1 Introduction and Summary

In this section, I'll first attempt to introduce all relevant mathametical notation and gas laws, then attempt summarize my solutions to the proposed problem:

> *Given a furnace F containing only* CO_2 *at some arbitrary initial conditions, a source H of relatively hot* CO_2 *and a source C of relatively cold* CO_2*, is there not only some way of constructing a mixture of H and C such that inserting it into F (possibly after removing from initial gas from F) yields exactly some target temperature and pressure in F, but a way in which minimizes how much from H and C we need to add?*

1.1 Notation

Firstly, I'll often write a closed mixture/volume of gasses as a single capital letter, like F for furnace, and where now F symbolically refers to all the conditions of the furnace. Formally:

Definition (M a gas mixture). Let M be a "mixture/volume" of gasses; we write:

- V_M for the total volume in liters of the system in which the mixture is contained,
- n_M for the total number of moles in the mixture,
- T_M for the temperature of the mixture,
- P_M for the total pressure in kilopascals, and
- C_M for the specific heat of the mixture.

Secondly, there are seven types of gasses in the game, each with a formula. I use the formulas as *indices* for the gasses. That is, if I say *the gas g*, then what I mean is *the gas indexed by g, an element of the set of gas indices*.

Definition (g a gas index). The set of gas indicies for Stationeers:

$$\mathcal{G} = \{H_2, N_2, O_2, X, CO_2, H_2O, N_2O\}.$$

We assume that the order of these indices is always the same; that is, the first element of $\mathcal{G}$ is always H_2, the second is always N_2, and so on.

Let M be a mixture of the gasses indexed by the elements of $\mathcal{G}$, and let $g \in \mathcal{G}$ (read, g is an element of $\mathcal{G} \Leftrightarrow g = H_2$, or $N_2, \ldots$, or N_2O). Then we write:

- $n_M(g)$ for the moles of g (the gas indexed by g) within mixture M,
- $P_M(g)$ for the partial pressure of g within mixture M, and
- C_g the specific heat (capacity) of a gas g.

Name	Formula (g)	Specific heat (C_g)
Hydrogen	H_2	20.4
Nitrogen	N_2	20.6
Oxygen	O_2	21.1
Pollutant	X	24.8
Carbon dioxide	CO_2	28.2
Water	H_2O	72
Nitrous oxide	N_2O	23

Figure 1: Table of gasses in Stationeers

(See see Figure 1 for a table of the gasses in Stationeers, their g formulas/indices, and their C_g specific heats.)

All but the last of these conditions can be determined without much calculation; in particular n_M the total moles, T_M temperature and P_M pressure can all be read directly from a furnace/pipe analyzer/tank in game, and $n_M(g)$ moles of the particular gas g can be calculated simply as the ratio of the gas times the total moles:

$$n_M(g) = (\text{ratio of } g \text{ in } M) \cdot n_M, \text{ where } 0 \leq \text{ratio of } g \text{ in } M \leq 1.$$

Note that if M consists of only one kind of gas (which I index as g^*):

$$\text{if } 1 = (\text{ratio of } g^* \text{ in } M) \Rightarrow n_M(g^*) = 1 \cdot n_M = n_M.$$

1.2 Gas laws

Gasses in Stationeers obey some of the ideal gas laws from real life. First, lets know how to calculate the specific heat of a mixture of gasses. Recall that a gas, indexed by $g \in \mathcal{G}$, has a specific heat C_g (Figure 1).

Definition (Specific heat capacity of a gas mixture)**.** Let M be a mixture of the $\mathcal{G}$ gasses. The specific heat of the mixture (C_M) is given by the sum of the products of the molar ratios of the gasses in the mixtures and their respective specific heats. We can notate this mathematically in a few ways:

$$C_M = \frac{n_M(H_2)}{n_M}C_{H_2} + \frac{n_M(N_2)}{n_M}C_{N_2} + \cdots + = \frac{n_M(N_2O)}{n_M}C_{N_2O} = \sum_{g \in \mathcal{G}} \frac{n_M(g)}{n_M}C_g$$

Or as the dot-product $\cdot$ of vectors:

$$C_M = (\boldsymbol{n}_M \cdot \boldsymbol{C}_\mathcal{G})\frac{1}{n_M} \tag{1}$$

Where $\boldsymbol{n}_M$ is the vector of the moles per gas in M, and where $\boldsymbol{C}_\mathcal{G}$ is the vector of the specific heats of each gas:

$$\boldsymbol{n}_M = \langle n_M(H_2), n_M(N_2), \ldots, n_M(N_2O) \rangle = \langle n_M(g) : g \in \mathcal{G} \rangle$$
$$\boldsymbol{C}_\mathcal{G} = \langle C_{H_2}, C_{N_2}, \ldots, C_{N_2O} \rangle = \langle C_g : g \in \mathcal{G} \rangle.$$

Note that if M consists of only one kind of gas g^*, then the ratio of $g^* \in \mathcal{G}$ is M is one (as noted before), and the ratio of $g \in \mathcal{G} \setminus g^*$ in M is zero. (We read $g \in \mathcal{G} \setminus g^*$ as: g is an element of the set $\mathcal{G} \setminus g^*$, where $\mathcal{G} \setminus g^*$ is the set $\mathcal{G}$ but with the g^* element removed.)

Example. Suppose F our furnace contains 10 moles of each kind of gas, of which there are seven.

Then $n_F = 70$, and:

$$C_F = \sum_{g \in \mathcal{G}} \frac{n_F(g)}{n_F} C_g = \frac{n_F(\mathrm{H_2})}{n_F} C_{\mathrm{H_2}} + \frac{n_F(\mathrm{N_2})}{n_F} C_{\mathrm{N_2}} + \cdots + \frac{n_F(\mathrm{N_2O})}{n_F} C_{\mathrm{N_2O}}$$

$$= \frac{10}{70}(20.4 + 20.6 + 21.1 + 24.8 + 28.2 + 72 + 23) = 30.01$$

Example. Suppose F our furnace contains 70 moles of only $g^* = \mathrm{CO_2}$.

Then $n_F = n_F(\mathrm{CO_2}) = 70$, and:

$$n_F(g) = \begin{cases} n_F(g) = n_F, & \text{if } g = \mathrm{CO_2} \\ n_F(g) = 0, & \text{if } g \neq \mathrm{CO_2} \end{cases},$$

$$C_F = (\boldsymbol{n_F} \cdot \boldsymbol{C_G}) \frac{1}{n_F}$$

$$= ((0,0,0,0,70,0,0) \cdot (C_{\mathrm{H_2}}, C_{\mathrm{N_2}}, C_{\mathrm{O_2}}, C_{\mathrm{X}}, C_{\mathrm{CO_2}}, C_{\mathrm{H_2O}}, C_{\mathrm{N_2O}})) \frac{1}{70}$$

$$= (70 \cdot C_{\mathrm{CO_2}}) \frac{1}{70} = C_{\mathrm{CO_2}}$$

This reduction of a volume's specific heat to a constant is one reason why I prefer a single gas ($\mathrm{CO_2}$) design; it reduces many of the later problems from problems of differential equations to simple problems of algebra. The other reason is that $\mathrm{CO_2}$ is plentiful, both as at atmospheric gas on planets like Mars and Venus and as a byproduct of combustion.

In terms of gas laws, the model in Stationeers obeys the following laws for a fixed volume of gas.

Definition (Gay-Lussac's law).
$$P_1 T_2 = P_2 T_1 \tag{2}$$
Change in pressure is directly proportional to change in temperature.

Note! This law is only *half*-obeyed! In game, changing the temperature in a fixed volume expectedly affects the pressure proportionally, but not vice-versa. That is: removing gas, i.e. "lowering its pressure", does *not* change its temperature.

Definition (Ideal/combined gas law).

$$PV = nRT \tag{3}$$

Where $R = 8.314$ (joules per mole kelvin) is the universal gas constant.

Definition (Dalton's law of partial pressures).

$$P = P(\mathrm{H_2}) + P(\mathrm{N_2}) + P(\mathrm{O_2}) + P(X) + P(\mathrm{CO_2}) + P(\mathrm{H_2O}) + P(\mathrm{N_2O}) = \sum_{g \in \mathcal{G}} P(g) \tag{4}$$

The pressure in the volume is the sum of partial pressures of the constituent gasses.

Definition (Energy of a volume gas). The energy of a volume of gas with temperature T, number of moles n, and (combined) specific heat C is given by TnC.

Furthermore, the game model satisfies these other equations. Given volumes A and B, and C the result of combining A and B:

$$n_C = n_A + n_B \qquad\qquad \text{(moles are additive)} \tag{5}$$

$$C_C = ((\boldsymbol{n_A} + \boldsymbol{n_B}) \cdot \boldsymbol{C_{\mathcal{G}}}) \frac{1}{n_C} \qquad\qquad \text{(combined specific heat)} \tag{6}$$

$$T_C n_C C_C = T_A n_A C_A + T_B n_B C_B \qquad\qquad \text{(combined energies)} \tag{7}$$

With all this notation and all these gas equations in mind, we move onto solving the proposed problem.

1.3 Summary

Drafting up the scenario:

(Initial conditions) A furnace already contains some amount of "hot" CO_2 as a starter. We add the reagents for some alloy, releasing some other gasses into the furnace mix. We then filter out all gases except for the CO_2, returning the CO_2 back into the furnace, leaving the furnace with only CO_2 gas but also the reagents for the alloy. For the alloy we require some specific temperature, T_T (target temperature), and pressure, P_T (target pressure); we'll consider these to be the minimum of whatever the alloy ranges are. Lastly, we have H the "hot" source of CO_2 that has temperature T_H of *at least* T_T kelvin, and have C the "cold" source of CO_2 that's *at most* T_T kelvin.

(The question now) How much of the furnace CO_2 do we need to remove (if any) in order for it to be possible to compose a *minimal* combination I of CO_2 from the H and C sources, that when added to F achieves T_T and P_T in the furnace.

There will be many variables in the following work (see Figure 2 for descriptions for all of the variables we will be encountering—note that I use *volume*, *mixture* and *composition*

Volume	Description
F	The initial furnace volume
T	The target furnace volume
R	The volume removed from the initial
I	The input volume
H	The hot source
C	The cold source

	Initial	Target	Removed	Input	Hot	Cold
Pressure	P_F	P_T	—	—	—	—
Temperature	T_F	T_T	—	T_I	T_H	T_C
Moles CO_2	n_F	n_T	n_R	n_I	n_H	n_C

Figure 2: Table of variables which we will encounter.

somewhat interchangeably when talking about some specific closed amount of gas, and that volume here has nothing to do with the actual volume of the container in which the closed amount of gas sits); we'll first lay out the relationships between these different variables with respect to the laws and equations defined above. We will use these relationships to establish a system of equations.

Calculating target moles

Given that P_T and T_T are our fixed targets, we can use the ideal gas law (Equation 3) to solve for n_T the target moles we will need. From the game we know the volume of a furnace V_F to be 1,000 liters, and R the universal gas constant is 8.314 joules per mole kelvin (and note that $1 \text{ kPa} = 1 \text{ J/m}^3 = 1 \text{ J/L}$). Then:

$$P_T \text{ J/L} \cdot V_F \text{ L} = n_T \text{ mol} \cdot R \text{ J/mol K} \cdot T_T \text{ K}$$
$$\Rightarrow n_T \text{ mol} = \frac{P_T \text{ J/L} \cdot V_F \text{ L}}{R \text{ J/mol K} \cdot T_T \text{ K}} = \frac{P_T \cdot V_F}{R \cdot T_T} \text{mol} \tag{8}$$

Simplifying energy equations (Equation 6 and Equation 7)

Firstly, consider Equation 6 for some volumes A, B and C of *only* CO_2, and where C is the combination of A and B. Then:

$$C_C = \left((\boldsymbol{n_A} + \boldsymbol{n_B}) \cdot \boldsymbol{C_G}\right)\frac{1}{n_C} = (n_A(CO_2) + n_B(CO_2))C_{CO_2}\frac{1}{n_C} = n_C C_{CO_2}\frac{1}{n_C} = C_{CO_2}.$$

As well as $C_A = C_B = C_C = C_{CO_2}$. Then Equation 7 can be simplified:

$$\begin{aligned} T_C n_C C_C &= T_A n_A C_A &+ T_B n_B C_B \\ \Rightarrow T_C n_C C_{CO_2} &= T_A n_A C_{CO_2} &+ T_B n_B C_{CO_2} \\ \Rightarrow T_C n_C &= T_A n_A &+ T_B n_B. \end{aligned} \tag{9}$$

And this is all we need for a one-gas, CO_2 system.

Bounds on moles

Consider that an amount of moles cannot be negative, $0 \leq n_M$, and that we can only remove as many moles as there are there initially, though we might remove none at all. This imposes a restriction on n_R:

$$0 \leq n_R \leq n_F.$$

Combinations of moles (applying Equation 5)

When we have theoretically reached our ideal target pressure and temperature, we will have removed some amount of moles n_R from the initial amount n_F in the furnace F, and will have input some amount of moles n_I into the furnace. The number of moles n_I of the input volume I is a function of n_H, some number of moles from the hot source, and n_C, some number of moles from cold source, that must satisfy Equation 5:

$$n_I = n_H + n_C. \tag{10}$$

The number of moles at the theoretical target must also satisfy this equation:

$$n_T = (n_F - n_R) + n_I$$

(Substituting Equation 10)

$$= (n_F - n_R) + (n_H + n_C).$$

Combinations of energies (applying Equation 9)

The temperature and number of moles for the input volume is a function of the temperature and number of moles from the hot and cold sources and must satisfy equation Equation 9:

$$T_I n_I = T_H n_H + T_C n_C. \tag{11}$$

Similarly, the temperature and number of moles for the target is a function of the initial temperature in the furnace, the number of moles remaining in the furnace after having removed some amount, and the temperature and number of moles of the input volume:

$$T_T n_T = T_F(n_F - n_R) + T_I n_I \tag{12}$$

(Substituting Equation 11)

$$= T_F(n_F - n_R) + (T_H n_H + T_C n_C). \tag{13}$$

This is the equation we're most interested in, as combines all the parts of the problem together and allows us to easily solve for either T_I or T_T.

Example. Suppose we have initial conditions $T_F = 500$, $P_F = 1000$, $T_C = 200$ and $T_H = 2500$, but we don't have a particular target pressure P_T or temperature T_T; we just want to see how things change as we remove some amount n_R from F, add some amount n_H from H into F, and/or add some amount n_C from C into F.

Lets call T_E, P_E and n_E our end results: the temperature, pressure and number of moles after having removed n_R and having added n_I. Our procedure is this: pick values for n_R, n_H and n_C, (a) calculate $n_E = (n_F - n_R) + n_I$ where $n_I = n_H + n_C$, (b) calculate T_E by solving Equation 13 for T_E, and then (c) calculate P_E by solving Equation 3 for P_E.

First we need to calculate n_F, the initial amount of moles in the furnace, using Equation 3 (recall that $V_F = 1000$ and $R = 8.314$):

$$n_F = \frac{P_F \cdot V_F}{R \cdot T_F} = \frac{1000 \cdot 1000}{8.314 \cdot 500} \doteq 240.56 \text{ moles } CO_2.$$

(Alternatively, since this is Stationeers we're talking about, we would have been able to find this value just by reading the number of moles from the furnace: `l nF furnace TotalMoles`.)

> *Example* ($n_R = 100$, $n_H = 200$ and $n_C = 0$).
>
> (a) $n_E = (240.56 - 100) + (200 + 0) = 340.56$
>
> (b) $T_E = (T_F(n_F - n_R) + (T_H n_H + T_C n_C))/n_E$
> $\quad = (500 \cdot 140.56 + (2500 \cdot 200 + 200 \cdot 0))/340.56 \doteq 1674.54 \text{ K}.$
>
> (c) $P_E = (n_E \cdot R \cdot T_E)/V_F$
> $\quad = (340.56 \cdot 8.314 \cdot 1674.54)/1000 \doteq 4741.32 \text{ kPa}.$

Bounds on temperatures

As setup in the paragraph of initial conditions, the hot source H has temperature at least T_T and the cold source C has temperature at most T_T. Thus we can always combine an amount from H and an amount from C into a volume I such that T_I equals T_T. So from this we get the bounds:

$$T_C \leq T_T \leq T_H, \text{ and } T_C \leq T_I \leq T_H.$$

1.4 Calculating minimal amounts

We have enough bounds here to establish a system of equations with which we can solve the problem; that is to be able to solve for a minimal amounts n_R to remove, n_H of H to add and n_C of C to add. It will be a matter of solving these equations for particular variables, and applying certain metrics to solve for these minimal values.

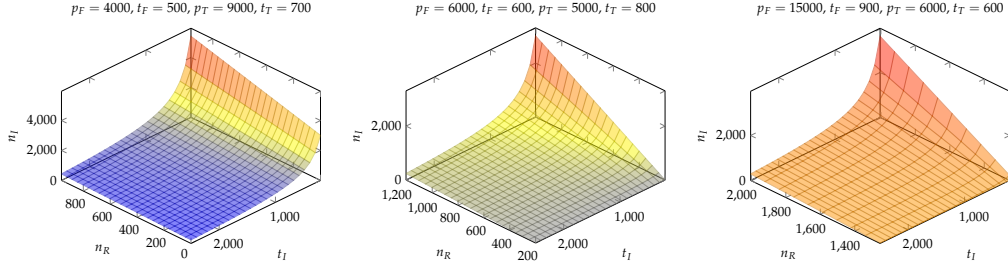First, solve Equation 12 for n_I. This yields n_I as a function of n_R and T_I; the remaining

Figure 3: The surface defined by (n_R, T_I, f) for various initial conditions.

variables (T_T, n_T, T_F and n_F) are all fixed:

$$f(n_R, T_I) = n_I = \frac{T_T n_T + T_F(n_F - n_R)}{T_I}. \tag{14}$$

We can plot this function f, being of two variables n_R and T_I, in three dimensions using the bounds on T_I and n_R: that $T_C \leq T_I \leq T_H$ and $0 \leq n_R \leq n_F$. Figure 3 visualizes this bounded surface for a few different initial conditions (in all cases I chose $T_C = 200$ and $T_H = 2500$). What this shows us is that any point (n_R, T_I, f) (where f is shorthand for $f(n_R, T_I)$) on the surface satisfies Equation 14.