Lecture 3: Differential forms on complex manifolds

1. Complex manifolds

X smooth manifold, $\dim X = 2n$

Defn: complex structure on X

$\{(U_i, \phi_i)_{i\in I}\}$ consisting of

(i) $X = \bigcup_{i\in I} U_i$ open cover

(ii) $\phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{C}^n$

diffeomorphisms

s.t. : $\phi_j \circ \phi_i^{-1}$ holomorphic

$\forall\, i,j \in I$

Then: $(X, \{(U_i, \phi_i)\}_{i\in I})$ complex manifold

X complex manifold

$\Rightarrow (X, I)$ almost complex manifold

- converse is false (see Van de Ven "Chern numbers of complex manifolds")

<u>Thm (Newlander–Nirenberg)</u>

(X, I) is <u>integrable</u> (i.e. I comes from a complex structure on X) iff

$$[T_X^{0,1}, T_X^{0,1}] \subset T_X^{0,1}$$

see Voisin, section 2

4. ∂ and $\bar{\partial}$ operators

- $\Omega_X^{1,0} = \mathrm{Hom}(T_X^{1,0}, \mathbb{C})$

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Examples: (i) $U \subseteq \mathbb{C}^n$ open

(ii) complex projective space

$$\mathbb{P}^n(\mathbb{C}) = \left(\mathbb{C}^{n+1} - \{0\}\right) / v \sim \lambda v, \quad \lambda \in \mathbb{C}^*$$

charts: $\tilde{U}_i = \{(z_1, \ldots, z_{n+1}) \mid z_i \neq 0\}$

$$U_i = \mathrm{Im}\left(\tilde{U}_i \longrightarrow \mathbb{P}^n(\mathbb{C})\right)$$

$$\phi_i : U_i \xrightarrow{\sim} \mathbb{C}^n$$

$$[z_1 : \ldots : z_{n+1}] \longmapsto \left(\frac{z_1}{z_i}, \frac{z_2}{z_i}, \ldots, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{n+1}}{z_i}\right)$$

check: $\phi_j \circ \phi_i^{-1}$ holomorphic

on $\phi_i(U_i \cap U_j)$

(iii) complex tori

$\Gamma \subset \mathbb{C}^n$ lattice (= free abelian subgroup, rank $2n$; $\mathrm{span}_{\mathbb{R}} \Gamma = \mathbb{C}^n$)

Then: $X := \mathbb{C}^n / \Gamma$

$$\pi : \mathbb{C}^n \longrightarrow X$$

Note: (ii), (iii) compact

Now X, Y complex manifolds

$f: X \to Y$ smooth

<u>Defn</u>: f is <u>holomorphic</u>

if $\phi_{j,Y} \circ f \circ \phi_{i,X}^{-1}$

chart on Y chart on X

is holomorphic, $\forall\, i \in I, j \in \mathcal{Y}$

2. Holomorphic vector bundles

Let $\pi_E : E \longrightarrow X$ complex manifold complex vector bundle

<u>Defn</u>: π_E <u>holomorphic vector bundle</u>

if $\exists$ local trivializations

$$\tau_i : \pi_E^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^m$$

s.t. the matrices

$\tau_{ij} := \tau_j \circ \tau_i^{-1}$ have holomorphic coefficients

Remark: E acquires complex structure s.t. π_E holomorphic

- holomorphic sections:

$\sigma : X \to E$ holomorphic

s.t. $\pi_E \circ \sigma = id$

- morphisms: holomorphic

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & \circlearrowleft & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd}$$

induces linear map

$$\varphi_x : \underset{\pi_E^{-1}(\{x\})}{\underset{\shortparallel}{E_x}} \longrightarrow \underset{\pi_F^{-1}(\{x\})}{\underset{\shortparallel}{F_x}}$$

Examples: (i) holomorphic tangent bundle $T_X \longrightarrow X$

$\{\phi_i : U_i \to V_i\}_{i\in I}$ holomorphic charts

Then: $T_X = \left(\coprod_{i\in I} U_i \times \mathbb{C}^n\right)/\sim$

$(u,v) \sim (u, \phi_{ij*}(v))$ $\qquad \phi_{ij} := \phi_j \circ \phi_i^{-1}$

hol. sections : "vector fields"

(ii) $\Omega_X := T_X^* = \mathrm{Hom}(T_X, \mathbb{C})$

holomorphic cotangent bundle

sections : hol. differential one-forms

$\rightsquigarrow \Omega_X^k = \Lambda^k \Omega_X$

sections: k-forms

(iii) E, F holomorphic vector bundles

$\rightsquigarrow E \otimes F$, $\mathrm{Hom}(E, F)$, E^*

are again holomorphic vector bundles

3. Almost complex structures

X complex manifold

$T_{X,\mathbb{R}}$ real tangent bundle

$$T_{U_i,\mathbb{R}} = U_i \times \mathbb{R}^{2n}$$

- Endomorphism: ("almost complex structure")

$$I : T_{X,\mathbb{R}} \longrightarrow T_{X,\mathbb{R}}$$

given by multiplication by $1 \times i$ $T_{U_i,\mathbb{R}}$

check: this is well-defined,

i.e. I_{X_i} "glue"

Facts: (i) $I^2 = -\mathrm{id}$

(ii) $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$

Then: $T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$

where $T_X^{1,0} = i$-eigenspace for I

$T_X^{0,1} = -i$-eigenspace for I

Explicitly: $T_X^{1,0/0,1}$ spanned by

$u \mp i \cdot Iu$, $u \in T_{X,\mathbb{R}}$

(iii) $T_X^{1,0} = T_X \subset T_{X,\mathbb{C}}$ as complex vector bundles

(iv) $\mathrm{Re}: T_X^{1,0} \xrightarrow{\simeq} T_{X,\mathbb{R}}$ as real vector bundles

Rmk: **Almost complex manifold**

(X, I) X = smooth manifold, $\dim X = 2n$

$I: T_{X,\mathbb{R}} \longrightarrow T_{X,\mathbb{R}}$, $I^2 = -\mathrm{id}$

α local section of $\Omega_X^{1,0}$/$0,1$

$$\alpha = \sum_{i=1}^{n} \alpha_i \, d\bar{z}_i \quad , \alpha_i \in C^\infty(X)$$

local coordinates

(p, q)-forms

• Also, $\Lambda^k \Omega_{X,\mathbb{C}} = \sum_{p+q=k} \Omega_X^{p,q}$

$$\Omega_X^{p,q} = \Lambda^p \Omega_X^{1,0} \otimes \Lambda^q \Omega_X^{0,1}$$

• local section α of $\Omega_X^{p,q}$

$$\alpha = \sum_{I,J} \alpha_{I,J} \, dz_I \wedge d\bar{z}_J$$

$$dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \quad , I = (i_1, \ldots, i_p)$$
$$i_1 < \ldots < i_p$$

Fact: $d\alpha = \sum_{I,J} d\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$

$$\Rightarrow d(\Omega_X^{p,q}) \subset \Omega_X^{p+1,q} \oplus \Omega_X^{p,q+1}$$

Dfn: $\partial : \Omega_X^{p,q} \longrightarrow \Omega_X^{p+1,q}$

$\bar{\partial} : \Omega_X^{p,q} \longrightarrow \Omega_X^{p,q+1}$

unique $\mathbb{C}$-linear / anti-linear maps

s.t. $d = \partial + \bar{\partial}$

Lemma: (i) $\partial, \bar\partial$ satisfy Leibniz rule, $\bar\partial(\alpha\wedge\beta) = \bar\partial(\alpha)\wedge\beta + (-1)^k \alpha\wedge\bar\partial(\beta)$

(ii) $\underbrace{\partial^2 = \bar\partial^2 = 0}_{\text{integrability}}$, $\partial\bar\partial + \bar\partial\partial = 0$

Proof: Follows from analogous properties of $d = \partial + \bar\partial$