## Abstract

Music and mathematics are two subjects more related to one-another than most people know. A single melody can be thought of as a sequence of changing pitch frequency intervals, along side a sequence of rhythmic durations. In this project we will first explore ways in which to translate musical melodies into real-valued numerical sequences, and second we will study these sequences by applying familiar fractal dimension metrics in an attempt to explore fractal patterns in music and build insight into what “fractal music” really means.


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## Introduction

Out of principle we must begin our discussion on fractals in music by mentioning the late Benoit Mandelbrot. Mandelbrot coined the phrase “fractal” in 1975 to describe objects that retained complexity and detail, at different scales; similar to how the photograph of a pile of rocks can look similar to the picture of a mountain, if there is nothing to provide a sense of scale{MIT}. Initially, Mandelbrot observed this in time series graphs of product prices in the economy, where if there was no scale it would be impossible to tell if the price changes were updated per minute, hour, day, etc. {MIT}. Mandelbrot also tended to speak of “roughness”{TED}. Something that was smooth would be akin to time series plot that looked like a smooth curve. A rough graph would look like a plot with many dynamic changes.

## Fractals

## Fractals in music

Mandelbrot believed in the power of the human eye to notice “roughness” {MIT}{TED}, but music provides a unique challenge, because in the moment, music is felt in a psychological sense, and is usually not observed as a whole, unless one acquires sheet music or other physical interpretation of the music as a composer intended. In fact, in the memoir book for Mandelbrot A life in Many Dimensions, Harlan Brothers wrote, “Benoit Mandelbrot always had a strong feeling that music could be viewed from a fractal perspective. However, without our eyes to guide us, how do we gain this perspective?”{ALMD} The question posed is an excellent one, and Brothers goes on to discuss that generally there are seven ways that fractals can appear in music.

Before we discuss these seven ways, we would like to mention that there are several misconceptions as to what fractal music is, which Brothers discusses on his webpage{Brothers}. The most common misconception is that converting fractal images into sound produces fractal music. In many cases these transformations can hardly be classified as music and simply as noise. Another misconception is to think that iterations always cause fractals in music. This is not true in the physical sense as the lodistic map illustrates {Brothers} and it does not hold in music either. The last misconception that Brothers talks about in regards to fractal music is that of self similarity. As with fractal diagrams, self similarity is a necessary but no sufficient condition{Brothers}. He gives the example that, " onions, spirals, and Russian dolls are not fractal; they do not contain a minimum of two matching or similar regions in which the arrangement of elements either mirrors or imitates the structure of the object as a whole." So, it is necessary that parts of a musical piece be similar to larger sections of the musical composition.

### Misconceptions

## A primer on music

### Pitch, notes, rhythm

### Intervals and melody

## Self-similarity scaling in music[1]

As a subject of research, fractals and self-similarity in music may be fairly niche, but there is no shortage of literature, new or old. One of the earliest attempts at mathematically quantifying musical self-similarity was conducted by Richard Voss and John Clarke, and in 1975 they published the article "$1/f$ noise in music and speech". They concluded that, within genres of music, a $1/f$ power-law scaling behavior is characteristic of musical components for pieces in the genre (though they were specifically concerned with the Baroque era compositions of J.S. Bach, or just "classical" in layman's terms). However, there may be many different ways in which measurable self-similarity within music can manifest; chapter 7 of the Mandelbrot text, written by Brothers, provides a few examples of how scaling within music has been quantified:

1. *Duration scaling*: the distribution of durations for individual notes is self-similar within a piece,
2. *Pitch scaling*: the distribution of pitches is statistically self-similar,
3. *Melodic interval scaling*: the distribution of melodic intervals is self-similar,

[1] R. F. Voss and J. Clarke, $1/f$ noise in music and speech, *Nature* Vol. 258, 1975; pages 317-318.

4. *Melodic moment scaling*: the distribution of the changes in melodic intervals is stylistically self-similar,
5. *Harmonic interval scaling*: the distribution of harmonic intervals is self-similar,
6. *Structural scaling*: the structure of the music from a compositional standpoint relies on nested or recursive patterns, and
7. *Motivic scaling*: a motif, melodic or rhythmic, is repeated simultaneously at different time scales (called augmentation or diminution).

Brothers also cautions "it is important to note that, regardless of the type of scaling under consideration, in order to fulfill a power law relation, any inherent pattern in a group of musical elements requires the presence of a minimum of three distinct levels of scaling. This requirement respects the fact that the log-log plot of a power law relation appears linear; at least three data points are needed to assert a linear relationship"[2].

# Fractal and multifractal dimension

# Structural scaling and motivic scaling: Bach and fractals

There exist many folklore about Bach and his impressive talent. There is such a story about his short musical piece called "The Little Fugue". One day Bach was challenged to a compitition by a fellow well renowned organist of the time. Bach and the fellow showed up for the competition at a church to play the orga

The first part of the fifth movement, the "Bourrée", from Johann Sebastian Bach's Cello Suite No. 3 in C Major, BWV 1009.

The paper by is dedicated entirely to examining the scaling characteristics within this single section of music, and the paper "Multifractal analyses of music sequences" by Zhi-Yuan Su and Tzuyin Wu also takes a look at this very same section (along with two other musical examples).

# Pitch scaling: Stochastic composition

- Describe the procedure used to compose Guapos and Nils
- Analyze Guapos and Nils
- Expand on what pitch scaling is

[2] H. J. Borthers, "Structural scaling in Bach's cello suite no. 3." *Fractals*, Vol. 15, No. 1, 2007; pages 89-95.

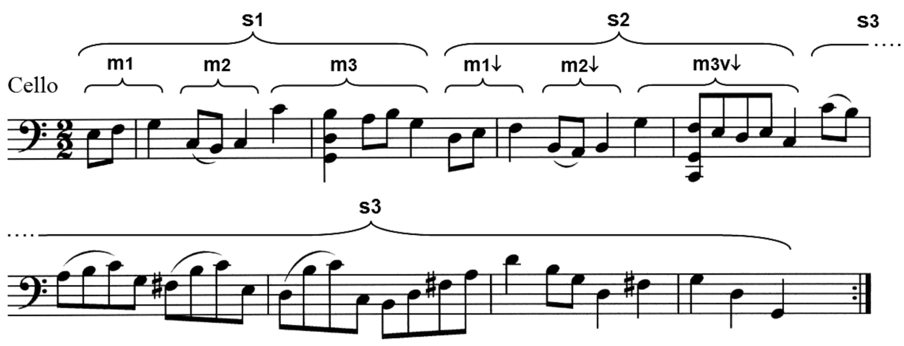


Figure 1: Bourrée notation structure

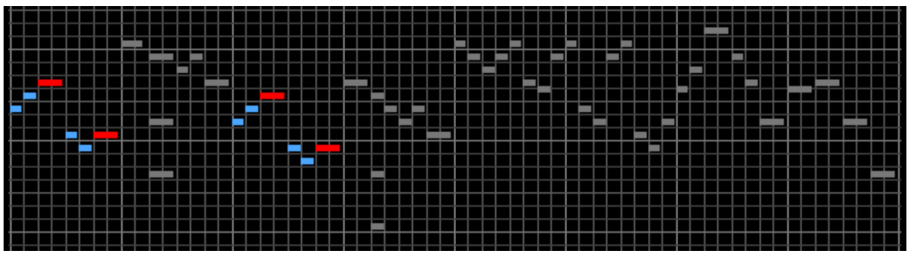

Figure 2: Bourrée MIDI structure 1

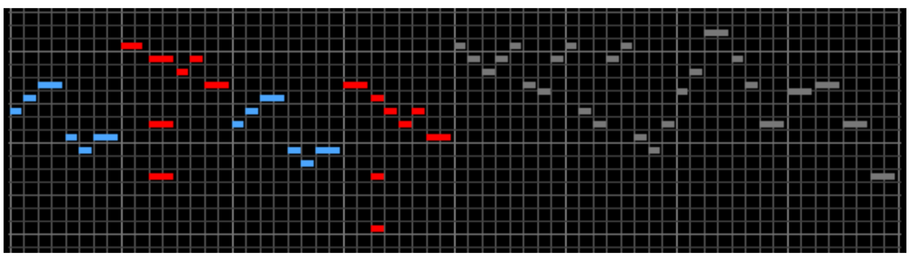

Figure 3: Bourrée MIDI structure 2

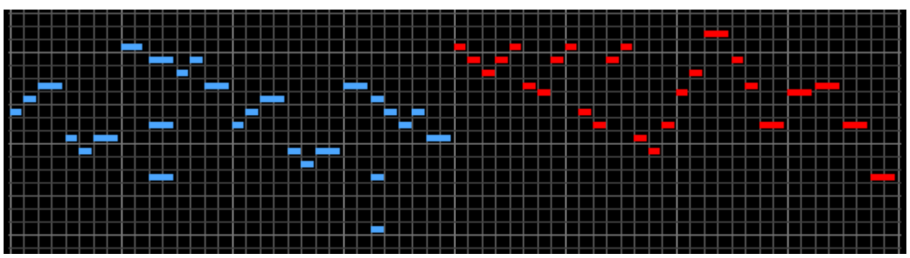

Figure 4: Bourrée MIDI structure 3

- Describe how Guapos and Nils provides an example of pitch scaling fracticality

Stochasitc pitch scaling can be generated using a method that involves rolling a number of dice, and the sum of the dice gives a note that was assigned a number. For example 7 could be assigned to the note D. Bulmer

| A | B | C | Total | Note |
|---|---|---|---|---|
| 2 | 6 | 1 | 9 | C |
| 2 | 6 | 5 | 13 | G |
| 2 | 2 | 5 | 9 | C |
| 2 | 6 | 2 | 10 | D |
| 4 | 6 | 2 | 12 | F |
| 1 | 6 | 3 | 10 | D |
| 2 | 4 | 3 | 9 | C |
| 6 | 2 | 1 | 9 | C |

# Melodic interval and duration scaling: implementing

## Converting a melodic line into point sequences

## Limitations of the method

The most dramatic limitation of this procedure and this metric of scaling in general is that it requires the musical material to be monophonic—only one note at a time—while the vast majority of modern music (and actually the majority of music since the 14th century) is polyphonic. Despite this, a piece of music containing simultaneous notes across simultaneous voices could potentially be reconstructed into standalone parts, where each roughly functions as standalone piece consisting of only a monophonic melody. Think of a Bach cantata for soprano, alto, tenor and bass four-part choir, where each voice is essentially its own piece, its own melody. On the other hand, not all music can be so simply reconstructed into parts. Even the Bach selection examined by Su and Wu (the Bourrée), despite being almost entirely monophonic, contains polyphony in measures 2, 4, and 28 (the last), but this may be an artifact of the editor, as some copies have a single G instead of the E chord. Either way, one can simplify a polyphonic melody by selecting a single constituent note whenever there's a simultaneous group of notes, as I did for simplifying the Bourrée.