Last time: X compact complex manifold, w. Hermitian metric

- $\Delta : A^k(X) \longrightarrow A^k(X)$ (d-) Laplacian

Then: $A^k(X) = \mathcal{H}^k \oplus \Delta(A^k(X))$

where $\mathcal{H}^k = \{\alpha \in A^k(X) \mid \Delta\alpha = 0\}$

- How to prove this?

Digression: H Hilbert space

$\Phi : H \longrightarrow H$ closed operator

- Elementary functional analysis:

$$H = \ker(\Phi) \oplus \ker(\Phi)^{\perp} = \ker(\Phi) \oplus \operatorname{Im}(\Phi^*)$$

If $\Phi = \Phi^*$, then

$$H = \ker(\Phi) \oplus \operatorname{Im}(\Phi)$$

- **Problem**: $A^k(X)$ <u>not</u> complete

- **But**: can extend Δ to

$A^k_{L^2}(X)$, closed operator

$A^k_{L^2}(X)$ = "uniform limits of smooth k-forms"

- $\ker(\Delta) \subset A^k_{L^2}(X)$
  = "weakly harmonic forms"

<u>key Fact</u>: (regularity theorem)

Let $\alpha, \beta \in A^k_{L^2}(X)$, $\Delta\alpha = \beta$

Then: If $\beta \in A^k(X)$, then

$\alpha \in A^k(X)$

- crucial: Δ is an "elliptic operator"
  (holds more generally for elliptic operators)

- Broadening?

Lecture 7: Hodge decomposition for Kähler manifolds

1. Kähler identities:

(X, ω) Kähler manifold

(not necessarily compact)

- $L: A^k(X) \longrightarrow A^{k+2}(X)$

$$\alpha \longmapsto \omega \wedge \alpha$$

$\Lambda: A^k(X) \longrightarrow A^{k-2}(X)$

formal adjoint of L

$$(L\alpha, \beta)_{L^2} = (\alpha, \Lambda\beta)_{L^2}$$

Fact: $\Lambda = *^{-1} \circ L \circ * = (-1)^k * \circ L \circ *$

Proof: $(L\alpha, \beta)\,\mathrm{vol} = L\alpha \wedge *\beta = (\omega \wedge \alpha) \wedge *\beta$

$$= \alpha \wedge (\omega \wedge *\beta)$$

$$= (\alpha, (*^{-1} \circ L \circ *)\beta) \cdot \mathrm{vol}$$

$\square$

Prop (Kähler identities):

$[\Lambda, \bar\partial] = -i \cdot \partial^*$, $[\Lambda, \partial] = i \cdot \bar\partial^*$

sketch of proof: Recall X Kähler

$\Rightarrow \forall x \in X$ $\exists$ local coord. $z_1, \dots, z_n$

$(h_{ij})_{i,j} = I_n + O(\sum_i |z_i|^2)$

Step 1: Reduce to constant metric

- L, Λ are $C^\infty(X)$-linear (depend on metric to 0-th order)
- $\partial^*, \bar\partial^*, \partial, \bar\partial$ depend on metric to first order

$\Rightarrow$ enough to prove for

$h = \sum_i dz_i \, d\bar z_i$ on $\mathbb{C}^n$

Step 2: Claim: Enough to show first order terms of $[\Lambda, \bar\partial]$ and $-i \cdot \partial^*$ agree

$\leadsto$ direct computation (messy)

$\Rightarrow [\Lambda, \bar{\partial}] = -i \cdot \partial^*$

- $[\Lambda, \bar{\partial}](\alpha) = -i \cdot \partial^*(\alpha)$

$\Leftrightarrow \overline{[\Lambda, \bar{\partial}]}(\bar{\alpha}) = \overline{-i \cdot \partial^*(\alpha)}$

$\Leftrightarrow [\Lambda, \partial](\bar{\alpha}) = i \cdot \bar{\partial}^*(\bar{\alpha})$

$\square$

2. Comparison of Laplacians

Thm: (X, ω) Kähler manifold.

Then: $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2} \Delta_d$

Proof: $\Delta_d = dd^* + d^*d$

$= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial})$

$\vdots$

$= \Delta_\partial + i\,\partial \circ [\Lambda, \bar{\partial}] + i\,[\Lambda, \bar{\partial}] \circ \partial$

$= 2\,\Delta_\partial$

$\Rightarrow \Delta_\partial = \frac{1}{2} \Delta_d$

- $\Delta_{\bar{\partial}} = \frac{1}{2} \Delta_d$ proved similarly

$\square$

Corollary: $\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$

where $\mathcal{H}^{p,q}(X) = \{ \alpha \in A^{p,q}(X) \mid \Delta_d \alpha = 0 \}$

Proof: "$\supseteq$" clear

"$\subseteq$": $\alpha \in A^k(X)$, $\alpha = \sum_{p+q=k} \alpha^{p,q}$, $\alpha^{p,q} \in A^{p,q}(X)$

since $\Delta_d \sim \Delta_{\bar\partial}$, have

$$\Delta_d \alpha = 0 \Rightarrow \Delta \alpha^{p,q} = 0$$

$\square$

3. Hodge decomposition etc.

Recall: X compact complex manifold

Thm (Hodge theorem):

$$H^k(X, \mathbb{C}) \cong \mathcal{H}^k(X)$$
$$(H^k_{dR}(X))$$

and $\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$

Thm (Hodge decomposition):

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X) = \mathrm{Im}(\mathcal{H}^{p,q}(X) \longrightarrow H^k(X, \mathbb{C}))$

Prop: Hodge decomposition independent of choice Kähler metric.

Proof; Let $K^{p,q} \subset H^k(X, \mathbb{C})$ subspace of classes, represented by a (p,q)-form.

Claim: $K^{p,q} = H^{p,q}$

· Indeed: $H^{p,q} \subseteq K^{p,q}$ obvious

conversely: $\alpha \in A^{p,q}(X)$ closed

write $\alpha = \beta + \Delta\gamma$ uniquely

β harmonic

· Δ bihomogeneous $\Rightarrow \alpha = \beta^{p,q} + \Delta\gamma^{p,q}$

$\beta^{p,q}$ harmonic

$\Rightarrow \Delta\gamma^{p,q} = (dd^* + d^*d)\gamma^{p,q}$ closed

$\Rightarrow d^*d\gamma^{p,q}$ closed

$\Rightarrow$ zero

$\Rightarrow [\alpha] = [\beta^{p,q}]$ in $H^k(X, \mathbb{C})$ □

4. Further consequences

<u>Corollary</u>: (i) Hodge symmetry:

$$\overline{H^{p,q}(X)} = H^{q,p}(X)$$

(ii) $b_{2k+1} := \dim H^{2k+1}(X, \mathbb{C})$

odd

(iii) ($\partial\bar{\partial}$-lemma): $\alpha \in A^k(X)$

$\partial \alpha = \bar{\partial} \alpha = 0$

Assume: α is d-exact or ∂-exact

or $\bar{\partial}$-exact

Then: $\exists\, \beta \in A^{k-2}(X)$ (locally on X)

s.t. $\alpha = \partial\bar{\partial}\beta$

(iv) $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$

<u>Proof</u>: (i) $\overline{\mathcal{K}^{p,q}} = \mathcal{K}^{q,p}$

(iii) computation similar to

previous proposition

(iv) $H^{p,q}(X) = \Delta_{\bar{\partial}}$-harmonic forms of type $(p,q) \cong H^q(X, \Omega^p_X)$
(see previous lecture)

Next lecture:

Lefschetz decomposition, Hard Lefschetz

Lemma: (X, ω) Kähler manifold

$L : A^k(X) \to A^{k+2}(X)$ Lefschetz operator

Λ formal adjoint.

Then: $[L, \Lambda] = (k-n) \cdot \mathrm{Id}$ on $A^k(X)$

$n = \dim X$