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Lecture 10

Prop 9.9: $F = \bar{F}$ field, X_i/F, $i=1,2$ affine algebraic sets.

Assume: X_1, X_2 have same dimension and number of connected components.

- $H \subset Aut(H_1) \cap Aut(H_2)$ subgroup, s.t. X_i no proper closed H-stable subsets, and $\mathcal{O}(X_i)^H = F$.

Then: If $\phi : X_1 \to X_2$ H-equiv. $\Rightarrow$ ϕ isom.

Proof: • $f : R_2 \longrightarrow R_1$, $R_i := \mathcal{O}(X_i)$ ring hom. associated to φ

• R_i has no proper H-stable ideals

$\Rightarrow \ker(f) = (0)$

H-stable

$\Rightarrow \phi$ is dominant

- $X_{1,0} \subset X_1$, connected component
  $X_{2,0} \subset X_2$, connected component
  $$X_{2,0} \supset \phi(X_{1,0})$$
- $H_0 := \mathrm{Stab}(X_{1,0}) \subset H$
  $h_1, \dots, h_n$ coset rep. of H_0 in H
  s.t. $X_{1,i} = h_i(X_{1,0})$, $i = 1, \dots, n$
- By equivariance,
  $$h\,\phi(X_{1,0}) = \phi(X_{1,0}), \quad h \in H_0$$
  $\Rightarrow H_0$ stabilizes $X_{2,0}$
- Now $\mathrm{stab}(X_{2,0}) \subset H$ has index
  n ($= \#$ conn. comp. of X_2),
  by Prop 9.8.(iii)
  and $H_0 \subset \mathrm{Stab}(X_{2,0})$
  $$\Rightarrow H_0 = \mathrm{Stab}(X_{2,0})$$
- Moreover, $X_{2,i} = h_i(X_{2,0})$
  same h_i as above
- Prop 9.8.(i) $\Rightarrow X_{1,0}, X_{2,0}$ have

as proper closed G-stable subsets

$\Rightarrow \phi : X_{1,0} \longrightarrow X_{2,0}$ dominant

- $\dim X_1 = \dim X_2$ and $h_i(X_{j,0}) = X_{j,i}$

$\Rightarrow \dim(X_{1,0}) = \dim(X_{2,0})$

ϕ dominant $\Rightarrow \phi$ generically finite
($\phi^{-1}(\eta)$ finite, η generic point)

$\Rightarrow \{x \in X_{1,0} \mid \dim \phi^{-1}(\phi(x)) > 0\}$
proper closed, and H_0-stable

$\Rightarrow$ empty

$\Rightarrow \phi$ quasi-finite ($\phi^{-1}(x)$ finite $\forall x \in X_{2,0}$)

- $X_{1,0}$, $X_{2,0}$ non-singular

Zariski's main theorem

$$\Rightarrow X_{1,0} \overset{i}{\hookrightarrow} \overline{X}_{1,0} \xrightarrow{\overline{\phi}} X_{2,0}$$

open immersion finite

- Can show (?): $\overline{\phi}$ is an iso. (How?)
(Would follow from ϕ gen. 1-1)

$\Rightarrow \phi : X_{1,0} \hookrightarrow X_{2,0}$ open immersion

- Complement of $\operatorname{im}(\phi)$ H_0-stable, closed

$\Rightarrow \phi$ isomorphism

$\Rightarrow \phi : X_1 \longrightarrow X_2$ isomorphism

$\square$

Will only need Prop 9.9. once

Notation: · $(L,\partial)/(K,\partial)$ PV-extension w. alg. closed field of constants k

· $G(L/K) := \mathrm{Aut}_\partial(L/K)$

· $T = T(L/K) = \{a \in L \mid \exists\, \ell \in K[\partial] \text{ s.t. } \ell(a) = 0\}$

"canonical domain"

Prop 10.1: T is a finitely generated K-algebra

Proof: · $G := G(L/K)$, $n := |G/G^\circ|$

· R k-algebra, $\{e_1, \dots, e_s\}$ minimal idempotents in R,

· Assume G acts "rationally" (i.e. through alg. hom.) on R

· $\mathrm{Stab}(e_1) \subset G$ alg. subgroup of index $\leq s$ (in fact $s \leq n$)

· Let $\overline{K}/K$ alg. closure

$S \subset L$ fin. gen. K-subalgebra

· Noether normalization

$\Rightarrow S/K[y_1, \dots, y_d]$ finite

$d = \dim S$

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- choose $S_1 \subset T(L/k)$ G-stable differential k-algebra, such that:
  - a) S_1/k fin. generated
  - b) $Q(S_1) = L$
  - c) $R_1 := S_1 \otimes_k \bar{k}$ has maximal number of min. idempotents (possible as their number is bounded by n) note: $\dim R_1 = \dim S_1$
- Cor 9.6. $\Rightarrow$ can assume $\mathfrak{S} \subset S_1$
  $\Rightarrow S_1$ has no non-trivial G-stable ideals
- Now $T(L/k) \supseteq S_2 \supseteq S_1$ be a second G-stable, fin. gen. diff. k-algebra
- Noether normalization
  $\Rightarrow \dim S_2 = \dim S_1$
- Also, $R_2 := S_2 \otimes_k \bar{k}$ has
  - a) $\dim R_2 = \dim R_1$
  - b) same number of minimal

idempotents as R_1 (by maximality)

- Prop 9.9 $\Rightarrow R_1 \hookrightarrow R_2$ is an isomorphism

$\Rightarrow R_1 = R_2$

$\Rightarrow T(L/k) = S_1$, so in particular fin. generated.

$\square$

<u>Thm 10.2 (Kolchin):</u>

Notation as above, let

- $G \subseteq \mathcal{G}(L/k)$ closed subgroup
- $T(L/L^G) = \{a \in L \mid \exists \delta \in L^G[\partial], \delta(a) = 0\}$

Then: (i) T is G-stable, fin. gen. differential k-algebra, $Q(T) = L$

(ii) There is a G-equiv. isom. $\bar{k} \otimes_k T \longrightarrow \bar{k} \otimes_k \mathcal{O}(G)$ of differential k-algebras.

<u>Proof</u>: L/L^G PV-extension

$\Rightarrow$ ... $L^G = k$ ...

$\Rightarrow$ can assume $L = k$

(i): follows from Prop 9.2
and 10.1

(ii) action of G on T

$\rightsquigarrow$ coaction $\Delta: T \longrightarrow T \otimes_k \mathcal{O}(G)$

$\rightsquigarrow$ base change $\bar{\Delta}: \bar{T} \longrightarrow \bar{T} \otimes_{\bar{k}} \overline{\mathcal{O}(G)}$

$$\bar{T} := \bar{k} \otimes_k T, \quad \overline{\mathcal{O}(G)} := \bar{k} \otimes_k \mathcal{O}(G)$$

- choose a morphism

$$\begin{tikzcd} f: T \arrow[r] \arrow[d, hook] & \bar{k} \\ \bar{T} \arrow[ur, dashed] & \end{tikzcd}$$

$$(f \otimes id) \circ \bar{\Delta}: \bar{T} \longrightarrow \overline{\mathcal{O}(G)}$$

$\rightsquigarrow$ $\varphi_x: \bar{G} \longrightarrow \bar{X}$

$$g \longmapsto gx \quad \text{(G-equivariant)}$$

where $\bar{G} := \bar{k} \otimes_k G$

$\bar{X} := \bar{k} \otimes_k X$ $\leftarrow$ affine variety of T

$x \in \bar{X} \leftrightarrow f: T \to \bar{k}$

- claim: φ_x is bijective

(a) surjectivity:

- Prop 9.4 + 9.7.(iii)

$\Rightarrow X$ has no proper G-stable closed subsets

- $\mathrm{im}(\varphi_x)$ dense + homogeneous

contains open subset, by Chevalley

$\Rightarrow \mathrm{im}(\varphi_x)$ open

$\Rightarrow \bar{X} \setminus \mathrm{im}(\varphi_x)$ proper closed, G-stable

$\Rightarrow \bar{X} = \mathrm{im}(\varphi_x)$

(b) injectivity: $H := \mathrm{stab}(x) \subset \bar{G}$

If $g \in H \cap G$, then Lemma 7.7

$\Rightarrow g = id$

- likewise $H^{g'} \cap G = \{id\}$, $g' \in \bar{G}$

where $H^{g'} := g' H (g')^{-1}$

($y := g'x$ has stabilizer $H^{g'}$

Lemma 7.7 $\Rightarrow H^{g'} \cap G = \{id\}$)

Fact: $H^{g'} \cap G = \{id\} \quad \forall\, g' \in \bar{G}$

$\Rightarrow H^{g'} = \{id\}$

(Proof below)

$\Rightarrow \varphi_x$ injective

$\Rightarrow \varphi_x : \bar{G} \longrightarrow \bar{X}$ bijective

- $\bar{X}$ nonsingular ($\bar{G}$ as well

(sing$(\bar{X})$ proper closed, G-stable.
$\Rightarrow$ sing$(\bar{X}) = \emptyset$)

- Zariski's main theorem
  $\Rightarrow \varphi_x$ isomorphism

$\square$

<u>Prop 10.3:</u> • k alg. closed,
K/k alg. closed extension
- G/k affine algebraic group
  $H \subset G(K)$ closed algebraic subgroup

Then: If $H \neq \{e\}$, then $\exists g \in G(K)$
s.t. $H^g \cap G(k) \neq \{e\}$

<u>Proof:</u> • Since $G \hookrightarrow Gl_n$, some n
can assume $G = Gl_n$
- choose $h \in H$, $h = h_u h_s$
  h_u unipotent, h_s semisimple
- Enough to show: a) $g h_u g^{-1} \in Gl_n(k)$
  or b) $g h_s g^{-1} \in Gl_n(k)$

For a): Jordan normal form
of h_u is in $Gl_n(k)$ ✓

b) can find $g \in GL_n(K)$ s.t.

$g h_s g^{-1} \in D_n(K)$, diagonal torus

Then $H' := \overline{\{g h_s g^{-1}\}} \cap D_n(k) \neq \emptyset$

Can now prove an analogue of the following fact from classical Galois theory:

"If L/k finite Galois extension, then: $\mathrm{Gal}(L/k) = [L:k]$"

Thm 10.4: Notation as above, assume that $\dim G = n$.

Then: (i) $\mathrm{trdeg}_{L^G}(L) = n$

(ii) If L^G algebraically closed, then $L \cong Q(\mathcal{O}(G_{L^G}))$

in particular: L/L^G is purely transcendental

Proof: Let $F := L^G$.

Noether normalization $\Rightarrow \mathrm{trdeg}_F L = \dim T$
$= \dim \overline{T}$

Thm 10.2 $\Rightarrow \dim T = \dim G = n$

This proves (i).

(ii) follows immediately from Thm 10.2

$\square$

- $G = G(L/k)$, L/k PV-extension, $G^\circ \subset G$ connected comp. of $id \in G$
  $L^\circ := L^{G^\circ}$ fixed field of G°
  $\Rightarrow \bar{G} := G/G^\circ$ acts on L° and $(L^\circ)^{\bar{G}} = k$
  $\Rightarrow L^\circ/k$ finite Galois extension
  $\bar{G} \longrightarrow Gal(L^\circ/k)$

<u>Prop. 10.5:</u> Notation as above.

Then: $L^\circ = \{a \in L \mid \exists f \in k[X],\ f(a) = 0\}$
"algebraic closure of k in L"

Moreover: G is connected
$\Leftrightarrow k = L^\circ$ "k algebraically closed in L"

<u>Proof:</u> Already know: L°/k algebraic

Thm 10.2 $\Rightarrow \bar{L^\circ} \otimes_{L^\circ} T(L/L^\circ) \cong \bar{L^\circ} \otimes_k \mathcal{O}(G^\circ)$

G^0 connected and smooth

$\Rightarrow G^0$ irreducible

$\Rightarrow \mathcal{O}(G^0)$ integral domain

$\Rightarrow \overline{L^0} \otimes_k \mathcal{O}(G^0) \cong \mathcal{O}(G^0_{\overline{L^0}})$ int. domain

so $\overline{L^0} \otimes_{L^0} T(L/L^0)$ int. domain

$\Rightarrow \overline{L^0} \otimes_{L^0} L$ int domain

$\Leftrightarrow L^0 \subset L$ algebraically closed

Since, in addition, L^0/K algebraic

$\Rightarrow L^0$ algebraic closure of K in L

- Second statement: Already know

G connected $\Rightarrow L^0 = K$

Conversely: $K \subset L$ alg. closed

$\Rightarrow \overline{K} \otimes_K L$ domain

$\Rightarrow \mathcal{O}(G)$ domain

$\Rightarrow G$ connected

$\square$

Thm. 10.2 has many more consequences

e.g.: (i) determination of

PV-extensions whose group

is $\mathbb{G}_m$, or $\mathbb{G}_a$

(ii) Main theorem of diff. Galois theory

Have seen: $\bar{k} \otimes_k T \xrightarrow{\sim} \bar{k} \otimes_k \mathcal{O}(G)$

Q: Is there an isomorphism over k?

A: If and only if

$$H^1(\bar{k}/k, G(\bar{k})) = 0 \qquad (*)$$

Galois cohomology

Not always the case!

(can fail for G finite)

However, $(*)$ holds if either:

(i) G is solvable

(ii) G is unipotent

(iii) $G = Sl_n, Gl_n$