KTH Teknikvetenskap

SF1624 Algebra och geometri
Exam
Thursday, 18 April 2019

Time: 08:00-11:00
No books/notes/calculators etc. allowed
Examiner: Danijela Damjanović, Tilman Bauer

This exam consists of six problems, each worth 6 points.

Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 3 points.

Part A

1. Consider the following system of equations:

$$\begin{cases} x + ay + z = -1 \\ ax + y + az = 1 \\ -2x + (1-2a)y - z = 3 \end{cases}$$

(a) For which values of the number a does the system have no solutions? **(2 p)**

(b) Find a value a such that the system have infinitely many solutions and determine these solutions. **(4 p)**

2. Let A be the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

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p.t.o.

Part B

3. The plane Π in 3-space contains the three points

$$(1,0,0),\ (2,3,0),\ (1,3,1).$$

(a) Determine an equation of the form $ax+by+cz=d$ for the plane Π. **(2 p)**

(b) Determine a parametric form for the line L passing through the origin and orthogonal to the plane Π. **(2 p)**

(c) Find the point of intersection of L and Π. **(2 p)**

4. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

be the matrix associated to a linear map T in the basis $\mathcal{B} = \{(1,1,1),(0,2,2),(0,0,3)\}$.

(a) Determine the dimension of the kernel $\ker(T)$ and the image $\operatorname{ran}(T)$. **(2 p)**

(b) Determine the standard matrix of T. **(4 p)**

Part C

5. The function $f(t) = Ce^{kt}$ describes the mass (in grams) of a radioactive substance at the time t (hours). The mass $f(t)$ has been measured at the following times:

- At the time $t=0$, the substance measured 1 gram.
- At the time $t=1$, the substance measured 0.8 grams.
- At the time $t=2$, the substance measured 0.5 grams.

Determine the constants C and k such that $f(t) = Ce^{kt}$ best fits the measurements in least-squares sense. **(6 p)**

6. Let $\vec{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. Assume that $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$. Consider the following matrix:

$$A = \vec{v}(\vec{w})^T = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$$

(a) Prove that A has rank 1 and that $\operatorname{col}(A) = \operatorname{span}(\vec{v})$. **(2 p)**

(b) Prove that $\vec{v}$ is an eigenvector to A with eigenvalue $\vec{v}\cdot\vec{w}$. **(1 p)**

(c) Under which constraints on $\vec{v}$ and $\vec{w}$ is A diagonalizable? **(3 p)**