Introduction to Hodge theory

Lecturers:

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Time: Mondays, 2-3

Thursdays, 10-11

Course website:

people.maths.ox.ac.uk/matthes/Hodge21

Plan for today:

1. What is Hodge theory?

2. Goals of this course

3. An illustrative example: Hodge theory of elliptic curves

1. What is Hodge theory?

- X = compact, complex manifold (think: smooth manifold + complex structure)
- $H^k(X, \mathbb{C})$ = singular / de Rham cohomology

topological invariant / does not "see" complex structure

- e.g.: $X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$

then $H^1(X) \cong H^1(X')$

<u>but</u> in general $X \not\cong X'$ as complex manifolds

<u>Basic idea</u>: $\exists$ extra structure on $H^k(X, \mathbb{C})$ that "sees" the complex structure on X

Main results

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• Assume: $\exists$ kähler metric ω on X

e.g.: $X \hookrightarrow \mathbb{P}^n(\mathbb{C})$

Theorem:

(i) Hodge decomposition:

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

"$dz_{i_1}\wedge\dots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\dots\wedge d\bar{z}_{j_q}$"

and $\overline{H^{p,q}(X)} \cong H^{q,p}(X)$

"Hodge symmetry"

(ii) Hard Lefschetz:

$$L^{n-k}: H^k(X,\mathbb{R}) \longrightarrow H^{2n-k}(X,\mathbb{R})$$

Lefschetz operator

is an isomorphism for $k \leq n$

$n = \dim X$

NB: (i) independent of choice of ω

Consequences I

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⟹ constraints on Betti numbers

$$b_k := \dim H^k(X, \mathbb{C})$$

<u>Corollary:</u>

(i) $b_{2k+1} \in 2\mathbb{Z}$

(ii) $b_0 \leq b_2 \leq b_4 \leq \dots \leq b_n$

$b_1 \leq b_3 \leq b_5 \leq \dots \leq b_n$

$n = \dim X$

<u>Example:</u> Hopf surface

$$S = \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{q^{\mathbb{Z}}}, \quad 0 \neq |q| < 1$$

$b_1(S) = 1$ (since $\pi_1(S) = \mathbb{Z}$)

$\Rightarrow$ S does not admit a Kähler metric

Main Principle (Hodge, Weil)

- Study harmonic differential forms on X

i.e.: solve Poisson's equation

$$\Delta \alpha = \varphi \quad \text{on } X$$

- This requires hard analysis (elliptic PDEs, Sobolev spaces, estimates, ...)

Net result: Each $[\alpha] \in H^k(X, \mathbb{C})$ has unique harmonic representative

$\Delta = 0$

(valid for all X compact complex manifolds) don't need Kähler!

This principle yields (for compact kähler):

a) Hodge decomposition

b) Hard Lefschetz

c) Lefschetz decomposition

primitive cohomology

$$\bigoplus_{k-2r\geq 0} H^{k-2r}(X,\mathbb{R})_{prim} \xrightarrow{\sim} H^k(X,\mathbb{R})$$

$$(\alpha_r) \longmapsto \sum_r L^r \alpha_r$$

d) Poincaré / Serre duality

for projective manifolds

e) "Vanishing theorems" à la Kodaira, Nakano, ...

not in this lecture

Algebraic case:

X projective manifold
($\Rightarrow X$ algebraic)

Facts: (i) $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$
analytic algebraic

(ii) Hodge decomposition
$\Leftrightarrow$ degeneration (at E^1) of
Frölicher spectral sequence

NB: Do **not** get Hodge
symmetry !!

All this is part of
"GAGA" = géometrie algébrique et
géometrie analytique

Further topics:

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Where to go from there?

1. (mixed) Hodge structures, weight filtration (Deligne)

2. Variation in families, period maps / domains (Griffiths, Schmid, ...)

3. Hodge theory and algebraic cycles $\rightsquigarrow$ Hodge conjecture (Hodge, Carlson, Voisin, ...)

+ much more

(p-adic Hodge theory, (Faltings, Scholze, ...)
Mumford-Tate groups,
non-abelian Hodge theory (Simpson), ...)

2. Goals of this course

(i) present and prove Hodge decomposition, Hard Lefschetz (reasonably detailed)

"lecture style"

(ii) Aspects / further topics in Hodge theory

"seminar-style"

(iii) Offer broadening topics (contact Tiago and myself)

3. Hodge theory of elliptic curves

- $E = \mathbb{C}/\Lambda$, $\Lambda \subset \mathbb{C}$ lattice

- **E is projective**, hence Kähler

Weierstrass function

$$\left(\begin{array}{ccc} E & \hookrightarrow & \mathbb{P}^2(\mathbb{C}) \\ z & \longmapsto & \begin{cases} [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1], & z \notin \Lambda \\ [0 : 1 : 0] & \text{else} \end{cases} \end{array} \right)$$

- $A^0(E) \xrightarrow{d^0} A^1(E) \xrightarrow{d^1} A^2(E) \xrightarrow{d^2} 0$

smooth de Rham complex

Definition: $H^k_{dR}(E) := \dfrac{\ker(d^k)}{\mathrm{Im}(d^{k-1})}$

(smooth) **de Rham cohomology**

Decomposition of de Rham complex

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Basic observation:

- $A^k(E) = \bigoplus_{p+q=k} A^{p,q}(E)$

$$\begin{array}{ccccc} A^0(E) & \longrightarrow & A^1(E) & \longrightarrow & A^2(E) \\ \| & & \| & & \| \\ A^{0,0}(E) & \longrightarrow & A^{1,0}(E) \oplus A^{0,1}(E) & \longrightarrow & A^{1,1}(E) \end{array}$$

- Also, $d = \partial + \bar{\partial}$

- E.g.: $\alpha = f(x,y)\,dx + g(x,y)\,dy \in A^1(E)$

Using $z = x + iy$, $\bar{z} = x - iy$

$\rightsquigarrow$ $\alpha = \left(\dfrac{f(x,y) - ig(x,y)}{2}\right) dz + \left(\dfrac{f(x,y) + ig(x,y)}{2}\right) d\bar{z}$

$\rightsquigarrow$ $d = \dfrac{\partial}{\partial x} dx + \dfrac{\partial}{\partial y} dy$

$$= \frac{1}{2}\underbrace{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)}_{=:\frac{\partial}{\partial z}} dz + \frac{1}{2}\underbrace{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}_{=:\frac{\partial}{\partial \bar{z}}} d\bar{z}$$

Definition: Dolbeault cohomology

$$H^{p,q}(E) := \frac{\ker\left(\bar\partial : A^{p,q}(E) \to A^{p,q+1}(E)\right)}{\operatorname{Im}\left(\bar\partial : A^{p,q-1}(E) \to A^{p,q}(E)\right)}$$

Note: $H^{0,0}(E) = \{\text{holomorphic fcts}\}$

$H^{1,0}(E) = \{$ —— " —— 1-forms $\}$

Relation to de Rham:

(i) $H^{0,0}(E) \xrightarrow{\sim} H^0_{dR}(E)$

both sides are <u>constant</u>

(ii) $j_2 : H^{1,1}(E) \longrightarrow H^2_{dR}(E)$

canonical map, induced by

$$A^{1,1}(E) \xrightarrow{id} A^2(E)$$

Hodge decomposition

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(iii) • $\sigma : H^{1,0}(E) \longrightarrow \overline{H^{0,1}(E)}$

$\alpha \longmapsto [\bar{\alpha}]$

isomorphism (will see later)

• $i : H^{1,0}(E) \longrightarrow H^1_{dR}(E)$ canonical

$\leadsto j_1 : H^{1,0}(E) \oplus H^{0,1}(E) \longrightarrow H^1_{dR}(E)$

$(\alpha, [\theta]) \mapsto i(\alpha) + \overline{i(\sigma^{-1}([\theta]))}$

Theorem 1 (Hodge decomposition):

(a) σ is an isomorphism
(Hodge symmetry)

(b) j_1, j_2 are isomorphisms

The Laplacian

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For proof, need the Laplacian

$$\Delta f = 2i \, \partial \bar{\partial} (f) \qquad , \; f \in A^0(E)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)(f) \, dx \wedge dy \in A^2(E)$$

Theorem 2: Let $\nu \in A^2(E)$

Then: $\exists \, f \in A^0(E)$ s.t.

$$\Delta f = \nu$$

$$\Longleftrightarrow \int_E \nu = 0$$

Remark: Theorems 1 and 2 are valid for general compact Riemann surfaces

(see Donaldson, "Riemann surfaces")

Proof of Theorem 1 (sketch):

a) $\sigma : H^{1,0}(E) \longrightarrow \overline{H^{0,1}(E)}$

$\alpha \longmapsto [\bar{\alpha}]$

surjectivity: given $[\theta] \in H^{0,1}(E)$

want: $f \in A^0(E)$ s.t.

$\theta' := \theta + \bar{\partial} f$ (*)

satisfies $\partial \theta' = 0$

(then $\overline{\theta'} \in H^{1,0}(E)$ and $\sigma(\overline{\theta'}) = [\theta]$)

Indeed: (*) $\iff \exists f \in A^0(E)$

$\Delta f = - \partial \theta$

Theorem 2 $\Rightarrow$ f exists

if and only if

$\int_E \partial \theta = 0$ (ok by Stokes) injectivity: use that

$$H^1_{dR}(E) \times H^1_{dR}(E) \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \longmapsto \int_E \alpha \wedge \bar{\beta}$$

is non-degenerate

$\Rightarrow$ (a)

Proof of (b) is similar $\square$

Proof of Theorem 2:

key fact: (i) every $f \in A^0(E)$

has a Fourier expansion

$$f(x,y) = \sum_{m,n \in \mathbb{Z}} f_{m,n} \exp(2\pi i m \cdot x + 2\pi i n \cdot y)$$

(ii) $\Delta f = -4\pi^2 \sum (m^2+n^2) f_{m,n} \exp(2\pi i m \cdot x + 2\pi i n \cdot y) \cdot dx \wedge dy$

(iii) $\int_E \nu = \nu_{0,0}$

$$\nu = \sum_{m,n\in\mathbb{Z}} \nu_{m,n} \exp(2\pi i m\cdot x + 2\pi i n\cdot y)\, dx \wedge dy$$

since $\iint_{[0,1]^2} \exp(2\pi i m.x)\exp(2\pi i n\cdot y)\, dx\, dy$

$$= \delta_{m,0}\cdot\delta_{n,0}$$

So, if $\int_E \nu = \nu_{0,0} = 0$

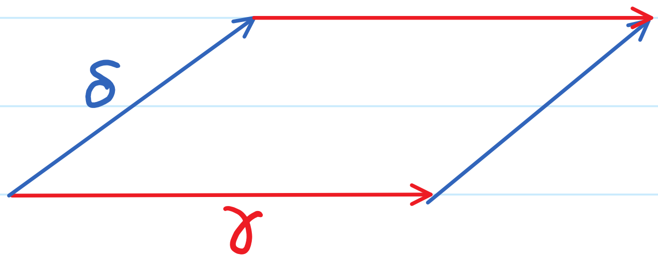
then $f = -\frac{1}{4\pi^2} \sum \frac{\nu_{m,n}}{m^2+n^2} \exp(2\pi i m\cdot x + 2\pi i n\cdot y)$

solves $\Delta f = \nu$ $\square$

Poisson's equation

- E elliptic curve

  γ, δ symplectic basis of $H_1(E;\mathbb{Z})$



- Hodge filtration

$$H^{1,0}(E) \hookrightarrow H^1_{dR}(E) \overset{\text{de Rham's theorem}}{\cong} H^1_{sing}(E;\mathbb{C})$$

$$\alpha \longmapsto \int_\gamma \alpha \cdot [\gamma]^\vee + \int_\delta \alpha \cdot [\delta]^\vee$$

"periods"

Define: $\tau(E,\gamma,\delta) = \dfrac{\int_\delta \alpha}{\int_\gamma \alpha} \in \mathfrak{h}$ — $\mathrm{Im} > 0$

independent of $\alpha \neq 0$

Prop: (E,γ,δ), (E',γ',δ') as above

Then: $(E,\gamma,\delta) \cong (E',\gamma',\delta')$

(i.e. $\exists f: E \xrightarrow{\sim} E'$, $f_*\gamma = \gamma'$, $f_*\delta = \delta'$)

$\iff \tau(E,\gamma,\delta) = \tau(E',\gamma',\delta')$

Idea of Proof:

"$\Rightarrow$" $\alpha \in H^{1,0}(E)$, $\alpha' \in H^{1,0}(E')$ non-zero

Then $f^*\alpha' = c \cdot \alpha$, $c \in \mathbb{C}$

Enough to prove: $c \neq 0$

Indeed, have

$$\int_E f^*\alpha' \wedge \overline{f^*\alpha'} = |c|^2 \int_E \alpha \wedge \bar{\alpha}$$

and $\displaystyle\int_E f^*\alpha' \wedge \overline{f^*\alpha'} \underset{\text{change of variables}}{=} \int_{f_*E} \alpha' \wedge \overline{\alpha'} \underset{f \text{ isomorphism}}{=} \int_{E'} \alpha' \wedge \overline{\alpha'} \neq 0$

$\Rightarrow c \neq 0$

"$\Leftarrow$" $\Lambda = \mathbb{Z} \cdot \int_\gamma \alpha + \mathbb{Z} \cdot \int_\delta \alpha$, $\Lambda' = \mathbb{Z} \cdot \int_{\gamma'} \alpha' + \mathbb{Z} \int_{\delta'} \alpha'$

$\tau(E, \gamma, \delta) = \tau(E', \gamma', \delta')$

$\Rightarrow \Lambda = k \cdot \Lambda'$, some $k \neq 0$

$\Rightarrow$ $\mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto k \cdot z$ induces $\underset{\cong}{E} \; \mathbb{C}/\Lambda \overset{\text{biholomorphism}}{\cong} \mathbb{C}/\Lambda' \; \underset{\cong}{E'}$

By construction, $f_* \gamma = \gamma'$, $f_* \delta = \delta'$

$\square$