Ink note

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Lecture 2:

Recap: • (K,∂) differential field

(always $\mathrm{char}(K)=0$)

k field of constants

• $\mathcal{L} = \partial^n + a_1\partial^{n-1} + \ldots + a_n\partial^0 \in K[\partial]$

linear differential operator

• $\exists$ Picard-Vessiot extension

$(L,\partial)/(K,\partial)$ for $\mathcal{L}$:

(i) $\dim_k \mathcal{L}^{-1}(0) = n$

(ii) L/k generated by $\mathcal{L}^{-1}(0)$

(iii) field of constants of (L,∂)

is exactly k

Thm 1.11: Notation as above, if

k is algebraically closed, then

Picard-Vessiot extensions

(i) exist.

(ii) unique up to non-canonical isomorphism

Leads to studying automorphisms of $(L,\partial)/(k,\partial)$

<u>Dfn 1.12:</u> $(L,\partial)/(k,\partial)$ Picard-Vessiot extension for $\mathcal{L}$. Define

$$\mathrm{Aut}_\partial(L/k) := \{\varphi : L \to L \mid \varphi \text{ field auto.}, \varphi|_k = \mathrm{id}, \varphi \circ \partial = \partial \circ \varphi\}$$

- $\mathrm{Aut}_\partial(L/k)$ acts on $\mathcal{L}^{-1}(0) =: V$

$\leadsto$ can view $\mathrm{Aut}_\partial(L/k) \subset GL(V)$

<u>Prop. 1.13:</u> $\mathrm{Aut}_\partial(L/k) \subset GL(V)$ is Zariski-closed, i.e. an algebraic subgroup.

$\leadsto$ $\mathrm{Aut}_\partial(L/k) = \mathrm{Gal}_\partial(L/k)(k)$

differential Galois group

Thm 1.14 (Main theorem of differential Galois theory):

(K,∂) differential field, k alg. closed, $(L,\partial)/(K,\partial)$ Picard-Vessiot extension for some $\mathcal{L} \in K[\partial]$.

Then: the maps

$$M \longmapsto \mathrm{Aut}_\partial(L/M), \quad H(k) \longmapsto L^{H(k)}$$

are order-reversing bijections

$$\left\{ \begin{matrix} \text{intermediate fields} \\ (L,\partial) \supset (M,\partial) \supset (K,\partial) \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{closed subgroups} \\ H(k) \subset \mathrm{Gal}_\partial(L/K)(k) \end{matrix} \right\}$$

Picard-Vessiot subextensions M/K $\mapsto$ closed normal subgroups.

and $\mathrm{Gal}_\partial(M/K) \cong \mathrm{Gal}_\partial(L/K)/H$

1.3: Examples

Example 1.15: (K,∂), $\mathcal{L} = \partial$.

Then $\mathcal{L}^{-1}(0) = k \subset K$

$\Rightarrow (K,\partial)$ itself is a PV-extension

for $\mathcal{L}$

$\Rightarrow \mathrm{Gal}_\partial(K/K) = \{e\}$

<u>Example 1.16:</u> $K = \mathbb{C}(x)$, $\partial = \frac{d}{dx}$

$k = \mathbb{C}$, $\mathcal{L} = \partial - id$

Consider $K' := \mathbb{C}((x))$ formal Laurent series

∂ extends by same formula

$e^x \in K'$ satisfies $\partial(e^x) = e^x$

Claim: $L := K(e^x) \subset K'$ is a PV-ext.

for $\mathcal{L}$

Exercise 4: Prove this!

Claim: $\mathrm{Gal}_\partial(L/K) \cong \mathbb{G}_m/\mathbb{C}$

Indeed, any $\varphi \in \mathrm{Aut}_\partial(L/K)$

maps $\underset{\underset{\mathbb{C}\cdot e^x}{\shortparallel}}{\mathcal{L}^{-1}(0)}$ to itself

$\Rightarrow \varphi(e^x) = \alpha \cdot e^x$, $\alpha \in \mathbb{C}^\times$

$\Rightarrow \varphi \mapsto \alpha$ gives an isomorphism

$\Rightarrow$ Claim

Now closed subgroups of $\mathbb{G}_m$ are

(i) $\mathbb{G}_m$

(ii) finite, even cyclic

main thm
and for each $n \geq 1$, get

subextension $L \supset M \supset K$
$$\begin{matrix} L & \supset & M & \supset & K \\ \| & & \| & & \| \\ K(e^x) & \supset & K(e^{nx}) & \supset & K \end{matrix}$$

corresponding operator is $\partial - n \cdot id$

Exercise 5: Repeat this for an arbitrary (K, ∂) (still $\mathcal{L} = \partial - id$), i.e. construct a PV-extension L and determine $\mathrm{Gal}_\partial(L/K)$ (answer depends on whether or not $\exists a \in K$ with $\partial(a) = n a$, $n \geq 1$)

Example 1.17: $K = \mathbb{C}(X)$, $\partial := X \frac{d}{dX}$

and equation $\partial(a) = 1$ multiply by ∂

and $\mathcal{L} = \partial^2$, want to solve $\mathcal{L}(a) = 0$

Consider $L := K(Y)$, Y indeterminate,

$\partial(Y) = 1$

Claim: $(L,\partial)/(K,\partial)$ is a PV-extension,

with $\mathrm{Galg}(L/K) \cong \mathbb{G}_a/_{\mathbb{C}}$

Briefly: (i) $\dim_{\mathbb{C}} \mathcal{L}^{-1}(0) = 2$

$\mathcal{L}^{-1}(0) \subset L$ $\quad$ ($1, y$ basis)

(ii) L gen. by $\mathcal{L}^{-1}(0)$

(iii) $\partial(h) = 0 \implies h \in \mathbb{C}$

$h \in K(y)$

Exercise 6: Prove (iii)

Hint: write $h = \frac{f}{g}$, $f, g \in K[Y]$

f, g coprime

$\overset{\partial(h)=0}{\leadsto} \partial(f) \mid f$, $\partial(g) \mid g$

As for $\mathrm{Galg}(L/K)$, each $\varphi \in \mathrm{Gal}(L/K)(\mathbb{C})$

maps $\underset{\mathbb{C} \oplus \mathbb{C}\cdot y}{\underset{\cong}{\mathcal{L}^{-1}(0)}} \longrightarrow \mathcal{L}^{-1}(0)$

$\leadsto \varphi(y) = \mu + \lambda \cdot y$ $\qquad$ $\mu \in \mathbb{C}$, $\lambda \in \mathbb{C}^\times$

In fact, $\lambda = 1$ (since $\varphi \circ \partial = \partial \circ \varphi$)

$\leadsto \varphi \longmapsto \mu$ bijection

$\mathrm{Aut}_\partial(L/K) \cong \mathbb{C}$

Fact: $\mathbb{G}_{a/\mathbb{C}}$ has no non-trivial alg. subgroups.

$\overset{\text{Main thm}}{\Longrightarrow}$ $(K(Y), \partial)/(K, \partial)$ has no non-trivial intermediate fields

Preview to Lecture 3:

- Rudiments of differential algebra
- Constants, in particular, how to ensure that $(L, \partial)/(K, \partial)$ has no new constants
- Wronskian ("descent")