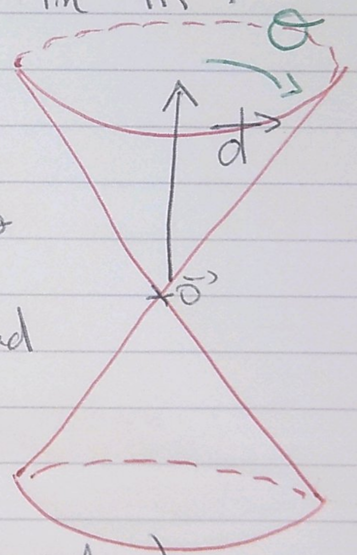
Interlude 1: On finding parametrization of conics

We are going to focus on a family of planar curves — conics — and give parametrizations for them.

WHAT ARE CONICS? (A GEOMETRIC APPROACH)

Consider a <u>revolution cone</u> in $\mathbb{R}^3$.



<u>Notice that:</u>

* any vector $\vec{x}$ that belongs to the cone can be rotated around the $\vec{d}$ axis, and remains on the cone (that's a <u>revolution</u> surface)
* any vector $\vec{x}$ can be multiplied by any scalar λ, and remains on the cone. (that's a <u>cone</u>)

NB: Not every cone is a revolution cone.

* For a cone of axis $\vec{d}$ and angle θ,

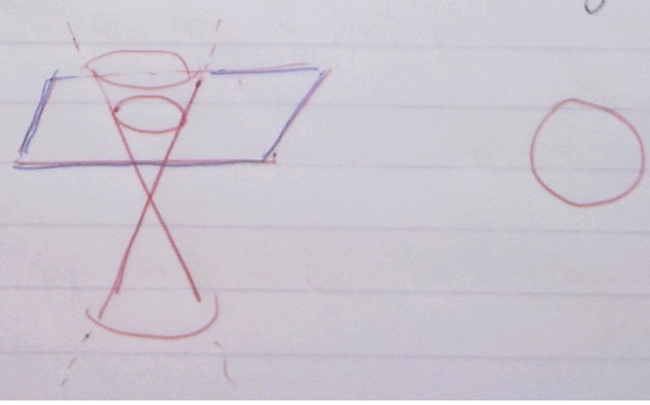
$$\vec{x} \in \text{Cone} \iff \underbrace{|\langle \vec{d}, \vec{x} \rangle| = \cos(\theta)\, \|\vec{d}\| . \|\vec{x}\|}_{\text{an intrinseque equation of the cone.}}$$

(Check-out the expression of an angle to understand that equation → Chapter 1)

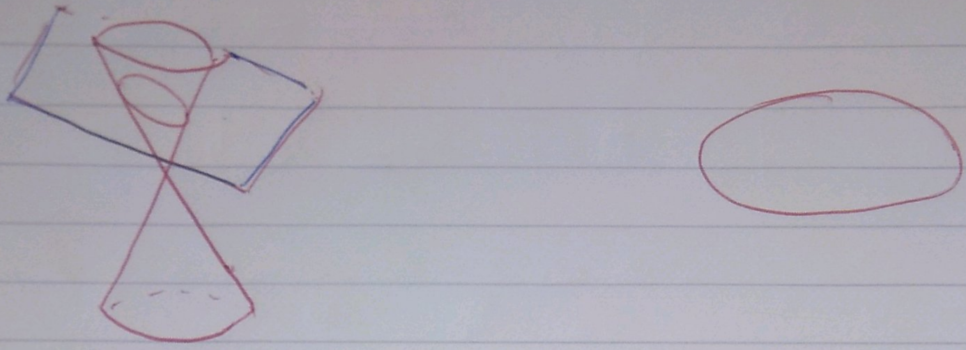
* Now if we intersect our cone with a plane, we obtain a conic section (or simply conic):

>>> Check-out TINYURL.COM/Y9DUVJHG if you want to see for yourself using a 3d visualization.

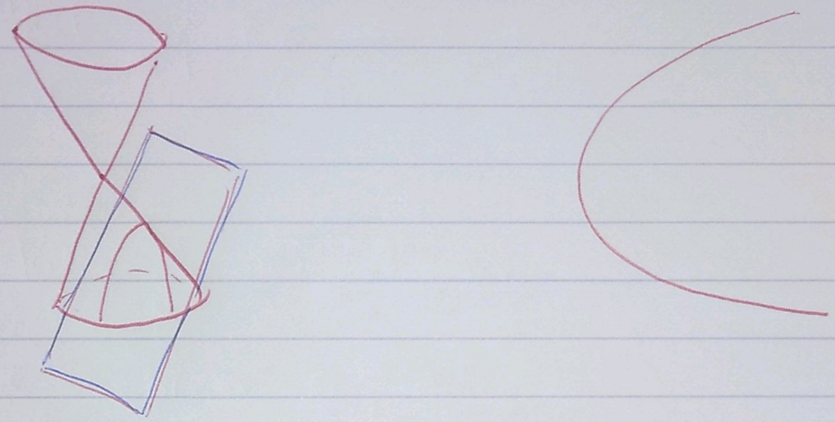
$\boxed{a}$ A circle (intersection with an "horizontal" plane)



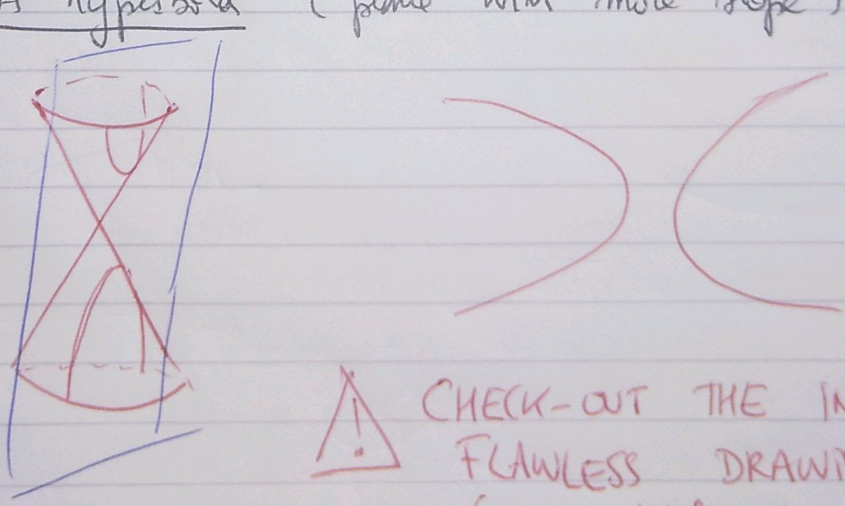
b <u>An ellipse</u> (plane with a slight slope)



c <u>A parabola</u> (plane parallel to a line going through $\vec{o}$ in the cone



d <u>A hyperbola</u> (plane with even more slope)

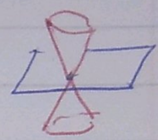


⚠ CHECK-OUT THE INTERNET FOR FLAWLESS DRAWINGS.
(e.g. Wikipedia)

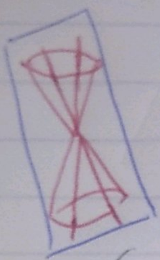
SOME REMARKS:

→ We didn't list degenerated conics.

↳ plane going through $\vec{0}$ gives "a point"



or "two intersecting lines": not very interesting



(The point is not really a curve, the intersecting line have a "singularity" in $\vec{0}$.)

→ The circle is a special case of the ellipse.

→ Ellipses are "closed" curves, parabolas are not closed but have only "one piece". Hyperbolas have "two pieces".

AN ALGEBRAIC APPROACH:

Conics can also be obtained by considering the general quadratic equation:

$$\boxed{Ax^2 + By^2 + Cxy + Dx + Ey + F = 0}$$

(The same classification arises from algebraic properties of the equation.)

Motivations:

→ Why are we even considering this odd family of curves?

1) ↳ They are the "next less complicated curve" after the line.
(From the algebraic point of view, we move from linear equation to quadratic).

2) ↳ They arise as trajectories of celestial bodies (see Kepler's laws).

3) ↳ They are central objects from point 1). Conics have applications in cryptography, related to algebraic geometry.

→ Now that we are into the topic, let us close it and give parametrizations of our conics.

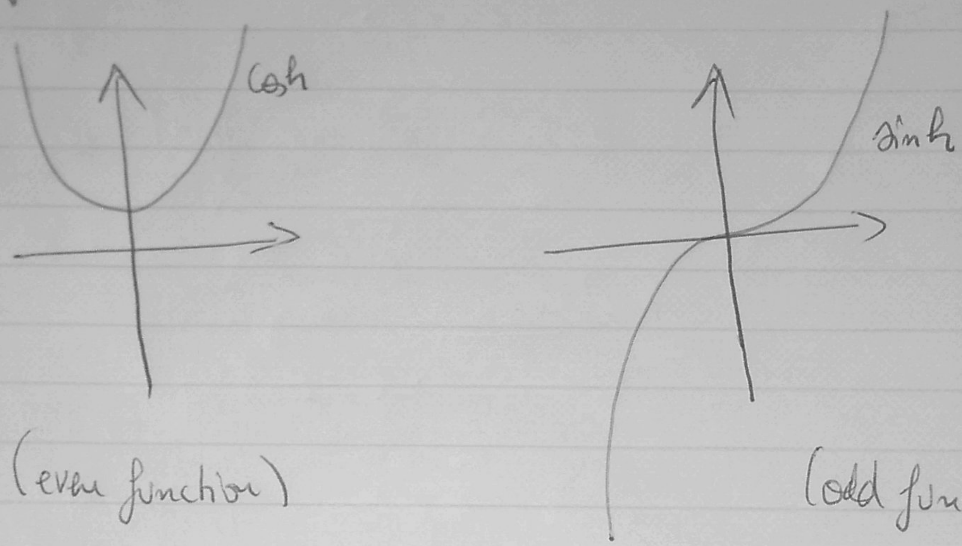
Definition: Hyperbolic cosine is the function

$$\cosh : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \frac{e^{x}+e^{-x}}{2}.$$

Hyperbolic sine is $\sinh : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \frac{e^{x}-e^{-x}}{2}$$

→ They look a bit like "amped" versions of $x \mapsto x^2$ and $x \mapsto x^3$, with <u>exponential growth</u> in $+\infty$ and $-\infty$.



(even function) (odd function)

<u>Property</u>: $\cosh^2(x) - \sinh^2(x) = 1$

(easy to check <u>from the definition</u>)

→ similar to $\cos^2(x) + \sin^2(x) = 1$

(A lot of trigonometric identities have hyperbolic counterparts).

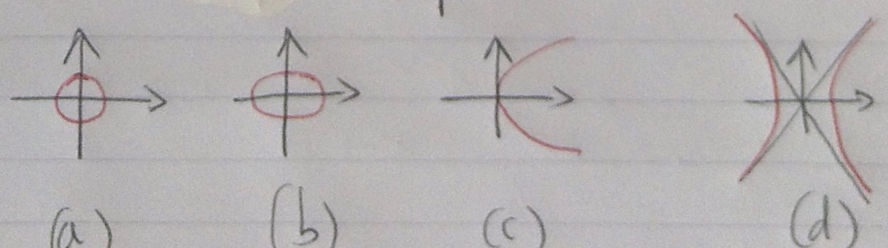
STANDARD INTRINSEQUE EQUATIONS AND PARAMETRIZATIONS OF CONICS

Intrinseque		Parametrization
circle:	$x^2+y^2=a^2$	$\begin{cases} x(\theta)=a\cos\theta \\ y(\theta)=a\sin\theta \end{cases}$
ellipse:	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$	$\begin{cases} x(\theta)=a\cos\theta \\ y(\theta)=b\sin\theta \end{cases}$
parabola:	$y^2=4ax$	$\begin{cases} x(t)=at^2 \\ y(t)=2at \end{cases}$
hyperbola:	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$	$\begin{cases} x(t)=\pm a\cosh(t) \\ y(t)=b\sinh(t) \end{cases}$

REMARKS: → a, b are positive parameters.

→ the variables $\theta, t \in \mathbb{R}$.

→ These equations correspond to conics that are "well aligned" with the axis. Check-out their graphs:



(a) (b) (c) (d)

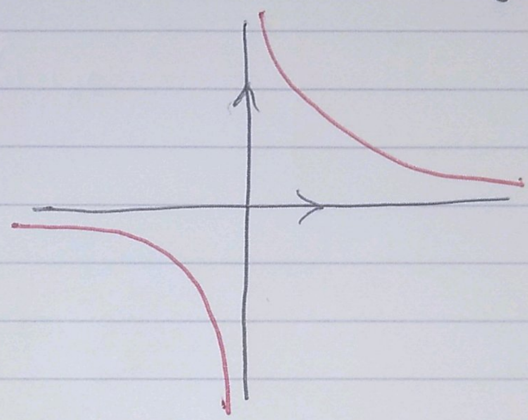
→ There is also a nice parametrization for the rectangle hyperbola (which is to the hyperbola what the circle is to the ellipse: take $a = b$ in the intrinseque equation).

$$\begin{cases} x(t) = ct \\ y(t) = \dfrac{c}{t} \end{cases},$$

and the intrinseque equation

$$xy = c^2$$

Graph:



(Notice the different orientation compared to drawing (d)).

→ ⚠ None of these intrinseque equations or parametrizations are unique, they are just very nice.

→ If we consider conics with any orientation and not centred in $\vec{O}$ we come back to the general equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$$

FROM STANDARD PARAMETRIZATIONS TO ANY ORIENTATION, AND TO SPACE.

A) Consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{rotation}} + \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\text{translation}}$$

→ $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$ is the matrix of the rotation of angle θ.

$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ is a product matrix

whose value is the vector $\begin{bmatrix} \cos(\theta) x + \sin(\theta) y \\ \sin(\theta) x - \cos(\theta) y \end{bmatrix}$.

(that is just $\begin{bmatrix} x \\ y \end{bmatrix}$ rotated by θ)

→ If we have a parametrization $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ of a conic,

$f\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$ allows us to obtain parametrization of any conic, rotated and translated.

B) From the parametrization

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) = 0 \end{bmatrix}$$ of a conic that is in $\mathbb{R}^3$,

but that stays in the $(\vec{x}, \vec{y})$ plane,

we obtain <u>any</u> conic by doing the same, with

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x, y, z) \longmapsto \begin{bmatrix} 3\times3 \\ \text{ROTATION} \\ \text{MATRIX} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Just remember they exist

TRANSLATION IN SPACE.

We have our goal, we now have parametrizations for any conic in the <u>plane</u>, or <u>space</u>.