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Lecture 11

Recall: $\cdot$ $(L,\partial)/(K,\partial)$ PV-extension

$\cdot$ k field of constants, alg. closed

$\cdot$ $G \subseteq \mathrm{Aut}_\partial(L/K)$ Zariski-closed subgroup

$\cdot$ $T(L/L^G) = \{a \in L \mid \exists \mathcal{L} \in L^G[\partial],\ \mathcal{L}(a) = 0\}$ canonical domain

Then: $T = T(L/L^G)$ is a fin. generated diff. K-algebra, $Q(T) = L$

and: $\bar{K} \otimes_K T \cong \bar{K} \otimes_k \mathcal{O}(G)$

G-equivariant isomorphism of $\bar{K}$-algebras

Now: Characterization of

PV-extensions with group:

a) $\mathbb{G}_a$, b) $\mathbb{G}_m$

Notation: $\cdot (K,\partial)$ differential field,

$\cdot (L,\partial)/(K,\partial)$ PV-extension,

w. alg. closed field of constants k

$\cdot G(L/K) := \mathrm{Aut}_\partial(L/K)$

a) $G(L/K) = \mathbb{G}_a$

Then: $\bar{K} \otimes_K T \cong \bar{K}[t]$, even

$T \cong K[t]$

(since $H^1(\bar{K}/K, \mathbb{G}_a(\bar{K})) = 0$, as

$\mathbb{G}_a$ is abelian)

$\cdot\ a \in G(L/K) \cong (k,+)$ acts as

$a \cdot t = t + a$

$\Rightarrow \exists\, y \in T$ s.t. $a \cdot y = y + a$

and $T = K[y]$

$\cdot$ Have $a \cdot \partial(y) = \partial(a \cdot y) = \partial(y)$

$\Rightarrow \partial(y) \in L^G = K$

"y is an integral"

- Explicitly, y is a zero of

$$\mathcal{L}(Y) = \frac{W(Y,y,1)}{W(y,1)} = \partial^2(Y) - \underbrace{\frac{\partial^2(y)}{\partial(y)}}_{\in K} \partial(Y)$$

- so $G(L/K) = \mathbb{G}_a$

$\Leftrightarrow T = K[y]$, $\partial(y) \in K$, $L = K(y)$

"adjunction of an integral"

b) $G(L/K) = \mathbb{G}_m$:

Then $\overline{K} \otimes_K T \cong \overline{K} \otimes_k \mathcal{O}(\mathbb{G}_m)$

$\cong \overline{K}[t, t^{-1}]$

Again: $T \cong K[t, t^{-1}]$

(since $H^1(\overline{K}/K, \mathbb{G}_m(\overline{K})) = 0$)

"Hilbert 90"

- $a \in G(L/K) \cong (k^\times, \cdot)$ acts as

$a \cdot t = at$

$\Rightarrow \exists y \in T$, s.t. $T = K[y, y^{-1}]$

and $a \cdot \partial(y) = \partial(a \cdot y) = a \partial(y)$

$\Rightarrow \partial(Y) \in kY$ "Y is an exponential"

- Explicitly: $\delta(Y)=0$ for

$$\delta(Y) = \partial(Y) - \underbrace{\frac{\partial(Y)}{Y}}_{\in k} \cdot Y$$

- So $G(L/k) = \mathbb{G}_m$

$\Longleftrightarrow T = k[Y, Y^{-1}]$, $\partial(Y) \in kY$, $L = k(Y)$

__Exc. 21:__ Generalize b) to the case $G(L/k) = \underbrace{\mathbb{G}_m \times \ldots \times \mathbb{G}_m}_{k}$

Final steps towards the main theorem of diff. Galois theory

__Prop 11.1:__ Let $H \subseteq G(L/k)$ be a Zariski-closed subgroup. Then $H = G(L/L^H)$

__Proof:__ - Have $H \subseteq G(L/L^H)$.

- Denote: $M = L^H$, $G = G(L/M)$

Prop 7.10 $\Rightarrow$ L/M PV-extension

Cor. 4.9 $\Rightarrow L^G = M$

- The inclusion $H \hookrightarrow G$ induces $h: \mathcal{O}(G) \twoheadrightarrow \mathcal{O}(H)$ and $H \cong G \Leftrightarrow h$ isomorphism
- Enough to prove:
  $\bar{M} \otimes h : \bar{M} \otimes_k \mathcal{O}(G) \twoheadrightarrow \bar{M} \otimes_k \mathcal{O}(H)$
  is an isomorphism
  (since $\ker(\bar{M} \otimes h) \cong \bar{M} \otimes \ker(h)$)
- Thm 10.2: $\exists$ isomorphisms
  $\bar{M} \otimes_M T \longrightarrow \bar{M} \otimes_k \mathcal{O}(G)$,
  $\bar{M} \otimes_M T \longrightarrow \bar{M} \otimes_k \mathcal{O}(H)$
  where $T = T(L/M)$
- These only depend on a choice of $f: T \longrightarrow \bar{M}$
- choosing the same f for both
  $\Rightarrow \bar{M} \otimes_k \mathcal{O}(G) \xrightarrow{\sim} \bar{M} \otimes_k \mathcal{O}(H)$
  $\Rightarrow H = G$

$\square$

Prop. 11.2: $H \trianglelefteq G(L/k)$ **normal** Zariski-closed subgroup.

Then: (i) L^H/k PV-extension

(ii) $r: G(L/k) \longrightarrow G(L^H/k)$ is surjective and $\ker(r) = H$

Proof: · Denote $G = G(L/k)$

$T = T(L/k)$

- Want to show: T^H fin. generated k-algebra, and $Q(T^H) = L^H$.
- If we know this, then: Prop 5.1 $\Rightarrow$ L^H/k PV-extension since $T^H \subset T$ G-stable, diff. k-subalgebra
- Clearly $Q(T^H) \subset L^H$

Conversely: $f \in L^H$, $f \neq 0$

$I = \{t \in T \mid tf \in T\}$ ideal of denominators

- $f \in L^H \Rightarrow I$ is H-stable

Let $s \in I \setminus \{0\}$, $s_1, \dots, s_n$ k-basis of $W = \mathrm{span}_k \{h(s) \mid h \in H\}$

- Prop 9.3 $\Rightarrow w = w(s_1, \dots, s_n)$ semi-invariant of H, weight $\chi = \det_W$ and $w \in I$

- $t := wf$ semi-invariant of weight χ

Claim: $X(T,H) = \{\chi \in X(H) \mid T_\chi \neq \{0\}\}$ is a subgroup of $X(H)$

- Assuming the claim, $\exists u \in T \setminus \{0\}$ semi-invariant of weight χ^{-1}

$\Rightarrow f = \frac{t}{w} = \frac{tu}{wu}$ $\in T^H$

$\Rightarrow$ (i)

Proof of claim:

<u>Step 1</u>: Every $\chi \in X(H)$ occurs as a semi-invariant in $\mathcal{O}(G)$

$H \subset \ker(\chi)$

• $H_0 := \bigcap_{\chi \in X(H)} \ker(\chi)$

• $H_0 \subset H$ characteristic subgroup

(i.e. stable by automorphisms)

$\Rightarrow H_0 \trianglelefteq H$ normal

• H/H_0 reductive (Ex. 22: Prove this!)

$\Rightarrow \mathrm{Rep}(H/H_0)$ semi-simple

Exc. 23: show that $H_0 \subseteq G$ is a normal subgroup.

(Hint: For each $g \in G$ and each $\chi \in X(H)$, show that

$$\chi^g : H \longrightarrow \mathbb{G}_m$$
$$h \longmapsto \chi(ghg^{-1})$$

is a character of H)

• $H/H_0 \hookrightarrow G/H_0$ inclusion

$\rightsquigarrow \mathcal{O}(G/H_0) \twoheadrightarrow \mathcal{O}(H/H_0)$

surjection of H/H_0-modules

• H/H_0 reductive

$\Rightarrow \mathcal{O}(G/H_0) \twoheadrightarrow \mathcal{O}(H/H_0)$

$$\Rightarrow \mathcal{O}(G/H_0)_\chi \twoheadrightarrow \mathcal{O}(H/H_0)_\chi$$

$$\forall \chi \in X(H) = X(H/H_0)$$

- Now $\chi \in \mathcal{O}(H/H_0)_\chi$

$$\Rightarrow \mathcal{O}(H/H_0)_\chi \neq 0$$

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$$\Rightarrow \underset{\substack{\cup\mid \\ \mathcal{O}(G/H_0)_\chi}}{\mathcal{O}(G)_\chi} \neq 0$$

- Step 2: Transfer to T

Thm 10.2 $\Rightarrow \bar{k} \otimes_k T \cong \bar{k} \otimes_k \mathcal{O}(G)$

Since G acts trivially on $\bar{k}$,

step 1 $\Rightarrow T_\chi \neq 0$

$\Rightarrow$ claim

- Thm 10.2

$$\Rightarrow \bar{k} \otimes_k T^H = (\bar{k} \otimes_k T)^H \cong (\bar{k} \otimes_k \mathcal{O}(G))^H$$
$$= \bar{k} \otimes_k \mathcal{O}(G)^H$$
$$= \bar{k} \otimes_k \mathcal{O}(G/H)$$

- $\mathcal{O}(G/H)$ fin. generated, G-stable

$\Rightarrow T^H$ ——— " ———

- $\exists\, V \subset T^H$, $\dim_k V < \infty$, $G(V) \subseteq V$ s.t. V generates T^H as a k-algebra

$\Rightarrow L^H = Q(T^H) = K\langle V\rangle$

- Also, L^H/K has no new constants

Prop 5.1 $\Rightarrow L^H/k$ PV-extension

$\Rightarrow$ (i)

- (ii) follows from Prop. 5.3. and Cor. 7.9

$\square$

Conversely:

<u>Prop 11.3:</u> $L/M/K$ intermediate extension. If M/K PV

$\Rightarrow G(L/M) \trianglelefteq G(L/K)$ normal subgroup

<u>Proof</u>: $\cdot\ \sigma_1, \sigma_2 : M \longrightarrow L$ k-diff. homomorphisms

$\Rightarrow \sigma_1(M) = \sigma_2(M) = M$ (Prop. 4.1)

- $G = G(L/K)$, $\quad H = G(L/M)$

  $\tau \in H$, $\sigma \in G$, $y \in M$

- $\gamma := \sigma^{-1}\tau\sigma$

- $\sigma|_M : M \longrightarrow M$ (by above)

  $\Rightarrow \sigma(y) \in M$

  $$\Rightarrow \gamma(y) = (\sigma^{-1}\tau\sigma)(y) = \sigma^{-1}(\tau(\sigma(y))) \overset{\tau\in H}{=} \sigma^{-1}(\sigma(y)) = y$$

  $\Rightarrow \sigma^{-1}\tau\sigma \in H$

$\square$

<u>Theorem 11.4</u> (Main theorem of differential Galois theory for PV-extensions):

Notation as above. Then:

(i) $G(L/K)$ is an algebraic group $/k$

(ii) There is an order-reversing bijection

$$\left\{ L \supseteq M \supseteq K \;\middle|\; \text{int. diff.} \right\} \longleftrightarrow \left\{ H \subseteq G(L/K) \;\middle|\; \text{closed} \right\}$$

{ fields } { subgroups }

$$M \longmapsto G(L/M)$$
$$L^H \longleftarrow\!\!\shortmid H$$

(iii) M/K PV-extension

$\Leftrightarrow G(L/M) \trianglelefteq G(L/K)$

We then have: $G(L^H/K) \cong G(L/K)/H$

(iv) $G^0(L/K) \subset G(L/K)$ connected comp.

L^0 corresponding diff. field

Then: a) $L^0 = \{ a \in L \mid \exists f \in K[x],\ f(a) = 0 \}$

(algebraic closure of K in L)

b) L^0/K finite Galois extension

w. $\mathrm{Gal}(L^0/K) \cong G(L/K)/G^0(L/K)$

c) $\mathrm{tr.deg}_{L^0} L = \dim(G^0(L/K))$

<u>Proof</u>: (i) was established in Lecture 7

(ii) $L \supseteq M \supseteq K$ intermediate diff. field

$\Rightarrow L/M$ PV (Prop 7.10)

$\Rightarrow G(L/M) \subseteq G(L/K)$ Zariski-closed

(follows from Cor. 7.9)

- $H \subseteq G(L/K)$ closed subgroup
  $\Rightarrow L \supseteq L^H \supseteq K$ intermediate diff. field
- Have $L^{G(L/K)} = K$ (Cor. 4.9)
  and $H = G(L/L^H)$ (Prop. 11.1)
- order reversal clear

(iii) "$\Rightarrow$" see Prop. 11.3

"$\Leftarrow$" see Prop. 11.2.(i)

- For $G(L^H/K) \cong G(L/K)/H$, see Prop. 11.2.(ii)

(iv).a) : proved in Prop. 10.5

b) follows from Prop. 10.5

c) proved in Thm 10.4

$\square$

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what next? Possibly:

(i) Integration in finite terms
(Liouvillian extensions)

ex: when is an anti-derivative

DV: when is an anti-derivative expressible in terms of elementary functions?

(ii) Inverse Galois problem

(given a linear algebraic group G/k, is it the diff. Galois group of some PV-extension L/k?)

Affirmative answer for $k = \mathbb{C}(t)$

uses Riemann-Hilbert corr.

(iii) Reinterpretation of PV-ext. using Tannakian categories