# Introduction to Hodge theory 17/06/2021

## Part 2 - Lecture 8 Oxford

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Last time:

- Divisors vs line bundles

  $M$ projective $\Rightarrow$ $\mathrm{Div}(M) \twoheadrightarrow \mathrm{Pic}(M)$

- Chern class $c_1(L) \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$

- Lefschetz $(1,1)$ theorem

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7) Cycle classes

Let $M$ be a projective complex manifold of dim $n$.

<u>Def</u> An analytic cycle of codimension $p$ in $M$ is a formal sum

$$Z = \sum_i n_i [Y_i], \qquad n_i \in \mathbb{Z},$$

where $Y_i \subset M$ are closed irreducible analytic subvarieties of codim $p$. Group: $Z^p(M)$.

$\hookrightarrow$ equiv: $\dim Y_i^{\text{smooth}} = n-p$

There is a natural $\mathbb{Z}$-linear map

$$cl : Z^p(M) \longrightarrow H^{2p}(M, \mathbb{Z})$$

given by

$[\tilde{Y}]$ fund class in $H_{2n-2p}(\tilde{Y}, \mathbb{Z})$

$\rightsquigarrow \varphi_* [\tilde{Y}] \in H_{2n-2p}(M, \mathbb{Z})$

$$Y \longmapsto PD(\varphi_* [\tilde{Y}])$$

where $\tilde{Y} \to Y$ is a $\underbrace{\text{resolution of singularities}}_{\text{Hironaka}}$.

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[dr, "\varphi"'] & Y \arrow[d, hook] \\ & M \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "{[\tilde{Y}]}"] \arrow[d, dotted, "cl(Y)"'] & H_{2n-2p}(\tilde{Y}, \mathbb{Z}) \arrow[r] \arrow[d, "\varphi_*"] & H^{2n-2p}(\tilde{Y}, \mathbb{Z})^\vee \arrow[d, "(\varphi^*)^\vee"] \\ H^{2p}(M, \mathbb{Z}) \arrow[r, "PD", "\sim"'] \arrow[rr, bend right, "*", "\alpha \mapsto (\beta \mapsto \langle \alpha \cup \beta, [M] \rangle)"'] & H_{2n-2p}(M, \mathbb{Z}) \arrow[r, "*"] & H^{2n-2p}(M, \mathbb{Z})^\vee \end{tikzcd}$$

$*$ not isom if there's torsion!

<u>Lemma</u> For every $\beta \in H^{2n-2p}_{dR}(M)$,

$$\int_M cl(Y) \wedge \beta = \int_{Y^{\text{smooth}}} \beta . \qquad \square$$

Note: $cl(Z) \in H^{2p}(M, \mathbb{Z}) \cap H^{p,p}(M)$.

Thm For $D \in \mathrm{Div}(M) = Z^1(M)$,

$$cl(D) = c_1(\mathcal{O}(D)) \qquad \text{in } H^2(M, \mathbb{Z})$$

Proof idea: Choose a Hermitian metric on $\mathcal{O}(D)$ and let $\omega$ be the corresponding Chern curvature (locally, $\mathcal{O}(D) = \mathcal{O}\, e$, $\omega = \frac{i}{2\pi} \partial\bar\partial \log \|e\|^2$).

Fact: $[\omega] = c_1(\mathcal{O}(D))$ in $H^2_{dR}(M)$ (use Čech).

Now apply Poincaré – Lelong: $f \in \mathcal{O}(U)$, $U \subset \mathbb{C}^n$,

$$\frac{i}{2\pi} \partial\bar\partial \log |f|^2 = \delta_{\{f=0\}}$$

as currents. □

Now, let $X$ be an algebraic variety $/\mathbb{C}$.

We can consider, similarly, algebraic cycles $Z^p(X)$.

By Lefschetz (1,1) and GAGA, we get:

Thm If $X$ is smooth and projective, then every $\alpha \in H^2(X^{an}, \mathbb{Z}) \cap H^{1,1}(X)$ is of the form $\alpha = cl(Z)$ for some $Z \in Z^1(X)$. □

8) The Hodge conjecture

<u>Conjecture</u> Let $X$ be a smooth projective variety $/\mathbb{C}$.
For any $0 \leq p \leq \dim X$,

$$cl \otimes \mathbb{Q} : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X^{an}, \mathbb{Q}) \cap H^{p,p}(X^{an})$$

is surjective.

Terminology: elts of $H^{2p}(X^{an}, \mathbb{Q}) \cap H^{p,p}(X^{an})$ are called <u>Hodge classes</u>

"Every Hodge class is algebraic".

Remarks:

- Always true for $p=1$ (Lefschetz $(1,1)$)
- False in general without $\otimes \mathbb{Q}$ (Atiyah - Hirzebruch, $p=2$) cf. Totaro
- False in general for Kähler (Zucker, non-algebraic complex tori)
- Not known: Künneth components of $cl(\Delta)$ in $H^{2n}(X \times X)$.

Fact: Lefschetz's theorem on hyperplane sections and the hard Lefschetz thm imply that if $2p \leq n$ and the Hodge conjecture is true in codim $p$, then it's true in codim $n-p$.

Coro. If $\dim X \leq 3$, then the Hodge conj. is true for $X$. $\square$

$H$ Hodge structure

$$\mathrm{Hdg}(H) \leq MT(H) \leq GL(H_{\mathbb{Q}})$$

" fixes Hodge classes in multilinear constructions ... "

Thm $H$ polarizable $\rightsquigarrow$ $MT(H)$ reductive

Ex $E$ elliptic curve without cplex mult.

$$\Rightarrow \quad MT(H_1(E)) = GL_{2,\mathbb{Q}}, \quad \mathrm{Hdg}(H_1(E)) = SL_{2,\mathbb{Q}}$$

Let $X = E^n = E \times \cdots \times E$. Hodge ring:

$$R = \bigoplus_{k \geq 0} H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$$

$H^1(X, \mathbb{Q}) = H^1(E, \mathbb{Q})^{\oplus n} \Rightarrow H^{2k}(X, \mathbb{Q}) = \Lambda^{2k} H^1(E, \mathbb{Q})^{\oplus n}$

$SL_{2,\mathbb{Q}}$ acts on $R' = \bigoplus_{k \geq 0} \Lambda^{2k} H^1(E, \mathbb{Q})^{\oplus m}$

and $R = R'^{SL_{2,\mathbb{Q}}}$.

Invariant theory $\Rightarrow$ $R$ is generated in deg 2

<u>Coro.</u> The Hodge conjecture holds for $X$. $\square$

(See Ribet "Hodge classes on certain ab. var."). //