Lecture 8: Hard Lefschetz

Last time: • (X, ω) Kähler manifold, $\dim X =: n$

• If X compact, then

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

$H^{p,q}(X)$ = classes represented by a (p,q)-form

$\Longleftrightarrow$ harmonic (p,q)-forms

"Hodge decomposition"

independent of choice of ω

1. Lefschetz decomposition on differential forms

$$L : \Omega^k_{X,\mathbb{R}} \longrightarrow \Omega^{k+2}_{X,\mathbb{R}}$$ Lefschetz operator

$$\alpha \longmapsto \omega \wedge \alpha$$

$$\Lambda : \Omega^k_{X,\mathbb{R}} \longrightarrow \Omega^{k-2}_{X,\mathbb{R}}$$ formal adjoint

$$\Lambda = *^{-1} \circ L \circ *$$

Lemma: $[L, \Lambda] = (k-n) \cdot \mathrm{id}_{\Omega^k}$

Idea of Proof: L, Λ are C^∞_X-linear

$\Rightarrow$ enough to do $(U, \omega_{\text{standard}})$

$$\omega_{\text{standard}} = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

$\Rightarrow$ direct computation $\square$

Rmk: $(L, \Lambda, *\mathrm{id})$ is an sl_2-triple

Lemma: $L^{n-k} : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}$

isomorphism of vector bundles

Proof: $\mathrm{rk}(\Omega^k_{X,\mathbb{R}}) = \binom{2n}{k} = \mathrm{rk}(\Omega^{2n-k}_{X,\mathbb{R}})$

$\Rightarrow$ enough to prove that

L^{n-k} is injective

claim: $[L^r, \Lambda] = (r(k-n) + r(r-1)) L^{r-1}$

(use induction on r + $r=1$ case)

Assume α section of $\Omega^k_{X,\mathbb{R}}$

s.t. $L^r(\alpha)=0$, $r\leq n-k$

$\Rightarrow (L^r\circ\Lambda)(\alpha)-(r(k-n)+r(r-1))L^{r-1}(\alpha)=0$

$\Rightarrow L^{r-1}(L\circ\Lambda-(r(k-n)+r(r-1))(\alpha)=0$

induction on r $\Rightarrow (L\circ\Lambda-\underbrace{(r(k-n)+r(r-1))}_{\neq 0}\cdot id)(\alpha)=0$

$\Rightarrow \exists\beta$ section of $\Omega^{k-2}_{X,\mathbb{R}}$ s.t.

$\alpha=L(\beta)$, $L^{r+1}(\beta)=0$

induction on k $\Rightarrow$ $\beta=0$, $\alpha=0$

$\Rightarrow L^r$ injective for $r\leq n-k$

$\square$

<u>Dfn</u>: α section of $\Omega^k_{X,\mathbb{R}}$, $k\leq n$

is <u>primitive</u>, if $L^{n-k+1}(\alpha)=0$

<u>Prop</u>: α as above.

$\exists!$ α_r primite sections of

$\Omega^{k-2r}_{X,\mathbb{R}}$, $k-2r\leq\inf(2n-k,k)$

s.t.: $\alpha=\sum_r L^r\alpha_r$

<u>Proof</u>: May assume $k \leq n$

<u>Uniqueness</u>: $\sum_r L^r d_r = 0$

Case (i): $d_0 = 0$

$$\Rightarrow L\left(\sum_r L^{r-1} d_r\right) = 0$$

$$\Rightarrow \sum_r L^{r-1} d_r = 0$$

$$\underset{\text{on } r}{\overset{\text{induct.}}{\Rightarrow}} \quad d_r = 0 \quad \forall r$$

Case (ii) $d_0 \neq 0$, $L^{n-k+1} d_0 = 0$

$$\Rightarrow L^{n-k+1}\left(\sum_{r>0} L^r d_r\right) = 0$$

$$= L^{n-k+2}\left(\sum_{r>0} L^{r-1} d_r\right)$$

$$\Rightarrow d_r = 0 \;, \quad r > 0$$

$$\Rightarrow d_0 = 0$$

$\Rightarrow$ Uniqueness

Existence: α section of $\Omega^k_{X,\mathbb{R}}$

Lemma $\Rightarrow \exists \beta$ section of $\Omega^{k-2}_{X,\mathbb{R}}$

s.t. $L^{n-k+2}\beta = L^{n-k+1}\alpha$

$\Rightarrow \alpha_0 := \alpha - L\beta$ primitive

$\Rightarrow \alpha = \alpha_0 + L\beta$

$\overset{\text{induction}}{\underset{\text{on } r}{\Rightarrow}} \alpha = \sum L^r \alpha_r$

$\Rightarrow$ existence

$\square$

- Using $[L,\Lambda] = (k-n)\cdot id$, can show

Fact: α is primitive $\Leftrightarrow \Lambda\alpha = 0$

"highest/lowest weight vectors"

2. Lefschetz decomposition for cohomology

- $L : H^k(X, \mathbb{R}) \longrightarrow H^{k+2}(X, \mathbb{R})$

$$[\alpha] \longmapsto [\alpha \wedge \omega]$$

cup product with $[\omega] \in H^2(X, \mathbb{R})$

Thm (Hard Lefschetz):

If X is compact, then

$$L^{n-k} : H^k(X, \mathbb{R}) \xrightarrow{\sim} H^{2n-k}(X, \mathbb{R})$$

for all $k \leq n$

Corollary (Lefschetz decomposition)

Every $\alpha \in H^k(X, \mathbb{R})$ admits unique decomposition

$$\alpha = \sum_r L^r \alpha_r \quad , \quad \alpha_r \in H^{k-2r}(X, \mathbb{R})$$

$$k - 2r \leq \inf(n, 2n-k)$$

s.t. $L^{n-k+2r+1} \alpha_r = 0$

Proof: Same as for differential forms (using Hard Lefschetz)

Need the following

Lemma: $[\Delta_d, L] = 0$

i.e. Laplacian commutes

with Lefschetz

Proof: $[\Delta_d, L] = [2\Delta_\partial, L]$

$= 2([\partial\partial^*, L] + [\partial^*\partial, L])$

$= 2(\partial[\partial^*, L] + [\partial^*, L]\partial)$

(use that $\partial\omega = 0$
$\Rightarrow [\partial, L] = 0$)

- Kähler identities $\Rightarrow [\partial^*, L] = -i\cdot\bar{\partial}$

$\Rightarrow [\Delta_d, L] = 2(\partial\circ(-i\bar{\partial}) + (-i\cdot\bar{\partial})\circ\partial) = 0$

as $\partial, \bar{\partial}$ anticommute

$\square$

Proof of Hard Lefschetz

Lemma $\Rightarrow L^{n-k}: \mathcal{H}^k(X) \longrightarrow \mathcal{H}^{2n-k}(X)$

$$\mathcal{H}^k(X) = \{ \alpha \mid \Delta_d \alpha = 0 \}$$

- Hodge theorem:

$$\dim_{\mathbb{C}} \mathcal{H}^k(X) = \dim_{\mathbb{R}} H^k(X, \mathbb{R}) \underset{\text{duality}}{\overset{\text{Poincaré}}{=}} \dim_{\mathbb{R}} H^{2n-k}(X, \mathbb{R}) = \dim_{\mathbb{C}} \mathcal{H}^{2n-k}(X)$$

$\Rightarrow$ enough to prove that

L^{n-k} injective

(already proved) $\square$

3. Hodge index theorem

- X compact kähler

  $\dim X = n$

- intersection form

$$Q : H^k(X, \mathbb{R}) \otimes H^k(X, \mathbb{R}) \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \longmapsto \int_X \omega^{n-k} \wedge \alpha \wedge \beta$$

symmetric / alternating

k even / odd

$\Rightarrow H_k(\alpha, \beta) = i^k Q(\alpha, \overline{\beta})$

Hermitian form on

$H^k(X, \mathbb{C})$

<u>Thm</u>: (i) $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$

is orthogonal for H_k

(ii) $(-1)^{\frac{k(k-1)}{2}} \cdot i^{p-q-k} H_k$ is

positive definite on

$H^{p,q}(X)_{prim} := H^k(X, \mathbb{C})_{prim} \cap H^{p,q}(X)$

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Thm (Hodge index theorem)

signature of Q on $H^k(X,\mathbb{R})$ equals

$$\sum_{p,q} (-1)^p h^{p,q}(X)$$

where $h^{p,q}(X) := \dim H^{p,q}(X)$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\updownarrow \qquad \updownarrow$$

$$L \qquad \Lambda$$