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Lecture 6: Automorphisms

Recall: $(L,\partial)/(k,\partial)$ extension

$$\mathrm{Aut}_\partial(L/k) = \left\{ \sigma : L \to L \;\middle|\; \begin{matrix} \sigma_{|k} = id \\ \sigma \circ \partial = \partial \circ \sigma \end{matrix} \right\}$$

Want to show: If L/k PV-extension $\Rightarrow$ $\mathrm{Aut}_\partial(L/k)$ is a linear algebraic group over k (field of constants)

__1st step:__ $\mathrm{Aut}_\partial(L/k)$ __embeds__ into a linear algebraic group

__Prop 6.1:__ $\cdot$ $(L,\partial)/(k,\partial)$ extension

$\cdot$ $\mathcal{L} \in k[\partial]$ monic differential operator

$\cdot$ $V = \mathcal{L}^{-1}(0) \subset L$, $G = \mathrm{Aut}_\partial(L/k)$

Then: (i) $G(V) \subseteq V$ and G acts on V via k-linear transformations

(ii) If $\dim_k(V) < \infty$, then the action of G on V gives rise to

$$\rho : G \longrightarrow Gl(V)$$

group homomorphism

(iii) If $L = K\langle V \rangle$, then ρ is injective

Rmk 6.2: In particular, if L/k PV, then the action of G on V defines a faithful representation of dim. equal to the order of $\mathcal{L}$.

Proof: (i) If $\mathcal{L}(y) = 0$, then

$$\mathcal{L}(\sigma(y)) = 0 \quad \forall \sigma \in G$$

$(\text{since } \sigma \circ \partial^i = \partial^i \circ \sigma, \text{ and } \sigma|_K = id)$

$\Rightarrow G(V) \subseteq V$.

$\mathcal{L} : L \longrightarrow L$ $\quad k$-linear

$\Rightarrow \sigma$ acts via linear

transformations on V

(ii) Clear by the above

(iii) If $L = K\langle V\rangle$, then each $\sigma \in G$ uniquely determined by $\sigma|V$. $\square$

<u>2nd step</u>: Identify $Gl(V)$ with a group of differential automorphisms of full universal solution algebra.

Notation: $\mathcal{L} = \partial^n - \sum_{i=0}^{n-1} a_i \partial^i \in K[\partial]$

- $S = K[Y_{ij}][W(Y_{0j})^{-1}]$, $0 \leq i \leq n-1$, $1 \leq j \leq n$

  w. derivation

$$\partial(Y_{ij}) = \begin{cases} Y_{i+1j}, & i < n-1 \\ \sum_{k=0}^{n-1} a_k Y_{kj}, & i = n-1 \end{cases}.$$

- write $Y_j^{(i)} := Y_{ij}$ and $Y_j := Y_{0j}$

  and $W := W(Y_1, \ldots, Y_n)$

- $V = \sum_{k=1}^{n} k \cdot Y_k \subset S$, $B = \{Y_1, \ldots, Y_n\}$ basis of V

· $\mathfrak{m} \subset S$ fixed maximal differential ideal, $\bar{S} := S/\mathfrak{m}$, $L := Q(\bar{S})$

· $\pi : S \longrightarrow \bar{S}$ canonical projection

$\bar{y}_i := \pi(y_i)$, $\bar{w} := \pi(w)$, $\bar{V} := \pi(V)$

$\bar{\mathcal{B}} := \{\bar{y}_1, \dots, \bar{y}_n\}$

· Note: $\pi : V \longrightarrow \bar{V}$ is an isom.

Construction:

$$\Phi : \mathrm{Gl}(V) \longrightarrow \mathrm{Aut}_\partial(S/k)$$

given by basis $y_1, \dots, y_n$ $\longrightarrow$ $\mathrm{Gl}(V) \cong \mathrm{Gl}_n(k)$

defined by $T \longmapsto \Phi_T$

where $\Phi_T : k[Y_j^{(i)}] \longrightarrow k[Y_j^{(i)}]$

$$Y_j^{(i)} \longmapsto \sum_{k=1}^{n} t_{kj} Y_k^{(i)}$$

$T = (t_{ij})$

<u>Prop. 6.3:</u> Φ is a well-defined embedding, and

$$\mathrm{Im}(\Phi) = \{ \sigma \in \mathrm{Aut}_\partial(S/k) \mid \sigma(V) \subseteq V \}.$$

<u>Proof:</u> (i) $\Phi_T \circ \partial = \partial \circ \Phi_T$

enough to show this on $Y_j^{(i)}$

(direct computation)

$\Rightarrow \Phi_T \in \mathrm{End}_g(S/k)$

(ii) $\Phi_U \circ \Phi_T = \Phi_{UT} \qquad \forall\, U, T \in Gl_n(k)$

enough to show this on Y_j

(direct computation)

Since $\Phi_{\mathbb{1}} = id$

$\Rightarrow \Phi_{T^{-1}} = (\Phi_T)^{-1}$

$\Rightarrow \Phi_T \in \mathrm{Aut}_g(S/k)$

Also: $\Phi : Gl_n(k) \longrightarrow \mathrm{Aut}_g(S/k)$

is a group homomorphism

(iii) $\Phi_T(w) = \det(T) \cdot w$

$$\Phi_T(w) \overset{(1)}{=} \det(\Phi_T(Y_j^{(i)}))$$
$$\overset{(2)}{=} \det(T) \cdot \det((Y_j^{(i)}))$$
$$= \det(T) \cdot w$$

(1): Φ_T alg. hom

(2): $\Phi_T(Y_j^{(i)}) = (Y_j^{(i)}) \cdot T$

(iii) $\Rightarrow \Phi_T$ extends to a map

$$S \longrightarrow S$$
$$y_j^{(i)} \longmapsto \sum_k t_{kj} y_k^{(i)}$$
$$w^{-1} \longmapsto \det(T)^{-1} \cdot w^{-1}$$

(iv) $\Phi_T(V) \subseteq V$, $\Phi_T|_V = T$

by definition of Φ_T

Also: If $\tau \in \mathrm{Aut}_\partial(S/k)$, $\sigma(V) \subseteq V$

$$\Rightarrow \tau|_V =: T \in Gl(V)$$

and $\tau = \Phi_T$

$$\Rightarrow \Phi : Gl(V) \hookrightarrow \mathrm{Aut}_\partial(S/k)$$

$$\mathrm{Im}(\Phi) = \{\tau \in \mathrm{Aut}_\partial(S/k) \mid \tau(V) \subseteq V\}$$

$\square$

Ex 12: · L/k extension of differential fields, no new constants.

· $V \subset L$ k-vector subspace, $\dim_k V < \infty$

- $z_1, \dots, z_n$ basis of V

- $w := w(z_1, \dots, z_n)$ Wronskian

· $\sigma \in \mathrm{Aut}_\partial(L/k)$, s.t. $\sigma(V) \subseteq V$

Show: $\sigma(w) = \chi(\sigma) w$, where

$$\chi(\sigma) := \det(\sigma|_V)$$

<u>3rd step:</u> From universal full solution algebra to Picard-Vessiot extensions

<u>Prop 6.4:</u> Restriction to $\bar{S} \hookrightarrow L$

$$\mathrm{Aut}_\partial(L/k) \longrightarrow \mathrm{Aut}_\partial(\bar{S}/k)$$

is an isomorphism of groups.

<u>Proof</u>: $\bar{S} = k\{\bar{V}\}[w^{-1}]$

diff. k-algebra generated by $\bar{V}$

Ex 12 $\Rightarrow$ $k \cdot w^{-1}$ $\mathrm{Aut}_\partial(L/k)$-stable

$\Rightarrow$ $\bar{S} \hookrightarrow L$ $\mathrm{Aut}_\partial(L/k)$-stable

Since $L = k\langle \bar{V} \rangle$, $\sigma \in \mathrm{Aut}_\partial(L/k)$ uniquely determined by $\sigma|_V$

$\Rightarrow$ restriction is injective.

surjectivity clear (automorphisms of $\bar{S}$ extend uniquely to $L = Q(\bar{S})$, compatible with ∂).

4th step: Relate $\mathrm{Aut}_\partial(S/k)$ with $\mathrm{Aut}_\partial(\bar{S}/k)$, $\bar{S} := S/m$

max. diff. ideal

Given $\tau \in \mathrm{Aut}_\partial(\bar{S}/k)$

$\rightsquigarrow T \in Gl(V) \simeq Gl_n(k)$

basis $y_1, \dots, y_n$ of V

$\rightsquigarrow \Phi_T \in \mathrm{Aut}_\partial(S/k)$

Lemma 6.5: The diagram

$$\begin{array}{ccc} S & \xrightarrow{\pi} & \bar{S} \\ \Phi_T \downarrow & & \downarrow \tau \\ S & \xrightarrow{\pi} & \bar{S} \end{array}$$

commutes, i.e.

$$\tau \circ \pi = \pi \circ \Phi_T$$

Idea of proof: construct "generic" automorphism Φ_Y, specializing to Φ_T

Φ_T

Construction:

- $\{X_{ij} \mid 1 \leq i,j \leq n\}$ indeterminates

  $d := \det(X_{ij})$

- $k[X_{ij}][d^{-1}]$, differential ring

  w. zero derivation

- $R := S[X_{ij}][d^{-1}] \cong S \otimes_k k[X_{ij}][d^{-1}]$

  differential K-algebra, X_{ij} constants

- $\Phi_T : R \longrightarrow R$ diff. autom.

  $$Y_j^{(i)} \longmapsto \sum_k X_{kj} Y_k^{(i)}$$

  $$W^{-1} \longmapsto (dW)^{-1}$$

  (restriction to R of $k(X_{ij})$-linear map

  $$k(X_{ij})[Y_j^{(i)}][W^{-1}] \longrightarrow k(X_{ij})[Y_j^{(i)}][W^{-1}]$$

  defined analogously)

- Also: $\bar{R} := \bar{S}[X_{ij}][d^{-1}] \cong \bar{S} \otimes_k k[X_{ij}][d^{-1}]$

  $$\Pi := \pi \otimes id : R \longrightarrow \bar{R}$$

  $$i : S \hookrightarrow R, \qquad \bar{i} : \bar{S} \hookrightarrow \bar{R}$$

• For $t=(t_{ij})\in Gl_n(k)$, evaluation maps

$$ev_t : k[x_{ij}][d^{-1}] \to k$$
$$x_{ij} \mapsto t_{ij}$$

$$EV_t : \bar{R} \longrightarrow \bar{S}$$
$$x_{ij} \longmapsto t_{ij}$$

Proof of Lemma 6.5:

$$EV_t \circ \Pi \circ \Phi_S \circ i = \pi \circ \Phi_T$$

by definition

On the other hand:

$\tau \in Aut_\partial(\bar{S}/k)$, $t \in Gl(\bar{V}) \cong Gl(V) \ni T$

Then $\tau(\pi(y_j^{(i)})) = \tau(\bar{y}_j^{(i)})$

$$= \sum_k t_{kj} \bar{y}_k^{(i)}$$

$$= (EV_t \circ \Pi \circ \Phi_S \circ i)(y_j^{(i)})$$

$\Rightarrow \tau \circ \pi = EV_t \circ \Pi \circ \Phi_S \circ i = \pi \circ \Phi_T$ $\square$

Lemma 6.6: - $\tau \in Aut_\partial(\bar{S}/k)$

- $t \in Gl_n(k)$ restriction of τ to $\bar{V}$ in basis $\bar{y}_1, \dots, \bar{y}_n$ of $\bar{V}$:

$$\left(\begin{array}{ccccc} Gl_n(k) & \xleftarrow{\sim} & Gl(\overline{V}) & \xrightarrow{\sim} & Gl(V) \\ t & \longleftarrow\!\shortmid & \tau|\overline{V} & \longmapsto & T \end{array}\right)$$

Then: (i) $\Phi_T(m) \subset m$

(ii) $\overline{\Phi}_T = \tau$, where

$\overline{\Phi}_T \in \mathrm{Aut}_\partial(\overline{S}/k)$ image

of $\Phi_T \in \mathrm{Aut}_\partial(S/k)$

<u>Proof</u>: (i) Lemma 6.5.

$\Rightarrow \{0\} = (\tau \circ \pi)(m) = (\pi \circ \Phi_T)(m)$

$\Rightarrow \Phi_T(m) \subset \ker(\pi) = m$

(ii) $\tau|\overline{V} = \overline{\Phi}_T|\overline{V}$ by construction

of $\overline{\Phi}_T$

$\Rightarrow \tau = \overline{\Phi}_T$ (since $\overline{S}$ generated

by $\overline{V}$ as a diff. k-algebra)

$\square$

<u>Theorem 6.7</u>: • L/k PV-extension

• Let $Gl(V)_m := \{T \in Gl(V) \mid \Phi_T(m) \subset m\}$

(stabilizer of m)

(stabilizer of m)

- Then the natural map
$$Gl(V)_m \longrightarrow Aut_\partial(L/K)$$
induced by $T \longmapsto \Phi_T \in Aut_\partial(S/K)$
is an isomorphism.

<u>Proof</u>: Prop 6.3:
$$\Phi : Gl(V) \hookrightarrow Aut_\partial(S/K)$$
$$T \longmapsto \Phi_T$$
$$Im(\Phi) = \{\tau \in Aut_\partial(S/K) \mid \tau(V) \subseteq V\}$$
$$\rightsquigarrow Gl(V)_m \hookrightarrow Aut_\partial(L/K)$$
elements autom. stabilize $\overline{V}$

Conversely: $\sigma \in Aut_\partial(L/K)$
$$\Rightarrow \sigma|_{\overline{V}} \in Gl(\overline{V}) \cong Gl(V) \ni T$$
Also: $\sigma|_{\overline{S}} = \overline{\Phi}_T$

Lemma 6.6 $\Rightarrow \Phi_T(m) \subset m$
$$\Rightarrow T \in Gl(V)_m$$
$\square$

Two points remain to be clarified:

(i) $Gl(V)_m \subset Gl(V)$

algebraic subgroup

(ii) Compare: $G \xrightarrow{\sim} Gl(V)_m \hookrightarrow Gl(V)$

(Thm 6.7)

with $G \hookrightarrow Gl(V)$

(Prop 6.1)

First step for (i)

$T \in Gl(V)$. When does $\Phi_T(m) \subset m$ hold?

Lemma 6.5: $\Phi_T(m) \subset m$

$\Rightarrow (\pi \circ \Phi_T)(m) = \{0\}$

$\Rightarrow Ev_t(\pi(\Phi_T(m))) = \{0\}$

$\leadsto$ study $\ker(Ev_t)$

<u>Notation:</u> $\mathcal{U} = \{u_i \mid i \in I\}$ k-basis of $\overline{S}$.

$\overline{R} = \overline{S} \otimes_k k[X_{ij}][d^{-1}]$ is then a free $k[X_{ij}][d^{-1}]$-module

For $f \in R$, write

$$f = \sum_{i \in I} f_i u_i \quad (\text{expansion in } \mathcal{U})$$

Dfn 6.8: For $X \subset \overline{R}$, define

$$J(X) = \underset{\nwarrow \text{ ideal gen. by}}{<} f_i \mid f \in X, i \in I > \subset k[x_{ij}][d^{-1}]$$

Lemma 6.9: · $X \subset \overline{R}$ subset

· $t \in Gl_n(k)$.

Then: $Ev_t(X) = \{0\}$

$\iff ev_t(J(X)) = \{0\}$

$\quad ev_t : k[x_{ij}][d^{-1}] \to k$

Proof: "$\Leftarrow$": ok since $X \subset \overline{R} \cdot J(X)$

"$\Rightarrow$": $Ev_t(f) = \sum_i ev_t(f_i) u_i$

If $Ev_t(f) = 0 \Rightarrow ev_t(f_i) = 0 \quad \forall i \in I$

Since $J(X)$ generated by f_i, $f \in X$

$\Rightarrow J(X) \subset \ker(ev_t)$ □

Consequence: $\Phi_T(m) \subset m$

$\iff ev_t$ annihilates $J(\pi(\Phi_T(m)))$.