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Lecture 4: Existence and uniqueness of Picard-Vessiot extensions

Fix: - (K,∂) differential field

$k \subset K$ field of constants

- $\mathcal{L} = \partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0\partial^0 \in K[\partial]$

4.1: Picard-Vessiot extensions are normal (needed for uniqueness)

Prop 4.1: $\cdot$ $(L_i,\partial)/(K,\partial)$, $i=1,2$, Picard-Vessiot extensions for $\mathcal{L}$

- $(L,\partial)/(K,\partial)$ with $\ker(\partial)=k$
- $\sigma_i : L_i \longrightarrow L$ embeddings of diff. K-algebras

Then $\sigma_1(L_1) = \sigma_2(L_2)$

Proof: $\left.\begin{array}{l} \cdot\ V_i := \mathcal{L}^{-1}(0) \subset L_i \\ \cdot\ V := \mathcal{L}^{-1}(0) \subset L \end{array}\right\}$ k-vector spaces

- L_i PV-extension $\Rightarrow \dim_k V_i = n$

  and $\dim_k V \leq n$

- σ_i are K-differential embeddings

  $\Rightarrow \sigma_i(V_i) \subseteq V$

  $\Rightarrow \sigma_i(V_i) = V$

- Axiom (ii) of PV-extensions:

  L_i generated by V_i as a differential K-algebra

  $\Rightarrow \sigma_1(L_1) = \sigma_2(L_2)$ $\square$

4.2: The universal full solution algebra

- $S := K[y_{ij} \mid 0 \leq i \leq n-1,\ 1 \leq j \leq n][w^{-1}]$

  $w = w(y_{01}, \dots, y_{0n})$ Wronskian

  finitely generated K-algebra,

  integral domain

$\partial(y_{ij}) = \{ y_{i+1\,j} \quad 0 \leq i < n-1$

- $\partial(y_{ij}) = \begin{cases} \\ -\sum_{k=0}^{n-1} a_k y_{kj} & i = n-1 \end{cases}$

Prop 4.2: $\mathfrak{p} \subset S$ differential prime ideal

$L := Q(S/\mathfrak{p})$, $k' \subset L$ field of constants

Then: $(L,\partial)/(k,\partial)$ satisfies

(i) $\dim_{k'} \mathcal{L}^{-1}(0) = n$

(ii) (L,∂) is generated by $\mathcal{L}^{-1}(0)$

as a differential k-algebra

Proof: $[y_{0j}] \in \mathcal{L}^{-1}(0)$, for $j = 1, \dots, n$

(by construction of ∂)

and $W([y_{01}], \dots, [y_{0n}]) = [W(y_{01}, \dots, y_{0n})] \neq [0]$

($W \in S$ unit)

$\Rightarrow$ (i)

For (ii): $y_{ij} \in k[\partial^i(y_{0j}) \mid 0 \leq i \leq n-1, 1 \leq j \leq n]$

$\Rightarrow S$ differentially generated by $\{y_{0j}\}$

$\Rightarrow S/\mathfrak{p}$ —— " —— $\underbrace{[y_{01}], \dots, [y_{0n}]}_{\text{these span } \mathcal{L}^{-1}(0)}$

$\Rightarrow L$ —— " ——

$\Rightarrow$ (ii) $\square$

$\Rightarrow$ (ii) $\square$

Rmk 4.3: In general $k' \supsetneq k$.

E.g.: $\mathcal{L} = \partial^2$, $S = K[Y_{01}, Y_{11}, Y_{02}, Y_{12}][W^{-1}]$

$$\partial(Y_{01}) = Y_{11}, \quad \partial(Y_{02}) = Y_{12}$$

$$\partial(Y_{11}) = \partial(Y_{12}) = 0$$

$$\Rightarrow Y_{11}, Y_{12} \in k' \setminus k$$

4.3: Existence of Picard-Vessiot extensions

Thm 4.4. S as above

$\mathfrak{m} \subset S$ maximal differential ideal

Then: (i) $S/\mathfrak{m}$ is an integral domain

(ii) If $k = K$ algebraically closed, then $L := Q(S/\mathfrak{m})$ is a Picard-Vessiot extension for $\mathcal{L}$.

Proof: • $S/\mathfrak{m}$ has no non-trivial differential ideals

• $K \subset S/\mathfrak{m}$, $\operatorname{char}(K) = 0$

Prop. 3.10 $\Rightarrow$ $S/\mathfrak{m}$ integral domain

- Prop 4.2 $\Rightarrow$ L satisfies axioms (i), (ii) of Picard-Vessiot extensions
- k algebraically closed, so:

  Cor. 3.8 $\Rightarrow$ $k' = k$

  $$k' := \ker(\partial) \subset L$$

  $\Rightarrow$ axiom (iii) satisfied

  $\Rightarrow$ $(L,\partial)/(k,\partial)$ Picard-Vessiot extension.

$\square$

<u>Exmp 4.5</u>: · $\partial^2 = 0$, $S = k[Y_{01}, Y_{02}, Y_{11}, Y_{12}][W^{-1}]$

- $\partial(Y_{01}) = Y_{11}$, $\partial(Y_{02}) = Y_{12}$

  $$\partial(Y_{11}) = \partial(Y_{12}) = 0$$

- $$W = \begin{vmatrix} Y_{01} & Y_{02} \\ Y_{11} & Y_{12} \end{vmatrix} = Y_{01}Y_{12} - Y_{02}Y_{11}$$

Exercise 9: (i) show that

$$\mathfrak{a} = (Y_{02}-1, Y_{11}-1, Y_{12}) \subset S$$

is a differential ideal.

(ii) show that $\mathfrak{a}$ is maximal if and only if $\nexists\, a \in k$ s.t.

$\partial(a)=1$.

(iii) Assume $\exists a \in K$ s.t. $\partial(a)=1$

show that

$\mathfrak{a}' =$ diff. ideal gen. by $\mathfrak{a}$ and

$Y_{01} - a$

is maximal.

4.4: Uniqueness

Thm 4.5: L_1, L_2 Picard-Vessiot extensions for $\mathcal{L}$.

If k is algebraically closed,

$\exists \ \sigma : L_1 \xrightarrow{\sim} L_2$

K-differential isomorphism.

Proof: Wlog, $L_1 = Q(S')$, $S' = S/\mathfrak{m}$

where: S full universal solution algebra

$\mathfrak{m} \subset S$ maximal differential ideal

- $R := S' \otimes_K L_2$ w. derivation

$\partial_R := \partial_{S'} \otimes id_{L_2} + id_{S'} \otimes \partial_{L_2}$

- R finitely generated k-algebra

$$\varphi_1 : S' \longrightarrow R \ , \quad \varphi_2 : L_2 \longrightarrow R$$
$$s \longmapsto s \otimes 1 \qquad b \longmapsto 1 \otimes b$$

- $P \subset R$ maximal differential ideal,

$$\tilde{\mathfrak{p}} := \varphi_1^{-1}(P) \subset S'$$

- If $\tilde{\mathfrak{p}} = S' \Rightarrow 1 \otimes 1 \in P$ $\lightning$

$\Rightarrow \tilde{\mathfrak{p}} = \{0\}$ (S' no non-trivial diff. ideals)

$\Rightarrow \quad \overline{\varphi}_1 : S' \longrightarrow R' := R/\mathfrak{p}$

is injective

- $\overline{\varphi}_2 : L_2 \longrightarrow R'$ also injective

$\Rightarrow \overline{\varphi}_1, \overline{\varphi}_2$ extend to embeddings

$$\sigma_i : L_i \longrightarrow L \ , \quad L := Q(R')$$

- Cor 3.8 $\Rightarrow$ field of constants

of L is equal to k

- Prop 4.1 $\Rightarrow \sigma_1(L_1) = \sigma_2(L_2)$

$\Rightarrow \sigma := \sigma_2^{-1} \circ \sigma_1$ is an isomorphism

$$L_1 \xrightarrow{\sim} L_2 \quad \square$$

4.5. Automorphisms

Dfn 4.6: $(L,\partial)/(K,\partial)$ extension

$$\mathrm{Aut}_\partial(L/K) := \{\varphi \in \mathrm{Aut}(L/K) \mid \varphi\circ\partial = \partial\circ\varphi\}$$

group of differential automorphisms

Want to show: $L^{\mathrm{Aut}_\partial(L/K)} = K$

Need:

Lemma 4.7: k perfect field,

A, B reduced k-algebras.

Then: $A \otimes_k B$ is reduced

Proof: • $c \in A \otimes_k B$ s.t. $c^n = 0$

Want to show: $c = 0$

• $c = \sum_i a_i \otimes b_i$ finite sum

$\Rightarrow$ may assume A, B fin. gen.

• $\{e_i\}_{i\in I}$ k-basis of B

$$c = \sum_i a_i \otimes e_i$$

• If $a_i = 0$ $\forall i$ $\Rightarrow$ done

- If $a_j \neq 0$, $j \in I$, choose $m \subset A$ max. s.t. $a_j \notin m$ (Possible since A reduced)
- Hilbert's Nullstellensatz:

  A/m alg. over k
- Now $0 \neq [c] \in A/m \otimes B$ nilpotent

  $\Rightarrow$ Can assume $A = L$ finite extension of k

  $\overset{\text{symmetry}}{\Rightarrow}$ $B = M/k$ finite
- k perfect $\Rightarrow M \cong k[x]/(f)$

  $f \in k[x]$ separable ("Primitive element")
- Then $L \otimes_k M \cong L[x]/(f)$

  reduced, since f separable
- $\lightning$ to $a_j \neq 0$ $\square$

<u>**Prop 4.8**</u>: $\cdot$ $(L, \partial)/(K, \partial)$ PV-extension for $\mathcal{L}$, k alg. closed.
- Then for each $a \in L \setminus K$

  $\exists\, \tau \in \mathrm{Aut}_\partial(L/K)$, s.t. $\tau(a) \neq a$

Cor 4.9: $L^{\mathrm{Aut}_\partial(L/K)} = K$

Proof: $K \subset L^{\mathrm{Aut}_\partial(L/K)}$ by definition

$L^{\mathrm{Aut}_\partial(L/K)} \subset K$ by Prop. 4.8

Proof of Prop 4.8: May assume:

$L = Q(S')$, $S' = S/m$ maximal diff. ideal

solution algebra

- $a = \frac{b}{c}$, $b, c \in S'$ $\Rightarrow$ $a \in S'[c^{-1}]$
- $R := S'[c^{-1}] \otimes_K S'[c^{-1}]$, $d := a \otimes 1 - 1 \otimes a \in R$

  $a \notin K \Rightarrow d \neq 0$

$\Rightarrow Q^{-1}R \neq \{0\}$, $Q = \{d^i \mid i \geq 0\}$

- $\tilde{m} \subset Q^{-1}R$ max. diff. ideal

  $K \subset Q^{-1}R$:

- Prop 3.10 $\Rightarrow T := Q^{-1}R/\tilde{m}$ domain
- (T, ∂) fin. gen. differential K-algebra
- Note: $[d] \neq 0 \in T$

Now have $S'[c^{-1}] \overset{\tau_1}{\underset{\tau_2}{\rightrightarrows}} T$

$\tau_1(s) = [s \otimes 1]$, $\tau_2(s) = [1 \otimes s]$

differential k-embeddings

- Claim: τ_1, τ_2 are embeddings

Indeed, S' no non-trivial diff. ideals

$\Rightarrow S'[c^{-1}]$ ———— " ————

$\Rightarrow \ker(\tau_1) = \ker(\tau_2) = \{0\}$

diff. ideals

- Claim $\Rightarrow \tilde{\tau}_i : L \hookrightarrow Q(T)$

- T no non-trivial diff. ideals

$+$ k alg. closed

$\overset{(\text{Ex. } 3.8)}{\Longrightarrow}$ k field of constants of $Q(T)$

- Prop 4.1. $\Rightarrow \tilde{\tau}_1(L) = \tilde{\tau}_2(L)$

- Also $\tilde{\tau}_1(a) - \tilde{\tau}_2(a) = [a \otimes 1 - 1 \otimes a] = [d] \neq 0$

$\Rightarrow \tau : L \rightarrow L$, $\tau := \tilde{\tau}_2^{-1} \circ \tilde{\tau}_1$

$\tau(a) \neq a$

$\square$