SF1625 Calculus in one variable
Solutions to the exam 2017-06-09

Part A

1. The half life of the radioactive isotope carbon 14 is approximately 5730 years. Living organisms keep an approximately constant level of carbon 14, but in dead organisms the amount of the substance is decreasing at a rate proportional to the amount remaining, that is the amount $y(t)$ satisfies a differential equation of the form $y' = ky$ for some constant k. A certain bone fragment contains 80% of the original amount of carbon 14. How old is the bone fragment? **(4 p)**

   Solution. The amount y of carbon 14 in the bone fragment at time t satisfies $y'(t) = ky(t)$ for some constant k, i.e. $y(t) = Ce^{kt}$ where C is the original amount. Since the half life is 5730 years we can determine k:

$$e^{k5730} = \frac{1}{2} \iff k = \frac{-\ln 2}{5730}.$$

   The age T of the bone fragment satisfies

$$Ce^{(-T\ln 2)/5730} = 0.8C \iff T = -5730\frac{\ln 0.8}{\ln 2} \ (\approx 1800) \ \text{år}.$$

$\square$

2. Let R denote the bounded area in the first quadrant that is above the curve $y = x^2$, and below the curve $y = 8 - x^2$. Determine the volume of the solid of revolution generated when the bounded area R is rotated around the y-axis. **(4 p)**

   Solution. The points of intersection between the curves satisfies $x^2 = 8 - x^2$ that is, the only point of intersection in the first quadrant is $x = 2$. The volumes of the solids of revolution can now be computed as

$$2\pi \int_0^2 x(8 - 2x^2)\,dx = 2\pi \left[4x^2 - \frac{1}{2}x^4\right]_{x=0}^{x=2} = 16\pi.$$

$\square$

3. The ellips E is the solutions to the equation $3x^2 + 4y^2 = 5$. Determine an equation for the line L that is tangent to E at the point $P = (1, \sqrt{2}/2)$. **(4 p)**

Solution. Implicit derivation gives $6x + 8y\frac{dy}{dx} = 0$. At the point $P = (1, \sqrt{2}/2)$ we obtain

$$\frac{dy}{dx}_{|P} = -\frac{6x}{8y}_{|P} = -\frac{3}{4} \cdot \frac{2}{\sqrt{2}} = -\frac{3\sqrt{2}}{4},$$

which the slope k of our sought tangent line. An equation is of the form $y = kx + m$, and the point P should satisfy the equation. This gives

$$\sqrt{2}/2 = -\frac{3\sqrt{2}}{4} \cdot 1 + m,$$

and that $m = 5\sqrt{2}/4$. □

PART B

4. Compute the integrals below.
   (a) $\displaystyle\int_1^e x^5 \ln x \, dx.$ **(2 p)**
   (b) $\displaystyle\int_3^4 \frac{6}{x^2 - x - 2} \, dx.$ **(2 p)**

Solution. a) Integration by parts yields

$$\begin{aligned}\int_1^e x^5 \ln x \, dx &= \left[\frac{x^6}{6} \ln x\right]_1^e - \int \frac{x^5}{6} \, dx \\ &= \frac{e^6}{6} - \left[\frac{x^6}{36}\right]_1^e = \frac{e^6}{6} - \frac{e^6}{36} + \frac{1}{36} \\ &= \frac{1}{36}(5e^6 + 1).\end{aligned}$$

b) We have that $x^2 - x - 2 = (x - 2)(x + 1)$, and consequently that

$$\frac{6}{x^2 - x - 2} = \frac{A}{x - 2} + \frac{B}{x + 1},$$

for some numbers A and B. These numbers are determined by the equation

$$6 = A(x + 1) + B(x - 2) = (A + B)x + (A - 2B) \cdot 1.$$

This gives us $A = -B$, and that $6 = A - 2B = -3B$. In other words we have

$$\int_3^4 \frac{6}{x^2 - x - 1} dx = \int_3^4 \frac{2}{x - 2} dx + \int_3^4 \frac{-2}{x + 1} dx.$$

The sought integral is

$$2 \left[\ln(x - 2)\right]_{x=3}^{x=4} - 2 \left[\ln(x + 1)\right]_{x=3}^{x=4} = 2 \ln(2) - 2 \ln(5) + 2 \ln(4).$$

□

5. Let $P(x)$ be the second order Taylor polynomial around $x = 0$ to the function $f(x) = e^x$.
   (a) Use $P(x)$ to give an approximation of $\sqrt{e}$. **(2 p)**
   (b) Determine whether the error is less or bigger than 0.02. **(2 p)**

Solution. Taylor expansions of e^x around origo gives

$$P(x) = 1 + x + \frac{x^2}{2}.$$

As $\sqrt{e} = e^{1/2}$ we get that

$$\sqrt{e} = e^{1/2} \approx P(1/2) = 1 + \frac{1}{2} + \frac{1}{8} = \frac{13}{8}.$$

b) The error in this approximation, at the point $x = 1/2$, is

$$E_3(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{e^6}{6} \cdot \left(\frac{1}{2}\right)^3,$$

for some $0 < c < 1/2$. As $e^c > 1$, we obtain that

$$\frac{e^c}{6} \cdot \left(\frac{1}{2}\right)^3 \geq \frac{1}{48} > 0.02$$

because $c \geq 0$. The error is bigger than 0.02 □

6. We study the function f given by $f(x) = \dfrac{x^3}{2x^2-1}$. Sketch the curve $y = f(x)$ after investigating where f is increasing and decreasing, respectively, what local extreme values f has, where the zeros of the function are, and what the asymptotes to the graph are.

(4 p)

Solution. We see that f is defined and continuous for all $x \neq \pm 1/\sqrt{2}$. Since f is unbounded when x approaches these points, we have found two vertical asymptotes, namely

$$x = \frac{1}{\sqrt{2}} \quad \text{and} \quad x = -\frac{1}{\sqrt{2}}.$$

In order to find asymptotes at $\pm\infty$ we rewrite $f(x)$ using polynomial division as $f(x) = \frac{x}{2} + \frac{x}{4x^2-2}$. We see that $y = x/2$ is oblique asymptote as $x \to \pm\infty$.

Now we look for local extreme values.

$$f'(x) = \frac{2x^4 - 3x^2}{(2x^2-1)^2}$$

and we see that

$$f'(x) = 0 \Leftrightarrow x = 0 \;\text{ or }\; x = \pm\sqrt{\frac{3}{2}}.$$

We study the sign of the derivative:

If $x < -\sqrt{3/2}$ then $f'(x) > 0$ and the function is strictly increasing.

If $-\sqrt{3/2} < x < -1/\sqrt{2}$ then $f'(x) < 0$ and the function is strictly decreasing.

If $-1/\sqrt{2} < x < 1/\sqrt{2}$ then $f'(x) \leq 0$ with equality only if $x = 0$ and the function is strictly increasing

If $1/\sqrt{2} < x < \sqrt{3/2}$ then $f'(x) < 0$ and the function is strictly decreasing

If $x > \sqrt{3/2}$ then $f'(x) > 0$ and the function is strictly increasing.

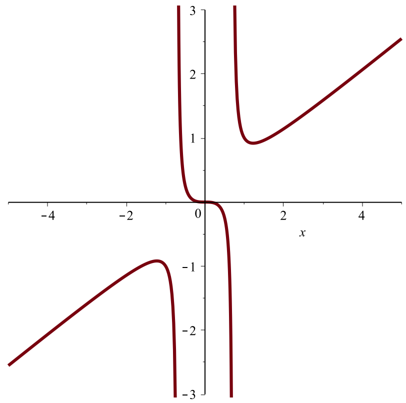
It follows that f has a local maximum at $x = -\sqrt{3/2}$ and a local minimum at $x = \sqrt{3/2}$ while $x = 0$ is not a local extreme point.

We have $f(x) = 0$ only if $x = 0$ and $\lim_{x\to\infty} f(x) = \infty$ och $\lim_{x\to-\infty} f(x) = -\infty$. Other relevant limits are

$$\lim_{x\to -1/\sqrt{2}^-} f(x) = -\infty, \qquad \lim_{x\to -1/\sqrt{2}^+} f(x) = \infty,$$

$$\lim_{x\to 1/\sqrt{2}^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 1/\sqrt{2}^+} f(x) = \infty$$

Now we can sketch the graph $y = f(x)$:


3
2
1
−1
−2
−3
−4
−2
0
2
4
x

□

PART C

7. (a) For what real x does $\sin(\arcsin x) = x$ hold? **(2 p)**
   (b) Use implicit differentation of the above identity to get a formula for the derivative of $\arcsin x$. **(2 p)**

Solution. a) The identity hos for all x such that $-1 \leq x \leq 1$. b) Implicit differentiation gives

$$(\cos(\arcsin x)) \cdot \frac{d}{dx}(\arcsin x) = 1.$$

Solving for the derivative of $\arcsin x$ gives

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}},$$

where last identity follows from $\cos^2(y) + \sin^2(y) = 1$. □

8. Prove, by computing an integral, the formula $A = \pi ab$ for the area A of an ellips with half axes a and b. **(4 p)**

Solution. In the usual coordinate system the ellips is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

It is symmetric with respect to each axis. The upper half is the graph

$$y = b\sqrt{1 - \frac{x^2}{a^2}}, \qquad -a \le x \le a.$$

The area of the ellips is twice the area between this curve and the x-axes. We get, using $x = a \sin t$ that

$$\begin{aligned} A = 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx &= 2ab\int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 t}\cos t\,dt \\ &= 2ab\int_{-\pi/2}^{\pi/2} \cos^2 t\,dt \\ &= \pi ab, \end{aligned}$$

where we at the last step used $\cos^2 t = (1 + \cos 2t)/2$.

$\square$

9. Prove the formula
$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$
for all integers $n \geq 2$. **(4 p)**

Solution. Put $I = \int_0^{\pi/2} \sin^n x\,dx$. We use integration by parts and get
$$I = \int_0^{\pi/2} \sin^n x\,dx = [-\cos x \sin^{n-1} x]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x\,dx.$$
Using $\cos^2 x = 1 - \sin^2 x$ we get
$$I = (n-1)\int_0^{\pi/2} \sin^2 x\,dx - (n-1)I$$
that is
$$I = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$
which is what we wanted to prove.

□