Introduction to Hodge theory 31/05/2021

Part 2 - Lecture 3 Oxford

Last time:

- Coherent analytic sheaves
  (Oka's theorem, ...)
- Analytification $\left\{ \begin{array}{l} (X, \mathcal{O}_X) \rightsquigarrow (X^{an}, \mathcal{O}_{X^{an}}) \\ \mathcal{F} \rightsquigarrow \mathcal{F}^{an} \end{array} \right.$

Fact: $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \hat{\mathcal{O}}_{X^{an},x}$

Prop (1) $\mathcal{F} \mapsto \mathcal{F}^{an}$ is exact

(2) $\mathcal{F}$ coherent $\Rightarrow$ $\mathcal{F}^{an}$ coherent

---

3) Analytification (cont.)

Proof of prop.:

(1) It suffices to show that $\mathcal{O}_x^{an}$ is flat over $\mathcal{O}_x$ ($\forall x \in X^{an}$). Given an $\mathcal{O}_x$-monomorphism

$$M \hookrightarrow N,$$

let $K = \ker\left(M^{an} = M \otimes_{\mathcal{O}_x} \mathcal{O}_x^{an} \to N^{an} \otimes_{\mathcal{O}_x} \mathcal{O}_x^{an} \right)$.

Since $\hat{\mathcal{O}}_x = \hat{\mathcal{O}}_x^{an}$ is flat over $\mathcal{O}_x$ (resp. $\mathcal{O}_x^{an}$) ↳ Artin-Rees lemma

$$\hat{K} = K \otimes_{\mathcal{O}_x^{an}} \hat{\mathcal{O}}_x = \ker(\hat{M} \to \hat{N}) = 0$$

But $K \hookrightarrow \hat{K}$, so that $K = 0$.

(2) Locally, there is an exact sequence

$$\mathcal{O}_X^{\oplus m} \to \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$$

Since $\mathcal{O}_X^{an} = \mathcal{O}_{X^{an}}$, and $\mathcal{F} \to \mathcal{F}^{an}$ is exact,

we get an exact sequence

$$\mathcal{O}_{X^{an}}^{\oplus m} \to \mathcal{O}_{X^{an}}^{\oplus n} \to \mathcal{F}^{an} \to 0$$

Since $\mathcal{O}_{X^{an}}$ is coherent, $\mathcal{F}^{an}$ is coherent. $\square$

---

4) GAGA (Géométrie algébrique et géométrie analytique)

Let X be an alg. var. and $\mathcal{F}$ be an $\mathcal{O}_X$-module.

Natural

$$\Gamma(X, \mathcal{F}) \to \Gamma(X^{an}, \mathcal{F}^{an})$$

induces

$$H^n(X, \mathcal{F}) \to H^n(X^{an}, \mathcal{F}^{an}) \quad \forall n$$

[use derived functors or Čech cohomology]

Let Coh_X (resp. $Coh_{X^{an}}$) be the category of coherent sheaves on X (resp. X^{an}).

<u>Thm</u> (Serre)

If X is a <u>projective</u> variety, then:

(1) $\forall\ k \geq 0$, $H^k(X, \mathcal{F}) \xrightarrow{\sim} H^k(X^{an}, \mathcal{F}^{an})$

(2) $Coh_X \to Coh_{X^{an}}$ is an equivalence

$\mathcal{F} \mapsto \mathcal{F}^{an}$ of categories.

⚠ Projective (or proper) is essential: $X = \mathbb{A}^1$

$\Rightarrow H^0(X, \mathcal{O}) = \mathbb{C}[x]$ but $H^0(X^{an}, \mathcal{O}) = \mathcal{O}(\mathbb{C}) \neq \mathbb{C}[x]$

Similarly, $I_{\mathbb{Z}} \subset \mathcal{O}_{X^{an}}$ not algebraic.

Applications :

(i) <u>Thm</u> (Chow) Every closed analytic subvariety Y of $\mathbb{P}^n(\mathbb{C})$ is algebraic.

Proof: $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}/I_Y$ is coherent $\Rightarrow \exists\ \mathcal{F}$ algebraic st $\mathcal{F}^{an} \simeq \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}/I_Y$.

Since $\mathrm{supp}(\mathcal{F}) = \mathrm{supp}(\mathcal{F}^{an}) = Y$, we conclude that Y is Zariski-closed. $\square$

<u>Coro.</u> X_1, X_2 projective $\Rightarrow$ any holomorphic $X_1^{an} \xrightarrow{\varphi} X_2^{an}$ is algebraic.

Proof: $\Gamma = \mathrm{Graph}(\varphi) \subset (X_1 \times X_2)^{an}$ closed analytic subvariety $\Rightarrow$ algebraic by Chow. $\square$

<u>Coro.</u> A projective analytic variety has a unique algebraic structure. $\square$

[$X \mapsto X^{an}$ conservative]

⚠ E elliptic curve, X = moduli of $(\mathcal{L}, \nabla)$

Non-projective, <u>non-affine</u> alg. surface.

but $X^{an} \simeq \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^2 = (\mathbb{G}_m^2)^{an}$.

(ii) If X is projective, then

$$H^q(X^{an}, \Omega^p_{X^{an}}) \simeq H^q(X, \Omega^p_{X/\mathbb{C}})$$

$$\Rightarrow \quad H^k(X^{an}, \mathbb{Z}) \otimes \mathbb{C} \simeq \bigoplus_{p+q=k} H^q(X, \Omega^p_{X/\mathbb{C}})$$

Coro. $X \subset \mathbb{P}^n$, $\sigma \in \mathrm{Aut}(\mathbb{C}) \Rightarrow b_k(X) = b_k(X^\sigma)$. $\square$

In general, we can define an <u>algebraic de Rham cohomology</u> $H^k_{dR}(X/\mathbb{C})$, which comes with a spectral seq

(*) $\quad E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Rightarrow E_\infty^k = H^k_{dR}(X/\mathbb{C})$

If X is projective, then

(**) $\quad H^k_{dR}(X/\mathbb{C}) \simeq H^k(X, \mathbb{Z}) \otimes \mathbb{C}$

and (*) degenerates at page 1.

<u>Rk</u> (1) $\exists$ purely alg. proof of the Hodge decomp. (Deligne - Illusie), but not of Hodge symmetry.

(2) (**) is true for any smooth X (Grothendieck's comparison theorem)