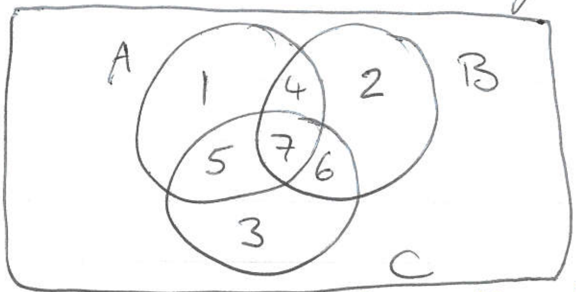
Inclusion - Exclusion for 3 sets

Consider the Venn diagram associated with A, B & C. Let's view $A \cup B \cup C$ as the union of the pairwise disjoint regions 1, 2, 3, 4, 5, 6, 7 :



(*) For simplicity, let's assume that $U - (A \cup B \cup C)$ is empty.

We note that region 1 is the set of elements in A that are not in B and not in C. Thus, region 1 is $A \cap \overline{B} \cap \overline{C}$.

Similarly, region 2 is $B \cap \overline{A} \cap \overline{C}$,
region 3 is $C \cap \overline{A} \cap \overline{B}$,
region 4 is $A \cap B \cap \overline{C}$
(as these elements are in A and B but not C),
region 5 is $A \cap C \cap \overline{B}$,
region 6 is $B \cap C \cap \overline{A}$
and region 7 is $A \cap B \cap C$.

We know that

$$|A \cup B \cup C| = |A \cap \overline{B} \cap \overline{C}| + |B \cap \overline{A} \cap \overline{C}| + |C \cap \overline{A} \cap \overline{B}| + |A \cap B \cap \overline{C}| + |A \cap C \cap \overline{B}| + |B \cap C \cap \overline{A}| + |A \cap B \cap C|.$$

Let's confirm that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

by showing that the expression on the right hand side counts each of the 7 regions once and only once.

We can use a table to do this:

	$A\cap\bar{B}\cap\bar{C}$	$B\cap\bar{A}\cap\bar{C}$	$C\cap\bar{A}\cap\bar{B}$	$A\cap B\cap\bar{C}$	$A\cap C\cap\bar{B}$	$B\cap C\cap\bar{A}$	$A\cap B\cap C$
$+\lvert A\rvert$	1	0	0	1	1	0	1
$+\lvert B\rvert$	0	1	0	1	0	1	1
$+\lvert C\rvert$	0	0	1	0	1	1	1
$-\lvert A\cap B\rvert$	0	0	0	-1	0	0	-1
$-\lvert A\cap C\rvert$	0	0	0	0	-1	0	-1
$-\lvert B\cap C\rvert$	0	0	0	0	0	-1	-1
$+\lvert A\cap B\cap C\rvert$	0	0	0	0	0	0	1
Total	1	1	1	1	1	1	1

Our final row of column totals shows that each of our 7 regions is counted once and only once.

Hence
$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|,$$

as desired.