Introduction to Hodge theory

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Part 2 - Lecture 1

Oxford

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1) Hodge structures

<u>Thm</u> If M is a compact Kähler manifold, then

(i) $H^k(M,\mathbb{Z})\otimes\mathbb{C} \simeq \bigoplus_{p+q=k} H^{p,q}(M)$,

$$\overline{H^{p,q}(M)} = H^{q,p}(M)$$

(ii) $H^{p,q}(M) \simeq H^q(M, \Omega^p_M)$

Notation: $h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}(M)$ Hodge numbers

Hodge diamond ($n = \dim_{\mathbb{C}} M$)

Hodge symmetry

$$\begin{array}{ccccccccc}
 & & & & H^{0,0} & & & & \\
 & & & H^{1,0} & & H^{0,1} & & & \\
 & & H^{2,0} & & H^{1,1} & & H^{0,2} & & \\
 & \cdot^{\cdot^{\cdot}} & & & & & & \ddots & \\
H^{n,0} & & H^{n-1,1} & & \cdots & & H^{1,n-1} & & H^{0,n} \\
 & \ddots & & & & & & \cdot^{\cdot^{\cdot}} & \\
 & & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,1} & & \\
 & & & H^{n,n-1} & & H^{n-1,n} & & & \\
 & & & & H^{n,n} & & & &
\end{array}
\quad \overset{\dim}{\longrightarrow} \quad
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{0,2} & & h^{1,1} & & h^{0,2} & & \\
 & \cdot^{\cdot^{\cdot}} & & & & & & \ddots & \\
h^{n,0} & & h^{n-1,1} & & \cdots & & h^{1,n-1} & & h^{0,n} \\
 & \ddots & & & & & & \cdot^{\cdot^{\cdot}} & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,1} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & &
\end{array}$$

Serre duality

$\underline{\text{Ex}}$ M = compact Riemann surface ($\dim_{\mathbb{C}} M = 1$)

[any Hermitian metric is Kähler]

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array} \qquad\qquad g = \dim H^0(M, \Omega^1_M)$$

$\underline{\text{Ex}}$ $M = \mathbb{P}^n$, ω = Fubini - Study

Fact: $H^k(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n \text{ even} \\ 0, & \text{otherwise} \end{cases}$

[Induction: $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

$$H^k(\mathbb{P}^n, \mathbb{P}^{n-1}) = H^k_c(\mathbb{P}^n \setminus \mathbb{P}^{n-1}) = H_{2n-k}(\underbrace{\mathbb{P}^n \setminus \mathbb{P}^{n-1}}_{\mathbb{C}^n}) = \begin{cases} 0, & k \neq 2n \\ \mathbb{Z}, & k = 2n \end{cases}$$

$$\cdots \to H^k(\mathbb{P}^n, \mathbb{P}^{n-1}) \to H^k(\mathbb{P}^n) \to H^k(\mathbb{P}^{n-1}) \to H^{k+1}(\mathbb{P}^n, \mathbb{P}^{n-1}) \to \cdots]$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & \cdot^{\cdot^{\cdot}} & & & \vdots & & & \ddots & \\ 0 & & \cdots & & 1 & & \cdots & & 0 \\ & \ddots & & & \vdots & & & \cdot^{\cdot^{\cdot}} & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Note:

1) $H^{p,p}(\mathbb{P}^n) = \mathbb{C}[\omega^p]$
$\qquad \cong$
$\qquad H^p(\mathbb{P}^n, \Omega^p)$

2) $H^q(\mathbb{P}^n, \Omega^p) \cong \begin{cases} \mathbb{C}, & p = q \leq n \\ 0, & \text{otherwise} \end{cases}$

Ex $M \subset \mathbb{P}^4$ quintic (e.g. $z_0^5 + z_1^5 + \cdots + z_4^5 = 0$)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$h^{1,1}(M) = 1$, $h^{1,2}(M) = 101$

$\hookrightarrow$ moduli of complex str.

$\hookrightarrow$ moduli of symplectic str.

Mirror W satisfies $h^{1,1}(W) = 101$, $h^{1,2}(W) = 1$.

Def A Hodge structure of wt $k \in \mathbb{Z}$ is a lattice H with a decomposition

$$H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q} \quad , \quad H^{p,q} = \overline{H^{q,p}}$$

Ex M cpct Kähler $\left(H^k(M,\mathbb{Z}), \left(H^{p,q}(M)\right)_{p,q}\right)$

- Topology of cpct Kähler
- $M \hookrightarrow \mathbb{P}^N$ projective $\rightsquigarrow$ algebraic geometry

"when is an elt of $H_k(M,\mathbb{Z})$ the fundamental class of an alg. cycle $\Gamma \subset M$?"

Hodge conjecture ...

- Functoriality :

$$\begin{tikzcd} \text{Smooth proj. alg. var}/\mathbb{C} \arrow[r] \arrow[d, hook] & \text{Hodge structures} \arrow[d, hook] \\ \text{Alg var }/\mathbb{C} \arrow[r] & \text{Mixed Hodge str.} \end{tikzcd}$$

Analogies with ℓ-adic cohomology ...

- Families $\rightsquigarrow$ variations of Hodge structures

Limiting MHS, period domains, Shimura varieties, ...

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2) Coherent analytic sheaves

Complex manifold : $(M, \mathcal{O}_M)$

- M Hausdorff paracompact
- $\mathcal{O}_M$ sheaf of complex fcts.

locally isom. to $(U, \mathcal{O}_{\mathbb{C}^n|U})$, $U \subset \mathbb{C}^n$ open

$\hookrightarrow$ holomorphic fcts.

Morphism $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$:

$f : M \to N$ continuous st $\forall\, U \subset N$ open

$$f^* : \mathcal{O}_N(U) \to \mathcal{O}_M(f^{-1}(U))$$
$$g \mapsto g \circ f$$

<u>Def</u> A subset $X \subset \mathbb{C}^n$ is <u>analytic</u> if $\forall\, x \in X$ $\exists\, U \subset \mathbb{C}^n$ open neighb. of x and $f_1, \dots, f_m \in \mathcal{O}_{\mathbb{C}^n}(U)$ st $X \cap U = \{ z \in U \mid f_1(z) = \dots = f_m(z) = 0 \}$.

It inherits a sheaf of hol. fct. $\mathcal{O}_X$ from $\mathcal{O}_{\mathbb{C}^n}$.

<u>Def</u> An <u>analytic variety</u> is a pair $(X, \mathcal{O}_X)$, X Hausdorff paracompact, locally isomorphic to $(Y, \mathcal{O}_Y)$, $Y \subset \mathbb{C}^n$ is an analytic subset.

<u>Rk:</u> A complex manifold is an analytic variety. Conversely, a <u>smooth</u> analytic variety (locally, $(\partial f_i / \partial z_j)_{i,j}$ has maximal rank) is a complex manifold by the "analytic implicit fct thm".

<u>Ex</u> (Subvariety of a complex manifold)

M complex manifold, $X \subset M$ closed st

$M = \bigcup_i U_i$, $\quad X \cap U_i = \{ p \in M \mid f_1(p) = \cdots = f_{n_i}(p) = 0,\ f_j \text{ holo} \}$

$\rightsquigarrow (X, \mathcal{O}_X)$ analytic variety.

e.g. • $X = \{ (z, e^z) \mid z \in \mathbb{C} \} \subset \mathbb{C}^2$

• $X = \{ (z_0 : \cdots : z_n) \mid F_i(z) = 0 \ \ \forall 1 \le i \le k \} \subset \mathbb{P}^n$

(Projective variety) //