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Notebook: DGT
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Lecture 5:

Last time: - existence and uniqueness of Picard-Vessiot extensions $(L,\partial)/(K,\partial)$ for $\mathcal{L}\in K[\partial]$

- $L^{\mathrm{Aut}_\partial(L/K)} = K$

5.1. Characterization of Picard-Vessiot extensions without fixing $\mathcal{L}$

Prop 5.1: • (K,∂) differential field,

• $k := \ker(\partial) \subset K$ algebraically closed.

Then: L/K Picard-Vessiot extension if and only if:

(1) $\exists$ finite-dimensional k-vector space $V\subset L$, s.t. $L = K\langle V\rangle$

i.e. L/K generated by V as a

differential field

(2) $\exists\, G \subset \mathrm{Aut}_\partial(L/k)$, $G(V) \subseteq V$ s.t.

$$L^G = k$$

(3) Let $k' := \ker(\partial) \subset L$. Then

$$k' = k$$

If (1), (2), (3) hold, $y_1, \dots, y_n \in V$ k-basis, then L/k is a Picard-Vessiot extension for

$$\mathcal{L}(Y) = \frac{w(Y, y_1, \dots, y_n)}{w(y_1, \dots, y_n)},$$

and $V = \mathcal{L}^{-1}(0)$.

<u>Ex 10:</u> (i) Prove that

$$\mathcal{L}(Y) = \sum_{i=0}^{n} b_i \partial^i(Y), \quad b_i := a_n^{-1} \cdot a_i$$

$$a_i = (-1)^i \det\left(\underline{y}^{(0)}, \underline{y}^{(1)}, \dots, \underline{y}^{(i-1)}, \underline{y}^{(i+1)}, \dots, \underline{y}^{(n)}\right)$$

$$\underline{y} := (y_1, \dots, y_n), \quad \underline{y}^{(i)} := \partial^i(\underline{y})$$

(ii) $z_i = \sum_j c_{ij} y_j$, $c_{ij} \in k$

Then:

$$\det\left(\underline{z}^{(k_1)}, \dots, \underline{z}^{(k_n)}\right) = \det(c_{ij}) \det\left(\underline{y}^{(k_1)}, \dots, \underline{y}^{(k_n)}\right)$$

for all $k_1, \dots, k_n \geq 0$

Proof of Prop 5.1:

"$\Rightarrow$" L/k Picard-Vessiot extension for $\mathcal{L}$. Then:

(1) ok for $V = \mathcal{L}^{-1}(0)$

(2) ok for $G = \mathrm{Aut}_\partial(L/K)$ (by Cor 4.9)

(3) ok ($k' = k$ since L/k PV-extension)

"$\Leftarrow$" (1), (2), (3) satisfied

Let $y_1, \dots, y_n \in V$ basis

$$\mathcal{L}(Y) = \frac{W(Y, y_1, \dots, y_n)}{W(y_1, \dots, y_n)}$$

Then $\mathcal{L}(y_i) = 0$, $1 \leq i \leq n$

(1) $\Rightarrow$ $L = k\langle \mathcal{L}^{-1}(0) \rangle$ (PV-axiom (ii))

$\dim_k \mathcal{L}^{-1}(0) = n$ (PV-axiom (i))

(3) k field of constants of L (PV-axiom (iii))

Remains to show:

$\sum^{n} \quad y^{(i)}$

$$\mathcal{L}(Y) = \sum_{i=0} b_i Y \quad , \quad b_i \in K, \; b_n = 1 \; !$$

- $\sigma \in G$, $\quad Z_i := \sigma(Y_i) = \sum_j \underset{\in k}{c_{ij}} Y_j$

  (possible since $G(V) \subseteq V$)

Ex 10.(ii) $\Rightarrow \sigma(a_k) = \det(c_{ij})\, a_k$

Ex 10(i) $\Rightarrow$
$$\begin{aligned}\sigma(b_k) &= \sigma(a_n^{-1} \cdot a_k)\\ &= \sigma(a_n)^{-1} \cdot \sigma(a_k)\\ &= \det(c_{ij})^{-1} \cdot a_n^{-1} \cdot \det(c_{ij})\, a_k\\ &= a_n^{-1} a_k\\ &= b_k\end{aligned}$$

- $L^G = k \quad \Rightarrow b_k \in K$
- $b_n = 1$ clear

$\square$

<u>Rmk 5.2:</u> Prop 5.1 useful to prove: composita of PV-extensions are PV-extensions

<u>5.2. extending differential automorphisms</u>

Prop 5.3: (K, ∂) differential field

Prop 5.3. $\cdot$ (k,∂) differential field, k algebraically closed,

- L/k PV-extension for
$$\mathcal{L} = \sum_{i=0}^{n} a_i \partial^i \in k[\partial], \quad a_n = 1$$

suppose: $\sigma \in \mathrm{Aut}_\partial(K)$ s.t. $\sigma(a_i) = a_i$ for all $0 \leq i \leq n$

Then: $\exists \bar{\sigma} : (L,\partial) \longrightarrow (L,\partial)$ s.t. $\bar{\sigma}|_K = \sigma$

Proof: $\cdot$ $R = k[Y_{ij}][W(Y_{0i})^{-1}]$

$S = k[Z_{ij}][W(Z_{0i})^{-1}]$

two copies of full universal solution algebra for $\mathcal{L}$

- Define $\tau : R \longrightarrow S$
$$Y_{ij} \longmapsto Z_{ij}$$
satisfies $\tau|_K = \sigma$

- τ is an isom. of algebras $\underbrace{\text{not of } k\text{-algebras}}$ and $\partial \circ \tau = \tau \circ \partial$ (since $\sigma(a_i) = a_i$)

$\Rightarrow$ τ is an isom. of differential algebras.

- $m_R \subset R$ maximal differential ideal

  $m_S := \tau(m_R) \subset S$ (also a maximal diff. ideal)

- $\tau : R \xrightarrow{\sim} S$

  map $\overline{\tau} : Q(R/m_R) \xrightarrow{\sim} Q(S/m_S)$

  PV-extensions for $\mathcal{L}$

Thm 4.5 $\Rightarrow \exists\, \rho_1 : Q(S/m_S) \xrightarrow{\sim} L$

$\exists\, \rho_2 : L \xrightarrow{\sim} Q(R/m_R)$

s.t. $\rho_1|_K = \rho_2|_K = id$

$\Rightarrow \overline{\sigma} := \rho_1 \circ \overline{\tau} \circ \rho_2$ differential automorphism of L, $\overline{\sigma}|_K = \sigma$

$\square$

5.3. Composite of PV-extensions

- $L_1, L_2 \subset (L, \partial)$ differential subfields

  $\Rightarrow L_1 \cdot L_2$ subfield

  compositum (smallest $M \subset L$ s.t. $L_1, L_2 \subset M$)

- $K := L_1 \cap L_2$. Natural map

  $\varphi : L_1 \otimes_K L_2 \longrightarrow L$

  morphism of diff. K-algebras

- Then $L_1 \cdot L_2 \cong Q(\mathrm{Im}(\varphi))$
  is a differential field.

Thm 5.4: - (K, ∂) differential field,
  $k \subset K$ algebraically closed.
- $\mathcal{L}_1, \dots, \mathcal{L}_m \in K[\partial]$, monic

Then: $\exists$ Picard-Vessiot extension
$$L/K$$
such that: for all $1 \leq i \leq m$
$\exists\, L_i/K$ PV-extension for $\mathcal{L}_i$
with $L \supseteq L_i \supseteq K$

Moreover: L is the compositum
of its subfields L_i.

Proof: - Inductively construct
$K = L_0 \subseteq L_1 \subseteq \dots \subseteq L_m = L$
s.t. L_i/L_{i-1} PV-extension for $\mathcal{L}_i$
- Want to verify (1)-(3) of
Prop. 5.1. for L/K
(3): L/K no new constants
(ok since L_i/L_{i-1} PV-extension)

(2): Let $G := \mathrm{Aut}_\partial(L/k)$

$a \in L \setminus k$. want $\tau \in G$ s.t.

$\tau(a) \neq a$

- $a \in L_i \setminus L_{i-1}$

Prop 4.8 $\Rightarrow \exists\, \tau_i \in \mathrm{Aut}_\partial(L_i/L_{i-1})$

s.t. $\tau_i(a) \neq a$

- Prop 5.3. $\Rightarrow \exists\, \tau \in \mathrm{Aut}_\partial(L)$

s.t. $\tau|_{L_i} = \tau_i$

$\Rightarrow \tau \in \mathrm{Aut}_\partial(L/k)$

$\tau(a) \neq a$

$\Rightarrow L^G = k$

- (1): $V_i := \mathcal{L}_i^{-1}(0) \subset L$

$n_i :=$ degree of $\mathcal{L}_i$

$n_i = \dim_k(V_i \cap L_i) \leq \dim_k V_i \leq n_i$

$\Rightarrow \dim_k V_i = n_i$, $V_i \subset L_i$

- Also $G(V_i) \subset V_i$, since $\mathcal{L}_i \in k[\partial]$

- Now let $V := V_1 + \dots + V_m$

Then: - $\dim_k V < \infty$

$G(V) \subset V$

$- \sigma(V) \subset V$

$- L \supseteq K\langle V\rangle \supseteq \underbrace{\underbrace{K\langle V_1\rangle}_{L_1}\langle V_2\rangle}_{L_2} \dots \langle V_m\rangle$

$= L$

$\Rightarrow L = K\langle V\rangle$

$\Rightarrow (1)$

$\Rightarrow L/K$ PV-extension

· Also: $L_i = K\langle V_i\rangle$ PV-extension for $\mathcal{L}_i$

and $L = L_1 \cdot L_2 \cdot \ldots \cdot L_m$

$\square$

5.4: Families of linear differential operators

Dfn 5.5: · (K, ∂) differential field

· $\mathcal{S} = \{\mathcal{L}_i \mid 1 \leq i \leq m\}$, $\mathcal{L}_i \in K[\partial]$, monic

Then L/K is a Picard-Vessiot extension for $\mathcal{S}$ if:

(1) $L = K\langle V_1 \cup \ldots \cup V_m\rangle$, $V_i := \mathcal{L}_i^{-1}(0) \subset L$

(2) For each $1 \leq i \leq m$, $\deg(\mathcal{L}_i) =: n_i$

$\exists\, y_{ij} \in V_i$, $1 \leq j \leq n_i$ s.t.

$w(y_{i1}, \dots, y_{in_i}) \neq 0$

(3) $k' = k$, where $k' := \ker(\partial) \subset L$

Thm 5.4 $\Rightarrow$ PV-extensions for $\mathcal{L}$ always exist (if k alg. closed)

<u>Thm 5.6</u>: Notation as above, assume $k \subset K$ algebraically closed.

- L/K, M/K PV-extensions for $\mathcal{L}$.

Then: $\exists$ K-differential isomorphism

$$\sigma : L \xrightarrow{\sim} M$$

<u>Proof</u>:
- $V_i := \mathcal{L}_i^{-1}(0) \subset L$

  $W_i := \mathcal{L}_i^{-1}(0) \subset M$
- $L_i := K\langle V_1 \cup \dots \cup V_i \rangle$

  $M_i := K\langle W_1 \cup \dots \cup W_i \rangle$

Then $L_{i+1} = L_i\langle V_{i+1}\rangle$, $M_{i+1} = M_i\langle W_{i+1}\rangle$

$\Rightarrow L_{i+1}/L_i$, M_{i+1}/M_i

PV-extensions for $\mathcal{L}_{i+1}$

- Now construct $\sigma_i : L_i \xrightarrow{\sim} M_i$

K-differential isomorphism by

K-differential isomorphism by

induction on i $(1 \leq i \leq m)$

- $i=1$: L_1, M_1 PV-extensions of

  K for $\mathcal{L}_1$

Thm 4.5 $\Rightarrow \exists \sigma_1 : L_1 \xrightarrow{\sim} M_1$

- suppose σ_i constructed

As in proof of Prop 5.3,

construct PV-extensions

$\tilde{L}_{i+1}/L_i$, $\tilde{M}_{i+1}/M_i$, for $\mathcal{L}_{i+1}$

and extend $\sigma_i : L_i \xrightarrow{\sim} M_i$

to $\tilde{\sigma}_{i+1} : \tilde{L}_{i+1} \xrightarrow{\sim} \tilde{M}_{i+1}$

- Thm 4.5 $\Rightarrow$ $\exists$ isomorphisms

$\tau_{i+1} : L_{i+1} \longrightarrow \tilde{L}_{i+1}$, $\rho_{i+1} : \tilde{M}_{i+1} \longrightarrow M_{i+1}$

$\tau_{i+1}|_{L_i} = id$, $\rho_{i+1}|_{M_i} = id$

$\Rightarrow$ $\sigma_{i+1} := \rho_{i+1} \circ \tilde{\sigma}_{i+1} \circ \tau_{i+1}$

is the desired extension

$\sigma_{i+1} : L_{i+1} \xrightarrow{\sim} M_{i+1}$

$\sigma_{i+1}|_{L_i} = \sigma_i$ $\square$