Ink note

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Lecture 16:

Tannakian categories and differential Galois theory

- k field

Recall from last time:

- A Tannakian category $\mathcal{C}$ is a rigid abelian k-linear tensor category, with $\mathrm{End}(\mathbb{1}) = k$, such that there exists a fibre functor

$$\omega : \mathcal{C} \longrightarrow \mathrm{Mod}_R$$

for some k-algebra $R \neq 0$

- $\mathcal{C}$ is neutral if $\exists$

$$\omega : \mathcal{C} \longrightarrow \mathrm{Vect}_k$$

Theorem (Deligne): If $k = \bar{k}$, then $\mathcal{C}$ is neutral.

- proved in "Catégories tannakiennes" if $\mathcal{C} = \langle X \rangle_{\otimes}$, i.e. $\mathcal{C}$ admits a tensor generator (Cor. 6.20)
- general case: see Deligne-Milne, (online version!), footnote 16

Moreover: If $\omega_1, \omega_2 : \mathcal{C} \to \mathrm{Vect}_k$ are two fibre functors, then $\exists$ k-linear isomorphism $\omega_1 \simeq \omega_2$

Reason: $\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)$ is a $\underline{\mathrm{Aut}}^{\otimes}(\omega_1)$-torsor (fpqc-locally)

- k algebraically closed
  $\Rightarrow \underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)(k) \neq \emptyset$
  "trivial torsor"

- Note: $\omega_1 \simeq \omega_2$ not unique, unless $\underline{\mathrm{Aut}}^{\otimes}(\omega_1) = \{id\}$

Now back to differential

Galois theory:

Galois theory.

- (k,∂) differential field ($\mathrm{char}(k)=0$)

  $\mathbb{k}=\ker(\partial)$ field of constants

- $\mathcal{T}$ = category of pairs (V,∇)

  - $V\in \mathrm{Vect}_K$ (finite-dimensional)
  - $\nabla: V\longrightarrow V$ additive,

    $\nabla(\lambda v)=\partial(\lambda)v+\lambda\nabla(v)$ Leibniz rule

    for all $\lambda\in k$, $v\in V$

A morphism $f:(V_1,\nabla_1)\longrightarrow(V_2,\nabla_2)$ is a k-linear map

$$f: V_1\longrightarrow V_2 \quad \text{s.t.}$$

$$\nabla_2\circ f=f\circ\nabla_1$$ "∇ is $\mathbb{k}$-linear"

- Note: (i) $\mathrm{Hom}_{\mathcal{T}}(X,Y)$ is a $\mathbb{k}$-vector space, $\forall X,Y\in \mathrm{Ob}(\mathcal{T})$

(ii) $(V_1,\nabla_1)\otimes(V_2,\nabla_2)=(V_1\otimes V_2,\nabla_1\otimes\nabla_2)$

$$(\nabla_1\otimes\nabla_2)(v_1\otimes v_2)=\nabla_1(v_1)\otimes v_2+v_1\otimes\nabla_2(v_2)$$

- In fact, $\mathcal{T}$ is a rigid abelian $\mathbb{k}$-linear tensor category.

Exercise: show that $\mathbb{1}=(K,\partial)$

and $\mathrm{End}(\mathbb{1}) = k$

- $\omega : \mathcal{T} \longrightarrow \mathrm{Vect}_K$ (not in Vect_k)

$$(V, \nabla) \longmapsto V$$

is a fibre functor

$\Rightarrow$ $\mathcal{T}$ is a (possibly non-neutral) Tannakian category over k

Rmk: • Previously, we considered

$$\mathcal{L} = \partial^n + a_1 \partial^{n-1} + \ldots + a_n \cdot \mathrm{id} \in K[\partial]$$

linear differential operator.

- Consider $(V, \nabla) \in \mathcal{T}$, with

$V = K^n$, $\nabla = \partial + A$

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & & & & \vdots \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & -a_1 \end{pmatrix}$$

then $a \in K$ satisfies

$$\mathcal{L}(a) = 0$$

$\Leftrightarrow$ $\vec{a} = (a, \partial(a), \ldots, \partial^{n-1}(a)) \in K^n$

satisfies $\nabla(\vec{a}) = 0$

- $\mathcal{J}$ is nice category (linear diff. operators form a "bad" category)
- $X = (V, \nabla) \in Ob(\mathcal{J})$

  $\langle X \rangle_{\otimes}$ = strictly full subcategory of $\mathcal{J}$, objects are sub-quotients of $X^{\otimes n} \otimes (X^{\vee})^{\otimes n}$
- $\langle X \rangle_{\otimes}$ is a Tannakian category

<u>Dfn</u>: A <u>Picard-Vessiot extension</u> of k for $X = (V, \nabla)$ is a diff. field extension $(L, \partial_L)/(k, \partial)$ such that

(i) $V \otimes_k L$ is generated, as an L-vector space, by

$$(V \otimes_k L)^{\nabla} = \{ v \in V \otimes_k L \mid \nabla(v) = 0 \}$$

(ii) L/k is generated by

$$\left\{ e_i^{\vee}(v) \;\middle|\; v \in (V \otimes_k L)^{\nabla},\ \{e_i\} \text{ } k\text{-basis of } V \right\}$$

(iii) $\ker(\partial_L) = k$

- Comparison with the previous definition: $(L, \partial_L)/(k, \partial)$ is PV-extension of k for $\mathcal{L} \in k[\partial]$.

monic, $\deg(\mathcal{L})=n$, if

(i'): $\dim_k \mathcal{L}^{-1}(0)=n$

(ii'): $\underline{(L,\partial)/(k,\partial)}$ generated by $\mathcal{L}^{-1}(0)$
(as a diff. field)

(iii'): $\ker(\partial_L)=k$

- $X=(k^n, \partial+A)$ as above
(corresponding to $\mathcal{L}$)

- $-(iii)\Leftrightarrow(iii')$ (obvious)

$-(ii)\Leftrightarrow(ii')$ (ok, since

$\mathcal{L}(a)=0 \Leftrightarrow \nabla((a,\partial(a),\dots,\partial^{n-1}(a)))=0$)

$-(i)\Leftrightarrow V\otimes_k L\cong L^n$ spanned

by $(L^n)^{\nabla}$ over L
k-vector space

$\Leftrightarrow \dim_k (L^n)^{\nabla}\geqslant n$

$\Leftrightarrow \dim_k \mathcal{L}^{-1}(0)=n$

($\dim_k \mathcal{L}^{-1}(0)\leqslant n$ holds always)

- Now consider the functor

$$\omega_L : \langle X\rangle_{\otimes} \longrightarrow \mathrm{Vect}_k$$

$$(V,\nabla)\longmapsto (V\otimes_k L)^{\nabla_L}$$

<u>Claim</u>: ω_L is a fibre functor.

Key: condition (i) is stable under tensor products/duals/subobjects/quotients

e.g. if $V_i \otimes L$ spanned L-linearly by $(V_i \otimes L)^{\nabla_i}$, $i = 1,2$

$\Rightarrow (V_1 \otimes V_2 \otimes L)^{\nabla_1 \otimes \nabla_2}$ is as well
(follows from def. of $\nabla_1 \otimes \nabla_2$)

Sketch of proof of claim:

- $(V_1 \otimes L)^{\nabla_1} \otimes_k (V_2 \otimes L)^{\nabla_2} \cong (V_1 \otimes V_2 \otimes L)^{\nabla_1 \otimes \nabla_2}$ (by the above)
  $\Rightarrow \omega_L$ compatible w. $\otimes$-product
- $\omega_L(V,\nabla) \otimes_k L \cong V \otimes_k L$
  $\Rightarrow \omega_L$ faithful
- for exactness:
  $0 \to Y_1 \to Y_2 \to Y_3 \to 0$
  s.e.s in $\langle X \rangle_\otimes$
  each object in $\langle X \rangle_\otimes$ satisfies (i) + faithfull flatness of L/k
  $\Rightarrow 0 \to \omega_L(Y_1) \to \omega_L(Y_2) \to \omega_L(Y_3) \to 0$
  exact

$\square$

- Conversely, let

$$\omega_0 : \langle X \rangle_\otimes \longrightarrow \mathrm{Vect}_k$$

be a fibre functor

- want: $\omega_0(V,\nabla) \cong (V \otimes L_0)^\nabla = \omega_{L_0}(V,\nabla)$ naturally in $(V,\nabla) \in \langle X \rangle_\otimes$ for some $(L_0,\partial)/(K,\partial)$ PV-extension of X

- For this, consider, for a given PV-extension L,

$$P = \underline{\mathrm{Isom}}^\otimes_K(\omega_L \otimes_k K, \omega|\langle X \rangle_\otimes)$$

which is a G_K-torsor,

$$G := \underline{\mathrm{Aut}}^\otimes(\omega_L), \quad G_K := G \otimes_k K$$

Fact: P is integral and

$$k(P) \cong L$$

Function field of P

sketch: · P is smooth

(since G_K is, as $\mathrm{char}(K)=0$)

and connected (see 9.3.(ii) ...

of Deligne)

$\Rightarrow P$ is integral

- since $(V \otimes L)^\nabla \otimes L \cong V \otimes L$

obtain $(V\otimes L)^\nabla \otimes k \longrightarrow V$

$\rightsquigarrow \omega_L \otimes k \longrightarrow \omega|\langle X\rangle_\otimes$

- dually

$$\mathcal{O}(P) \longrightarrow L$$

k-algebra

- can show: this is injective and property (ii) of PV-extension proves the fact $\square$

More generally, similar considerations show that

$$P_0 = \underline{\mathrm{Isom}}^\otimes_k(\omega_0 \otimes_k k, \omega|\langle X\rangle_\otimes)$$

is integral and that

should have a derivation!

$k(P_0)$ (is a PV-extension of (k,∂))

crucial part of argument

Rmk: Both P, P_0 (hence $k(P), k(P_0)$) inherit a derivation from $\omega|\langle X\rangle_\otimes$, which in turn inherits a connection from $\omega(Y)$, for each Y (natural in Y)

(for this, need to interpret a connection $V \to V$ as an $\exp(\varepsilon\partial)$-linear automorphism

$$V \otimes k[\varepsilon] \longrightarrow V \otimes k[\varepsilon]$$

where $k[\varepsilon] = k[X]/(X^2)$

- All in all:

Thm: For each $X \in Ob(\mathcal{J})$, have a natural bijection

$$\left\{ \begin{matrix} \text{fibre functors} \\ \omega_0 : \langle X \rangle_\otimes \to \mathrm{Vect}_k \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{PV-extensions} \\ (L,\partial)/(K,\partial) \text{ for } X \end{matrix} \right\}$$

Combined with the Theorem above we get:

Corollary (Deligne): $\cdot (K,\partial)$ diff. field $k = \ker(\partial)$ algebraically closed

- $X \in Ob(\mathcal{J})$

Then: $\exists$ PV-extension

$(L,\partial)/(K,\partial)$ for X

$(L,\partial)/(k,\partial)$ for X,

unique up to (non-canonical) isomorphism

- Now fix a PV-extension

$(L,\partial)/(k,\partial)$ for X

- Recall: $\mathrm{Aut}_\partial(L/k)$ affine algebraic group $/k$: "differential Galois group"

$$\mathrm{Aut}_\partial(L/k) = \{\varphi : (L,\partial) \to (L,\partial) \mid \varphi|_k = \mathrm{id}\}$$

- On the other hand, $\underline{\mathrm{Aut}}^{\otimes}(\omega|\langle X\rangle_\otimes)$ ''

$$\langle X\rangle_\otimes \xrightarrow{\sim} \mathrm{Rep}_k(G)$$

for some affine algebraic group (see last lecture)

Theorem: If k is alg. closed, then there exists an isomorphism

$$G \cong \mathrm{Aut}_\partial(L/k)$$

of affine algebraic groups

Idea of proof: • Have fibre functor

$\omega_1 : \langle X\rangle_\otimes \longrightarrow \mathrm{Vect}_k$

$$(V,\nabla) \longmapsto (V\otimes_k L)^{\nabla_L}$$

- $Aut_\otimes(L/k)$ acts on L $\Rightarrow$ it acts on $(V\otimes_k L)^{\nabla_L}$, functorially in (V,∇) $\rightsquigarrow$

$$\begin{tikzcd}
\langle X\rangle_\otimes \arrow[rr, dashed, "\tilde\omega_L"] \arrow[dr, "\omega_L"'] & & Rep_k(Aut_\otimes(L/k)) \arrow[dl] \\
& Vect &
\end{tikzcd}$$

similarly, $Aut_\otimes(L/k)(R)$ acts on $(V\otimes_k L\otimes_k R)^{\nabla_{L\otimes_k R}}$

(holds without assuming $k=\bar k$)

- Goal: $\tilde\omega_L$ is an equivalence

(a) full faithfulness (= isom. on hom-sets)

$(V_1,\nabla), (V_2,\nabla) \in \langle X\rangle_\otimes$

tensor unit

(*) $Hom((V_1,\nabla),(V_2,\nabla)) \cong Hom((k,\partial),(V_1^\vee\otimes_k V_2,\nabla))$

(**) $Hom((V_1\otimes L)^\nabla,(V_2\otimes L)^\nabla) \cong Hom(k,(V_1^\vee\otimes_k V_2\otimes_k L)^{\nabla_L})$

in $Rep_k(Aut_\otimes(L/k))$ trivial rep

$\Rightarrow$ can assume $(V_1,\nabla)=(k,\partial)$

- Now $(*) = \{v\in V_2 \mid \nabla(v)=0\} = V_2^\nabla$

$(**) = \{a\in (V_2\otimes L)^\nabla \mid g.a=a,\ \forall g\in Aut_\otimes(L/k)\}$

- Corollary 4.9 $\left(L^{Aut_\otimes(L/k)}=k\right)$

$\Rightarrow (**) = V^\nabla$

$\Rightarrow (**) = v_2$

$\Rightarrow$ full faithfulness

- Essential surjectivity

By results of lecture 6

(mutatis mutandis)

$\omega_L(V,\sigma) = (V \otimes_k L)^{\sigma_L}$ faithful

representation of $\mathrm{Autg}(L/k)$

$\mathrm{Autg}(L/k)(k) \longrightarrow Gl(V)(k)$

- Want to see: $W \in \mathrm{Rep}_k(\mathrm{Autg}(L/k))$ comes

from a G-representation.

Need (cf. Szamuely, Lemma 6.5.16)

Lemma k field, $G \subset Gl_n/k$

closed algebraic subgroup,

$W \in \mathrm{Rep}_k(G)$ n-dim standard rep.

Then $\langle W \rangle_\otimes \subset \mathrm{Rep}_k(G)$

is an equivalence of cats.

- Lemma $\Rightarrow$ essential surjectivity

$\Rightarrow \langle X \rangle_\otimes \simeq \mathrm{Rep}_k(\mathrm{Autg}(L/k))$

$\Rightarrow$ $G \cong \mathrm{Autg}(L/k)$ $\square$