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Lecture 14

- $\mathcal{U} = \mathbb{P}^1_{\mathbb{C}} \setminus S$, S finite set

Goal: Prove the Riemann-Hilbert correspondence for $\mathcal{U}$:

$$\left\{ \begin{array}{l} \text{hol. vector bundles} \\ \text{w. connection on } \mathcal{U} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{local systems} \\ \text{on } \mathcal{U} \end{array} \right\}$$

- also, each hol. vector bundle w. connection on $\mathcal{U}$ is algebraic, w. regular singularities on S

Notation and terminology:

- "vector bundle" := "locally free sheaf of finite rank"
- $\mathcal{O}$: sheaf of holomorphic functions on $\mathcal{U}$
- Ω^1: sheaf of holomorphic diff.

One-forms on U

Dfn: A <u>sheaf of $\mathcal{O}$-modules</u> on U is a sheaf of abelian groups $\mathcal{F}$ together with, for each $V \subset U$ open, an $\mathcal{O}(V)$-module structure, s.t. $\forall\, W \subset V$ open:

$$\begin{tikzcd} \mathcal{O}(V) \times \mathcal{F}(V) \arrow[r] \arrow[d, "\text{restr.}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}(V) \arrow[d, "\text{restr.}"] \\ \mathcal{O}(W) \times \mathcal{F}(W) \arrow[r] & \mathcal{F}(W) \end{tikzcd}$$

- $\mathcal{F}$ is <u>locally free</u> (of rank n) if $\exists$ open cover $U = \bigcup_{i \in I} V_i$ s.t. $\mathcal{F}|_{V_i} \cong \mathcal{O}^n|_{V_i}$, for some $n \geq 0$
- $\mathcal{F}$ is <u>free</u> if $\mathcal{F} \cong \mathcal{O}^n$

Dfn: A <u>holomorphic connection</u> is a pair $(\mathcal{E}, \nabla)$,

$\mathcal{E}$: locally free sheaf (of rank n)

$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}} \Omega^1$

morphism of abelian sheaves

s.t. $\nabla(fs) = f\nabla(s) + s \otimes df$

s.t. $\boxed{\nabla(fs) = f\nabla(s) + s\otimes df}$

for all local sections

$f \in \mathcal{O}(V)$, $s \in \mathcal{E}(V)$

∇ = "connection map"

- A morphism of connections

$$(\mathcal{E},\nabla) \longrightarrow (\mathcal{E}',\nabla')$$

is an $\mathcal{O}$-linear map $\phi: \mathcal{E} \to \mathcal{E}'$

s.t.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\nabla"] \arrow[d, "\phi"'] & \mathcal{E}\otimes\Omega^1 \arrow[d, "\phi\otimes id"] \\ \mathcal{E}' \arrow[r, "\nabla'"] & \mathcal{E}'\otimes\Omega^1 \end{tikzcd}$$

commutes

Fact: If $\mathcal{E} \cong \mathcal{O}^n$, then

$\nabla = d + \omega$ $\quad d: \mathcal{O} \to \Omega^1$, for some

$\omega \in Mat_{n\times n}(\Omega^1(U))$ "connection matrix"

- Indeed, $f \in \mathcal{E}(V)$, $f = (f_1,\dots,f_n) \in \mathcal{O}(V)^n$

and $\nabla - d: \mathcal{O}^n \longrightarrow \mathcal{O}^n \otimes_{\mathcal{O}} \Omega^1 \cong (\Omega^1)^n$

is $\mathcal{O}$-linear, given by

a matrix of global sections of

Ω^1

Dfn: • $(\mathcal{E}, \nabla)$ holomorphic connection on U.

• Then define a subsheaf

$$\mathcal{E}^{\nabla} \subset \mathcal{E} \quad \text{by}$$

$$\mathcal{E}^{\nabla}(V) = \{ s \in \mathcal{E}(V) \mid \nabla(s) = 0 \}$$

Rmk: If $\mathcal{E}$ is free, $\nabla = d - A(z)dz$ then $f = (f_1, \dots, f_n)$ satisfies

$$\nabla(f) = 0 \iff f' = A \cdot f$$

Fact: Each locally free sheaf on U $(S \neq \emptyset)$ is free (holds more generally for non-compact Riemann surfaces, or even Stein manifolds)

Dfn: • A **local system** $\mathcal{L}$ on U is a locally constant sheaf of $\mathbb{C}$-vector spaces on U.

• the **rank** of $\mathcal{L}$ is

$$\dim \mathcal{L}_x$$ stalk of $\mathcal{L}$ at x

(well-defined since U connected)

(well-defined since $\mathcal{U}$ connected)

Lemma: $\mathcal{E}^\nabla$ is a local system of rank $= \mathrm{rk}(\mathcal{E})$

Sketch of Proof: Fact about $\Rightarrow \mathcal{E}$ free

$\Rightarrow$ enough to show that

$$\{ f \in \mathcal{O}^n \mid f' = Af \}, \quad \nabla = d - A(z)dz$$

is a local system of rank n

- This follows from local! existence and uniqueness of holomorphic solutions to holomorphic ODEs.

$\square$

Proposition: The functor

$$\left\{ \begin{array}{l} \text{holomorphic} \\ \text{connections on } \mathcal{U} \end{array} \right\} \xrightarrow{(\mathcal{E}, \nabla) \mapsto \mathcal{E}^\nabla} \left\{ \begin{array}{c} \text{local systems} \\ \text{on } \mathcal{U} \end{array} \right\}$$

is an equivalence of categories.

Sketch of proof: · Construction of a quasi-inverse:

- $\mathcal{L}$ local system

$\leadsto \quad V \mapsto \mathcal{L}(V) \otimes_{\mathbb{C}} \mathcal{O}(V)$

defines a locally free sheaf $\mathcal{E}_\mathcal{L}$

• $\nabla_\mathcal{L} : \mathcal{E}_\mathcal{L} \longrightarrow \mathcal{E}_\mathcal{L} \otimes_\mathcal{O} \Omega^1$

defined locally by

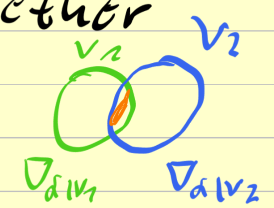
$$\nabla_{\mathcal{L}|V}(s_i \otimes f) = s_i \otimes df \quad , f \in \mathcal{O}(V)$$

where $\mathcal{L}|V \cong \mathbb{C}^n$, $\{s_i\}$ $\mathbb{C}$-basis of $\mathcal{L}|V$

• $\nabla_{\mathcal{L}|V}$ does not depend on choice of basis (since the change of basis is annihilated by d)

$\Rightarrow$ $\nabla_{\mathcal{L}|V}$ patch together

$\Rightarrow$ $\nabla_\mathcal{L}$ well-defined



$(\nabla_{\mathcal{L}|V_1})|_{V_1 \cap V_2} = (\nabla_{\mathcal{L}|V_2})|_{V_1 \cap V_2}$

Exc: show that

$$\mathcal{L} \longmapsto (\mathcal{E}_\mathcal{L}, \nabla_\mathcal{L})$$

is quasi-inverse to

$$\mathcal{E} \longmapsto \mathcal{E}^\nabla$$

• Now we address the problem of algebraicity of $(\mathcal{E}, \nabla)$

Idea: • want to extend $(\mathcal{E}, \nabla)$ to $\mathbb{P}^1$

to $\mathbb{P}_{\mathbb{C}}$

- Need to allow "connections with singularities", but those will be mild ("regular singular")

- $U = \mathbb{P}^1_{\mathbb{C}} \setminus S$, $S = \{x_1, \dots, x_n\}$

  $j : U \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$ open embedding

- $\mathcal{F}$ sheaf of $\mathcal{O}_U$-modules

  $\rightsquigarrow$ $j_* \mathcal{F}$ sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules

  $$j_* \mathcal{F}(V) := \mathcal{F}(\underbrace{V \cap U}_{= j^{-1}(V)}) \quad V \subset \mathbb{P}^1_{\mathbb{C}} \text{ open}$$

<u>Dfn</u>: $\Omega^1_{\mathbb{P}^1}(\log S) \subset j_* \Omega^1_U$

locally free $\mathcal{O}_{\mathbb{P}^1}$-submodule, generated (locally) by

$$f\, dz, \quad f \in \mathcal{O}_{\mathbb{P}^1}(V) \ \ \overset{V \ni x_i}{\text{at}}$$

most simple pole at x_i

- A <u>logarithmic connection</u> (with poles along S) is a pair $(\mathcal{E}, \nabla)$, $\mathcal{E}$ locally free sheaf on $\mathbb{P}^1_{\mathbb{C}}$

$\bar{\nabla} : \bar{\mathcal{E}} \longrightarrow \bar{\mathcal{E}} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1}(\log S)$

connection map "$\nabla(s) = s \otimes \frac{dz}{z}$"

<u>Rmk</u>: $\bar{\mathcal{E}}$ is not necessarily free! If it is, then $(\bar{\mathcal{E}}, \bar{\nabla})$ is called <u>Fuchsian</u>.

(e.g. $\mathcal{O}(n)$ = line sheaf of deg n homogeneous poly $\mathbb{C}[X,Y]$)

<u>Proposition</u>: • $(\mathcal{E}, \nabla)$ holomorphic connection on U

• $\exists\ (\bar{\mathcal{E}}, \bar{\nabla})$ logarithmic connection such that $(\bar{\mathcal{E}}, \bar{\nabla})|_U \cong (\mathcal{E}, \nabla)$

Fact: sheaves on a topological space X can be "patched", i.e. if $X = \bigcup_{i \in I} U_i$ open cover

$\mathcal{F}_i$ sheaf (of sets) on U_i s.t.

$\exists$ isomorphisms: $\theta_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$

$\forall\, i, j$ which satisfy

$\theta_{jk} \circ \theta_{ij} = \theta_{ik}$ over $U_i \cap U_j \cap U_k$

$\Rightarrow \exists!$ sheaf of sets $\mathcal{F}$ on X

s.t. $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$

s.t. $\mathcal{F}|U_i = \mathcal{F}_i$

- if all $\mathcal{F}_i$ locally free sheaves of $\mathcal{O}_{U_i}$-modules (X complex mfd)
  $\Rightarrow$ $\mathcal{F}$ locally free sheaf of $\mathcal{O}_X$-modules

<u>Exc</u>: Prove the fact

(Hint: define

$$\mathcal{F}(U) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U\cap U_i) \mid s_j|_{U\cap U_i\cap U_j} = \theta_{ij}(s_i|_{U\cap U_i\cap U_j})\}$$)

<u>Proof of proposition (sketch):</u>

- $S = \{x_1, \dots, x_n\}$, D_i = small disk at x_i
  $D_i \cap D_j = \emptyset$, $i \neq j$
- $U \cap D_i = D_i \setminus \{x_i\} =: D_i'$
- $(\mathcal{E}, \nabla)|D_i' \longleftrightarrow \pi_1(D_i') \longrightarrow Gl_n(\mathbb{C})$
  $\pi_1(D_i') \cong \mathbb{Z}$

  $\Rightarrow$ determined by $B_i \in Gl_n(\mathbb{C})$
- $\exists\, A_i \in Mat_{n\times n}(\mathbb{C})$ s.t.
  $\exp(2\pi i A_i) = B_i$

z = local coord. at x_i

$\Rightarrow (\mathcal{O}_{D_i^\circ}, d - A_i \frac{dz}{z})$ has same monodromy as $(\mathcal{E}, \nabla)|_{D_i^\circ}$

- Also, extends to D_i with regular singular point at x_i
- now patch $(\mathcal{O}^n_{D_i^\circ}, d - A_i \frac{dz}{z})$ with $(\mathcal{E}, \nabla)|_{D_i^\circ}$

$\rightsquigarrow$ extension of $(\mathcal{E}, \nabla)|_{D_i^\circ}$ to $(\overline{\mathcal{E}}, \overline{\nabla})|_{D_i}$

$U \cup \{x_i\}$

$\square$

Rmk: (i) $\overline{\mathcal{E}}$ in general not free

(ii) uniqueness of $(\overline{\mathcal{E}}, \overline{\nabla})$ subtle and only holds in a weak sense $(\overline{\mathcal{E}}_1, \overline{\nabla}_1), (\overline{\mathcal{E}}_2, \overline{\nabla}_2)$ two extensions of $(\mathcal{E}, \nabla)$ $\Rightarrow \exists\, (\overline{\mathcal{E}}_1, \overline{\nabla}_1) \to (\overline{\mathcal{E}}_2, \overline{\nabla}_2)$ meromorphic on $\mathbb{P}^1$

(iii) If one restricts to <u>unipotent</u> local systems, both problems (i), (ii) disappear

Point: If $(\mathcal{E}, \nabla)$ has unipotent monodromy, then $(\overline{\mathcal{E}}, \overline{\nabla})$ is unique up to isomorphism

and $\overline{\mathcal{E}}$ is free (holds bc.

and $\mathcal{E}$ is free ($H(\mathbb{P}, \mathcal{O}_{\mathbb{P}^1}) = 0$)

(see Deligne, "Equations différentiels ...

or (for an exposition)

Conrad "classical motivation for Riemann-Hilbert")

iv) Algebraicity of $(\overline{\mathcal{E}}, \overline{\nabla})$ follows from (a variant of) Serre's GAGA (= géometric algébrique et géometric analytique) principle (Deligne, loc. cit.)

Tannakian categories:

- k field, G/k (pro-) affine algebraic group
- $\mathrm{Rep}_k(G)$ = category of representations of G in fin.-dim. k-vector spaces
- Objects are pairs: (V, ρ), V k-vector space, $\dim V < \infty$
  $\rho : G \longrightarrow GL(V)$

algebraic group morphism

· Forgetful functor

$$\omega: \mathrm{Rep}_k(G) \longrightarrow \mathrm{Vect}_k$$
$$(V,\rho) \longmapsto V$$

- exact (maps short exact sequences to short exact sequences)
- compatible with tensor product

$(\omega((V_1,\rho_1)\otimes(V_2,\rho_2)) \cong \omega((V_1,\rho_1))\otimes\omega((V_2,\rho_2)))$

· Want to recover G from $(\mathrm{Rep}_k(G), \omega)$!

· Define a functor

$$\underline{\mathrm{Aut}}^{\otimes}(\omega): k\text{-alg} \longrightarrow \mathrm{Groups}$$
$$R \longmapsto \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$$

where $\mathrm{Aut}^{\otimes}(\omega)(R) = \{(\lambda_X) \mid X \in \mathrm{Ob}(\mathrm{Rep}_k(G))$

$$\lambda_X: \omega(X)\otimes R \longrightarrow \omega(X)\otimes R$$

R-linear automorphism s.t.

(i) $\lambda_{\mathbb{1}} = \mathrm{id}$ $\qquad \mathbb{1} = (k, e)$ (trivial rep)

(ii) $\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}$

(iii) for all $(\varphi : X \longrightarrow Y) \in \mathrm{Mor}(\mathrm{Rep}_k(G))$

$$\begin{tikzcd}
\omega(X)\otimes R \arrow[r, "\lambda_X"] \arrow[d, "\omega(\varphi)\otimes id"'] \arrow[dr, phantom, "\circlearrowleft"] & \omega(X)\otimes R \arrow[d, "\omega(\varphi)\otimes id"] \\
\omega(Y)\otimes R \arrow[r, "\lambda_Y"] & \omega(Y)\otimes R
\end{tikzcd}$$

- Natural map

$$G \xrightarrow{\sim} \underline{\mathrm{Aut}}^{\otimes}(\omega)$$

$$g \in G(R) \longmapsto (\gamma_X) \quad \text{(left action by } g\text{)}$$

Fact: This is an isomorphism of functors of k-algebras

- Conversely: $\mathcal{C}$ category, is $\mathcal{C} \simeq \mathrm{Rep}_k(G)$, for some G?

$\rightsquigarrow$ "Tannakian reconstruction" (next time)

- Also, applications to differential Galois theory (existence and uniqueness of PV-extensions)