Ink note

Notebook:	DGT		
Created:	6/2/2020 4:51 PM	Updated:	6/4/2020 10:59 AM
Author:	Nils Matthes		

Lecture 12: Liouvillian extensions

Prop 12.1:
- (K,∂) diff. field
- $k := \ker(\partial) \subset K$ alg. closed
- $(L,\partial)/(K,\partial)$ no new constants extension
- $K \subseteq M \subseteq L$ intermediate diff. field
- $K \subseteq N \subseteq L$ —— " ——
- Assume: M/K PV-extension

Then: MN/N PV-extension

and $G(MN/N) \hookrightarrow G(M/K)$

whose image has Zariski closure

equal to $G(M/M\cap N)$

Proof: $MN \cong Q(\mathrm{Im}(M \otimes_K N \to L))$
$$m\otimes n \mapsto mn$$

$\Rightarrow MN$ diff. field

• $V = \mathcal{L}^{-1}(0) \subset M$, $\dim_k V < \infty$ and

$M = K\langle V\rangle$ (PV for some $\mathcal{L}$)

$\Rightarrow MN = N\langle V\rangle$

• Also, MN/N no new constants since L/K ——"——

$\Rightarrow MN/N$ PV-extension for $\mathcal{L}$

• $\sigma \in G(MN/N)$. Prop 4.1

$\mathrm{Im}(\sigma_{|M}) = M$

$\Rightarrow G(MN/N) \longrightarrow G(M/K)$

$\sigma \longmapsto \sigma_{|M}$

well-defined

• also injective since $\sigma|M$ determines σ

• $H := \mathrm{Im}\left(G(MN/N) \hookrightarrow G(M/K)\right)$

Then $M^H \subset MN^{G(MN/N)} = N$

$\Rightarrow M^H \subset M \cap N$

Conversely, $M \cap N \subset M^H$

$\Rightarrow M^H = M \cap N$

- Exc. 24

Zariski closure

$\Rightarrow \overline{\mathrm{Im}(G(MN/N) \hookrightarrow G(M/k))} = G(M/M\cap N)$

$\square$

<u>Exc. 24:</u> In the context of Thm 11.4.(ii), show that if $H \subseteq G(L/k)$ any subgroup, then $G(L/L^H) = \overline{H}$

Before treating Liouvillian extensions, need some facts about virtually solvable groups.

<u>Dfn 12.2:</u> A linear algebraic group G over a field is <u>virtually solvable</u> if G^0 is solvable

- Note $1 \to G^0 \to G \to \underbrace{G/G^0}_{\text{finite}} \to 1$

Fact: Let G/k virtually solvable, k alg. closed.

$U \trianglelefteq G$ unipotent radical

Then: (i) $G^\circ = U \cdot T$, for any maximal torus T of G°. Get

$\{e\} \trianglelefteq U \trianglelefteq G^\circ \trianglelefteq G$

series of characteristic subgroups

(ii) U has a normal series whose successive quotients are $\mathbb{G}_a$'s

(iii) T has a normal series whose successive quotients are $\mathbb{G}_m$'s

· Combined with the main theorem, we get:

<u>Prop 12.3</u>: · $(L,\partial)/(K,\partial)$ PV-extension

· $k = \ker(\partial)$ alg. closed

Assume: $G := G(L/K)$ virtually solvable

unipotent radical

· $n := \dim(G)$, $m := \dim(U)$

Then: $\exists$ chain of subfields

Then: $\exists$ chain of subfields

$$K = K_0 \subseteq K_1 \subseteq \dots \subseteq K_{n+1} = L$$

with $K_{i+1} = K_i(a_i)$

where: (1) a_0 algebraic over K

(2) a_i purely transcendental

$\partial(a_i)/a_i \in K_i$, $1 \leq i \leq n-m$

(3) a_i purely transcendental

$\partial(a_i) \in K$ $n-m+1 \leq i \leq n$

<u>Idea of Proof:</u> $\{e\} \trianglelefteq U \trianglelefteq G^\circ \trianglelefteq G$

$\Rightarrow L \supseteq L^U \supseteq L^{G^\circ} \supseteq L^G = K$

chain of PV-extensions

w. groups G/G°, G°/U, U

(1) L^{G°/K algebraic , G/G° finite

Primitive element thm:

$\Rightarrow \exists a_0 \in L^{G^\circ} : L^{G^\circ} = K(a_0)$

(2) $G^\circ/U \cong T$ maximal torus

$\Rightarrow G^\circ/U$ has normal series

whose quotients are $\mathbb{G}_m$

$\Rightarrow L^{U}/L^{G^\circ}$ obtained by iteratively adding a_i, $\partial(a_i)/a_i \in K_i$

(3) U has normal series whose quotients are $\mathbb{G}_a$

$\Rightarrow L/L^{U}$ obtained by iteratively adding a_i, $\partial(a_i) \in K_i$ $\square$

- So if L/K PV-extension, $G(L/K)$ virtually solvable, then L/K obtained from three elementary operations:

(1) adding a root

(2) adding an exponential

(3) adding a primitive

Conversely:

<u>Thm 12.4</u>: $\cdot$ (K,∂) diff. field

- $k = \ker(\partial)$ alg. closed

Let M/K diff. field extension

with no new constants,

such that

$K = K_0 \subseteq K_1 \subseteq \dots \subseteq K_{n+1} = M$

$K_{i+1} = K_i(a_i)$

where for all i, either:

(1) a_i algebraic over K_i

or (2) $\partial(a_i)/a_i \in K_i$

or (3) $\partial(a_i) \in K_i$

Then: If $M \supseteq L \supseteq K$, L/K PV

then $G(L/K)$ virtually solvable.

__Proof:__ Induction on n. $n=0$ ✓

- $N = K_1 = K(a_0)$

Prop 12.1 $\Rightarrow$ LN/N PV-extension

contained in $M = N(a_1, \dots, a_n)$

Induction hypothesis

$\Rightarrow$ $G(LN/N)$ virtually solvable

Prop 12.1 $\Rightarrow$ $G(L/L\cap N)$ virt. solvable

- Now consider $N \supseteq L \cap N \supseteq K$

- $N = K(a_0)/K$ is PV,

$$G(N/K) = \begin{cases} \text{finite} & (1) \\ \mathbb{G}_m & (2) \\ \mathbb{G}_a & (3) \end{cases}$$

- Case (1) $G(N/K)$ finite

$\Rightarrow L \cap N / K$ finite alg.

$\Rightarrow G(L/L \cap N) \subseteq G(L/K)$

finite index

$\Rightarrow G(L/K)$ virtually solv.

- Case (2), (3): Then $L \cap N / K$

PV-extension by Thm 11.4

(each subgroup of $\mathbb{G}_a, \mathbb{G}_m$ is normal))

- Have

$$1 \to \underbrace{G(L/L \cap N)}_{\text{virt. solv.}} \to G(L/K) \to \underbrace{G(L \cap N/K)}_{\text{abelian}} \to 1$$

$\Rightarrow G(L/K)$ virtually solvable.

$\square$

Dfn 12.5: M/K diff. field

extension is Liouvillian if

it is of the form described

it is of the form described in Thm 12.4.

- Most extensions we've seen so far are Liouvillian.

Non-example: PV-extension for the Airy equation

$$\mathcal{L} = \partial^2 - x \cdot \mathrm{id}$$

$K = \mathbb{C}(x)$

- This has group Sl_2 not solvable

Integration in finite terms

classical problem: Given a function f satisfying some linear diff. equation, is it an elementary function?

- Elementary function: a function which is a composite of

1) rational functions

2) exponential, 3) logarithm

More precisely:

Dfn 12.6: k alg. closed field

$\cdot (K, \partial) = (k(x), d/dx)$.

i) $K(a_1, \dots, a_n; b_1, \dots, b_m) \supseteq K$ is a

field of elementary functions

if for $K_j = K(b_1, \dots, b_{j-1})$:

1) $\partial(a_i) \in K$, $1 \leq i \leq n$

2) for $1 \leq j \leq m$, either $\partial(b_j)/b_j \in K_j$

or b_j algebraic over K_j.

ii) An **elementary function** is

an element of a field as in i)

Prop 12.7: Notation as in 12.6,

let L/K be a PV-extension,

$L \subseteq K(a_1, \dots, a_n; b_1, \dots, b_m)$.

Then: $G^0(L/K)$ is abelian

Idea of proof: $\cdot$ $K(a_1, \dots, a_n)/K$

is a PV-extension for a family of lin. diff. operators (namely $\delta_i = \partial^2 - \frac{\partial^2(a_i)}{\partial(a_i)}\partial$, if $\partial(a_i) \neq 0$)

- Can show: $G_u = G(k(u_1,\dots,u_n)/k)$ embeds into $\mathbb{G}_a^n$

  $\Rightarrow G_u$ is a vector group

  (i.e., is connected, abelian, unipotent)

- On the other hand:

  $T := G(L/k(a_1,\dots,a_n))$ is such that

  T° is a torus

  and is normal in G°

  (since $k(u_1,\dots,u_n)/k$ PV)

- $$\begin{tikzcd}
1 \arrow[r] & T \arrow[r] & G(L/k) \arrow[r] & G_u \arrow[r] & 1 \\
1 \arrow[r] & T \cap U^\circ \arrow[u, hook] \arrow[r] & U^\circ \arrow[u, hook] \arrow[r] & G_u \arrow[u, equal] \arrow[r] & 1
\end{tikzcd}$$

  $T \cap U^\circ = \{e\}$ since T torus and U° unipotent

$\Rightarrow U^\circ \cong G_u$

$\Rightarrow U^\circ$ normalizes (even centralizes) T°

U°, T° abelian

$\Rightarrow U^\circ T^\circ \cong U^\circ \times T^\circ$

- Finally, $U^\circ T^\circ \subseteq G^\circ(L/k)$ finite index

$\Rightarrow U^\circ T^\circ = G^\circ(L/k)$

$\Rightarrow G^\circ(L/k)$ abelian $\square$

Exc. 25: Notation as in Dfn. 12.6, show that

$$y = \int e^{-x^2} dx$$

is <u>not</u> an elementary function over $k = \mathbb{C}$ w. trivial derivation.

a) Verify that $(K, \partial) = (k(x), d/dx)$ is a PV-extension of $(k, 0)$ with $G(K/k) = \mathbb{G}_a$

b) Show that $\mathcal{L}(y) = 0$, where

$$\mathcal{L} = \partial^2 + 2x\partial \in K[\partial].$$

Deduce that, if L/K is a PV-extension for $\mathcal{L}$, then

$V_i = \mathcal{L}^{-1}(0) = \mathbb{C} \oplus \mathbb{C}y$

c) Let $w = w(1, y)$, $M := k(w)$

show that $\partial(w) = -2x\, w$

and that $w \notin k$

(Hint: If $w \in k$, $w = \frac{p}{q}$, $p, q \in k[X]$

choose p of minimal degree

and differentiate $p = wq$)

d) show that $y \notin M$

(Hint: argue similarly as in c))

e) Use Exmp. 8.6 to deduce

that $G(L/k) \cong \mathbb{G}_a \rtimes \mathbb{G}_m$

$$\cong \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & \mu \end{pmatrix} \;\middle|\; \begin{matrix} \alpha \in \mathbb{G}_a \\ \mu \in \mathbb{G}_m \end{matrix} \right\}.$$

<u>Abelian extensions:</u>

Fact: If G/k commutative

algebraic group, then

$G \cong \mathbb{G}_a^p \times \mathbb{G}_m^q \times \mathcal{F}$

$\mathcal{F}$: finite group

$\mathcal{F}$: finite group, $p, q \geq 0$

Thm 12.8: (K, ∂) diff. field,

- $k = \ker(\partial)$ alg. closed
- L/K PV-extension

Suppose $G(L/K) \cong \mathbb{G}_a^p \times \mathbb{G}_m^q \times \mathcal{F}$ finite group

Then: $L = K(a_1, \ldots, a_p; b_1, \ldots, b_q; c)$

where:
1) $\partial(a_i) \in K$
2) $\partial(b_i)/b_i \in K$
3) c alg. over K, $\deg(c) = |\mathcal{F}|$
4) $a_1, \ldots, a_p, b_1, \ldots, b_q$ transcendental over $K(c)$.

Proof: Omitted (see Magid, Thm 6.15)