Project 3, TMA4320: Planetary Motion

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1 Introduction

In this document, we present our answers and results to the questions in Project 3. Furthermore, we will discuss whether our results adhered to our expectations and are reasonable.

2 Planetary Motion

2.1 Trajectory of Earth for fixed sun

The orbital trajectory of the Earth was calculated with the Euler-Cromer algorithm (EC) and Runge-Kutta 4th order algorithm (RK4). Euler Cromer is an implicit method which takes in a function f and initial conditions $x_0, y_0, v_{x,0}, v_{y,0}$ and computes first

$$v_{x,i+1} = v_{x,i} + f(x_i)\Delta t, \quad v_{y,i+1} = v_{y,i} + f(y_i)\Delta t, \tag{1}$$

and then computes

$$x_{i+1} = x_i + v_{x,i+1}\Delta t, \quad y_{i+1} = y_i + v_{y,i+1}\Delta t, \tag{2}$$

for $i = 0, 1, .., n$ [1]. *The* Runge-Kutta method is an explicit method which takes in a function f and a parameter w which the function works on. The algorithm we used computes

$$\begin{aligned} s_1 &= f(w) \\ s_2 &= f(w + s_1\Delta t/2) \\ s_3 &= f(w + s_2\Delta t/2) \\ s_4 &= f(w + s_3\Delta t) \\ w &+ \Delta t(s_1 + 2s_2 + 2s_3 + s_4)/6, \end{aligned} \tag{3}$$

and returns the last line of equation (3) [2]. The two algorithms were applied on the equations of motion

$$\frac{dx}{dt} = v_x, \quad \frac{dv_x}{dt} = \frac{F_{G,x}}{M_e}, \quad \frac{dy}{dt} = v_y, \quad \frac{dv_y}{dt} = \frac{F_{G,y}}{M_e}, \tag{4}$$

where F_G is the gravitational force acting on the planet,

$$F_G = \frac{GMm}{r^2}, \tag{5}$$

where G is the gravitational constant $G = 6.67 \times 10^{-11}$, M and m are the masses of the bodies the force is acting on and r is the distance between the centers of the bodies.
Good initial conditions for a given planet in our solar system are $x_0 = r$ and $y_0 = 0$, where r is the distance from the sun in astronomical units. For the velocity we have $v_{x,0} = 0$ and $v_{y,0} = 2\pi r/T$ AU/Yr, where T is the period of the planet in Earth years. The reason for this is that when we decided to place the sun at the origin $(0, 0)$, it is easiest for the planet to start on the x-axis. That makes is simple to know the planet's position because the planet is always a distance r from the sun. To find its initial velocity is simple because we know from mechanics that the velocity is always perpendicular to the acceleration in circular motion. When the planet is at position $(r, 0)$, we know that the velocity has no component in the x-direction, because the acceleration is parallel to the x-axis. This gives us that the velocity has a component solely in the y-direction. We know that for Earth, the

initial velocity is $v_{y,0} = 2\pi$ AU/Yr. So for a different planet, we have to adjust the initial velocity such that we end up with the correct value.
Figure (1) shows the orbit of Earth around the Sun using both algorithms and equation (4) and (5). As we can see on the right figure, for a larger time step $\tau = 0.05$ Yr, both of the algorithms give what resembles a polygon with many sides instead of a circle. However, the Runge-Kutta method is better at calculating the orbit than the Euler Cromer method. The Euler Cromer plot does not result a full circle, neither does it resemble a perfect circle. On the left figure, we see that for a small time step $\tau = 0.001$ years, the two algorithms give very similar and precise results.

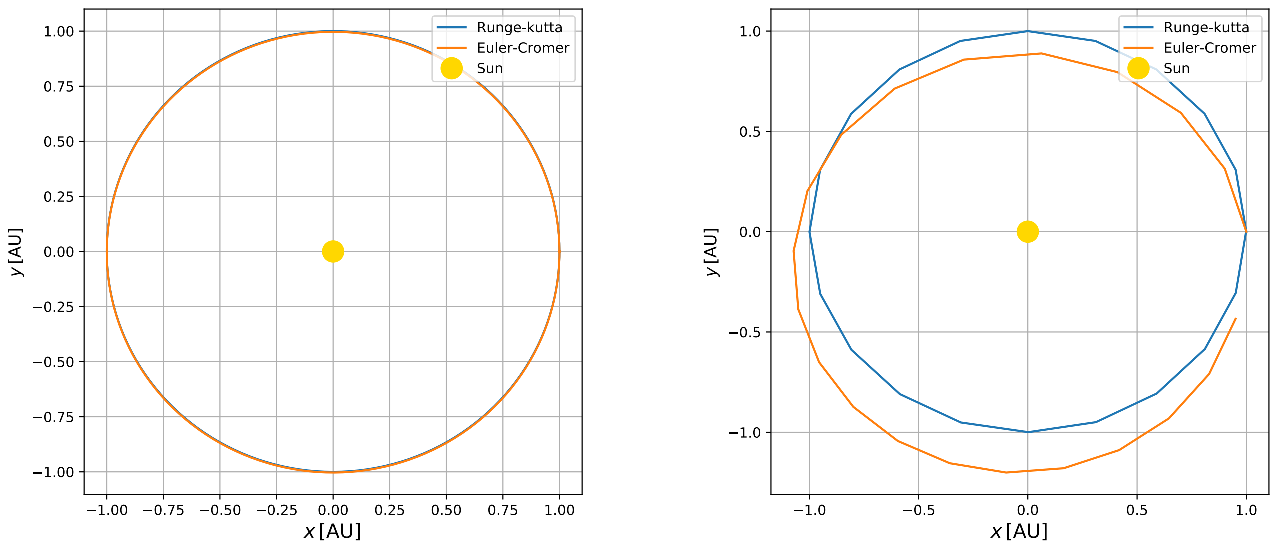

Figure 1: The position of Earth plotted as a function of time using the 4th order Runge-Kutta method and the Euler Cromer method, *Left:* with time step = 0.001 Yr. *Right:* with time step = 0.05 Yr.

2.2 Convergence test

When it comes to finding a good initial time step, we can perform a convergence test. The purpose of a convergence test is to understand the effect the time step size has on the trajectory of a planet. From what we saw in Figure (1), a good initial guess for a suitable time step for Earth is $\tau = 0.001$ Yr for both Runge-Kutta and Euler Cromer, which is what we used in the remaining questions. Any value of τ that is 10 times greater or more, gives a significant error even for Runge-Kutta since it is not a circle, as we can see on the right of Figure (1). To optimize the time step for Earth, we used the requirement of total energy conservation for RK4 and EC. We calculated the total energy of Earth during one orbit for different values of τ, and calculated $\Delta E_{tot} = E_{tot,start} - E_{tot,end}$ which was then plotted in Figure (2). When it comes to what behaviour we expected, if the methods had no errors, we would expect ΔE_{tot} to equal to zero, but that is not the case when one looks at Figure (2). What we expected was that the methods would have some errors and that as the time step decreases, the error would also decrease. In addition, we expected the RK4 method to converge faster than the EC method. Our expectations are confirmed in Figure (2), because one can see that when the number of steps $N = 1/\tau$ increases, the error decreases, and that the RK4 method converges faster than the EC method. The reason for this is because RK4 is a method of higher order than EC [2]. RK4 is a 4th order algorithm while EC is a 1st order algorithm.

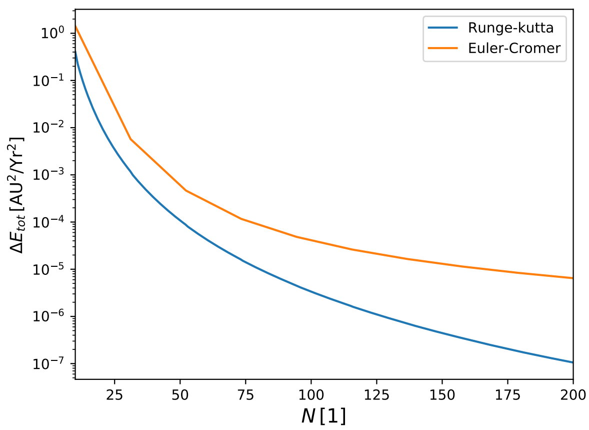

Figure 2: The change in Earth's mechanical energy after one orbit, $\Delta E_{tot} = E_{tot,start} - E_{tot,end}$, plotted as a function of the number of steps $N = 1/\tau$ using the 4th order Runge-Kutta method and the Euler Cromer method.

2.3 Kepler's third law

Kepler's third law states that the square of the period of any planet is proportional to the cube of the semimajor axis of its orbit [3]. When using non-astronomical units this law can be formulated as,

$$T^2 = \frac{4\pi^2}{GM}a^3, \tag{6}$$

where T is the period of any planet, a is the semi-major axis of its orbit and GM is the gravitational constant multiplied with the mass of the sun. However, when we use astronomical units in equation (6), GM would then equal to $4\pi^2$ and we get,

$$T^2 = a^3, \tag{7}$$

where T is expressed in Earth years and a is measured in AU. To check if our code follows Kepler's third law for a given planet, we calculated each side of equation (7) separately for Venus, Earth and Mars and plotted one side as a function of the other in Figure (3). We did not plot for other planets with nearly circular orbits because then it would be difficult to see how Venus, Earth and Mars compare. They would just be scaled down into a single point in the left corner. The graph shows us that there is a linear correlation between T^2 and a^3 for the three planets, so we can conclude that we have managed to demonstrate Kepler's Third Law.

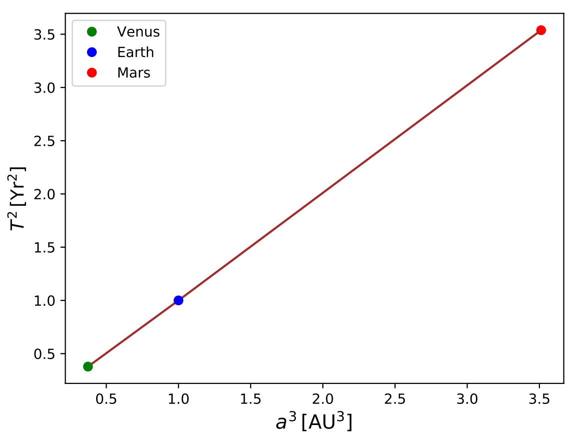

Figure 3: The period of Venus, Earth and Mars squared, T^2, plotted as a function of the semi-major axis of the planets' elliptical orbit cubed, a^3.

3 Perihelion of Mercury á la Einstein

3.1 Case without precession

A orbital trajectory in polar coordinates is given by

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{L^2}F_G(r), \quad r = r(\theta) \tag{8}$$

where $\mu = Mm/(M+m)$ is the reduced mass and $L = \mu r v$ is the angular momentum. In a system of two bodies with mass M and m, if $M >> m$ then $\mu \approx m$. Equation (8) was solved with the RK4 method to simulate the elliptical orbit of Mercury around the Sun. To solve the second order differential equation, it was split into two first order differential equations

$$u_1 = \frac{1}{r}, \quad u_2 = u_1', \quad u_2' = -\frac{\mu r^2}{L^2}F(r) - u_1 = \frac{GM_S}{(rv)^2} - u_1, \tag{9}$$

that were solved with RK4 with initial conditions $u_1 = 1/r_p$ and $u_2 = 0$, where r_p is the distance between the sun and the perihelion [2]. Since $1/r_p$ is a global maximum for u, the derivative must be zero for this value. Since $L/\mu = rv$ is preserved in an orbit, we chose $r = r_p$ and $v = v_{max}$ which is the velocity of Mercury at the perihelion given by

$$v_{max} = \sqrt{GM_S\frac{(1+e)}{a(1+e)}\left(1 + \frac{M_M}{M_S}\right)}, \tag{10}$$

where a is the semi-major axis of Mercury's orbit, e is the eccentricity of Mercury and M_M is the mass of Mercury. The resulting plot of the orbit can be seen in Figure (4), where it is compared to the analytical solution of equation (8). As we can see from the plot, they are identical so we can conclude that the numerical solution is a sufficient model for the elliptical orbit of Mercury.

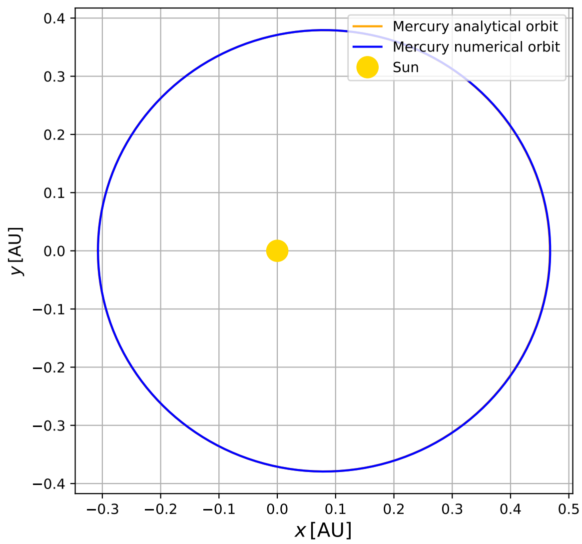

Figure 4: The position of Mercury plotted analytically and numerically as a function of time using the 4th order Runge-Kutta method with time step $\tau = 0.001$ Yr.

3.2 Einstein's correction

Mercury has a 43 arcsecond precession of the perihelion per century explained by Einstein's theory of general relativity. The force law was then corrected for general relativity

$$F_G \approx \frac{GM_S M_M}{r^2}\left(1 + \frac{\alpha}{r^2}\right), \tag{11}$$

where α is a constant. The degree of precession changes with this constant. Einstein derived the value of α to be 1.1×10^{-8} AU^2, which is too small to see a difference when we plot. Instead a larger α was chosen to investigate the effect. Precession with $\alpha = 10^{-4}$ AU^2 can be seen in Figure (6), which results in a 108° precession per century. To find the correct α, the degree of precession was plotted along different values of α ranging from 10^{-4} to 10^{-3}. The resulting plot can be seen in Figure (5). The plot is linear which allows us to find the slope $k = \mathrm{d}\theta/\mathrm{d}\alpha$ with linear interpolation and solve $\alpha k = 43$. With this method α was found to be $\alpha = 43/k \approx 1.076 \times 10^{-8}$ AU^2, which is close to Einstein's value of 1.1×10^{-8} AU^2.

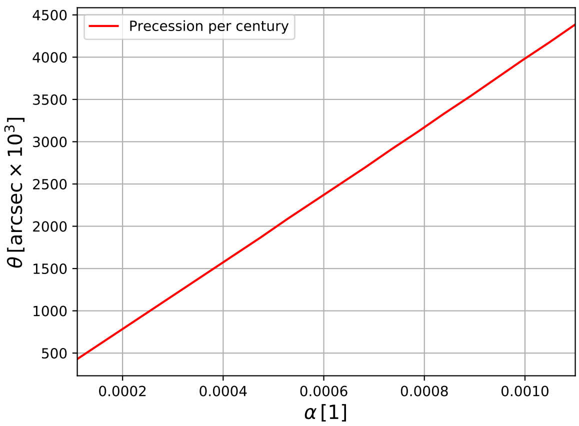

Figure 5: Mercury's precession per century, θ, plotted as a function of α.

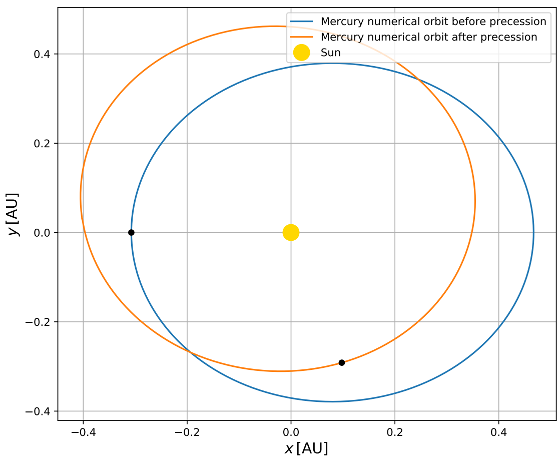

Figure 6: The position of Mercury plotted before and after a century of precession as a function of time using the 4th order Runge-Kutta method with $\alpha = 10^{-4}$ AU^2 and time step $\tau = 0.001$ Yr. The black dots represent the position of Mercury's perihelion.

4 Earth satellite orbits

4.1 Satellite trajectory in LEO

The planetary orbit program from section (2.1) was applied to calculate the trajectory of a satellite in low earth orbit (LEO). The units were changed to SI units, the mass of the Sun, M_S, was replaced with the mass of the Earth, M_E, and the orbital radius was changed to a low earth orbit, r. The velocity was found using equation (10) with $e = 0$ and $a = r$. The resulting plot can be seen in Figure (7). It seems that we have managed to successfully plot a satellite in LEO, as there seems to be no significant errors in the plot.

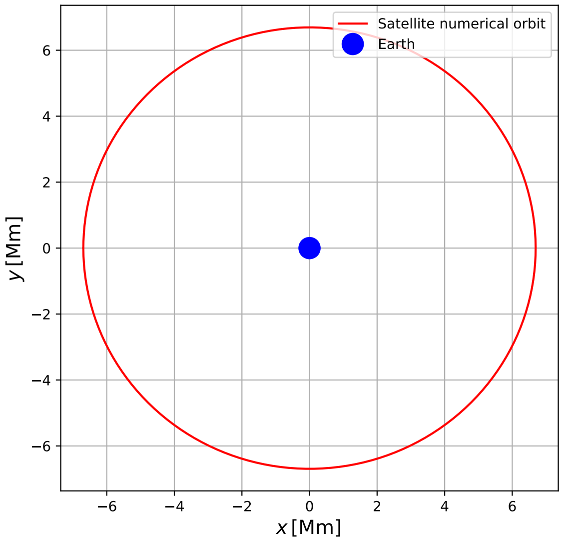

Figure 7: The position of a satellite in low earth orbit plotted as a function of time using the 4th order Runge-Kutta method with time step $\tau = 0.001$ revolutions.

4.2 The most fuel-efficient maneuver from LEO to GEO

To change the orbit of the satellite, its velocity has to be increased in the correct positions. The most fuel-efficient maneuver is the Hohmann transfer orbit using only two velocity increases

$$\Delta v_1 = \sqrt{\frac{GM_E}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right), \quad \Delta v_2 = \sqrt{\frac{GM_E}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1+r_2}}\right), \tag{12}$$

where r_1 is the initial orbital radius and r_2 is the final radius [4]. The time it takes to change from one circular orbit, r_1 into another larger circular orbit, r_2, is given by

$$t = \pi\sqrt{\frac{(r_1+r_2)^3}{8GM_E}} \quad [4]. \tag{13}$$

The first velocity increase pushes the satellite into a elliptical orbit, with the aphelion $r_A = r_2$. When the satellite arrives at the aphelion of the orbit, the second velocity increase is applied. This makes the satellite stay at this distance, r_2, in a circular orbit. To demonstrate this, the program was applied to calculate a Hohmann transfer from LEO to geostationary orbit (GEO). The velocity changes were applied by increasing the velocity factor in the angular momentum $L = \mu r(v + \Delta v_i)$ for $i = 1, 2$. The resulting plot can be seen on the left in Figure (8). This was repeated with two Hohmann transfer orbits with a intermediate orbit between LEO and GEO. Here the angular momentum $L = \mu r(v + \Delta v_i)$ was changed for $i = 1, 2, 3, 4$. The resulting plot can be seen on the right in Figure (8). The total velocity increase for one Hohmann transfer with two velocity increases was $\Delta v = 3883$ m/s, with transfer time $t = 18939$ s. For two transfers with four velocity increases, the total velocity increase was $\Delta v = 4240$ m/s and the transfer time was $t = 42078$ s. Since the fuel consumption of the satellite is directly proportional with the net velocity increase, this demonstrates that the method with two velocity increases is the most fuel-efficient maneuver, because its net velocity increase, Δv, is less than the net velocity increase for the second method.

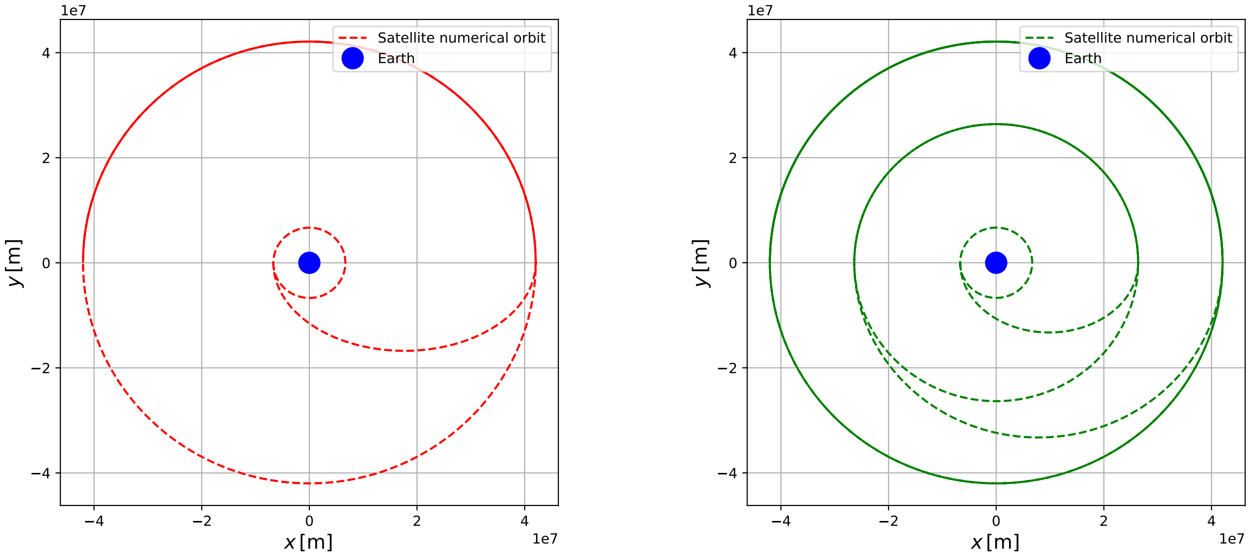

Figure 8: The position of a satellite in orbit plotted as a function of time using the 4th order Runge-Kutta method with time step $\tau = 0.001$ revolutions, *Left:* with one Hohmann transfer from low earth orbit to geostationary orbit. *Right:* one Hohmann transfer from low earth orbit to an intermediate orbit and then another transfer to geostationary orbit.

5 Three-body problem: Earth and Mars

5.1 Interaction between Earth and Mars

In a three-body problem, we have to take into consideration the gravitational force between Earth and Mars, the force between Earth and the Sun, and the force between Mars and the Sun. The Sun is fixed in position at the origin. This changes our equations of motion from section (2.1). For Earth, we have to compute

$$\frac{dx_e}{dt} = v_{x,e}, \quad \frac{dv_{x,e}}{dt} = -\frac{GM_S x_e}{r_{ES}^3} - \frac{GM_M(x_e - x_m)}{r_{EM}^3}, \quad \frac{dy_e}{dt} = v_{y,e}, \quad \frac{dv_{y,e}}{dt} = -\frac{GM_S y_e}{r_{ES}^3} - \frac{GM_M(y_e - y_m)}{r_{EM}^3}, \tag{14}$$

where $GM_S = 4\pi^2$, $GM_M = 4\pi^2(M_M/M_S)$ is a ratio between the mass of Mars and the Sun, r_{ES} is the distance between Earth and the Sun, and r_{EM} is the distance between Earth and Mars. The equations of motion for Mars would then be

$$\frac{dx_m}{dt} = v_{x,m}, \quad \frac{dv_{x,m}}{dt} = -\frac{GM_S x_m}{r_{MS}^3} - \frac{GM_E(x_m - x_e)}{r_{EM}^3}, \quad \frac{dy_m}{dt} = v_{y,m}, \quad \frac{dv_{y,m}}{dt} = -\frac{GM_S y_m}{r_{MS}^3} - \frac{GM_E(y_m - y_e)}{r_{EM}^3}, \tag{15}$$

where $GM_E = 4\pi^2(M_E/M_S)$ is a ratio between the mass of Earth and the Sun, and r_{MS} is the distance between Mars and the Sun. When we implement the RK4 method on equations (14) and (15), we have to calculate both sets at the same time. When we plot the positions of Earth and Mars, $(x_{e,i}, y_{e,i})$ and $(x_{m,i}, y_{m,i})$ for $i = 1, 2, ..., n$, the resulting plot can be seen in Figure (9). Here we see that both planets have stable orbits, so we can conclude that Mars has a negligible effect on Earth and vice versa.

5.2 Hypothetical test

In our first hypothetical test, we assumed that Earth had the mass of Jupiter. The sun is still fixed at the origin. The resulting trajectory after 10 orbits can be seen on the left in Figure (10). The orbit of Mars is still somewhat stable, but we see several rings instead of just one ring like in Figure (9). This shows us that when Earth's mass increases to the size of Jupiter's, the gravitational force acting between Earth and Mars is no longer negligible. The same can be said when we reverse the situation - we assume that Mars has the mass of Jupiter. The center plot in Figure (10) shows the same result as the left plot. We see many similar rings, and

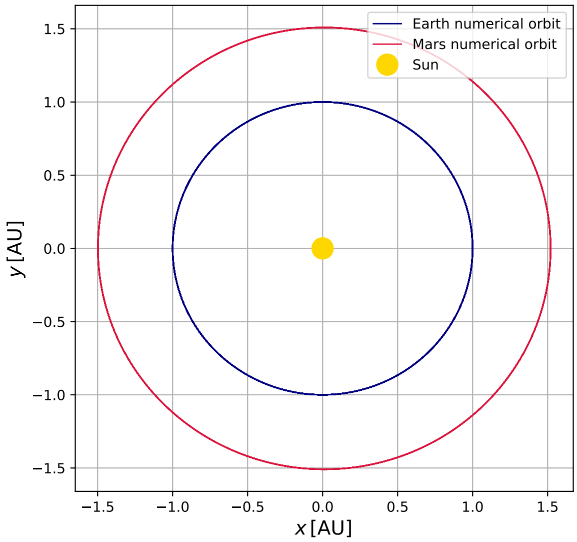

Figure 9: The position of Earth and Mars plotted as a function of time during 10 orbits using the 4th order Runge-Kutta method when the mass of Earth is $M_E = 1$, the mass of Mars is $M_M = 0.1074$ and the time step is $\tau = 0.001$ Yr.

even though Earth's orbit is still somewhat stable, the gravitational force is no longer negligible between the two planets. The right plot in Figure (10) shows what happens when both Earth and Mars have the mass of Jupiter. We then get a plot that is a merging of the left and center plot, which leads us to the same conclusion.

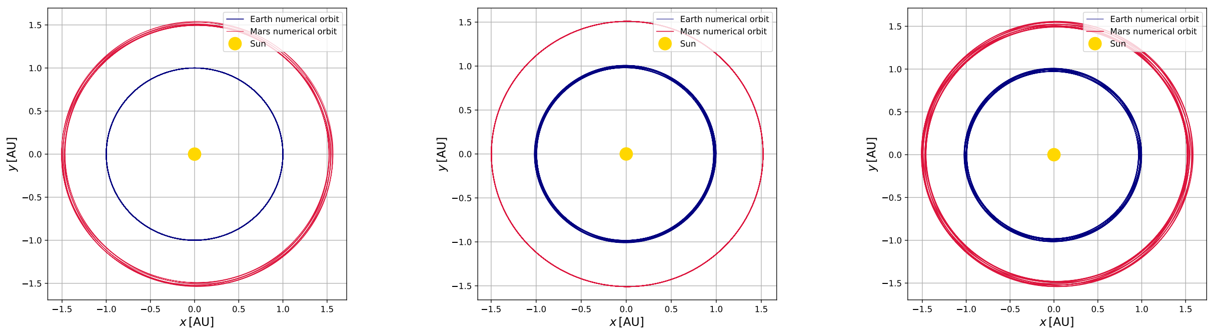

Figure 10: The position of Earth and Mars plotted as a function of time during 10 orbits using the 4th order Runge-Kutta method with time step $\tau = 0.001$ Yr, *Left:* when Earth has the mass of Jupiter $M_E = M_J = 317.89$ and the mass of Mars is $M_M = 0.1074$. *Middle:* when Mars has the mass of Jupiter $M_M = M_J = 317.89$ and the mass of Earth is $M_M = 1$. *Right:* when both Earth and Mars has the mass of Jupiter $M_E = M_M = M_J = 317.89$.

In our second hypothetical test, we first assume that Earth has 10^4 times its original mass. This gives us a special result which is shown in the left plot of Figure (11). The orbit of Mars has drastically changed and does not seem stable anymore. When we increased the Earth's mass to 10^5 its original mass, Mars got ejected from the solar system. In the next test, we then assumed that Mars has 10^4 its original mass. This result can be seen in Figure (11) on the right. The plot is not as special as the left one, but we can see that Earth's orbit is not that stable anymore either. When we multiplied Mars' mass by 10^5, Earth also got ejected from the solar system. This is sufficient to conclude that when either Earth or Mars has a large enough mass, it will disturb each other's orbits greatly and we should be grateful that this is not the case at the moment.

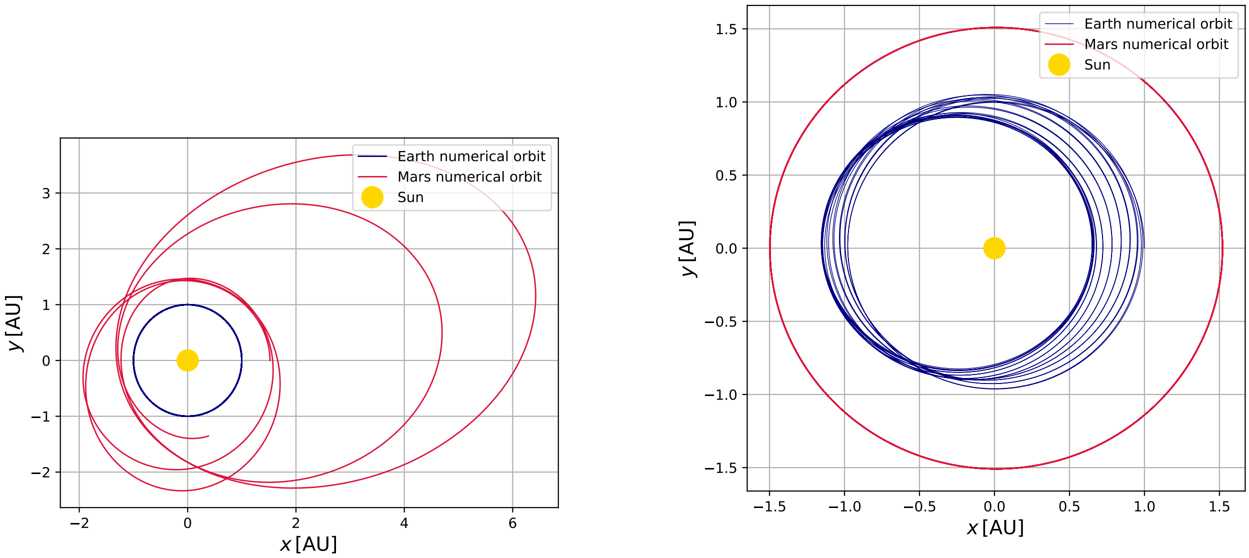

Figure 11: The position of Earth and Mars plotted as a function of time during 10 orbits using the 4th order Runge-Kutta method with time step $\tau = 0.001$ Yr, *Left:* when the mass of Earth is $M_E = M_E \times 10^4$ and the mass of Mars is $M_M = 0.1074$. *Right:* when the mass of Mars is $M_M = M_M \times 10^4$ and the mass of Earth is $M_E = 1$.

References

[1] "Semi-implicit Euler method", Wikipedia. URL: https://en.wikipedia.org/wiki/Semi-implicit_Euler_method (Accessed: 22nd April 2018)

[2] T. Sauer, Numerical Analysis, Addison-Wesley Publishing Company, 2nd ed., 2011. (Pages 304, 316 and 317)

[3] "Kepler's Laws", Hyper Physics. URL: http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html (Accessed: 22nd April 2018)

[4] "Hohnmann transfer orbit", Wikipedia. URL: https://en.wikipedia.org/wiki/Hohmann_transfer_orbit (Accessed: 22nd April 2018)