KTH Teknikvetenskap

SF1625 Calculus in one variable
Solutions to the exam 2017-03-17

PART A

1. (a) Compute the integral $\int_0^1 e^{2x} \cos e^x \, dx$. **(2 p)**

   (b) Compute the limit **(2 p)**

$$\lim_{x \to 0} \frac{\sin(2x)(1-x^2)}{x+x^2}.$$

Solution. (a) With the substitution $u = e^x$ we have $du = e^x \, dx$, and the new interval of integration is from 1 to e. This gives

$$\begin{aligned} \int_0^1 e^{2x} \cos e^x \, dx &= \int_1^e u \cos u \, du \\ &= [u \sin u]_1^e - \int_1^e \sin u \, du \\ &= e \sin e - \sin 1 + \cos e - \cos 1. \end{aligned}$$

(b) We have that $(1-x^2) = (1-x)(1+x)$, so

$$\frac{\sin(2x)(1-x^2)}{x+x^2} = (1-x)\frac{\sin(2x)}{x}$$

As $\lim_{x \to 0}(1-x) = 1$, we need only to determine

$$\lim_{x \to 0} \frac{\sin(2x)}{x}.$$

The l'Hôpital rule gives

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2\cos(2x)}{1} = 2.$$

$\square$

2. The power P (Watt) in a resistor of resistance R (Ohm) is a function of the voltage U (Volt). This function $P = P(U)$ satisfies $P'(220) = 440/R$. Use the derivative to approximate the amount of change in power when the voltage is increased from 220 to 230 volt? **(4 p)**

   Solution. Using linear approximation (or the definition of the derivative) we get

$$P(230) - P(220) \approx P'(220)(230 - 220) = \frac{440}{R} \cdot 10 = \frac{4400}{R}.$$

   The power is changed by approximately $4400/R$ Watt.

$\square$

3. (a) Write down an integral that gives the area between the t-axis and the curve $y = (\arctan t)^2$ on the interval $[0, x]$. **(2 p)**
   (b) Determine the rate of change of the area in problem a) at the point $x = 1$. **(2 p)**

Solution. (a) The area is given by

$$\int_0^x (\arctan t)^2 \, dt$$

(b) The rate of change is given by the derivative that can be computed using the Fundamental Theorem of Calculus. At the point x we get the rate of change

$$\frac{d}{dx} \int_0^x (\arctan t)^2 \, dt = (\arctan x)^2.$$

At $x = 1$ the rate of change is therefore $\pi^2/16$.

□

PART B

4. Newton's law of cooling says that an object cools at a rate proportional to the difference in temperature to the surrounding medium. Let $y(t)$ denote the temperature in a water bowl at the time t minutes. When the water boils the bowl is put outside where the temperature is -20°. The temperature $y(t)$ satisfies the differential equation $y'(t) = k(y(t) + 20)$. We also know that the temperature is 40° after 10 minutes.
   (a) Solve the differential equation (hint: Substitute $u(t) = y(t) + 20$). **(3 p)**
   (b) When is the temperature 25°? **(1 p)**

Solution. Let $u(t) = y(t) + 20$. Then we have that

$$\frac{du}{dt} = \frac{dy}{dt} = k \cdot u(t),$$

whose solutions are of the form $u(t) = Ce^{kt}$, some constant C. This implies that

$$y(t) = u(t) - 20 = Ce^{kt} - 20.$$

The initial condition $y(0) = 100$ (since the water was boiling when it was put out in the cold air). This gives $C = 120$, and we have that $y(t) = Ce^{kt} - 20$. As we have that $y(10) = 40$, we get that

$$y(10) = 40 \iff 120e^{10k} - 20 = 40 \iff k = \frac{-\ln 2}{10}.$$

The temperature of the water, in centigrades C at time t minutes is hence given by

$$y(t) = 120e^{-(t\ln 2)/10} - 20.$$

(b) Now we seek the time when the temperature of the water is 25° C, that is

$$y(t) = 25 \iff 120e^{-(t\ln 2)/10} - 20 = 25 \iff t = -\frac{10\ln\frac{45}{120}}{\ln 2} = 10\frac{\ln 8 - \ln 3}{\ln 2}$$

which is about 14 minutes.

$\square$

5. Sketch the function graph of $f(x) = \dfrac{x^2+x-1}{x^2-3}$. Your solution should show where the function is increasing, and decreasing, and which local extreme values, zeros, and asymptotics it has. **(4 p)**

Solution. The function $f(x)$ is defined for all $x \neq \pm\sqrt{3}$. The zeros of the function are given as the solutions of $x^2+x-1=0$. These two solutions are $x = \frac{1}{2}(\sqrt{5}-1)$ and $x = -\frac{1}{2}(\sqrt{5}+1)$. The derivative of $f(x)$ is

$$(2x+1)(x^2-3)^{-1} - (x^2+x-1)(x^2-3)^{-2}2x = \frac{-1}{(x^2-3)^2(x^2+4x+3)}.$$

The polynomial $x^2+4x+3 = (x+3)(x+1)$, and it follows that the function f has local extreme values at $x=-3$ and $x=-1$. The value of f at these points is $f(-3) = \frac{9-4}{9-3} = \frac{5}{6}$ and $f(-1) = \frac{1-2}{1-3} = \frac{1}{2}$. We have, furthermore, that

$$\lim_{x\to\infty}(f(x)) = \lim_{x\to-\infty}(f(x)) = 1.$$

A study of the sign of the derivative of f now shows the following. If we move from $-\infty$ and to the right. The value of the function is below one, strictly decreasing untill the local extreme value at $x=-3$. Thereafter strictly increasing, and unbounded when apporaching the vertical asymptotic at $x=-\sqrt{3}$. Then the function continues being strictly increasing, crosses the x-axis and has a local maximum at $x=-1$. Then it continues being strictly decreasing, crossed the x-axis, and becomes negatively unbounded when approaching the vertical asymptotic at $x=\sqrt{3}$. Follows being strictly decreasing, approaching the value $y=1$ from above.

$\square$

6. Linear approximation of the function $f(x) = x^{1/3}$ around the point $a = 8$ gives the error term $E(x)$. For each x there exists a number $s = s(x)$ such that $E(x) = \frac{f''(s)}{2}(x-8)^2$, where $8 < s < x$.
   (a) Show that $|E(x)| < \frac{1}{9\cdot 32}$ on the interval $8 \le x \le 9$. **(2 p)**
   (b) Show that $|9^{1/3} - \frac{25}{12}| < \frac{1}{9\cdot 32}$. **(2 p)**

Solution. (a) As $8 < s$ we have that

$$|f''(s)| = \frac{2}{9} \cdot \frac{1}{s^{5/3}} < \frac{2}{9} \cdot \frac{1}{32}.$$

Then we have that

$$|E(x)| < \frac{9 \cdot 32}{(} x - 8)^2 \le \frac{1}{9 \cdot 32} \cdot 1,$$

as $8 \le x \le 9$.

(b) We compute the degree one Taylor polynomial $P_1(x)$ of $f(x) = x^{1/3}$ around $x = 8$. We have that $f'(x) = \frac{1}{3}x^{-2/3}$. Hence

$$P_1(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8),$$

and in particular we have that $P_1(9) = \frac{25}{12}$. The difference between $f(9)$ and the approximation $P_1(9)$ is measured by the error term $E(9)$. From a) above we have that

$$|9^{1/3} - \frac{25}{12}| = |E(9)| < \frac{1}{9 \cdot 32}.$$

$\square$

PART C

7. We study the function f given by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Show that f is differentiable at the origin, and determine $f'(0)$. **(2 p)**

(b) Is f:s derivative continuous at the origin? **(2 p)**

Solution. (a) We use the definition of the derivative and have that

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h}.$$

As $|\sin(1/h)| \leq 1$ for any h, it follows that the limit $\lim_{h \to 0} h \sin(1/h) = 0$. This shows that f is differentiable at the origin and that the derivative is 0 at the origin.

(b) To understand continuity of f' around origo, we first not that for $x \neq 0$ we have that

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

In order for the derivative to be continuous at the origin we must have

$$\lim_{x \to 0} f'(x) = f'(0).$$

But,

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} (2x \sin \frac{1}{x} - \cos \frac{1}{x})$$

does not exist (the first term has zero as the limit, the second term osculates between -1 and 1). The derivative is therefore not continuous at the origin.

$\square$

8. The reasoning "As $-1/x$ is a primitive function of $1/x^2$ we have that
$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -2."$$
is wrong. Explain what is wrong in the reasoning, and determine then the correct value of the integral above. **(4 p)**

Solution. The errorness claim is $-1/x$ is an anti-derivative to $1/x^2$ in any interval containing the origin, since $-1/x$ is neither defined nor differentiable there. And our interval of integration contains the origin.

The integral is an improper integral since the integrand is unbounded when $x \to 0$. We need to write
$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$
and only if both integrals on the write hand side are convergent our integral is convergent. But
$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{c\to 0^-} \int_{-1}^{c} \frac{1}{x^2}\,dx = \lim_{c\to 0^-} [-1/x]_{-1}^{c} = \infty$$
and in the same way it is shown that the other integral on the write hand side is divergent.

The conclusion is that the integral $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ is divergent.

$\square$

9. The cardioid curve is parametrized by

$$x(t) = \frac{1}{2}\cos t + \frac{1}{4}\cos 2t \qquad y(t) = \frac{1}{2}\sin t + \frac{1}{4}\sin 2t, \qquad t \in [0, 2\pi].$$

(a) Compute the length of the curve. **(2 p)**

(b) Determine the smallest distance to the origin from the curve. **(2 p)**

Solution. (a) The length L of a parametric curve is given by

$$L = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

In our case

$$x'(t)^2 + y'(t)^2 = \frac{1}{2} + \frac{1}{2}(\sin t \sin 2t + \cos t \cos 2t) = \frac{1}{2}(1 + \cos t) = \cos^2 \frac{t}{2}$$

and so the length of the curve is

$$\int_0^{2\pi} \sqrt{\cos^2 \frac{t}{2}}\, dt = \int_0^{2\pi} \left|\cos \frac{t}{2}\right| dt = 2\int_0^{\pi} \cos \frac{t}{2}\, dt = 4.$$

The length is 4 length units.

(b) In order to minimize the distance to the origin we need to minimize $x^2 + y^2$ for points (x, y) on the curve. We seek the minimum value of

$$f(t) = \left(\frac{1}{2}\cos t + \frac{1}{4}\cos 2t\right)^2 + \left(\frac{1}{2}\sin t + \frac{1}{4}\sin 2t\right)^2, \quad t \in [0, 2\pi].$$

Using trigonometric identities we can write f as

$$f(t) = \frac{5}{16} + \frac{1}{4}\cos t,$$

and clearly this is minimized when $t = \pi$. The minimum distance is $\sqrt{1/16} = 1/4$.

□