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Lecture 7

Last time: $(L,\partial)/(k,\partial)$ PV-extension

for $\mathcal{L} \in k[\partial]$ (monic)

Wlog: $L = Q(\bar{S})$, $\bar{S} := S/m$

univ. full sol. algebra

maximal diff. ideal

Then: $Gl(V)_m \longrightarrow Aut_\partial(L/k) =: G$

$\mathcal{L}^{-1}(0) \subset L$

$T \longmapsto \Phi_T$

is an isomorphism of groups

Also, $\Phi_T : S \longrightarrow S$ satisfies

$\Phi_T(m) \subset m$

$\Longleftrightarrow t \in Gl(\bar{V}) \cong Gl(V)$

image of T

vanishes on $\underline{J(\pi(\Phi_T(m))}$)

7.1. Algebraic group structures

Goals: 1. $Gl(V)_m \subset Gl(V)$ algebraic subgroup

2. $G \longrightarrow Gl_n(k) \simeq Gl(V)$ (cf. Prop 6.1) image is algebraic subgroup

3. These algebraic group structures "coincide" (more precise statement later)

Notation: • (K,∂) differential field, $k = \ker(\partial) \subset K$ alg. closed

• $\mathcal{L} \in K[\partial]$ monic, $\mathcal{L} = \partial^n - \sum_{i=0}^{n-1} a_i \partial^i$

• $S = K[Y_{ij}][W^{-1}]$ universal full solution algebra, $m \subset S$ max. diff. ideal

• $L = Q(\bar{S})$, $\bar{S} := S/m$

$V := \mathcal{L}^{-1}(0) \subset L$

• Write $S = K \otimes_k k[Y_{ij}][W^{-1}]$

- Write $S = K \otimes_k \underbrace{k[y_{ij}][W\ ]}_{\mathcal{O}(Gl_n/k)}$

- $Gl_n \times \mathcal{O}(Gl_n) \longrightarrow \mathcal{O}(Gl_n)$

$$(t, f) \longmapsto (A \mapsto f(A \cdot t))$$

left regular representation

- For $f = y_{ij}$ coordinate function

$$t \cdot y_{ij}(A) = y_{ij}(A \cdot t) = \sum_k y_{ik}(A) \cdot t_{kj}$$

$$\Rightarrow t \cdot y_{ij} = \sum_k t_{kj} y_{ik}$$

$\rightsquigarrow$ This is $T \longmapsto \Phi_T$!

(after base change to

$S = \underset{\nwarrow\, Gl_n \text{ acts trivially}}{K} \otimes \mathcal{O}(Gl_n)$)

$\Rightarrow$ S is a Gl_n-module

via the left regular representation

<u>Fact</u>: • H/k algebraic group,

- M k-vector space,

$$\rho: H \longrightarrow Gl(M)$$

representation

representation

- $N \subset M$ k-linear subspace.

Then: $H_N := \{\tau \in H \mid \tau(N) \subseteq N\} \subset H$

is an algebraic subgroup

- Summarizing:

<u>Thm 7.1:</u> Notation as above, the action

$$Gl_n \times S \longrightarrow S$$
$$(t, y_{ij}) \longmapsto \sum_k t_{kj} y_{ik}$$

is algebraic, and

$(Gl_n)_m \subset Gl_n$ is an algebraic subgroup.

Thm 7.1. & Thm 6.7

$\Rightarrow G \cong Gl(V)_m$ is an algebraic group over k

$\Rightarrow$ Goal 1

<u>7.2. The second algebraic structure</u>

Thm 7.2: Notation as above,

$\exists$ ideal $J \subset \mathcal{O}(Gl(\bar{V})) \cong k[x_{ij}][d^{-1}]$

such that the image

$\rho: G = \mathrm{Aut}_\partial(L/k) \hookrightarrow Gl(\bar{V})$

is $V(J)$ ← vanishing locus of J.

In particular: $\mathrm{Im}(\rho) \subset Gl(\bar{V})$

is an algebraic subgroup.

Proof: This follows from

Lemma 6.9 in the last lecture.

$\Rightarrow$ Goal 2

7.3. Comparison

Prop 7.3. The composite

$\Psi: Gl(V)_m \xrightarrow{\sim} G \hookrightarrow Gl(\bar{V})$

is a morphism of algebraic groups.

Proof: • Enough to show:

$\mathcal{O}(Gl(\bar{V})) \longrightarrow \mathcal{O}(Gl(V)_m)$

is a ring homomorphism

- $X_{ij} \in \mathcal{O}(Gl(\bar{V}))$ coordinate function for $\bar{\mathcal{B}} = \{\bar{y}_1, \dots, \bar{y}_n\}$ basis of $\bar{V}$
- $z_{ij} \in \mathcal{O}(Gl(V))$ coordinate function for $\mathcal{B} = \{y_1, \dots, y_n\}$ basis of V
- Let $T \in Gl(V)_m$. Then

$$X_{ij}(\Psi(T)) = t_{ij}, \qquad \begin{array}{ccc} (t_{ij}) & \overset{\bar{\mathcal{B}}}{\longleftrightarrow} & T \\ \cap & & \cap \\ Gl_n(k) & & Gl(V) \end{array}$$

and

$$Z_{ij}(T) = T_{ij} \qquad \begin{array}{ccc} (T_{ij}) & \overset{\mathcal{B}}{\longleftrightarrow} & T \\ \cap & & \cap \\ Gl_n(k) & & Gl(V) \end{array}$$

$\Rightarrow \mathcal{O}(Gl(\bar{V})) \longrightarrow \mathcal{O}(Gl(V)_m)$

is given by $X_{ij} \longmapsto Z_{ij}$

$\square$

<u>Rmk 7.4.</u> Prop 7.3 implies that the diagram

$$\begin{tikzcd} Gl(V)_m \arrow[r, "T \mapsto \Phi_T"] \arrow[d, hook] & Aut_g(S/k)_m \arrow[r] & Aut_g(L/k) \arrow[d, hook] \\ Gl(V) \arrow[rr, "\sim"] & & Gl(\bar{V}) \end{tikzcd}$$

is a commutative diagram of algebraic groups

7.4. Dependence on choice of $\mathfrak{m}$

Thm 7.5: Notation as above, let $\sigma_i : S \longrightarrow L$, $i = 1,2$ differential k-algebra homs.

Then: $\exists\ T \in Gl(V)$ s.t.

$$\sigma_2 = \sigma_1 \circ \Phi_T$$

In particular, if $\mathfrak{m}_i \subset S$, $i = 1,2$ maximal diff. ideals then: $\exists T \in Gl(V)$ s.t.

$$\mathfrak{m}_2 = \Phi_T(\mathfrak{m}_1)$$

and $\quad Gl(V)_{\mathfrak{m}_1} = T^{-1} Gl(V)_{\mathfrak{m}_2} T$

Proof: $w \in S$ unit

$\Rightarrow \sigma_i(w) \neq 0$

$\Rightarrow \sigma_i : V \xrightarrow{\sim} \mathcal{L}^{-1}(0) \subset L$

k-linear isomorphism

k-linear isomorphism

Therefore: $\exists\, t \in Gl_n(k)$ s.t.

$$\sigma_2(Y_k) = \sigma_1(t \cdot Y_k) \quad , \quad k = 1, \dots, n$$

$T \in Gl(V)$ corresponding to t

$\Rightarrow \sigma_2 \ , \ \sigma_1 \circ \Phi_T$ coincide on V

$\Rightarrow$ they coincide on S

<u>For the second part:</u>

- $\overline{S}_i = S/m_i \quad , \quad i = 1,2$
- Then $Q(\overline{S}_i)$ PV-extension for L

$\rightsquigarrow \sigma_i : S \longrightarrow \overline{S}_i \hookrightarrow L$

(by uniqueness of PV-extensions)

$\ker(\sigma_i) = m_i$

- First part $\Rightarrow \exists\, T \in Gl(V)$ s.t.

$$\sigma_2 \circ \Phi_T = \sigma_1$$

$\Rightarrow \Phi_T(m_1) = m_2$

<u>For the third part:</u>

$$Gl(V)_{\Phi_T(m_1)} = \Phi_T^{-1} \cdot Aut_\partial(S/k)_{\Phi_T(m_1)} \cdot \Phi_T$$

$$= \Phi\left(T^{-1} Gl(V)_{m_2} \cdot T\right)$$

and $Gl(V)_{\Phi_T(m_1)} = \Phi(Gl(V)_{m_1})$

$\Rightarrow T^{-1} Gl(V)_{m_2} \cdot T = Gl(V)_{m_1}$ $\square$

<u>Rmk 7.6.</u> All maximal diff. ideals of S are $Gl(V)$-conjugate to one another (by Thm 7.5).

<u>7.5. Automorphisms acting on prime ideals</u>

- $m \subset p \subset S$, p prime ideal (<u>not</u> differential)
- $T \in Gl(V)_m$ s.t. $\Phi_T(p) \subset p$

$\rightsquigarrow S/p \longrightarrow S/p$ autom. of k-algebras (<u>not</u> of diff. k-algebras)

- $(Gl(V)_m)_p = \{ T \in Gl(V)_m \mid \Phi_T(p) \subset p \}$

<u>Lemma 7.7.</u> $M \supseteq k$ field extension (not necessarily differential)

$f: S \longrightarrow M$ k-algebra hom.

If $\sigma \in \mathrm{Aut}_\partial(L/k)$, $f\sigma = f$

stabilizer $\bar{S}$

$\Rightarrow$ $\sigma = id$

<u>**Proof:**</u> $\sigma \in \mathrm{Aut}_\partial(L/k) \cong Gl(V)_m \ni T$

and (t_{ij}) matrix of T in $\mathcal{B}$

Then $\sigma(\bar{y}_{ij}) = \sum_k t_{kj} \bar{y}_{ik}$ $\forall\, i,j$

$\overset{\text{Apply } f}{\Rightarrow}$ $(f(\bar{y}_{ij})) = (f(\bar{y}_{ij})) \cdot (t_{ij})$

Now $\det(f(\bar{y}_{ij})) = f(\bar{w})$ unit in M

$\Rightarrow (f(\bar{y}_{ij}))$ invertible

$\Rightarrow (t_{ij}) = \mathbb{1}$

$\square$

7.6. Independence of $\mathcal{L}$

Recall from previous lecture:

$(L,\partial)/(k,\partial)$ is PV-extension for some $\mathcal{L}$

$\Leftrightarrow$ (i) $L = K\langle V\rangle$, for some V, $\dim_k V < \infty$

(ii) $\exists\, G \subset \mathrm{Aut}_\partial(L/k)$ s.t. $L^G = k$

(iii) L has no new constants

Prop 7.8: $\cdot\ (L,\partial)/(K,\partial)$ PV-extension for some $\mathcal{L} \in K[\partial]$

- $k = \ker(\partial) \subset K$ alg. closed
- $W \subset L$ $\mathrm{Aut}_\partial(L/K)$-stable, $\dim_k W < \infty$

Then: $\mathrm{Aut}_\partial(L/K) \longrightarrow Gl(W)$ is a morphism of alg. groups over k.

Proof: $V = \mathcal{L}^{-1}(0) + W$, $G = \mathrm{Aut}_\partial(L/K)$ satisfy (i)-(iii) above

$\Rightarrow L = K\langle V\rangle$ is PV-extension for some $\mathcal{L}$

$\Rightarrow$ can assume $\mathcal{L}^{-1}(0) \supseteq W$

Thm 7.2 $\Rightarrow G \longrightarrow Gl(V)$ morphism of alg. groups

$\Rightarrow G \longrightarrow Gl(V) \longrightarrow Gl(W)$ morphism of alg. groups $\square$

Cor 7.9. $(L,\partial)/(K,\partial)$ PV-extension for $\mathcal{L}$. Then the algebraic group structure on $\mathrm{Aut}_\partial(L/K)$ is independent of $\mathcal{L}$.

Proof: • $\mathcal{L}_i \in K[\partial]$ monic diff. operator

L/K PV-extension for $\mathcal{L}_i$, $i=1,2$

• $V_i = \mathcal{L}_i^{-1}(0)$

• $\rho_i : \mathrm{Aut}_\partial(L/K) \longrightarrow \mathrm{Gl}(V_i)$

induced representations

• $G_i := \mathrm{Im}(\rho_i) \cong \mathrm{Aut}_\partial(L/K)$ (as groups)

Thm 7.2 $\Rightarrow$ G_i alg. group

Prop. 7.8 ($\mathcal{L}=\mathcal{L}_1$, $W=V_2$)

$$\Rightarrow \underbrace{\mathrm{Aut}_\partial(L/K)}_{\text{alg. group structure from } \rho_1} \longrightarrow \mathrm{Gl}(V_2)$$

is a morphism of alg. groups. $\square$

7.8. Algebraicity of restriction maps

$M/L/k$ extensions of diff. fields

s.t. M/k, L/k PV-extensions

Prop 5.3 $\Rightarrow$

$$r : \mathrm{Aut}_\partial(M/k) \twoheadrightarrow \mathrm{Aut}_\partial(L/k)$$

Cor 7.9. r is a morphism of algebraic groups.

Proof: • $\mathcal{L} \in k[\partial]$ s.t. L/k is PV-extension for $\mathcal{L}$.

• Then $W := \mathcal{L}^{-1}(0) \subset M$ is $\underbrace{\mathrm{Aut}_\partial(M/k)}_{=:G}$-stable and $\dim_k W < \infty$

• Prop 7.8 $\Rightarrow$ $G \longrightarrow Gl(W)$ algebraic

• $L = k\langle W \rangle$ $\Rightarrow$

$$\begin{tikzcd} G \arrow[r] \arrow[d, "r"'] & Gl(W) \\ \mathrm{Aut}_\partial(L/k) \arrow[ur, dashed] & \end{tikzcd}$$

morphisms of alg. groups

$\Rightarrow$ r algebraic $\square$

7.9. Intermediate extensions of PV-

extensions

Prop 7.10. • $(L,\partial)/(K,\partial)$ PV-extension for $\mathcal{L} \in K[\partial]$ of order n

• $L \supseteq M \supseteq K$ intermediate diff. field

Then: $(L,\partial)/(M,\partial)$ PV-extension for $\mathcal{L} \in M[\partial]$.

Proof: • $V := \mathcal{L}^{-1}(0) \subset L$, $\dim_k V = n$

• $L = K\langle V \rangle = M\langle V \rangle$

$\Rightarrow$ axioms (i), (ii) of PV-extensions

• L/M has no new constants

$\Rightarrow$ axiom (iii) of PV-extensions

$\square$