MS321 Algebra, tutorial 7, question 4

4. Suppose G is a group and define the set

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } \ g \in G\},$$

that is, the subset of G consisting of those elements which commute with all elements of G. Show that $Z(G)$ is a subgroup of G. Show that $Z(G)$ is normal in G.

(a) The element $e \in Z(G)$ since, for all $g \in G$, $ge = g$ and $eg = g$ so that $eg = ge$.
(b) Suppose $x, y \in Z(G)$ and let $g \in G$. Then

$$(xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy),$$

where the second equality follows from $y \in Z(G)$ and the fourth equality follows from $x \in Z(G)$. Thus $xy \in Z(G)$.
(c) Suppose $x \in Z(G)$ and let $g \in G$. Then

$$x^{-1}g = (g^{-1}x)^{-1} = (xg^{-1})^{-1} = gx^{-1},$$

where the second equality follows from $x \in Z(G)$. Thus $x^{-1} \in Z(G)$.
(a), (b) and (c) give $Z(G) < G$.

Finally, if $x \in Z(G)$ and $g \in G$, then $g^{-1}xg = g^{-1}gx = x \in Z(G)$, so that $Z(G)$ is normal in G.