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Lecture 3: 3.1: Basic constructions.

Conventions:

- All rings are commutative.
- All fields have characteristic zero.

Dfn 3.1: (i) A *differential ring* is a pair (R, ∂) consisting of a ring R and a derivation $\partial: R \to R$

(ii) A *morphism of differential rings* $(R, \partial_R) \longrightarrow (S, \partial_S)$ is a ring homomorphism $f: R \longrightarrow S$ such that

$$f \circ \partial_R = \partial_S \circ f.$$

If $R \subset S$, f the canonical embedding (S, ∂_S) is called an *extension* of (R, ∂_R).

(iii) (R, ∂_R) differential ring. A <u>differential (R, ∂_R)-algebra</u> is a pair (A, ∂_A) consisting of an R-algebra A and a derivation $\partial_A : A \longrightarrow A$ such that

$$\partial_A(ra) = \partial_R(r)a + r\partial_A(a)$$

for all $r \in R$ and $a \in A$.

<u>Remark 3.2</u>: (i) Morphisms of differential algebras are defined analogously to morphisms of differential rings.

(ii) Differential rings = Differential $\mathbb{Z}$-algebras (since ∂ annihilates the image $\mathbb{Z} \longrightarrow R$)

(iii) $\ker(\partial) \subset R$ is a subring, the <u>ring of constants</u> of (R, ∂).

(iv) Let R be a ring. Then:

$$\left\{\begin{array}{c}\text{derivations}\\ \partial: R \to R\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{ring homs.}\\ \varphi: R \to R[\varepsilon]\\ \text{s.t. } \pi\circ\varphi = \mathrm{id}_R\end{array}\right\}$$

where $R[\varepsilon] := R[X]/(X^2)$ <u>ring of dual numbers</u>, and

$$\pi: R[\varepsilon] \longrightarrow R$$
$$r_1 + r_2\cdot\varepsilon \longmapsto r_1$$

$$\partial: R \to R \longleftrightarrow \varphi: R \to R[\varepsilon]$$
$$r \longmapsto r + \partial(r)\cdot\varepsilon$$

<u>Dfn 3.3</u>: Let (R,∂) be a differential ring. A <u>differential ideal</u>

$$\mathfrak{a} \subset R$$

is an ideal s.t. $\partial(\mathfrak{a}) \subset \mathfrak{a}$.

A <u>maximal differential ideal</u>

$$\mathfrak{m} \subset R$$

is a differential ideal $\neq R$ s.t.:

if $\mathfrak{m} \subsetneq \mathfrak{a}$, $\mathfrak{a}$ differential ideal

then $\mathfrak{a} = R$.

<u>Rmk 3.4</u> (i) If (R, ∂) is a differential ring, $\mathfrak{a} \subset R$ a differential ideal then $(R/\mathfrak{a}, \bar{\partial})$, $\bar{\partial} : R/\mathfrak{a} \longrightarrow R/\mathfrak{a}$ is a differential ring

(ii) $\{0\}$, R are always differential ideals of (R, ∂) ("trivial diff. ideals")

(iii) If $R \neq \{0\}$, then maximal differential ideals exist (use Zorn's lemma).

<u>3.1.1</u>: Tensor Product

- (R, ∂_R) differential ring
- (A, ∂_A), (B, ∂_B) differential R-algebras

Then $(A \otimes_R B, \partial)$ with

$$\partial(a \otimes b) := \partial_A(a) \otimes b + a \otimes \partial_B(b)$$

is a differential R-algebra.

Exercise 7: check that ∂ is well

Exercise 4. Check that ∂ is well-defined.

3.1.2: Localization

- (R, ∂) differential ring, $Q \subset R$ multiplicative subset
- $R \longrightarrow Q^{-1}R$ localization of R at Q
- want to extend ∂ to $Q^{-1}R$ uniquely
- $$\begin{aligned} \varphi: R &\longrightarrow R[\varepsilon] \longrightarrow (Q^{-1}R)[\varepsilon] \\ r &\longmapsto r + \partial(r)\cdot\varepsilon \longmapsto \tfrac{r}{1} + \tfrac{\partial(r)}{1}\cdot\varepsilon \end{aligned}$$
  ring homomorphism
- $r + s\cdot\varepsilon \in R[\varepsilon]$ unit $\iff$ $r \in R$ unit
  $\Rightarrow$ $\varphi(Q)$ consists of units
- Universal property of localization

$$\Rightarrow \quad \begin{tikzcd} R \arrow[r] \arrow[dr, "\varphi"'] & Q^{-1}R \arrow[d, dashed, "\exists!\ \text{ring hom. } \tilde{\varphi}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & (Q^{-1}R)[\varepsilon] \end{tikzcd}$$

- $\tilde{\varphi}: Q^{-1}R \longrightarrow (Q^{-1}R)[\varepsilon]$

- $\varphi : Q^{-1}R \longrightarrow (Q^{-1}R)[\varepsilon]$

$$\longleftrightarrow \tilde{\partial} : Q^{-1}R \longrightarrow Q^{-1}R \text{ derivation}$$

$$(\text{extends } \partial : R \longrightarrow R)$$

- explicitly: for $q \in Q$, have

$$0 = \partial(1) = \tilde{\partial}\left(\frac{q}{q}\right) = q\,\tilde{\partial}\left(\frac{1}{q}\right) + \frac{\tilde{\partial}(q)}{q}$$

$$\Rightarrow \tilde{\partial}\left(\frac{1}{q}\right) = -\frac{\partial(q)}{q^2}$$

- Leibniz rule

$$\Rightarrow \tilde{\partial}\left(\frac{r}{q}\right) = \frac{\tilde{\partial}(r)q - r\tilde{\partial}(q)}{q^2}$$

for all $r \in R$, $q \in Q$

3.2: Constants

Lemma 3.5: k field, R integral domain, $S \subset k$ infinite set.

Assume R is a finitely generated k-algebra.

Then, for every $r \in R$, either:

(i) r is algebraic over k

(ii) $\exists s \in S$ and $x \in R$ [illegible]

(ii) $\exists s \in S$ s.t. $r - s \in R$ not a unit

Proof: $\cdot$ $\bar{k}/k$ algebraic closure

- $V \subset \bar{k}^n$ affine algebraic variety corresponding to R
- $r \in R \rightsquigarrow f: V \longrightarrow \mathbb{A}^1_{\bar{k}}$ regular function
- $f(V)$ finite $\Rightarrow f(V) = \{pt\}$ since V irreducible $\Rightarrow r \in \bar{k}$
- $f(V)$ infinite $\Rightarrow U \subset f(V)$, $U \subset \mathbb{A}^1_{\bar{k}}$ open and dense (Chevalley's theorem)

$S \subset k$ infinite $\Rightarrow S \cap f(V) \neq \emptyset$

$\Rightarrow \exists v \in V, s \in S$ s.t. $f(v) = s$

$\Rightarrow f - s$ vanishes somewhere on V

$\Rightarrow r - s$ not a unit $\square$

Rmk 3.6: Lemma 3.5 also holds if R is not an integral domain

(see Mag.a, Lemma 1.16)

Thm 3.7: • (K,∂) differential field,

$k \subset K$ field of constants

• (R,∂) differential integral domain,

finitely generated K-algebra.

• $L = Q(R)$ field of fractions

• $d \in L \setminus k$ constant

If R contains no non-trivial

differential ideal, then

d is algebraic over k.

Proof: • $\mathfrak{a} = \{ r \in R \mid rd \in R \}$

• $\mathfrak{a}$ is a differential ideal

(since d is constant)

• $\mathfrak{a} \neq (0)$ (it contains the denominator

of d)

$\Rightarrow \mathfrak{a} = R$

$\Rightarrow d \in R$

• For every $c \in k$, $(d-c) = R$

$\Rightarrow d-c$ is a unit

Lemma 3.5 $\Rightarrow d$ is algebraic over K

- $P(X) = X^n + a_1 X^{n-1} + \dots + a_0 \in K[X]$

  minimal polynomial of d

- $0 = \partial(P(d)) = \partial(a_1) d^{n-1} + \dots + \partial(a_n)$

  $\Rightarrow d$ is also a zero of

  $$P^\partial(X) = \partial(a_1) X^{n-1} + \partial(a_2) X^{n-2} + \dots + \partial(a_n)$$

  $\Rightarrow P^\partial \equiv 0$

  $\Rightarrow \partial(a_i) = 0 \quad , 1 \leq i \leq n$

  $\Rightarrow P(X) \in k[X]$

$\square$

<u>Cor. 3.8:</u> (K, ∂) differential field

- R differential integral domain

  finitely generated over K

Assume: (i) R has no non-trivial differential ideals

(ii) $k \subset K$ algebraically closed

Then each constant of $L = Q(R)$

is contained in K

Rmk 3.9: Corollary 3.8 important in the construction of Picard-Vessiot extensions

- How to construct R as in Corollary 3.8?
- (R,∂) differential ring
  $\mathfrak{m} \subset R$ maximal differential ideal
  $\Rightarrow R/\mathfrak{m}$ has no non-trivial differential ideals

In fact: If $K \subset R$, $\mathrm{char}(K)=0$, then $R/\mathfrak{m}$ is a domain!

Prop. 3.10: (R,∂) $\mathbb{Z}$-torsion free differential ring

If R has no non-trivial diff. ideals, then R is a domain ideals, then R is a domain.

Proof: • 1st step: Every zero divisor in R is nilpotent.

- $a, b \in R$, $ab = 0$, $a \neq 0$ (so b zero div.)
- $\mathfrak{a} = (a) \subset R$ differential ideal generated by a.
- By assumption, $\mathfrak{a} = R$

$\Rightarrow 1 = \sum_{k=0}^{n} r_k \partial^k(a)$, for some $r_k \in R$

- Since $ab = 0$, can show that

$$\partial^n(a) b^{n+1} = 0$$

(induction on n and

$$^n(ab) = \sum_{k=0}^{n} \binom{n}{k} \partial^k(a) \partial^{n-k}(b)$$)

$\Rightarrow b^{n+1} = b^{n+1} \sum_{k=0}^{n} r_k \partial^k(a) = 0$

$\Rightarrow b$ nilpotent

2nd step: $\sqrt{(0)}$ (the nilradical) is a differential ideal.

- $a \in \sqrt{(0)}$, $a^n = 0$, $a^{n-1} \neq 0$

$\Rightarrow 0 = \partial(a^n) = n\,\partial(a)\,a^{n-1}$

- R $\mathbb{Z}$-torsion free $\Rightarrow \partial(a)$ zero divisor

1st step $\Rightarrow \partial(a)$ nilpotent

- Have shown: $a \in R$ zero divisor

$\Rightarrow a \in \sqrt{(0)}$

and $\sqrt{(0)}$ is a differential ideal

$\Rightarrow \sqrt{(0)} = (0)$, and $a = 0$

$\Rightarrow R$ is a domain $\square$

3.3. The Wronskian.

(K, ∂) differential field.

Dfn 3.11: For $Y_1, \dots, Y_n \in K$,

define the **Wronskian**

$$W(Y_1,\dots,Y_n) := \begin{vmatrix} Y_1 & Y_2 & \dots & Y_n \\ \partial(Y_1) & \partial(Y_2) & \dots & \partial(Y_n) \\ \vdots & & & \\ \partial^{n-1}(Y_1) & \partial^{n-1}(Y_2) & \dots & \partial^{n-1}(Y_n) \end{vmatrix}$$

Exercise 8: show that

$W(aY_1, \dots, aY_n) = a^n W(Y_1, \dots, Y_n)$

$W(ay_1,\dots,ay_n) = a^n W(y_1,\dots,y_n)$

for all $a \in K$, $y_1,\dots,y_n \in K$

Lemma 3.12: $y_1,\dots,y_n \in K$ k-linearly dependent

$\Longleftrightarrow W(y_1,\dots,y_n) = 0$

Proof: "$\Rightarrow$": $\sum_{i=1}^{n} c_i y_i = 0$, $c_i \in k$

$$\Rightarrow \sum_{i=1}^{n} c_i \partial^j(y_i) = 0 \quad , \forall j \geq 0$$

$$\Rightarrow W(y_1,\dots,y_n) = 0$$

"$\Leftarrow$": · Induction on $n \geq 1$

· If $n=1$, have $y_1 = 0$

$\Rightarrow$ linear dependence

· Assume for all $1 \leq m < n$, have

$W(y_1,\dots,y_m) = 0 \Rightarrow y_1,\dots,y_m$ k-lin. dependent

· If $y_n = 0 \Rightarrow y_1,\dots,y_n$ k-linearly dependent

· If not, then

$$0 = W(y_1,\dots,y_n) = (y_n)^n W(y_1/y_n, \dots, y_{n-1}/y_n, 1)$$

$$= (-1)^{n-1} (y_n)^n W(\partial(y_1/y_n), \dots, \partial(y_{n-1}/y_n))$$

• Induction hypothesis

$\Rightarrow \exists\, c_1, \dots, c_{n-1} \in k$, not all zero, s.t.

$$c_1 \partial(y_1/y_n) + \dots + c_{n-1} \partial(y_{n-1}/y_n) = 0$$

$$\Rightarrow \sum_{i=1}^{n-1} c_i \frac{y_i}{y_n} \in k$$

$\Rightarrow y_1, \dots, y_n$ k-linearly dependent

$\square$

<u>Thm 3.13:</u> $\mathcal{L} = \partial^n + a_1 \partial^{n-1} + \dots + a_n \cdot \mathrm{id} \in K[\partial]$

Then $\dim_k \mathcal{L}^{-1}(0) \leq n$

<u>Proof</u>: Let $y_1, \dots, y_{n+1} \in \mathcal{L}^{-1}(0)$

Have $\partial^n(y_j) = -\sum_{i=0}^{n-1} a_i \partial^i(y_j)$, $1 \leq j \leq n+1$

$\Rightarrow W(y_1, \dots, y_{n+1}) = 0$

Lemma 3.12 $\Rightarrow y_1, \dots, y_{n+1}$

k-linearly dependent

$\Rightarrow \dim_k \mathcal{L}^{-1}(0) \leq n$

$\square$