Contents

Todo

- Add motivations for calculation specif heat capacities and energies. That is, what do these quantities mean?

1   Introduction and Summary

In this section, I'll first attempt to introduce all relevant mathematical notation; units, fundamental concepts and gas laws in physics; and then attempt summarize my solutions to the proposed problem:

> *Given a furnace F containing only* CO_2 *at some arbitrary initial conditions, a source C of relatively cold* CO_2 *and a source H of relatively hot* CO_2*: is there not only some way of constructing a mixture of H and C such that inserting it into F yields exactly some target temperature* T_T *and pressure* P_T *in F, but a way in which (a) we minimize the amount* n_R *which is removed from the initial amount of gas, and (b) we minimize the amounts* n_C *from C and* n_H *from H that we are adding?*

1.1   Notation, basic physics, and thermodynamics

Firstly, I'll often write a closed mixture/volume of gasses as a single capital letter, like *F* for furnace, and where I can now use *F* to symbolically refer to any of the conditions of the furnace. Formally:

> **Definition 1** (*M* a gas mixture)**.** Let *M* be a mixture/volume/composition of gasses; we write:
>
> - V_M for the total volume in liters of the system in which the mixture is contained,
> - n_M for the total number of moles in the mixture,
> - T_M for the temperature of the mixture,
> - P_M for the total pressure of the mixture,
> - C_M for the specific heat (capacity) of the mixture, and
> - Q_M for the total energy of the mixture.

Note. I use *volume*, *mixture* and *composition* somewhat interchangeably when talking about some specific closed amount of gas, and that "volume" here has nothing to do with the actual volume in liters of the container in which the closed amount of gas sits.

For example, "a volume M of gas" versus "the volume (in liters) of the mixture/composition/volume M."

Note. Seeing as we're going to be working with equations involving quantities of various units, it's going to be important to understand what a particular unit of measurement means and how these units relate to one-another. But to understand units is also to have some basic understanding of physics and thermodynamics. (It's also imperative to understand unit prefixes; like *kilo k* for "a thousand of", *mega M* for "a million of", or *milli m* for "a millionth of".)

Here's sort of a summary introduction to all of these things:

Definition (Moles, volume, temperature, pressure). A *mole* (mol) is the base unit of an amount of a substance. A mole of a substance contains a particular number of particles of the substance, just as gram of a substance contains a certain number of particles, and so the relationship between moles and mass is some constant scaling factor (that is, if I have a sample of a substance that is a moles and b grams, then there is some number c such that $a = cb$ or $a/c = b$ where c is the scaling factor).

Volume is the quantity of an enclosed space. Where a m meter is the base unit of distance, a L liter is the base unit of volume. A single liter is defined as equivalent to three cubic meters ($1 \cdot \mathrm{L} = 1 \cdot \mathrm{m}^3$). Particles of gasses are special in that they prefer to be evenly spaced apart from one-another while inside some volume. Take some amount of gas (some number of moles of gas perhaps) and put it in a volume: a larger volume means more space between particles, while a smaller volume means less space between particles.

Temperature is the quantity measured in K kelvin that expresses hot or cold, where hot and cold are manifestations of the transfer of thermal energy between particles (of which all contain some amount of thermal energy). That is, an object is hot relative to its colder surroundings if it has more thermal energy that its surroundings, and as a result can impart some of its thermal energy to its surroundings: hot things impart energy and become colder, cold things absorb energy and become hotter. Considering an amount of gas within a volume, though the particles of gas are spaced apart from one-another they are not stationary: they move or vibrate relative to their temperature. These movements may cause the particles to collide with one-another and with the walls of the container.

Pressure is the quantity that expresses the average amount of force that is being exerted upon the area of a surface. A force (measured in newtons) is an influence that can causes matter to change velocity (i.e. to push), the actual act of imparting force is called work (measured in joules), and the capacity for something to do work is called energy (also measured in joules). As mentioned for gas particles in a container, collisions occur between the particles and the walls of the container. However much these particles end up on average colliding with, or pushing against, the walls of the container is the pressure of the gas within the volume, measured as pascals (Pa). A single pascal is

defined to be a single joule per cubic meter (J/m^3, or equivalently J/L since $1 \cdot L = 1 \cdot m^3$).

Definition (Specific heat, total energy)**.** Informally, the specific heat (capacity) of a substance is the amount of heat/energy (in joules) required to increase its temperature by a single degree, and is quantified as joules per mole kelvin ($J/_{mol\,K}$). Particular substances each have their own specific heat (fig. 1). These quantities are important in that they relate energy to substance amount and temperature. Given an amount and temperature of a substance we can calculate the energy; given the energy and amount of a substance we can calculate the temperature; and given the energy and temperature of a substance we can calculate the amount.

The total energy then for some amount of a particular kind of gas is simply the specific heat multiplied by the amount and the temperature. For example, one-thousand moles of CO_2 at 273.15 kelvin has a total energy of $1000 \cdot 273.15 \cdot 27.2 = 742kJ$.

Secondly, there are seven types of gasses in the game, each with a formula. I use the formulas as *indices* for the gasses. That is, if I say *the gas g*, then what I mean is *the gas indexed by g, an element of the set of gas indices*. I can then use this index g to refer to the conditions, within a mixture of gasses, of just this one gas indexed by g.

Definition 2 (g a gas index)**.** Define the set of gas indices for Stationeers:

$$\mathcal{G} = \{H_2, N_2, O_2, X, CO_2, H_2O, N_2O\}.$$

We assume that the order of these indices is always the same; that is, the first element of $\mathcal{G}$ is always H_2, the second is always N_2, and so on.

Let M be a mixture of the gasses indexed by the elements of $\mathcal{G}$, and let $g \in \mathcal{G}$ (read, g is an element of $\mathcal{G} \Leftrightarrow g = H_2$, or N_2, ..., or N_2O). Then we write:

- $n_M(g)$ for the moles of g (the gas indexed by g) within mixture M,
- $P_M(g)$ for the partial pressure of g within mixture M, and
- C_g the specific heat (capacity) of a gas g.

And note that we don't write $T_M(g)$ for the temperature of g in M, because every particular gas in mixture M has the same temperature. Though, this often *not* true for the pressures of each individual constituent gas of a mixture (see Dalton's law, def. 6).

See see fig. 1 for a table of the gasses in Stationeers, their g formulas/indices, and their C_g specific heats.

Note (Sets in math)*.* A *set* in math is simply a collection of things that are distinct but of the same type, are notated with their elements or their definition surrounded by braces, { and }, and are often represented by a widely accepted single character or symbol. I say

Name	Formula (g)	Specific heat (C_g in J/mol K)
Hydrogen	H_2	20.4
Nitrogen	N_2	20.6
Oxygen	O_2	21.1
Pollutant	X	24.8
Carbon dioxide	CO_2	28.2
Water	H_2O	72
Nitrous oxide	N_2O	23

Figure 1: Table of gasses in Stationeers

or for how they are notated because we can define sets in various but equivalent ways: *semantically*, as a *finite* or *infinite roster*, or using what's called *set-builder notation*.

To give examples of these different ways of construction, consider that numbers in general are members of sets of other numbers, and often are members of various sets. And often sets are *subsets* of other sets; smaller and more restrictive than the outer set. Take first for example the *set of integers*, often represented by the symbol $\mathbb{Z}$. See the semantic definition of an integer in both the semantic and set-builder definitions below:

$$\mathbb{Z} = \text{ the set of numbers with no fractional part}$$
$$= \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \{n : n \text{ is a number with no fractional part}\}.$$

Now consider the set $\mathbb{W}$ of *whole*, or *counting* numbers:

$$\mathbb{W} = \{1, 2, 3, \ldots\} = \{n : n \in \mathbb{Z}, n > 0\}.$$

The colon (:) defines rule(s) for set-builder definition, and the symbol $\in$ means "element tof": that what's on the left is taken to be an element of the set on the right. The set we construct by the above rules is then simply the set of all the positive integers—a subset of the integers.

In either example above the roster definitions are infinite roster definitions; we cannot list every integer of these sets because there is an infinite number of them. On the other hand, a set $\{1, 2, 3\} = \{n : n \in Z, 0 < n < 4\}$ is finite in its size/in its roster definition.

All but the last of these conditions can be determined without much calculation; in particular n_M the total moles, T_M temperature and P_M pressure can all be read directly from a furnace/pipe analyzer/tank in game, and $n_M(g)$ moles of the particular gas g can be calculated simply as the ratio of the gas times the total moles:

$$n_M(g) = (\text{ratio of } g \text{ in } M) \cdot n_M, \text{ where } 0 \leq \text{ratio of } g \text{ in } M \leq 1.$$

Note that if M consists of only one kind of gas (which I index as g^*):

$$\text{if } 1 = (\text{ratio of } g^* \text{ in } M) \Rightarrow n_M(g^*) = 1 \cdot n_M = n_M.$$

1.2 Gas laws

Gasses in Stationeers obey some of the ideal gas laws from real life. First, lets know how to calculate the specific heat of a mixture of gasses. Recall that a gas, indexed by $g \in \mathcal{G}$, has a specific heat C_g (fig. 1).

> **Definition 3** (Specific heat capacity of a gas mixture). Let M be a mixture of the $\mathcal{G}$ gasses. The specific heat of the mixture (C_M) is given by the sum of the products of the molar ratios of the gasses in the mixtures and their respective specific heats. We can notate this mathematically in a few ways:
>
> $$C_M = \frac{n_M(\mathrm{H_2})}{n_M}C_{\mathrm{H_2}} + \frac{n_M(\mathrm{N_2})}{n_M}C_{\mathrm{N_2}} + \cdots + \frac{n_M(\mathrm{N_2O})}{n_M}C_{\mathrm{N_2O}} = \sum_{g \in \mathcal{G}} \frac{n_M(g)}{n_M}C_g$$
>
> Or as the dot-product ($\cdot$) of vectors:
>
> $$C_M = (\boldsymbol{n}_M \cdot \boldsymbol{C}_\mathcal{G})\frac{1}{n_M} \tag{1}$$
>
> Where $\boldsymbol{n}_M$ is the vector of the moles per gas in M, and where $\boldsymbol{C}_\mathcal{G}$ is the vector of the specific heats of each gas:
>
> $$\boldsymbol{n}_M = \langle n_M(\mathrm{H_2}), n_M(\mathrm{N_2}), \ldots, n_M(\mathrm{N_2O}) \rangle = \langle n_M(g) : g \in \mathcal{G} \rangle$$
> $$\boldsymbol{C}_\mathcal{G} = \langle C_{\mathrm{H_2}}, C_{\mathrm{N_2}}, \ldots, C_{\mathrm{N_2O}} \rangle = \langle C_g : g \in \mathcal{G} \rangle.$$

> *Note* (If you're unfamiliar with summation notation). A sum exactly what you think it is: $a + b$, two things added together. If you want to add/sum more than two things you can of course write it out: $a + b + c + d$. And if there's a lot of things to sum up, and there's some sort of logical progression in the sequence of things you're summing as well as a logical end point, then you can use ellipses: $1 + 2 + 3 + \cdots + 99 + 100$.
>
> But another, more compact, way to notate this is is *summation notation*. Summation notation uses the symbol capital sigma $\sum$ to denote a summation, and uses some kind of an *index* to designate where we start in the sum, and sometimes where to end. For example, the following sum says that we're letting i be our index, that we start with i being 1, we stop when i is 100, and after each i we move onto the next i ($i = 1, 2, 3, \ldots, 99, 100$). What we're summing are the numbers x_i indexed by i:
>
> $$\sum_{i=1}^{100} x_i \quad \text{(for whatever number x_i is).}$$
>
> In English we would say "take the sum of the values x_i for each i in the range 1 to 100 inclusive," where *inclusive* means we include x_{100} in the sum, as opposed to *exclusive* where we would exclude x_{100}. We also might write this range of 1 to 100 inclusive as $[1, 100]$; versus $[1, 100)$ meaning we don't include 100, $(1, 100]$ meaning we don't include 1 but *do* include 100, and $(1, 100)$ where we don't include either 1 nor 100.

For a less abstract example, consider the sum of the first 100 positive integers:

$$\sum_{x=1}^{100} x = 1 + 2 + 3 + \cdots + 99 + 100.$$

However, the index need not be a number. A common way of expressing indices is via sets, and is to say "take the sum of this expression indexed by every element of the given set." For example: $\sum_{g \in \mathcal{G}} C_g$ = the sum of the specific heats for each gas indexed by g in the set of gas indices $\mathcal{G}$.

Note (If you're unfamiliar with vectors and dot-products). Consider a vector to be a list of numbers. For example, here are two vectors, x and y, of size 3:

$$x = \langle x_0, x_1, x_2 \rangle, \quad y = \langle y_0, y_1, y_2 \rangle.$$

The dot product ($\cdot$) of x and y (written $x \cdot y$) is given as the sum of the products of the corresponding entries of x and y:

$$x \cdot y = \langle x_0, x_1, x_2 \rangle \cdot \langle y_0, y_1, y_2 \rangle = x_0 y_0 + x_1 y_1 + x_2 y_2.$$

Note that if M consists of only one kind of gas g^*, then the ratio of $g^* \in \mathcal{G}$ in M is one, and the ratio of $g \in \mathcal{G} \setminus g^*$ in M is zero. And so:

$$n_F(g) = \begin{cases} n_F(g) = n_F, & \text{if } g = g^* \\ n_F(g) = 0, & \text{if } g \neq g^* \end{cases}.$$

(We read $g \in \mathcal{G} \setminus g^*$ as: g is an element of the set $\mathcal{G} \setminus g^*$, where $\mathcal{G} \setminus g^*$ is the set $\mathcal{G}$ but with the g^* element removed.)

Example (Using summation notation). Suppose F our furnace contains 10 moles of each kind of gas, of which there are seven.

Then $n_F = 70$, and:

$$C_F = \sum_{g \in \mathcal{G}} \frac{n_F(g)}{n_F} C_g = \frac{n_F(\text{H}_2)}{n_F} C_{\text{H}_2} + \frac{n_F(\text{N}_2)}{n_F} C_{\text{N}_2} + \cdots + \frac{n_F(\text{N}_2\text{O})}{n_F} C_{\text{N}_2\text{O}}$$

$$= \frac{10}{70} (20.4 + 20.6 + 21.1 + 24.8 + 28.2 + 72 + 23) = 30.01$$

Example (Using vector notation). Suppose F our furnace contains 70 moles of only $g^* = \text{CO}_2$.

Then $n_F = n_F(CO_2) = 70$, and:

$$n_F(g) = \begin{cases} n_F(g) = n_F, & \text{if } g = CO_2 \\ n_F(g) = 0, & \text{if } g \neq CO_2 \end{cases},$$

$$C_F = (\boldsymbol{n_F} \cdot \boldsymbol{C_G}) \frac{1}{n_F}$$

$$= (\langle 0, 0, 0, 0, 70, 0, 0 \rangle \cdot \langle C_{H_2}, C_{N_2}, C_{O_2}, C_X, C_{CO_2}, C_{H_2O}, C_{N_2O} \rangle) \frac{1}{70}$$

$$= (70 \cdot C_{CO_2}) \frac{1}{70} = C_{CO_2}$$

This reduction of a volume's specific heat to a constant is one reason why I prefer a single gas (CO_2) design; it reduces many of the later problems from problems of differential equations to simple problems of algebra. The other reason is that CO_2 is plentiful, both as at atmospheric gas on planets like Mars and Venus and as a byproduct of combustion.

In terms of gas laws, the model in Stationeers obeys the following laws for a fixed volume of gas.

Definition 4 (Gay-Lussac's law).
$$P_1 T_2 = P_2 T_1 \tag{2}$$

Change in pressure is directly proportional to change in temperature.

Note. In Stationeers this law is *only half-obeyed*! Pressure is dependent on temperature (i.e. changing the temperature in a fixed volume expectedly affects pressure proportionally), but temperature is *not* dependent on temperature (i.e. changing the pressure/removing gas does *not* affect temperature).

(Also note that we wont actually be using this law for solving the problem, but it is very important to understand in general; especially the above caveat.)

Definition 5 (Ideal/combined gas law).

$$PV = nRT \tag{3}$$

Where $R = 8.314$ (joules per mole kelvin) is the universal gas constant.

Definition 6 (Dalton's law of partial pressures).

$$P = P(H_2) + P(N_2) + P(O_2) + P(X) + P(CO_2) + P(H_2O) + P(N_2O) = \sum_{g \in \mathcal{G}} P(g) \tag{4}$$

The pressure in the volume is the sum of partial pressures of the constituent gasses.

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> **Definition 7** (Energy of a volume gas). The energy of a volume of gas with temperature T, number of moles n, and (combined) specific heat C is given by TnC.

Furthermore, the game model satisfies these other equations. Given volumes A and B, and C the result of combining A and B:

$$n_C = n_A + n_B \qquad\qquad \text{(moles are additive)} \tag{5}$$

$$C_C = ((\boldsymbol{n}_A + \boldsymbol{n}_B) \cdot \boldsymbol{C}_{\mathcal{G}}) \frac{1}{n_C} \qquad \text{(combined specific heat of a single mixture)} \tag{6}$$

$$T_C n_C C_C = T_A n_A C_A + T_B n_B C_B \qquad \text{(combined energies of two mixtures)} \tag{7}$$

> *Note* (Why we need to consider combined energies). When we mix a volume of gas A with another B into a combined volume C, the resulting combined energy is entirely dependent on not only the temperatures of A and B, but also the compositions of A and B. There is a lot more energy in one mole of H_2O at 273.15 kelvin than there is in one mole of CO_2 (precisely, 72 joules to 28.2 joules). When we combine A and B we create a new composition, but the

With all this notation and all these gas equations in mind, we move onto solving the proposed problem.

Volume	Description
F	The initial furnace volume
T	The target furnace volume
R	The volume removed from the initial
I	The input volume
H	The hot source
C	The cold source

	Initial	Target	Removed	Input	Hot	Cold
Pressure	P_F	P_T	—	—	—	—
Temperature	T_F	T_T	—	T_I	T_H	T_C
Moles CO_2	n_F	n_T	n_R	n_I	n_H	n_C

Figure 2: Table of variables which we will encounter.

1.3 Solving the problem

To draft up the scenario:

(Initial conditions) A furnace already contains some amount of "hot" CO_2 as a starter. We add the reagents for some alloy, releasing some other gasses into the furnace mix. We then filter out all gases except for the CO_2, returning the CO_2 back into the furnace, leaving the furnace with only CO_2 gas but also the reagents for the alloy. For the alloy we require some specific temperature, T_T (target temperature), and pressure, P_T (target pressure); we'll consider these to be the minimum of whatever the alloy ranges are. Lastly, we have H the "hot" source of CO_2 that has temperature T_H of *at least* T_T kelvin, and have C the "cold" source of CO_2 that's *at most* T_T kelvin.

(The question now) How much of the furnace CO_2 do we need to remove (if any) in order for it to be possible to compose a *minimal* combination I of CO_2 from the H and C sources, that when added to F achieves T_T and P_T in the furnace.

There will be many variables in the following work (see fig. 2 for descriptions for all of the variables we will be encountering); we'll first lay out the relationships between these different variables with respect to the laws and equations defined above. We will use these relationships to establish a system of equations.

Calculating target moles

Given that P_T and T_T are our fixed targets, we can use the ideal gas law (eq. (3)) to solve for n_T the target moles we will need. From the game we know the volume of a furnace V_F to be 1,000 liters, and R the universal gas constant is 8.314 joules per mole kelvin (and note

that $1\,\text{kPa} = 1\,\text{J/m}^3 = 1\,\text{J/L}$). Then:

$$P_T\,\text{J/L} \cdot V_F\,\text{L} = n_T\,\text{mol} \cdot R\,\text{J/mol K} \cdot T_T\,\text{K}$$

$$\Rightarrow n_T\,\text{mol} = \frac{P_T\,\text{J/L} \cdot V_F\,\text{L}}{R\,\text{J/mol K} \cdot T_T\,\text{K}} = \frac{P_T \cdot V_F}{R \cdot T_T}\text{mol} \tag{8}$$

Simplifying energy equations (eq. (6) and eq. (7))

Firstly, consider eq. (6) for some volumes A, B and C of *only* CO_2, and where C is the combination of A and B. Then:

$$C_C = \left((n_A + n_B) \cdot C_{\mathcal{G}}\right)\frac{1}{n_C} = (n_A(CO_2) + n_B(CO_2))C_{CO_2}\frac{1}{n_C} = n_C C_{CO_2}\frac{1}{n_C} = C_{CO_2}.$$

As well as $C_A = C_B = C_C = C_{CO_2}$. Then eq. (7) can be simplified:

$$\begin{aligned} T_C n_C C_C &= T_A n_A C_A &&+ T_B n_B C_B \\ \Rightarrow T_C n_C C_{CO_2} &= T_A n_A C_{CO_2} &&+ T_B n_B C_{CO_2} \\ \Rightarrow T_C n_C &= T_A n_A &&+ T_B n_B. \end{aligned} \tag{9}$$

And this is all we need for a one-gas, CO_2 system.

Bounds on moles

Consider that an amount of moles cannot be negative, $0 \leq n_M$, and that we can only remove as many moles as there are there initially, though we might remove none at all. This imposes a restriction on n_R:

$$0 \leq n_R \leq n_F.$$

Combinations of moles (applying eq. (5))

When we have theoretically reached our ideal target pressure and temperature, we will have removed some amount of moles n_R from the initial amount n_F in the furnace F, and will have input some amount of moles n_I into the furnace. The number of moles n_I of the input volume I is a function of n_H, some number of moles from the hot source, and n_C, some number of moles from cold source, that must satisfy:

$$n_I = n_H + n_C. \tag{10}$$

The number of moles at the theoretical target must also satisfy this equation:

$$n_T = (n_F - n_R) + n_I \tag{11}$$

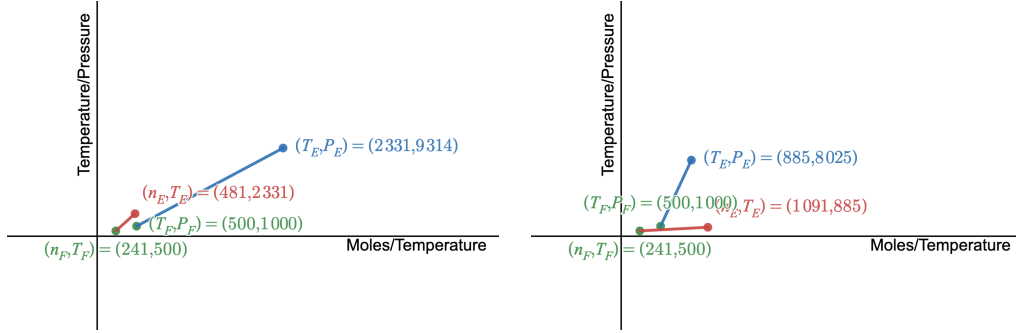(Substituting eq. (10))

$$= (n_F - n_R) + (n_H + n_C). \tag{12}$$

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Left plot:
- Temperature/Pressure (y-axis), Moles/Temperature (x-axis)
- $(T_E, P_E) = (2331, 9314)$
- $(n_E, T_E) = (481, 2331)$
- $(T_F, P_F) = (500, 1000)$
- $(n_F, T_F) = (241, 500)$

Right plot:
- Temperature/Pressure (y-axis), Moles/Temperature (x-axis)
- $(T_E, P_E) = (885, 8025)$
- $(T_F, P_F) = (500, 1000)$
- $(n_E, T_E) = (1091, 885)$
- $(n_F, T_F) = (241, 500)$

Figure 3: Desmos visualization of the affect of n_R, n_H and n_C on the final temperature T_E and pressure P_E.

Combinations of energies (applying eq. (9))

The temperature and number of moles for the input volume is a function of the temperature and number of moles from the hot and cold sources and must satisfy equation eq. (9):

$$T_I n_I = T_H n_H + T_C n_C. \tag{13}$$

Similarly, the temperature and number of moles for the target is a function of the initial temperature in the furnace, the number of moles remaining in the furnace after having removed some amount, and the temperature and number of moles of the input volume:

$$T_T n_T = T_F (n_F - n_R) + T_I n_I \tag{14}$$

We'll find this to be one of the most useful equations since it combines all the parts of the problem together and allows us to solve for all sorts of variables (e.g. T_I and T_T) and quantities (e.g. $T_I n_I$).

Bounds on temperatures

As setup in the paragraph of initial conditions, the hot source H has temperature at least T_T and the cold source C has temperature at most T_T. Thus we can always combine an amount from H and an amount from C into a volume I such that T_I equals T_T. So from this we get the bounds:

$$T_C \leq T_T \leq T_H, \text{ and } T_C \leq T_I \leq T_H.$$

Example. Suppose we have initial conditions $T_F = 500$, $P_F = 1000$, $T_C = 200$ and $T_H = 2500$, but we don't have a particular target pressure P_T or temperature T_T; we just want to see how things change as we remove some amount n_R from F, add some amount n_H from H into F, and/or add some amount n_C from C into F.

Lets call T_E, P_E and n_E our end results: the temperature, pressure and number of moles after having removed n_R and having added n_I. Our procedure is this: pick values for n_R, n_H and n_C, (a) calculate $n_E = (n_F - n_R) + n_I$ where $n_I = n_H + n_C$, (b) calculate

T_E by solving $T_E n_E = T_F(n_F - n_R) + T_I n_I$ (eq. (9)) for T_E, and then (c) calculate P_E by solving eq. (3) for P_E.

First we need to calculate n_F, the initial amount of moles in the furnace, using eq. (3) (recall that $V_F = 1000$ and $R = 8.314$):

$$n_F = \frac{P_F \cdot V_F}{R \cdot T_F} = \frac{1000 \cdot 1000}{8.314 \cdot 500} \doteq 240.56 \text{ moles } CO_2.$$

(Alternatively, since this is Stationeers we're talking about, we would have been able to find this value just by reading the number of moles from the furnace: `l nF furnace TotalMoles`.)

Example ($n_R = 100$, $n_H = 200$ and $n_C = 0$).

(a) $n_E = (240.56 - 100) + (200 + 0) = 340.56$

(b) $T_E = (T_F(n_F - n_R) + (T_H n_H + T_C n_C))/n_E$
$= (500 \cdot 140.56 + (2500 \cdot 200 + 200 \cdot 0))/340.56 \doteq 1674.54$ K.

(c) $P_E = (n_E \cdot R \cdot T_E)/V_F$
$= (340.56 \cdot 8.314 \cdot 1674.54)/1000 \doteq 4741.32$ kPa.

Example ($n_R = 0$, $n_H = 0$, and $n_C = 300$).

(a) $n_E = (240.56 - 0) + (0 + 300) = 540.56$

(b) $T_E = (500 \cdot 240.56 + (2500 \cdot 0 + 200 \cdot 300))/540.56 \doteq 333.51$ K.

(c) $P_E = (540.56 \cdot 8.314 \cdot 333.51)/1000 \doteq 1498.84$ kPa.

From these examples (and playing around on Desmos) we can see that it should be possible to construct a decent variety of target pressures and temperatures.

Check out this quick graph in Desmos to help visualize what's going on (or see fig. 3).

1.4   Calculating minimal amounts

We have enough bounds here to establish a system of equations with which we can solve the problem; that is to be able to solve for a minimal amounts n_R to remove, n_H of H to add and n_C of C to add. It will be a matter of solving these equations for particular variables, and applying certain metrics to solve for these minimal values. Recall that what we want is for our furnace to end up at some target temperature T_T and pressure P_T. We will find there there's a whole lot of potential solutions in terms of n_R, n_H and n_C (technically an infinite amount, even within the bounds we establish), but we can simplify these solutions by selecting the ones which either minimize the amount removed from the initial furnace volume, minimize the amount added from only H, or minimize the amount added from only C.
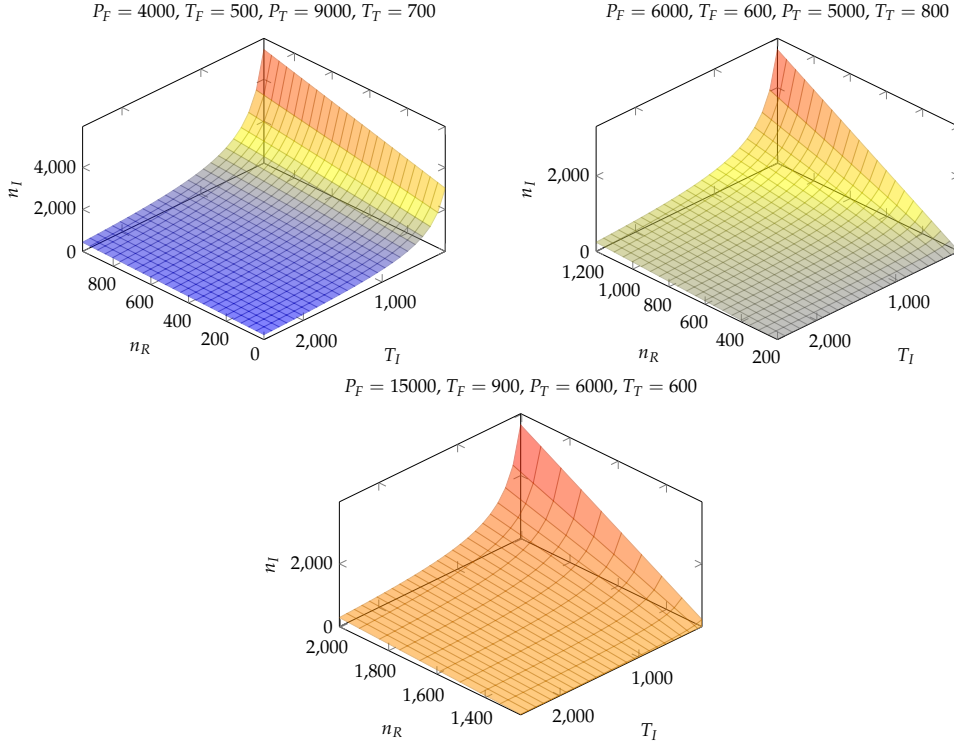
Figure 4: The surface defined by (n_R, T_I, f) for various initial conditions.

First, solve eq. (13) for n_I. This yields n_I as a function of n_R and T_I; the remaining variables $(T_T, n_T, T_F$ and $n_F)$ are all fixed:

$$f(n_R, T_I) = n_I = \frac{T_T n_T - T_F(n_F - n_R)}{T_I}. \tag{15}$$

We can plot this function f, being of two variables n_R and T_I, in three dimensions using the bounds on T_I and n_R: that $T_C \leq T_I \leq T_H$ and $0 \leq n_R \leq n_F$. fig. 4 visualizes this bounded surface for a few different initial conditions (and in these examples I chose $T_C = 200$ and $T_H = 2500$).

> *Note.* This surface we define by (n_R, T_I, f) is in-fact a two-dimensional surface, although it is *embedded* within three-dimensional space! That is, imagine that you are a point on this surface, and that you are not allowed to leave it. You can walk across this surface, but you cannot jump up off of it, and you cannot get below it, but you can still see above and below: you still stand within three-dimensional space but you cannot leave this two dimensional surface. This is what we mean by point solutions to equation eq. (15): a point in three dimensional space $(x, y, z) = (n_R, T_I, n_I)$ sits upon this surface if and only if $z = f(x, y) \Rightarrow n_I = f(n_R, T_I)$ is satisfied (that is, if after evaluating f at n_R and T_I, the equality $n_I = f(n_R, T_I)$ is indeed *true*).

Second, consider solving $n_T = (n_F - n_R) + n_I$ for n_I. We get $n_I = n_T - n_F + n_R$. This expresses n_I as a function of merely n_R, since both n_F and n_T are fixed:

$$g(n_R) = n_I = n_T - n_F + n_R. \tag{16}$$

Then lets take f (eq. (15)) and, instead of solving for n_I, solve for T_I:

$$n_I = \frac{T_T n_T + T_F(n_F - n_R)}{T_I} \Rightarrow T_I = \frac{T_T n_T + T_F(n_F - n_R)}{n_I}$$

With this we keep n_R as a free-variable, but replace n_I with g (eq. (16)). We'll call this newly constructed equation, of *only one variable*, h:

$$h(n_R) = T_I = \frac{T_T n_T + T_F(n_F - n_R)}{n_I} = \frac{T_T n_T + T_F(n_F - n_R)}{g(n_R)}. \tag{17}$$

Consider the surface we previously defined by (n_R, T_I, f) (fig. 4). We have to pick both n_R and T_I together, a *pair* of points, from a mesh of points (the Cartesian product of the n_R and T_I domains). However, now we have an expression $h(n_R) = T_I$ that is dependent on only this only variable n_R. What we have then is that we can solve for T_I in terms of of n_R (eq. (17), $T_I = h(n_R)$), and then can solve for n_I in terms of n_R and $T_I = h(n_R)$ (eq. (15), $n_I = f(n_R, g(n_R))$).

Effectively, we've reduced our previously two dimensional surface down to a one dimensional *curve* (a curve which is embedded within the surface, with the surface embedded within three dimensional space). And again we can plot this; see fig. 5 in which we can see the curve embedded within the surface.

So we now have values for both T_I and n_I in terms of whatever we pick for n_R. However n_I is not just some amorphous gas mixture, but a particular combination of moles from C and moles from H—which we can now solve for. Consider the equations $n_I = n_H + n_C$ (eq. (10)) and $T_I n_I = T_H n_H + T_C n_C$ (eq. (13)); solve the first in terms of n_C:

$$n_I = n_H + n_C \Rightarrow n_C = n_I - n_H$$

For the second, recognize that T_I, n_I, T_H and T_C are all fixed; all except for n_H and n_C. But we solved the first equation for n_C. Thus, substitute n_C with $n_I - n_H$ in the second equation and solve for n_H:

$$T_I n_I = T_C n_C + T_H n_H = T_C(n_I - n_H) + T_H n_H \Rightarrow n_H = \frac{n_I(T_C - T_I)}{T_C - T_H}. \tag{18}$$

So that now we have n_H as a function of everything we previously calculated, and n_C is simply $n_I - n_H$, *and to an extent we have completely solved the proposed problem! Given our initial conditions and chosen target pressure and temperature, we can now choose any n_R in the range 0 to n_F with which we can calculate exactly the number of moles n_C from C and n_H from H to combine into I so that when we add I into F we precisely hit our target temperature and pressure.*

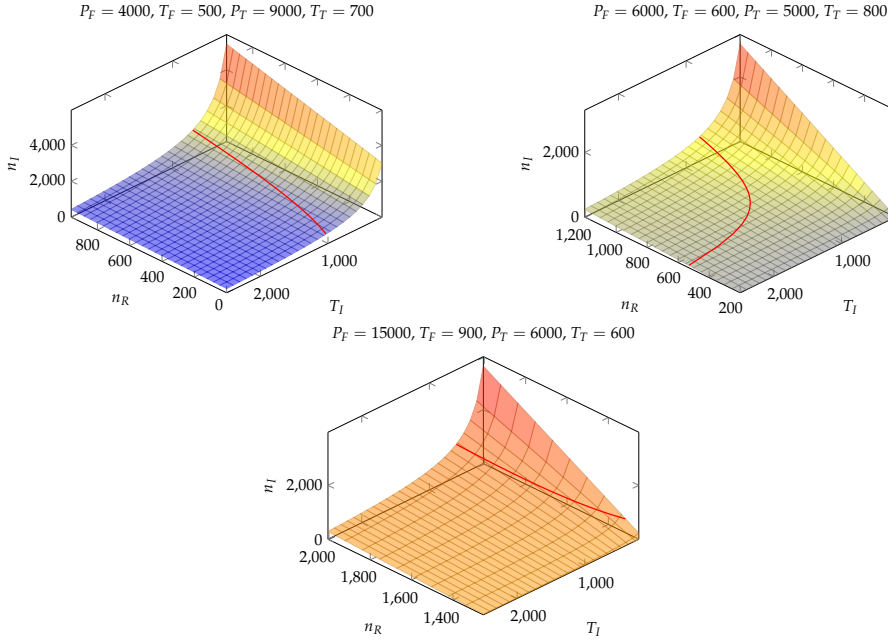*That being said, we're not quite done. . .*

Figure 5: The curve in red defined by (n_R, g, f) embedded within the surface defined by (n_R, T_I, f) (for the same various initial conditions).

> *Example.* Lets consider the third example of fig. 5, where our initial conditions are $P_F = 15000$ and $T_F = 900$, and target conditions are $P_T = 6000$ and $T_T = 600$.
>
> Using Desmos again, I've sketched up the model so far (fig. 6) and plugged in these initial conditions. By moving the n_R slider, notice that calculated values T_E and P_E (our final temperature and pressure) are exactly T_T and P_T (our targets), always.
>
> However, try lowering the value of n_R and observe the values of n_H and n_C; one or both of them go negative! These values are *mathematically correct*, but intuitively it makes no sense to add a *negative* amount of C or H...

As the example above demonstrates, we're not actually done. We've solved for all real-solutions to the problem, but we've neglected the bound that we cannot add a negative amount of C or H. Looking back at the third example of fig. 5, notice that the curve exits to the right of the graph. This shows us that not all values of n_R are valid for our model, because for some values (where the curve leaves the graph) we are adding a negative amount of C or H.

To address this, consider the energy quantity $T_I n_I$. So far we have not bounded this quantity, but in fact T_I is bounded by $T_C \leq T_I \leq T_H$. We can use these inequalities to solve for a true lower bound to n_R by substituting $n_I = n_T - n_F + n_R$ (eq. (10)) and $T_I n_I = T_T n_T - T_F(n_F - n_R)$ (from solving eq. (14) for $T_I n_I$).
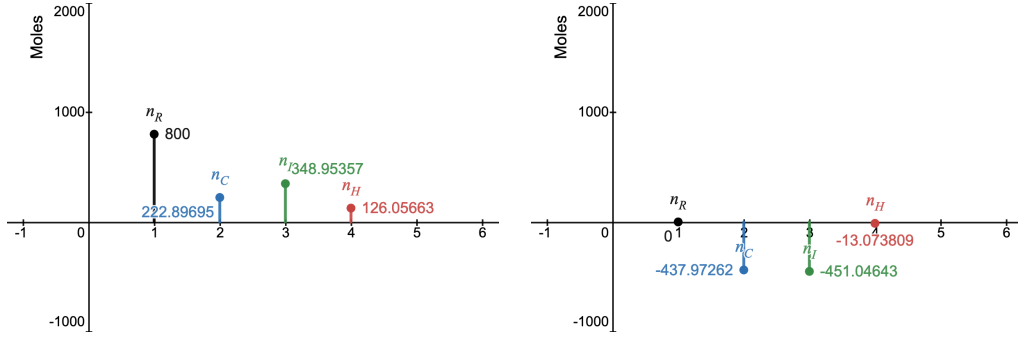
Figure 6: Desmos visualization of the n_R, n_I, n_H and n_C relationships.

First solving for n_R via $T_C n_I \leq T_I n_I$:

$$T_C n_I \leq T_I n_I$$
$$\Rightarrow \quad T_C(n_T - n_F + n_R) \leq T_T n_T - T_F(n_F - n_R)$$
$$\Rightarrow \quad T_C n_T - T_C n_F + T_C n_R \leq T_T n_T - T_F n_F + T_F n_R$$
$$\Rightarrow T_C n_T - T_T n_T + T_F n_F - T_C n_F \leq T_F n_R - T_C n_R$$
$$\Rightarrow \quad n_T(T_C - T_T) + n_F(T_F - T_C) \leq n_R(T_F - T_C)$$

$$\Rightarrow n_{R_H} = \frac{n_T(T_C - T_T)}{T_F - T_C} + n_F \leq n_R. \tag{19}$$

And second, solving for n_R via $T_I n_I \leq T_H n_I$:

$$T_I n_I \leq T_H n_I$$
$$\Rightarrow \quad T_T n_T - T_F(n_F - n_R) \leq T_H(n_T - n_F + n_R)$$
$$\Rightarrow \quad T_T n_T - T_F n_F + T_H n_F \leq T_H n_T - T_H n_F + T_H n_R$$
$$\Rightarrow T_T n_T - T_H n_T + T_H n_F - T_F n_F \leq T_H n_R - T_F n_R$$
$$\Rightarrow \quad n_T(T_T - T_H) + n_F(T_H - T_F) \leq n_R(T_H - T_F)$$

$$\Rightarrow n_{R_C} = \frac{n_T(T_T - T_H)}{T_H - T_F} + n_F \leq n_R. \tag{20}$$

The names n_{R_H} and n_{R_C} that I have chosen might be confusing and appear to be backwards, but this is what these two lower bounds are actually telling us about n_R:

- The lower bound n_{R_C}, despite being a function of T_H, tells us that if we remove an amount less than n_{R_C} (i.e. if $n_R < n_{R_C}$), then n_C becomes negative; and...

- The lower bound n_{R_H}, despite being a function of T_C, tells us that if we remove an amount less than n_{R_H} (i.e. if $n_R < n_{R_H}$), then n_H becomes negative.

So since we know that we are not allowed to remove an amount less than either of these bounds, nor are we allow to remove less-than zero, we can simplify the bounds on n_R once-and-for-all:

$$n_{R_{\min}} \leq n_R \leq n_F, \quad \text{where } n_{R_{\min}} = \max(0, n_{R_C}, n_{R_H}). \tag{21}$$
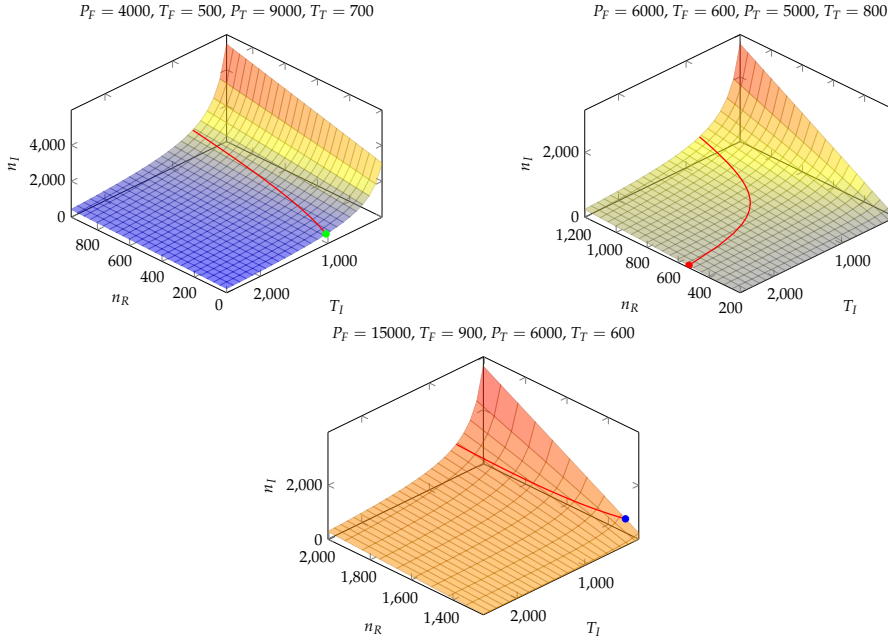
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Figure 7: The surface (n_R, T_I, f), the curve (n_R, g, f), and a point (n_R, g, f) where n_R is determined to be $\max(0, n_{R_C}, n_{R_H})$.

So we have fixed the range for n_R but we haven't actually said what value to choose. Now, there's some analysis we could apply to help study this curve (and that might be something I relearn later to include here), but from the visualization we can also clearly see that where this curve will always intersect the boundaries of the graph fig. 7 is actually the point at which not only n_R is minimal, but so is n_C and n_H (by virtue of minimizing n_I).

Therefore, $n_{R_{\min}}$ doesn't just give us the range for n_R but it actually gives us the minimal point!

We have, in the end, a final theorem to use in the application of solving the problem within Stationeers:

Theorem 1 (Acquiring T_T and P_T with minimal n_R, n_C and n_H). *Fix a single gas furnace system F with initial temperature T_F and pressure P_F, a target temperature T_T and target pressure P_T, a cold source C with temperature $T_C \leq T_T$, and a hot source H with temperature $T_H \geq T_T$.*

The point (n_R, n_C, n_H) which satisfies $T_T n_T = T_F(n_F - n_R) + T_I n_I$ that minimizes the values of n_R, n_H and n_C is given by the following algorithm:

(a) $n_T = \dfrac{P_T V_F}{R T_T},$

(b) $n_{R_C} = \dfrac{n_T(T_T - T_H)}{T_H - T_F} + n_F,$

(c) $n_{R_H} = \dfrac{n_T(T_C - T_T)}{T_F - T_C} + n_F,$

(d) $n_R = n_{R_{\min}} = \max(0, n_{R_C}, n_{R_H}),$

(e) $n_I = n_T - n_F + n_R,$

(f) $T_I = \dfrac{T_T n_T - T_F(n_F - n_R)}{n_I},$

(g) $n_H = \dfrac{n_I(T_C - T_I)}{T_C - T_H},$

(h) $n_C = n_I - n_H.$