Introduction to Hodge theory — 10/06/2021

Part 2 - Lecture 6 — Oxford

Last time:

- Complex tori are Kähler
- $H^{p,q}(X) = \Lambda^p(\mathrm{Lie}\,X)^\vee \otimes \Lambda^q(\overline{\mathrm{Lie}\,X})^\vee$
- Cat. of complex tori $\xrightarrow[H_1]{\text{equiv.}}$ torsion-free Hodge str of wt -1

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5) Complex tori and abelian varieties (cont.)

<u>Def</u> An <u>abelian variety</u> (over $\mathbb{C}$) is a connected smooth projective group variety over $\mathbb{C}$.

If X is an abelian variety, then X^{an} is a complex torus. By GAGA a complex torus is (the analytification of) an abelian variety iff it embeds in some $\mathbb{P}^N$.

Ex Every complex torus of dim $=1$ is an abelian variety: $\mathbb{C}/L \hookrightarrow \mathbb{P}^2$, $z \mapsto \begin{cases} [\wp_L(z) : \wp_L'(z) : 1], & z \notin L \\ [0:1:0], & z \in L \end{cases}$

Ex $X = \mathbb{C}^2/L$, where

$$L = \mathbb{Z}(1+\sqrt{-2}, \sqrt{-3}) + \mathbb{Z}(\sqrt{-5}, 1+\sqrt{-7}) + \mathbb{Z}(\sqrt{-11}, \sqrt{-13}) + \mathbb{Z}(\sqrt{-17}, \sqrt{-19})$$

isn't projective!

$$\begin{pmatrix} 1+\sqrt{-2} & \sqrt{-5} & \sqrt{-11} & \sqrt{-17} \\ \sqrt{-3} & 1+\sqrt{-7} & \sqrt{-13} & \sqrt{-19} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{5} & \sqrt{11} & \sqrt{17} \\ 0 & 1 & 0 & 0 \\ \sqrt{3} & \sqrt{7} & \sqrt{13} & \sqrt{19} \end{pmatrix}$$

Back to general theory...

Recall: Fubini-Study metric on $\mathbb{P}^N(\mathbb{C})$

$\rightsquigarrow \omega_{FS} \in A^{1,1}(\mathbb{P}^N(\mathbb{C}))$

Restriction to $U_0 = \mathbb{C}^N \subset \mathbb{P}^N(\mathbb{C})$:

$$\omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log(1 + |z_1|^2 + \cdots + |z_N|^2)$$

Lemma Under the de Rham isomorphism $H^2_{dR}(\mathbb{P}^N(\mathbb{C})) \simeq H^2(\mathbb{P}^N(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C}$, we have $[\omega_{FS}] \in H^2(\mathbb{P}^N(\mathbb{C}), \mathbb{Z})$.

Proof: $H_2(\mathbb{P}^N(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}\,\delta$, δ = fundamental class of $\{[z_0:z_1:0:\dots:0]\in\mathbb{P}^N(\mathbb{C}) \mid [z_0:z_1]\in\mathbb{P}^1(\mathbb{C})\} \cong \mathbb{P}^1(\mathbb{C})$

We have: $\displaystyle\int_\delta \omega_{FS} =$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\frac{4r\,dr\,d\theta}{(1+r^2)^2} = 1 \qquad . \qquad \square$$

If $i : M \hookrightarrow \mathbb{P}^N(\mathbb{C})$, then $\omega := i^*(\omega_{FS})$ is a Kähler form st $[\omega]\in H^2(M,\mathbb{Z})$

<u>Thm</u> (Kodaira) A compact complex manifold M is projective if and only if it admits a Kähler form ω st. $[\omega]\in H^2(M,\mathbb{Z})$. $\square$

Back to complex tori ...

By Kodaira, a complex torus X is (the analytification of) an abelian variety iff there is a Kähler form $\omega \in A^{1,1}(X)$ st $[\omega]\in H^2(X,\mathbb{Z})$.

Note:

$$H^2(X,\mathbb{Z}) \cong \Lambda^2 H^1(X,\mathbb{Z}) \cong \Lambda^2\,\mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$$

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\begin{tikzcd}
H^2(X,\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, hook] & \Lambda^2 H^1(X,\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, hook] & \Lambda^2 \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) \arrow[d, hook] \\
H^2(X,\mathbb{Z})\otimes\mathbb{C} \arrow[r, "\simeq"] & \Lambda^2 H^1(X,\mathbb{Z})\otimes\mathbb{C} \arrow[r, "\simeq"] & \Lambda^2 \mathrm{Hom}_{\mathbb{Z}}(H_1(X,\mathbb{Z}),\mathbb{C}) \arrow[d, equal] \\
 & \mathcal{H}^2(X) \arrow[ul, "dR", "\sim"'] \arrow[r, "\simeq"] & \Lambda^2 \mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}\,X,\mathbb{C}) \\
 & \mathcal{H}^{1,1}(X) \arrow[u, hook] \arrow[r, "\simeq"] & (\mathrm{Lie}\,X)^\vee \otimes \overline{(\mathrm{Lie}\,X)}^\vee \arrow[u, hook]
\end{tikzcd}
$$

So that ω corresponds to a <u>Riemann</u> <u>form</u> on the lattice $H_1(X,\mathbb{Z}) \subset \mathrm{Lie}\,X$.

<u>Def</u> Given $L \subset V$, a Riemann form is a Hermitian product $H : V \times V \to \mathbb{C}$ st the symplectic pairing $E = \mathrm{Im}\, H : V \times V \to \mathbb{R}$ satisfies $E(L,L) \subset \mathbb{Z}$.

<u>Ex</u> $\tau \in \mathfrak{h}_g = \{ M \in M_{g\times g}(\mathbb{C}) \mid M^T = M,\ \mathrm{Im}(M) > 0 \}$ ↝ Siegel upper half-space

$L_\tau = \mathbb{Z}^g + \tau\mathbb{Z}^g \subset \mathbb{C}^g$ lattice

$H_\tau(z,w) = z\,(\mathrm{Im}\,\tau)^{-1}\,\overline{w}$ Riemann form on L_τ

$\rightsquigarrow$ $\mathbb{C}^g / L_\tau$ is an abelian variety.

Rk H_τ is principal : E_τ induces $L_\tau \xrightarrow{\sim} L_\tau^\vee$.

Exercise Show that every complex torus with a principal Riemann form is isomorphic to some $(\mathbb{C}^g/L_\tau, H_\tau)$ and $\tau, \tau' \in \mathfrak{h}_g$ define isomorphic tori (with Riemann form) if and only if $\exists\, \gamma \in Sp_{2g}(\mathbb{Z})$ st $\tau' = \gamma\tau$

$\left[\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \gamma\tau = (A\tau + B)(C\tau + D)^{-1} \right]$

Note: $\dim \mathfrak{h}_g = \frac{g(g+1)}{2}$ (dim "space of tori" $= g^2$)

Def A polarization of a Hodge structure H of wt k is a morphism of Hodge structures

$$Q : H \otimes H \to \mathbb{Z}(-k)$$

(where $\mathbb{Z}(-k) = (2\pi i)^{-k}\mathbb{Z}$ carries a HS of wt $2k$ with $\mathbb{Z}(-k)^{k,k} = \mathbb{C}$) st

$$\begin{aligned} H_{\mathbb{R}} \otimes H_{\mathbb{R}} &\to \mathbb{R} \\ x \otimes y &\mapsto (2\pi i)^k Q(x, Cy) \end{aligned}$$

Weil operator : $Cx = i^{p-q}x$ for $x \in H^{p,q}$

is symmetric and positive definite.

Ex H Riemann form on a complex torus X

then $Q := 2\pi i \operatorname{Im} H : H_1(X,\mathbb{Z}) \otimes H_1(X,\mathbb{Z}) \to 2\pi i \mathbb{Z}$

is a polarization on $H_1(X)$.

Thm The functor $A \mapsto H_1(A^{an})$ is an equivalence between the category of abelian varieties and the category of polarizable torsion-free Hodge structures of wt -1. $\square$