KTH Teknikvetenskap

SF1624 Algebra och geometri
Exam
Friday, April 17, 2020

Time: 08:00-11:00
Help in any form (such as books/notes/computers/person etc.) is NOT allowed
Any plagiarism found in solutions will be reported

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This exam consists of six problems, each worth 6 points.

Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

Instruktions

To obtain points on a problem, it is required that:

- The solution is well presented and easy to follow.
- The solutions are neatly written with a handwriting which is easy to read.
- The notations introduced are to be clearly defined, the logical structure clearly described in words or symbols. The reasoning should be well motivated and clearly explained. All steps in all calculations are to be presented clearly and should be easy to follow.

Solutions and answers without correct, detailed and clear justifications will receive no points.

PART A

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) Determine the eigenvalues of A. **(3 p)**

(b) Determine an orthogonal basis to $\mathbf{R}^3$ consisting of eigenvectors to A. **(3 p)**

2. For which values of a and b does the system below (with unknowns x, y and z) have infinitely many solutions? Solve the system in this case. **(6 p)**

$$\begin{cases} x + y + 2z = & 1 \\ 2x + 3y + 5z = & b \\ 3x + 4y + az = & 5 \end{cases}$$

PART B

3. Let $u = (1, 2, -2)$ and $v = (1, 1, 0)$.

(a) Find a vector w satisfying **all three** of the following conditions:

- The angle between u and w is $60°$
- The angle between v and w is $45°$
- The norm of w is 2.

(4 p)

(Remember that $\cos 45° = 1/\sqrt{2}$ and $\cos 60° = 1/2$.)

(b) Find the volume of the parallelepiped spanned by the three vectors u, v, och w. **(2 p)**

4. (a) Let A be the matrix

$$A = \begin{bmatrix} 2 & 6 & 1 & 0 \\ 1 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a1) Find a basis for the image subspace $\operatorname{im}(A)$. **(2 p)**

(a2) Find a basis of the null space $\ker(A)$. **(2 p)**

(b) Let B be a matrix with 7 rows and 19 columns.

(b1) What is the largest possible value of the rank of B? **(1 p)**

(b2) What is the smallest possible dimension of $\ker(B)$? **(1 p)**

PART C

5. We consider the equation

$$f(n+2) = 5f(n+1) - 6f(n) \qquad (*)$$

where $n \geq 0$ are integers and $f(n)$ is an unknown function.

(a) Let $X(n) = \begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix}$ and write the equation $(*)$ as an equation of matrices $X(n+1) = AX(n)$ where A is a matrix (with constant entries). **(1 p)**

(b) Determine the eigenvalues and eigenvectors of the matrix A. **(2 p)**

(c) Determine the solution to $(*)$ that satisfies $f(0) = 1$ and $f(1) = 4$. **(3 p)**

6. Suppose that the 2×2 matrix A has the characteristic polynomial $p(\lambda) = (\lambda + 1)(\lambda + 2)$. Show that $A - A^2$ is invertible and determine the eigenvalues to the inverse. **(6 p)**