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Notebook:	DGT		
Created:	5/25/2020 9:20 AM	Updated:	5/26/2020 3:13 PM
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Lecture 9

Goal (for the next lectures):

- More conceptual approach to Picard-Vessiot extensions (recall: their construction involved several choices, independence only a posteriori)
- More precisely:

a) Define a canonical diff. K-algebra

$$T = T(L/K) \subset L$$

for any PV-extension $(L,\partial)/(K,\partial)$

s.t. $Q(T) = L$

b) Prove Kolchin's theorem:

$$\bar{K} \otimes_K T \xrightarrow{\sim} \bar{K} \otimes_k \mathcal{O}(G)$$

where $G := \mathrm{Aut}_\partial(L/k)$

(viewed as an algebraic group)

i.e.; "T is a G-equivariant

$\bar{K}/K$-form of $\mathcal{O}(G)$"

Notation: $\cdot$ $(L,\partial)/(k,\partial)$ PV-extension

$\cdot$ field of constants k alg. closed

$\cdot$ $G(L/k) := \mathrm{Aut}_\partial(L/k)$, alg. group$/_k$

<u>Dfn 9.1:</u> The <u>canonical domain</u>

$T = T(L/k)$ is the subset

$$T := \{ a \in L \mid \mathcal{L}(a) = 0, \text{ for some } \mathcal{L} \in k[\partial] \}$$

<u>Prop 9.2:</u> The following hold:

(i) $T = \left\{ a \in L \;\middle|\; \begin{array}{l} \dim_k V(a) < \infty, \\ V(a) := \sum_{g \in G(L/k)} k \cdot g(a) \end{array} \right\}$

(ii) T is a differential k-subalgebra

of L, stable under $G(L/k)$

and $Q(T) = L$.

Proof: (i)

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"$\subset$": $a \in T$, $\mathcal{L} \in K[\partial]$ s.t.

$$\mathcal{L}(a) = 0$$

Then $a \in V := \mathcal{L}^{-1}(0)$, $\dim_K V < \infty$

and V is $G(L/K)$-stable

$\Rightarrow V(a) \subset V$

$\Rightarrow \dim_K V(a) < \infty$

"$\supset$": $a \in L$, $\dim_K V(a) < \infty$

$a_1, \dots, a_n$ basis of $V(a)$

Consider: $\mathcal{L}(Y) := \dfrac{w(Y, a_1, \dots, a_n)}{w(a_1, \dots, a_n)}$

Prop 5.1 $\Rightarrow \mathcal{L} \in K[\partial]$

Also: $\mathcal{L}|_{V(a)} = 0$

(ii) Description of T in (i)

$\Rightarrow T \subset L$ K-subalgebra

Since $\partial(V(a)) = V(\partial(a))$

$\Rightarrow T \subset L$ differential K-subalg.

Finally: $L = K\langle V \rangle$, for some

$V := \mathcal{L}^{-1}(0)$

By definition $V \subset T$

$$\Rightarrow Q(T) = L$$

$\square$

- $G/_k$ algebraic group

$X(G) := \mathrm{Hom}(G, \mathbb{G}_m)$ character group

- W a G-module

(not necessarily rational)

- For $\chi \in X(G)$, let

$$W_\chi = \{ w \in W \mid g(w) = \chi(g) \cdot w \quad \forall g \in G \}$$

χ-eigenspace of W, k-vector space

- $w \in W_\chi$ is a <u>semi-invariant</u> (of weight χ)

<u>Exc. 19:</u> Suppose W is a k-algebra, and $\underbrace{G \text{ acts on } W \text{ by algebra morph.}}_{\text{"G acts rationally"}}$

⚠ Abuse of termin.

Show that:

(i) $W_{\chi_1} W_{\chi_2} \subseteq W_{\chi_1 \chi_2}$, $\forall \chi_1, \chi_2 \in X(G)$

(ii) $w \in W_\chi$ unit

$$\Rightarrow w^{-1} \in W_{\chi^{-1}}$$

(iii) W integral domain

$$\Rightarrow X(W,G) := \{ \chi \in X(G) \mid W_\chi \neq \{0\} \}$$

closed under multiplication

(iv) W field

$\Rightarrow X(W,G) \subset X(G)$ subgroup

Now $G \subseteq G(L/k)$ algebraic subgroup

s.t. $L^G = k$ (later: $\Rightarrow G = G(L/k)$)

Exc 20: (i) If $v \in W_\chi \setminus \{0\}$ then

$a := \frac{v'}{v} \in K$.

(ii) Deduce that $v \in T(L/k)$.

Partial converse to this:

Prop 9.3: · $a \in T$,

· $a_1, \dots, a_n$ k-basis of $V(a)$,

· $w = w(a_1, \dots, a_n)$ wronskian

· $\chi: G(L/k) \xrightarrow{\text{restr.}} Gl(V(a)) \xrightarrow{\det} \mathbb{G}_m$

Then: $w \in W_\chi$, i.e. w is a semi-invariant of weight χ.

Proof: Immediate from Exc. 12 (Lecture 6)

Prop 9.4: $\cdot$ $G \subseteq G(L/k)$ alg. subgroup

s.t. $L^G = k$.

$\cdot$ $S \subseteq L$ differential k-subalgebra

s.t. $G(L/k)$ acts rationally on S

(in particular $S \subseteq T$ by Prop 9.2.)

Then: $\cdot$ If $\{0\} \neq I \subset S$ G-stable ideal

$\Rightarrow \exists\ 0 \neq w \in I$ semi-invariant

$\cdot$ In particular: T has no non-trivial

G-stable ideals.

Proof: $\cdot$ $0 \neq a \in I$, $a_1, \dots, a_n$ basis of $V(a)$

$\cdot$ Prop 9.3.

$\Rightarrow w := w(a_1, \dots, a_n) \neq 0$ semi-invariant

$\cdot$ Have $w \in I$ (cofactor expansion

along first row)

$\cdot$ For the last statement:

$w^{-1} \in L$ semi-invariant

(by Exc 19.(ii))

$\Rightarrow w^{-1} \in T$ (by Exc. 20)

$\Rightarrow I = T$

$\Rightarrow I = 1$ $\square$

<u>Prop 9.5</u>: $\cdot\, G \subseteq G(L/k)$ alg. subgroup w. $L^G = k$.

$\cdot\, \mathcal{S} = \mathcal{S}(L)$ set of non-zero G-semi-invariants in L

Then: (i) $\mathcal{S} \subset L^\times$ subgroup w. $k^\times \subset \mathcal{S}$

(ii) $\exists\, \mathcal{S}' \subset \mathcal{S}$ finitely generated subgroup such that $\mathcal{S} = k^\times \cdot \mathcal{S}'$

<u>Proof</u>: (i) $k^\times \subset \mathcal{S}$ is clear since G fixes k

$\cdot$ Exc 19 $\Rightarrow \mathcal{S} \subset L^\times$ subgroup

(ii) $\cdot\, X(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ f.g. abelian group

$\Rightarrow X(L,G) \subset X(G)$ —— " ——

$\cdot\quad \chi_1, \dots, \chi_n$ generators of $X(L,G)$

$\updownarrow \qquad \updownarrow$

$a_1, \dots, a_n \in L^\times$ semi-invariants

$\cdot\, \mathcal{S}' := \langle a_1, \dots, a_n \rangle \subset \mathcal{S}$

subgroup generated by $a_1, \dots, a_n$

- If $a \in L^\times$ semi-invariant for
$$x = \prod_{i=1}^n x_i^{k_i}, \quad k_i \in \mathbb{Z}$$
define $a' := \prod_{i=1}^n a_i^{-k_i} \in L^\times$
- $g(aa') = aa' \quad \forall g \in G$
by construction
$\Rightarrow aa' \in K^\times$
$\Rightarrow \mathcal{S} = K^\times \cdot \mathcal{S}'$

$\square$

Keep the notation of Prop. 9.5

Cor. 9.6: $S \subset L$ diff. K-algebra, fin. generated + $G(L/K)$-stable

Then: $S[\mathcal{S}] \subset L$ fin. generated, G-stable, diff. K-subalgebra which has no non-trivial G-stable ideals.

Proof: $\cdot K \cdot \mathcal{S}$ G-stable
$\cdot \mathcal{S} \cup \{0\}$ stable under ∂

both from definition of semi-inv.

Prop 9.5 $\Rightarrow S[\mathcal{S}] = S[\mathcal{S}']$ fin. gen.

Prop 9.4 $\Rightarrow S[\mathcal{S}]$ no non-trivial

G-stable ideals (J closed under taking inverses)

$\square$

Next some technical results about base-change to $\overline{k}$

<u>Prop. 9.7</u>: $\cdot$ $G \leq \mathcal{G}(L/k)$ alg. subgroup

$\cdot$ $S \subset L$ G-stable diff. k-algebra

$\cdot$ $\overline{k}/k$ an algebraic closure

$\cdot$ $\overline{S} := S \otimes_k \overline{k}$

Assume: $S^G = k$

Then: (i) $\overline{S}$ is reduced

(ii) $G = G \otimes 1$ acts rationally on $\overline{S}$, with $\overline{S}^G = \overline{k}$

(iii) S no non-trivial G-stable ideals

$\Rightarrow \overline{S}$ —— $''$ ——

<u>Proof</u>: (i) Have $\overline{S} \subset L \otimes_k \overline{k}$

conclude by Lemma 4.7

(ii) $S^G = \ker\left(S \longrightarrow \prod_{g \in G} S,\ s \mapsto (g(s)-s)_g\right)$

$k \hookrightarrow \overline{k}$ flat

$\Rightarrow S^G \otimes \overline{k} = \ker\left(S \otimes_k \overline{k} \longrightarrow (\prod S) \otimes_k \overline{k}\right)$

$$\Rightarrow (S \otimes_k \overline{k})^G = \ker\Big(S \otimes_k \overline{k} \to \prod_{g \in G} (S \otimes_k \overline{k})\Big)$$

$$\Rightarrow \overline{S}^G = \overline{k}$$

(iii) $\{0\} \neq I \subset \overline{S}$ G-stable ideal

- Let $0 \neq x = \sum_{i=1}^{n} a_i \otimes b_i \in I$ s.t.

(a) $\{b_i\}$ k-linearly independent

(b) n minimal ($\Rightarrow a_i \neq 0 \ \forall i$)

- Now $J = \{c_1 \in S \mid \exists c_2, \dots, c_n, \sum_{i=1}^{n} c_i \otimes b_i \in I\}$

Have $\{0\} \neq J$ (since $a_1 \in J$)

and J is G-stable, since I is

$\Rightarrow 1 \in J$

$\Rightarrow \exists\ 0 \neq y = 1 \otimes b_1 + \sum_{i=2}^{n} c_i \otimes b_i \in I$

- $y - g(y) = \sum_{i=2}^{n} (c_i - g(c_i)) \otimes b_i \in I$

for all $g \in G$

minimality $\Rightarrow$ $y = g(y)$ $\forall g \in G$

$\Rightarrow y \in \overline{k}^\times \subset \overline{S}$

$\Rightarrow I = \overline{S}$

$\square$

Want to "geometrize" the

Preceding result (rings and varieties)

Prop 9.8: $\cdot\ F = \bar{F}$ field,

- X/F affine algebraic set
- $R := \mathcal{O}(X)$ coordinate ring of X
- $H \subset \mathrm{Aut}(X)$ subgroup s.t.
  - (a) X has no proper closed H-stable subsets
  - (b) $R^H = F$

Then: $\exists\ H_0 \subset H$ subgroup finite index

$X_0 \subset X$ connected component:

(1) X_0 non-singular, H_0-stable has no proper H_0-stable closed subsets, $\mathcal{O}(X_0)^{H_0} = F$

(2) $H_0 = \{h \in H \mid h(X_0) = X_0\}$

(3) $\exists\ h_1, \dots, h_n$ coset representatives for H_0 in H s.t.

$$\{h_i(X_0) \mid 1 \leq i \leq n\}$$

are the connected components of X

conn. comp.

Proof: $X = \bigcup_{i=1} X_i$, $X_i := V(e_i)$ min. idempotent of R

Have $\sum_{i=1}^{n} e_i = 1$, $e_i e_j = 0$, $i \neq j$

- H acts transitively on $\{e_i\}$

  (since $\sum_{h' \in He_1} h'$ is H-invariant idempotent $\Rightarrow \in F$)

- choose $e_0 \in \{e_i\}$, $h_i \in H$, s.t. $h_i(e_0) = e_i$

  Let $H_0 := \mathrm{Stab}(e_0)$, $X_0 := V(e_0)$

- Then (ii),(iii) hold by construction

- For (i), let $C \subset X_0$ H_0-stable and closed

  $\Rightarrow \bigcup_i h_i(C) \subset X$ H-stable, closed

  $\Rightarrow C = \begin{cases} \emptyset \\ X_0 \end{cases}$

- $\mathrm{sing}(X_0) \subsetneq X_0$ closed and H_0-stable

  $\Rightarrow \mathrm{sing}(X_0) = \emptyset$

- Finally, show that $\mathcal{O}(X_0)^{H_0} = F$

  Let $f_0 \in \mathcal{O}(X_0)^{H_0}$, and define

  $f \in \mathcal{O}(X)$, via $f(h_i(x_0)) = f_0(x_0)$

Claim: $f \in \mathcal{O}(X)^H$

- Let $x \in X$, $h \in H$.

  $x \in X_i$, $h(x) \in X_j$, some i, j

  $x = h_i(x_0)$, some $x_0 \in X_0$

- Now $h_0 := h_j^{-1} h h_i \in H_0$

  Have $f(x) = f(h_i(x_0)) = f_0(x_0)$

  and $f(h(x)) = f(hh_i(x_0)) = f(h_j h_0(x_0))$

  $= f_0(h_0(x_0))$

- $f_0 \in \mathcal{O}(X_0)^{H_0} \Rightarrow f \in \mathcal{O}(X)^H$ (Claim)

- Since $f \in F$, have $f_0 \in F$ $\square$

<u>Prop 9.9</u>: $F = \bar{F}$ field, X_i / F, $i = 1, 2$ affine algebraic sets.

Assume: X_1, X_2 have same dimension and number of connected components.

- $H \subset \mathrm{Aut}(H_1 \cup H_2)$ subgroup, s.t.

  X_i no proper closed H-stable subsets, and $\mathcal{O}(X_i)^H = F$.

Then: If $\phi : X_1 \to X_2$ H-equiv. $\Rightarrow \phi$ isom.