Last lecture:

$$H^q(X, \underline{\mathbb{R}}) \cong H^q_{dR}(X) := H^q(\Gamma(X, \mathcal{A}^\bullet))$$

$$\mathcal{A}^\bullet = (\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \to \mathcal{A}^n \to 0)$$

- X complex manifold

  E/X holomorphic vector bundle

  $\rightsquigarrow \mathcal{E}$ sheaf of holomorphic sections of E

- Compute $H^q(X, \mathcal{E})$ via

  <u>Dolbeault resolution</u>

  $\mathcal{A}^{0,q}(E)$ = sheaf of smooth sections of $\Omega_X^{0,q} \otimes E$

- $\mathcal{A}^{0,0}(E) \xrightarrow{\bar\partial_E} \mathcal{A}^{0,1}(E) \xrightarrow{\bar\partial_E} \mathcal{A}^{0,2}(E) \to \cdots$

- $\bar\partial$-Poincaré lemma $\Rightarrow \mathcal{A}^{0,\bullet}(E)$ resolution of $\mathcal{E}$ ✓

Thm: $H^q(X, E) \cong H^q(\Gamma(X, \mathcal{A}^{0,\bullet}(E)))$ ← Dolbeault cohomology

proved as before, i.e., for

$$H^q(X, \underline{\mathbb{R}}) \cong H^q_{dR}(X)$$

4. Comparison with singular cohomology

Thm: X locally contractible topological space.

Then: $H^q_{sing}(X, \mathbb{Z}) \cong H^q(X, \underline{\mathbb{Z}})$

Idea: Consider presheaf on X

$U \mapsto C^\bullet(U, \mathbb{Z})$ singular cochain complex

"resolution of $\mathbb{Z}$"

(can be made precise)

Cor (de Rham): X smooth manifold

Then: $H^q_{sing}(X, \mathbb{R}) \cong H^q_{dR}(X)$

Lecture 6: Harmonic forms

1. The Hodge star operator

- X compact smooth manifold
  g Riemannian metric on X
  (only need symmetric, non-degenerate)
  $\leadsto$ metrics $(\cdot,\cdot)$ on $\Omega^k_{X,\mathbb{R}}$
- Assume (X,g) oriented
  vol_X = volume form

Thm: L^2-metric on $A^k(X)$

$$(\alpha,\beta)_{L^2} := \int_X (\alpha,\beta)\, vol_X$$

$\alpha,\beta\in A^k(X)$

$\leadsto (A^k(X), (\cdot,\cdot)_{L^2})$ Pre-Hilbert space
(not complete)

Now: $m : \Omega^k_{X,x} \cong \mathrm{Hom}(\Omega^k_{X,x}, \mathbb{R})$

$P : \Omega^{n-k}_{X,x} \cong \mathrm{Hom}(\Omega^k_{X,x}, \mathbb{R})$

<u>Dfn</u>: (Hodge Star operator)

$* := P^{-1} \circ m : \Omega^k_X \xrightarrow{\sim} \Omega^{n-k}_X$

isom. of vector bundles

$\rightsquigarrow * : A^k(X) \xrightarrow{\sim} A^{n-k}(X)$

- can be made explicit in local coordinates (messy, signs)

<u>Lemma</u>: (i) $(\alpha, \beta)_{L^2} = \int_X \alpha \wedge * \beta$

for all $\alpha, \beta \in A^k(X)$

(ii) $*^2 = (-1)^{k(n-k)}$

<u>Proof</u>: (i) $(\alpha, \beta) \cdot \mathrm{vol}_X = \alpha \wedge * \beta$

(ii) $\alpha \wedge * \beta = (\alpha, \beta) \mathrm{vol}_X = (* \alpha, * \beta) \mathrm{vol}_X$

$= (* \beta, * \alpha) \mathrm{vol}_X = * \beta \wedge * * \alpha$

$= (-1)^{k(n-k)} * * \alpha \wedge * \beta$ $\square$

<u>Rmk</u>: $\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C}$

$$(\alpha, \beta)\, vol_X = \alpha \wedge * \overline{\beta}$$

extend to Hermitian metric

2. Formal adjoint of d

<u>Dfn</u>: $d^* : A^k(X) \longrightarrow A^{k-1}(X)$

$$d^* = (-1)^k *^{-1} \circ d \circ *$$

(makes sense even if X not compact)

<u>Lemma</u>: $(\alpha, d^*\beta)_{L^2} = (d\alpha, \beta)_{L^2}$

For all $\alpha \in A^{k-1}(X)$, $\beta \in A^k(X)$

<u>Proof</u>: $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{k-1} \alpha \wedge d*\beta$

$$\overset{\text{Stokes}}{\Longrightarrow} (d\alpha, \beta)_{L^2} = \int_X d\alpha \wedge *\beta = (-1)^k \int_X \alpha \wedge d*\beta$$

$$= (\alpha, d^*\beta)_{L^2}$$

$\square$

<u>Rmk</u>: If n even, then

$$d^* = (-1) \cdot * \circ d \circ *, \text{ since } *^2 = (-1)^{k^2}$$

3. Formal adjoints of $\partial, \bar\partial$

- Now X compact, complex mfd.

$$d = \partial + \bar\partial, \quad \partial : A^{p,q}(X) \longrightarrow A^{p+1,q}(X)$$
$$\bar\partial : A^{p,q}(X) \longrightarrow A^{p,q+1}(X)$$

<u>Dfn</u>: $\partial^* = (-1) \cdot * \circ \partial \circ *$

$\bar\partial^* = (-1) \circ * \circ \bar\partial \circ *$

<u>Lemma</u>: $\bar\partial^*$ adjoint to $\bar\partial$:

$$(\bar\partial \alpha, \beta)_{L^2} = (\alpha, \bar\partial^* \beta)_{L^2}$$

<u>Proof</u>: Exercise (use that

$\bar\partial = d$ on $A^{n,n-1}(X)$, $n = \dim X$)

- similar story if E/X holomorphic vector bundle (w. Hermitian metric)

<u>Have isomorphisms</u>:

$\mathbb{C}$-antilinear

$$*_E : \Omega_X^{0,q} \otimes E \xrightarrow{\sim} (\Omega_X^{0,q} \otimes E)^*$$
$$\xrightarrow{\sim} \Omega_X^{n,n-q} \otimes E^*$$
$$\xrightarrow{\sim} \Omega_X^{0,n-q} \otimes K_X \otimes E^*$$

where $K_X := \Omega_X^{n,0} = \Lambda^n \Omega_X^{1,0}$

holomorphic vector bundle

Dfn: (i) L^2-metric on $A^{0,q}(E)$

$$(\alpha,\beta) = \int_X \overset{\text{on } \Omega_X^{0,q}\otimes E}{(\alpha,\beta)} \, \mathrm{vol}_X$$

(ii) $\bar\partial_E^* : A^{0,q}(E) \longrightarrow A^{0,q-1}(E)$

$$\bar\partial_E^* = (-1)^q *_E^{-1} \circ \bar\partial_{K_X\otimes E^*} \circ *_E$$

Lem: $\bar\partial_E^*$ formal adjoint of $\bar\partial_E$

Proof: Exercise

4. The Laplacian and harmonic forms

Dfn: $\Delta_d := d\circ d^* + d^*\circ d \,;\, A^k(X) \longrightarrow A^k(X)$

similarly $\Delta_{\bar\partial}, \Delta_\partial, \Delta_{\bar\partial_E}$

Fact: $\ker \Delta_d = \ker d \cap \ker d^*$

(similarly for $\Delta_\partial, \Delta_{\bar\partial}, \Delta_{\bar\partial_E}$)

Proof: use adjunction

$$(\alpha, \Delta_d \alpha)_{L^2} = (d\alpha, d\alpha)_{L^2} + (d^*\alpha, d^*\alpha)_{L^2} \quad \square$$

<u>Dfn</u>: $\alpha \in A^k(X)$ (Δ_d-) harmonic

if $\Delta_d(\alpha) = 0$ ($\Leftrightarrow d\alpha = d^*\alpha = 0$)

Denote $\mathcal{H}^k \subset A^k(X)$ subspace

of harmonic forms.

<u>Deep fact</u>: (i) $A^k(X) = \mathcal{H}^k \oplus \Delta(A^k(X))$

(ii) $\dim_{\mathbb{C}} \mathcal{H}^k < \infty$

<u>Theorem (Hodge)</u>: X compact

complex manifold. <u>Then</u>:

(i) $\mathcal{H}^k \xrightarrow{\sim} H^k(X, \mathbb{C}) \cong H^k_{dR}(X)$

$\alpha \longmapsto [\alpha]$

(ii) E holomorphic vector bundle on X

$\mathcal{H}^{0,q}(E) := \ker \Delta_{\bar{\partial}_E} \xrightarrow{\sim} H^q(X, \mathcal{E})$

$\alpha \longmapsto [\alpha]$

proof: only do (i), (ii) is similar

(i) surjective: $\beta \in A^k(X)$ closed

$\overset{\text{drop}}{\underset{\text{fact}}{\Rightarrow}} \beta = \alpha + \Delta\gamma$, $\Delta\alpha = 0$

$= \alpha + dd^*\gamma + d^*d\gamma$

$\Rightarrow d^*d\gamma \in \ker d \cap \operatorname{Im} d^* = \{0\}$
$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{(\ker d)^\perp}$

$\Rightarrow [\beta] = [\alpha]$

injective: $\beta \in \mathcal{H}^k$ exact

$\Rightarrow \beta \in \operatorname{Im} d \cap \ker d^* = \{0\}$

$\Rightarrow \beta = 0$

$\square$

Corollary: (i) $\dim_{\mathbb{C}} H^k(X, \underline{\mathbb{C}}) < \infty$

(ii) $\dim_{\mathbb{C}} H^k(X, E) < \infty$