KTH Teknikvetenskap

SF1684/SF1624 Algebra and geometry
Exam
January 11, 2021

Time: 08:00-11:00
Help in any form (such as books/notes/computers/person etc.) is NOT allowed.
Any plagiarism found in solutions will be reported

Examiner: Danijela Damjanović

This exam consists of six assignments, each worth 6 points.

Part A comprises the first two assignments. The bonus points from the seminars will be automatically added to the total score of the assignment 1, which however cannot exceed 6 points.

The next two assignments constitute part B, and the last two assignments part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

Instruktions

To obtain points on a problem, it is required that:

- The solution is well presented and easy to follow.
- The solutions are neatly written with a handwriting which is easy to read.
- The notations introduced are to be clearly defined, the logical structure clearly described in words or symbols. The reasoning should be well motivated and clearly explained. All steps in all calculations are to be presented clearly and should be easy to follow.

Solutions and answers without correct, detailed and clear justifications will receive no points.

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0. Honor code. See Assignment 0 in Canvas. Honor code is mandatory and the exam will not be graded unless you have uploaded the signed honor code.

PART A

1. The lines L_1 and L_2 are represented by the parametric equations

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R},$$

$$L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

(a) Find the skalar equation of the plane $\mathcal{P}$ that contains the line L_1 and is parallel to the line L_2. **(3 p)**

(b) Find the shortest distance between lines L_1 and L_2. (Hint: the shortest distance between lines L_1 and L_2 is the same as the distance of an arbitrary point on the line L_2 to the plane $\mathcal{P}$). **(3 p)**

2. Let A be the matrix

$$\begin{bmatrix} 2 & 2 & 7 \\ 0 & 2 & 1 \\ 3 & -14 & 2 \end{bmatrix}.$$

(a) Find a vector $\vec{v}$ such that the equation $A\vec{x} = \vec{v}$ does not have any solutions. **(3 p)**

(b) Find a vector $\vec{w} \neq \vec{0}$ such that the equation $A\vec{x} = \vec{w}$ has infinitely many solutions. **(3 p)**

PART B

3. Let $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Let F be an transformation that satisfies:

$$F(\vec{u}) = \vec{u}, \quad F(\vec{v}) = 2\vec{v}, \quad F(\vec{w}) = -\vec{w}.$$

(a) Find the matrix A for the transformation F with respect to the standard basis. **(4 p)**

(b) Diagonalize matrix A. **(2 p)**

4. Let $V = \text{span}(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix})$.

(a) Determine if $\vec{v} = [\ 1 \quad 1 \quad 1\ ]^T$ lies in V. **(2 p)**

(b) Find the dimension $\dim(V)$ of V. **(1 p)**

(c) Find an orthonormal basis for V. **(3 p)**

PART C

5. The matrix A is a symmetric $3{\times}3$ matrix. It has two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 1 and an eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. The eigenvalue $\lambda_2 = -1$ has the algebraic multiplicity 2.

(a) Find the matrix A. **(3 p)**

(b) Find $A^{99}\vec{w}$, where $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. **(3 p)**

6. Let U be a finite dimensional vector space, and let V and W be two subspaces of V. Then $U = V \oplus W$ is said to be the *inner direct product* of V and W if every vector $\vec{u} \in U$ can be written in a *unique* way as $\vec{u} = \vec{v} + \vec{w}$, where $\vec{v} \in V$ and $\vec{w} \in W$.

Show that $U = V \oplus W$ if and only if the following two assertions hold: **(6 p)**

(1) $V \cap W = \vec{0}$,

(2) $\dim(V) + \dim(W) = \dim(U)$.