KTH Teknikvetenskap

SF1624 Algebra and geometry
Exam
January 11, 2021

Time: 14:00-17:00
Help in any form (such as books/notes/computers/person etc.) is NOT allowed.
Any plagiarism found in solutions will be reported

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This exam consists of six assignments, each worth 6 points.

Part A comprises the first two assignments. The bonus points from the seminars will be automatically added to the total score of the assignment 1, which however cannot exceed 6 points.

The next two assignments constitute part B, and the last two assignments part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

Instruktions

To obtain points on a problem, it is required that:

- The solution is well presented and easy to follow.
- The solutions are neatly written with a handwriting which is easy to read.
- The notations introduced are to be clearly defined, the logical structure clearly described in words or symbols. The reasoning should be well motivated and clearly explained. All steps in all calculations are to be presented clearly and should be easy to follow.

Solutions and answers without correct, detailed and clear justifications will receive no points.

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0. Honor code. See Assignment 0 in Canvas. Honor code is mandatory and the exam will not be graded unless you have uploaded the signed honor code.

Part A

1. Let the following system of equations be given:

$$\begin{aligned} x_1 - 3x_2 + 4x_3 + 5x_4 &= 2 \\ 2x_2 + 3x_3 + 4x_4 &= 1 \\ -3x_1 + 10x_2 - 6x_3 - 7x_4 &= -4 \end{aligned}$$

(a) Determine all solutions to this system of equations. **(3 p)**

(b) Let V denote the subspace of $\mathbb{R}^4$ defined as $V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -3 \\ 10 \\ -6 \\ -7 \end{bmatrix}.$$

Find a basis for $V^{\perp}$. **(3 p)**

2. Compute the determinant $\det A$ for **(6 p)**

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

PART B

3. Let V be the plane in $\mathbb{R}^3$ defined by the equation $2x - y + 3z = 0$, and let L denote the line in $\mathbb{R}^3$ described by the parametric equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \qquad t \in \mathbb{R}.$$

(a) Determine the intersection of the plane V and the line L. **(3 p)**
(b) Determine a system of equations whose set of solutions is L. **(3 p)**

4. Let the following two vectors be given:

$$\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Find a vector $\vec{v}_3$ such that $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ form an orthonormal basis $\mathcal{B}$ for $\mathbb{R}^3$. **(2 p)**
(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that maps $\vec{v}_1$ to $\vec{v}_3$ and $\vec{v}_3$ to $\vec{v}_1$, and for which $\vec{v}_2$ is an eigenvector corresponding to the eigenvalue $\lambda = -5$. Find the coordinate matrix $[T]_{\mathcal{B}}$ that describes the map T with respect to the basis $\mathcal{B}$. **(4 p)**

PART C

5. Two $n \times n$-matrices A and B are said to be *simultaneously diagonalizable* if there exists a common basis consisting of eigenvectors, i.e. if there exists an invertible $n \times n$ matrix S such that both $S^{-1}AS$ and $S^{-1}BS$ are diagonal matrices.

(a) Show that $AB = BA$ if A and B are simultaneously diagonalizable. **(3 p)**
(b) Show that A and B are simultaneously diagonalizable if we assume that $AB = BA$ and that A has n distinct eigenvalues. **(3 p)**

6. If A is a real symmetric 3×3 matrix let $q_A(\vec{x}) = \vec{x}^T A\vec{x}$ denote the quadratic form associated to A.

(a) Let λ be an eigenvalue of A and let $\vec{v}$ be the corresponding eigenvector which is normalised so that $\|\vec{v}\| = 1$. Compute $q_A(\vec{v})$. **(1 p)**
(b) Suppose that the smallest eigenvalue of A is 1. Prove that $q_A(\vec{x}) \geq 1$ for any vector $\vec{x}$ such that $\|\vec{x}\| = 1$. (Hint: show first that $q_A(\vec{x}) \geq 1$ for any unit vector $\vec{x}$ in the case when A is diagonal). **(5 p)**