Solving First-Order Non-Homogeneous Linear Recurrence Relations

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Introduction

In this handout, we look at first-order non-homogeneous linear recurrence relations (try saying that three times fast!), that is, the recurrence relation of the form

$$x_{n+1} = p(n)x_n + q(n),$$

where p and q are each functions of n such that p does not have any roots when n is positive.

Theorem. *Given that x_1 is the initial term of the sequence (x_n) that satisfies the recurrence relation $x_{n+1} = p(n)x_n + q(n)$, the explicit solution for x_n is given by*

$$x_n = \left(x_1 + \sum_{k=1}^{n-1} \frac{q(k)}{\prod_{m=1}^{k} p(m)} \right) \prod_{k=1}^{n-1} p(k).$$

Proof. It suffices to show that the solution satisfies the recursion since x_1 and the relation itself uniquely determines the sequence (x_n). We can do so by applying the given formula to x_{n+1} and recreating the recurrence relation.

$$\begin{aligned}
x_{n+1} &= \left(x_1 + \sum_{k=1}^{n} \frac{q(k)}{\prod_{m=1}^{k} p(m)} \right) \prod_{k=1}^{n} p(k) \\
&= \left(x_1 + \sum_{k=1}^{n-1} \frac{q(k)}{\prod_{m=1}^{k} p(m)} + \frac{q(n)}{\prod_{m=1}^{n} p(m)} \right) \prod_{k=1}^{n} p(k) \\
&= \left(x_1 + \sum_{k=1}^{n-1} \frac{q(k)}{\prod_{m=1}^{k} p(m)} \right) \prod_{k=1}^{n} p(k) + \left(\frac{q(n)}{\prod_{m=1}^{n} p(m)} \right) \prod_{k=1}^{n} p(k) \\
&= \left(x_1 + \sum_{k=1}^{n-1} \frac{q(k)}{\prod_{m=1}^{k} p(m)} \right) \left(\prod_{k=1}^{n-1} p(k) \right) p(n) + \left(\frac{q(n)}{\prod_{k=1}^{n} p(k)} \right) \prod_{k=1}^{n} p(k) \\
&= p(n)x_n + q(n)
\end{aligned}$$

□

However, even if we know that the formula works, it is much more meaningful to derive the formula from the recursion itself. We do this as follows.

Define a new sequence (a_n) such that $p(n)a_n = a_{n+1}$. Now consider dividing the recurrence relation by a_{n+1}. We get the following:

$$\begin{aligned}\frac{x_{n+1}}{a_{n+1}} &= \frac{p(n)x_n}{a_{n+1}} + \frac{q(n)}{a_{n+1}} \\ &= \frac{p(n)x_n}{p(n)a_n} + \frac{q(n)}{p(n)a_n} \\ &= \frac{x_n}{a_n} + \frac{q(n)}{p(n)a_n}.\end{aligned}$$

Notice how we get $\frac{x_{n+1}}{a_{n+1}}$ on the left side of the equality and $\frac{x_n}{a_n}$ on the right. This gives us a recurrence relation that is additive (sort of like an arithmetic sequence), so applying this recurrence $n-1$ times, we get

$$\frac{x_n}{a_n} = \frac{x_1}{a_1} + \sum_{k=1}^{n-1} \frac{q(k)}{a_k p(k)}.$$

Therefore, we have that

$$x_n = a_n \left(\frac{x_1}{a_1} + \sum_{k=1}^{n-1} \frac{q(k)}{a_k p(k)} \right).$$

Similarly, from the fact that $a_{n+1} = p(n)a_n$, we know that

$$a_n = a_1 \prod_{k=1}^{n-1} p(k).$$

This allows us to replace a_n with the product giving us

$$\begin{aligned}x_n &= a_1 \prod_{k=1}^{n-1} p(k) \left(\frac{x_1}{a_1} + \sum_{k=1}^{n-1} \frac{q(k)}{a_k p(k)} \right) \\ &= \left(x_1 + \sum_{k=1}^{n-1} \frac{a_1 q(k)}{a_k p(k)} \right) \prod_{k=1}^{n-1} p(k).\end{aligned}$$

Finally, to remove (a_n) from the picture completely, we can combine the definition of (a_n) with the recursion $a_n = a_1 \prod_{k=1}^{n-1} p(k)$ from above. We have that $p(k)a_k = a_{k+1}$, and by a change of variables, we also have that $a_{k+1} = a_1 \prod_{m=1}^{k} p(m)$. Therefore,

$$\frac{a_1}{a_k p(k)} = \frac{1}{\prod_{m=1}^{k} p(m)},$$

and so,

$$\begin{aligned}
x_n &= \left(x_1 + \sum_{k=1}^{n-1} \frac{a_1 q(k)}{a_k p(k)} \right) \prod_{k=1}^{n-1} p(k) \\
&= \left(x_1 + \sum_{k=1}^{n-1} \frac{q(k)}{\prod_{m=1}^{k} p(m)} \right) \prod_{k=1}^{n-1} p(k),
\end{aligned}$$

which is what we have above.

The key step in this derivation was taking the recursion to the form $c_{n+1} = c_n + f(n)$ by dividing the equation using a convenient new sequence. This allowed us to simplify the problem greatly which, combined with the geometric nature of the new sequence, allowed us to completely represent the recursion in terms of the original functions and the initial term.

Examples

Now, it turns out that knowing the solution to this recursion is quite useful due to the generality in the functions of p and q. We follow this with a couple of examples.

Example 1. *A sequence (a_n) satisfies $na_{n+1} = (n+1)a_n + 1$ for $n \geq 2$ and $a_1 = 3$. Find an explicit formula for a_n in terms of n.*

Solution: We divide both sides of the recurrence relation by n and apply our formula. We have that $a_1 = 3$, $p(n) = \frac{n+1}{n}$, and $q(n) = \frac{1}{n}$, so we obtain

$$\begin{aligned}
a_n &= \left(a_1 + \sum_{k=1}^{n-1} \frac{q(k)}{\prod_{m=1}^{k} p(m)} \right) \prod_{k=1}^{n-1} p(k) \\
&= \left(3 + \sum_{k=1}^{n-1} \frac{1/k}{\prod_{m=1}^{k} \frac{m+1}{m}} \right) \prod_{k=1}^{n-1} \frac{k+1}{k} \\
&= \left(3 + \sum_{k=1}^{n-1} \frac{1/k}{k+1} \right) n \\
&= \left(3 + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \right) n \\
&= \left(3 + 1 - \frac{1}{n} \right) n \\
&= 4n - 1.
\end{aligned}$$

Therefore, $a_n = \boxed{4n-1}$.

Example 2 (WOOT). *An infinite sequence of real numbers (a_n) satisfies $a_{n+1} = 2^n - 3a_n$ for $n \geq 0$. Find the set of real numbers a_0 such that (a_n) is strictly increasing.*

Solution: We can adjust for the indices (since it starts at a_0) and apply our formula. We have $p(n) = -3$, and $q(n) = 2^n$, so we obtain

$$\begin{aligned} a_n &= \left(a_0 + \sum_{k=0}^{n-1} \frac{q(k)}{\prod_{m=0}^{k} p(m)}\right) \prod_{k=0}^{n-1} p(k) \\ &= \left(a_0 + \sum_{k=0}^{n-1} \frac{2^k}{\prod_{m=0}^{k} -3}\right) \prod_{k=0}^{n-1} -3 \\ &= \left(a_0 - \frac{1}{3} \sum_{k=0}^{n-1} \left(-\frac{2}{3}\right)^k\right) (-3)^n \\ &= \left(a_0 + \frac{-1 + \left(-\frac{2}{3}\right)^n}{3 - (-2)}\right) (-3)^n \\ &= a_0(-3)^n + \frac{-(-3)^n + 2^n}{5} \\ &= \left(a_0 - \frac{1}{5}\right)(-3)^n + \frac{1}{5} \cdot 2^n. \end{aligned}$$

For (a_n) to be strictly increasing, the coefficient $a_0 - \frac{1}{5}$ must be 0 or else the term with $(-3)^n$ will fluctuate between positive and negative depending on the parity of n. Thus, $a_0 = \boxed{1/5}$.

Problems

Problem 1 (2005 AIME). *Let m be a positive integer, and let $a_0, a_1, \ldots, a_m$ be a sequence of reals such that $a_0 = 37, a_1 = 72, a_m = 0$, and $a_{k+1} = a_{k-1} - \frac{3}{a_k}$ for $k = 1, 2, \ldots, m-1$. Find m.*

Problem 2 (RoKuluro96). *A sequence (a_n) satisfies $a_{n+1} = 5^n a_n^2$ for $n \geq 2$ and $a_1 = \sqrt{5}$. The quantity a_{2020} can be expressed in the form k^m for positive integers k and m. If the last digit of m is M, find $k + M$.*

Problem 3 (Tower of Hanoi).

Problem 4 (1980 Putnam). *A sequence (b_n) is defined by the recursion $b_{n+1} = 2b_n - n^2$ for $n \geq 1$ and $b_0 = \beta$. For which real values of β are the numbers in the sequence always positive?*

Further Inquiry

In this section, we try to extend the knowledge given above to more general questions and problems. These are not meant to be solved per say, but are meant to be curiosities that may potentially spark new ideas relating to the topic.

Question 1. *Can we generalize our method to solve for linear recurrence relations of the form*

$$x_{n+1} = p(n)x_n + q(n)x_{n-1} + r(n)?$$

If so, how? Can we generalize it even further to the form

$$x_{n+1} = p_0(n)x_n + p_1(n)x_{n-1} + p_2(n)x_{n-2} ... + p_k(n)x_{n-k} + r(n)?$$

If not, why not? What are the things that are preventing us from generalizing?

Question 2. *Can we generalize our method to solve for recurrences of the form*

$$x_{n+1} = p(n)x_n^2 + q(n)?$$

If so, how? Can we generalize it even further to the form

$$x_{n+1} = p(n)x_n^k + q(n)$$

or even

$$x_{n+1} = p(n)r(x_n) + q(n)$$

where r is any function of x_n?
If not, why not? What are the things that are preventing us from generalizing?

Question 3. *Instead of thinking about a sequence x_n, consider a function $f : \mathbb{R} \to \mathbb{R}$ that maps n to $f(n)$. If we have*

$$f(n + 1) = p(n)f(n) + q(n),$$

Can we still say that our theorem holds? How does this relate to the topic of functional equations?