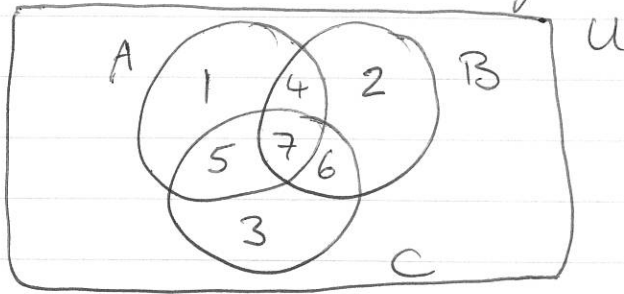


## Inclusion - Exclusion for 3 sets

Consider the Venn diagram associated with  $A, B$  &  $C$ .  
Let's view  $A \cup B \cup C$  as the union of the pairwise disjoint regions 1, 2, 3, 4, 5, 6, 7:



We note that region 1 is the set of elements in  $A$  that are not in  $B$  and not in  $C$ .  
Thus, region 1 is  $A \cap \bar{B} \cap \bar{C}$ .

Similarly, region 2 is  $B \cap \bar{A} \cap \bar{C}$ ,  
region 3 is  $C \cap \bar{A} \cap \bar{B}$ ,  
region 4 is  $A \cap B \cap \bar{C}$   
(as these elements are in  $A$  and  $B$  but not  $C$ ),  
region 5 is  $A \cap C \cap \bar{B}$ ,  
region 6 is  $B \cap C \cap \bar{A}$   
and region 7 is  $A \cap B \cap C$ .

We know that

$$|A \cup B \cup C| = |A \cap \bar{B} \cap \bar{C}| + |B \cap \bar{A} \cap \bar{C}| + |C \cap \bar{A} \cap \bar{B}| \\ + |A \cap B \cap \bar{C}| + |A \cap C \cap \bar{B}| + |B \cap C \cap \bar{A}| + |A \cap B \cap C|.$$

Let's confirm that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

by showing that the expression on the right hand side counts each of the 7 regions once and only once.

We can use a table to do this:

	$A\bar{B}\bar{C}$	$B\bar{A}\bar{C}$	$C\bar{A}\bar{B}$	$A\bar{B}C$	$A\bar{C}B$	$B\bar{C}A$	$A\bar{B}C$
$+  A $	1	0	0	1	1	0	1
$+  B $	0	1	0	1	0	1	1
$+  C $	0	0	1	0	1	1	1
$-  A \cap B $	0	0	0	-1	0	0	-1
$-  A \cap C $	0	0	0	0	-1	0	-1
$-  B \cap C $	0	0	0	0	0	-1	-1
$+  A \cap B \cap C $	0	0	0	0	0	0	1
Total	1	1	1	1	1	1	1

Our final row of column totals shows that each of our 7 regions is counted once and only once.

$$\text{Hence } |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|,$$

as desired.