The Electric Potential of a Square Hollow Tube

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Introduction

The electric potential and field for a long, square, hollow tube was calculated by solving the Laplace equation in two dimensions. With four boundary conditions for each side of the square tube the solution takes the form of a Fourier series that was calculated numerically with a fixed number of Fourier coefficients.

1. Theory

The two dimensional potential in a square hollow tube can be found by solving the Laplace equation

$$\nabla^2 V = 0 \tag{1}$$

together with four boundary conditions for the walls of the tube. We will look at the special case with these following boundary conditions:

$$\begin{aligned} i)\ & V(x=0,y)=0 \\ ii)\ & V(x=L,y)=0 \\ iii)\ & V(x,y=0)=0 \\ iv)\ & V(x,y=L)=V_0(x), \end{aligned} \tag{2}$$

Where $V_0(x)$ is an arbitrary function. The two dimensional potential is a product of two independent functions $X(x)$ and $Y(y)$

$$V(x,y) = X(x)Y(x). \tag{3}$$

Equation (1) then results in two independent ODEs

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0, \tag{4}$$

with the following solutions

$$V(x,y) = C\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) \quad , \ n = 1,2,3,\ldots\ , \tag{5}$$

satisfying the three first boundary conditions. The last boundary condition requires that the solution is a linear combination of all the solutions. This gives us the following Fourier sine series

$$V(x\prime, y\prime) = \sum_n C_n \sin(n\pi x\prime)\sinh(n\pi y\prime) \tag{6}$$

where C_n are Fourier coefficients, $x\prime = x/L$ and $y\prime = y/L$. By using "Fourier's trick" [1] and the last boundary condition the solution for the coefficients is derived to be

$$C_n = \frac{2}{\sinh(n\pi)}\int_0^1 \sin(n\pi x\prime)V_0(x\prime L)dx\prime. \tag{7}$$

Inserting equation (7) into equation (6) gives the desired solution for the potential. The electric field is found by taking the negative gradient of the potential

$$\vec{E}(x,y) = -\nabla V(x,y). \tag{8}$$

2. Results and Discussion

Three functions were chosen for the fourth boundary condition (2): $V_0(x) = \sin(3\pi x)$, $V_0(x) = 1-(x-1/2)^4$ and $V_0(x) = \theta(x-L/2)\theta(3L/4-x)$, where θ is the Heaviside step function. The three potential with the different choices of boundary condition (iv), were calculated with $n = 100$ Fourier coefficients, where n represents the order the Fourier series is summed up to. The surface plots can be seen respectively in figure (1), (2) and (3).

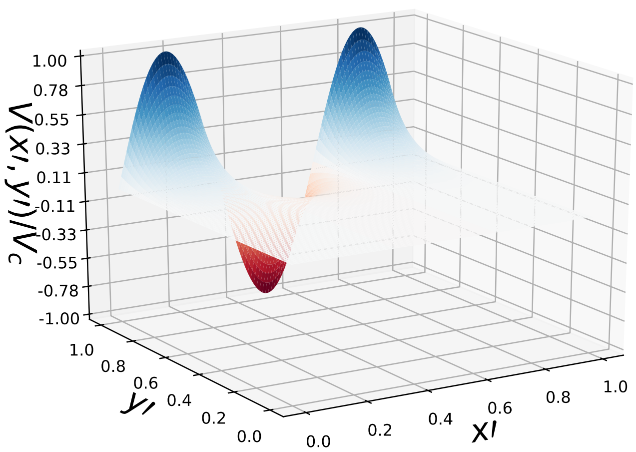

Figure 1: Plot of potential $V(x,y)$ with $V_0(x) = sin(3\pi x)$.

From a quick observation all the 3D plots seem to fulfil all of the boundary conditions, except for figure (2) and (3) which seem to have some oscillations on the edge $y = L$.

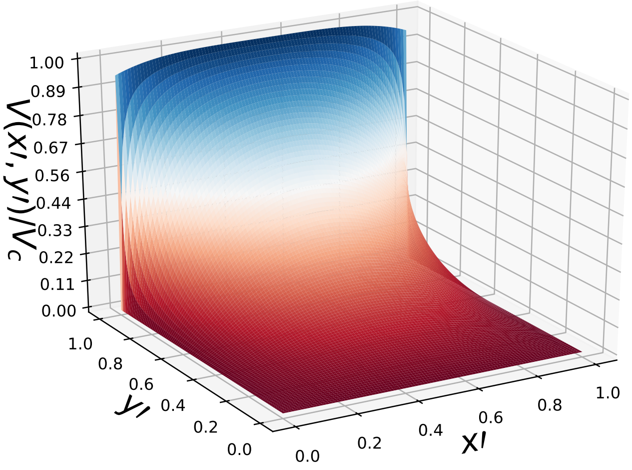

Figure 2: Plot of potential $V(x, y)$ with $V_0(x) = 1-(x-1/2)^4$.

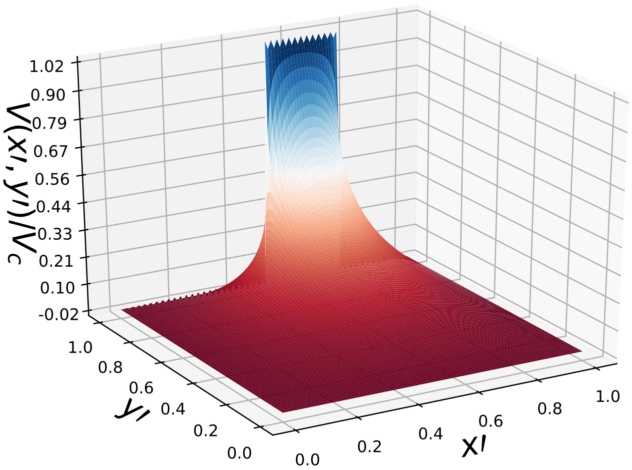

Figure 3: Plot of potential $V(x, y)$ with $V_0(x) = \theta(x - L/2)\theta(3L/4 - x)$.

To verify if the calculated potentials fulfilled their boundary conditions $V_0(x)$ was plotted along with $V(x, y = L)$, which can be seen in figures (4), (6) and (8). To see how fast the calculation converges to the exact solution, the absolute error $max|V_0(x) - V(x, y = L)|$ was plotted against n, which can be seen in figures (5), (7) and (9). As seen in figure (4) the calculated potential is a very good approximation when $V_0(x)$ is a sinus function. In the error plot (5) we see that $V(x, y = L)$ converges very rapidly for increasing n. Only a few orders are necessary for a good approximation where the error is $\sim 10^{-15}$. These results are expected since the solution for the Laplace equation (1) takes the form of a Fourier sine series (6) and of course a sine function is also considered to be a Fourier series. For around $n = 190$ the error rises back to the same error

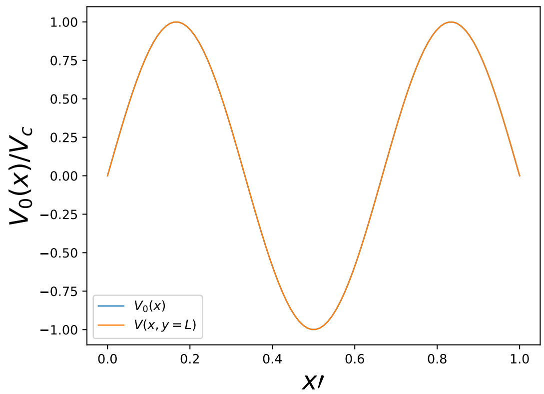

Figure 4: $V_0(x) = \sin(3\pi x)$ plotted with $V(x, y = L)$ and order 100 of Fourier coefficients.

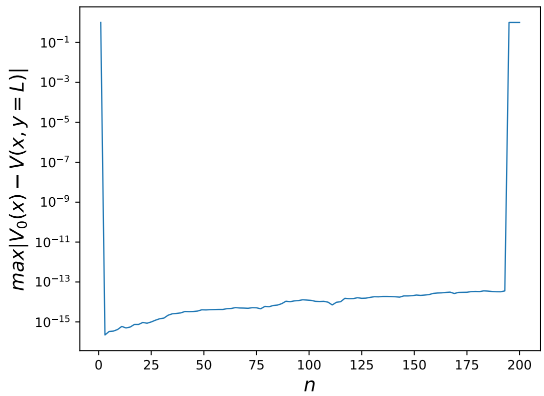

Figure 5: Plot of $max|\sin(3\pi x) - V(x, y = L)|$ against order of Fourier coefficients.

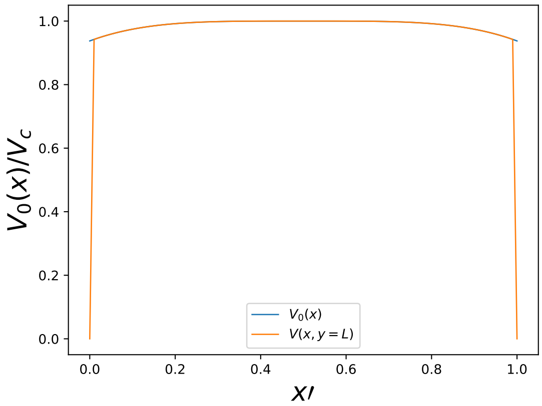

Figure 6: $V_0(x) = 1 - (x - 1/2)^4$ plotted with $V(x, y = L)$ and order 100 of Fourier coefficients.

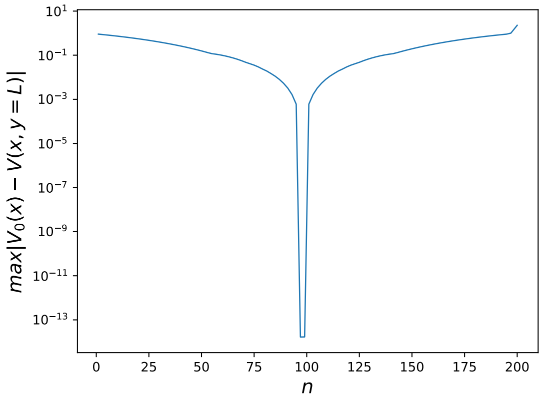

Figure 7: Plot of $max|1-(x-1/2)^4 - V(x, y = L)|$ against order of Fourier coefficients.

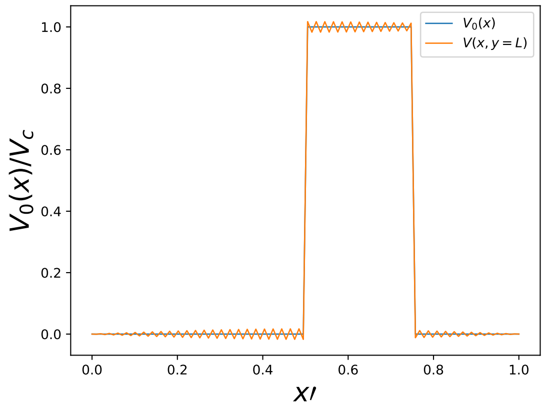

Figure 8: $V_0(x) = \theta(x - L/2)\theta(3L/4 - x)$ plotted with $V(x, y = L)$ and order 100 of Fourier coefficients.

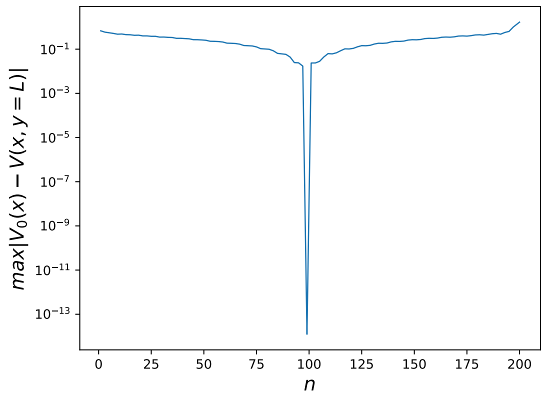

Figure 9: Plot of $max|\theta(x - L/2)\theta(3L/4 - x) - V(x, y = L)|$ against order of Fourier coefficients in the logarithmic scale.

as $n = 1$. This is not expected and might suggest that the method is flawed. Another explanation is that since sine is a Fourier series it converges for only a few n and increasing n only adds up unnecessary numerical inaccuracies. For orders below $n = 190$, the numerical solution serves as a very accurate description of the potential. In figure (6) and (8) we see that the approximation is less accurate as there are oscillations in certain regions. The error is greatest near the edges where the functions have continuity jumps. In the error convergence plots (7) and (9), we see that it takes a substantial amount of orders of Fourier coefficients to get a accurate solution. This is to be expected when using Fourier series to approximate non-periodic, sharp functions with discontinuities. At $n = 99$ orders the error is at it's minimum of $\sim 10^{-13}$, and yields a very good approximation. The error does not converge to zero for higher orders, instead it increases back to the error for $n = 1$ resulting in a symmetric plot. These observations are in line with the "Gibbs phenomenon" [2], which is the oscillation effect the nth partial sum of the Fourier series has around continuity jumps. In M. Bôcher's *Introduction to the Theory of Fourier's Series, Chap 9. Gibbs Phenomenon* (1906,p.123) it is stated: "In an interval which includes or reaches up to a Point where $f(x)$ is discontinuous the Fourier development cannot converge uniformly, since a uniformly convergent series of continuous functions necessarily represents a continuous function" [2]. This explains why these functions do not converge when increasing n. The electric field is found with equation (8). The plots of the electric fields for the three different potentials can be respectively seen in figures (10), (11) and (12). The resulting plots seem very reasonable when compared to the corresponding potential surface plots (1), (2) and (3). The field vectors point in the direction of descent and the magnitudes are proportional to the slopes of the surface plots. The plot goes to zero on the sides corresponding to the first three boundary conditions and the characteristics of the fourth boundary condition can be observed on the final side $y = L$ just as expected.

3. Conclusion

The numerically calculated Fourier series serves as a good approximation for the electric potential and electric field. Depending on the choice of boundary conditions the deviance between the numerical solution a the boundary and the boundary condition can be reduced to order $\sim 10^{-13}$ or $\sim 10^{-15}$ for sufficient orders of Fourier coefficients.

[1] D. J. Griffths, *Introduction to Electrodynamics, 4th Edition*, (2017), p. 131-138, Cambridge University Press, Cambridge, United Kingdom, 2017

[2] M. Bôcher, *Introduction to the theory of Fourier's series, Annals of Mathethematics, second series*, (April 1906), p.123–132, Mathematics Department, Princeton University 1906

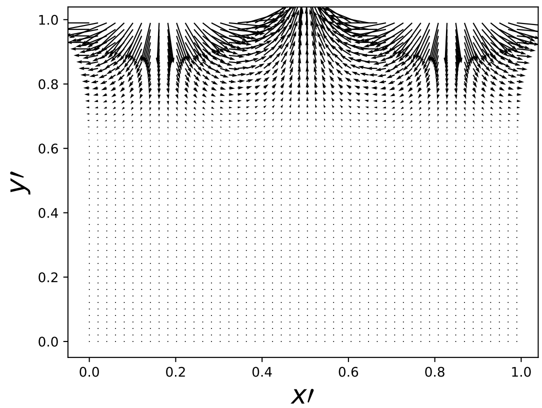

Figure 10: Plot of the electric field $\vec{E}(x, y)$ with $V_0(x) = \sin(3\pi x)$.

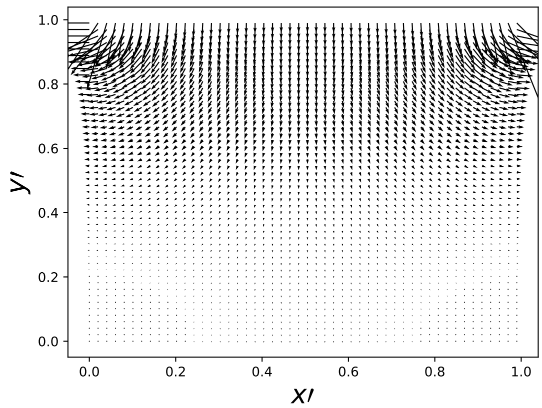

Figure 11: Plot of the electric field $\vec{E}(x, y)$ with $V_0(x) = 1 - (x - 1/2)^4$.

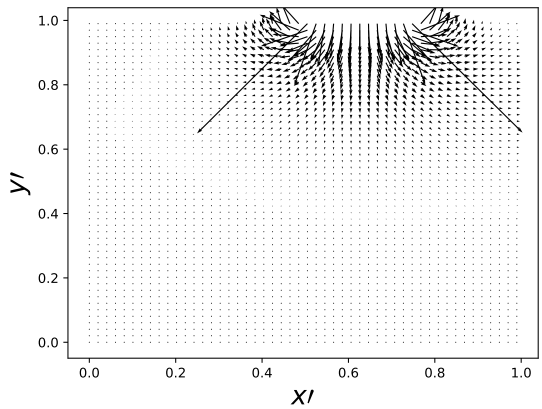

Figure 12: Plot of the electric field $\vec{E}(x, y)$ with $V_0(x). = \theta(x - L/2)\theta(3L/4 - x)$.