KTH VETENSKAP OCH KONST

KTH Teknikvetenskap

SF1684 Algebra and geometry
Exam
April 9, 2021

Time: 08:00-11:00
Help in any form (such as books/notes/computers/person etc.) is NOT allowed.
Any plagiarism found in solutions will be reported

Examiner: Danijela Damjanovic

This exam consists of six assignments, each worth 6 points.

Part A comprises the first two assignments. The bonus points from the seminars will be automatically added to the total score of the assignment 1, which however cannot exceed 6 points.

The next two assignments constitute part B, and the last two assignments part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

Instruktions

To obtain points on a problem, it is required that:

- The solution is well presented and easy to follow.
- The solutions are neatly written with a handwriting which is easy to read.
- The notations introduced are to be clearly defined, the logical structure clearly described in words or symbols. The reasoning should be well motivated and clearly explained. All steps in all calculations are to be presented clearly and should be easy to follow.

Solutions and answers without correct, detailed and clear justifications will receive no points.

0. Honor code. See Assignment 0 in Canvas. Honor code is mandatory and the exam will not be graded unless you have uploaded the signed honor code.

PART A

1. Let $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(a) Compute the transition matrix P which transforms from the stadard basis $\mathcal{E}$ to the basis $\mathcal{B} = \{\vec{u}, \vec{v}\}$. That is, compute P such that $P[\vec{x}]_{\mathcal{E}} = [\vec{x}]_{\mathcal{B}}$ for all $[\vec{x}]$. **(3 p)**

(b) Compute the coordinates of the vector $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the basis $\{\vec{u}, \vec{v}\}$. **(3 p)**

2. Find the parameters a and b such that the system of linear equations

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 9 & 3 \\ -1 & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix}$$

(a) has a unique solution. **(2 p)**
(b) has infinitely many solutions. **(2 p)**
(c) has no solution. **(2 p)**

Part B

3. Let H be the plane in $\mathbb{R}^3$ defined by the equation $x + 2y + 2z = 0$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto H.

(a) Determine the standard matrix for the map T. **(3 p)**

(b) Determine a basis of $\mathbb{R}^3$ in which the matrix representation of T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. **(3 p)**

4. An equilateral triangle ABC is contained in the plane $x - y + 2z = 0$. The triangle has vertex A at the origin, the second vertex B is at the point $(1, 1, 0)$, and the third vertex C has positive x-coordinate. Find the coordinates of the vertex C of the triangle. **(6 p)**

Part C

5. (a) Let V_1 and V_2 be two subspaces of $\mathbb{R}^n$. Prove that the intersection $W = V_1 \cap V_2$ is a subspace of $\mathbb{R}^n$. **(2 p)**

(b) Let $V_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}\right)$ and $V_2 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}\right)$. Let $W = V_1 \cap V_2$.

The linear transformation T is defined as the orthogonal projection of vectors in $\mathbb{R}^4$ onto the subspace W. Compute the standard matrix $[T]$ of the linear transformation T. **(4 p)**

6. Let A be an $n \times n$ matrix and let A^T be the transpose of A. Prove the following statements:

(a) A and $A^T A$ have the same null space. **(3 p)**

(b) If $n \geq 2$ there exist n orthogonal unit vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in $\mathbb{R}^n$ such that $A\vec{u}_1, A\vec{u}_2 \ldots, A\vec{u}_n$ are also orthogonal. **(3 p)**