CROSS DERIVATIVES:

Now that we have

$$\frac{\partial f}{\partial x} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(x,y,z) \longmapsto \left.\frac{\partial f}{\partial x}\right|_{(x,y,z)} ,$$

we are allowed to consider

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) : \mathbb{R}^3 \longrightarrow \mathbb{R} ,$$

and we denote this new function by

$$\boxed{\frac{\partial^2 f}{\partial y \partial x}} \longleftarrow$$ cross-derivative with respect to x and y.

In a similar way we consider

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \text{ denoted by } \boxed{\frac{\partial^2 f}{\partial x^2}} .$$

(double derivative with respect to x)

<u>THEOREM</u>: For $f: \mathbb{R}^3 \to \mathbb{R}$ a "nice function", we have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

<u>Remarks</u>: For "nice enough" functions, we can consider any combination like $\frac{\partial^3 f}{\partial x \partial y \partial z}$, and the order in which we differentiate doesn't matter.

E.g:

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial x^2 \partial y}$$

DIRECTIONAL DERIVATIVES

So far we have been considering derivatives in very specific directions, parallel to the x, y, z axis.

More generally for every $\vec{u} \neq \vec{0}$ we define

$$\boxed{\left.\frac{\partial f}{\partial \vec{u}}\right|_{\vec{x_0}} = \lim_{\Delta t \to 0} \frac{f(\vec{x_0} + \Delta t \vec{u}) - f(\vec{x_0})}{\Delta t},}$$

the <u>directional derivative</u> at $\vec{x_0}$ along the vector $\vec{u}$.

<u>REMARK:</u> → We are fixing a line that follows the direction of $\vec{u}$ at point $\vec{x_0}$, looking at f "on this line only" to obtain a function $g : \mathbb{R} \to \mathbb{R}$ <u>and we differentiate</u>.

* **Scaling:** Consider $\vec{u} \neq \vec{0}$ and $\lambda \neq 0$.

We have

$$\left.\frac{\partial f}{\partial(\lambda\vec{u})}\right|_{\vec{x_0}} = \lim_{\Delta t \to 0} \frac{f(\vec{x_0} + \Delta t \lambda \vec{u}) - f(\vec{x_0})}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \lambda \frac{f(\vec{x_0} + \lambda \Delta t \vec{u}) - f(\vec{x_0})}{\lambda \Delta t}$$

$$= \lambda \lim_{\Delta t \to 0} \frac{f(\vec{x_0} + \lambda\Delta t \vec{u}) - f(\vec{x_0})}{\lambda \Delta t}$$

$$= \lambda \lim_{\Delta t \to 0} \frac{f(\vec{x_0} + \Delta t \vec{u}) - f(\vec{x_0})}{\Delta t}$$

$$= \lambda \left.\frac{\partial f}{\partial \vec{u}}\right|_{\vec{x_0}}$$

↳ Scaling $\vec{u}$ by λ also scales the "speed" $\frac{\partial f}{\partial \vec{u}}$, which makes sense.

REMARK: ⊛ The notation $\frac{\partial f}{\partial \vec{u}}$ with $\vec{u}$ in the bottom is a bit weird: it is just a notation, by no mean are we dividing by $\partial\vec{u}$ here, or $\vec{u}$.

⊛ We will often consider $\frac{\partial f}{\partial \vec{u}}$ for $\|\vec{u}\| = 1$, because it is enough, and natural.

In this case we talk about the derivative in the direction $\vec{u}$.

THEOREM: For $\vec{u} = \begin{bmatrix} x_{\vec{u}} \\ y_{\vec{u}} \\ z_{\vec{u}} \end{bmatrix} \neq \vec{0}$

and $f : \mathbb{R}^3 \to \mathbb{R}$ a "nice" function,

$$\boxed{\left.\frac{\partial f}{\partial \vec{u}}\right|_{\vec{x_0}} = x_{\vec{u}} \left.\frac{\partial f}{\partial x}\right|_{\vec{x_0}} + y_{\vec{u}} \left.\frac{\partial f}{\partial y}\right|_{\vec{x_0}} + z_{\vec{u}} \left.\frac{\partial f}{\partial z}\right|_{\vec{x_0}}}$$

REMARKS:

⊛ This is the formula we use to compute a directional derivative!

⊛ <u>In matrix notation:</u>

$$\boxed{\left.\frac{\partial f}{\partial \vec{u}}\right|_{\vec{x_0}} = \left.\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]\right|_{\vec{x_0}} \cdot \begin{bmatrix} x_{\vec{u}} \\ y_{\vec{u}} \\ z_{\vec{u}} \end{bmatrix}}.$$

This comes from the fact that

$$\left.\frac{\partial f}{\partial \vec{u}}\right|_{\vec{x_0}} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\vec{u} \longmapsto \left.\frac{\partial f}{\partial \vec{u}}\right|_{\vec{x_0}}$$

⚠ Here $\vec{x_0}$ is fixed and $\vec{u}$ is the variable!

is a <u>linear mapping</u>.