KTH Teknikvetenskap

# SF1624/SF1684 Algebra and geometry
## Exam
## Wednesday, 10 January 2018

Time: 08:00-11:00
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of six problems, each worth 6 points.

Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

| Grade | A | B | C | D | E | Fx |
|---|---|---|---|---|---|---|
| Total sum | 27 | 24 | 21 | 18 | 16 | 15 |
| of which in part C | 6 | 3 | – | – | – | – |

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 3 points.

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## Part A

**1.** We are given the matrices

$$A = \begin{bmatrix} 4 & -1 \\ 15 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}, \quad C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{och} \quad D = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

For this problem, only answers are required, no explanations. Each part gives 1 point for three correct answers and 2 points for four correct answers.

(a) Which matrices are invertible and which aren't? **(2 p)**
(b) Which matrices are their own inverses and which aren't? **(2 p)**
(c) Which matrices are orthogonal and which aren't? **(2 p)**

**2.** Four points in $\mathbb{R}^3$ are given by

$$P = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \quad Q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad R = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; \quad S = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

The line $l_1$ passes through $P$ and $Q$, and the line $l_2$ through $R$ and $S$.

(a) Show that $l_1$ and $l_2$ do not intersect. **(3 p)**
(b) Find a parametric equation for a plane $\Pi$ in $\mathbb{R}^3$ then intersects neither $l_1$ nor $l_2$ and such that $l_1$ and $l_2$ lie on opposite sides of $\Pi$. **(3 p)**

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## PART B

**3.** Let $L$ be the projection onto the plane $x + 2y - z = 0$ in $\mathbb{R}^3$.

(a) Determine the matrix for the map $L$. **(3 p)**

(b) Determine the eigenvalues and eigenvectors of the map. **(3 p)**

**4.** The linear map $L\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is given by the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & -4 & 4 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and three of the vectors in the set

$$M = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

are eigenvectors of $L$.

(a) Decide which of the vectors in $M$ ar eigenvectors of $L$, and give the eigenvalues for them. **(2 p)**

(b) Use the result from (a) to decide whether $A$ is diagonalizable. (Hint: use that the trace is the sum of the eigenvalues.) **(4 p)**

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## PART C

**5.** In $\mathbb{R}^5$, the vector space $V$ is given as solution set of the system of equations

$$\begin{cases} x_1 - x_2 - x_3 - 3x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_5 &= 0 \end{cases}$$

The map $T\colon V \to \mathbb{R}^3$ maps an arbitrary vector $\vec{x}$ in $V$ to

$$T(\vec{x}) = T\left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 + x_3 + x_5 \\ x_1 + 2x_3 + x_4 - x_5 \\ x_1 + 2x_2 + x_4 + x_5 \end{bmatrix}.$$

(a) Determine a basis for the range $\mathrm{Range}(T)$. **(3 p)**

(b) The nullspace of $T$ is a line $L$. Determine a parametric equation for $L$. **(3 p)**

**6.** Let $Q$ be an indefinite quadratic form on $\mathbb{R}^n$. Show that there is a vector $\vec{v} \neq 0$ in $\mathbb{R}^n$ such that $Q(\vec{v}) = 0$. **(6 p)**