SF1685 (SF1625) Calculus in one variable
Solutions to exam 09.03.2020

DEL A

1. (a) Evaluate $\displaystyle\int_1^e \frac{(1+\ln(x))^2}{x}dx.$ **(3 p)**

   (b) Find all primitive functions to $f(x) = x\sin(x)$. **(3 p)**

Lösning. (a) We use the substitution $u = 1 + \ln x$, and get that $du = \frac{1}{x}dx$ with the new integration limits being $u(1) = 1$ and $u(e) = 2$. Substituting it in we get

$$\int_1^e \frac{(1+\ln(x))^2}{x}dx = \int_1^2 u^2 du = \left[\frac{u^3}{3}\right]_1^2 = 8/3 - 1/3 = 7/3.$$

(b) The primitive functions of $f(x)$ are given by $\int x \sin x dx$. Partial integration gives

$$\int x\sin x dx = \{U = x, dU = dx \text{ och } dV = \sin x dx, V = -\cos x\} =$$

$$= -x\cos x - \int (-\cos x)dx = -x\cos x + \sin x + C$$

where C is an arbitrary constant.

Answer: (a) $7/3$, (b) The primitive functions are $\sin x - x\cos x + C$ where C is a constant. □

2. (a) Let L be the tangent line to the curve $y = \arctan(x^2)$ at the point on the curve that has x-coordinate 1. Find an equation of the line L. **(3 p)**

   (b) Let $f(x) = e^{-x}$ and let $P(x)$ be the Taylor polynomial of degree 1 for f about $x = 0$. Show that $0 < f(1/3) - P(1/3) < 1/10$. **(3 p)**

Lösning. (a) Let $g(x) = \arctan(x^2)$. An equation for L is given by $y = g(1)+g'(1)(x-1)$. We have $g(1) = \arctan(1) = \pi/4$ and

$$g'(x) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}.$$

Therefore

$$y = g(1) + g'(1)(x-1) = \pi/4 + (x-1)$$

is an equation for the line L.

(b) We see that $f'(x) = -e^{-x}$ and $f''(x) = e^{-x}$. Using Taylors formula we have that

$$f(1/3) - P(1/3) = \frac{f''(c)}{2}(1/3)^2 = \frac{e^{-c}}{18}$$

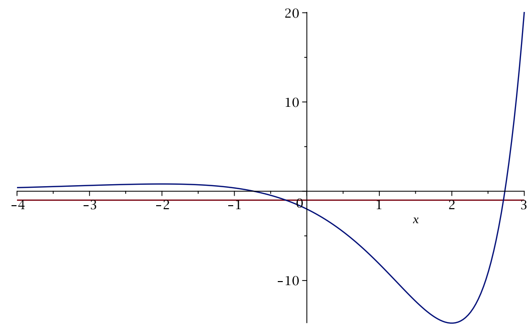
for some number c such that $0 < c < 1/3$. Since the function e^{-x} is positive and always decreasing, and since $0 < c$ we have that $0 < e^{-c} < e^0 = 1$, that is

$$0 < \frac{e^{-c}}{18} < \frac{1}{18} < 1/10.$$

Therefore, $0 < f(1/3) - P(1/3) < 1/10$.

Answer: (a) $y = \pi/4 + (x-1)$, (b) see above.

$\square$



FIGUR 1. Figure of Exercise 3.

DEL B

3. Let $f(x) = (x^2 - 2x - 2)e^x$.

   (a) Does the function f have absolute maximum or absolute minimum values? Find these values if they exist. **(4 p)**

   (b) How many solutions does the equation $f(x) = -1$ have? **(2 p)**

Lösning. (a) We start by looking at what happens to $f(x)$ as $x \to \pm\infty$. We have

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

Therefore f can not have a maximum value (since $f(x)$ can become arbitrarily large).

To determine if f can have a minimum values we first find the derivative. We have

$$f'(x) = (2x - 2)e^x + (x^2 - 2x - 2)e^x = (x^2 - 4)e^x.$$

Since $x^2 - 4 = (x - 2)(x + 2)$ and $e^x > 0$ for all x, we see that the critical points are $x = 2$ and $x = -2$. Further, $f'(x) > 0$ for $x < -2$ and for $x > 2$, and $f'(x) < 0$ for $-2 < x < 2$. This means that f is strictly increasing on the intervals $(-\infty, -2]$ and $[2, \infty)$, and strictly decreasing on the interval $[-2, 2]$. We also see that $f(-2) = 6e^{-2} > 0$ and $f(2) = -2e^2 < -2$.

Using the information we received from the limit values, the intervals of increasing and decreasing, and values of the function at $2, -2$, we can now sketch the graph of $y = f(x)$ (see figure). We specifically note that $f(x) > 0$ for all $x \leq -2$. From this we can draw the conclusion that f has a minimum value at the point $x = 2$ and that its value is $f(2) = -2e^2$.

b) Since $f(x) > 0$ for all $x \leq -2$ there does not exist a solution to the equation in the interval $(-\infty, -2]$. On the interval $[-2, 2]$ the function is strictly decreasing. Since $f(-2) > 0$ and $f(-2) < -1$, and f is continuous, there exists exactly one number $t_1 \in (-2, 2)$ such that $f(t_1) = -1$. Further, since f is strictly increasing on the interval $[-2, \infty)$ and since $f(-2) < -1$ and $f(x) > 0$ for all $x > 3$ there exists exactly one number $t_2 \in (2, \infty)$ such that $f(t_2) = -1$. Therefore, the equation has exact two solution. See the figure.

Answer: a) the maximum does not exist while the minimum is $f(2) = -2e^2$. b) The equation has exactly two solutions. $\square$

4. (a) Determine whether the integral $\displaystyle\int_0^\infty \frac{2+\sin(x)}{1+x^2}dx$ converges or diverges. **(4 p)**

(b) Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \cos(1/k^2)$ converges or diverges. **(2 p)**

Lösning. a) We have that $0 \leq 2+\sin(x) \leq 3$ for all x. Since the integrand is non-negative we get that by the comparison test the integral is convergent if the (larger) integral

$$\int_0^\infty \frac{3}{1+x^2}dx$$

is convergent. We check now if this new integral is convergent. We have that $\arctan(x)$ is a primitive function of $1/(1+x^2)$, and hence that

$$\lim_{N\to\infty}\int_0^N \frac{3}{1+x^2}dx = \lim_{N\to\infty}(3(\arctan(N) - \arctan(0)) = \lim_{N\to\infty} 3\arctan(N) = 3\pi/2,$$

that is, the integral is convergent. We can therefore draw the conclusion (as above) that the given integral is convergent.

b) A necessary condition for convergence is that the terms of the series tend to 0 as $k \to \infty$. In our case, we have the terms $a_k = \cos(1/k^2)$. Since $1/k^2 \to 0$ as $k \to \infty$ we have that $a_k = \cos(1/k^2) \to \cos(0) = 1$ as $k \to \infty$. Therefore, the limit of the sequence is not 0 and so the series must diverge.

Answer: a) The integral is convergent. b) The series is divergent. $\square$

Del C

5. Let $n \geq 1$ be an integer, and let P_n be the partition of the interval $[2,3]$ into n subintervals, each of length $1/n$. Let $L(P_n)$ denote the lower Riemann sum for the function $f(x) = 3x$ on the interval $[2,3]$ with respect to the given partition P_n.
   (a) Find a formula, only depending on n, for $L(P_n)$. **(4 p)**
   (b) Find the limit $\lim_{n\to\infty} L(P_n)$. **(2 p)**

Lösning. Since the integral $[2,3]$ should be divided into n different subintervals of length $1/n$ we see that P_n is the partition

$$x_0 = 2,\ x_1 = 2 + 1/n,\ x_2 = 2 + 2/n, \ldots,\ x_n = 2 + n/n = 3.$$

Since we are looking for the lower Riemann sum (with respect to the partition P_n) we must find the function f's smallest value on each subinterval $[x_{i-1}, x_i]$ $(1 \leq i \leq n)$. The function $f(x) = 3x$ is strictly increasing, so the minimum value (in the interval $[x_{i-1}, x_i]$) is always at the left end point x_{i-1}. Hence, we have

$$L(P_n) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{n} 3(2 + (i-1)/n)\frac{1}{n} = \sum_{i=1}^{n} 6/n + \sum_{i=1}^{n} 3(i-1)/n^2 =$$

$$= n \cdot (6/n) + \frac{3}{n^2}\sum_{i=1}^{n}(i-1).$$

Since we know the formula for each sum (arithmetic sums)

$$\sum_{i=1}^{n}(i-1) = 0 + 1 + 2 + \ldots + (n-1) = \frac{(n-1)n}{2}$$

we can then write

$$L(P_n) = n \cdot (6/n) + \frac{3}{n^2}\sum_{i=1}^{n}(i-1) = 6 + \frac{3}{n^2} \cdot \frac{n^2 - n}{2} = 6 + \frac{3}{2} - \frac{3}{2n}.$$

b) From the formula above we get now

$$\lim_{n\to\infty} L(P_n) = \lim_{n\to\infty}\left(6 + \frac{3}{2} - \frac{3}{2n}\right) = 6 + \frac{3}{2} = \frac{15}{2}.$$

We notice that this is good because the Riemann sum $L(P, n)$ should approximate the integral

$$\int_2^3 f(x)dx = \int_2^3 3x\, dx = 3\left[x^2/2\right]_2^3 = 3(9/2 - 4/2) = 15/2,$$

and converges to this value as $n \to \infty$ (as the partitions P_n become finer and finer).

□

6. Assume that the function f is differentiable with the derivative $f'(x) = 0$ for all real numbers x.
   (a) Show that f is continuous on the whole real line (that is, show that differentiability implies continuity). **(3 p)**
   (b) Use the mean value theorem (from differential calculus) to show that f is constant. **(3 p)**

Lösning. (a)See the proof of Theorem 1, section 2.3, in the course book.

(b) See the proof of Theorem 13, section 2.8, in the course book. Keep in mind that the continuity (shown in part a) is needed in order to be able to apply the mean value theorem.

$\square$