5. $\bar\partial$-Poincaré Lemma (general case)

Prop: X complex manifold

α smooth (p,q)-form, $q>0$

Assume: $\bar\partial\alpha = 0$

Then: locally on X, $\exists\beta$ $(p,q-1)$

s.t. $\alpha = \bar\partial\beta$

Proof (sketch): First, reduce $p=0$

Assume: $\alpha = f\cdot d\bar z_1\wedge\ldots\wedge d\bar z_q$

Then: $\bar\partial\alpha = 0 \iff \frac{\partial f}{\partial \bar z_\ell} = 0$, $\ell > q$

$\overset{\text{Variation of one var. } \bar\partial\text{-Lemma}}{\Rightarrow}$ $\exists g$ smooth, s.t.

$$\frac{\partial g}{\partial \bar z_q} = f, \quad \frac{\partial g}{\partial \bar z_\ell} = 0, \ \ell > q$$

$$\Rightarrow \beta = (-1)^{q-1} g\cdot d\bar z_1\wedge\ldots\wedge d\bar z_{q-1}$$

- In general, use induction on largest integer k s.t. $\exists\, J$ $k\in J$, $\alpha_J \neq 0$ $\square$

6. Dolbeault complex

- E holomorphic vector bundle (over X)

Dfn: $A^{0,q}(E)$ = smooth sections of $\Omega_X^{0,q} \otimes E$

Want: $\bar{\partial}_E : A^{0,q}(E) \longrightarrow A^{0,q+1}(E)$

- choose $X = \bigcup_{i \in I} U_i$ open cover s.t. $E|_{U_i}$ trivial

Prop/Dfn: (i) $\exists\, \bar{\partial}_E : A^{0,q}(E) \longrightarrow A^{0,q+1}(E)$ defined by $\bar{\partial}_{U_i} : A^{0,q}(E|_{U_i}) \to A^{0,q+1}(E|_{U_i})$

$$(\alpha_1, \dots, \alpha_m) \longmapsto (\bar{\partial}\alpha_1, \dots, \bar{\partial}\alpha_m)$$

(ii) Independent of choice of cover

(iii) satisfies Leibniz rule, $\bar{\partial}_E^2 = 0$ + Poincaré lemma for $\bar{\partial}_E$

<u>Idea of proof:</u> Check:

$$\bar\partial_{U_i} \alpha|_{U_i \cap U_j} = \bar\partial_{U_j} \alpha|_{U_i \cap U_j}$$

uses that τ_{ij} transition functions are <u>holomorphic</u>

$\Rightarrow$ (i), (ii)

(iii) All statements local on X so follow from analogous statements for $\bar\partial$

$\square$

<u>Dfn:</u> $A^{0,0}(E) \xrightarrow{\bar\partial_E} A^{0,1}(E) \xrightarrow{\bar\partial_E} A^{0,2} \longrightarrow \dots$

<u>Dolbeault complex</u>

- locally trivial cohomology by $\bar\partial$-Poincaré Lemma

Lecture 4: Kähler metrics

1. Kähler forms

(V, h) Hermitian space, i.e.

V = fin. dim. $\mathbb{C}$-vector space

$h: V \times V \longrightarrow \mathbb{C}$ Hermitian form

in particular: $h(u,v) = \overline{h(v,u)}$

- $h = g - i \cdot \omega$, $g = \mathrm{Re}(h)$, $\omega = -\mathrm{Im}(h)$

Let $W = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$

- $W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$

Dfn: $W^{1,1} := \mathrm{Im}(W^{1,0} \otimes W^{0,1} \longrightarrow \Lambda^2 W_{\mathbb{C}})$

$W^{1,1}_{\mathbb{R}} := W^{1,1} \cap \Lambda^2 W_{\mathbb{R}}$

Lemma: $\left\{ \begin{array}{l} \text{Hermitian forms} \\ h: V \times V \to \mathbb{C} \end{array} \right\} \xleftrightarrow{1:1} W^{1,1}_{\mathbb{R}}$

$$h \longmapsto \omega$$

$$g - i \cdot \omega \longleftarrow\!\shortmid\ \omega$$

$$g(u,v) := \omega(u, Iv)$$

Proof (sketch):

1. $\omega \in W^{1,1}_{\mathbb{R}}$

Only non-trivial thing:

$\omega \in W^{1,1}$

i.e. $\omega(u,v) = 0 \quad \forall\, u,v \in V^{1,0}$
$\qquad u,v \in V^{0,1}$

Indeed, $\omega(\tilde{u},\tilde{v}) = 0$ (exercise)

for $\tilde{u} = u \mp i \cdot I u$
$\tilde{v} = v \mp i \cdot I v$ $\qquad u,v \in V$

2. Verify: $h = g - i \cdot \omega$ is Hermitian (easy)

3. Constructions are inverse (clear) $\square$

Dfn: $\omega \in W^{1,1}_{\mathbb{R}}$ **positive** if corresponding h is positive definite (will assume this from now on)

Thursday, May 6, 2021 10:34 AM

Rmk: (i) $e_1, \dots, e_n$ basis of V

$$z = \sum t_i \cdot e_i, \quad z' = \sum t_i' \cdot e_i$$

Then: $h(z, z') = \sum_{i,j} \underbrace{h_{ij}}_{h(e_i, e_j)} t_i \overline{t_j'}$

$$\omega(z, z') = \frac{i}{2} \sum h_{ij} \, z_i \wedge \overline{z_j}$$

dual basis of W

(ii) $g = \mathrm{Re}\, h$ is symmetric bilinear form, analogous Lemma for g instead of ω

2. Kähler metrics

T_X holomorphic tangent bundle

Dfn: (i) h Hermitian metric on X (T_X)

$= (h_x)_{x \in X}$ positive definite hermitian forms on $T_{X,x}$

s.t. : $X \longrightarrow \mathbb{C}$

$x \longmapsto h_x\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$

z: local coord at x

is <u>smooth</u>

(ii) $\omega = -\mathrm{Im}\, h \in \Omega^{1,1}_X \cap \Omega^2_{X,\mathbb{R}}$

<u>kähler form</u> of h

(iii) h is kähler metric, if

$d\omega = 0$

<u>Rmk</u>: Existence of kähler metric puts constraints on cohomology of X

<u>Fact</u>: X compact kähler.

Then: ω^k <u>not exact</u>

for all $1 \leq k \leq n = \dim X$

(also that $\frac{\omega^n}{n!} = \mathrm{vol}_X$)

Corollary: If $H^{2k}_{dR}(X)=0$

for some $1\leq k\leq n$

then $\nexists$ Kähler metric on X

3. Levi-Civita and Chern connection

Prop: (X,h) h Hermitian metric

($\Rightarrow$ $g=\mathrm{Re}\, h$ Riemannian metric)

Then: $\exists$ unique connection

$$\nabla : A^0(T_{X,\mathbb{R}})\longrightarrow A^1(T_{X,\mathbb{R}})$$

s.t. (i) $d(g(X,Y))=g(X,\nabla Y)$

$$+g(\nabla X,Y)$$

"metric"

(ii) $\nabla_X Y-\nabla_Y X=[X,Y]$

"torsion-free"

Levi-Civita connection

Now E holomorphic vector bundle E on X

- $\nabla : A^0(E) \longrightarrow A^1(E)$ connection

$$\nabla^{0,1} : A^0(E) \xrightarrow{\nabla} A^1(E) \xrightarrow{pr} A^{0,1}(E)$$

<u>Prop</u>: $\exists$ unique connection ∇

s.t. (i) $d(h(\sigma,\tau)) = h(\nabla\sigma,\tau) + h(\sigma,\nabla\tau)$

$F: d \text{ mod } \bar{\partial}$

(ii) $\nabla^{0,1} = \bar{\partial}_E$

<u>Chern Connection</u>

<u>Proof</u>: (i), (ii) imply

$$\partial(h(\sigma,\tau)) = h(\nabla^{1,0}\sigma,\tau) + h(\sigma,\bar{\partial}_E\tau)$$

If $\sigma_1,\dots,\sigma_m$ <u>holomorphic</u> trivialization

then $i\,\partial(h(\sigma_i,\sigma_j)) = h(\nabla^{1,0}\sigma_i,\sigma_j)$

$\overset{h}{\underset{\text{pos. definite}}{\Longrightarrow}}$ determines ∇ uniquely $\square$

<u>Thm</u>: TFAE for (X, h)

(i) h is Kähler

(ii) $I : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ is <u>flat</u> for LC-connection,

$$\nabla(I \cdot X) = I \cdot \nabla(X)$$

(iii) LC-connection on $T_{X,\mathbb{R}}$ and Chern-connection on T_X agree;

$$\begin{tikzcd} A^0(T_X) \arrow[r, "\nabla_{Chern}"] & A^1(T_X) \\ A^0(T_{X,\mathbb{R}}) \arrow[u, "\mathrm{Re}^* \wr"] \arrow[r, "\nabla_{LC}"] \arrow[ur, phantom, "\circlearrowleft"] & A^1(T_{X,\mathbb{R}}) \arrow[u, "\mathrm{Re}^* \wr"] \end{tikzcd}$$