MS321 Algebra, tutorial 9, question 2

2. If G is a finite abelian group and p is a prime factor of $|G|$, prove that G has an element of order p.

We know $G \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \ldots \times \mathbb{Z}_{p_n^{k_n}}$ so that $|G| = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$. Since p is a divisor of $|G|$, one of these p_i must be p. Thus $\mathbb{Z}_{p^k}$ is one of the factors with $k \geq 0$. This factor is the cyclic group $\mathbb{Z}_{p^k}$ with generator 1. Thus $p^k = 0$ in this factor so that $p(p^{k-1}) = 0$ and $p^{k-1} \neq 0$. This p^{k-1} is the element we want. So an element in G of order p is the element of G which corresponds under the isomorphism

$$G \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \ldots \times \mathbb{Z}_{p_n^{k_n}}$$

to the element $(0, \ldots, 0, p_i^{k_i - 1}, 0, \ldots, 0)$.