Abstract

Music and mathematics are two subjects more related to one-another than most people know. A single melody can be thought of as a sequence of changing pitch frequency intervals, along side a sequence of rhythmic durations. In this project we will first explore ways in which to translate musical melodies into real-valued numerical sequences, and second we will study these sequences by applying familiar fractal dimension metrics in an attempt to explore fractal patterns in music and build insight into what "fractal music" really means.

Fractals

Fractals in music

Misconceptions

A primer on music

Pitch, notes, rhythm

Intervals and melody

Self-similarity scaling in music

One of the earliest attempts at mathematically quantifying musical self-similarity was conducted by Richard Voss and John Clarke. From their results they concluded that within genres of music a $1/f$ power-law scaling behavior is characteristic of musical components for pieces in the genre (though they were specifically concerned with the Baroque era compositions of J.S. Bach, or just "classical" in layman's terms). However, there may be many different ways in which measurable self-similarity within music can manifest; chapter 7 of the Mandelbrot text, written by Brothers, provides a few examples of how scaling within music has been quantified:

1. *Duration scaling*: the distribution of durations for individual notes is self-similar within a piece,
2. *Pitch scaling*: the distribution of pitches is statistically self-similar,
3. *Melodic interval scaling*: the distribution of melodic intervals is self-similar,
4. *Melodic moment scaling*: the distribution of the changes in melodic intervals is stylistically self-similar,

5. *Harmonic interval scaling*: the distribution of harmonic intervals is self-similar,
6. *Structural scaling*: the structure of the music from a compositional standpoint relies on nested or recursive patterns, and
7. *Motivic scaling*: a motif, melodic or rhythmic, is repeated simultaneously at different time scales (called augmentation or diminution).

Bach and fractals

The first part of the fifth movement, the "Bourrée", from Johann Sebastian Bach's Cello Suite No. 3 in C Major, BWV 1009.

The paper by is dedicated entirely to examining the scaling characteristics within this single section of music, and the paper "Multifractal analyses of music sequences" by Zhi-Yuan Su and Tzuyin Wu also takes a look at this very same section (along with two other musical examples).

Figure 1: Cello Suite No. 3 in C Major, BWV 1009, V. Bourrée I.

Implementing melodic interval and duration scaling measurement

Converting a melodic line into point sequences

Limitations

The most dramatic limitation of this procedure is that it requires the musical material to be monophonic—only one note at a time—while the vast majority of modern music (and actually the majority of music since the 14th century) is polyphonic. Despite this, a piece of music containing simultaneous notes across simultaneous voices could potentially be reconstructed into standalone parts, where each roughly functions as standalone piece consisting of only a monophonic melody. Think of a Bach cantata for soprano, alto, tenor and bass four-part choir, where each voice is essentially its own piece, its own melody. On the other hand, not all music can be so simply reconstructed into parts. Even the Bach selection examined by Su and Wu (the Bourrée), despite being almost entirely monophonic, contains polyphony in measures 2, 4, and 28 (the last), but this may be an artifact of the editor, as some copies have a single G instead of the E chord. Either way, one can simplify a polyphonic melody by selecting a single constituent note whenever there's a simultaneous group of notes, as I did for simplifying the Bourrée.

Figure 2: French Suite No. 5 in G Major, BWV 1009, III. Gavotte