Ink note

Notebook:	DGT		
Created:	5/20/2020 7:44 AM	Updated:	5/21/2020 11:13 AM
Author:	Nils Matthes		

Lecture 8: Examples

Prop 8.1: • $(L,\partial)/(K,\partial)$ PV-extension

• k alg. closed

• $(L,\partial) \supseteq (M,\partial) \supseteq (K,\partial)$

Then: $\mathrm{Aut}_\partial(L/M) \subseteq \mathrm{Aut}_\partial(L/K)$

algebraic subgroup

Proof: Prop 7.10

$\Rightarrow (L,\partial)/(M,\partial)$ PV-extension

$\Rightarrow \mathrm{Aut}_\partial(L/M)$ algebraic group

• $\mathcal{L} \in K[\partial]$ monic diff. operator s.t. $(L,\partial)/(K,\partial)$ PV-ext. for $\mathcal{L}$

• $S = K[Y_{ij}][W^{-1}]$ full univ. sol. algebra

• $S_M := M \otimes_K S$ —— " —— over M (identify S with image in S_M)

• $V \subseteq S_M \cap S$ $\quad V := \mathrm{span}_k \{Y_{0j} \mid 1 \leq j \leq n\}$

- If $\mathfrak{a} \subset S$ proper differential ideal

$\Rightarrow \mathfrak{a}_M := M \otimes_k \mathfrak{a} \subset S_M$

proper differential ideal

- If $\mathfrak{m} \subset S$ max. diff. ideal

$\mathfrak{m}' \subset S_M$ —— " ——

$\Rightarrow \mathfrak{m}' \cap S = \mathfrak{m}$

- $L = Q(S/\mathfrak{m}) \cong Q(S_M/\mathfrak{m}')$

Thm 7.1 $\Rightarrow \mathrm{Aut}_\partial(L/k) \cong Gl(V)_\mathfrak{m}$

$\mathrm{Aut}_\partial(L/M) \cong Gl(V)_{\mathfrak{m}'}$

- $S \subset S_M$ $Gl(V)$-submodule and

$\mathfrak{m} = S \cap \mathfrak{m}'$

$\Rightarrow Gl(V)_{\mathfrak{m}'} \subseteq Gl(V)_\mathfrak{m} \subseteq Gl(V)$

algebraic subgroups $\square$

<u>Ex. 13</u>: $(L,\partial)/(k,\partial)$ PV-extension

for $\mathcal{L} = \partial^n + \sum_{i=1}^{n} a_i \partial^{n-i} \in k[\partial]$

- Assume k alg. closed
- $V = \mathrm{span}_k \{y_1, \dots, y_n\}$, $y_1, \dots, y_n$ full set

of solutions of $\mathcal{L}=0$

- $w = w(y_1,\dots,y_n)$ Wronskian
- Then: $M = K(w)$ is a PV-extension for $\partial + a_1\partial^0$

Hint: show that $\partial(w) = -a_1 w$

<u>Prop 8.2.</u> Notation as in Ex. 13

we have $\mathrm{Aut}_\partial(L/M) = \mathrm{Aut}_\partial(L/K) \cap \mathrm{Sl}(V)$

where we view $\mathrm{Aut}_\partial(L/K) \subseteq \mathrm{Gl}(V)$

<u>Proof:</u> Let $\sigma \in \mathrm{Aut}_\partial(L/K)$

Ex. 12 $\Rightarrow \sigma(w) = \det(\sigma)\cdot w$

$\Rightarrow \mathrm{Aut}_\partial(L/K)_w = \{\sigma \in \mathrm{Aut}_\partial(L/K) \mid \det(\sigma)=1\}$

$\Rightarrow \mathrm{Aut}_\partial(L/M) = \mathrm{Aut}_\partial(L/K) \cap \mathrm{Sl}(w)$ $\square$

Now for the examples:

- Fix (K,∂) differential field
- $k \subset K$ field of constants algebraically closed

<u>Exmp 8.3.</u> · $\mathcal{L} = \partial - a\cdot\mathrm{id} \in K[\partial]$

- $(L,\partial)/(K,\partial)$ PV-extension for $\mathcal{L}$

- $V = \mathcal{L}^{-1}(0) = k \cdot Y \subset L$

$\Rightarrow G := \mathrm{Aut}_\partial(L/k) \subseteq \mathbb{G}_m(k)$

algebraic subgroup

$\Rightarrow G = \begin{cases} C_n \leftarrow \text{cyclic of order } n \\ \mathbb{G}_m \end{cases}$

(depending on $Y^n \in k$ or not)

- case where $\deg(\mathcal{L}) = 1$ done

Now $\deg(\mathcal{L}) = 2$

Exmp 8.4. $\cdot$ $a \in k^\times$, $b := \frac{\partial(a)}{a}$

- $\mathcal{L} = \partial^2 - b\partial$, $(L,\partial)/(k,\partial)$ PV-extension
- $k \subset V := \mathcal{L}^{-1}(0)$, $\dim_k V = 2$
- $Y \in V \setminus k$, $V = k \oplus kY \subset L$

Then $G := \mathrm{Aut}_\partial(L/k) \subseteq \mathrm{Gl}(V) \simeq \mathrm{Gl}_2(k)$

- $\sigma \in G \longleftrightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathrm{Gl}_2(k)$

basis $\{1, Y\}$

- $\sigma|_k = \mathrm{id} \Rightarrow c = 1, \ e = 0$

Ex. 14 (i) Show that $\frac{\partial(Y) - a}{a} \in k$

(ii) deduce that $f = 1$

$d = \sigma(Y) - Y$

$u = v(y) = y$

- Ex. 14.(ii)

$$\Rightarrow G \subseteq \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \,\middle|\, \alpha \in k \right\} \cong \mathbb{G}_a(k)$$

$$\Rightarrow G = \begin{cases} \{0\} & y \in K \\ \mathbb{G}_a & \text{else} \end{cases}$$

<u>Exmp 8.5:</u> (Cauchy equation)

- $K = k(X)$, $\partial(X) = 1$
- For $a, b \in k$, let

$$\mathcal{L} = \partial^2 + \frac{a}{X}\partial + \frac{b}{X^2}\,\mathrm{id}$$

(Cauchy operator divided by X^2)

- $m_i \in k$, $i = 1,2$, solutions of

$$Z^2 + (a-1)Z + b = 0$$

- $\mathcal{L}_i := \partial - \frac{m_i}{X} \cdot \mathrm{id} \in K[\partial]$

$(L_i, \partial)/(K, \partial)$ PV-extension for $\mathcal{L}$

<u>Ex. 15:</u> Show that if $y_i \in L_i$

$$\mathcal{L}_i(y_i) = 0$$

then $\mathcal{L}(y_i) = 0$

Hint: differentiate $X\partial(y_i) = m_i y_i$

- $(L_0, \partial)/(k, \partial)$ PV-extension for $\{\mathcal{L}_1, \mathcal{L}_2\}$

  $L_i \subseteq L_0$ and $L_0 = k(y_1, y_2)$

  (compositum of $L_1, L_2 \subseteq L_0$)

- $V = ky_1 + ky_2 \subset L_0$

  $\mathcal{L}(V) = 0$

- $W(y_1, y_2) = (m_1 - m_2)\frac{y_1 y_2}{x}$

Two cases:

(i) $m_1 \neq m_2$

$\Rightarrow V = ky_1 \oplus ky_2$

$\Rightarrow L_0/k$ PV-extension for $\mathcal{L}$

$G := \mathrm{Aut}_\partial(L_0/k) \subseteq Gl_2(k)$

preserving $ky_i \subseteq V$

$\Rightarrow G \subseteq \mathbb{G}_m(k) \times \mathbb{G}_m(k)$

$\sigma \in G$ acts as $\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$

on (y_1, y_2)

- when is $G = \mathbb{G}_m(k) \times \mathbb{G}_m(k)$?

Claim: This happens if and only if

Y_1, Y_2 algebraically independent

over K

Proof: "$\Rightarrow$" If Y_1, Y_2 alg. independent

then, for all $c_1, c_2 \in k^\times$,

$\sigma : L_0 \longrightarrow L_0$

$Y_i \longmapsto c_i Y_i$

$\sigma|_K = id$

is in G

"$\Leftarrow$" Assume Y_1, Y_2 alg. dependent

$\exists\, \phi \in K(Y_2)[X] \setminus \{0\}$ s.t.

$\phi(Y_1) = 0$

- If $\sigma \in G$ s.t. $\sigma(Y_1) = cY_1$, $\sigma(Y_2) = Y_2$

  then $\phi(cY_1) = 0$

- $c \longmapsto \phi(cY_1)$ finitely many zeros

  $\Rightarrow G \subset \mathbb{G}_m(k) \times \mathbb{G}_m(k)$ proper

  subgroup

Case (iv): $m_1 = m_2 \Rightarrow L_1 = L_2 = L_0$

$\Rightarrow L_0$ not a PV-extension for $\mathcal{L}$

- L/k PV-extension for $\mathcal{L}$

  $V = kY \oplus kZ \subset L$ full set of solutions for $\mathcal{L} = 0$

Ex. 16: Show that $L = k\langle Y, u\rangle$, where $\partial(u) = \frac{1}{x}$, in the following steps:

(i) show that $\tilde{u} = \frac{Z}{Y}$ satisfies

$$\partial^2(\tilde{u}) + \frac{1}{x}\partial(\tilde{u}) = 0$$

(Hint: look at $0 = \mathcal{L}(\tilde{u}Y)$)

(ii) show that $\tilde{u}$ satisfies

$$\partial(\tilde{u}) = \frac{c}{x}, \quad c \in k^\times$$

- Therefore $L = k\langle Y, u\rangle$

$$\partial(Y) = \frac{m}{x} \cdot Y, \quad \partial(u) = \frac{1}{x}$$

- $k\langle Y\rangle/k$, $k\langle u\rangle/k$ are both PV-subextensions w. groups

  $G_1 \subseteq \mathbb{G}_m(k)$, $G_2 \subseteq \mathbb{G}_a(k)$

  (in fact: $G_2 = \mathbb{G}_a(k)$, since $u \notin \underset{\overset{\shortparallel}{k}}{k(x)}$)

$\Rightarrow G := \mathrm{Aut}_\partial(L/k) \subseteq \mathbb{G}_m(k) \times \mathbb{G}_a(k)$

and $\quad G \twoheadrightarrow G_2 \quad$ (by Prop 5.3)

Exmp 8.6 (Generalization of 8.4)

- $\mathcal{L} = \partial^2 + a\partial \quad , \quad a \in K$
- $(L,\partial)/(k,\partial)$ PV-extension

  $G := \mathrm{Aut}_\partial(L/k) \quad , \quad V = k \oplus ky \subset L$
- $w = w(1,y) = \partial(y) \quad , \quad \partial(w) = -ay$

  $M := k(w)$
- $(M,\partial)/(k,\partial)$ PV-subextension

  Prop 8.2 $\Rightarrow$

  $$\ker(G \to \mathrm{Aut}_\partial(M/k)) = G \cap Sl(V)$$
- $\sigma \in G$ acts trivially on $k \subset V$

  $\sigma(y) = c + dy$
- $dw = d\partial(y) = \partial(\sigma(y)) = \sigma(w)$

  $\Rightarrow d = \frac{\sigma(w)}{w} \quad , \quad c = \sigma(y) - \frac{\sigma(w)}{w} y$

  $$\Rightarrow \sigma \longleftrightarrow \begin{pmatrix} 1 & \sigma(y) - \frac{\sigma(w)}{w} y \\ 0 & \frac{\sigma(w)}{w} \end{pmatrix} \in Gl_2(k)$$
- $\sigma \in G \cap \mathrm{Aut}_\partial(L/M)$

$\Leftrightarrow \sigma(w) = w$

$\Rightarrow \mathrm{Aut}_\partial(L/M) \subseteq \mathbb{G}_a(k)$

- All in all, get

$$G \cong \underset{\mathbb{G}_a(k)}{\underset{\cap}{\mathrm{Aut}_\partial(L/M)}} \rtimes \underset{\mathbb{G}_m(k)}{\underset{\cap}{\mathrm{Aut}_\partial(M/k)}}$$

Exmp 8.7. • $\mathcal{L} = \partial^2 + a\partial + a' \cdot \mathrm{id}$ $\quad (a' := \partial(a))$

- $V := \mathcal{L}^{-1}(0) = k \cdot y_1 \oplus k y_2 \subset L$
- Let $v_i := \partial(y_i) + a y_i \quad , \quad i = 1,2$

$\partial(v_i) = \partial^2(y_i) + a\partial(y_i) + a' y_i = 0$

$\Rightarrow v_i \in k$

- On the other hand:

$\underbrace{w(y_1, y_2)}_{\neq 0} = v_2 y_1 - v_1 y_2 \in V$

wlog: $v_1 \neq 0$, $\Rightarrow \{w, y\}$, $y := y_1$

basis of V

- As before: $\partial(w) = -aw$

$\Rightarrow k(w)/k$ PV-extension for

$\mathcal{L}_a := \partial + a \cdot \mathrm{id}$

$\Rightarrow G := \mathrm{Aut}_\partial(L/k)$ acts on

- $\sigma \in G := \mathrm{Aut}_\partial(L/k)$ acts on $\{w, y\}$ as $\begin{pmatrix} \frac{\sigma(w)}{w} & f \\ 0 & h \end{pmatrix}$

<u>Ex. 17:</u> Show that $h = 1$ and $f = \sigma(y) - y$

(Hint: compare $\partial(\sigma(y))$ with $\sigma(\partial(y))$)

- Have a short exact sequence

$$1 \to \mathrm{Aut}_\partial(L/k(w)) \to G \to \mathrm{Aut}_\partial(k(w)/k) \to 1$$

$$\mathrm{Aut}_\partial(L/k(w)) \subseteq \mathbb{G}_a(k), \quad \mathrm{Aut}_\partial(k(w)/k) \subseteq \mathbb{G}_m(k)$$

- $\mathrm{Aut}_\partial(L/k(w)) = 0 \iff y \in k$
  $\iff G \cong \mathrm{Aut}_\partial(k(w)/k)$

- Otherwise $\mathrm{Aut}_\partial(L/k(w)) = \mathbb{G}_a(k)$

- In any case:

$$G \cong \mathrm{Aut}_\partial(L/k(w)) \rtimes H$$

$$H := \{\tau \in G \mid \tau(y) = y\}$$

- Now for an example with a non-solvable automorphism group

Exmp 8.8. • $K = k(x)$, $\partial(x) = 1$

• $\mathcal{L} = \partial^2 - x \cdot \mathrm{id}$ (Airy equation)

($k = \mathbb{C}$, solutions are "Airy functions")

• $V := \mathcal{L}^{-1}(0) = ky \oplus kz \subset L$ ← PV-extension

• $w := w(y, z)$ wronskian

• Have: $\partial(w) = 0$

$\Rightarrow w \in k$

• can assume $w = 1$ (otherwise rescale y)

• Prop 8.2 $\Rightarrow G := \mathrm{Aut}_\partial(L/K) \subseteq Sl_2(k)$

• <u>Claim: $G = Sl_2(k)$</u>

Proof sketch ($k = \mathbb{C}$): • $G^\circ \subseteq G$ connected component of id

• Either $G^\circ = G$ or G° solvable (every 2-dim Lie algebra is solvable)

• Assume G° solvable

Lie-Kolchin $\Rightarrow G^\circ$ triangularisable

$\exists\, v \in V \setminus \{0\}$ s.t. $v \in L^{G^\circ}$ $\Rightarrow u := \frac{v'}{v} \in L^{G^\circ}$

• Fund. thm of diff. Galois theory:

$$G^0 \trianglelefteq G \longleftrightarrow M/k \quad \text{PV-subextension}$$

$$M = L^{G^0} \ni V$$

also $\mathrm{Aut}_\partial(M/k) \cong G/G^0$ ← finite

$\Rightarrow$ u is algebraic over $\mathbb{C}(x)$

$\Rightarrow$ u has finitely many poles

(uses that Airy functions are analytic)

• However, $u(x) = \frac{v'(x)}{v(x)}$ satisfies

$$u'(x) = x - u(x)^2 \quad \text{(Riccati eqn.)}$$

<u>Ex. 18* (for analysts only):</u>

Prove that each solution to the Riccati equation has infinitely many poles.

(Why believe that?

$\tan'(x) = 1 - \tan(x)^2$ and

$\tan(x)$ has infinitely many poles

$\Rightarrow$ u not algebraic over $\mathbb{C}(x)$ ↯

$\Rightarrow G^0 = G = Sl_2(\mathbb{C})$