KTH Teknikvetenskap

**SF1624 Algebra och geometri**
**Exam**
**Friday, 20 October 2017**

Time: 08:00-11:00
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of six problems, each worth 6 points.

Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

| Grade | A | B | C | D | E | Fx |
|---|---|---|---|---|---|---|
| Total sum | 27 | 24 | 21 | 18 | 16 | 15 |
| of which in part C | 6 | 3 | – | – | – | – |

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 3 points.

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## PART A

**1.** Consider the homogeneous linear system of equations $A\vec{x} = \vec{0}$, where

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 4 & -1 & 1 & k \end{bmatrix}$$

where $k$ is a constant.

(a) Solve the system of equations for $k = 3$. **(3 p)**

(b) Determine the value of the constant $k$ such that the solution set of $A\vec{x} = \vec{0}$ is a two-dimensional subspace of $\mathbb{R}^4$ and construct a basis for that subspace. **(3 p)**

**2.** The matrix $A$ has the eigenvalues $-1$ och $2$ with corresponding eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) Is $A$ diagonalizable? **(2 p)**

(b) Compute $A$. **(4 p)**

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*p.t.o.*

## PART B

**3.** Let

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

be the standard basis for $\mathbb{R}^2$, and let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^2$. **(1 p)**

(b) Compute the coordinate vector $[\vec{v}]_{\mathcal{B}}$ for the vector $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. **(2 p)**

(c) Compute matrices $M$ and $N$ such that

$$[\vec{x}]_{\mathcal{E}} = M\,[\vec{x}]_{\mathcal{B}} \quad \text{och} \quad [\vec{x}]_{\mathcal{B}} = N\,[\vec{x}]_{\mathcal{E}}$$

for all vectors $\vec{x}$ i $\mathbb{R}^2$. **(3 p)**

**4.** Let

$$A = \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Determine all vectors which lie in both $\mathrm{Col}(A)$ and $\mathrm{Col}(B)$. Explain why all vectors which lie in both $\mathrm{Col}(A)$ and $\mathrm{Col}(B)$ form a subspace of $\mathbb{R}^3$, and compute its dimension. **(4 p)**

(b) Give a vector in $\mathrm{Col}(A)$ which does not lie in $\mathrm{Col}(B)$. **(2 p)**

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## PART C

**5.** Show that the composition of two reflections around different lines through the origin in $\mathbb{R}^2$ is a rotation around the origin. **(6 p)**

**6.** Let $V$ be the vector space of all $2 \times 2$-matrices and $f\colon V \to V$ the linear map given by

$$f(M) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} M.$$

Choose an arbitrary basis $\mathcal{B}$ of $V$. Let $A$ be the matrix $[f]_{\mathcal{B}}$ for $f$ with respect to the basis $\mathcal{B}$. Compute the determinant of $A$. WARNING: it is not the determinant of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ that needs to be computed! **(6 p)**