times n$-matrix, then the matrix $A - I_n$ is invertible. **(2 p)**

(c) If $\vec{b}$ is a vector in $\mathbb{R}^n$ and A is an $n \times n$-matrix, we define a mapping $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$f(\vec{x}) = A\vec{x} + \vec{b}, \qquad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

Show that is A is expansive, then f must have a unique fixed point. **(2 p)**
