Flows

Proseminar Algorithmen auf Graphen

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RWTH Aachen University

May 21, 2024

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Consider the following problem:

- production facility s

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- rail network through cities A, B, C, D, s

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- production facility s
- rail network through cities A, B, C, D, s, and t
- destination t

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- limited amount of goods per day from one city to another

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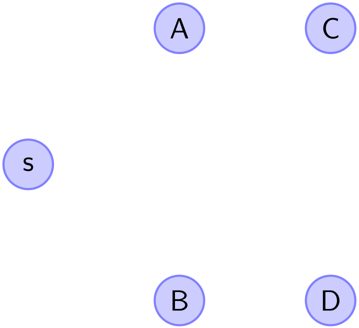

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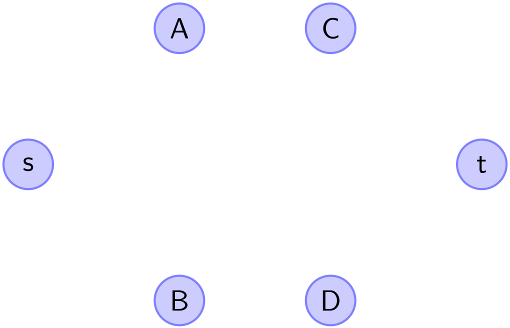

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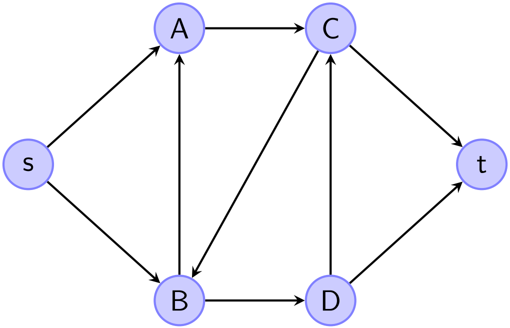

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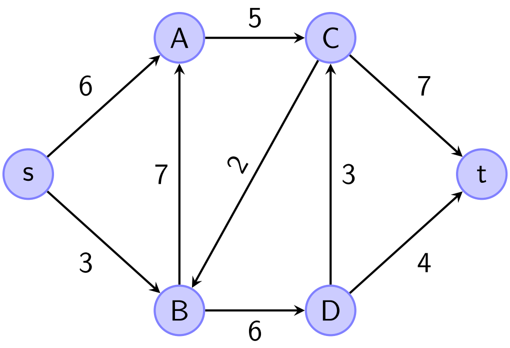

Consider the following problem:

- production facility s
- rail network through cities A, B, C, D, s, and t (flow network)
- destination t
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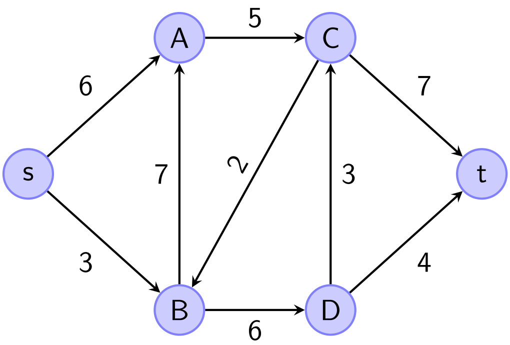

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- production facility s (source)
- rail network through cities A, B, C, D, s, and t *(flow network)*
- destination t (sink)
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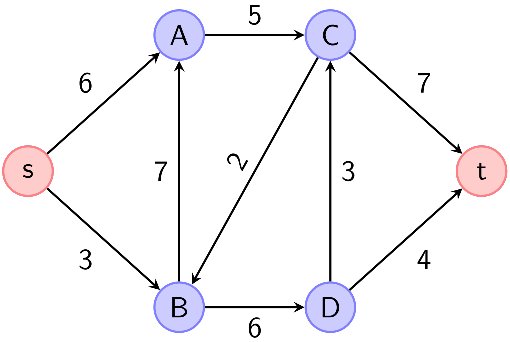

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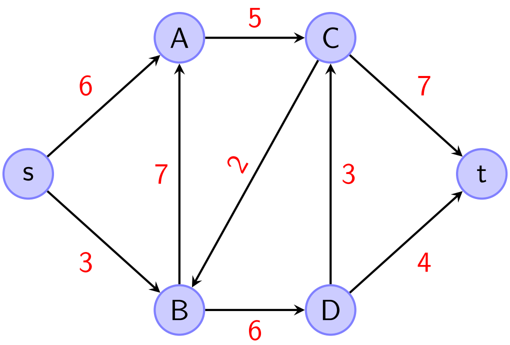
- production facility s *(source)*
- rail network through cities A, B, C, D, s, and t *(flow network)*
- destination t *(sink)*
- limited amount of goods per day from one city to another (capacity)
- How many goods can be transported from s to t per day?



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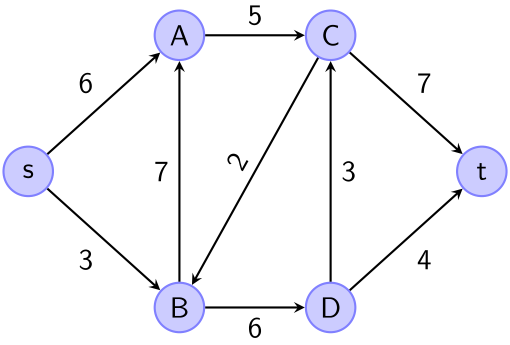
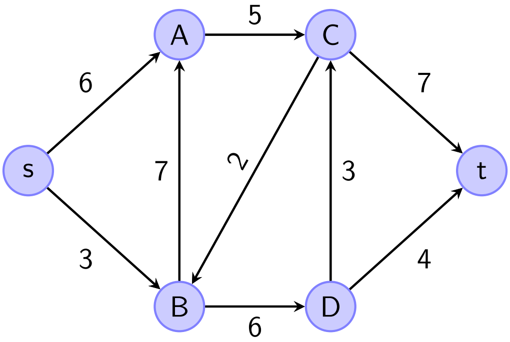
- production facility s *(source)*
- rail network through cities A, B, C, D, s, and t *(flow network)*
- destination t *(sink)*
- limited amount of goods per day from one city to another *(capacity)*
- How many goods can be transported from s to t per day? (maximum flow)



Definition (Flow Network)

A flow network $G = (V, E)$ is a directed graph.



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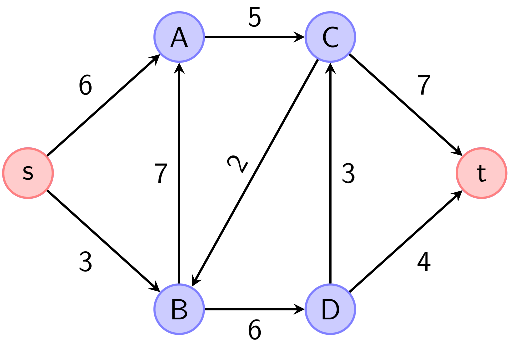
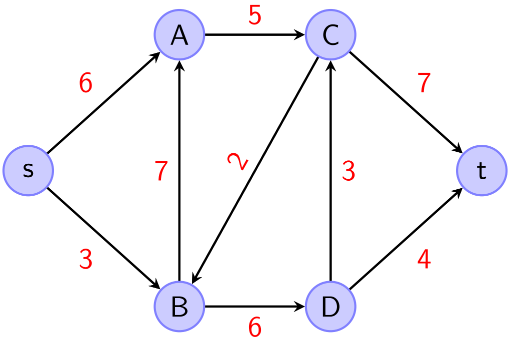
The vertices $s, t \in V$ are called the source and the sink of G.

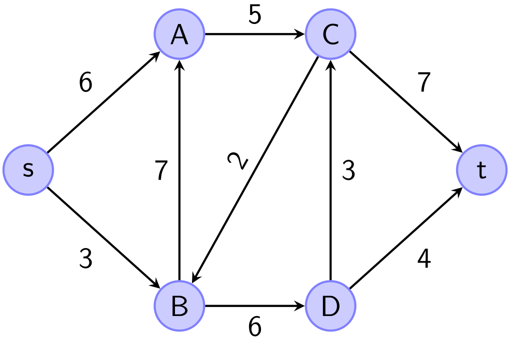


Definition (Flow Network)

A flow network $G = (V, E)$ is a directed graph.
The vertices $s, t \in V$ are called the source and the sink of G.
The capacity function $c : V \times V \to \mathbb{R}_{\geq 0}$ assigns capacities to the edges of G where $c(u, v) = 0$ for all $(u, v) \notin E$.



Definition (Flow and Value of Flow)



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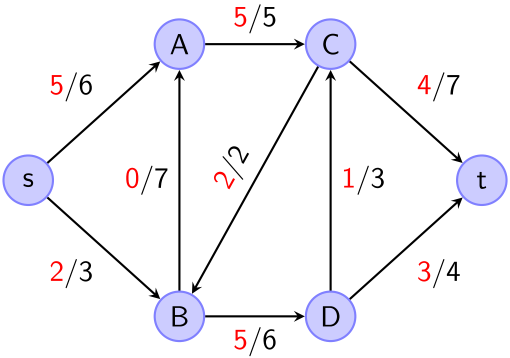

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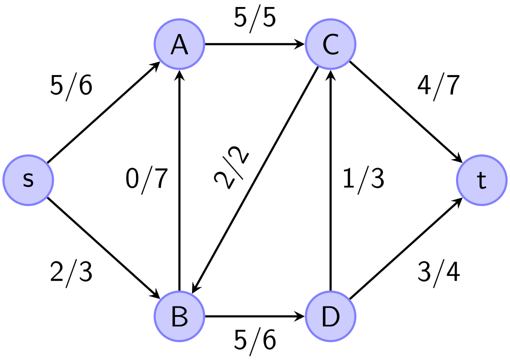
A function $f : V \times V \to \mathbb{R}$ is called a flow in the network $G = (V, E)$ if it satisfies the following properties:



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A function $f : V \times V \to \mathbb{R}$ is called a flow in the network $G = (V, E)$ if it satisfies the following properties:

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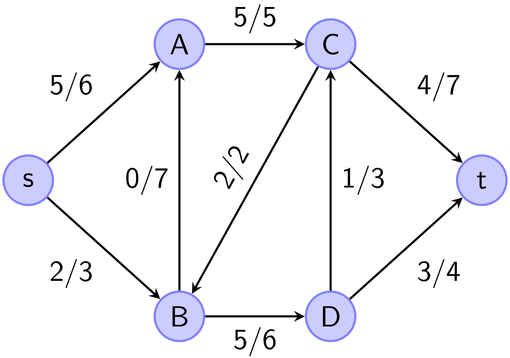

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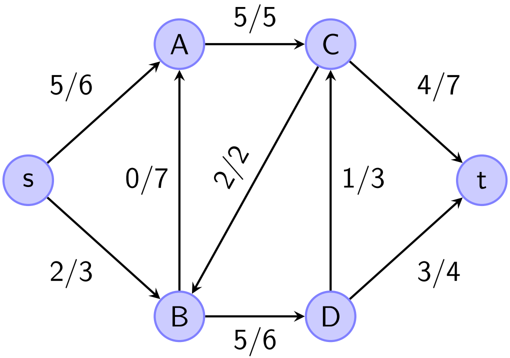

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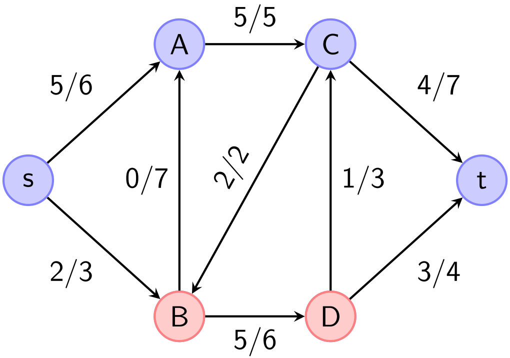

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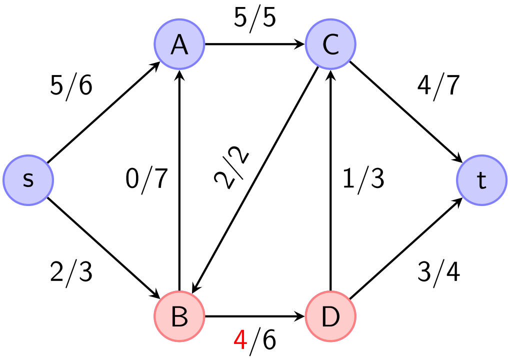

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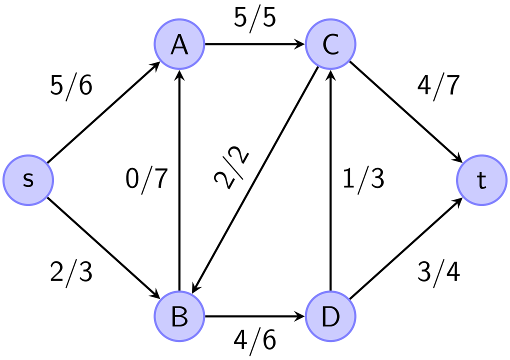
The value of a flow $|f|$ is given by

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s).$$



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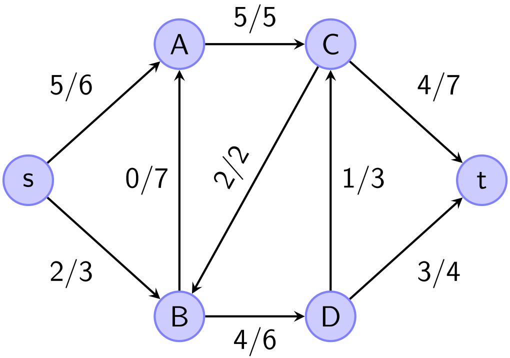

$$\begin{aligned} |f| &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) \\ &= \quad 5 + 2 \quad - \quad 0 \quad = 7 \end{aligned}$$

Maximum-Flow Problem

Problem

Given a flow network G with source s, sink t, and a capacity function c.
Find a flow f in G of maximum value.

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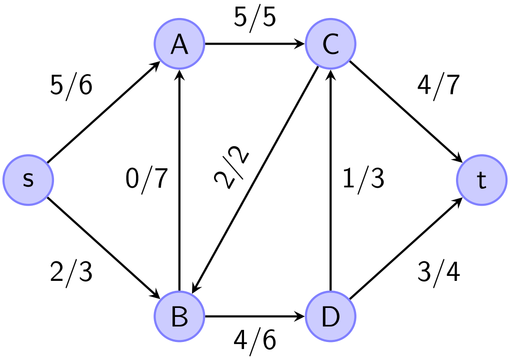

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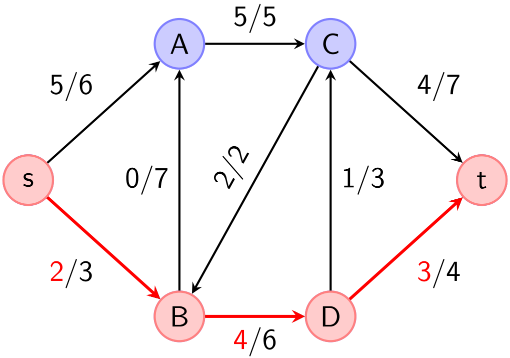

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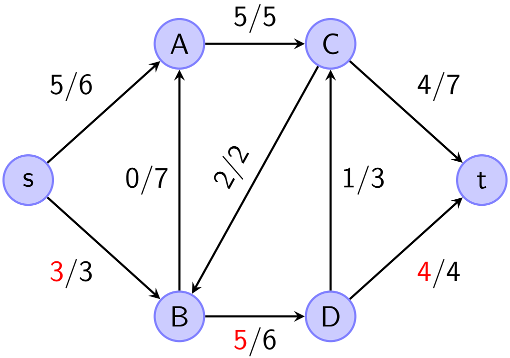

Ford-Fulkerson Method

- method to solve the maximum-flow problem, i.e. find a maximum flow

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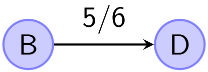
Ford-Fulkerson Method

initialize flow f with 0
while there exists an augmenting path p in the residual network G_f **do**
  augment flow f along p
end while
return f

Residual Networks

edge in flow network G



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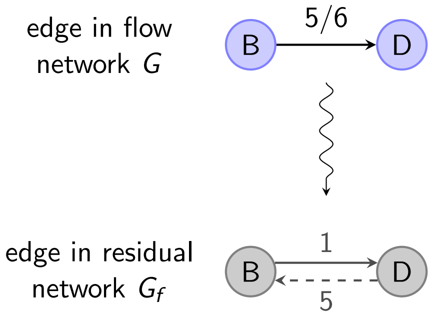
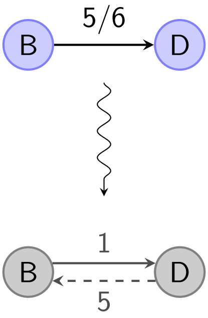


Residual Networks

edge in flow network G



edge in residual network G_f

Definition (Residual Network)

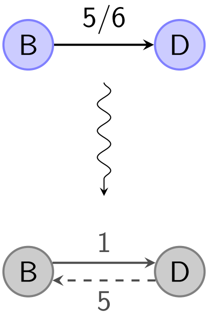
the residual capacity c_f is given by

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

Residual Networks

edge in flow network G

edge in residual network G_f



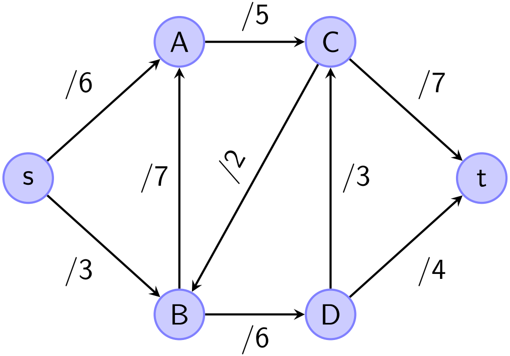
Definition (Residual Network)

A residual network of G induced by f is $G_f = (V, E_f)$ with

$$E_f = \{(u,v) \in V \times V \mid c_f(u,v) > 0\},$$

where the residual capacity c_f is given by

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$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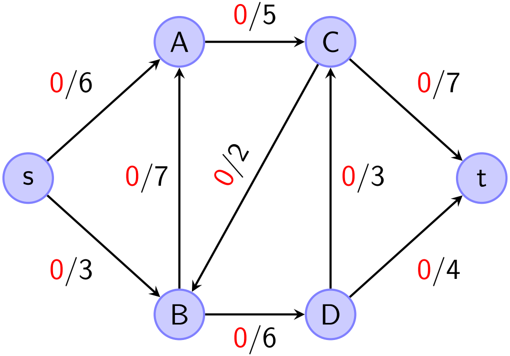
Demonstration of Ford-Fulkerson Method



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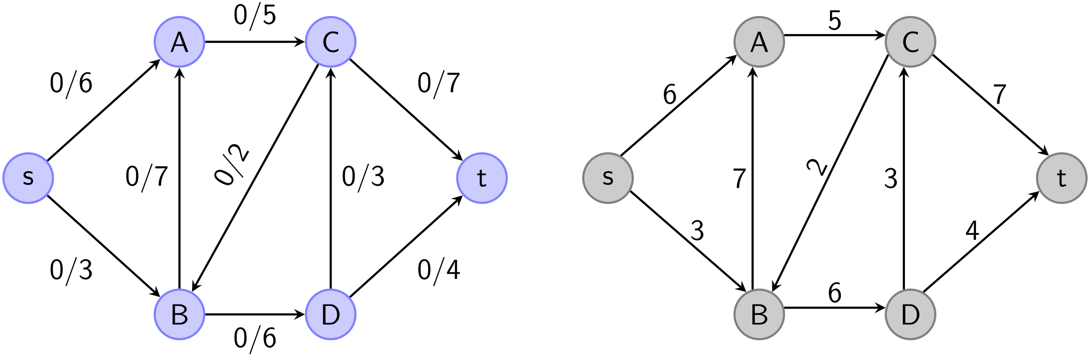

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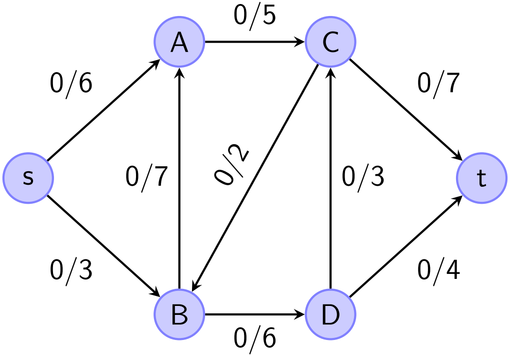

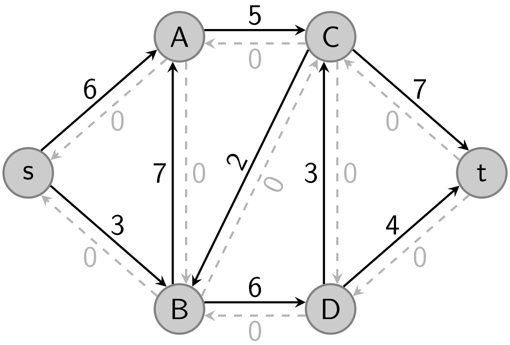

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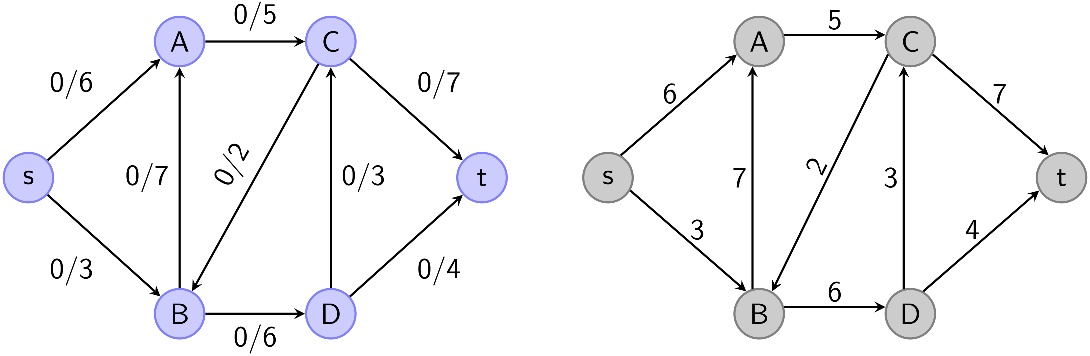

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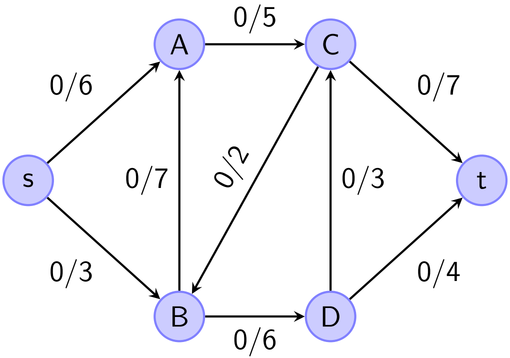

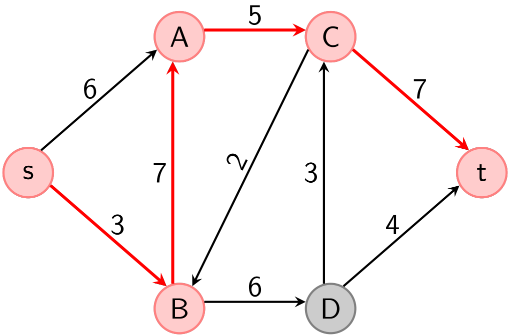

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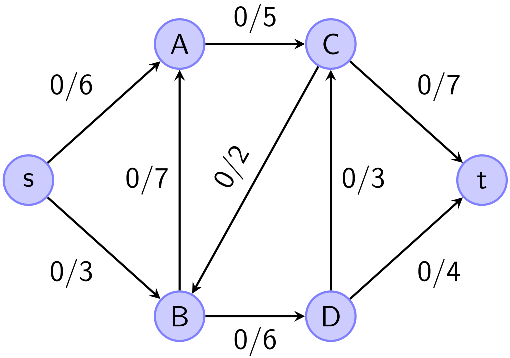

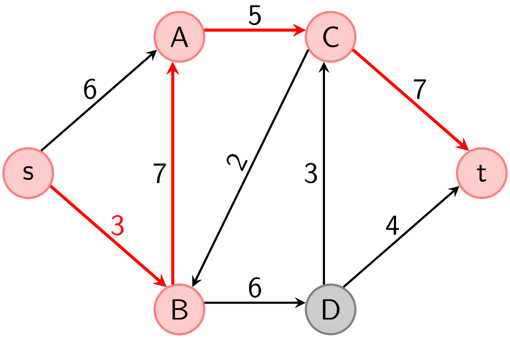

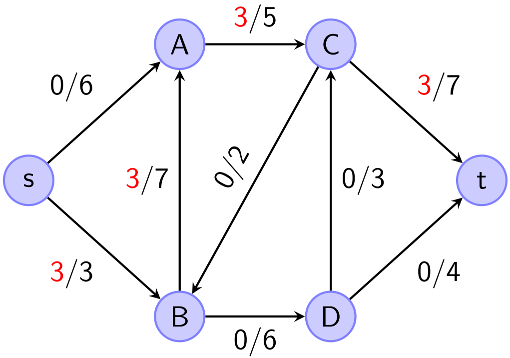
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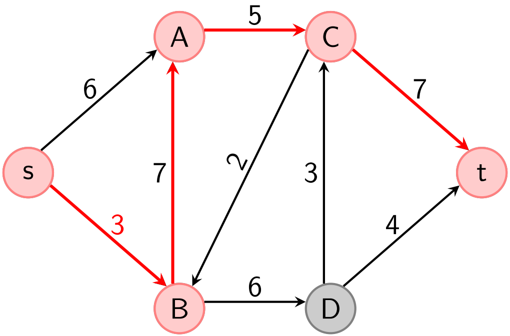
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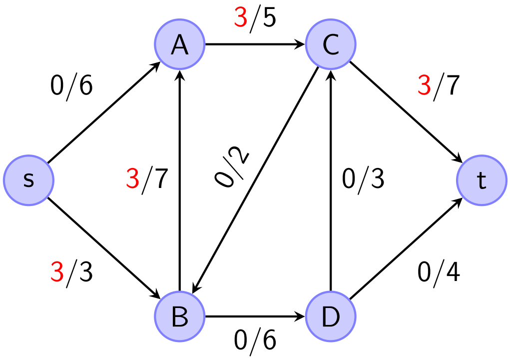
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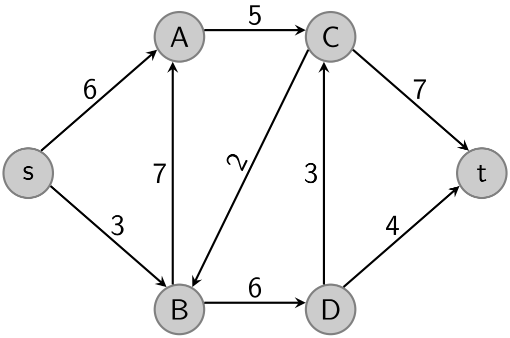
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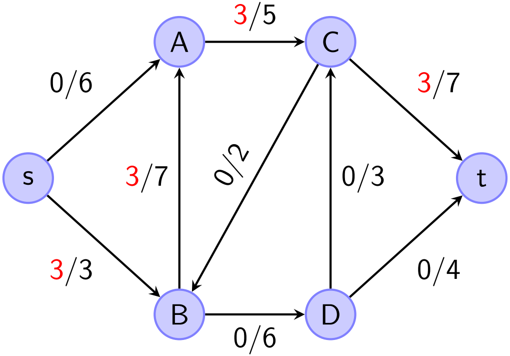

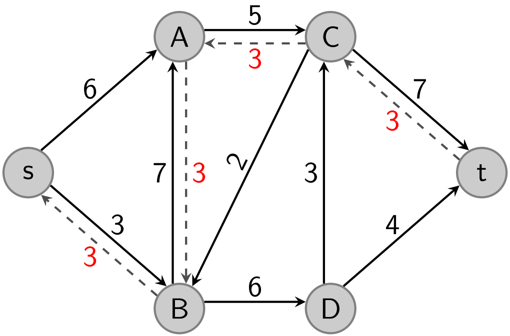

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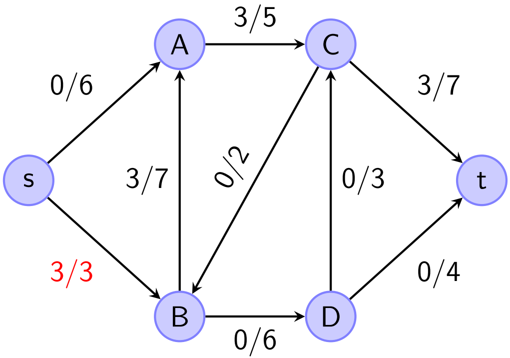

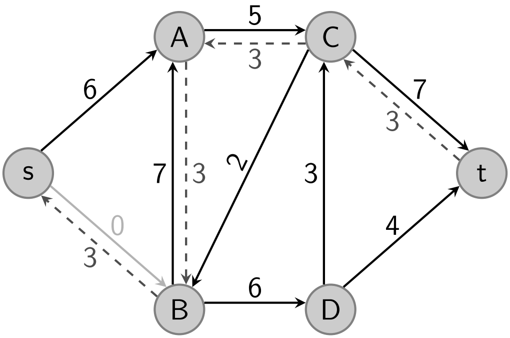

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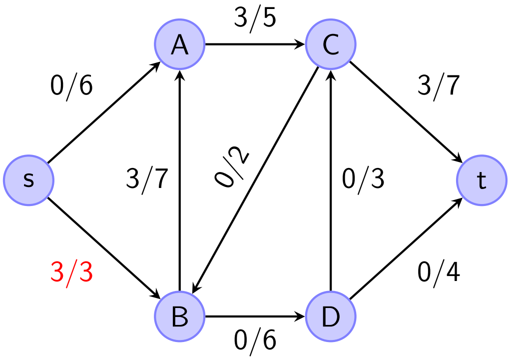

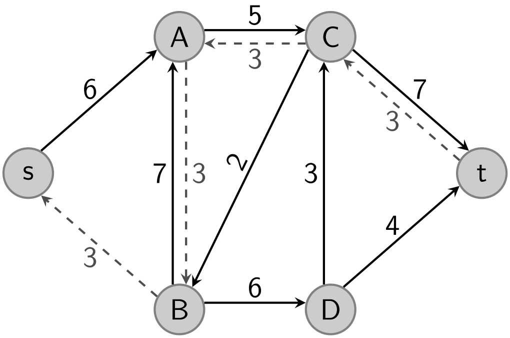

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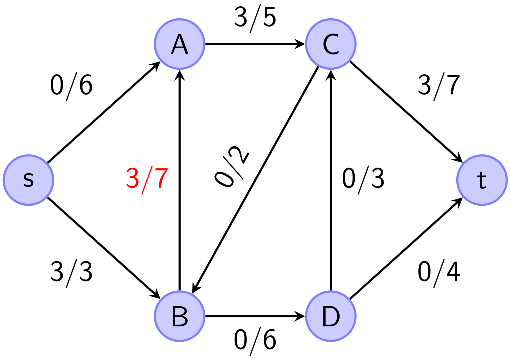

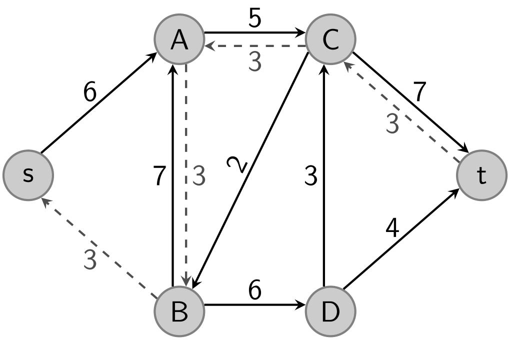

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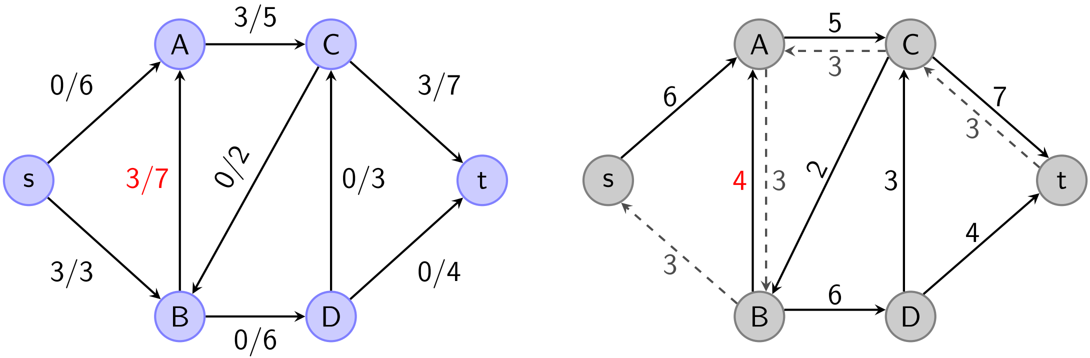

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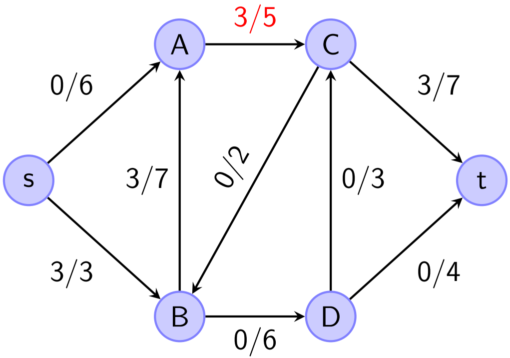

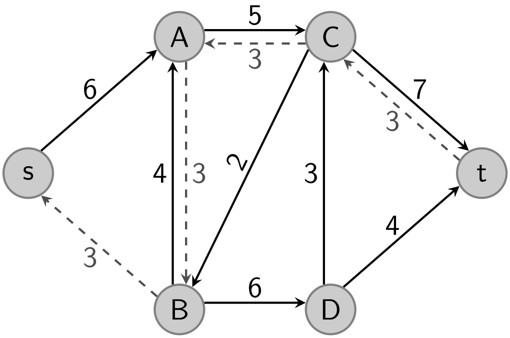

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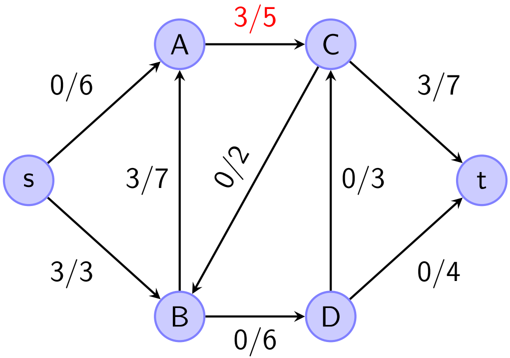

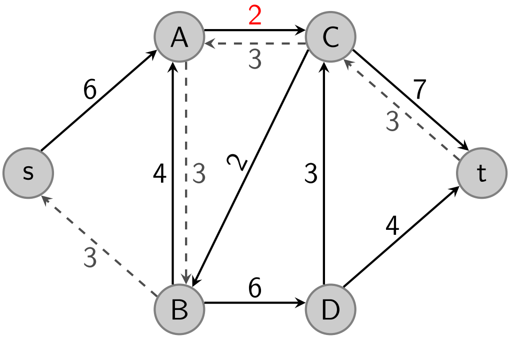

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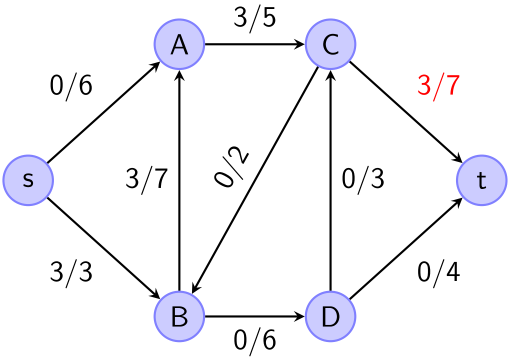

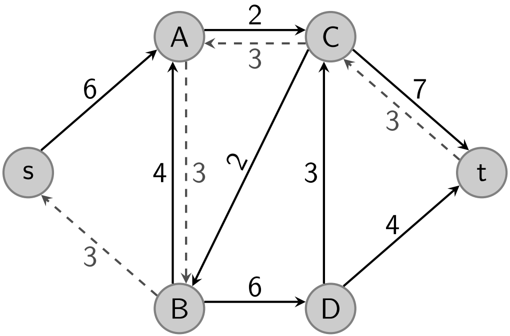

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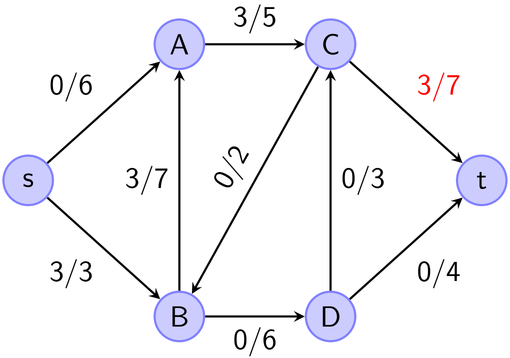

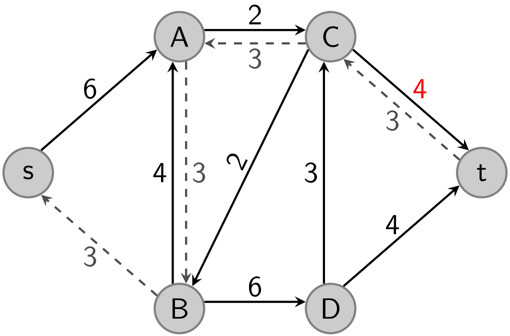

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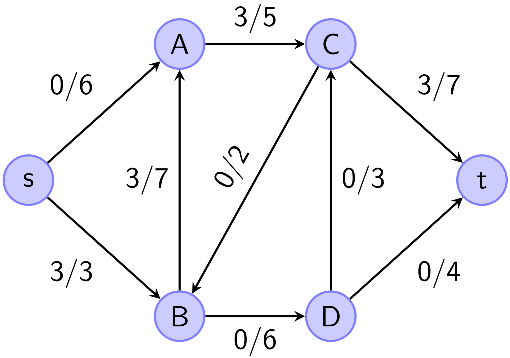

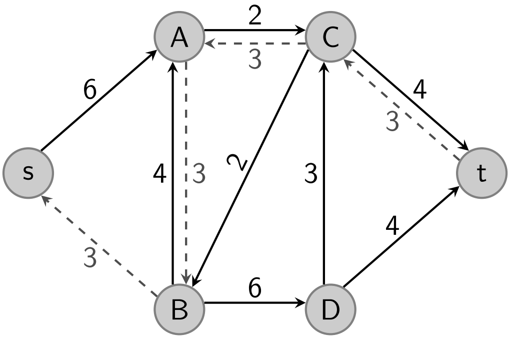

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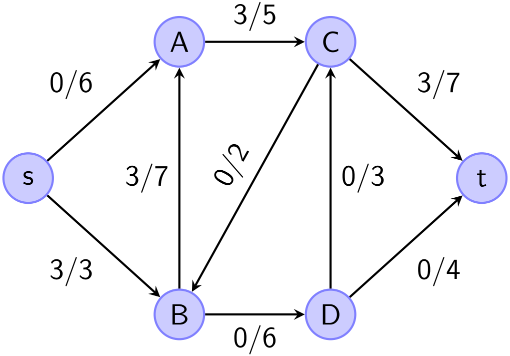

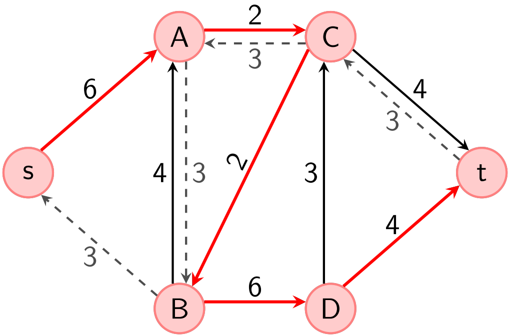

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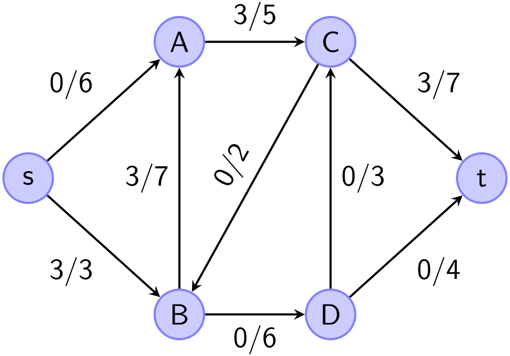

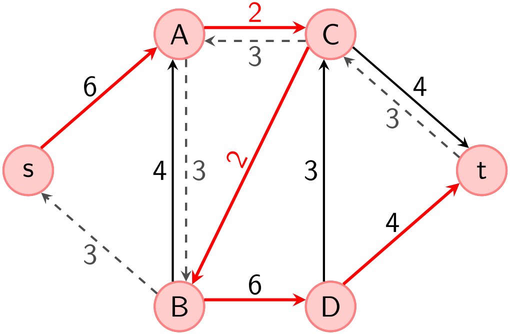

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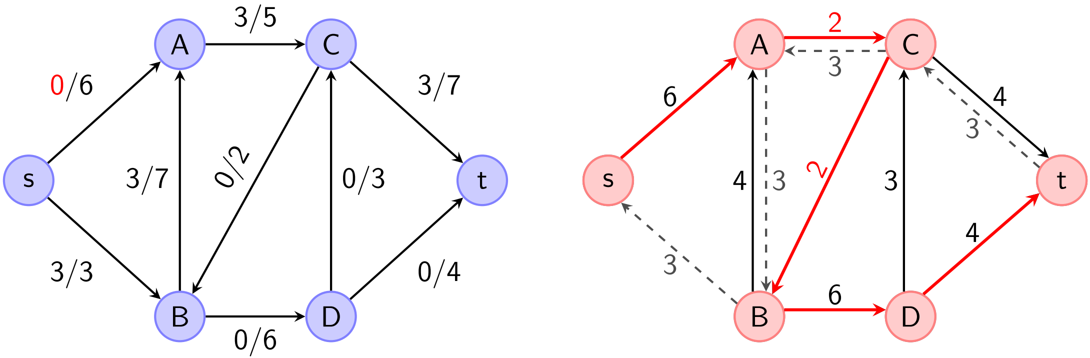

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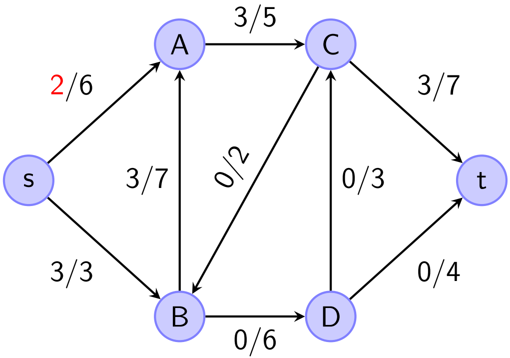

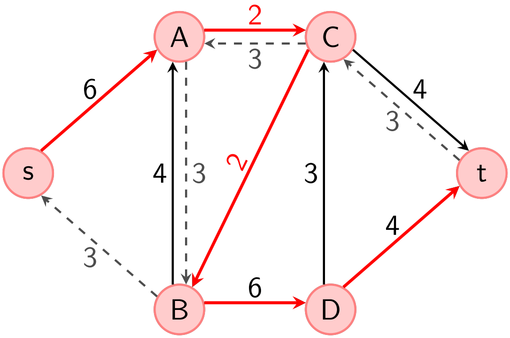

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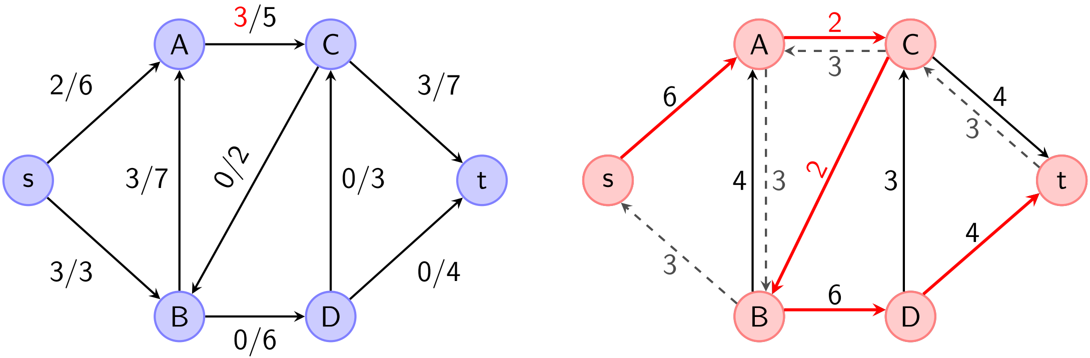

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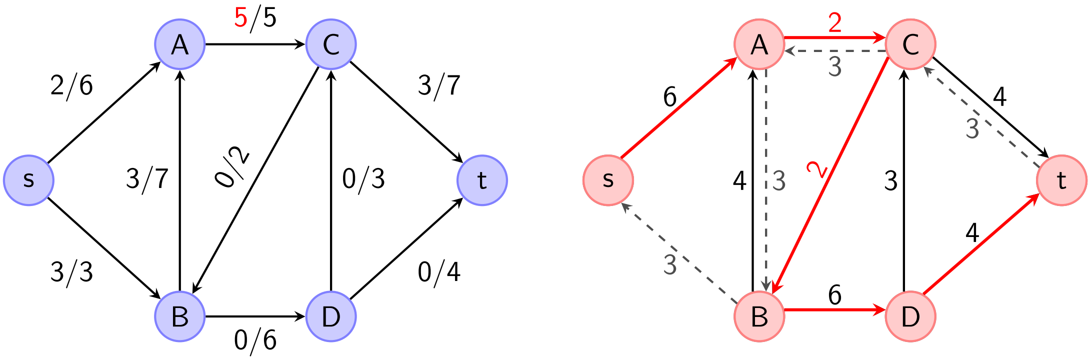

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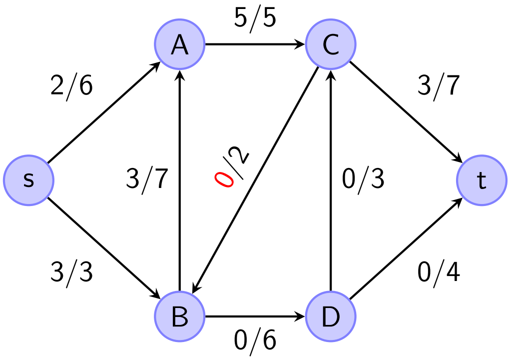

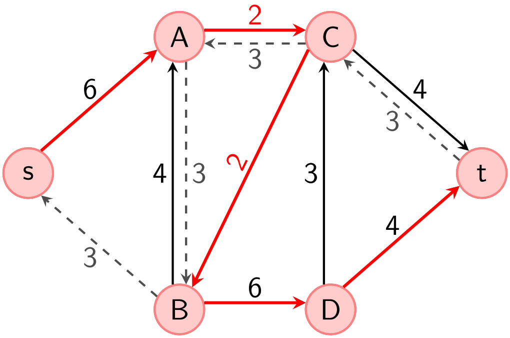

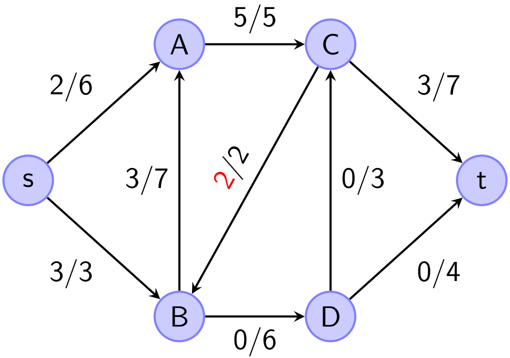
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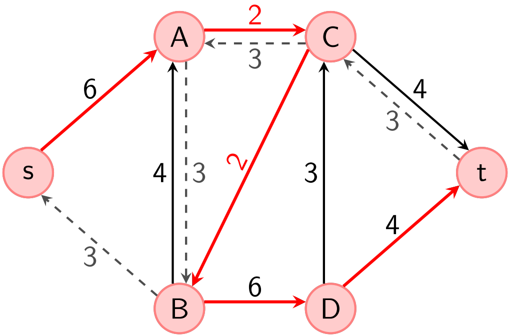
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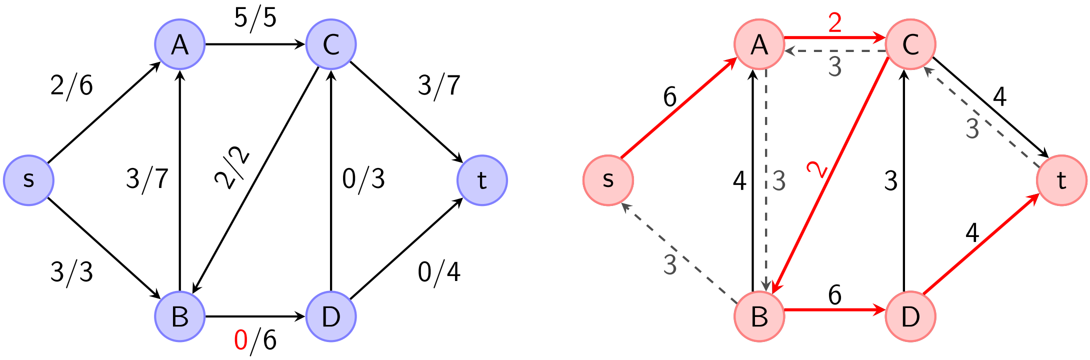

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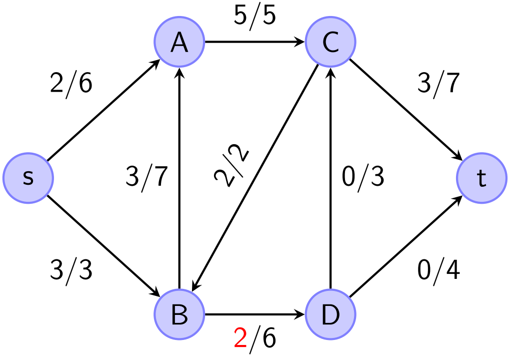

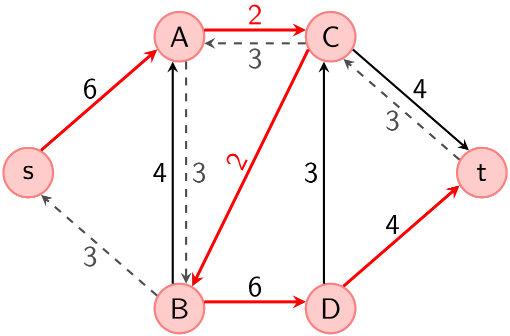

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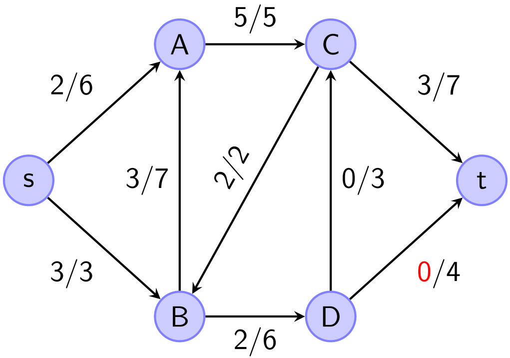

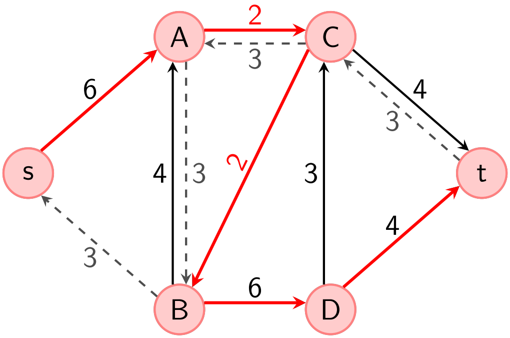

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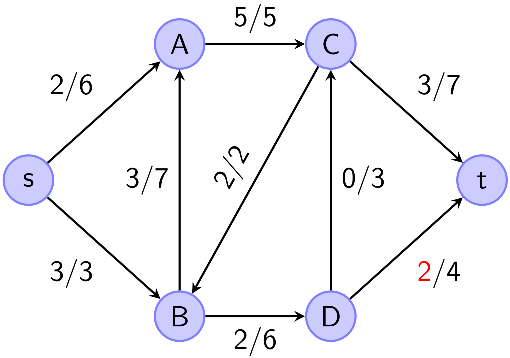

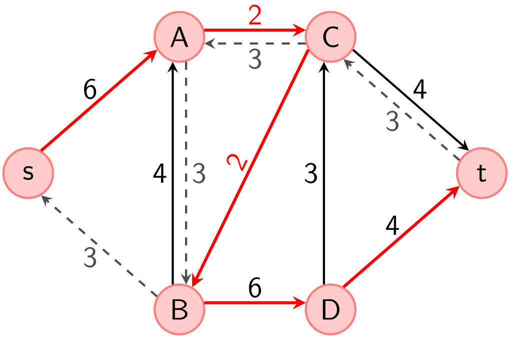

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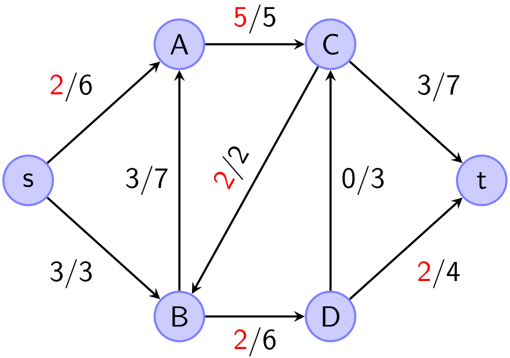

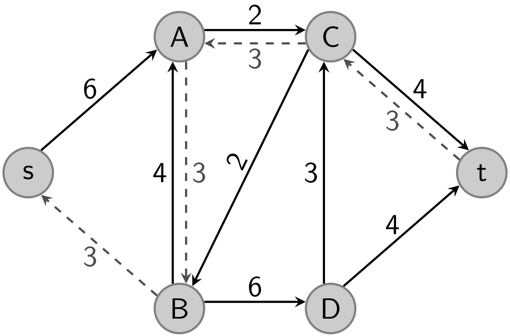

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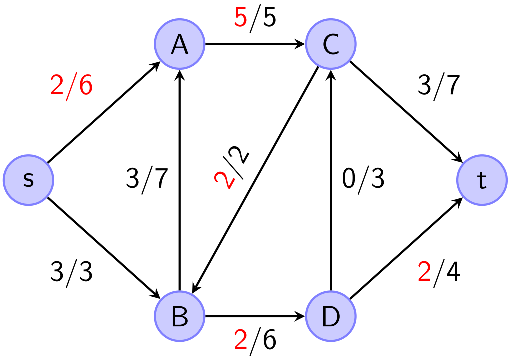

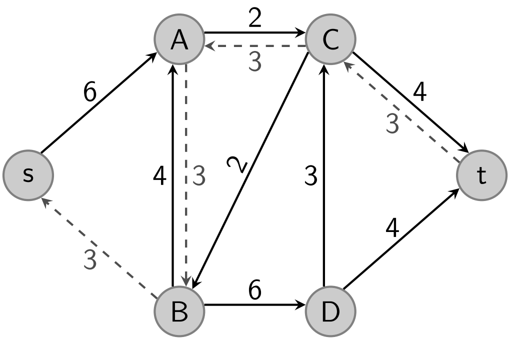

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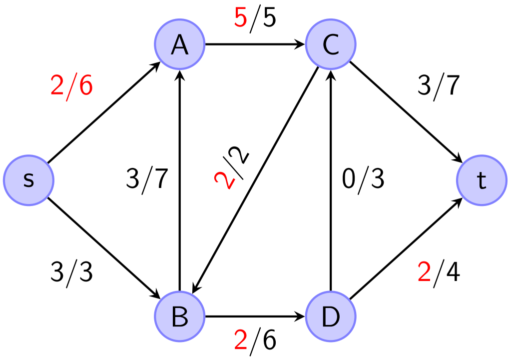

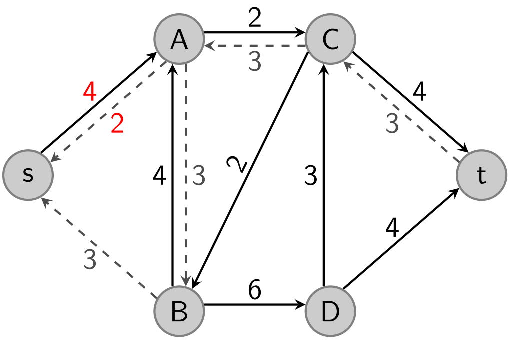

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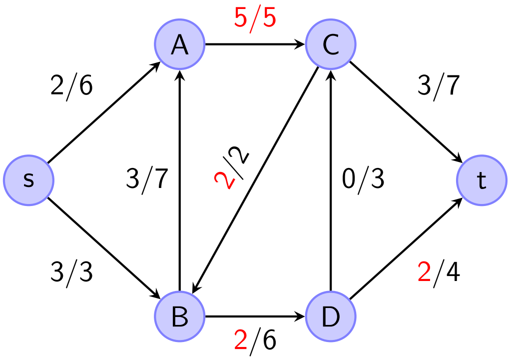

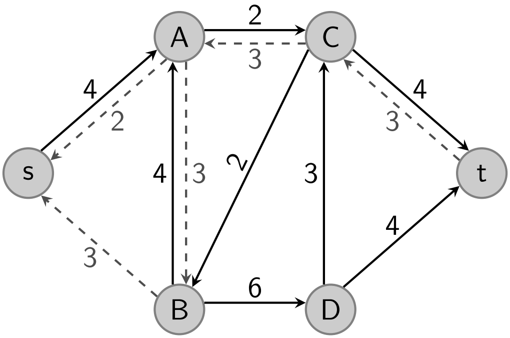

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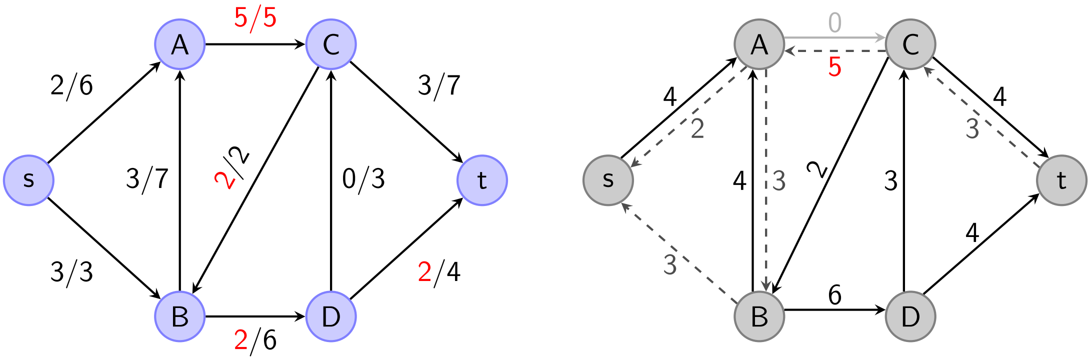

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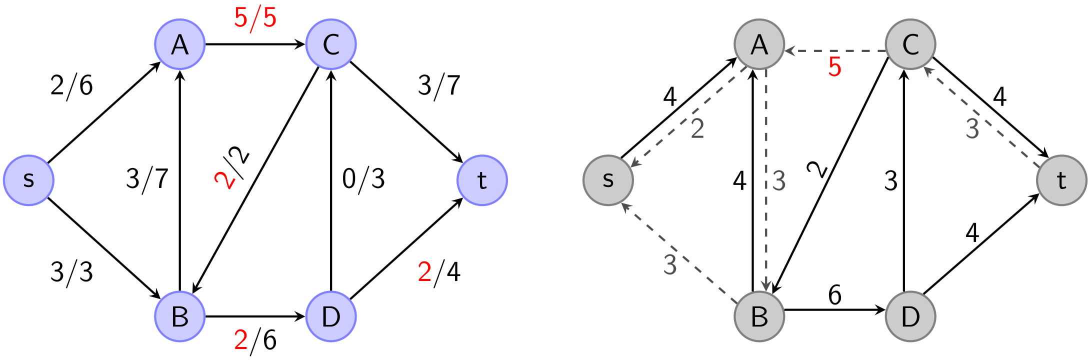

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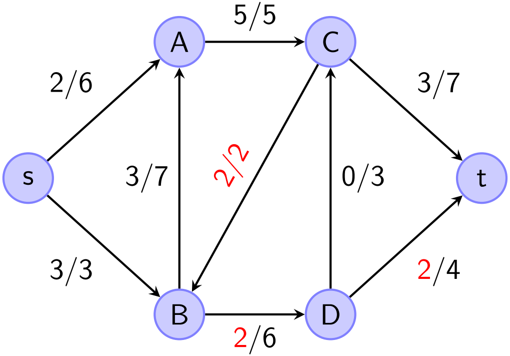

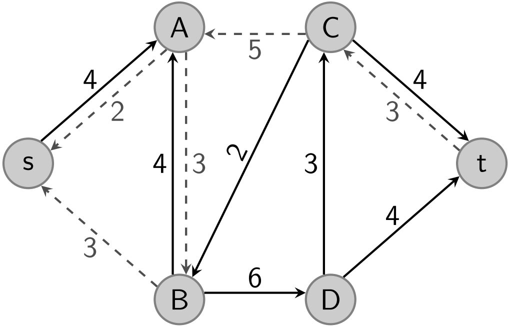

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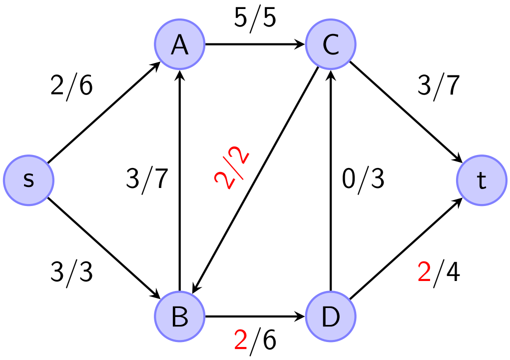

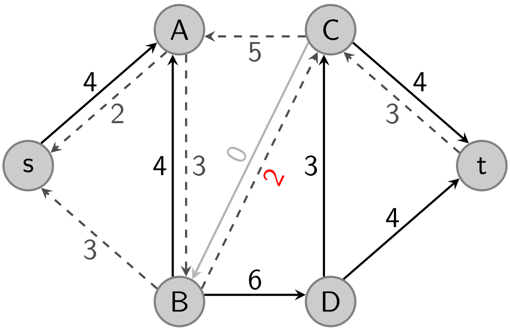

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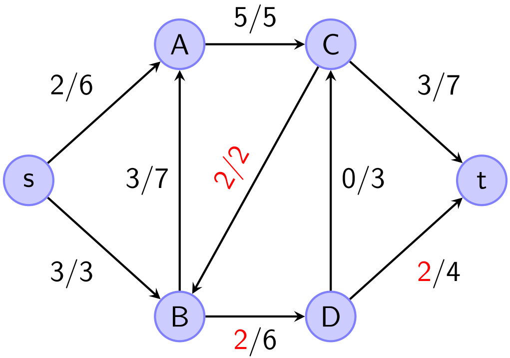

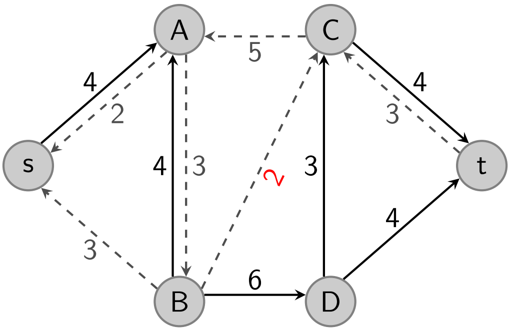

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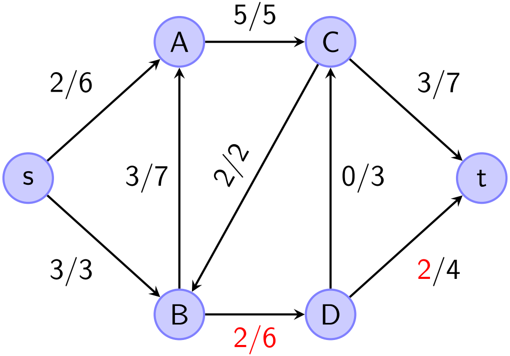

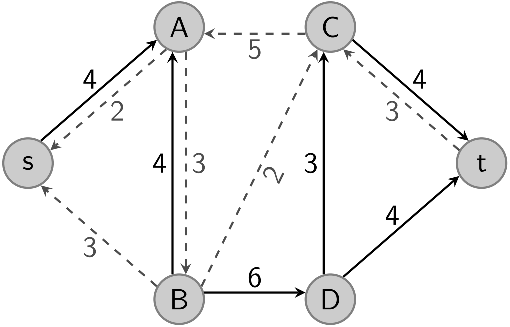

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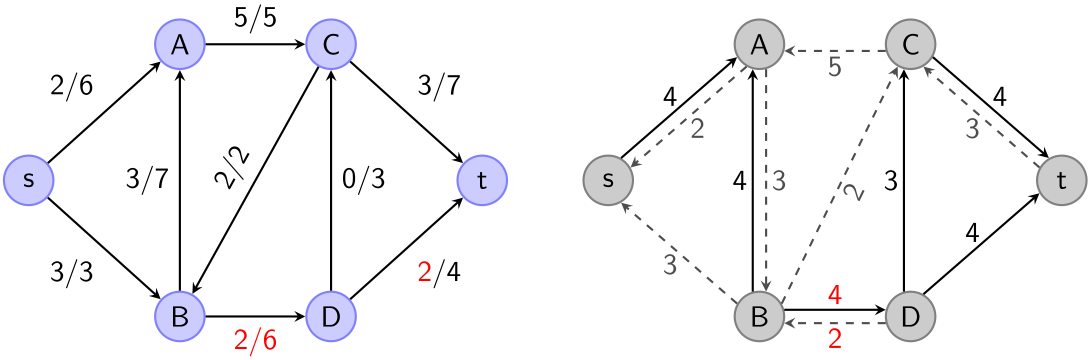

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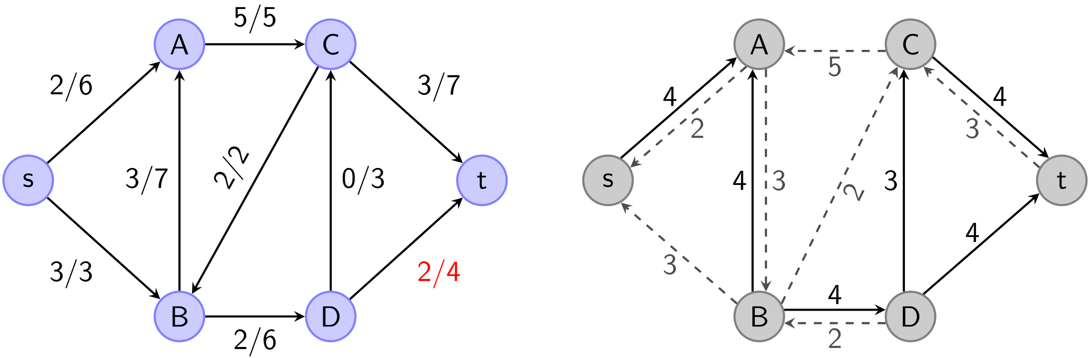

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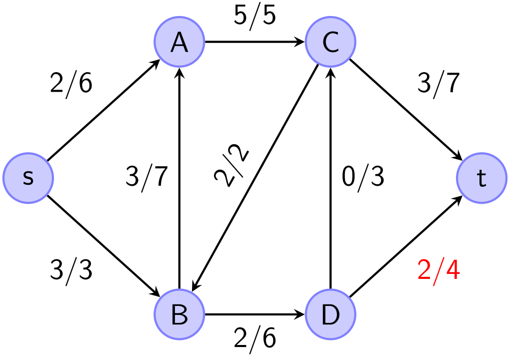

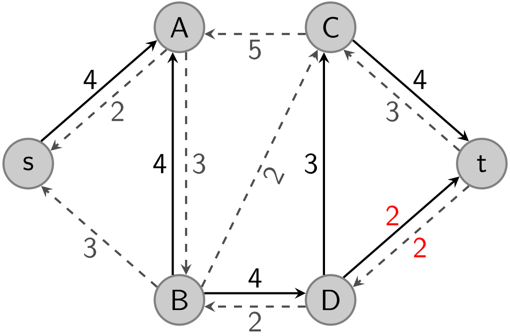

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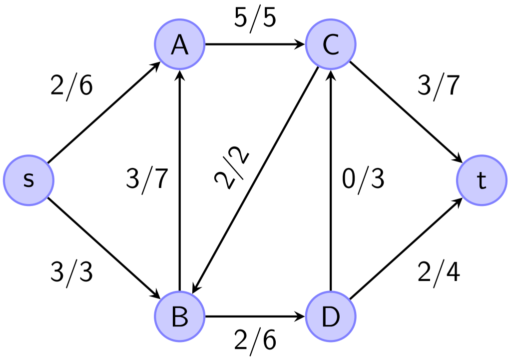

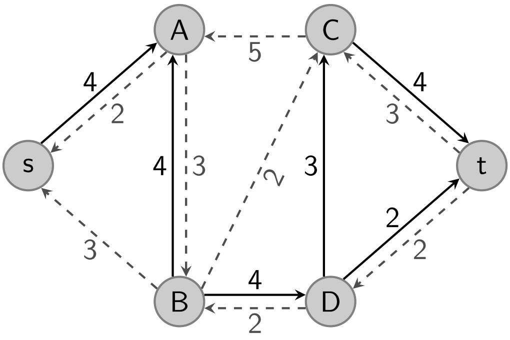

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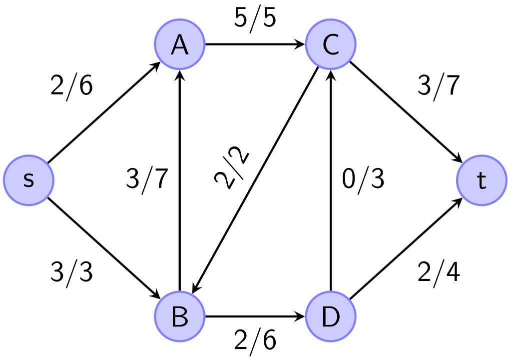

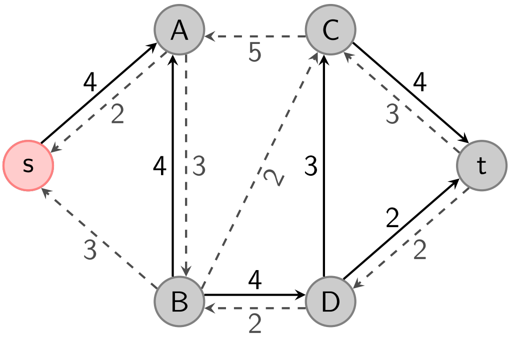

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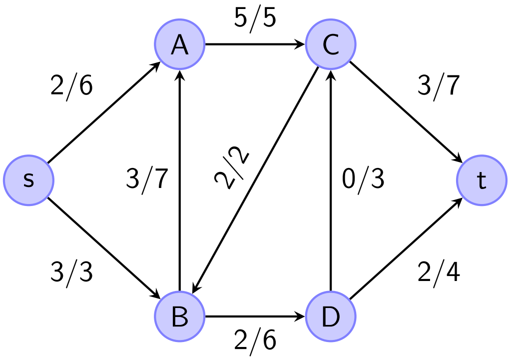

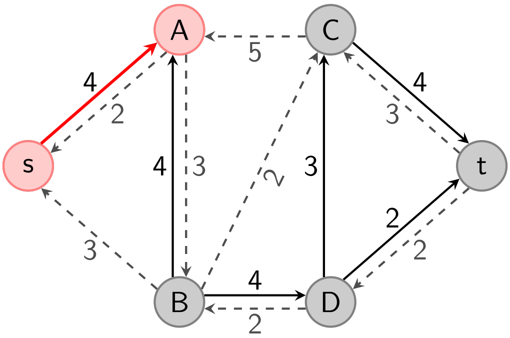

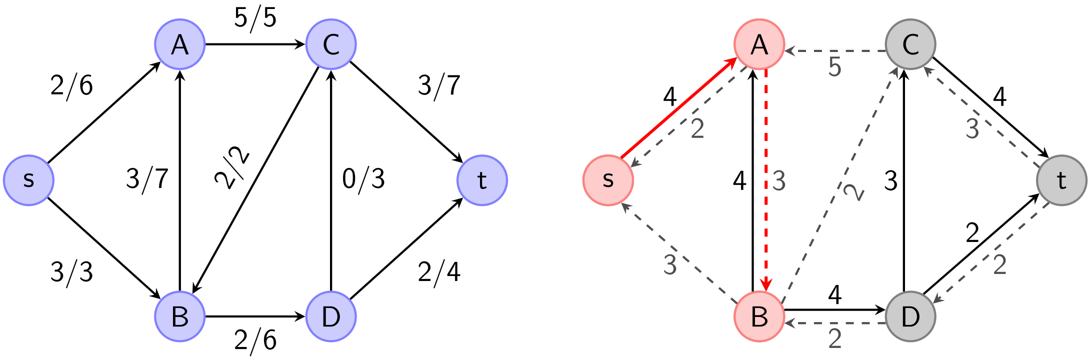
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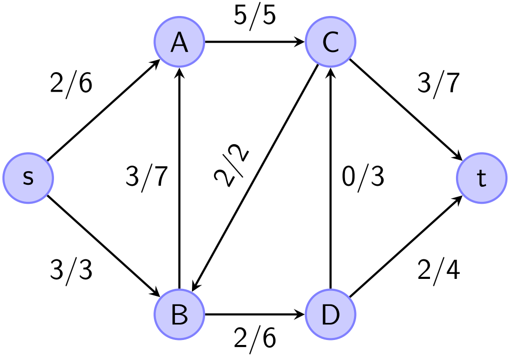

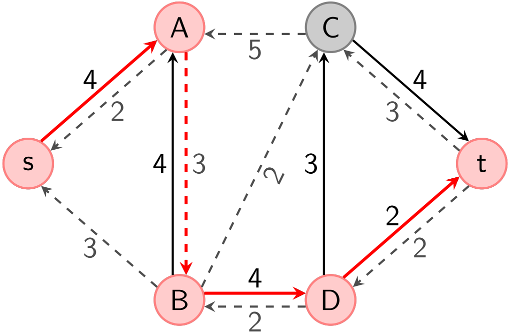

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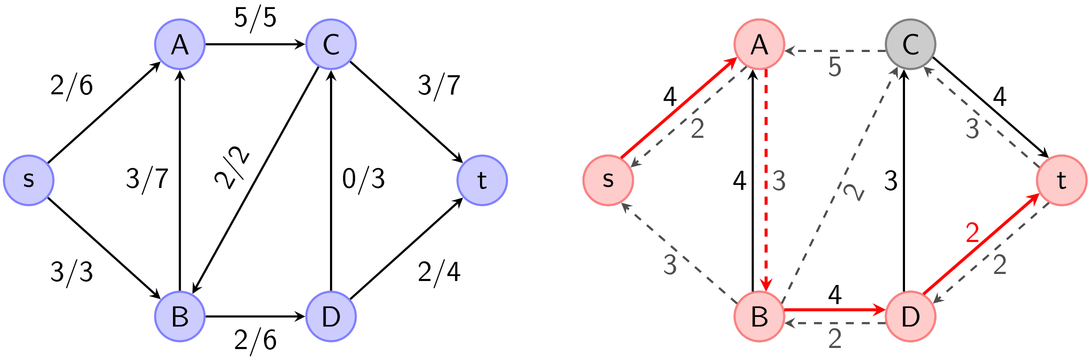

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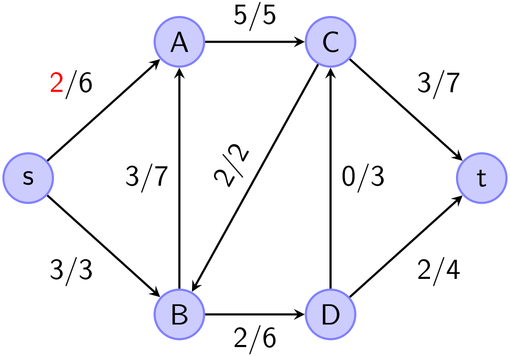

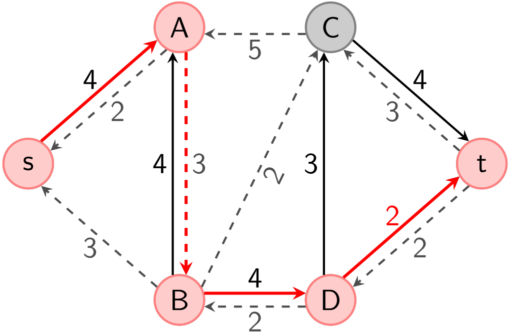

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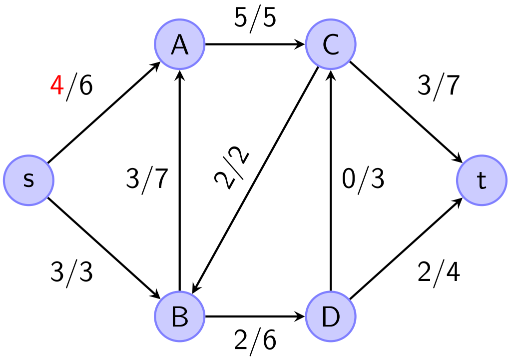
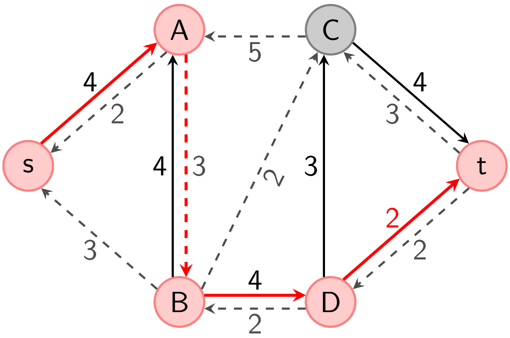
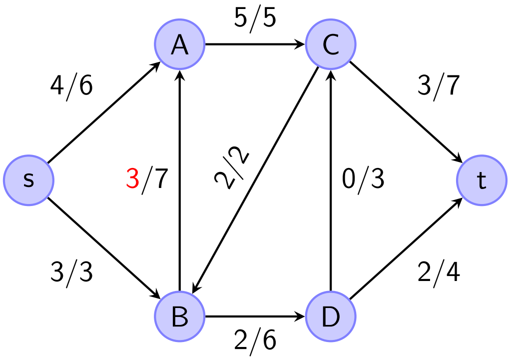
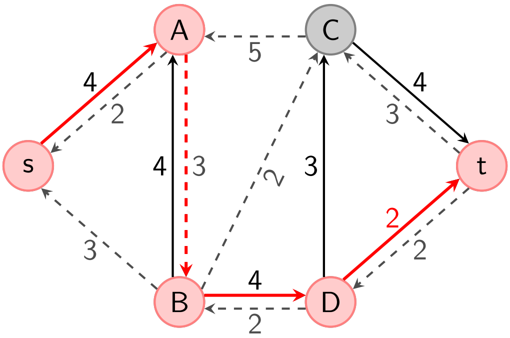




Demonstration of Ford-Fulkerson Method





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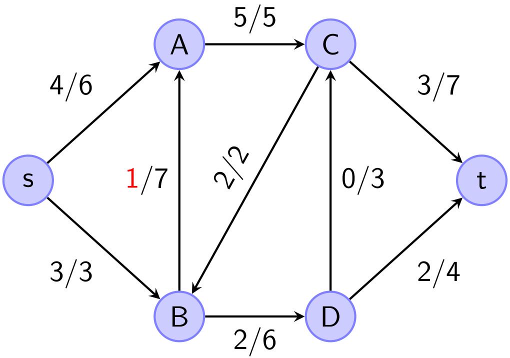

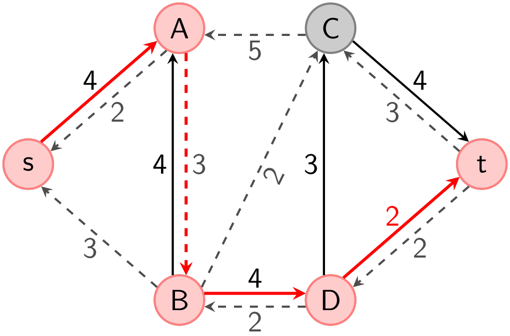

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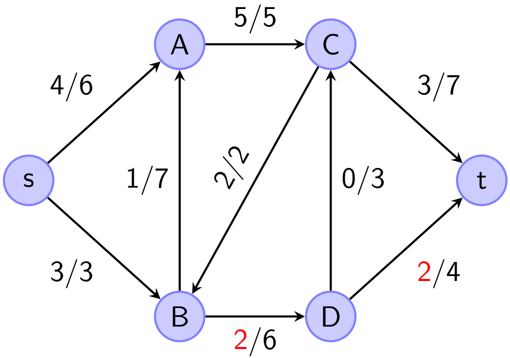

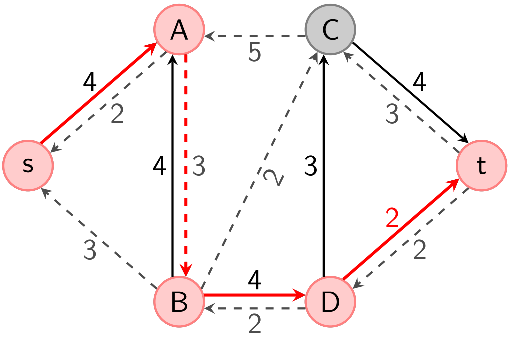

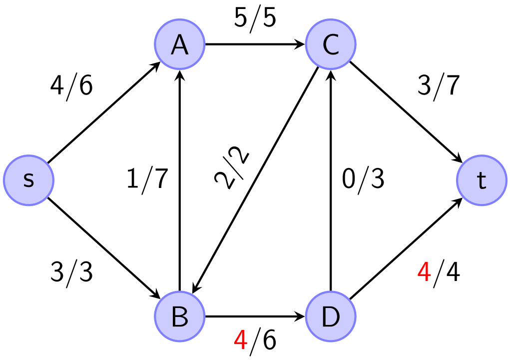
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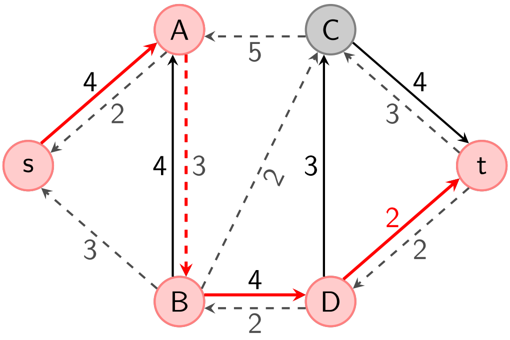
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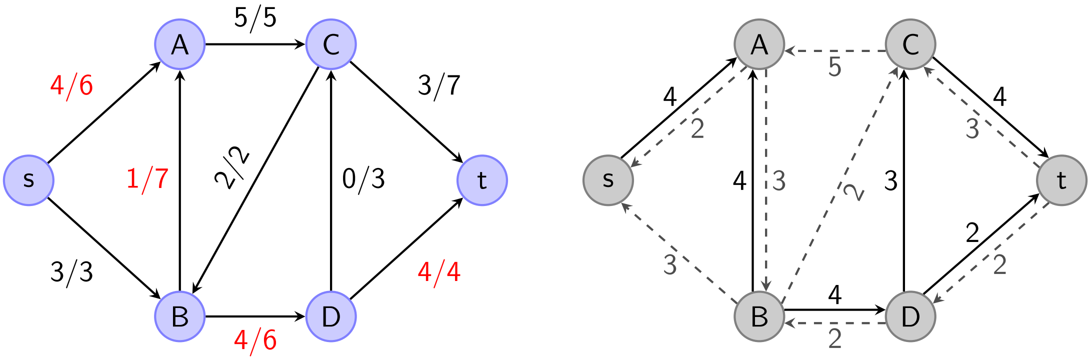
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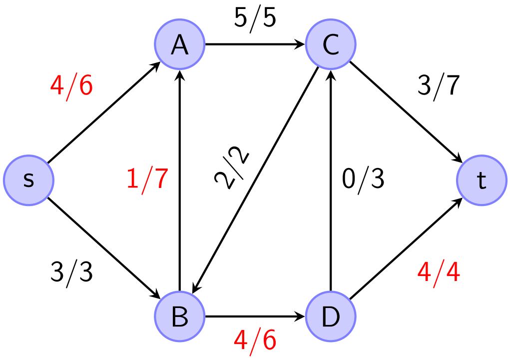

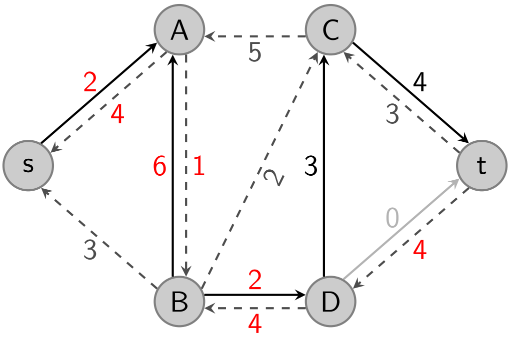

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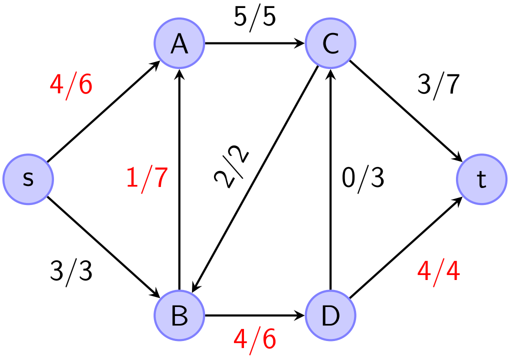

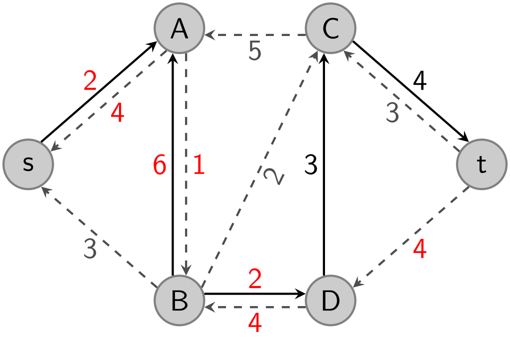

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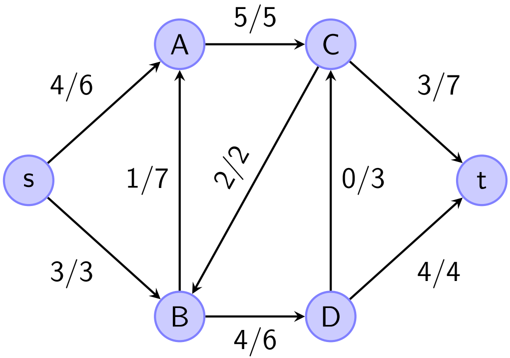

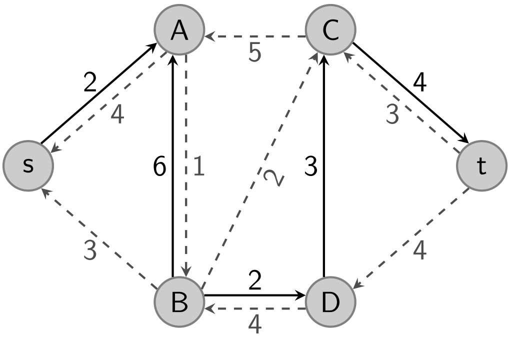

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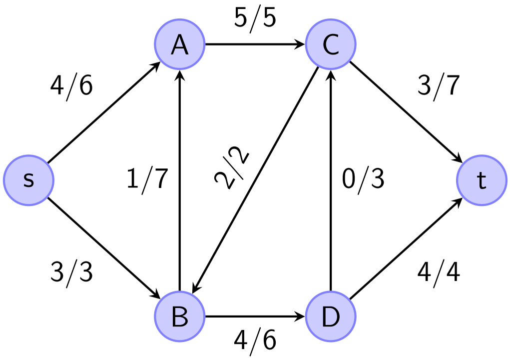

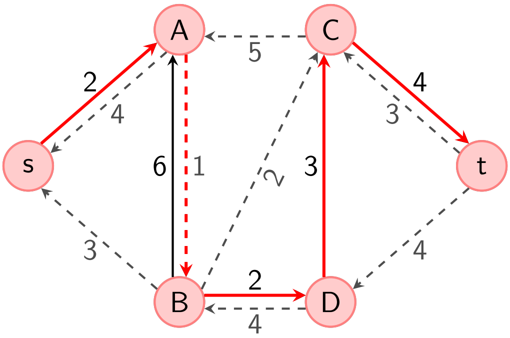

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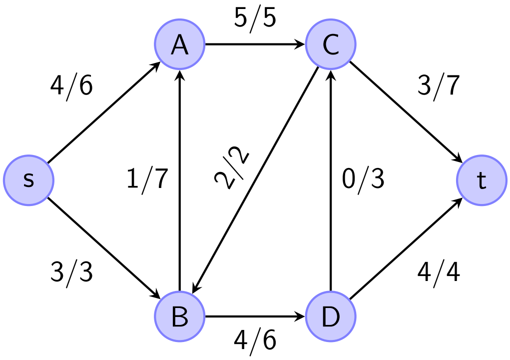

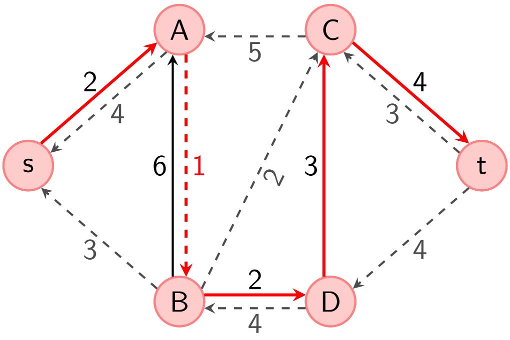

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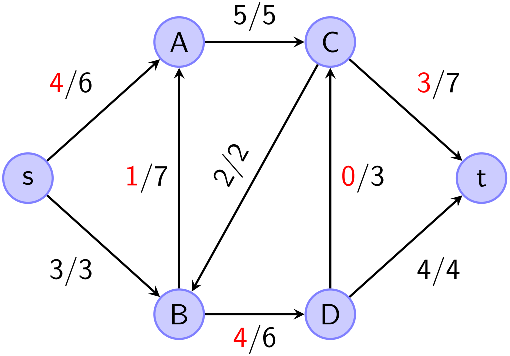

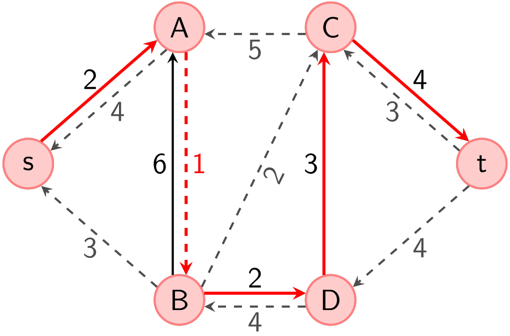

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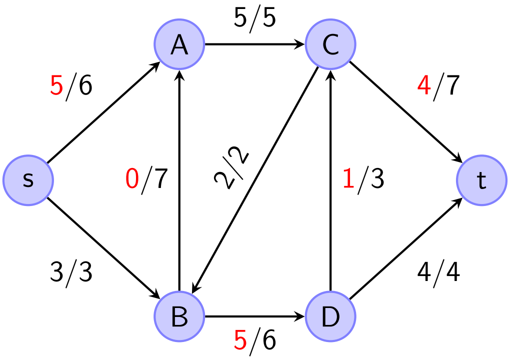

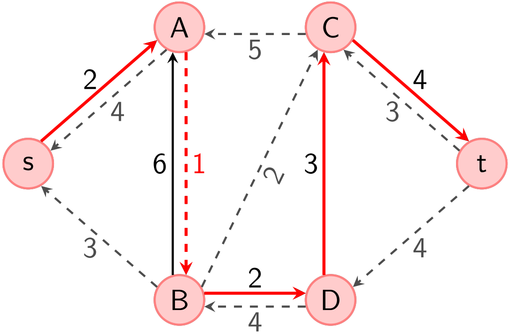

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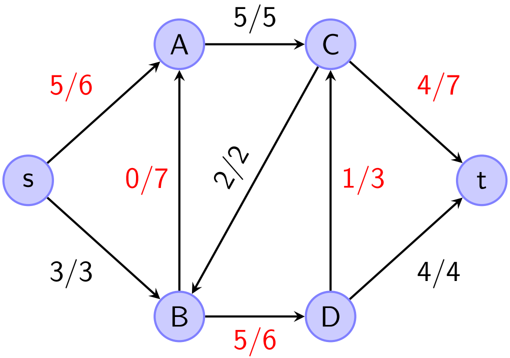

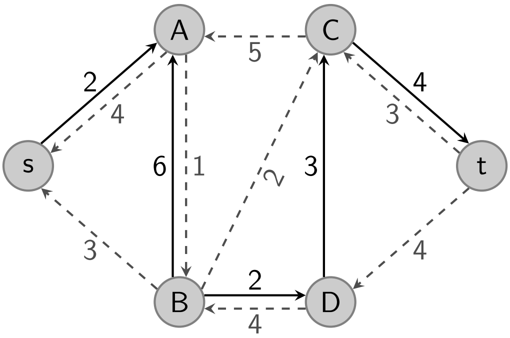

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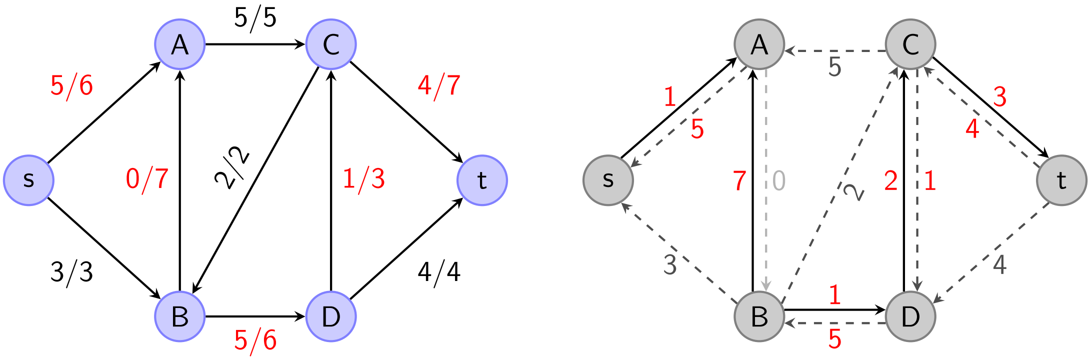

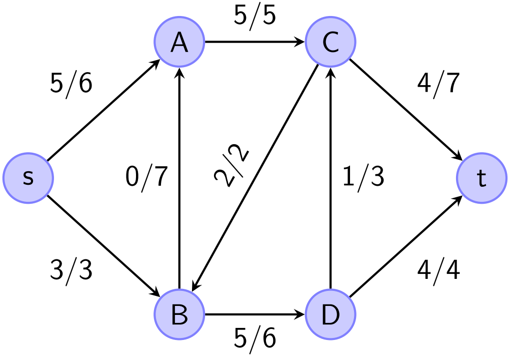
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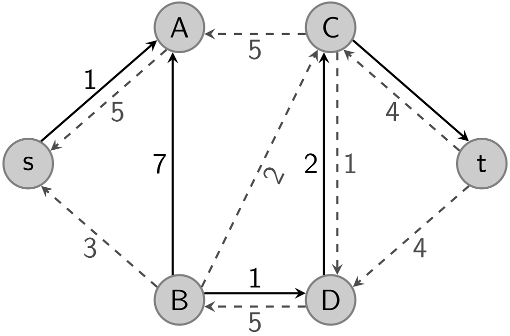
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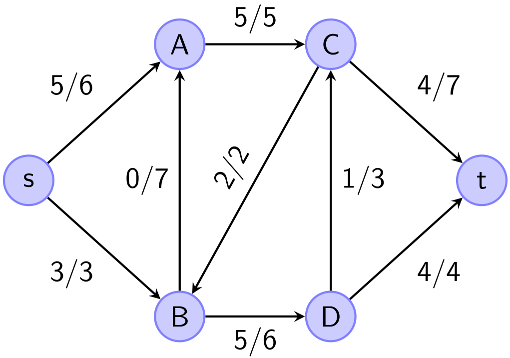

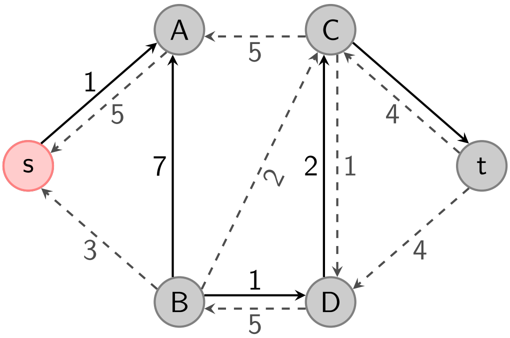

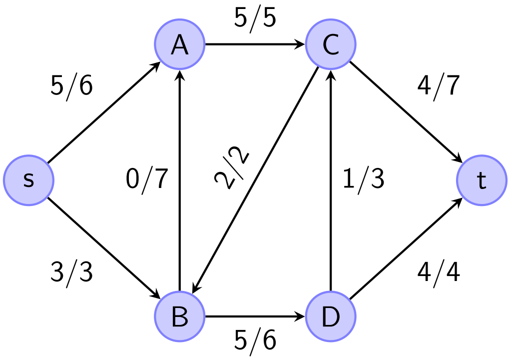
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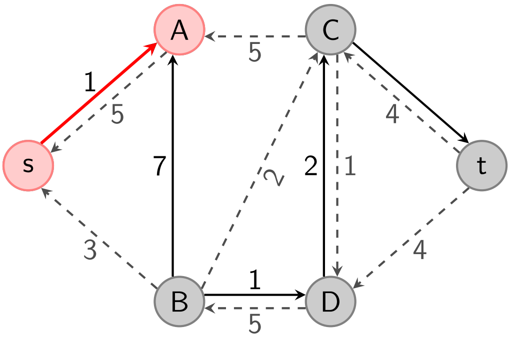
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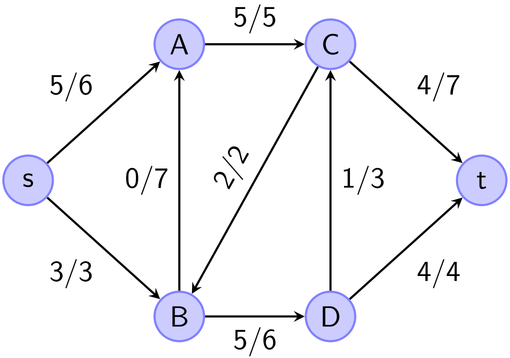
value of flow:
$8 = 5 + 3$



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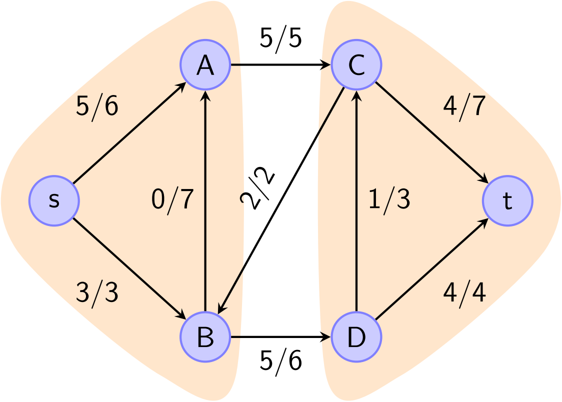
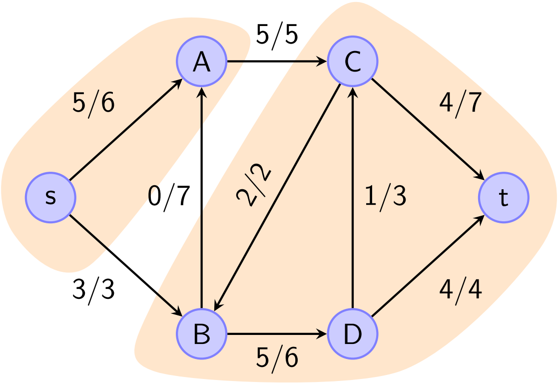


net flow across cut: $8 = 5 - 2 + 5$

capacity of cut: $11 = 5 + 6$

value of flow:
$8 = 5 + 3$



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capacity of cut:

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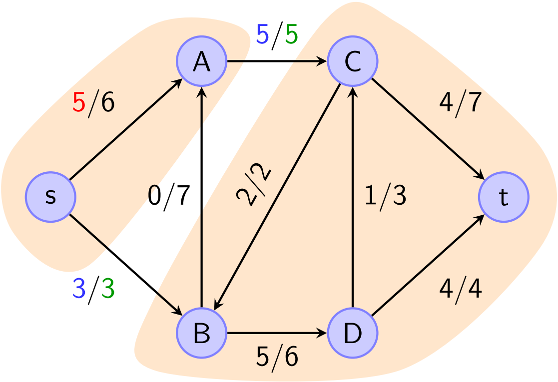

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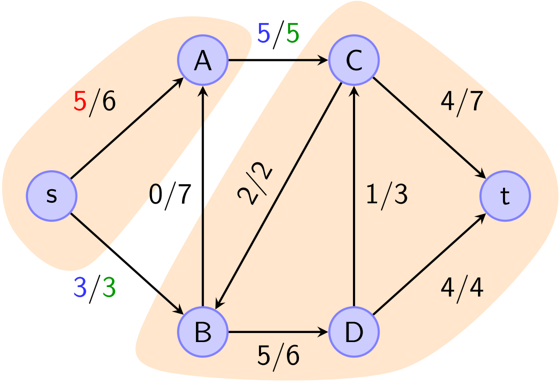
Partial Correctness

Max-Flow Min-Cut Theorem

For a flow f in $G = (V, E)$ the following conditions are equivalent

1. f is a maximum flow
2. there is no augmenting path in G_f
3. the value of flow equals capacity of some cut of G



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Partial Correctness and Termination

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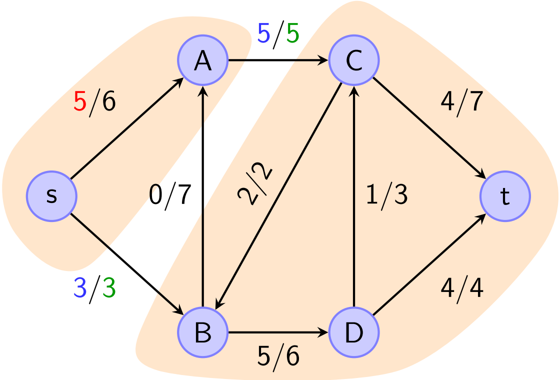
Termination:
in each iteration the flow is increased by at least one unit



value of flow: $8 = 5 + 3$

net flow across cut: $8 = 5 + 3$

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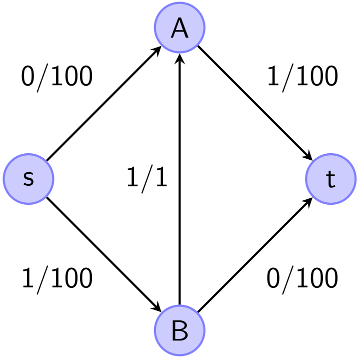

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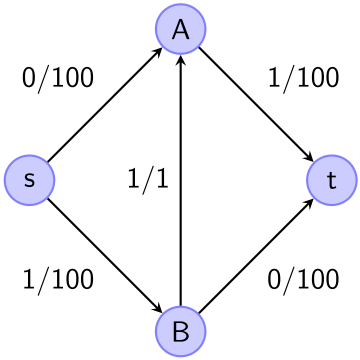
$\Rightarrow$ $|f|$ iterations of $O(|E|)$ yields $O(|f| \cdot |E|)$ in total



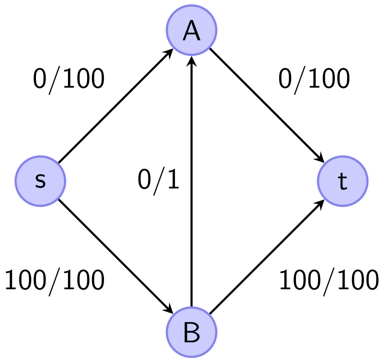
Optimizing Ford-Fulkerson Method

Idea: vary method of finding augmenting paths by

- doing Breadth-First Search (BFS)
- choosing paths of high bottle neck value first
- using so-called level graph to find shortest augmenting paths

Edmonds-Karp Algorithm

- use BFS to find augmenting paths
- shortest-path distance increases monotonically with flow augmentation
- maximum number of iterations: $O(VE)$
  - maximum number of augmenting paths of same length: $|E|$
  - maximum length of any path: $|V|$
- in total: $O(VE^2)$ which is independent of maximum flow value



Capacity Scaling

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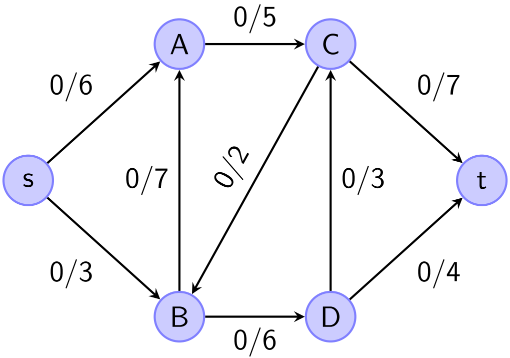

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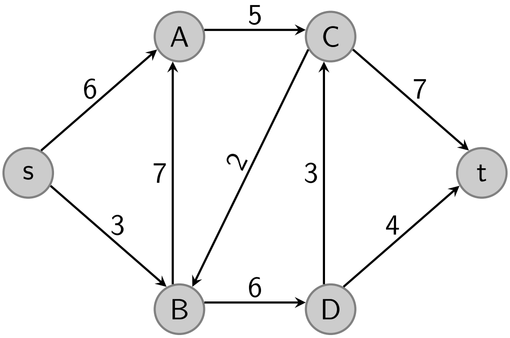

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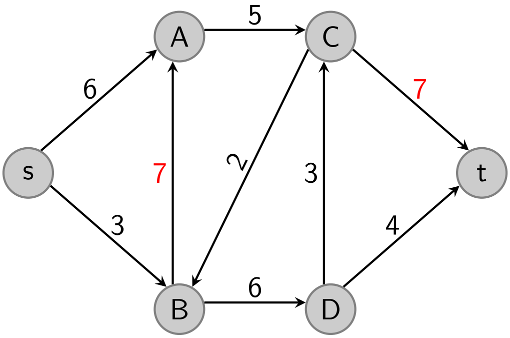

$C_{max} = 7$

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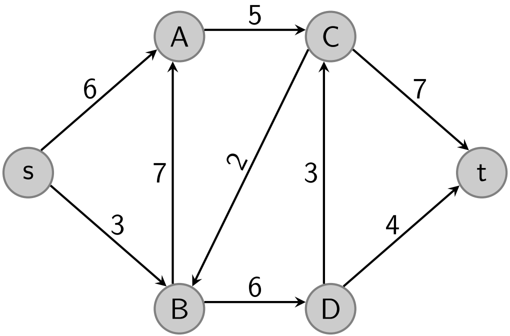

$\Delta = 4 = 2^2 \leq C_{max} < 2^3$

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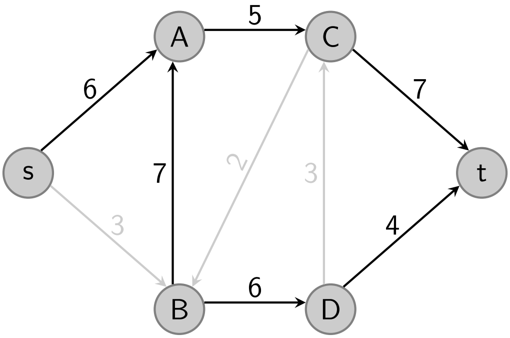

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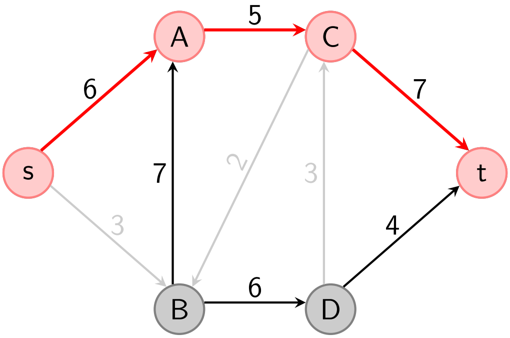

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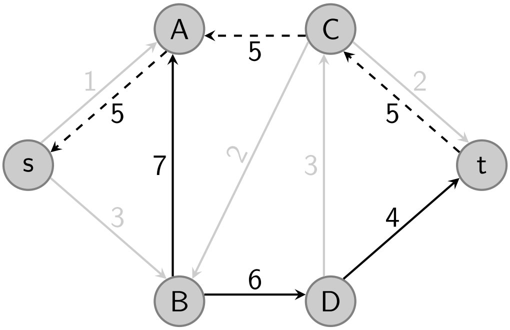

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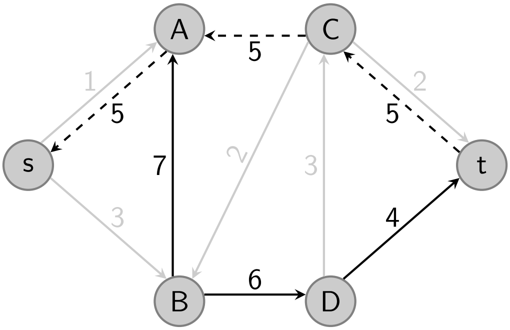

$\Delta = 2 = 4/2$

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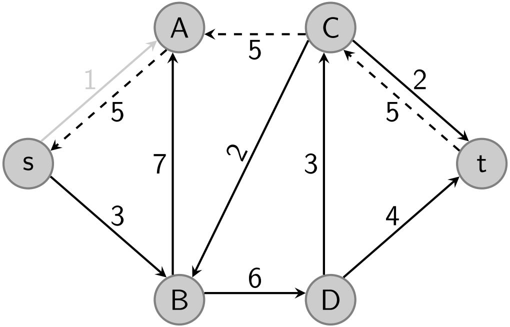

$\Delta = 2$

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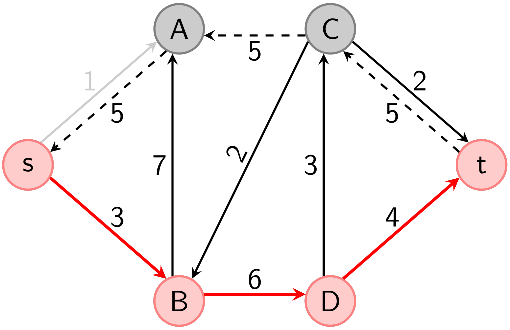
- decrease Δ if there are no more paths:
$$\Delta \leftarrow \Delta/2$$



$\Delta = 2$

Capacity Scaling

- heuristic to choose paths of high capacity first
- highest capacity of all edges C_{max}
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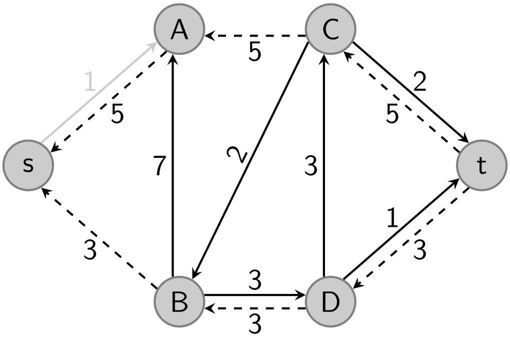

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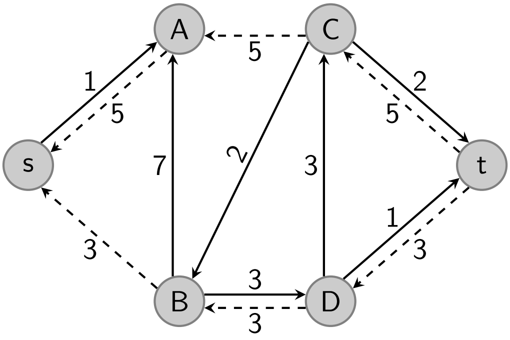

$\Delta = 1 = 2/2$

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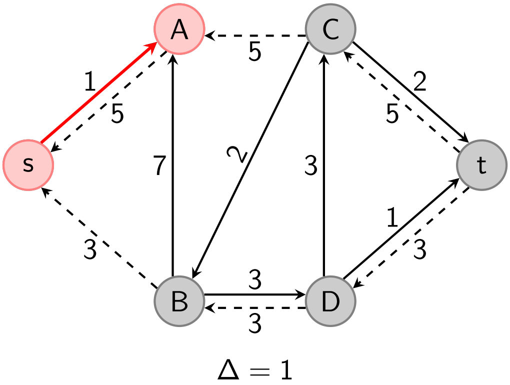

Time Complexity of Capacity Scaling

Ford-Fulkerson Method (with BFS) / Edmonds-Karp Algorithm

$C_{max} \leftarrow$ max capacity edge; $\Delta \leftarrow 2^{\lfloor \log_2(C_{max}) \rfloor}$
initialize flow f with 0
while $\Delta \geq 1$ **do** $O(\log(C_{max}))$ iterations
    $G_f(\Delta) \leftarrow$ residual graph with edges of capacity $\geq \Delta$
    while there exists an augmenting path p in $G_f(\Delta)$ **do**
        augment flow f along p and update $G_f(\Delta)$
    end while
    $\Delta \leftarrow \Delta/2$
end while
return f

DFS: $O(E^2 \log(C_{max}))$ BFS: $O(EV \log(C_{max}))$

Dinic's Algorithm

- uses shortest augmenting paths (like Edmonds-Karp Algorithm)
- find augmenting flows in level graph created from BFS
- time complexity: $O(V^2E)$

Dinic's Algorithm

initialize flow f with 0
while there exists an s-t-path in the residual network G_f **do**
    $G_L \leftarrow$ level graph from BFS on G_f
    find blocking flow f' in G_L by DFS
    augment flow f by f'
end while
return f

Definition (Level Graph)

The level graph of a residual network $G_f = (V, E_f)$ is a graph $G_L = (V, E_L)$ with

$$E_L = \{(u, v) \in E_f \mid \mathrm{dist}(v) = \mathrm{dist}(u) + 1\}$$

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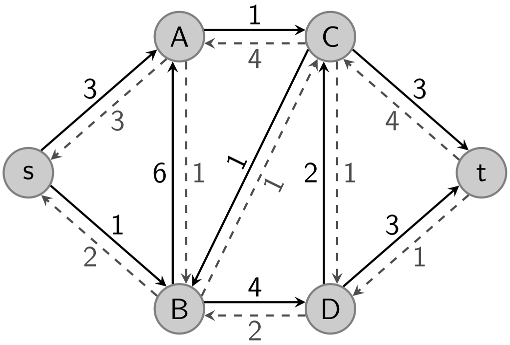

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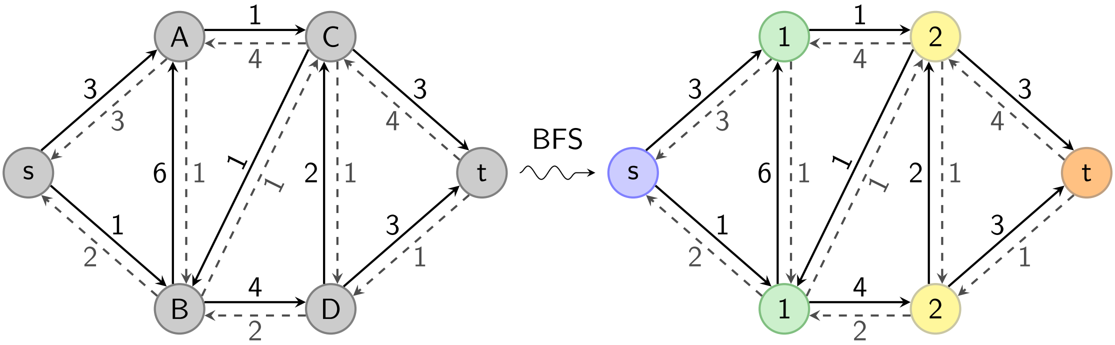

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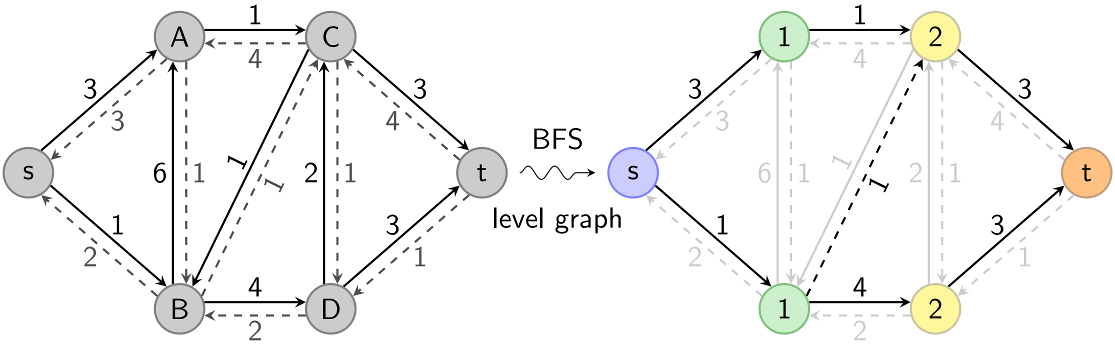

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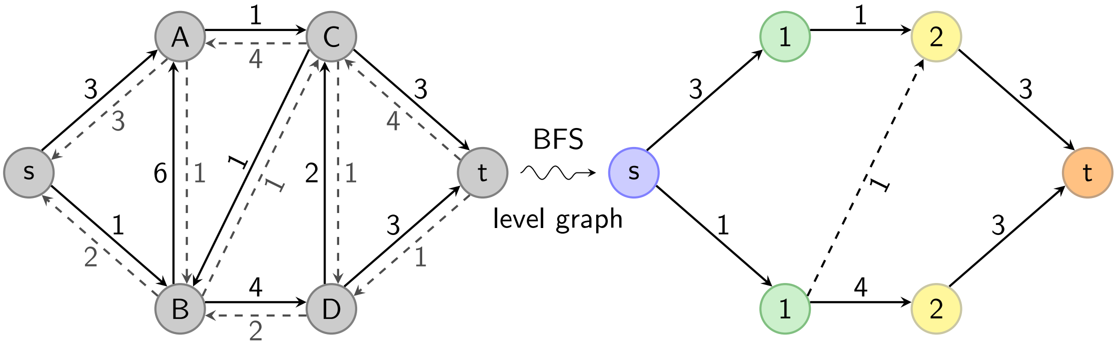

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- each iteration increases s-t distance by 1, so only $O(V)$ blocking flows

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$\Rightarrow$ time complexity of $O(V(E + EV)) = O(V^2E)$

Algorithms & Time Complexity (Summary)

Algorithm	Time Complexity	Comment
Ford-Fulkerson Method	$O(\lvert f \rvert E)$	using DFS
Edmonds-Karp	$O(VE^2)$	Ford-Fulkerson + BFS
Capacity Scaling + DFS	$O(E^2 \log(C_{max}))$	lower capacity bound Δ
Capacity Scaling + BFS	$O(EV \log(C_{max}))$	lower capacity bound Δ
Dinic's Algorithm	$O(V^2 E)$	level graph + DFS
Modification of Dinic's	$O(V^3)$	new approach to find blocking flows

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