Ink note

Notebook:	DGT		
Created:	6/14/2020 5:08 PM	Updated:	6/16/2020 3:08 PM
Author:	Nils Matthes		

Lecture 15

Tannakian categories

- k field (fixed throughout)
- when is a category equivalent to $\mathrm{Rep}_k(G)$, for some (pro-)algebraic group G/k?

Dfn: A neutral Tannakian category is a rigid, abelian, k-linear tensor category $\mathcal{C}$ with $\mathrm{End}(\mathbb{1}) = k$, such that there exists an exact, faithful k-linear tensor functor

overlooked for some time

$$\omega : \mathcal{C} \longrightarrow \mathrm{Vect}_k$$

finite-dim. k-vector spaces

"Fibre functor"

- (Rough) dictionary:

- "rigid" $\longleftrightarrow$ dual objects exist,

$$X \cong (X^\vee)^\vee$$

- "abelian" $\longleftrightarrow$ kernels / cokernels exist
  $f = \ker(\operatorname{coker}(f))$ (f mono)
  $f = \operatorname{coker}(\ker(f))$ (f epi)

- "k-linear" $\longleftrightarrow$ $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a k-vector space

- "tensor category" $\longleftrightarrow$ $\exists$ $\otimes$-product, associative + commutative (up to isomorphism), $\exists$ $\mathbb{1} \in \mathcal{C}$ unit object

- "exact" $\longleftrightarrow$ takes short exact sequences to short exact sequences

- "faithful" $\longleftrightarrow$ injective on hom-sets

- "tensor functor" $\longleftrightarrow$ $F(X \otimes Y) \cong F(X) \otimes F(Y)$

- tensor functor $\Leftrightarrow F(X \otimes Y) = F(X) \otimes F(Y)$

$$F(\mathbb{1}) = \mathbb{1}' = k$$

- "k-linear" $\Leftrightarrow \mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X),F(Y))$ is k-linear

Examples: (i) Vect_k, $\omega = id$, $\mathbb{1} = k$

(ii) Vect_k^{Gr} = graded vector spaces

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

$$\omega((V, \text{grading})) = V \quad , \quad \mathbb{1} = (k, 0)$$

(iii) X connected topological space

$\mathrm{Loc}_k(X)$ = category of local systems of fin.-dim. k-vector spaces

each $x \in X$ gives a fibre functor

$$\mathcal{L} \mapsto \mathcal{L}_{(x)} \quad , \quad \mathbb{1} = \underline{k}$$

(iv) X/k smooth, geometrically connected scheme ($\mathrm{char}(k) = 0$), $(\mathrm{End}(\mathcal{O}_X, d) = k)$

$X(k) \neq \emptyset$

$\mathrm{MIC}(X/k)$ = category of vector bundle w. integrable conn, regular singular at $\bar{X} \setminus X$

each $x \in X(k)$ gives a fibre functor

$$(\mathcal{F}, \nabla) \mapsto \mathcal{F}_{(x)} \quad , \quad \mathbb{1} = (\mathcal{O}_X, d)$$

(v) $\mathrm{Rep}_k(G)$, G (affine) group scheme/k

$$\omega((V, \rho)) = V \quad , \quad \mathbb{1} = (k, 0)$$

Dfn: $\mathcal{C}$ neutral Tannakian cat.

$\omega : \mathcal{C} \to \mathrm{Vect}_k$ fibre functor

Then define: $\underline{\mathrm{Aut}}^{\otimes}(\omega) : k\text{-alg} \to \mathrm{Grps}$

by $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R) = \{ (\lambda_X)_{X \in \mathrm{Ob}(\mathcal{C})} \mid$

$\lambda_X : \omega(X) \otimes R \to \omega(X) \otimes R$

nat. transformation

such that $\lambda_{X \otimes Y} = \lambda_X \otimes \lambda_Y$,

$\lambda_{\mathbb{1}} = \mathrm{id}_R \}$

R k-algebra

- The following is the main theorem of Tannakian categories

Thm (Tannakian reconstruction)

$\mathcal{C}$ neutral Tannakian category,

ω fibre functor

Then: (i) $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is representable

by an affine group scheme G

(ii) The functor

$$\mathcal{C} \longrightarrow \mathrm{Rep}_k(G)$$

$$X \longmapsto \omega(X) \circlearrowleft G \simeq \underline{\mathrm{Aut}}^{\otimes}(\omega)$$

defined by ω is an

equivalence of tensor

categories

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\simeq"] \arrow[dr, "\omega"'] & & \mathrm{Rep}_k(G) \arrow[dl, "\text{forget}"] \\ & \mathrm{Vect}_k & \end{tikzcd}$$

<u>**Example**</u>: (i) $\mathcal{C} = \mathrm{Vect}_k$, $G = \{id\}$

(ii) $\mathcal{C} = \mathrm{Vect}_k^{gr}$, $G = \mathbb{G}_m/k$

Exercise: show that to give

a grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$ on a

finite-dimensional vector space V

is equivalent to giving

$$\rho : \mathbb{G}_m \longrightarrow GL(V) \quad \text{morphism of alg. groups}$$

(iii) $\mathcal{C} = \mathrm{Loc}_k(X)$, $\omega_x(\mathcal{L}) = \mathcal{L}_{(x)}$

X locally arc-connected

$G = \pi_1(X, x)^{alg} = \lim H$

$$G = \pi_1(X,x) = \varprojlim H$$

$\pi_1(X,x) \to H$ Zariski dense ← linear alg./k

(iv) $k = \mathbb{C}$, $\mathcal{C} = MIC(X/k)$, ω_x

$$G = \pi_1(X(\mathbb{C}), x)^{alg}$$

(define: $\pi_1^{alg}(X,x)$ Tannakian group of $(MIC(X/k), \omega_x)$)

NOT compat. with k-str

(consequence of Riemann-Hilbert)

- For the proof, need notion of coalgebra/comodule

<u>Dfn</u> (i) A <u>k-coalgebra</u> is a triple (C, Δ, ε) where C is a k-vector space,

"comultiplication"

$$\left.\begin{array}{l} \Delta : C \to C \otimes C \\ \varepsilon : C \to k \end{array}\right\} k\text{-linear}$$

"counit"

such that the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes id"] & C \otimes C \otimes C \end{tikzcd}$$

"coassociative"

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d] \arrow[dr, "id"] & C \otimes C \arrow[d, "id \otimes \varepsilon"] \\ {} & {} \end{tikzcd}$$

"counital"

$$C \otimes C \xrightarrow{\varepsilon \otimes id} k \otimes C \cong C \cong C \otimes k$$

commute

(ii) Given a k-coalgebra (C, Δ, ε) a (right) C-comodule is a pair (V, ρ), V k-vector space $\rho : V \longrightarrow V \otimes C$ k-linear such that the diagrams

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V \otimes C \arrow[d, "id \otimes \Delta"] \\ V \otimes C \arrow[r, "\rho \otimes id"] & V \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[dr, "id"'] & V \otimes C \arrow[d, "id \otimes \varepsilon"] \\ & V \otimes k \cong V \end{tikzcd}$$

commute

Fact: If (A, m, u) finite-dimensional !

k-algebra, with

$m : A \otimes A \to A$ multiplication

$m: A \otimes A \longrightarrow A$ multiplication

$u: k \longrightarrow A$ unit

Then: $(A^\vee, m^\vee, u^\vee)$ is a coalgebra

Converse also true <u>without</u>

$\dim_k C < \infty$

Exercise: V = finite-dimensional

k-vector space. Show:

There exists a natural bijection

$$\left\{ \begin{matrix} \rho: V \to V \otimes C \\ C\text{-comodule structure} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \mathrm{End}(V)^\vee \longrightarrow C \\ \text{morphism of coalgebras} \end{matrix} \right\}$$

Δ dual to matrix multiplication

$\varepsilon(\varphi) = \varphi(id)$

<u>Sketch of Proof of Theorem</u> (Deligne–Milne Section 2)

- First, for any G/k affine group scheme, have

$$\mathrm{Rep}_k(G) \simeq \mathrm{Comod}_k(\mathcal{O}(G))$$

and $\mathcal{O}(G)$ is a (commutative) <u>Hopf algebra/k</u>

(Hopf algebra $= (H, m, u, \Delta, \varepsilon, S)$ s.t.

- (H, m, u) k-algebra
- (H, Δ, ε) k-coalgebra

"H is a

- Δ, ε are morphisms of k-algebras $\Longleftrightarrow$ m, u are morphisms of k-coalgebras — H is a bialgebra
- $S : H \longrightarrow H$ k-linear s.t. ("antipode")

$$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = u \circ \varepsilon$$

- e.g., $G = \mathbb{G}_m$, then

$\mathcal{O}(G) = k[X, X^{-1}]$ Hopf algebra,

$$\Delta(X^{\pm}) = X^{\pm} \otimes X^{\pm},$$

$$\varepsilon(X^{\pm}) = 1, \quad S(X^{\pm}) = X^{\mp}$$

(Exercise: show that this defines a Hopf algebra structure on $\mathcal{O}(\mathbb{G}_m)$ dual to the group structure on $\mathbb{G}_m$)

- First step: "Reconstruct the coalgebra structure on $\mathcal{O}(G)$"

$\mathcal{C}, \omega$ as above, $X \in Ob(\mathcal{C})$

- For $V \in Vect_k$, define

$$\underline{Hom}(V, X) := \underbrace{V^{\vee} \otimes X}_{(= X^n, \text{ if } V = k^n, \ldots)} \in Ob(\mathcal{C})$$

general case similar)

<u>Fact 1</u>: $\exists !$ smallest subobject

$$P_X \subset \underline{\mathrm{Hom}}(\omega(X), X)$$

s.t. $\omega(P_X) \subset \mathrm{Hom}(\omega(X), \omega(X))$

contains $\mathrm{id}_{\omega(X)}$

<u>Fact 2</u>: $A_X := \omega(P_X)$ is the largest k-subalgebra of $\mathrm{End}(\omega(X))$ stabilizing $\omega(Y)$, for all $Y \subset X^n$

- Let $\langle X \rangle \subset \mathcal{C}$ be the <u>strictly</u> (closed under isomorphism) full subcategory, whose objects are subquotients of X^n, $n \geq 0$

$$\begin{tikzcd} \langle X \rangle \arrow[rr, "\omega|\langle X \rangle"] \arrow[dr, dashed] & & \mathrm{Vect}_k \\ & \mathrm{Mod}_k(A_X) \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

More precisely:

<u>Lemma 1</u>: For each $Y \in \langle X \rangle$, the k-algebra A_X acts on

$\omega(Y)$ and ω defines an equivalence of categories

$$\begin{tikzcd} \langle X \rangle \arrow[rr, "\sim"] \arrow[dr, "\omega|\langle X\rangle"'] & & \mathrm{Mod}_k(A_X) \arrow[dl, "\text{forget}"] \\ & \mathrm{Vect}_k & \end{tikzcd}$$

Moreover, $A_X = \mathrm{End}(\omega|\langle X\rangle)$

(e.g. if $M \in \mathrm{Mod}_k(A_X)$, then

$P_X \otimes_{A_X} M := \mathrm{coker}(P_X \otimes A_X \otimes M \rightrightarrows P_X \otimes M)$

lies in $\langle X \rangle$, since P_X does and $\omega(P_X \otimes_{A_X} M) \simeq M$, because

$\omega(P_X) =: A_X$

$\Rightarrow$ essential surjectivity)

- Define $B_X := A_X^\vee = \mathrm{Hom}_k(A_X, k)$ k-coalgebra (ok since A_X finite-dimensional)
- Have $\mathrm{Mod}_k(A_X) \simeq \mathrm{Comod}_k(B_X)$ ($\mathrm{Hom}(V \otimes A_X, V) \simeq \mathrm{Hom}(V, V \otimes B_X)$ gives the equivalence)

- B_X, $X \in Ob(\mathcal{C})$ form a filtered diagram in k-coalgebras

($X, Y \in Ob(\mathcal{C})$, $X' = X \oplus Y$

$$\rightsquigarrow \begin{tikzcd} B_X \arrow[r] & B_{X'} \\ B_Y \arrow[ur] & \end{tikzcd}$$
)

Fact 3: $\mathcal{C} \simeq \mathrm{Comod}_k(B)$

$$B := \varinjlim_{X \in Ob(\mathcal{C})} B_X$$

equivalence of categories defined by ω

- key point: Each C-comodule V is a filtered colimit of its finite-dimensional C-subcomodules

$$V_i \subset V$$

$v \in V$, $\Delta(v) = \sum v_i \otimes c_i$
finitely many c_i, v_i
$\Rightarrow v$ generates a finite-dim. k-subcomodule

$\Rightarrow$ first step

- second step: "Reconstruct the algebra structure on $\mathcal{O}(G)$"

Fact 4: There is a one-one corr.

$$\left\{ \begin{matrix} B \otimes_k B \to B \\ \text{commutative} \\ k\text{-alg structure} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathrm{Comod}_B \times \mathrm{Comod}_B \to \mathrm{Comod}_B \\ \text{functors, satisfying} \\ \text{associativity, comm.} \\ \text{and unit constraints} \end{matrix} \right\}$$

- In our case, $\mathrm{Comod}_B \simeq \mathcal{C}$ has a tensor product (ACU)

$\Rightarrow$ B is a commutative k-algebra

in fact a k-bialgebra

Fact 5: For each $R \in k\text{-alg}$,

$$\underline{\mathrm{End}}(\omega)(R) \cong \mathrm{Hom}_{k\text{-alg}}(B, R)$$

- Fact 5 $\Rightarrow$ B represents

$$\underline{\mathrm{End}}(\omega) : k\text{-alg} \longrightarrow \text{Monoids}$$

- Third step: B is a Hopf algebra

Fact 6: $F, G : \mathcal{C} \to \mathcal{C}'$ tensor functors, $\mathcal{C}, \mathcal{C}'$ tensor categories. If $\mathcal{C}, \mathcal{C}'$ are <u>rigid</u>, then each natural transformation

$$\lambda : F \longrightarrow G$$

is an isomorphism

• Fact 6 $\Rightarrow \underline{\mathrm{End}}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$

$\Rightarrow B$ is a Hopf-algebra

$\Rightarrow$ Theorem

Non-neutral Tannakian categories

• R k-algebra

$\omega : \mathcal{C} \longrightarrow \mathrm{Mod}_R$ fibre functor,

if ω exact, faithful, tensorial

and $\omega(X) \in \mathrm{Proj}_R$, $\forall X \in \mathrm{ob}(\mathcal{C})$

Dfn: $\mathcal{C}$ rigid abelian tensor cat.

$\mathcal{C}$, $\mathrm{End}(\mathbb{1}) = k$. $\mathcal{C}$ is Tannakian

if $\exists$ $\omega : \mathcal{C} \longrightarrow \mathrm{Mod}_R$ fibre

functor, for some $0 \neq R \in k$-alg.

• Main theorem of "non-neutral"

Tannakian theory:

$\mathcal{C} \simeq \mathrm{Rep}_k(\mathcal{G})$, $\mathcal{G}$ certain affine gerbe

(c.f. "Catégories Tannakiennes P. Deligne")