Idea of proof:

(iii) => (ii) easy

use that ∇_{chern} is $\mathbb{C}$-linear

(ii) => (i) $g(u,v) = \omega(u, Iv)$

∇ Levi-Civita

$\Rightarrow d(g(x,\psi)) = g(\nabla x, \psi) + g(x, \nabla\psi)$

$\nabla \circ I = I \circ \nabla$

$\Rightarrow d(\omega(x,\psi)) = \omega(\nabla x, \psi) + \omega(x, \nabla\psi)$

Direct computation shows that:

$d\omega = 0$

(i) => (iii) hardest part

Key fact: h Kähler $\Leftrightarrow \forall x \in X$

$\exists$ hol. coord. $z_1, \dots, z_n$ s.t.

$$(h_{ij})_{1 \leq i,j \leq n} = I_n + O(\sum |z_i|^2)$$

Now use that ∇_{chern}, ∇_{LC} depend

on h to first order

=> reduces to standard metric $\square$

4. The Fubini-Study metric

- $X = \mathbb{P}^n(\mathbb{C})$, $\mathcal{O}_{\mathbb{P}^n}(1)$ = dual of tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$
- Hermitian metric h on $\mathcal{O}_{\mathbb{P}^n}(-1)$ via $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}$ and Hermitian metric h^* on $\mathcal{O}_{\mathbb{P}^n}(1)$
- Now $U_i = \{[z_1 : \ldots : 1 : \ldots : z_{n+1}]\} \subset \mathbb{P}^n(\mathbb{C})$

  σ_i canonical trivialization of $\mathcal{O}_{\mathbb{P}^n}(-1)$

<u>Dfn</u>: Chern form ω of h^* locally by $\omega_i = \frac{1}{2\pi i} \partial\bar\partial \log h^*(\sigma_i^*)$

(1,1)-form on $\mathbb{P}^n(\mathbb{C})$

<u>Check</u>: This is well-defined

$$\omega_j|_{U_i \cap U_j} = \omega_i|_{U_i \cap U_j}$$

Hint: Real parts of analytic fcts are harmonic

Explicitly: $\omega_i = \frac{1}{2\pi i} \partial\bar\partial \log\left(\frac{1}{1+\sum_{j\neq i} |z_j|^2}\right)$

Facts: (i) ω is positive

Indeed, $\omega_{[0:\ldots:1:0\ldots:0]} = \frac{i}{2\pi} \sum_{j\neq i} dz_j \wedge d\bar z_j$

(ii) ω is closed

(use that $\partial^2 = \bar\partial^2 = 0$, $\partial\bar\partial + \bar\partial\partial = 0$)

$\Rightarrow$ (X, ω) compact Kähler manifold

$\Rightarrow$ $Y \hookrightarrow X$ closed manifold

Then: (Y, ω) compact Kähler manifold

Lecture 5: The de Rham theorem via sheaves

1. Sheaves:

X topological space

$\mathcal{F}$ sheaf (of abelian groups) on X:

$\mathcal{F}$ presheaf

$$U \longmapsto \underbrace{\mathcal{F}(U)}^{\text{ab. group}} \qquad U \subset X \text{ open}$$

s.t. if $V \subset U$, then have

$$\rho_{UV}: \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

+ compatibility if $W \subset V \subset U$

$$\mathcal{F}(\emptyset) = 0$$

+ sheaf condition:

$U \subset X$ open, $U = \bigcup_i V_i$ open cover

Then: $\prod \rho_{UV_i}: \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(V_i)$

should be an isomorphism

onto $\{ (\sigma_i)_{i \in I} \mid \sigma_i|_{V_i \cap V_j} = \sigma_j|_{V_i \cap V_j} \}$

- $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of (pre-) sheaves:

$$= \left(\phi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U), \quad \rho^{\mathcal{G}}_{UV} \circ \phi_U = \phi_V \circ \rho^{\mathcal{F}}_{UV} \right)$$

<u>Fact</u>: ("sheaves are local")

$\phi : \mathcal{F} \longrightarrow \mathcal{G}$ monomorphism/epimorphism/isomorphism

$\Leftrightarrow \phi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ —— " ——

$\forall x \in X$, $\mathcal{F}_x = \operatorname{colim}_{U \ni x} \mathcal{F}(U)$

- $Ab(X)$ = category of sheaves on X is abelian category, "enough injectives"

$\Rightarrow$ can do cohomology

- Basic observation:

$$\Gamma : Ab(X) \longrightarrow Ab$$
$$\mathcal{F} \longmapsto \mathcal{F}(X)$$

left-exact, but not right exact

e.g. X complex manifold

$$\exp: \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X^*$$

(since logarithms exist locally)

but in general:

$$\mathcal{O}_X(X) \not\twoheadrightarrow \mathcal{O}_X^*(X)$$

2. Sheaf cohomology.

- $\{H^q(X,\mathcal{F})\}_{q\geq 0}$ sequence of abelian groups

(i) choose injective resolution

$$\mathcal{I}^\bullet = (\mathcal{I}^0 \xrightarrow{\phi^0} \mathcal{I}^1 \xrightarrow{\phi^1} \mathcal{I}^2 \longrightarrow \dots)$$

$i: \mathcal{F} \longrightarrow \mathcal{I}^0$, s.t.

(a) $\mathcal{I}^q$ injective

(b) $\mathcal{I}^{q-1} \xrightarrow{\phi^{q-1}} \mathcal{I}^q \xrightarrow{\phi^q} \mathcal{I}^{q+1}$ exact

and i monomorphism

$$i(\mathcal{F}) = \ker(\phi^0)$$

(ii) Apply Γ to $\mathcal{I}^\bullet$

⤳ complex $\Gamma(X, \mathcal{I}^\bullet)$ of abelian groups

(iii) $H^q(X, \mathcal{F}) := H^q(\Gamma(X, \mathcal{I}^\bullet))$

Need to check: (a) $\mathcal{I}^\bullet$ exists

(b) $H^q(X, \mathcal{F})$ independent of choice of $\mathcal{I}^\bullet$

- $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$

<u>In Practice</u>: can compute $H^q(X, \mathcal{F})$ using either

- flasque/flabby resolution (Gur surjective)
- fine resolution (partitions of unity exist)

- <u>not</u> obvious

3. De Rham cohomology:

- X smooth manifold, $\dim X = n$

  $\underline{\mathbb{R}}$ = constant sheaf, $\underline{\mathbb{R}}_x = \mathbb{R}$

  warning: $\underline{\mathbb{R}}(U) \neq \mathbb{R}$

- de Rham complex

  $$\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \to \dots \to \mathcal{A}^n \to 0$$

  sheaves of C^∞-modules

- Poincaré lemma $\Rightarrow$ resolution of $\underline{\mathbb{R}}$

__Fact__: (i) $\mathcal{A}^k$ is fine

(ii) (i) $\Rightarrow$ $H^q(X, \mathcal{A}^k) = 0$, $q > 0$

(iii) $H^q(X, \underline{\mathbb{R}}) \cong H^q(\Gamma(X, \mathcal{A}^\bullet))$

usual de Rham cohomology