Ink note

Notebook:	DGT		
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5.5. Finite extensions:

- (k,∂) differential field

  $P(X) = \sum_{i=0}^{n} a_i X^i \in k[X]$ irreducible
- Want $\partial_L : k[X] \longrightarrow k[X]$, $\partial_{L|k} = \partial$

  s.t. $(P) \subset k[X]$ differential ideal
- $P^{\partial}(X) := \sum_{i=0}^{n} \partial(a_i) X^i$

  $h \in k[X]$ s.t. $h \cdot P' \equiv 1 \ (P)$
- Define: $\partial_L(X) = -h \cdot P^{\partial}$

Claim: $\partial_L(P) \in (P)$

Indeed, $\partial_L(P) = P^{\partial} + P' \cdot \partial_L(X)$

$$= P^{\partial}(1 - h \cdot P')$$
$$= P^{\partial}(1 - (1 - \tilde{h} \cdot P))$$
$$= P^{\partial} \cdot \tilde{h} \cdot P$$

$\Rightarrow \partial_L(P) \in (P)$

Claim $\Rightarrow (P) \subset k[X]$ diff. ideal

$\Rightarrow (L, \partial_L) = (K[X]/(p), \partial_L)$ extension of (K, ∂)

Ex. 11: - Show that, if $\alpha \in L$

$q(X)$ minimal polynomial of α

then $\partial_L(\alpha) = - \dfrac{q^{\partial}(\alpha)}{q'(\alpha)}$

- Conclude that $\partial_L : L \longrightarrow L$ is the unique derivation extending $\partial : K \longrightarrow K$

- Want to show: L/K Galois $\Rightarrow (L, \partial_L)/(K, \partial)$ PV

- Need following lemma

Lemma 5.7: $(L, \partial)/(K, \partial)$ differential field extension, $x_1, \dots, x_n \in k' := \ker(\partial) \subset L$ algebraically dependent $/K$.

Then: $x_1, \dots, x_n$ algebraically dependent over $k = \ker(\partial) \subset K$

In particular, if k alg. closed

$\Rightarrow k' = k$

<u>Proof:</u> $\mathcal{B} = \{e_i\}_{i \in I}$ k-basis of K

$\Rightarrow \mathcal{B}$ also $k[X_1, \dots, X_n]$-basis for $K[X_1, \dots, X_n]$

- Algebraic dependence of $x_1, \dots, x_n$

$\Rightarrow \exists f \in K[X_1, \dots, X_n] \setminus \{0\}$ s.t. $f(x_1, \dots, x_n) = 0$

- Write $f = \sum_{i=1}^{m} h_i \cdot e_i$, $e_i \in \mathcal{B}$, $h_i \in k[X_1, \dots, X_n]$

- Lemma 3.12 $\Rightarrow w(e_1, \dots, e_i) \neq 0$

$\overset{\text{Lemma 3.12}}{\Longrightarrow} e_1, \dots, e_i$ linearly independent over k'

- $0 = f(x_1, \dots, x_n) = \sum_{i=1}^{m} \underbrace{h_i(x_1, \dots, x_n)}_{\in k'} \cdot e_i$

$\Rightarrow h_i(x_1, \dots, x_n) = 0 \quad \forall\, 1 \le i \le m$

$\Rightarrow x_1, \dots, x_n$ algebraically dependent over k. $\square$

<u>Prop 5.8:</u> • (K, ∂) differential field,

$k \subset K$ algebraically closed

L/k finite Galois extension

Then: $(L, \partial_L)/(k, \partial)$ PV-extension

w. ∂_L as above.

<u>Proof:</u> $G = \mathrm{Gal}(L/k)$

Have $\sigma \circ \partial_L = \partial_L \circ \sigma$, $\forall \sigma \in G$

(both $\sigma^{-1} \circ \partial_L \circ \sigma$ and ∂_L extend

$\partial : k \to k$ to L

Ex. 11 $\Rightarrow \sigma^{-1} \circ \partial_L \circ \sigma = \partial_L$)

$\Rightarrow G \subset \mathrm{Aut}_\partial(L/k)$

- Let $P(X) \in k[X]$ s.t.

  $L =$ splitting field of $P(X)$

- $X = \{a \in L \mid P(a) = 0\}$

  $V = \mathrm{span}_k X$, $\dim_k V = |X| < \infty$

Then: (1) $L = k\langle V \rangle$

(2) $G(V) \subseteq V$, $L^G = k$

(3) L/k has no new constants

(Lemma 5.4)

$\Rightarrow (L, \partial_L)/(k, \partial)$ PV-extension

$\square$

Note: Can construct $\mathcal{L}$ for which L/k is PV, by Prop. 5.1.