Introduction to Differential Equations (MS211)

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Outline

Quick recapitulation of previous lectures

Recap

So at this point of our lectures we:

- Focused exclusively on order 1 differential equations

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- Focused exclusively on order 1 differential equations
- Know how to solve them explicitely in a few special cases
- Have a good idea of the structure of the space of solutions in the linear case
- Have some powerful existence and uniqueness results (The Picard-Lindelöf theorems)

Order 1 equations only

Apart from the introduction chapter where we briefly mentioned equations such as the pendulum equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0,$$

all the equations we considered were order 1 differential equations, of the form

$$y' = F(y, t),$$

that is to say none of the equations involved higher order derivatives such as $y'', y''', \ldots$

Special cases where we know how to solve explicitely

The separable case:

$$g(y)y' = f(t).$$

The order 1 linear case:

$$y' + p(t)y = g(t).$$

We have a good idea of the structure of the space of solutions in the linear case

- When we have an order 1 linear equation, even if we can't solve it, we know that all the solutions are under the form

$$y = y_H + y_0,$$

where y_H is a solution of the associated homogeneous equation and y_0 any particular (fixed) solution of the equation.

- The set of the solutions of the homogeneous equation is a vector subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, and the set of the solutions of the linear equation is an affine subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

And some theoretical results (the Picard-Lindelöf theorems)

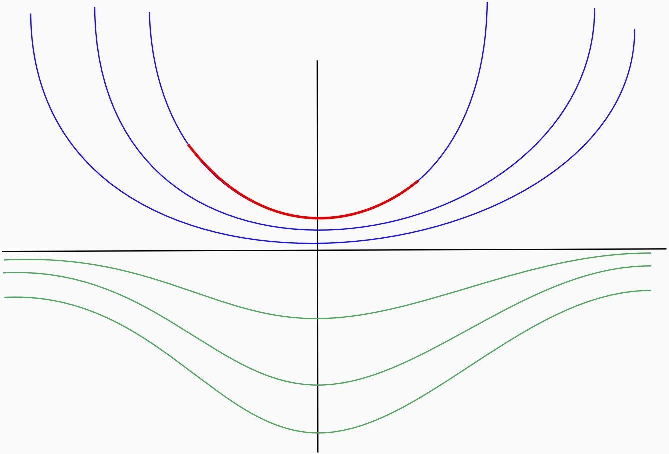
For an equation $y' = F(y, t)$ and f defined over $I \times J$ we have:

Theorem 2.6 (Picard-Lindelöff) If F is $\mathcal{C}^1$, then for every (y_0, t_0), there exist a unique maximal solution y such that $y(t_0) = y_0$.

Theorem 2.7 (Global Picard-Lindelöff) If furthermore $\frac{\partial F}{\partial y}$ is bounded, then the solutions are global (defined for every $t \in J$).

Theorem 2.8 (Linear Picard-Lindelöff) For a linear equation $y' + p(t)y = g(t)$, p and g continuous is enough to have all the conclusions above (existence and uniqueness of global solutions).

Solutions of the equation $y' = ty^2$



In red, a non-maximal solution. In blue, maximal non-global solutions. In green, global solutions.

Back to Theorem 2.9 (explosion of solutions)

And we had just arrived to this theorem

Theorem 2.9 (Explosion of solutions) Assume F is $\mathcal{C}^1$ defined over $\mathbb{R} \times \mathbb{R}$. For every (y_0, t_0), we have a unique maximal solution y such that $y(t_0) = y_0$, defined over some interval $I_y = (a, b)$, with $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$.

1. If a is finite, then $\lim_a |y| = +\infty$.
2. If y is bounded in the neighborhood of a, then $a = -\infty$.
3. If b is finite, then $\lim_b |y| = +\infty$.
4. If F is bounded in the neighborhood of b, then $b = +\infty$.
5. If F is bounded on $\mathbb{R}$, then $I_y = \mathbb{R}$ (the solution is global).

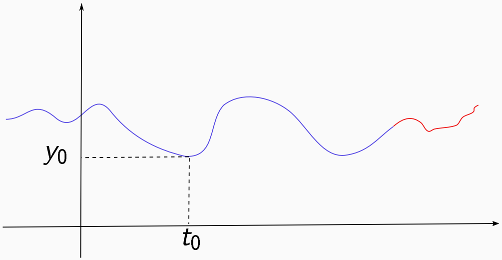
Idea of the proof

Remember that y is a maximal solution defined on (a, b). Let us proove:

3 . If y is bounded in the neighborhood of b, then $b = +\infty$.

4 . If b is finite, then $\lim_b |y| = +\infty$.

We admit that in our context, $\lim_b y$ exists.



An example of application

Let us consider the equation

$$y' = (y-1)(y-2) \quad (E)$$

- Clearly, Theorem 2.6 (Picard-Lindelöff) applies: the function $F : (t, y) \mapsto (y-1)(y-2)$ is $\mathcal{C}^1$, since

$$\frac{\partial F}{\partial t}(y, t) = 0$$

and

$$\frac{\partial F}{\partial y}(y, t) = -(y-1) - (y-2)$$

are continuous on $\mathbb{R}^2$.

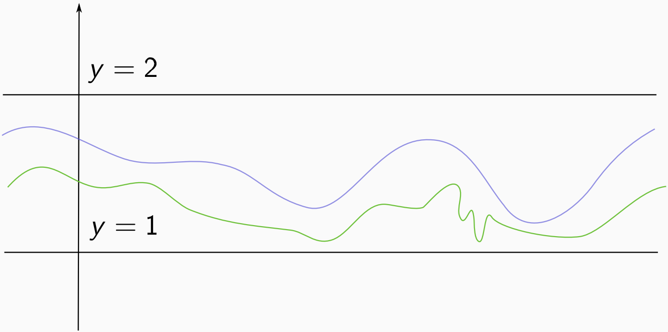
$$y' = (y-1)(y-2) \quad (E)$$

- Now the constant functions $t \mapsto 1$ and $t \mapsto 2$ are obvious solutions of (E).
- From Theorem 2.6 (Picard-Lindelöff), we know that any maximal solution that verifies $y(t_0) = y_0 \in [1, 2]$ for some t_0 will verify, for every t at which it is defined:

$$1 \leq y(t) \leq 2.$$

- In particular such a solution is bounded, which by Theorem 2.9 (explosion of solutions) shows that y is defined over $\mathbb{R}$: it is a global solution.

A drawing of our findings



We have no idea what the solutions are doing, but if they are at some time in the horizontal band between $y = 1$ and $y = 2$, they stay there at all time, and they are defined over $\mathbb{R}$.

Some comments

- We've just deduced some properties of the solutions without solving the equation.
- Doing that is called the qualitative study of a differential equation.
- They are precious since we don't know how to solve most differential equations.

2.4 From order n ODEs to order 1 systems

2.4.1 Systems of ordinary differential equations

Definition: A system of ordinary differential equations of order 1 and size d has d unknowns functions $y_1, \cdots, y_d$, that are functions of the variable t (time if we want). It also has d equations of the form:

$$\begin{cases} y_1' = F_1(y_1, \cdots, y_d, t) \\ y_2' = F_2(y_1, \cdots, y_d, t) \\ \quad\vdots \\ y_d' = F_d(y_1, \cdots, y_d, t). \end{cases}$$

Example:

$$\begin{cases} y_1' = t + y_2 y_1 \\ y_2' = t^2 - y_2 y_1 \end{cases}$$

is a system of order 1 and size 2.

Remarks

- Notice that in general, y_1' depends on t, y_1, but also the others y_i: it is impossible to successively any equation without solving the others. We say that the equations are coupled. Check-it out on our previous example:

$$\begin{cases} y_1' = t + y_2 y_1 \\ y_2' = t^2 - y_2 y_1 \,. \end{cases}$$

- On the other hand, if every equation has the form $y_i' = F(y_i, t)$, we say that the system is uncoupled. The following system

$$\begin{cases} y_1' = t + y_1 \\ y_2' = t^2 - y_2 \end{cases}$$

is uncoupled.

Rewriting our system

Our system of equations

$$\begin{cases} y_1' = F_1(y_1, \cdots, y_d, t) \\ y_2' = F_2(y_1, \cdots, y_d, t) \\ \quad\vdots \\ y_d' = F_d(y_1, \cdots, y_d, t). \end{cases}$$

can be rewritten $y' = F(y, t)$, with

$$\begin{aligned} y : \mathbb{R} &\to \mathbb{R}^d \\ t &\mapsto (y_1(t), \cdots, y_d(t)), \end{aligned}$$

and

$$\begin{aligned} F : \mathbb{R}^{d+1} &\to \mathbb{R}^d \\ (y_1, \cdots, y_d) &\mapsto (F_1(y_1, \cdots, y_d, t), \cdots, F_d(y_1, \cdots, y_d, t)). \end{aligned}$$

An order 1 system is just a generalization of an order 1 ode

Remember that in the definition of an order 1 ode we also have

$$y' = F(y, t),$$

with

$$y : \mathbb{R} \to \mathbb{R} : t \mapsto y(t)$$

and

$$F : \mathbb{R}^2 \to \mathbb{R} : (y, t) \mapsto F(y, t).$$

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to obtain a system we just replace y and F by

$$y : \mathbb{R} \to \mathbb{R}^d : t \mapsto (y_1(t), \cdots, y_d(t))$$

and

$$\begin{aligned} F : \mathbb{R}^{d+1} &\to \mathbb{R}^d \\ (y_1, \cdots, y_d) &\mapsto (F_1(y_1, \cdots, y_d, t), \cdots, F_d(y_1, \cdots, y_d, t)). \end{aligned}$$

Where we are now

- We have defined order 1 systems.
- We have seen that they are somehow "big" (for real, vectorial,

or multidimensional) order 1 differential equations.

- We will now see that every ode of order n is equivalent to a system of order 1 and size n so that essentially...
- Every differential equation can be rewritten as some "big" order 1 differential equations,
- which is why order 1 equations are so important.

2.4.2 Equivalence between order n odes and order 1 systems

Let us demonstrate the idea we are following on an example first:

Consider the order 2 equation

$$y'' = t^2 y' + y. \qquad (E)$$

If we call $y = y_1$ and $y' = y_2$, we can write

$$\begin{cases} y_1' = y_2 \\ y_1'' = t^2 y_1' + y_1 \end{cases} \Leftrightarrow \begin{cases} y_1' = y_2 \\ y_2' = t^2 y_2 + y_1 \end{cases} \qquad (S)$$

We now have the equivalence:
y is a solution of equation (E) if and only if (y, y') are solutions of the system (S).

In general we have the following result

Theorem 2.10: y is a solution of the order n ode

$$y^{(n)} = F\left(y, y', \cdots, y^{(n-1)}, t\right) \quad (E)$$

if and only if $(y, y', \cdots, y^{(n)})$ are solution of the system:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \quad \vdots \\ y_{n-1}' = y_n \\ y_n' = F(y_1, y_2, \cdots, y_n, t). \end{cases} \qquad (S)$$

Another example

A function y is solution of the differential equation

$$y''' = y' + t$$

if and only if (y, y', y'') are solutions of the system

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_2 + t. \end{cases}$$