Lecture 2:

Review of complex analysis

1. Holomorphic functions in one variable

- $U \subset \mathbb{C} \cong \mathbb{R}^2$ open

  $f : U \longrightarrow \mathbb{C}$ smooth

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- $u \in U$, $df_u : T_{U,u} = \mathbb{C} \longrightarrow \mathbb{C}$

  $\mathbb{R}$- linear

Defn: f is <u>holomorphic</u> if

df_u is $\mathbb{C}$-linear, $\forall u \in U$

- write $z = x + i \cdot y$, $\bar{z} = x - iy$

now $df = \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right)}_{\frac{\partial f}{\partial z}} dz + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i \cdot \frac{\partial f}{\partial y}\right)}_{\frac{\partial f}{\partial \bar{z}}} d\bar{z}$

Fact: (i) f holomorphic

$\iff \frac{\partial f}{\partial \bar{z}}(u) = 0 \quad \forall u \in U$

(Cauchy–Riemann equations)

(ii) f, g holomorphic

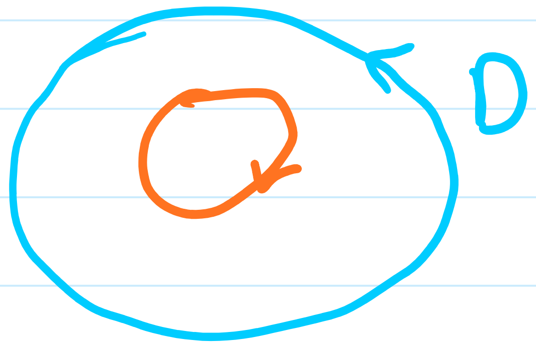
$\Rightarrow f+g,\ f \cdot g,\ f \circ g$ holomorphic

if $f(u) \neq 0 \quad \forall u \in U$

$\Rightarrow \frac{1}{f}$ holomorphic

2. Cauchy's formula

- Let $D \subset U$ closed disk at $z_0 \in U$
- $D_\varepsilon \subset D$, radius $\varepsilon > 0$
- $\partial(D - D_\varepsilon) = \partial D \cup \partial D_\varepsilon$ opposite orientation



Theorem (Cauchy):

$f : U \to \mathbb{C}$ holomorphic

Then: $f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} d\zeta$

Proof: $\frac{f(\zeta)}{\zeta - z_0} d\zeta$ closed

i.e. $d\left(\frac{f(\zeta)}{\zeta - z_0} d\zeta\right) = 0$

Stokes $\Rightarrow \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z_0} d\zeta$ natural orientation

Now: $\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z_0} d\zeta$

$= \lim_{\varepsilon \to 0} \int_0^1 f(z_0 + \varepsilon \cdot e^{2\pi i t}) dt$

$= \int_0^1 f(z_0) dt$

$= f(z_0)$ $\square$

3. Several complex variables

$U \subset \mathbb{C}^n$ open

$f : U \longrightarrow \mathbb{C}$ smooth

Defn: f is <u>holomorphic</u>

if $df_u : \mathbb{C}^n \longrightarrow \mathbb{C}$

is $\mathbb{C}$-linear, $\forall u \in U$

Theorem: TFAE

(i) f is holomorphic

(ii) For all $z_0 \in U$, have

$$f(z_0 + z) = \sum_I \alpha_I z^I$$

$I = (i_1, \dots, i_n)$, $i_k \geq 0$

$z^I = z_1^{i_1} \dots z_n^{i_n}$, $\alpha_I \in \mathbb{C}$

such that: $\exists\, R_1, \dots, R_n > 0$

$\sum_I |\alpha_I| r^I$ converges, for all $0 \leq r_1 < R_1, \dots, 0 \leq r_n < R_n$

(iii) If $D = \{ (\zeta_1, \dots, \zeta_n) \mid |\zeta_i - a_i| \leq \alpha_i \}$

poly-disk, then:

$\forall \ z = (z_1, \dots, z_n) \in D^\circ$, have:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\prod_{i=1}^{n} |\zeta_i - a_i| = \alpha_i} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \dots \wedge \frac{d\zeta_n}{\zeta_n - z_n}$$

<u>Idea of Proof</u>:

(iii) => (ii) => (i) OK

(i) => (iii) Careful application

of Stokes' theorem gives:

$$\frac{1}{(2\pi i)^n} \int_{\prod_{i=1}^{n} |\zeta_i - a_i| = \alpha_i} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \dots \wedge \frac{d\zeta_n}{\zeta_n - z_n}$$

$$= \frac{1}{(2\pi i)^n} \int_{\prod_{i=1}^{n} |\zeta_i - a_i| = \varepsilon} \text{———} \ '' \ \text{———}$$

- Now take $\lim_{\varepsilon \to 0}$ $\square$

4. $\bar{\partial}$-Poincaré lemma (in one variable)

Theorem: $f: U \to \mathbb{C}$, $U \subset \mathbb{C}$ open

Then: locally on U,

$\exists$ g smooth function

$$\frac{\partial g}{\partial \bar{z}} = f$$

Proof: May assume that

f has compact support

Set $$g(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{C} - D_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

D_ε disk centered at z

well-defined, since $(\zeta - z)^{-1}$ is

locally integrable

$\Rightarrow$ g well-defined

- $g(z)$ is also smooth

(differentiate under integral sign)

Now

change of variables

$$\frac{\partial g}{\partial \bar z}(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{C} - D_\varepsilon} \frac{\partial f}{\partial \bar\zeta}(\zeta) \frac{d\zeta \wedge d\bar\zeta}{\zeta - z}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{C} - D_\varepsilon} -d\left(\frac{f(\zeta)}{\zeta - z} d\zeta \right)$$

Stokes

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= f(z)$$

$\square$

<u>Next time:</u>

- complex manifolds

holomorphic vector bundles

- differential forms