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Notebook:	DGT		
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Lecture 13: The inverse problem in differential Galois theory

- Classical inverse Galois problem:

  k field, G finite group

Q: Does there exist L/k Galois, $G(L/k) \cong G$?

- $k = \mathbb{Q}$: not known,
  - every solvable G arises (Shafarevich, '58)
  - many sporadic groups arise, e.g. the monster group
- $k = \mathbb{C}(t)$: Yes! (cf. Szamuely)

More generally, "Yes" for

$K = k(t)$, $\mathrm{char}(k) = 0$, $k = \bar{k}$

$K = k(t)$, $\mathrm{char}(k) = 0$, $k = k$

- Now (K, ∂) differential field,
  $k = \ker(\partial) \subset K$ field of constants,
  $G/_k$ linear algebraic group

Q: Does there exist a Picard-Vessiot extension $(L, \partial)/(K, \partial)$ s.t. $G(L/K) \cong G$?

<u>Theorem (Tretkoff-Tretkoff, '79):</u>
Yes for $(K, \partial) = (\mathbb{C}(t), \frac{d}{dt})$!

Proof has 3 ingredients:

1. Identification of $G(L/K)$ as the Zariski closure of the monodromy group of $\mathcal{L} \longleftrightarrow (L, \partial)/(K, \partial)$
2. $\exists$ finitely generated subgroup $H \subset G$, s.t. $\overline{H} = G$
3. Solution of the Riemann-

finite

Hilbert problem: $U=\mathbb{P}^1_{\mathbb{C}}\setminus S$, $p\in U$

$\rho: \pi_1(U,p) \longrightarrow Gl_n(\mathbb{C})$

representation. Then there exists a regular singular diff. equation

$$\frac{d}{dt} - A(t) = 0 \quad , \quad A(t)\in M_{n\times n}(\mathbb{C}(t))$$

whose monodromy representation is ρ and whose singular locus is S

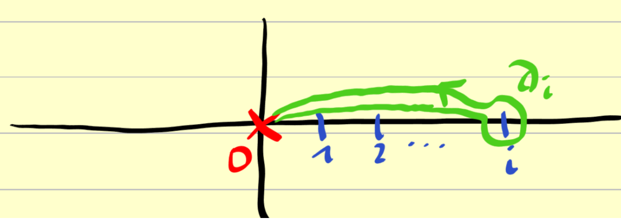
<u>Proof:</u> · Take $g_1,\dots,g_k \in G$ s.t. $\langle g_1,\dots,g_k\rangle \subset G$ is Zariski dense.

· $S=\{1,2,\dots,k,\infty\}$, $U=\mathbb{P}^1\setminus S$

$\pi_1(U,0)$ = free group on

$$\{\gamma_1,\dots,\gamma_k\}$$

γ_i loop winding once around i



- $\rho : \pi_1(U, 0) \longrightarrow Gl_n(\mathbb{C})$

  $$\gamma_i \longmapsto g_i$$

  well-defined since γ_i free

- Riemann-Hilbert

  $\Rightarrow \exists\ \frac{d}{dt} - A(t) = 0$ whose monodromy group is G

- G Zariski closed (by assumption)

  $\Rightarrow G = G(L/K)$, where $(L, \partial)/(k, \partial)$ obtained by adding linearly independent solutions of

  $$\frac{d}{dt} - A(t) = 0$$

  $\square$

- Now some details about the three ingredients

1. <u>Differential Galois groups and monodromy</u>

- $U = \mathbb{P}^1 \setminus S$, S finite

- $A(t) \in M_{n\times n}(\mathbb{C}(t))$, regular on U (i.e. for each $p \in U$, $A(t)$ is similar over $\mathbb{C}((t-p))$ to some $\tilde{A}(t) \in M_{n\times n}(\mathbb{C}[[t-p]])$)

and regular singular at $p \in S$ (i.e. $A(t)$ similar to some $\tilde{A}(t) \in M_{n\times n}\left(\frac{1}{t-p}\mathbb{C}[[t-p]]\right)$)

- consider the ODE

$$(*) \quad \frac{df}{dt} - A(t)\cdot f = 0 \qquad f = (f_1, \dots, f_n)^t$$

Note: If
$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & & \\ & & & \ddots & 1 \\ a_n(t) & \dots & & & a_1(t) \end{pmatrix}$$

Then $f = (f_1(t), \dots, f_n(t))^t$ solves $(*)$

$\Leftrightarrow$ $f_1(t)$ satisfies

$$\left(\frac{d}{dt}\right)^n (f) + a_1(t)\left(\frac{d}{dt}\right)^{n-1}(f) + \dots + a_n(t)\cdot f = 0$$

- Conversely, each $A(t)$ can be brought into Frobenius normal form $\begin{pmatrix} C_1 & & \\ & C_2 & 0 \\ 0 & & \ddots \end{pmatrix}$

$$\begin{pmatrix} \ddots & & \\ 0 & & C_k \end{pmatrix}$$

where $C_i = \left.\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ a_i(t) & \cdots & & \cdots & -\hat{a}_n(t) \end{pmatrix}\right\} i$

$\Rightarrow$ little difference between between n-th order linear ODE's and rank n linear homogeneous diff. eqn.

- In particular, $\exists$ PV-extension for $(*)$ with same uniqueness properties, diff. Galois group of $(*)$, ...
- $V_p = \mathbb{C}$-vector space of germs at p of holomorphic solutions of $(*)$

$\dim V_p = n$, $\forall\, p \in U$

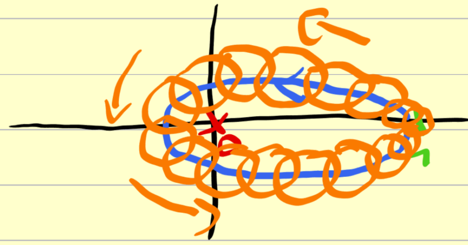
- $\lambda : [0,1] \longrightarrow U$ continuous path

  $\lambda(0) = \lambda(1) = p$
- analytic continuation along λ

  $\rightsquigarrow M(\lambda) : V_p \longrightarrow V_p$

  $\mathbb{C}$-linear automorphism

$\mathbb{C}$-linear automorphism



- $\mathrm{Fact}: M(\lambda)$ depends only on the homotopy class of λ
- If $S = \{s_1, \ldots, s_k\}$, $p \in U$, can choose loops $\lambda_1, \ldots, \lambda_k$ such that $\pi_1(U, p) = \langle \lambda_1, \ldots, \lambda_k \mid \lambda_1 \cdot \ldots \cdot \lambda_k = 1 \rangle$

$\Rightarrow M : \pi_1(U, p) \longrightarrow Gl(V_p)$

monodromy representation

<u>Theorem</u>: $G/_{\mathbb{C}}$ differential Galois group of $(*)$. Then:

$$\overline{\mathrm{Im}(M)} = G$$

<u>Sketch of proof</u>: • Coordinates of V_p generate a PV-extension $\mathbb{C}(t) \subset L \subset \mathcal{M}_p$ meromorphic germs of functions at p

- For each $\lambda \in \pi_1(U, p)$,

$M(a) \in Gl(V_p)$ maps solutions to solutions $\Rightarrow$ commutes with $\frac{d}{dt}$

- Can extend $M(a)$ to a diff. automorphism $L \longrightarrow L$
  $\Rightarrow M(a) \subset G$
- Main theorem of DGT
  $\Rightarrow$ enough to show: if $f \in L$ invariant under monodromy
  $\Rightarrow f \in \mathbb{C}(t)$
- If f invariant under monodromy
  $\Rightarrow f$ extends to a meromorphic function on $U = \mathbb{P}^1 \setminus S$
- Fact: $A(t)$ regular singular
  $\Rightarrow f$ extends to a meromorphic function on $\mathbb{P}^1$ (since holomorphic solutions to $(*)$ have at most logarithmic growth near

most logarithmic growth near

$p \in S$)

$\Rightarrow f \in \mathbb{C}(t)$

$\square$

2. $\exists$ finitely generated subgroup $H \subset G$

such that $\overline{H} = G$

<u>Lemma</u>: If every element of

$G \subset Gl_n(k)$ has finite order (alg. closed, char$(k)=0$)

then G is finite

<u>Sketch of proof</u>:

- $g \in G$ finite order, $\forall g \in G$

$\Rightarrow$ G contains a commutative

subgroup of finite index (Schur)

$\Rightarrow G^0$ commutative (a)

- Also, $g \in G$ finite order

$\Rightarrow g$ semisimple (b)

(its minimal polynomial

divides $X^n - 1$)

B...

• put a), b) together

$\Rightarrow G^0 \cong \mathbb{G}_m^r$ torus

$\Rightarrow G^0 = \{e\}$ (i.e. $r=0$)

$\Rightarrow G$ finite $\square$

<u>Sketch of proof of 2:</u>

• WLOG, $G = G^0$ (ok since G/G^0 finite)

• induction on $\dim G$

$\dim G = 0$ trivial

$\dim G = 1$ ok by Lemma

• In general, select $H \subsetneq G$ closed, connected subgroup of maximal dimension

Case (i): H normal in G

$\exists R \subset H$, $S \subset G/H$ Zariski dense, generated by $\{h_1, \dots, h_r\}$, $\{\bar{g}_1, \dots, \bar{g}_s\}$ resp.

• $T \subset G$ subgroup generated

by $\{h_1, \dots, h_r, g_1, \dots, g_s\}$

is Zariski dense

Case (ii): $\exists\, g \in G$ s.t. $gHg^{-1} \neq H$

$R \subset H$ as above

$T \subset G$ subgroup generated by

R and $\{g\}$ (fin. gen.)

- $H, gHg^{-1} \subset \overline{T}$

H, gHg^{-1} connected

$\Rightarrow H, gHg^{-1} \subset \overline{T}^\circ$

- If $\dim H = \dim \overline{T}^\circ$

$\overset{\text{maximality}}{\Rightarrow} H = \overline{T}^\circ$ $\lightning$

- so $\dim H < \dim \overline{T}^\circ$

$\Rightarrow \dim \overline{T}^\circ = \dim G^\circ$

$\Rightarrow \overline{T}^\circ = G^\circ$

$\square$

3. The Riemann–Hilbert problem

(see A. Beauville: "Monodromie des systèmes différentiels ..."

Séminaire Bourbaki)

History: Hilbert's 21st problem

Prove that there always exists a linear differential equation of Fuchsian type with prescribed monodromy representation

Variant: $S \subset \mathbb{P}^1$ finite, $U = \mathbb{P}^1 \setminus S$

$$\rho : \pi_1(U, *) \longrightarrow Gl_n(\mathbb{C})$$

Does there exist a linear differential equation $\frac{df}{dt} = A(t) f(t)$

with $A(t) = \sum_{s \in S} \frac{A_s}{t-s}$

$A_s \in Gl_n(\mathbb{C})$ (constant!)

whose <u>monodromy rep.</u> is ρ?

Yes, <u>if</u>:

1. At least one $\rho(\gamma_i)$ diagonalizable

(Plemelj, 1908; Birkhoff 1913)

2. $\rho(\gamma_i)$ sufficiently close to id_n

(Lappo-Danilevskii, 1928)

3. $n=2$ (Dekkers, 1979: relying on general results of Deligne, 1970)

4. ρ irreducible (Kostov, 1992; Bolibruch, 1992)

In general, the answer is no!

Counterexample for $n=3$, $|S|=4$ (Bolibruch, 1992)

- Recall that we only need a <u>much</u> weaker statement: $\exists\, A(t) \in M_{n\times n}(\mathbb{C}(t))$ which is regular on U and has regular singularities along S
- Difference: "regular singular" is a local notion, whereas the variant above asks for "globally regular singular"

- Riemann-Hilbert correspondence (Deligne 1970)

$X/\mathbb{C}$ smooth algebraic variety.

$$\left\{\begin{array}{l}\text{vector bundles } /X \\ \text{w. integrable connection} \\ \text{regular singular}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{local systems} \\ \text{on } X^{an} \text{ in} \\ \text{fin. dim. } \mathbb{C}\text{-vector spaces}\end{array}\right\}$$

$$\longleftrightarrow \left\{\begin{array}{l}\text{representations} \\ \pi_1(X^{an}, *) \to GL_n(\mathbb{C})\end{array}\right\}$$

equivalence of categories

- generalizations due to Kashiwara, Mebkhout (~1980s)
- major open problem: remove the regular singular assumption (keywords: algebraic D-modules, perverse sheaves, ...)
- mod p / p-adic versions (Katz, Emerton-Kisin, Bhatt-Lurie, ...)