Introduction to Hodge theory 27/05/2021

Part 2 - Lecture 2 Oxford

Last time:

- Hodge structures, Hodge diamonds, examples
- Analytic varieties: $(X, \mathcal{O}_X)$ locally given by $f_1 = \cdots = f_m = 0$, $f_i \in \mathcal{O}_{\mathbb{C}^n}(U)$.

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2) Coherent analytic sheaves (cont.)

Def. An analytic sheaf on an analytic variety X is a sheaf of $\mathcal{O}_X$-modules.

Ex M complex manifold, $E \xrightarrow{p} M$ holomorphic vector bundle. Sheaf of holomorphic sections:

$$\mathcal{E}(U) = \{ s : U \to E \mid p \circ s = \mathrm{id}, \ s \text{ holo.} \}$$

Then $\mathcal{E}$ is an analytic sheaf.

Ex $X \subset M$ closed subvariety $\rightsquigarrow \mathcal{I}_X \subset \mathcal{O}_X$, given by $f \in \mathcal{O}_X(U)$ st $f|_X = 0$, is an analytic sheaf on M.

Def. An analytic sheaf $\mathcal{F}$ is coherent if:

1) $\mathcal{F}$ is of finite type: $X = \bigcup_i U_i$,

$$\mathcal{O}_X|_{U_i}^{\oplus n_i} \twoheadrightarrow \mathcal{F}|_{U_i} \quad \forall i$$

2) $\forall\ U \subset X$, $\forall\ f : \mathcal{O}_X|_U^{\oplus n} \to \mathcal{F}$, $\ker(f)$ is of finite type.

Thm (Oka) If M is a complex manifold, then $\mathcal{O}_M$ is coherent. $\square$

Note: $\mathcal{O}(\mathbb{C})$ not Noetherian!

Coro. 1) E holomorphic vb $\rightsquigarrow \mathcal{E}$ is coherent.

2) $X \subset M$ closed subvariety $\rightsquigarrow \mathcal{I}_X$ coherent.

3) $\mathcal{O}_X$ coherent $\forall$ analytic variety X. $\square$

3) Analytification

<u>Def</u> A (complex) affine alg. variety is a pair $(X, \mathcal{O}_X)$, where

- $X = \{ z \in \mathbb{C}^n \mid f_1(z) = \cdots = f_m(z) = 0 \}$, for some $f_i \in \mathbb{C}[z_1, \dots, z_n]$, with the Zariski topology.
- $\mathcal{O}_X$ = sheaf of regular fcts

  $f \in \mathcal{O}_X(U) \iff \forall x \in U, \exists g, h \in \mathbb{C}[x_1, \dots, x_n]$ $h(x) \neq 0$, $f = g/h$ on $U \cap \{h \neq 0\}$.

(Reduced affine sch. of f.t. / $\mathbb{C}$)

<u>Def</u> A pre-variety is a pair $(X, \mathcal{O}_X)$ such that $X = \bigcup_{i=1}^{n} U_i$, $(U_i, \mathcal{O}_X|_{U_i}) \simeq$ affine alg. var. A pre-variety is an <u>alg. variety</u> if it's separated ($\forall (Y, \mathcal{O}) \underset{g}{\overset{f}{\rightrightarrows}} (X, \mathcal{O})$, $\{f - g = 0\} \subset Y$ closed).

(Separated scheme of f.t. / $\mathbb{C}$)

Ex • $V(f_1, \dots, f_m) \subset \mathbb{A}^n$ ($= \mathbb{C}^n$ as alg. var.)

• $V_+(F_1, \dots, F_m) \subset \mathbb{P}^n$, $F_i \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous. Projective varieties.

Analytification functor:

$$\text{Alg. Var.}/\mathbb{C} \longrightarrow \text{Analytic var.}$$
$$(X, \mathcal{O}_X) \longmapsto (X^{an}, \mathcal{O}_{X^{an}})$$

Given by:

Lemma (1) The Zariski topology on $\mathbb{C}^n$ is coarser than the analytic topology.

(2) Every morphism $f: X \to Y$ of affine alg. var. is holomorphic.

Proof: Polynomials are holomorphic fcts. $\square$

Rk. X smooth alg. var. $\rightsquigarrow X^{an}$ cplx manifold

Exercise (not trivial!)

$D \subset \mathbb{C}$ and $\{(z, e^z) \mid z \in \mathbb{C}\} \subset \mathbb{C}^2$

are not algebraic varieties (not in ess. image of anal. functor)

Natural morphism of locally ringed spaces

$$(\mathrm{id}, \theta) : (X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X)$$

(identity on the level of top. spaces)

<u>Prop.</u> θ_x induces isom. $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \hat{\mathcal{O}}_{X^{an},x}$

[(A, m) local ring, $\hat{A} = \lim_i A/m^{i+1}$]

<u>Ex</u> • $\mathcal{O}_{\mathbb{A}^n, 0} = \mathbb{C}[z_1, \dots, z_n]_{m_0}$

e.g. $z_1 / (1 + z_2^2)$

• $\mathcal{O}_{\mathbb{C}^n, 0} = \mathbb{C}\{z_1, \dots, z_n\}$ convergent power series

e.g. $z_1 - z_1 z_2^2 + z_1 z_2^4 - \dots$, $\exp(z_1)$, ...

Completion : $\mathbb{C}[[z_1, \dots, z_n]]$ formal power series

<u>Coro.</u> $\dim X = \dim X^{an}$. (Krull dimension)

Proof: $\dim X = \sup_{x \in X} \dim \mathcal{O}_{X,x}$

$\dim \mathcal{O}_{X,x} = \dim \hat{\mathcal{O}}_{X,x} = \dim \hat{\mathcal{O}}_{X^{an},x} = \dim \mathcal{O}_{X^{an},x}$. □

Note: M complex manifold $\leadsto$ Krull dimension

= topological dimension.

We have a functor

$$\mathcal{O}_X\text{-mod} \longrightarrow \mathcal{O}_{X^{an}}\text{-mod}$$

$$\mathcal{F} \longmapsto \mathcal{F}^{an} := \theta^* \mathcal{F}$$

$$\left[\, x \in X^{an}, \quad \mathcal{F}^{an}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{an},x} \,\right]$$

Prop 1) $\mathcal{F} \mapsto \mathcal{F}^{an}$ is exact

2) $\mathcal{F}$ coherent $\Rightarrow$ $\mathcal{F}^{an}$ coherent. //