Two time-scale Stability Analysis

Niloufar Yousefi

May 26, 2025

1 Problem Formulation

We consider a two time-scale LTI plant

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \epsilon \dot{x}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ 0 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}}_{E} w \\
y &= \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
\end{aligned} \tag{1}$$

where $\epsilon > 0$, with slow state $x_1 \in \mathbb{R}^{n_1}$, fast state $x_2 \in \mathbb{R}^{n_2}$, control input $u \in \mathbb{R}^m$, measured output $y \in \mathbb{R}^p$, and where $w \in \mathbb{R}^q$ is a *constant* disturbance.

The problem of interest is to design a controller which drives the output and control input of the LTI system (1) towards the solution of the time-invariant steady-state optimization problem

$$\begin{aligned}
\underset{\bar{y},\bar{u}}{\text{minimize}} \quad & f(\bar{u}) + g(\bar{y}) \\
\text{subject to} \quad & \bar{y} = \Pi_u \bar{u} + \Pi_w w
\end{aligned} \tag{2}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^p \to \mathbb{R}$ are costs on the steady-state input and output, respectively. Note that the optimization is over equilibrium values compatible with the plant (1). We place the following technical assumptions on the costs.

Assumption 1.1 (Cost Assumptions) *The input cost function f is continuously differentiable, μ_f-strongly convex, and its gradient ∇f is globally ℓ_f-Lipschitz continuous, where $\mu_f, \ell_f > 0$. The output cost function g is continuously differentiable, convex, and its gradient ∇g is globally ℓ_g-Lipschitz continuous.* □

Under Assumption 1.1, for each $w \in \mathbb{R}^q$ the problem (2) possesses a unique minimizer $(\bar{u}^\star, \bar{y}^\star)$. Note that the linear equality constraint in (2) can be eliminated, yielding the unconstrained problem

$$\min_{\bar{u}} \quad f(\bar{u}) + g(\Pi_u \bar{u} + \Pi_w w), \tag{3}$$

which may then be solved from any initial condition $u(0)$ via the *gradient flow*

$$\tau \dot{u} = -\nabla f(u) - \Pi_u^{\mathsf{T}} \nabla g(\Pi_u u + \Pi_w w) \tag{4}$$

where $\tau > 0$ is a tuning parameter. The algorithm 4 is unconditionally stable, in the sense that its convergence speed to $\bar{u}^\star$ can be made arbitrarily fast by decreasing τ.

A feedback controller can be obtained from (4) by replacing the steady-state output value $\bar{y} = \Pi_u u + \Pi_w w$ with the *measured* output value of $y(t)$, yielding the controller

$$\tau \dot{u} = -\nabla f(u) - \Pi_u^{\mathsf{T}} \nabla g(y). \tag{5}$$

An advantage of the controller (5) is that it requires only the DC gain Π_u of the plant (1) as model information.

2 Estimator-Based Performance Enhancement of FBO

We obtain the *reduced model*

$$\begin{aligned} \dot{x}_1^r &= A_0 x_1^r + B_0 u + E_0 w \\ y^r &= C_0 x_1^r + D_0 w, \end{aligned} \tag{6}$$

where

$$\begin{aligned} A_0 &:= A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad B_0 := B_1 \\ C_0 &:= C_1 - C_2 A_{22}^{-1} A_{21} \\ D_0 &:= -C_2 A_{22}^{-1} E_2 \\ E_0 &:= E_1 - A_{12} A_{22}^{-1} E_2 \end{aligned}$$

Since A and A_{22} are Hurwitz, Tikhonov's Theorem implies that A_0 is Hurwitz. Moreover

$$\Pi_u = -C_0 A_0^{-1} B_0, \qquad \Pi_w = -D_0 A_0^{-1} E_0.$$

We obtain the augmented model

$$\begin{aligned} \begin{bmatrix} \dot{x}_1^r \\ \dot{w} \end{bmatrix} &= \underbrace{\begin{bmatrix} A_0 & E_0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}}_{:=\mathbf{A}_{\text{aug0}}} \begin{bmatrix} x_1^r \\ w \end{bmatrix} + \underbrace{\begin{bmatrix} B_0 \\ \mathbb{0} \end{bmatrix}}_{:=\mathbf{B}_{\text{aug0}}} u \\ y^r &= \underbrace{\begin{bmatrix} C_0 & D_0 \end{bmatrix}}_{:=\mathbf{C}_{\text{aug0}}} \begin{bmatrix} x_1^r \\ w \end{bmatrix} \end{aligned} \tag{7}$$

Assumption 2.1 (Non-Resonance) *The matrix* $\left[\begin{smallmatrix} A_0 & E_0 \\ C_0 & D_0 \end{smallmatrix}\right]$ *has full column rank.*

The proposed EE-FBO design based on the reduced model is

$$\begin{bmatrix} \dot{\hat{x}}_1^r \\ \dot{\hat{w}} \end{bmatrix} = \mathbf{A}_{\mathrm{aug}0} \begin{bmatrix} \hat{x}_1^r \\ \hat{w} \end{bmatrix} + \mathbf{B}_{\mathrm{aug}0} u - \mathbf{L}_0 (y - \hat{y}^r) \tag{8a}$$

$$\hat{y}^r = \mathbf{C}_{\mathrm{aug}0} \begin{bmatrix} \hat{x}_1^r \\ \hat{w} \end{bmatrix} \tag{8b}$$

$$\hat{\bar{y}} = \Pi_u u + \Pi_w \hat{w} \tag{8c}$$

$$\tau \dot{u} = -\nabla f(u) - \Pi_u^{\mathsf{T}} \nabla g(y - (\hat{y}^r - \hat{\bar{y}})), \tag{8d}$$

where $\mathbf{L}_0 \in \mathbb{R}^{(n_1+q)\times p}$ is an estimator gain to be designed and $\tau > 0$.

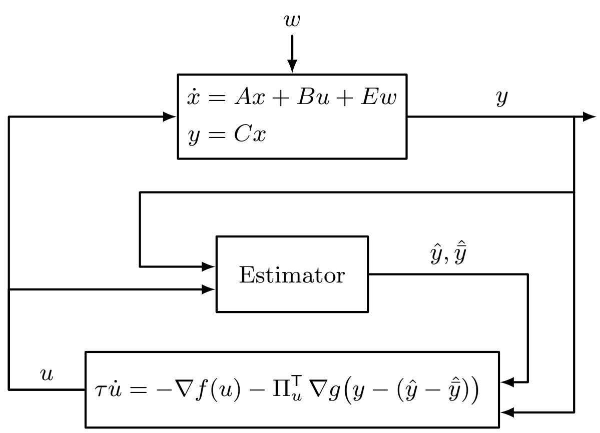


Figure 1: Feedback-based optimization with an estimator. The estimator predicts output and its steady state, feeding these into the controller.

3 Stability Analysis

Theorem 3.1 (Unconditional Stability of EE-FBO Based on Reduced Model) *Assume that A and A_{22} are Hurwitz, that Assumptions 1.1 and 2.1 hold, and select $\mathbf{L}_0$ such that $\mathbf{A}_{\mathrm{aug}0}+\mathbf{L_0}\mathbf{C}_{\mathrm{aug}0}$ is Hurwitz. Then for each $w \in \mathbb{R}^q$, the closed-loop system* (1), (8) *possesses a unique globally exponentially stable equilibrium point $(x, \hat{x}, \hat{w}, u) = (\bar{x}, \bar{x}, w, \bar{u}^\star)$ with corresponding output $\bar{y}^\star$.*

Proof of Theorem 3.1:

Applying the PBH test and using Assumption 2.1, it is straightforward to show that (7) is detectable, and thus $\mathbf{L}$ can be selected such that $\mathcal{A} \coloneqq \mathbf{A}_{\mathrm{aug}0} + \mathbf{L}\mathbf{C}_{\mathrm{aug}0}$ is Hurwitz.

First, write the plant in new coordinates defined as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \begin{pmatrix} x_1 \\ e_2 = x_2 - \bar{x}_r = x_2 - \big(-A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} E_2 w\big) \end{pmatrix}$$

Therefore:

$$x_2 = e_2 + \big(-A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}E_2w\big),$$

$$\begin{aligned}\dot{x}_1 &= A_{11}x_1 + A_{12}\Big(e_2 + (-A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}E_2w)\Big) + B_1u + E_1w\\ &= \underbrace{\big(A_{11} - A_{12}A_{22}^{-1}A_{21}\big)}_{A_0}x_1 + \underbrace{\big(E_1 - A_{12}A_{22}^{-1}E_2\big)}_{E_0}w + \underbrace{B_1}_{B_0}u + A_{12}e_2\end{aligned}$$

$$\begin{aligned}\dot{e}_2 &= \dot{x}_2 - \big(-A_{22}^{-1}A_{21}\,\dot{x}_1\big) = \dot{x}_2 + A_{22}^{-1}A_{21}\dot{x}_1\\ &= \frac{1}{\varepsilon}A_{21}x_1 + \frac{1}{\varepsilon}A_{22}x_2 + \frac{1}{\varepsilon}E_2w + A_{22}^{-1}A_{21}\big(A_{11}x_1 + A_{12}x_2 + B_1u + E_1w\big)\\ &= \frac{1}{\varepsilon}A_{21}x_1 + \frac{1}{\varepsilon}A_{22}\Big(e_2 + (-A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}E_2w)\Big)\\ &\quad + A_{22}^{-1}A_{21}\Big(A_{11}x_1 + A_{12}\big(e_2 + (-A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}E_2w)\big)\Big) + A_{22}^{-1}A_{21}B_1u + A_{22}^{-1}A_{21}E_1w\\ &= A_{21}x_1 + \frac{1}{\varepsilon}A_{22}e_2 - A_{21}x_1 - E_2w + E_2w\\ &\quad + A_{22}^{-1}A_{21}A_{11}x_1 + A_{22}^{-1}A_{21}A_{12}e_2 - A_{22}^{-1}A_{21}A_{22}A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}A_{21}A_{22}A_{22}^{-1}E_2w\\ &\quad + A_{22}^{-1}A_{21}A_{12}B_1u + A_{22}^{-1}A_{21}A_{12}E_1w\\ &= \big(A_{22}^{-1}A_{21}A_{11} - A_{22}^{-1}A_{21}A_{12}A_{22}^{-1}A_{21}\big)x_1 + \big(\tfrac{1}{\varepsilon}A_{22} + A_{22}^{-1}A_{21}A_{12}\big)e_2\\ &\quad + A_{22}^{-1}A_{21}B_1u + \big(A_{22}^{-1}A_{21}E_1 - A_{22}^{-1}A_{21}A_{12}A_{22}^{-1}E_2\big)w\end{aligned}$$

$$\begin{aligned}\dot{e}_2 &= A_{22}^{-1}A_{21}\underbrace{\big(A_{11} - A_{12}A_{22}^{-1}A_{21}\big)}_{A_0}x_1 + \big(\tfrac{1}{\varepsilon}A_{22} + A_{22}^{-1}A_{21}A_{12}\big)e_2\\ &\quad + A_{22}^{-1}A_{21}\underbrace{B_1}_{B_0}u + A_{22}^{-1}A_{21}\underbrace{\big(E_1 - A_{12}A_{22}^{-1}E_2\big)}_{E_0}w\end{aligned}$$

The two time-scale system is:

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + A_{12}e_2 + B_1u + (E_1 - A_{12}A_{22}^{-1}E_2)w,$$

$$\varepsilon\dot{e}_2 = A_{22}e_2 + \varepsilon A_{22}^{-1}A_{21}\dot{x}_1,$$

And in the matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{e}_2\end{bmatrix} = \begin{bmatrix}A_0 & A_{12}\\ A_{22}^{-1}A_{21}A_0 & \frac{1}{\varepsilon}A_{22} + A_{22}^{-1}A_{21}A_{12}\end{bmatrix}\begin{bmatrix}x_1\\ e_2\end{bmatrix} + \begin{bmatrix}B_0\\ A_{22}^{-1}A_{21}B_0\end{bmatrix}u + \begin{bmatrix}E_0\\ A_{22}^{-1}A_{21}E_0\end{bmatrix}w$$

Writing the matrix in upper triangular form:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \varepsilon\dot{e}_2\end{bmatrix} &= \left(\begin{bmatrix}A_0 & A_{12}\\ 0 & A_{22}\end{bmatrix} + \varepsilon\begin{bmatrix}0 & 0\\ A_{22}^{-1}A_{21}A_0 & A_{22}^{-1}A_{21}A_{12}\end{bmatrix}\right)\begin{bmatrix}x_1\\ e_2\end{bmatrix}\\ &\quad + \left(\begin{bmatrix}E_0\\ 0\end{bmatrix} + \varepsilon\begin{bmatrix}0\\ A_{22}^{-1}A_{21}E_0\end{bmatrix}\right)w + \left(\begin{bmatrix}B_0\\ 0\end{bmatrix} + \varepsilon\begin{bmatrix}0\\ A_{22}^{-1}A_{21}B_0\end{bmatrix}\right)u\end{aligned}$$

$$
\begin{aligned}
y &= C_1x_1 + C_2x_2 = C_1x_1 + C_2\big(e_2 + (-A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}E_2w)\big) \\
&= (C_1 - C_2A_{22}^{-1}A_{21})x_1 + C_2e_2 - C_2A_{22}^{-1}E_2w = C_0x_1 + D_0w + C_2e_2 \\
&= \begin{bmatrix} C_0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ e_2 \end{bmatrix} + D_0w
\end{aligned}
$$

So the two time-scale system

$$
\begin{bmatrix} \dot{x}_1 \\ \varepsilon\,\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} w,
$$

$$
y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

With reduced model:

$$
\begin{aligned}
\dot{x}_1^r &= A_0x_1^r + B_0u + E_0w \\
y^r &= C_0x_1^r + D_0w
\end{aligned}
$$

in new coordinated is

$$
\begin{bmatrix} \dot{x}_1 \\ \varepsilon\,\dot{e}_2 \end{bmatrix} = \left(\begin{bmatrix} A_0 & A_{12} \\ \varepsilon M_1 & A_{22} \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbb{0} \\ M_2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} B_0 \\ \varepsilon\, M_3 \end{bmatrix} u + \begin{bmatrix} E_0 \\ \varepsilon\, M_4 \end{bmatrix} w
$$

$$
y = \begin{bmatrix} C_0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ e_2 \end{bmatrix} + D_0\, w
$$

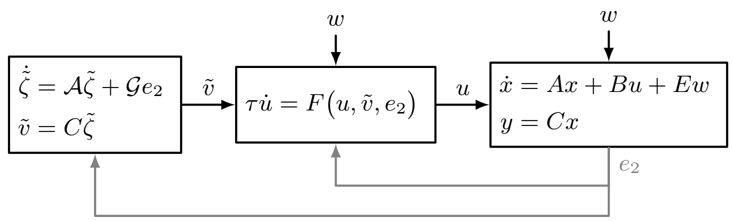
where

$$
\begin{aligned}
M_1 &\coloneqq A_{22}^{-1}A_{21}A_0 \\
M_2 &\coloneqq A_{22}^{-1}A_{21}A_{12} \\
M_3 &\coloneqq A_{22}^{-1}A_{21}E_0 \\
M_4 &\coloneqq A_{22}^{-1}A_{21}B_0
\end{aligned}
$$

Now, writing the estimator error coordinates as:

$$
\tilde{x}_1 = x_1 - \hat{x}_1^r, \quad \tilde{w} = w - \hat{w}^r, \quad \tilde{y} = y - \hat{y}^r
$$

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \dot{x}_1 - \dot{\hat{x}}_1^r \\
&= A_0x_1 + B_0u + E_0w + A_{12}e_2 - \Big(A_0\hat{x}_1^r + B_0u + E_0\hat{w}^r + L_1\big(C_0x_1 + D_0w + C_2e_2 - C_0\hat{x}_1^r - D_0\hat{w}^r\big)\Big) \\
&= (A_0 - L_1C_0)\,(x_1 - \hat{x}_1^r) + (E_0 - L_1D_0)\,(w - \hat{w}^r) + (A_{12} - L_1C_2)\,e_2 \\
&= (A_0 - L_1C_0)\tilde{x}_1 + (E_0 - L_1D_0)\tilde{w} + (A_{12} - L_1C_2)\,e_2
\end{aligned}
$$



$$\begin{aligned}\dot{\tilde{w}} &= \dot{w} - \dot{\hat{w}}^r = -L_2\,(y - \hat{y}^r) = -L_2 C_0\,(x_1 - \hat{x}_1^r) - L_2 D_0\,(w - \hat{w}^r) - L_2 C_2\, e_2 \\ &= -L_2 C_0 \tilde{x}_1 - L_2 D_0 \tilde{w} - L_2 C_2 e_2\end{aligned}$$

$$\begin{aligned}\tilde{y} &= y - \hat{y}^r = [\,C_0 \;\; D_0\,] \begin{bmatrix} x_1 - \hat{x}_1^r \\ w - \hat{w}^r \end{bmatrix} + C_2\, e_2 \\ &= [\,C_0 \;\; D_0\,] \begin{bmatrix} \tilde{x}_1 \\ \tilde{w} \end{bmatrix} + C_2\, e_2\end{aligned}$$

Therefore, the estimator in error coordinates is:

$$\begin{aligned}\underbrace{\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{w}} \end{bmatrix}}_{\dot{\tilde{\zeta}}} &= \underbrace{\begin{bmatrix} A_0 - L_1\, C_0 & E_0 - L_1\, D_0 \\ -\,L_2\, C_0 & -\,L_2\, D_0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \tilde{x}_1 \\ \tilde{w} \end{bmatrix} + \underbrace{\begin{bmatrix} A_{12} - L_1\, C_2 \\ -\,L_2\, C_2 \end{bmatrix}}_{\mathcal{G}} e_2 \\ \tilde{y} &= [\,C_0 \;\; D_0\,] \begin{bmatrix} \tilde{x}_1 \\ \tilde{w} \end{bmatrix} + C_2\, e_2 \\ \hat{\bar{y}} &= \Pi_u\, u + \Pi_w\, \hat{w} = \Pi_u\, u + \Pi_w\,(w - \tilde{w})\end{aligned}$$

Now modifying the controller accordingly:

$$\begin{aligned}\tau\dot{u} &= -\nabla f(u) \;-\; \Pi_u^T \nabla g\big(y - \hat{y}^r - \hat{g}\big) \\ &= -\nabla f(u) \;-\; \Pi_u^T \nabla g\big(\tilde{y} + \Pi_u\, u + \Pi_w\,(w - \tilde{w})\big) \\ &= -\nabla f(u) \;-\; \Pi_u^T \nabla g\big(C_0\, \tilde{x}_1 + D_0\, \tilde{w} + C_2\, e_2 + \Pi_u\, u + \Pi_w\, w - \Pi_w\, \tilde{w}\big) \\ &= -\nabla f(u) + \Pi^T\, \nabla g\big(\underbrace{\Pi_u u + \Pi_w w}_{\bar{y}} + \underbrace{C_0 \tilde{x}_1 + (D_0 - \Pi_w)\tilde{w}}_{\tilde{v}} + C_2 e_2\big) = F\big(u, \tilde{v}, e_2\big)\end{aligned}$$

Define:

$$\tilde{v} = \underbrace{\big[C_0 \quad D_0 - \Pi_w\big]}_{\mathcal{C}} \begin{bmatrix} \tilde{x}_1 \\ \tilde{w} \end{bmatrix}$$

Now, we study the (exponential) input-to-state stability of the function F

$$
\begin{aligned}
V_c &= \| \, u - \bar{u}^\star)\|^2, \\
\dot{V}_c &= (u - \bar{u}^\star)^\top \dot{u} \\
&= -\frac{1}{\tau}(u - \bar{u}^\star))^\top \Big(-\nabla f(u) - \Pi_u^\top \nabla g\big(\Pi_u u + \Pi_w w + \tilde{v} + C_2 e_2\big)\Big)
\end{aligned}
$$

Adding and subtracting $\frac{1}{\tau}(u-\bar{u}^\star)^\top \Big(-\nabla f(u) - \Pi_u^\top \nabla g\big(\Pi_u u + \Pi_w \tilde{w}\big)\Big)$ and adding zero which is :

$$
\begin{aligned}
\dot{V}_c = &- \frac{1}{\tau}(u - \bar{u}^\star)^\top \Big(-\nabla f(u) - \Pi_u^\top \nabla g\big(\Pi_u u + \Pi_w w + \tilde{v} + C_2 e_2\big)\Big) \\
&+ \frac{1}{\tau}(u - \bar{u}^\star)^\top \Big(-\nabla f(u) - \Pi_u^\top \nabla g\big(\Pi_u u + \Pi_w w\big)\Big) \\
&- \frac{1}{\tau}(u - \bar{u}^\star)^\top \Big(-\nabla f(u) - \Pi_u^\top \nabla g\big(\Pi_u u + \Pi_w w\big)\Big) \\
&+ \frac{1}{\tau}(u - \bar{u}^\star)^\top \Big(-\nabla f(\bar{u}^\star) - \Pi_u^\top \nabla g\big(\Pi_u \bar{u}^\star + \Pi_w w\big)\Big)
\end{aligned}
$$

Regrouping the terms, we have:

$$
\begin{aligned}
\dot{V}_c = &\underbrace{-\frac{1}{\tau}\,(u - \bar{u}^\star)^\top \big(\nabla f(u) - \nabla f(\bar{u}^\star)\big)}_{\leq -\frac{\mu_f}{\tau}\|u-\bar{u}^\star\|^2} \\
&+ \underbrace{\frac{1}{\tau}\,(u - \bar{u}^\star)^\top \Pi_u^\top \Big(\nabla g(\Pi_u u + \Pi_w w + \tilde{v} + C_2 e_2) - \nabla g(\Pi_u u + \Pi_w w)\Big)}_{-\frac{\|\Pi_u\|\ell_g}{\tau}\|u-\bar{u}^\star\|\|\tilde{v}+C_2e_2\|} \\
&\underbrace{-\frac{1}{\tau}(u - \bar{u}^\star)^\top \Pi_u^\top \Big(\nabla g(\Pi_u u + \Pi_w w) - \nabla g(\Pi_u \hat{u}^r + \Pi_w w)\Big)}_{-\frac{\mu_g}{\tau}\|u-\bar{u}^\star\|^2} \\
&\leq -\frac{\mu_f + \mu_g}{\tau}\,\|u - \bar{u}^\star\|^2 + \frac{\|\Pi_u\|\,L_g}{\tau}\,\|u - \bar{u}^\star\|\,\|\tilde{v} + C_2 e_2\|
\end{aligned}
$$

Using Peter Paul's inequality:

$$
\begin{aligned}
\dot{V}_c &\leq -\frac{\mu_f + \mu_g}{\tau}\|u - \bar{u}^\star\|^2 + \frac{\|\Pi_u\|\,L_g}{2\,\tau}\Big(\tfrac{1}{\varepsilon\,c}\|u - \bar{u}^\star\|^2 + \varepsilon_c\,\|\tilde{v} + C_2 e_2\|^2\Big) \\
&= -\underbrace{\frac{\mu_f + \mu_g + \frac{\|\Pi_u\|\,L_g}{2\,\varepsilon_c}}{\tau}}_{\alpha_c}\,\|u - \bar{u}^\star\|^2 + \underbrace{\frac{\|\Pi_u\|\,L_g\,\varepsilon_c}{2\,\tau}}_{\beta_c}\,\|\tilde{v}\|^2 + \underbrace{\frac{\|\Pi_u\|\,L_g\,\|C_2\|^2}{2\,\tau}}_{\gamma_c}\,\|e_2\|^2
\end{aligned}
$$

Therefore, we have:

$$\dot{V}_c \leq -\alpha_c \|u - \bar{u}^\star\|^2 + \beta_c \|\tilde{v}\|^2 + \gamma_c \|e_2\|^2 \tag{9}$$

and the controller $F = (u, \tilde{v}, e_2)$ is ISS with respects to inputs $\tilde{v}$ and e_2.

Similarly, construct Lyapunov functions for the estimator. Considering the estimator is designed to be Hurwitz:

$$\dot{\tilde{\zeta}} = \mathcal{A}\tilde{\zeta} + Ge_2$$

Assume positive semidefinite matrices $P_e, Q_e \succ 0$ such that $P_e\mathcal{A} + \mathcal{A}^\mathsf{T} P_e = -Q_e$, consider Lyapunov function:

$$V_e = \tilde{\zeta}^T P_e \tilde{\zeta}$$

$$\begin{aligned}
\dot{V}_e &= 2\tilde{\zeta}^\mathsf{T} P_e \dot{\tilde{\zeta}} \\
&= 2\tilde{\zeta}^\top P_e(\mathcal{A}\tilde{\zeta} + Ge_2) \\
&= \tilde{\zeta}^\top (2P_e\mathcal{A})\tilde{\zeta} + 2\tilde{\zeta}^\top P_e G e_2 \\
&= \tilde{\zeta}^\top (P_e\mathcal{A} + \mathcal{A}^\top P_e)\tilde{\zeta} + 2\tilde{\zeta}^\top P_e G e_2 \\
&= -\tilde{\zeta}^\top Q_e \tilde{\zeta} + 2\tilde{\zeta}^\top P_e G e_2 \\
&\leq -\lambda_{\max}(Q_e)\|\tilde{\xi}\|^2 + 2\|P_e G\| \|\tilde{\zeta}\| \|e_2\| \\
&\leq -\lambda_{\min}(Q_e)\|\tilde{\zeta}\|^2 + 2\|P_e G\| \left(\frac{\|\tilde{\xi}\|^2}{2\varepsilon_e} + \frac{\varepsilon_e \|e_2\|^2}{2} \right)
\end{aligned}$$

Selecting $\varepsilon_e > 0$ such that $\lambda_{max}(Q_e) - \frac{2\|P_e\|\|G\|}{\epsilon_e} > 0$

$$\dot{V}_e \leq -\underbrace{\left(\lambda_{\max}(Q_e) - \frac{2\|P_e\|\|G\|}{2\varepsilon_c} \right)}_{\alpha_e} \|\tilde{\zeta}\|^2 + \underbrace{(\|P_e\|\|G\|\varepsilon_c)}_{\gamma_e} \|e_2\|^2$$

Therefore, we have:

$$\dot{V}_e \leq -\alpha_e \|\tilde{\zeta}\|^2 + \gamma_e \|e_2\|^2 \tag{10}$$

Now, combining V_c snf V_e we get:

$$V_1 = V_c + \delta V_e$$

$$\begin{aligned}
\dot{V} &= \dot{V}_C + \delta \dot{V}_e \\
&\leq -\alpha_c \|u - \bar{u}^*\|^2 + \beta_c \|\tilde{v}\|^2 + \gamma_c \|e_2\|^2 - \delta\alpha_e \|\tilde{\zeta}\|^2 + \delta\gamma_e \|e_2\|^2 \\
&\leq -\alpha_c \|u - \bar{u}^*\|^2 + \beta_c \|\mathcal{C}\|^2 \|\tilde{\zeta}\|^2 - \delta\alpha_e \|\tilde{\zeta}\|^2 + (\gamma_c + \delta\gamma_e) \|e_2\|^2 \\
&\leq -\alpha_c \|u - \bar{u}^*\|^2 - (\delta\alpha_e - \beta_c \|\mathcal{C}\|^2) \|\tilde{\zeta}\|^2 + (\gamma_c + \delta\gamma_e) \|e_2\|^2
\end{aligned}$$

Choose δ such that $\delta\alpha_e - \beta_c\|\mathcal{C}\|^2 < 0$ Therefore we have

$$\dot{V}_1 \le -\alpha_1\|u-\bar{u}^*\|^2 - \beta_1\|\tilde{\zeta}\|^2 + \underbrace{(\frac{\nu_c}{\tau}+\nu_e)}_{\gamma_1}\|e_2\|^2$$

where

$$\begin{aligned}
\alpha_1 &:= \frac{\mu_f+\mu_g+\frac{\|\Pi_u\|\,L_g}{2\,\varepsilon_c}}{\tau} > 0\\
\beta_1 &:= \delta\alpha_e - \beta_c\|\mathcal{C}\|^2 = \delta\left(\lambda_{\max}(Q_e) - \frac{2\|P_e\|\|G\|}{2\varepsilon_c}\right) - \frac{\|\Pi_u\|\,L_g\,\varepsilon_c}{2\,\tau}\|\mathcal{C}\|^2 > 0\\
\nu_c &:= \frac{\|\Pi_u\|\,L_g\,\varepsilon_c}{2}\\
\nu_e &:= \delta\|P_e\|\|G\|\varepsilon_c\\
\gamma_1 &:= \gamma_c + \delta\gamma_e = \frac{\|\Pi_u\|\,L_g\,\varepsilon_c}{2\,\tau} - \delta\|P_e\|\|G\|\varepsilon_c
\end{aligned}$$

Therefore, combination of the estimator and the controller is ISS with respect input e_2.

Now, we work on Lyapunov function of the plant.

$$\begin{aligned}
\dot{x}_1 &= A_0x_1 + A_{12}e_2 + B_0u,\\
\epsilon\,\dot{e}_2 &= \epsilon\,M_1\,x_1 + \big(A_{22}+\epsilon\,M_2\big)e_2 + \epsilon\,M_3\,u,
\end{aligned}$$

Since A_0 and A_{22} are Hurwitz, there exist symmetric $P_0, P_{22} \succ 0$ and $\alpha_0, \alpha_{22} > 0$ such that

$$\begin{aligned}
P_0A_0 + A_0^\top P_0 &\prec -\alpha_0\mathbb{I}\\
P_{22}A_{22} + A_{22}^\top P_{22} &\prec -\alpha_{22}\mathbb{I}
\end{aligned}$$

Define Lyapunov function as:

$$V(x_1,e_2) = x_1^\top P_0\,x_1 \;+\; e_2^\top P_{22}\,e_2$$

$$\begin{aligned}
\dot{V} &= 2\,x_1^\top P_0\,\dot{x}_1 \;+\; 2\,e_2^\top P_{22}\,\dot{e}_2\\
&= 2\,x_1^\top P_0\big(A_0x_1 + A_{12}e_2 + B_0u\big) + 2\,e_2^\top P_{22}\,\frac{1}{\epsilon}\Big(\epsilon\,M_1x_1 + (A_{22}+\epsilon M_2)e_2 + \epsilon M_3u\Big)\\
&= 2\,x_1^\top P_0A_0x_1 + 2\,x_1^\top P_0A_{12}e_2 + 2\,x_1^\top P_0B_0u\\
&+ 2\,e_2^\top P_{22}M_1x_1 + \tfrac{2}{\epsilon}\,e_2^\top P_{22}A_{22}e_2 + 2\,e_2^\top P_{22}M_2e_2 + 2\,e_2^\top P_{22}M_3u\\
&\le\; -\alpha_0\|x_1\|^2 \;-\; \frac{\alpha_{22}}{\epsilon}\,\|e_2\|^2 \;+\; 2\,e_2^\top P_{22}M_2e_2 \;+\; 2\,x_1^\top\big(P_0A_{12}+M_1^\top P_{22}\big)e_2 \;+\; 2\,x_1^\top P_0B_0u \;+\; 2\,e_2^\top P_{22}M_3u
\end{aligned}$$

Let

$$Q = P_0A_{12} + M_1^\top P_{22}.$$

Then for any $\delta_1, \delta_2, \delta_3 > 0$, using Young's inequality we have

$$
\begin{aligned}
2x_1^\top Q e_2 &\leq \delta_1 \|x_1\|^2 + \frac{\|Q\|^2}{\delta_1}\|e_2\|^2, \\
2x_1^\top P_0 B_0 u &\leq \frac{1}{\delta_2}\|x_1\|^2 + \frac{\|P_0 B_0\|^2}{\delta_2}\|u\|^2, \\
2e_2^\top P_{22} M_3 u &\leq \frac{1}{\delta_3}\|e_2\|^2 + \frac{\|P_{22} M_3\|^2}{\delta_3}\|u\|^2.
\end{aligned}
$$

Putting all terms together:

$$
\dot{V} \leq -\alpha_0\|x_1\|^2 - \Big(\frac{\alpha_{22}}{\epsilon} - 2\|P_{22}M_2\| - \frac{\|Q\|^2}{\delta_1} - \delta_3\Big)\|e_2\|^2 + \frac{1}{\delta_2}\|x_1\|^2 + \frac{\|P_0 B_0\|^2}{\delta_2}\|u\|^2 + \frac{1}{\delta_3}\|e_2\|^2 + \frac{\|P_{22}M_3\|^2}{\delta_3}\|u\|^2.
$$

Collecting similar terms we have

$$
\dot{V} \leq -(\alpha_0 - \delta_1 - \delta_2)\|x_1\|^2 - \left(\tfrac{\alpha_{22}}{\varepsilon} - 2\|P_{22}M_2\| - \tfrac{\|Q\|^2}{\delta_1} - \delta_3\right)\|e_2\|^2 + \left(\tfrac{\|P_0 B_0\|^2}{\delta_2} + \tfrac{\|P_{22}M_3\|^2}{\delta_3}\right)\|u\|^2.
$$

Should choose $\delta_1, \delta_2, \delta_3$ such that:

$$
\alpha_0 > \delta_1 + \delta_2 \quad \text{and} \quad \frac{\alpha_{22}}{\varepsilon} > 2\|P_{22}M_2\| + \frac{\|Q\|^2}{\delta_1} + \delta_3
$$

focusing only on e_2:

$$
\dot{V}_2 \leq -\left(\tfrac{\alpha_{22}}{\varepsilon} - \alpha_2\right)\|e_2\|^2 + \gamma_2\,\|u\|^2
$$

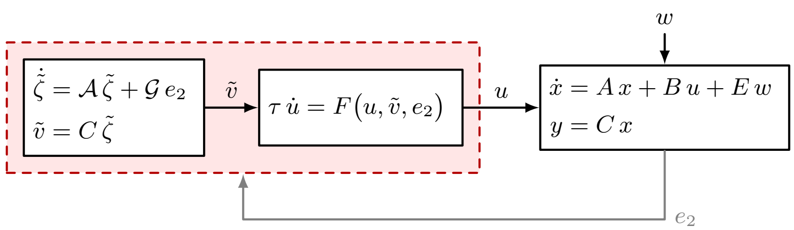
where

$$
\begin{aligned}
\alpha_2 &\coloneqq 2\|P_{22}M_2\| + \tfrac{\|Q\|^2}{\delta_1} + \delta_3 \\
\gamma_2 &\coloneqq \frac{\|P_0 B_0\|^2}{\delta_2} + \frac{\|P_{22}M_3\|^2}{\delta_3}
\end{aligned}
$$

Rewriting

$$
\dot{V}_2 \leq -\|e_2\|^2 + \frac{\gamma_2}{\frac{\alpha_{22}}{\varepsilon} - \alpha_2}\,\|u\|^2
$$

In particular, as $\varepsilon \to 0$ (making the fast subsystem infinitely fast), the coefficient of the input goes to zero.



Now we have these two interconnected ISS systems. Their Lyapunov functions are:

$$\dot{V}_1 \leq -\alpha_1\|u - \bar{u}^*\|^2 - \beta_1\|\tilde{\zeta}\|^2 + (\frac{\nu_c}{\tau} + \nu_e)\|e_2\|^2$$
$$\dot{V}_2 \leq -\|e_2\|^2 + \frac{\gamma_2}{\frac{\alpha_{22}}{\varepsilon} - \alpha_2}\|u\|^2$$

Based on ISS small-gain Theorem, to have a stable interconnected system, we should have:

$$\Big(\frac{\nu_c}{\tau} + \nu_e\Big)\Big(\frac{\gamma_2}{\frac{\alpha_{22}}{\varepsilon} - \alpha_2}\Big) < 1$$
$$\frac{\nu_c}{\tau} + \nu_e < \frac{\frac{\alpha_{22}}{\varepsilon} - \alpha_2}{\gamma_2}$$
$$\frac{\nu_c}{\tau} < \frac{\frac{\alpha_{22}}{\varepsilon} - \alpha_2 - \gamma_2\nu_e}{\gamma_2}$$

Therefore, we have:

$$\tau > \frac{\nu_c\gamma_2}{\frac{\alpha_{22}}{\varepsilon} - \alpha_2 - \gamma_2\nu_e}$$

It can be seen that as $\epsilon \to 0$ the right-hand side of the above inequality goes to zero, resulting in a smaller bound over τ. This allows making the controller faster by decreasing τ.

□