Introduction to Hodge theory 07/06/2021

Part 2 - Lecture 5 Oxford

Last time:

- Proof of GAGA
- Complex tori: connected, compact cplx Lie gp
  $\iff$ of the form V/L

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5) Complex tori and abelian varieties (cont.)

<u>Lemma</u> Every complex torus is Kähler.

Proof: $X \cong \mathbb{C}^g/L$, $\omega = \sum_{i=1}^{g} dz_i \wedge d\bar{z}_i$. $\square$

⚠ In fact, "most" complex tori are not projective!

Note: $A^{p,q}(X) = A^{p,q}(\mathbb{C}^g)^L$ ← forms invariant under L

$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J}\, dz_I \wedge d\bar{z}_J$, $\alpha_{I,J}(z+l) = \alpha_{I,J}(z)$ $\forall l \in L$

Lemma $\alpha \in A^{p,q}(X)$ is harmonic if and only if $\alpha_{I,J} \in \mathbb{C}$ $\forall\, I, J$.

[Recall: given Riemannian metric $\rightsquigarrow$ $(\, , \,)$ L^2 metric $\rightsquigarrow$ $*$ operator $\rightsquigarrow$ Laplacian $\Delta = d \circ d^* + d^* d$

$\alpha \in A^k(X)$ is <u>harmonic</u> if $\Delta \alpha = 0$.]

Proof: Equiv, $\alpha \in A^k(X)$ is harmonic iff has constant coeffs. Indep. of choice of basis. Can assume $X \simeq \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ with standard flat metric so that $\Delta = -\sum_{j=1}^{2g} \frac{\partial^2}{\partial x_j^2}$. Then:

$$\alpha = \sum_{|I|=k} \overset{\rightsquigarrow\, \mathbb{Z}^{2g}\text{-inv.}}{\alpha_I} \, dx_I \overset{\text{Fourier}}{=} \sum_{n \in \mathbb{Z}^{2g}} \sum_I a_{I,n} e^{2\pi i n \cdot x} \, dx_I$$

$$= \sum_I a_{I,0} \, dx_I + \Delta \left(\sum_I \sum_{n \neq 0} \frac{a_{I,n}}{4\pi^2 |n|^2} e^{2\pi i n \cdot x} \, dx_I \right). \square$$

Coro. If X is a complex torus, then

$$H^{p,q}(X) \simeq \Lambda^p (\mathrm{Lie}\, X)^\vee \otimes \Lambda^q \overline{(\mathrm{Lie}\, X)}^\vee . \square$$

Rk. Since $X \overset{\text{homeo}}{\simeq} S^1 \times \cdots \times S^1$, we have

$$H^k(X, \mathbb{Z}) \simeq \wedge^k H^1(X, \mathbb{Z}) \quad \text{(use Künneth)}$$

Compatible w/ Hodge decomp.

Coro. If $\dim X = g$, then $h^{p,q}(X) = \binom{g}{p}\binom{g}{q}$

$g = 1$

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

$g = 2$

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

$g = 3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 9 & & 3 & \\ 1 & & 9 & & 9 & & 1 \\ & 3 & & 9 & & 3 & \\ & & 3 & & 3 & & \\ & & & 1 & & & \end{array}$$

Recall: Hodge structure of wt k

$(H, (H^{p,q})_{p+q=k})$, $\quad H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$, $\quad \overline{H^{p,q}} = H^{q,p}$

A morphism is a $\mathbb{Z}$-linear $f : H_1 \to H_2$ st

$f_{\mathbb{C}}(H_1^{p,q}) \subset H_2^{p,q}$.

($f \neq 0 \Rightarrow H_1$ and H_2 have same weight).

If H_1, H_2 carry Hodge str. of wt k_1, k_2, then $L = \mathrm{Hom}_{\mathbb{Z}}(H_1, H_2)$ carries Hodge str. of wt $k_2 - k_1$:

$$L^{p,q} = \{ \varphi \in L_{\mathbb{C}} \mid \varphi(H_1^{r,s}) \subset H_2^{r+p,s+q} \ \forall r,s \}.$$

$H_2 = \mathbb{Z}(0) \rightsquigarrow H_1^\vee$ dual Hodge str., wt $= -k_1$.

<u>Ex</u> X = complex torus

$$H^1(X) = \left(H^1(X,\mathbb{Z}), \left(H^{1,0} = (\mathrm{Lie}\, X)^\vee, H^{0,1} = \overline{(\mathrm{Lie}\, X)^\vee} \right) \right)$$

Dual:

$\rightsquigarrow$ cplx conj. in $H_1(X,\mathbb{Z}) \otimes \mathbb{C}$

$$H_1(X) = \left(H_1(X,\mathbb{Z}), \left(H^{-1,0} \cong \mathrm{Lie}\, X, H^{0,-1} = \overline{H^{0,-1}} \right) \right).$$

$\frac{v \otimes 1 + i v \otimes i}{2} \leftarrow\!\shortmid v$ $\qquad \in H_1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = (\mathrm{Lie}\, X) \otimes_{\mathbb{R}} \mathbb{C}$

<u>Thm</u> The functor $X \mapsto H_1(X)$ is an equivalence between the category of complex tori and the category of torsion-free Hodge structures of weight -1.

Proof: – Fully faithful: follows from

$$0 \to H_1(X,\mathbb{Z}) \to \mathrm{Lie}\, X \xrightarrow{\exp} X \to 0$$

– Ess. surjective: $H \otimes_{\mathbb{Z}} \mathbb{C} = H^{-1,0} \oplus H^{0,-1}$

$C: x \mapsto i^{p-q} x$ on $H^{p,q}$ gives cplx structure on $H \otimes_{\mathbb{Z}} \mathbb{R} \rightsquigarrow X = H \otimes \mathbb{R} / H$. $\square$

Note: $H^{-1,0} \xrightarrow{\sim} H \otimes \mathbb{R}$, $x \mapsto -\mathrm{Re}(x) = -\frac{x - \bar{x}}{2}$

Isom of $\mathbb{C}$ - spaces! //