~~The~~ Calculating the min of valuations of lattice index Jacobi forms 23-10-2015

Let $\lambda : \underline{L} \hookrightarrow \mathbb{Z}^N$ is an isometric embedding, let l be the ~~level of $\underline{L}$~~ smallest positive integer such that lG is integral, where G is a Gram matrix of $\underline{L}$. Finally let

$$f_\lambda(x) := B(\lambda_1(x)) + \dots + B(\lambda_N(x)).$$

Proposition

The function $f_\lambda(x)$ takes its minimum value at a rational x whose denominators are bounded by $2l$.

~~proof~~ Suppose $\underline{L} \simeq \mathbb{Z}^n$ and $\lambda : x \mapsto xM$ with a suitable matrix M. Clearly, since λ is isometric

$$MM^t = G$$

and G is the Gram matrix of $L = \mathbb{Z}^n$ (with respect to the canonical basis).

Let H be the "half lattice", i.e.

$$H = (\tfrac{1}{2}, \dots, \tfrac{1}{2}) + \mathbb{Z}^N.$$

Then f_λ takes its minimum at a point x where xM is closest to a point h in H. In particular $h - xM$ is perpendicular to $\mathbb{Z}^n M$, i.e.

$$0 = (h - xM)M^t = \cancel{Mh^t - MM^t x^t}$$
$$= hM^t - xG,$$

that is

$$x = hM^t G^{-1}.$$

From this the proposition is obvious. $\square$