Abstract

Music and mathematics are two subjects more related to one-another than most people know. A single melody can be thought of as a sequence of changing pitch frequency intervals, along side a sequence of rhythmic durations. In this project we will first explore ways in which to translate musical melodies into real-valued numerical sequences, and second we will study these sequences by applying familiar fractal dimension metrics in an attempt to explore fractal patterns in music and build insight into what "fractal music" really means.

Introduction

Out of principle we must begin our discussion on fractals in music by mentioning the late Benoit Mandelbrot. Mandelbrot coined the phrase "fractal" in 1975 to describe objects that retained complexity and detail, at different scales; similar to how the photograph of a pile of rocks can look similar to the picture of a mountain, if there is nothing to provide a sense of scale{MIT}. Initially, Mandelbrot observed this in time series graphs of product prices in the economy, where if there was no scale it would be impossible to tell if the price changes were updated per minute, hour, day, etc. {MIT}. Mandelbrot also tended to speak of "roughness"{TED}. Something that was smooth would be akin to time series plot that looked like a smooth curve. A rough graph would look like a plot with many dynamic changes.

A primer on music

Pitch, notes, rhythm

Intervals and melody

Mandelbrot believed in the power of the human eye to notice "roughness" {MIT}{TED}, but music provides a unique challenge, because in the moment, music is felt in a psychological sense, and is usually not observed as a whole, unless one acquires sheet music or other physical interpretation of the music as a composer intended. In fact, in the memoir book for Mandelbrot A life in Many Dimensions, Harlan Brothers wrote, "Benoit Mandelbrot always had a strong feeling that music could be viewed from a fractal perspective. However, without our eyes to guide us, how do we gain this perspective?"{ALMD} The question posed is an excellent one, and Brothers goes on to discuss that generally there are seven ways that fractals can appear in music.

Misconceptions

Before we discuss these seven ways, we would like to mention that there are several misconceptions as to what fractal music is, which Brothers discusses on his webpage{Brothers}. The most common misconception is that converting fractal images into sound produces fractal music. In many cases these transformations can hardly be classified as music and simply as noise. Another misconception is to think that iterations always cause fractals in music. This is not true in the physical sense as the lodistic map illustrates {Brothers} and it does not hold in music either. The last misconception that Brothers talks about in regards to fractal music is that of self similarity. As with fractal diagrams, self similarity is a necessary but no sufficient condition{Brothers}. He gives the example that, " onions, spirals, and Russian dolls are not fractal; they do not contain a minimum of two matching or similar regions in which the arrangement of elements either mirrors or imitates the structure of the object as a whole." So, it is necessary that parts of a musical piece be similar to larger sections of the musical composition.

A primer on music

Before introducing the different ways in which fractal patterns arise in music it is essential that some basic building-blocks of music be introduced.

Pitch, intervals, and rhythm

A single key on a piano, when struck, produces a single pitch. Pitch is one of the most fundamental properties of notes in music, but what exactly is pitch? The sound that is produced by the piano is really the combination of several strings vibrating, each producing different frequencies of sound. Together this combination forms the characteristic sound of the piano (also called timbre, the specific quality to the sound of, in this case, the piano). The strongest frequency in this collection is called the fundamental frequency of the "note" being played, and to this frequency a name is assigned. In the case of a piano, and of all Western music, these frequency names come from the alphabet: A, B, C, D, E, F, and G. Though pitches are unique unto their frequency, some pitches are considered equivalent with respect to the ratio of their frequencies. For example, the pitch A4 (concert A) sounds at 440 cycles per second (hertz), while A5 sounds at 880 hertz: a ratio of 1:2. This ratio is called an octave, and thus we have names for distances between frequencies which we call *intervals*. Within Western music notation there are a finite number of named intervals; here's a few: perfect octaves, perfect fifths, diminished fifths, augmented fourths (equivalent to diminished fifths), major thirds, minor thirds, etc. Between any two notes (and even between a note and itself) there is an interval between them. (These intervals between notes are the concern of a later section of this paper.) Back to octaves: despite having unique frequencies, any two pitches which are an interval of an octave apart (or multiple octaves apart, e.g. C2 and C5) from one another, are considered equivalent. Lastly, If you are familiar with the piano then you will know there are actually are more than seven keys in between octaves, twelve precisely. These are just smaller subdivisions of the frequency ranges between octaves.

The other most fundamental property of notes is duration, also called rhythmic value (or just value). Music as listened to is inherently ephemeral: in the moments in which we experience music by listening to it, it exists, but then it is gone. How long we experience the sound of a pitch is called its duration, and we might measure the duration of a pitch by how long it is played in seconds. Establishing a baseline duration is important in the composition of music; some pieces of music consist of pitches that are long in duration, while others consist of pitches that exist only for a split-second. But not all pitches need have the same duration, and so we have rhythm: the ratios between pitches in terms of duration. For example we might play one pitch for twice as long as the one before, and then next two pitches for a quarter as long. Like with pitch, different rhythmic values also have names within Western music notation, some of the most basic of which being whole notes 𝅝, half notes 𝅗𝅥, quarter notes ♩, and eighth notes ♪. Notice that the names all have to do with fractions. A piece of music might set a baseline speed by stating that the duration of a quarter note should be a certain number of seconds—then the duration of a half note would be twice as long as the duration of the quarter note, a whole note four times as long, an eighth note half as long, a sixteenth note a quart as long, etc.

It is these two fundamental properties that are a part of every note in a piece of music.

Intervals, melody and harmony

Self-similarity scaling in music[1]

As a subject of research, fractals and self-similarity in music may be fairly niche, but there is no shortage of literature, new or old. One of the earliest attempts at mathematically quantifying musical self-similarity was conducted by Richard Voss and John Clarke, and in 1975 they published the article "$^{1}/_{f}$ noise in music and speech". They concluded that, within genres of music, a $^{1}/_{f}$ power-law scaling behavior is characteristic of musical components for pieces in the genre (though they were specifically concerned with the Baroque era compositions of J.S. Bach, or just "classical" in layman's terms). However, there may be many different ways in which measurable self-similarity within music can manifest;

[1] R. F. Voss and J. Clarke, $^{1}/_{f}$ noise in music and speech, *Nature* Vol. 258, 1975; pages 317-318.

chapter 7 of the Mandelbrot text, written by Brothers, provides a few examples of how scaling within music has been quantified:

1. *Duration scaling*: the distribution of durations for individual notes is self-similar within a piece,
2. *Pitch scaling*: the distribution of pitches is statistically self-similar,
3. *Melodic interval scaling*: the distribution of melodic intervals is self-similar,
4. *Melodic moment scaling*: the distribution of the changes in melodic intervals is stylistically self-similar,
5. *Harmonic interval scaling*: the distribution of harmonic intervals is self-similar,
6. *Structural scaling*: the structure of the music from a compositional standpoint relies on nested or recursive patterns, and
7. *Motivic scaling*: a motif, melodic or rhythmic, is repeated simultaneously at different time scales (called augmentation or diminution).

Brothers also cautions "it is important to note that, regardless of the type of scaling under consideration, in order to fulfill a power law relation, any inherent pattern in a group of musical elements requires the presence of a minimum of three distinct levels of scaling. This requirement respects the fact that the log-log plot of a power law relation appears linear; at least three data points are needed to assert a linear relationship"[2].

Fractal and multifractal dimension

Structural scaling and motivic scaling: Bach and fractals

There are many tales of Bach's impressive talents as a composer and virtuoso musician.A biographer of Bach's also recounts the time that Bach arrived to a town in Prussia as a stranger. Upon arrival King Frederick the Great invited him to the Royal Palace to try the new pianofortes. "After he had gone on for some time, he asked the King to give him a subject for a Fugue, in order to execute it immediately without any preparation."{H.T} Speaking about this same event, Douglas Hofstadter in his book Godel, Escher, Bach: An Eternal Golden Braid states, "the ten canons in the musical offering are among the most sophisticated canons Bach ever wrote. However, curiously enough, Bach himself never wrote them out in full. This was deliberate. They were posed as puzzles to King Frederick. It was a familiar musical game of the day to give a single theme, together with some more or less tricky hints, and to let the canon based on the theme be 'discovered' by someone else."{D.R} This shows that Bach truly

[2] H. J. Borthers, "Structural scaling in Bach's cello suite no. 3." *Fractals*, Vol. 15, No. 1, 2007; pages 89-95.

incorporated math into his music and that his compositions were much more than a creative thought; that they contained a mathematical complexity that retained beauty. So, it is possible to see that his music contained fractal.

H.T. David and A. Mendel, The Bach Reader, pp.305-306. D.R. Hofstadter, Godel, Escher, Bach: An Eternal Golden Braid (Basic Books, 1980).

Specifically, The first part of the fifth movement, the “Bourrée”, from Johann Sebastian Bach’s Cello Suite No. 3 in C Major, BWV 1009.

The paper Structural Scaling of Bachs Cello Suite No.3 by Harlan Brothers is dedicated entirely to examining the scaling characteristics within this single section of music, and the paper “Multifractal analyses of music sequences” by Zhi-Yuan Su and Tzuyin Wu also takes a look at this very same section (along with two other musical examples).
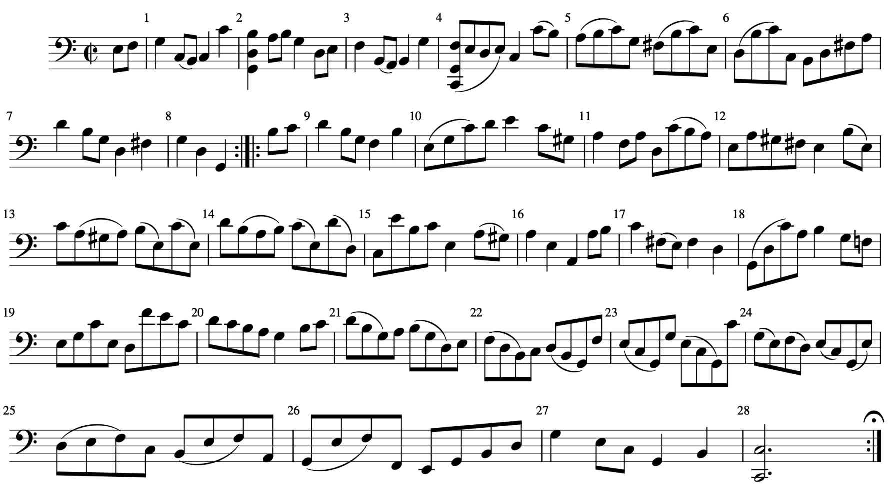
Figure 1: Cello Suite No. 3 in C Major, BWV 1009, V. Bourrée I.

Pitch scaling: Stochastic composition

- Describe the procedure used to compose Guapos and Nils
- Analyze Guapos and Nils
- Expand on what pitch scaling is
- Describe how Guapos and Nils provides an example of pitch scaling fracticality

Stochasitc pitch scaling can be generated using a method that involves rolling a number of dice, and the sum of the dice gives a note that was assigned a number. For example 7 could be assigned to the note D. Bulmer used six dice to generate
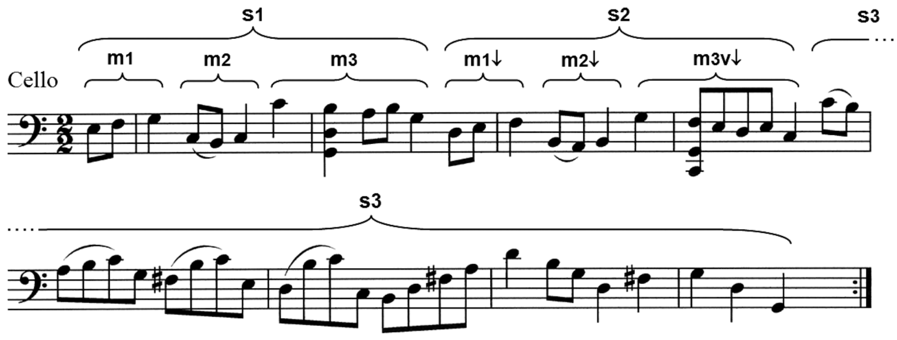
Figure 2: Bourrée notation structure

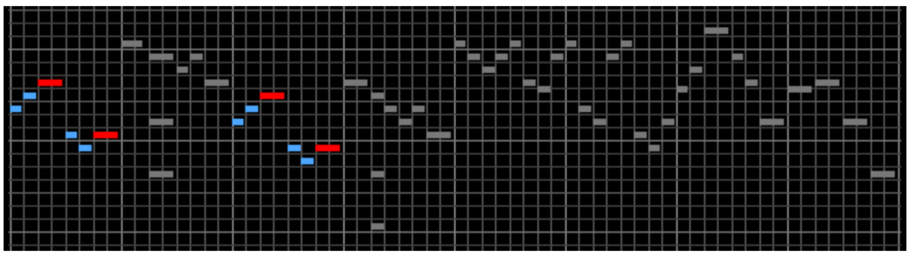

Figure 3: Bourrée MIDI structure 1

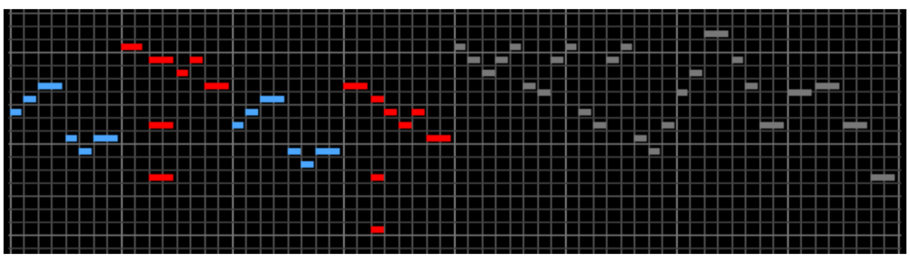

Figure 4: Bourrée MIDI structure 2

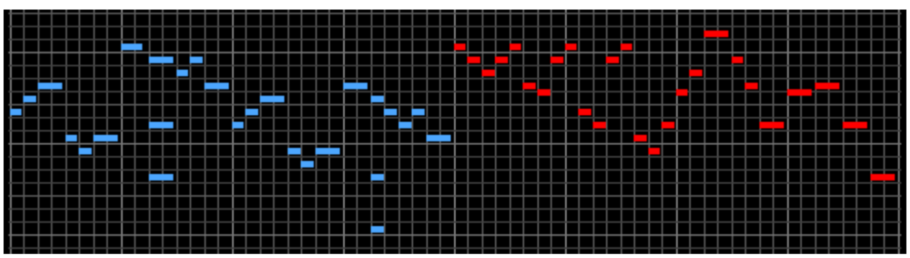

Figure 5: Bourrée MIDI structure 3

a scale of notes that when listened to has a static like sound. The noise that is termed for this is white noise, which music that is considered white noise has an atroious sound to it due to the disconnection of flow when going from note to note. White noise is considered be too unpredictable, where brown noise is considered to be very predictable. Brown noise has a Brownian motion which can be see as being too predicitable because the notes have an erratic motion back and forth. White and brown noise are the two extremes for random music.

Pink noise has been seen to achieve both of these extremes which brings a balance which is considered the $\frac{1}{f}$ noise. {Voss and Clarke(1978)} Achieving a true balance between these extremes is very difficult. {Mandelbrot(1971)} Pink noise has been seen to have a self-similiar process with a pattern that is long-range dependent and exhibits short-term randomness.

Voss approximates pink noise using a variation of the dice method that was described earlier. Using n dice, for our experiment we used n = 3, we generated 2^n notes. the table below was constructed by rolling three dice that are labeled A, B, and C. We sum of the three dice and the total gave us a note which each note was assigned to a value. Binary digits were used to generate the next note which changes from row to row in the table. The table below is the results from our experiment. The notes below provide a stabbiliy in the sequence while haivng long-range dependence which is need to be considered pink noise.

A	B	C	Total	Note
2	6	1	9	C
2	6	5	13	G
2	2	5	9	C
2	6	2	10	D
4	6	2	12	F
1	6	3	10	D
2	4	3	9	C
6	2	1	9	C

Time plot of pink noise (Bulmer)

When graphing a time plot that was produced using the dice method created by Voss, we are able to see that the series of notes exhibit long-range dependence. There is more stability due to the high digits have less change.

When describing noise we can use auocorrelation to describe the notion of music notes being “related” over time. Autocorrelation is composed of two data sets, r_k, which is the values in the noise sequence and k which is called the lag.

$$r_k = \frac{\sum_{t=1}^{N-k}(x_t - \overline{x}(x_{t+k-\overline{x}})}{\sum_{t=1}^{N}(X_t - \overline{x})^(2)}$$

The formula above can be used to construct correlograms, a plot of autocorrelation against lag. This will give a visual into the autocorrelation structure of the noise.

Correlogram for pink noise (Bulmer)

The correlogram above shows that the correlation over time does not disappear to even over a longer period of time. Pink noise exhibits the 1/f phenomena which is attributed to the pitch scaling. We can see that that the extremes of pitch scaling can lead to some dissonance in sounds. When finding a balance in the scaling the results can began to exhibit fractal characteristics. The compositions of songs are reminiscent of the Koch snowflake which posess long-range dependence and short-term randomness which was observed in pink noise. The results is a composition that is self-similar which like snowflakes are considered fractals.

Melodic interval and duration scaling: implementing

Converting a melodic line into point sequences

Limitations of the method

The most dramatic limitation of this procedure and this metric of scaling in general is that it requires the musical material to be monophonic—only one note at a time—while the vast majority of modern music (and actually the majority of music since the 14th century) is polyphonic. Despite this, a piece of music containing simultaneous notes across simultaneous voices could potentially be reconstructed into standalone parts, where each roughly functions as standalone piece consisting of only a monophonic melody. Think of a Bach cantata for soprano, alto, tenor and bass four-part choir, where each voice is essentially its own piece, its own melody. On the other hand, not all music can be so simply reconstructed into parts. Even the Bach selection examined by Su and Wu (the Bourrée), despite being almost entirely monophonic, contains polyphony in measures 2, 4, and 28 (the last), but this may be an artifact of the editor, as some copies have a single G instead of the E chord. Either way, one can simplify a polyphonic melody by selecting a single constituent note whenever there's a simultaneous group of notes, as I did for simplifying the Bourrée.