Fractal Patterns and Music

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1 Abstract

In this paper we will explore the history and some existing research in the area of fractals in music. We include a description of fractals and how they can be perceived in music, as well as a primer on pitch, rhythm, melody and harmony (the basic building-blocks of music). We discuss how the note frequencies follow a power law relation across many genres of western music. We present the Michael Bulmers technique for creating music that is close to true pink noise. There is also an overview of how fractals have occurred in musical compositions in the past, including the specific example in Bachs Cello Suite No. 3. Then we attempted to reproduce the results of a paper in calculating a particular fractal dimension within a musical example.

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2 Fractals

Out of principle we must begin our discussion on fractals in music by mentioning the late Benoit Mandelbrot. Mandelbrot coined the phrase fractal in 1975 to describe objects that retained complexity and detail, at different scales; similar to how the photograph of a pile of rocks can look similar to the picture of a mountain, if there is nothing to provide a sense of scale[3]. Initially, Mandelbrot observed this in time series graphs of product prices in the economy, where if there was no scale it would be impossible to tell if the price changes were updated per minute, hour, day, etc.[3]. Mandelbrot also tended to speak of roughness[4]. Something that was smooth would be akin to time series plot that looked like a smooth curve. A rough graph would look like a plot with many dynamic changes.

Mandelbrot believed in the power of the human eye to notice roughness[3, 4], but music provides a unique challenge, because in the moment, music is felt in a psychological sense, and is usually not observed as a whole, unless one acquires sheet music or other physical interpretation of the music as the composer intended. In fact, in the memoir Benoit Mandelbrot A life in Many Dimensions, Harlan Brothers wrote, Benoit Mandelbrot always had a strong feeling that music could be viewed from a fractal perspective. However, without our eyes to guide us, how do we gain this perspective?[13] The question posed is an excellent one, and Brothers goes on to discuss that generally there are seven ways that fractals can appear in music.

Before we discuss these seven ways, we would like to mention that there are several misconceptions as to what fractal music is of which Brothers discusses[5]. The most common misconception is that converting fractal images into sound produces fractal music. In many cases these transformations can hardly be classified as music and simply as noise. Another misconception is to think that iterations always cause fractals in music. This is not true in the physical sense as the logistic map illustrates[5] and it does not hold in music either. The last misconception that Brothers talks about in regard to fractal music is that of self-similarity. As with fractal diagrams, self-similarity is a necessary but not sufficient condition[5]. He gives the example that, onions, spirals, and Russian dolls are not fractal; they do not contain a minimum of two matching or similar regions in which the arrangement of elements either mirrors or imitates the structure of the object as a whole.[5]. So, it is necessary that parts of a musical piece be similar to larger sections of the musical composition.

3 A primer on music

Before introducing the different ways in which fractal patterns arise in music it is essential that some basic building-blocks of music be introduced.

3.1 Pitch, intervals, and rhythm

A single key on a piano, when struck, produces a single pitch. Pitch is one of the most fundamental properties of notes in music, but what exactly is pitch? The sound that is produced by the piano is really the combination of several strings vibrating, each producing different frequencies of sound. Together this combination forms the characteristic sound of the piano (also called timbre, the specific quality to the sound of, in this case, the piano). The strongest frequency in this collection is called the fundamental frequency of the note being played, and to this frequency a name is assigned. In the case of a piano, and of all Western music, these frequency names come from the alphabet: A, B, C, D, E, F, and G.

Though pitches are unique unto their frequency, some pitches are considered equivalent with respect to the ratio of their frequencies. For example, the pitch A4 (concert A) sounds at 440 cycles per second (hertz), while A5 sounds at 880 hertz: a ratio of 1:2. This ratio is called an octave, and thus we have names for distances between frequencies which we call intervals. Within Western music notation there are a finite number of named intervals; here are a few: perfect octaves, perfect fifths, diminished fifths, augmented fourths (equivalent to diminished fifths), major thirds, minor thirds, etc. Between any two notes (and even between a note and itself) there is an interval between them, and is measured aptly by counting the number of keys of the piano it takes to get from one note to the other where these steps are called semitones. For example, the interval between any F and the nearest B on the piano consists of six semitones. As an aside, this interval is known as a tritone and is one of the most dissonant intervals producible on the piano.

Returning to octaves, despite having unique frequencies, any two pitches which are an interval of an octave apart (or multiple octaves apart, e.g. C2 and C5) from one another, are considered equivalent. Lastly, If you are familiar with the piano then you will know there are actually are more than seven keys in between octaves, twelve precisely. These are just smaller subdivisions of the frequency ranges between octaves.

The other most fundamental property of notes is duration, also called rhythmic value (or simply value). Music as listened to is inherently ephemeral: in the moments in which we experience music by listening to it, it exists, but then it is gone. How long we experience the sound of a pitch is called its duration, and we might measure the duration of a pitch by how long it is played in seconds. Establishing a baseline duration is important in the composition of music; some pieces of music consist of pitches that are long in duration, while others consist of pitches that exist only for a split-second.

However, not all pitches need have the same duration, and so we have rhythm: the ratios between pitches in terms of duration. For example we might play one pitch for twice as long as the one before, and the next two pitches for a quarter as long. Like with pitch, different rhythmic values also have names within Western music notation, some of the most basic of which being whole notes, half notes, quarter notes, and eighth notes. Notice that the names all have to do with fractions. A piece of music might set a baseline speed by stating that the duration of a quarter note should be a certain number of secondsthen the duration of a half note would be twice as long as the duration of the quarter note, a whole note four times as long, an eighth note half as long, a sixteenth note a quart as long, etc.

In summary, it is these two fundamental properties, pitch and rhythmic value, that are a part of every note in a piece of music.

3.2 Melody and harmony

Melody and harmony are in some ways the two sides to a coin: melody is how notes move one at a time in a single line horizontally across the page, while harmony is how notes sound simultaneously. Colloquially we call harmony the experience of multiple things working together in tandem in a pleasing manner. In music, working in tandem might mean how good two or more notes sound together. The notes C E and G played simultaneously form a kind of harmony that is familiar to anyone that is accustomed to Western music: a C major chord. Chords similar to this one are characterized often emotively, but also intervalically (that is by ratios of the distances between the notes forming the chord), and since it is hard to quantify emotion, analysis of music usually pertains to intervals. Melodic analysis pertains to observing the intervals between sequential notes that are not simultaneous. Just as before, the notes C E and G form a C major chord, but if played sequentially instead of simultaneously we have melody instead of harmony and possibly a different emotional experience altogether (this breaking up of harmony is called arpeggiation).

4 Self-similarity scaling in music

As a subject of research, fractals and self-similarity in music may be fairly niche, but there is no shortage of literature, new or old. One of the earliest attempts at mathematically quantifying musical self-similarity was conducted by Richard Voss and John Clarke, and in 1975 they published the article "$1/f$ noise in music and speech". They concluded that, within genres of music, a $1/f$ power-law scaling behavior is characteristic of musical components for pieces in the genre (though they were specifically concerned with the Baroque era compositions of J.S. Bach, or just "classical" in layman's terms).

However, there may be many different ways in which measurable self-similarity within music can manifest; chapter 7 of the Mandelbrot text, written by Brothers, provides a few examples of how scaling within music has been quantified:

- *Duration scaling:* the distribution of durations for individual notes is self-similar within a piece,
- *Pitch scaling:* the distribution of pitches is statistically self-similar,
- *Melodic interval scaling:* the distribution of melodic intervals is self-similar,
- *Melodic moment scaling:* the distribution of the changes in melodic intervals is stylistically self-similar,
- *Harmonic interval scaling:* the distribution of harmonic intervals is self-similar,
- *Structural scaling:* the structure of the music from a compositional standpoint relies on nested or recursive patterns, and
- *Motivic scaling:* a motif, melodic or rhythmic, is repeated simultaneously at different time scales (called augmentation or diminution).

A great difficulty in analyzing fractal patterns in music is that the majority of these scaling examples rely on a deep understanding of how music works at a compositional level, something that computers are likely to struggle with. Furthermore, some scalings may be present in a piece of music while others may be absent. Many of the movements from Johann Sebastian Bachs cello suites are great examples of highly melodic music, and so we might readily examine such a piece using duration, pitch, or any of the melodic scaling methods. However, there may be a distinct lack of harmony throughout these suites, and so an analysis of harmonic interval scaling would be inappropriate. Additionally, Brothers warns that "it is important to note that, regardless of the type of scaling under consideration, in order to fulfill a power law relation, any inherent pattern in a group of musical elements requires the presence of a minimum of three distinct levels of scaling. This requirement respects the fact that the log-log plot of a power law relation appears linear; at least three data points are needed to assert a linear relationship"[2]. Fractal and multifractal dimension

5 Structural scaling and motivic scaling (Bach and fractals)

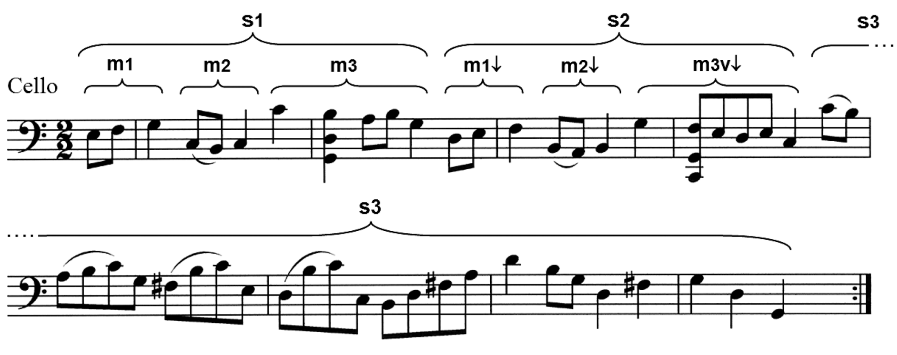

Figure 1: Bach Cello Suite No. 3 in C Major, V. Burrée I.

There are many tales of Bach's impressive talents as a composer and virtuoso musician. A biographer of Bach recounts that Bach arrived to the main town in Prussia as a stranger. Upon arrival King Frederick the Great invited him to the Royal Palace to try the new pianofortes. "After he had gone on for some time, he asked the King to give him a subject for a Fugue, in order to execute it immediately without any preparation."[6]. Speaking about this same event, Douglas Hofstadter in his book Godel, Escher, Bach: An Eternal Golden Braid states, "the ten canons in the musical offering are among the most sophisticated canons Bach ever wrote. However, curiously enough, Bach himself never wrote them out in full. This was deliberate. They were posed as puzzles to King Frederick. It was a familiar musical game of the day to give a single theme, together with some more or less tricky hints, and to let the canon based on the theme be 'discovered' by someone else"[7]. This shows that Bach truly incorporated math into his music and that his compositions were much more than a creative thought; that they contained a mathematical complexity that retained beauty. So, it is possible to see that his music contained fractal.

The paper Structural Scaling of Bachs Cello Suite No.3 by Harlan Brothers is dedicated entirely to examining the scaling characteristics within the Bourrée Part 1 (the score of which is displayed in Figure **??**). The structural scaling that Brothers noticed in his Bachs Cello Suite No. 3 can be described by the four figures he provided below. In the section m1 and m2 of the first figure we see two eighth notes followed by a quarter note. This structure of Short-Short-Long is repeated throughout this section of the piece by Bach. The section m3 is also Short-Short-Long but scaled twice as long as the m1, and m2 section so it contains two quarter notes followed by a section of the duration of a half note. Since m3 is twice as long as m2 and m1, in section s1 we again see the pattern of Short-Short-Long. Continuing in the same manner we can see how s1, s2 and s3 form a Short-Short-Long pattern as well. In Figure 2 we can see again that the self-similar pattern resembles the section as a whole in three different levels.

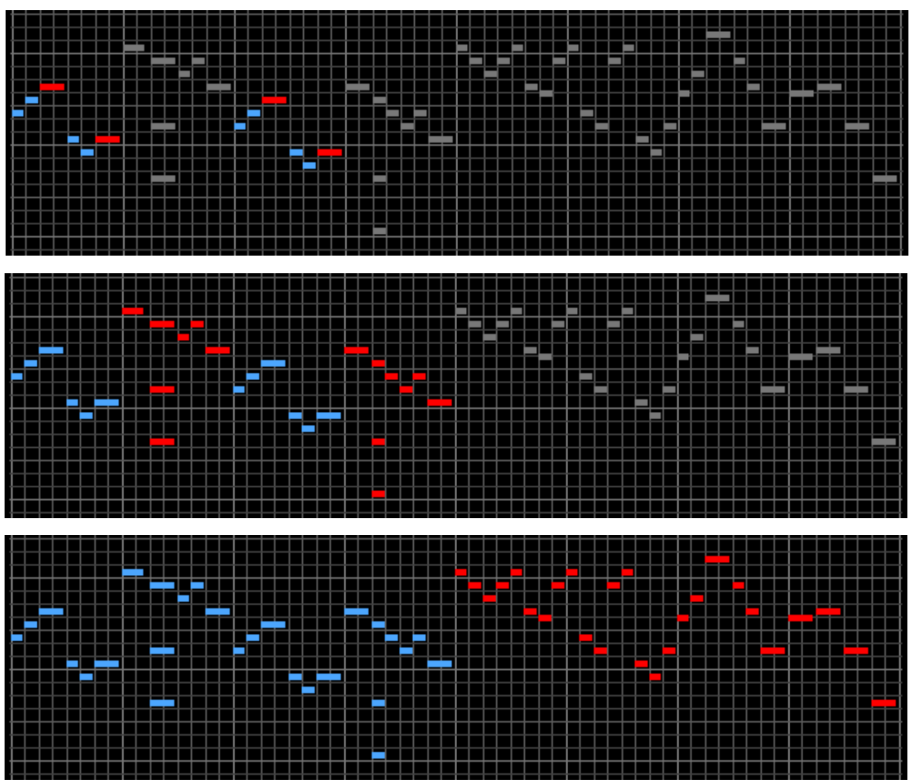

Figure 2: Structurally self similar patterns at different scales.

6 Pitch scaling and stochastic composition

Stochastic pitch scaling can be generated using a method that involves rolling a number of dice, and the sum of the dice gives a note that was assigned a number. For example 7 could be assigned to the note D. Bulmer used six dice to generate a scale of notes that when listened to has a static like sound. The noise that is termed for this is white noise, which music that is considered white noise has an atrocious sound to it due to the disconnection of flow when going from note to note. White noise is considered to be too unpredictable, where brown noise is considered to be very predictable. Brown noise has a Brownian motion which can be seen as being too predictable because the notes have an erratic motion back and forth. White and brown noise are the two extremes for random music.

Pink noise has been seen to achieve both of these extremes which brings a balance which is considered the $1/f$ noise[1]. Achieving a true balance between these extremes is very difficult, as described by Mandelbrot (1971)[8,9]. Pink noise has been seen to have a self-similar process with a pattern that is long-range dependent and exhibits short-term randomness.

Voss approximates pink noise using a variation of the dice method that was described earlier. Using n dice, for our experiment we used $n = 3$, we generated 2^n notes. The table below was constructed by rolling three dice that are labeled A, B, and C. We sum of the three dice and the total gave us a note which each note was assigned to a value. Binary digits were used to generate the next note which changes from row to row in the table. The table below is the results from our experiment. The notes below provide a stability in the sequence while having long-range dependence which is needed to be considered pink noise.

A	B	C	Total	Note
2	6	1	9	C
2	6	5	13	G
2	2	5	9	C
2	6	2	10	D
4	6	2	12	F
1	6	3	10	D
2	4	3	9	C
6	2	1	9	C

Figure 3 shows a time plot produced using the dice method created by Voss, we are able to see that the series of notes exhibit long-range dependence. There is more stability due to the high digits have less change.

When describing noise we can use autocorrelation to describe the notion of music notes being "related" over time. Autocorrelation is composed of two data sets, r, which is the values in the noise sequence and k which is called the lag.

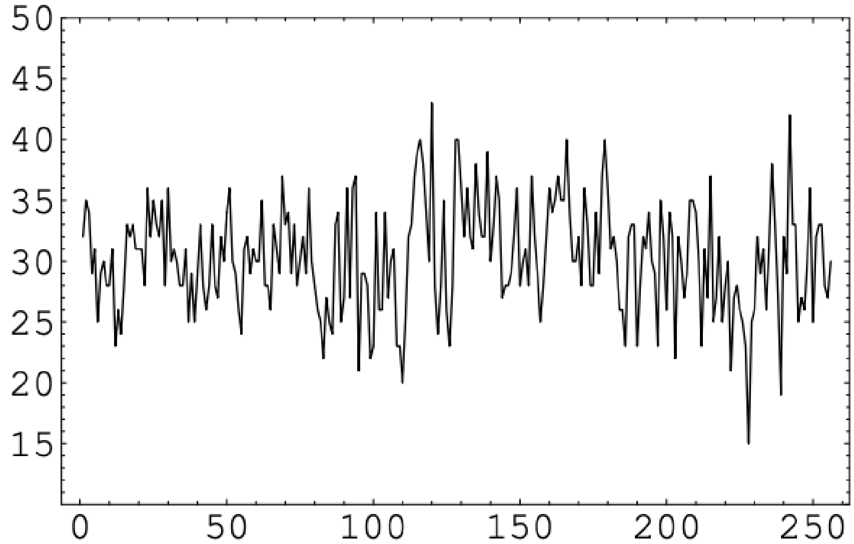

Figure 3: Time plot of pink noise[8]

$$r_k = \frac{\sum_{t=1}^{N-k}(x_t - \bar{x})(x_{t+k} - \bar{x})}{\sum_{t=1} N(x_t - \bar{x})^2}$$

The formula above can be used to construct correlograms, a plot of autocorrelation against lag. This will give a visual into the autocorrelation structure of the noise.

Figure 4 shows that within the correlation over time does not disappear to even over a longer period of time. Pink noise exhibits the 1/f phenomena which is attributed to the pitch scaling. We can see that that the extremes of pitch scaling can lead to some dissonance in sounds. When finding a balance in the scaling the results can begin to exhibit fractal characteristics. The compositions of songs are reminiscent of the Koch snowflake which possesses long-range dependence and short-term randomness which was observed in pink noise. The results is a composition that is self-similar which like snowflakes are considered fractals.

7 Melodic interval and duration scaling

In their article Multifractal analyses of music sequences[12], Zhi-Yuan Su and Tzuyin Wu concerned themselves with the analysis of music using two of the seven previously mentioned methods of structural scaling: melodic interval

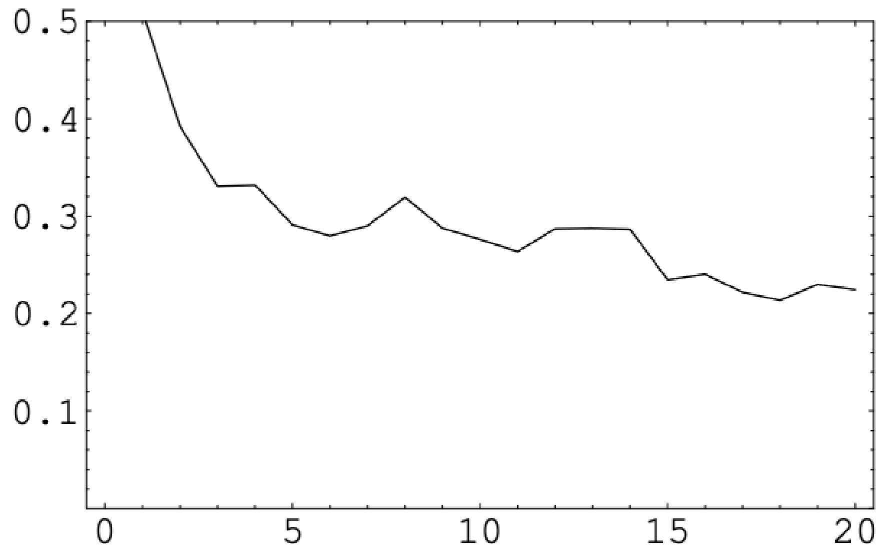

Figure 4: Correlogram for pink noise[8]

scaling and duration scaling. Following a summary of music as 1/f noise and fractal geometry in music as described by the literature of authors like Mandelbrot and duo Voss and Clarke, authors Su and Wu note that many previous investigations of scaling exponents refer only to mean properties of musical sequences. However, they argue that fractal geometries are rarely accurately characterized by any single scaling exponent, that fractal geometries and phonomena change and develop throughout a piece of music, and that such clustering patterns are certainly not uniform. A more granular approach was necessary. The authors loosely define a multifractal as an interwoven set constructed from sub-sets with different local fractal dimensions[12]. Su and Wu provide a formalization of the local scaling exponent of a point distribution (the Hölder exponent) α given by the equation

$$\alpha = \lim_{r \to 0} \frac{\log p_i(r)}{\log r}.$$

Where r is the width of sub-covers of the point distribution and $p_i(r) = N_i(r)/N$ the portion of points that fall within the ith sub-cover. The generalized dimension of this point distribution is given by

$$D_q = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \sum_i p_i(r)^q}{\log r}$$

For q is a given weight (or moment in terms of the exponent). They note that common approaches to calculating the local Hölder exponent first calculate the generalized dimension and utilize a relation between the generalized dimension D_q and q via a Legendre transformation. However, for signals which are sampled or discrete this method fails, as D_q must be a smooth function. Instead the multifractal spectrum $f(\alpha)$ can be obtained directly from the weighted $p_i(r)$ as per the following equations

$$\mu_i(r,q) = p_i(r)^q / \sum_i p_i(r)^q$$
$$\alpha(q) = \lim_{r\to 0} \frac{\sum_i \mu_i(r,q) \log p_i(r)}{\log r}$$
$$f(q) = \lim_{r\to 0} \frac{\sum_i \mu_i(r,q) \log \mu_i(r,q)}{\log r}.$$

Following these formal descriptions, Su and Wu describe an entire methodology for analyzing the multifractal spectrum of melodic musical passages by converting the notes of the passage into two point sequences: one from the absolute intervals between sequential notes, and one from the relative rhythmic values of each note.

7.1 Melodic interval sequence

The first note of the passage becomes the first point of a sequence of integers with value zero. The second point of the sequence then is the absolute value of the intervallic distance between the first and second notes of the piece in terms of semitones, plus the value of the previous, plus one (in the case that two neighboring notes are actually the same note). As in the example from the music primer, the interval from F to B consists of 6 semitones, thus the absolute intervallic distance in either direction (from F to B or from B to F) is 6 semitones. Repeating this process for the entirety of the passage we generate the point sequence of integers corresponding to absolute intervallic distance we desire. We can then consider these values to be the integer positions on a numberline at which our points sit, and thus we generate a sequence of points upon which we can measure sub-coverings.

As an example consider a passage of three notes, F B C. The first value is always zero, the second is $6 + 0 + 1 = 7$, and the last is $1 + 7 + 1 = 9$ (B to C is a single semitone distance). Thus we have a sequence of points at 0, 7 and 9.

7.2 Rhythmic sequence

For the construction of the rhythmic sequence. For our sequence of points, we mark the 0 position as the starting point. Then we determine the shortest note duration of the entire passage. We then divide the rhythmic value of every note in the passage by this shortest duration. Then similar to the interval sequence, every successive value accumulates the values of those that came before it. And finally these values become positions with the sequence of points.

As an example consider a passage of two quarter notes and two eighth notes. We start with 0, and find eighth notes to be the smallest rhythmic value. Two eighth notes divide a quarter note, so the first quarter note is mapped to $\frac{1/4}{1/8} + 0 = 2$. Similarly the second quarter note is mapped to 4, and then the eighth notes are mapped to 5 and 6. Thus we have a sequence of points at 0, 2, 4, 5 and 6.

7.3 Local Hölder exponent calculation

Going forward, the point sequence used is generically either the melodic interval sequence or the rhythmic sequence. Since both are simply sequences of integer points on a numberline the process is identical. The general method of obtaining the scaling exponent is very similar to the methods of determining the box-counting dimension within the material of the class. We determine a range of values for r, the width of sub-coverings, from 2 to one-tenth of N the total length of the sequence. Given a center position i on the sequence we count the number of points lying within an open ball of radius r at the ith position. The local Hölder exponent α is determined to be the slope of a linear fitting of data points given by $\log r$ versus $\log p_i(r)$. We then vary i within the middle 1/10 to 9/10 portion of the sequence to determine the variation of the local Hölder exponent for different localities within the sequence.

7.4 Reproduction of results

(All the material of this project can be found at `https://github.com/nilsso/math538-project`, including all source code and the paper itself.)

The Python language was chosen for the attempt at reproducing the results in the Su and Wu paper, and the first stage was to develop a way to represent a melody in code. There is a plethora of music notation software out there in the world, but the especially niche LilyPond system allows one to notate music in plain text. Due to familiarity with this system, LilyPond notation was adopted.

First a few helper classes, `Note` and `Melody`, were written to encapsulate the data about pitch and rhythmic values for notes and to encapsulate data about the notes in a melody, as well as facilitate the parsing of LilyPond notation (abbreviated Ly). In the end, constructing a melody can be accomplished like in the following example:

```python
notes = [ "c'4", "g", "e8", "c" ]
M = Melody(notes)

for n in M.notes:
    print(n)
```

Which results in the formatted output:

```
C5 quarter-note
G4 quarter-note
E4 eighth-note
C4 eighth-note
```

Then chosen from the three examples in the article was Gossec's *Gavotte*, which the melody was simplified and notated using the Ly format:

```
notes = [
        # 1
        "a'8", "b", "a", "fis", "g", "a", "g", "e"
        # 2
        "d4", "b'", "d,2",
        # 3
        "g8", "a", "g", "e", "fis", "g", "fis", "d",
        # etc ...
```

Finally functions for calculating the point distribution $p_i(r)$, the local Hölder exponent α, and the Hölder variation $f(\alpha)$ were developed, as well as plotting utilities. The end result of these steps are Figures 5, 6 and 7 that begin to approach those illustrated in the Su and Wu article.

7.5 Limitations of the method

The most dramatic limitation of this procedure and this metric of scaling in general is that it requires the musical material to be monophonic—only one note at a time—while the vast majority of modern music (and actually the majority of music since the 14th century) is polyphonic. Despite this, a piece of music containing simultaneous notes across simultaneous voices could potentially be reconstructed into standalone parts, where each roughly functions as standalone piece consisting of only a monophonic melody. Think of a Bach cantata for soprano, alto, tenor and bass four-part choir, where each voice is essentially its own piece, its own melody. On the other hand, not all music can be so simply reconstructed into parts. Even the Bach selection examined by Su and Wu (the Bourrée), despite being almost entirely monophonic, contains polyphony in measures 2, 4, and 28 (the last), but this may be an artifact of the editor, as some copies have a single G instead of the E chord. Either way, one can simplify a polyphonic melody by selecting a single constituent note whenever there's a simultaneous group of notes, as was done for simplifying the Bourrée.

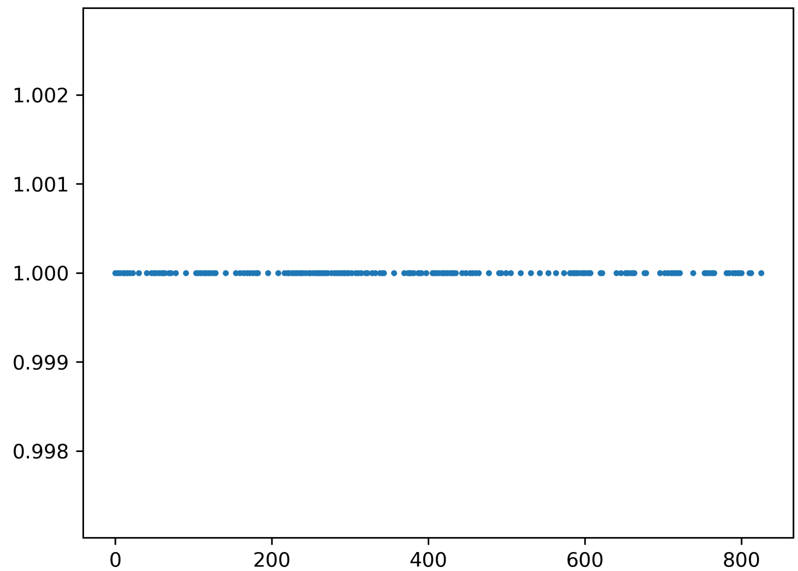

Figure 5: Melodic interval sequence of the Gossec's *Gavotte*.

8 Conclusions

In conclusion, there is quite a lot of insight into the form and structure of music that can be gained by studying music with an appreciation of fractals. Fractal geometry patterns have a way of appearing in most of nature, so despite (most) music being intelligently designed it should not be all too surprising that fractals also show up in music, especially when there exist psychoacoustic explanations as to why humans have been composing music in certain genres and with certain sensibilities for melodic and harmonic consonance and dissonance for centuries. The composition of music is also accessible in a very mathematically intuitive and intriguing way by applying stochastic and self-scaling patterns. Lastly, there is much room for further research and improvement of self-scaling pattern analysis techniques, with applications in classifying individual pieces or even entire genres of music based on their different fractal dimensions.

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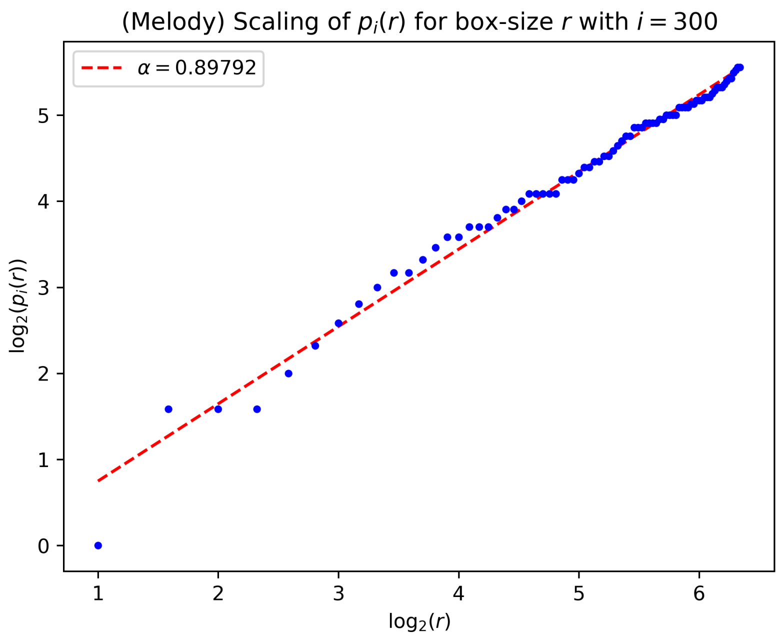

Figure 6: Scaling of $p_i(r)$ with sub-cover size r of *Gavotte*.

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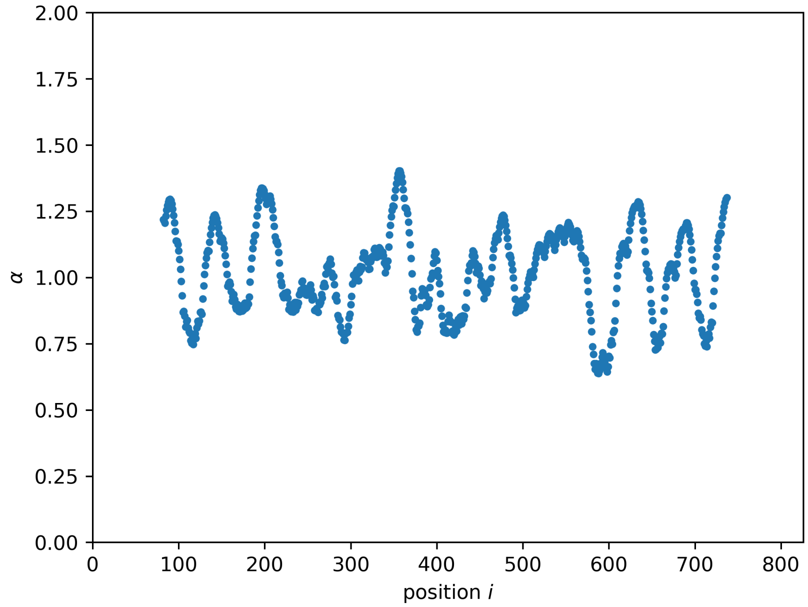

Figure 7: Variation of Hölder exponents along the melody sequence converted from the *Gavotte*.

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