Introduction to Hodge theory

03/06/2021

Part 2 - Lecture 4

Oxford

Last time:

- $\mathcal{F} \mapsto \mathcal{F}^{an}$ exact, coherent $\mapsto$ coherent
- GAGA theorem:

  If X is a projective variety, then:

  (1) $\forall\, k \geq 0$, $H^k(X, \mathcal{F}) \xrightarrow{\sim} H^k(X^{an}, \mathcal{F}^{an})$

  (2) $\mathrm{Coh}_X \to \mathrm{Coh}_{X^{an}}$ is an equivalen
  $\mathcal{F} \mapsto \mathcal{F}^{an}$ of categories. ce

- Applications: Chow, Betti $\otimes\, \mathbb{C}$ is algebraic

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4) GAGA (cont.)

Proof of GAGA (outline):

(1) $H^k(X, \mathcal{F}) \xrightarrow{\sim} H^k(X^{an}, \mathcal{F}^{an})$

- Can assume $X = \mathbb{P}^N$: given $i : X \hookrightarrow \mathbb{P}^N$,

  $H^k(X, \mathcal{F}) = H^k(\mathbb{P}^N, i_* \mathcal{F})$

  $\hookrightarrow$ extension of $\mathcal{F}$ by 0

- Claim holds for $\mathcal{F} = \mathcal{O}$:

$$H^0(\mathbb{P}^N, \mathcal{O}) = H^0(\mathbb{P}^N(\mathbb{C}), \mathcal{O}) = \mathbb{C}$$

$$H^k(\mathbb{P}^N, \mathcal{O}) = H^k(\mathbb{P}^N(\mathbb{C}), \mathcal{O}) = 0 \quad \forall\, k \geq 1$$

- Claim holds for $\mathcal{F} = \mathcal{O}(n)$:

D = hyperplane $\simeq \mathbb{P}^{N-1}$

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

$$0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \to \mathcal{O}_D(n) \to 0$$

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(\mathbb{P}^N, \mathcal{O}(n-1)) \arrow[r] \arrow[d] & H^k(\mathbb{P}^N, \mathcal{O}(n)) \arrow[r] \arrow[d] & H^k(D, \mathcal{O}(n)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^k(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(n-1)) \arrow[r] & H^k(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(n)) \arrow[r] & H^k(D^{an}, \mathcal{O}(n)) \arrow[r] & \cdots
\end{tikzcd}$$

Induction on N + 5-lemma.

- For general $\mathcal{F}$, use

$$0 \to \mathcal{G} \to \mathcal{O}(n)^{\oplus r} \to \mathcal{F} \to 0$$

and 5-lemma with <u>decending</u> induction on k.

(2.1) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}^{an}, \mathcal{G}^{an})$ (fully faithfulness)

- $\mathcal{H}om(\mathcal{F}, \mathcal{G})^{an} \xrightarrow{\sim} \mathcal{H}om(\mathcal{F}^{an}, \mathcal{G}^{an})$, by flatness
- $H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Use (1).

(2.2) $\mathcal{M} \in Coh(X^{an}) \rightsquigarrow \exists\, \mathcal{F} \in Coh(X)$, $\mathcal{M} \simeq \mathcal{F}^{an}$

- Reduce to $X = \mathbb{P}^N$
- Show that $\mathcal{M}(n)$ is generated by its global sections $\forall\, n \gg 0$

  Idea: induction on dimension

  $$0 \to K \to \mathcal{M}(-1) \to \mathcal{M} \to \mathcal{M}|_D \to 0$$

  Twist by $\mathcal{O}(n)$, and prove

  $$H^0(\mathbb{P}^N(\mathbb{C}), \mathcal{M}(n)) \twoheadrightarrow H^0(D^{an}, \mathcal{M}(n))$$

  $\forall\, n \gg 0$. [One needs: $\dim H^0(X^{an}, \mathcal{M}) < \infty$]
- Use $\mathcal{O}(d_1)^{\oplus n_1} \to \mathcal{O}(d_2)^{\oplus n_2} \to \mathcal{M}^{an} \to 0$.

$\square$

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5) Complex tori and abelian varieties

<u>Def</u> A <u>complex</u> <u>torus</u> is a connected compact <u>complex</u> <u>Lie</u> <u>gp</u>.

$\hookrightarrow$ G complex manifold with holomorphic

$$\mu : G \times G \to G \quad , \quad i : G \to G$$

Ex V = finite dimensional $\mathbb{C}$-vector space

$L \subset V$ lattice (cocompact discrete subgp)

$\Rightarrow X = V/L$ is a complex torus.

Thm Any complex torus is of the above form.

Proof (sketch): Consider

$$\exp : \mathrm{Lie}\, X \longrightarrow X \qquad \text{(bihol. at neighb. of } 0\text{)}$$

Compactness $\Rightarrow$ gp law is commutative $\Rightarrow$ exp is a morphism $\Rightarrow \begin{cases} (1) & \exp \text{ is surjective} \\ (2) & \ker(\exp) \text{ is discrete} \end{cases}$

so that $X \simeq (\mathrm{Lie}\, X)/\ker(\exp)$. $\square$

Note:

$$\mathrm{Lie}\, X \simeq \Gamma(X, \Omega^1_X)^\vee$$

$$\left[\begin{array}{ll} (\mathrm{Lie}\, X)^\vee \otimes \mathcal{O}_X \xrightarrow{\sim} \Omega^1_X & \omega \in \Omega^1_X(e) \rightsquigarrow \tilde{\omega} \in \Gamma(X, \Omega^1_X) \\ \quad \| & \\ (T_e X)^\vee = \Omega^1_X(e) & \tilde{\omega}(x) = \tau_{-x}^* \omega \\ & \\ \text{Then, use } \Gamma(X, \mathcal{O}_X) = \mathbb{C} & \end{array}\right]$$

Then,

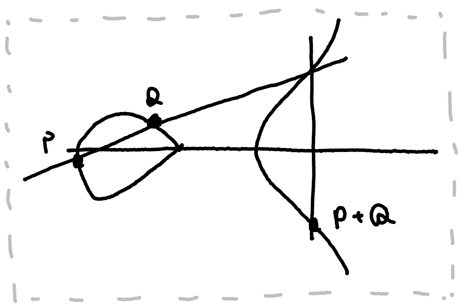
$$\ker(\exp) \simeq H_1(X, \mathbb{Z}) \hookrightarrow \operatorname{Lie} X \simeq \Gamma(X, \Omega^1)^\vee$$
$$\gamma \mapsto \int_\gamma : \omega \mapsto \int_\gamma \omega$$

so that we have an exact seq.

$$0 \to H_1(X, \mathbb{Z}) \to \operatorname{Lie} X \xrightarrow{\exp} X \to 0$$
$$0 \to L \to V \to V/L \to 0$$

<u>Ex</u> $E \subset \mathbb{P}^2$ elliptic curve : $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$

$g_2, g_3 \in \mathbb{C}, \quad g_2^3 - 27 g_3^2 \neq 0$



$$E^{an} \xrightarrow{\sim} \Gamma(E, \Omega^1)^\vee / H_1(E^{an}, \mathbb{Z})$$
$$P \mapsto \left(\omega \mapsto \int_0^P \omega\right)$$

$$\Gamma(E, \Omega^1) = \mathbb{C}\, dx/y$$
$$L = \left\{ \int_\gamma \frac{dx}{y} \;\middle|\; \gamma \in H_1(E^{an}, \mathbb{Z}) \right\} \rightsquigarrow E^{an} \xrightarrow{\sim} \mathbb{C}/L$$

Inverse (exponential) given by elliptic fcts

$$\begin{aligned} \mathbb{C} &\to E^{an} \\ z &\mapsto \begin{cases} [\wp_\Lambda(z) : \wp_\Lambda'(z) : 1], & z \notin L \\ [0:1:0] = O, & z \in L \end{cases} \end{aligned}$$ //