

Introduction to Hodge theory

24/05/2021

Part 2 - Lecture 1

Oxford

1) Hodge structures

Thm If M is a compact Kähler manifold, then

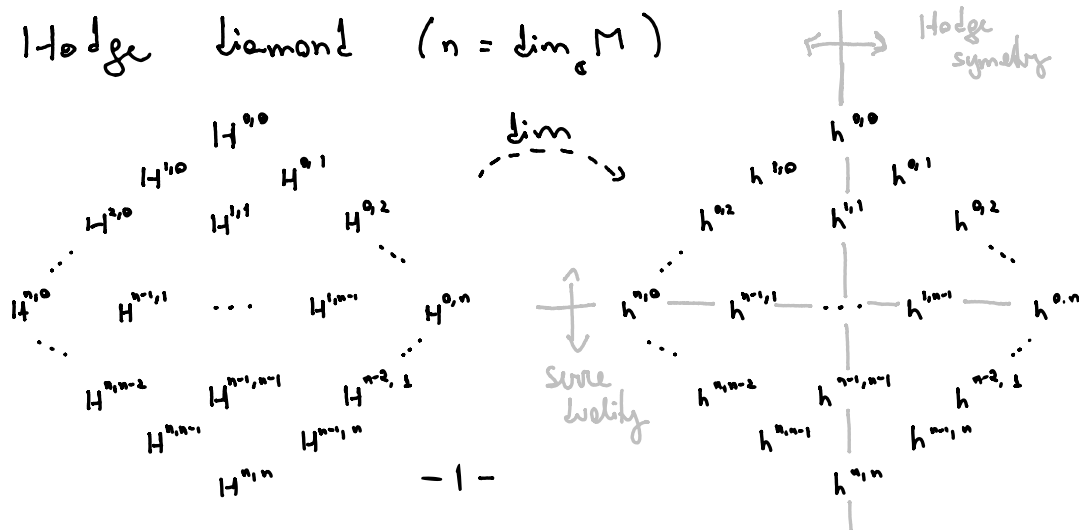
$$(i) \quad H^k(M, \mathbb{Z}) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}(M),$$

$$\overline{H^{p,q}(M)} = H^{q,p}(M)$$

$$(ii) \quad H^{p,q}(M) \cong H^q(M, \Omega^p_M)$$

Notation: $h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}(M)$ Hodge numbers

Hodge diamond ($n = \dim_{\mathbb{C}} M$)



Ex $M =$ compact Riemann surface ($\dim_{\mathbb{C}} M = 1$)

[any Hermitian metric is Kähler]

$$\begin{array}{ccc} & \downarrow & \\ g & \perp & g \end{array} \quad g = \lim H^0(M, \Omega^n)$$

Ex $M = \mathbb{P}^n$, $\omega =$ Fubini-Study

$$\text{Fact: } H^k(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

[Induction: $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$]

$$H^k(\mathbb{P}^n, \mathbb{P}^{n-1}) = H_c^k(\mathbb{P}^n \setminus \mathbb{P}^{n-1}) = H_{2n-k}^{\mathbb{C}^n}(\mathbb{P}^n \setminus \mathbb{P}^{n-1}) = \begin{cases} 0, & k < n \\ \mathbb{Z}, & k = n \end{cases}$$

$$\cdots \rightarrow H^k(\mathbb{P}^n, \mathbb{P}^{n-1}) \rightarrow H^k(\mathbb{P}^n) \rightarrow H^k(\mathbb{P}^{n-1}) \rightarrow H^{k+1}(\mathbb{P}^n, \mathbb{P}^{n-1}) \rightarrow \cdots$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \circ & & \circ & & \\ & \circ & & 1 & & \circ & \\ & \vdots & & \vdots & & \vdots & \\ \circ & \cdots & & 1 & & \cdots & \circ \\ & \vdots & & \vdots & & \vdots & \\ & \circ & & 1 & & \circ & \\ & \circ & & \circ & & \circ & \\ & & & 1 & & & \end{array}$$

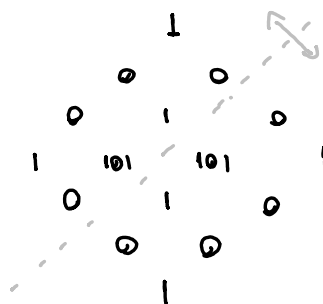
Note:

$$1) H^{p,p}(\mathbb{P}^n) = \mathbb{C}[\omega^p]$$

$$H^p(\mathbb{P}^n, \Omega^p)$$

$$2) H^q(\mathbb{P}^n, \Omega^p) \cong \begin{cases} \mathbb{C}, & p=q \leq n \\ 0, & \text{otherwise} \end{cases}$$

Ex $M \subset \mathbb{P}^4$ quintic (e.g. $z_0^5 + z_1^5 + \dots + z_4^5 = 0$)



$$h^{1,1}(M) = 1, \quad h^{1,2}(M) = 101$$

↳ moduli of complex str.
↳ moduli of symplectic str.

Mirror W satisfies $h^{1,1}(W) = 101, \quad h^{1,2}(W) = 1.$

Def A Hodge structure of wt $k \in \mathbb{Z}$ is a lattice H with a decomposition

$$H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}, \quad H^{p,q} = \overline{H^{q,p}}$$

Ex M cpt Kähler $(H^k(M, \mathbb{Z}), (H^{p,q}(M))_{p,q})$

- Topology of cpt Kähler
- $M \hookrightarrow \mathbb{P}^n$ projective \rightarrow algebraic geometry
- " when is an elt of $H_k(M, \mathbb{Z})$ the fundamental class of an alg. cycle $\Gamma \subset M$?"
- Hodge conjecture...

- Functoriality :

$$\begin{array}{ccc}
 \text{Smooth proj. alg. var} / \mathbb{C} & \longrightarrow & \text{Hodge structures} \\
 \downarrow & & \downarrow \\
 \text{Alg var} / \mathbb{C} & \longrightarrow & \text{Mixed Hodge str.}
 \end{array}$$

Analogies with ℓ -adic cohomology...

- Families \rightsquigarrow variations of Hodge structures
Limiting MHS, period domains, Shimura varieties,...

2) Coherent analytic sheaves

Complex manifold: (M, \mathcal{O}_M)

- M Hausdorff paracompact
- \mathcal{O}_M sheaf of complex fcts.

locally isom. to $(U, \mathcal{O}_{\mathbb{C}^n|_U})$, $U \subset \mathbb{C}^n$ open
 \hookrightarrow holomorphic fcts.

Morphism $(M, \mathcal{O}_M) \rightarrow (N, \mathcal{O}_N)$:

$f: M \rightarrow N$ continuous st $\forall U \subset N$ open

$$f^*: \mathcal{O}_N(U) \rightarrow \mathcal{O}_M(f^{-1}(U))$$

$$g \mapsto g \circ f$$

Def A subset $X \subset \mathbb{C}^n$ is analytic if $\forall x \in X$
 $\exists U \subset \mathbb{C}^n$ open neigh. of x and $f_1, \dots, f_m \in \mathcal{O}_{\mathbb{C}^n}(U)$
st $X \cap U = \{z \in U \mid f_1(z) = \dots = f_m(z) = 0\}$.

It inherits a sheaf of hol. fun. \mathcal{O}_X from $\mathcal{O}_{\mathbb{C}^n}$.

Def An analytic variety is a pair (X, \mathcal{O}_X) ,
 X Hausdorff paracompact, locally isomorphic to
 (Y, \mathcal{O}_Y) , $Y \subset \mathbb{C}^n$ is an analytic subset.

Def: A complex manifold is an analytic variety.

Conversely, a smooth analytic variety (locally,
 $(\partial f_i / \partial z_j)_{i,j}$ has maximal rank) is a complex
manifold by the "analytic implicit fun thm".

Ex (Subvariety of a complex manifold)

M complex manifold, $X \subset M$ closed \hookrightarrow

$$M = \bigcup_i U_i, \quad X \cap U_i = \{p \in M \mid f_1(p) = \dots = f_{n_i}(p) = 0, \text{ } f_j \text{ hol}\}$$

$\leadsto (X, \mathcal{O}_X)$ analytic variety.

e.g. • $X = \{(z, e^z) \mid z \in \mathbb{C}\} \subset \mathbb{C}^2$

• $X = \{(z_0 : \dots : z_n) \mid F_i(z) = 0 \ \forall 1 \leq i \leq k\} \subset \mathbb{P}^n$

(Projective variety) //