Introduction to Hodge theory — 14/06/2021

Part 2 - Lecture 7 — Oxford

Last time:

- Abelian varieties
- Kodaira's thm
- Polarizable Hodge structures of wt -1

6) Lefschetz (1,1) theorem

<u>Thm</u> (Lefschetz)

X smooth, projective
$\dim X = 2$

If X is an algebraic surface, then a class $\delta \in H_2(X^{an}, \mathbb{Z})$ is algebraic iff $\int_\delta \omega = 0$ $\forall\, \omega \in \Gamma(X, \Omega^2_X)$.

$\delta = cl(C)$, where

- $C \subset X$ algebraic curve
- $\int_X cl(C) \wedge \alpha = \int_{C^{smooth}} \alpha$, $\forall\, [\alpha] \in H^2_{dR}(X^{an})$

Rk 1) Lefschetz also proved: algebraic equivalence = homological equivalence ($\Rightarrow$ Picard nb is a top. invariant).

2) $PD(\gamma) \in H^2(X^{an}, \mathbb{Z})$, $\int_\gamma \alpha = \int_{X^{an}} PD(\gamma) \wedge \alpha$

$\int_\gamma \omega = 0 \ \forall \omega \in H^{2,0}(X) \iff PD(\gamma) \in H^{1,1}(X)$.

↳ Exercise!

Let M be a compact complex manifold.

Def A divisor on M is a formal sum

$$D = \sum_{i=1}^{r} n_i [Y_i] \quad , \quad n_i \in \mathbb{Z}$$

where $Y_i \subset M$ are irreducible closed analytic subvarieties of codim 1. Group: $\mathrm{Div}(M)$.

↳ i.e. locally given by $f=0$, f holo.

Def A meromorphic function over $U \overset{\text{open}}{\subset} M$ is a collection $f = \{f_x\}_{x \in U}$, $f_x \in \mathrm{Frac}(\mathcal{O}_{M,x})$, "locally given by g/h", with g, h holomorphic.

Sheaf: $\mathcal{K}_M$

We define a $\mathbb{Z}$-linear map

$$\Gamma(M, \mathcal{K}_M^\times / \mathcal{O}_M^\times) \longrightarrow \mathrm{Div}(M)$$

$$\varphi \mapsto \mathrm{div}(\varphi) = \sum_Y n_Y [Y]$$

where n_Y is given locally by

$$f = g^{n_Y} u$$

where $[f] = \varphi$, $\{g=0\} = Y$, $u \in \mathcal{O}^\times$.

<u>Lemma</u> the map $\varphi \mapsto \mathrm{div}(\varphi)$ is an isomorphism.

"Proof" Inverse map $D = \sum_i n_i [Y_i] \mapsto \varphi_D$ locally given by $\varphi_D = [f]$, $f = \prod_i g_i^{n_i}$, g_i local eqn for Y_i. $\square$

From

$$0 \to \mathcal{O}_M^\times \to \mathcal{K}_M^\times \to \mathcal{K}_M^\times / \mathcal{O}_M^\times \to 0$$

we get a map

$$\mathrm{Div}(M) \simeq \Gamma(M, \mathcal{K}_M^\times / \mathcal{O}_M^\times) \longrightarrow H^1(M, \mathcal{O}_M^\times) \simeq \mathrm{Pic}(M)$$

line bundles $/\sim$

$$D \mapsto [\mathcal{O}(D)]$$

where $\mathcal{O}(D) \subset \mathcal{K}_M^\times$ is locally generated by f^{-1}, where $D = [f]$.

<u>Prop.</u> If M is projective, then $\mathrm{Div}(M) \twoheadrightarrow \mathrm{Pic}(M)$.

Proof: Note: if $s \in \Gamma(M,L)\setminus\{0\}$, then

$\mathrm{div}(s) = \{x \in M \mid s(x)=0\} \in \mathrm{Div}(M)$ and $L \simeq \mathcal{O}(\mathrm{div}(s))$.

Let $[L] \in \mathrm{Pic}(M)$ and $\mathcal{O}(1)$ given by $M \hookrightarrow \mathbb{P}^N$.

$\mathcal{O}(1)$ ample $\overset{\text{GAGA}}{\Longrightarrow}$ $\exists\ n \gg 0$ st

$$\Gamma(M, L(n)) \neq 0, \quad \Gamma(M, \mathcal{O}(n)) \neq 0$$

$\Rightarrow$ $[L(n)]$, $[\mathcal{O}(n)] \in \mathrm{im}(\mathrm{Div}(M) \to \mathrm{Pic}(M))$

$\Rightarrow$ $[L] = [L(n)] - [\mathcal{O}(n)] \in \mathrm{im}(\mathrm{Div}(M) \to \mathrm{Pic}(M))$. $\square$

<u>Rk</u>: Not true that $H^1(M, \mathcal{K}_M^\times) = 0$!

(cf. Chen - Kerr - Lewis '10)

The exponential sequence

$$0 \longrightarrow \mathbb{Z}_M \to \mathcal{O}_M \xrightarrow{\exp(2\pi i\,\cdot)} \mathcal{O}_M^\times \to 0$$

induces a map

$$\mathrm{Pic}(M) \cong H^1(M, \mathcal{O}_M^\times) \to H^2(M, \mathbb{Z})$$

$$[L] \mapsto c_1(L)$$

<u>Lemma</u> If M is compact Kähler, then the maps

$H^k(M, \mathbb{C}) \to H^k(M, \mathcal{O}_M)$ given by $\mathbb{C}_M \hookrightarrow \mathcal{O}_M$ and by the projection onto $H^{0,2}(M)$ coincide.

Proof: The diagram

$$\begin{tikzcd}
\mathbb{C}_M \arrow[r] \arrow[d, hook] & A^0_M \arrow[r, "d"] \arrow[d, equal] & A^1_M \arrow[r, "d"] \arrow[d] & A^2_M \arrow[r] \arrow[d] & \cdots \\
\mathcal{O}_M \arrow[r] & A^0_M \arrow[r, "\bar\partial"] & A^{0,1}_M \arrow[r, "\bar\partial"] & A^{0,2}_M \arrow[r] & \cdots
\end{tikzcd}$$

commutes. $\square$

Since $c_1(L)$ maps to 0 in $H^2(M,\mathcal{O})$, and it is invariant under cplx conjugation in $H^2(M,\mathbb{Z})\otimes\mathbb{C}$, it follows from Hodge symmetry that $c_1(L) \in H^{1,1}(M)$.

<u>Thm</u> If M is compact Kähler, then every class in $H^{1,1}(M) \cap H^2(M,\mathbb{Z})$ is of the form $c_1(L)$ for some $[L] \in \mathrm{Pic}(M)$. $\square$

If M is projective, then $[L] = [\mathcal{O}(D)]$ for some divisor $D \hookrightarrow M$ (algebraic by GAGA).

<u>Thm</u> If $D = \sum_i n_i [Y_i]$, then

$$\int_M c_1(\mathcal{O}(D)) \wedge \alpha = \sum_i n_i \int_{Y_i^{\text{smooth}}} \alpha$$

$\forall\, \alpha \in H^{n-2}(M,\mathbb{C})$. //

"Poincaré–Lelong"

Next time